Local and global bifurcations in a simple model of digitally controlled systems

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Summary. Chaotic vibrations that are originated from the digital implementation of control are known for more than 20 years [1]. The consideration of the effects of sampling and processing delay is a routine procedure among control engineers, but the quantization in the analogue-digital converters is usually neglected. However, this latter effect frequently leads to deterministic but small scale chaotic behaviour – so-called micro-chaos [2, 3] – which is often considered simply as stochastic noise in the practice. In the present contribution a simple but quite general PD-controlled system is considered. It is shown that micro-chaotic oscillations may have a considerable influence on the accuracy and settling time of control systems. We focus on the numerical and analytical examination of global bifurcations since these may lead to the abrupt change of the aforementioned properties of the control system.

Introduction

Consider a linear oscillator with negative stiffnesswhose equation of motion is linearized about the upper equilibrium:

$$M\ddot{x}(t) + c\dot{x}(t) - kx(t) = 0, \tag{1}$$

where M denotes (generalized) mass, c is the damping coefficient, while $k > 0$ is the (generalized) stiffness. PD control is applied on the system in order to stabilize the position $[x, \dot{x}] = [0, 0]$, according to the output of a digital control system. The control force is assumed to be kept constant between the jth and $(j + 1)$st samping instants. The samping time is denoted by τ and we assume that the control force is sent out without delay. Due to the sampling of the state and the quantisation of the control force with resolution h, the equation of motion of the controlled system assumes the form

$$\ddot{x}(t) + \frac{c}{M}\dot{x}(t) - \frac{k}{M}x(t) = -\frac{h}{M}\mathrm{Int}\left(\frac{\tilde{P}x_j + \tilde{D}\dot{x}_j}{h}\right), \quad t \in [j\tau, (j+1)\tau), \tag{2}$$

where function $\mathrm{Int}()$ rounds toward zero. Introducing dimensionless variables one arrives at the equation

$$y''(T) + 2\beta y'(T) - \alpha^2 y(T) = -\mathrm{Int}\left(Py_j + Dy'_j\right), \tag{3}$$

where $T \in [j, (j + 1))$ and $y' \equiv dy/dT$. Equation (3) can be solved between the successive sampling instants. Consequently, the solutions can be given by the following mapping – the micro-chaos map:

$$\mathbf{y_{j+1}} = \mathbf{U}\mathbf{y_j} + \mathbf{b}\,\mathrm{Int}\left(Py_j + Dy'_j\right) \equiv \mathbf{U}\mathbf{y_j} + \mathbf{b}\,m, \tag{4}$$

where $m = \mathrm{Int}\left(Py_j + Dy'_j\right)$ and $\mathbf{y_j} \equiv \mathrm{col}[y_j, y'_j]$. By introducing the notations $s \equiv \sinh(\gamma)$, $c \equiv \cosh(\gamma)$, $e \equiv \exp(\beta)$, and $\gamma^2 \equiv \alpha^2 + \beta^2$, one obtains

$$\mathbf{U} = \frac{1}{e\gamma}\begin{bmatrix} \gamma c + \beta s & s \\ \alpha^2 s & \gamma c - \beta s \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{e-c}{e\alpha^2} - \frac{\beta s}{e\gamma\alpha^2} \\ -\frac{s}{e\gamma} \end{bmatrix}. \tag{5}$$

Phase-space structure

According to the value of the control force m, the phase-space can be divided into parallel bands that are separated by switching lines:

$$y'_j = \frac{m - Py_j}{D}.$$

The initial points of possible periodic orbits of length p – that visit the bands $m_1, m_2 \ldots m_p$ – can be found by the following formula:

$$\mathbf{y_1} = (\mathbf{I} - \mathbf{U}^p)^{-1}\left(m_1\mathbf{U}^{p-1} + m_2\mathbf{U}^{p-2} + \cdots + m_p\mathbf{U}^0\right)\mathbf{b}.$$

Note, that $\mathbf{y_i}$ must be in band m_i. For example, the fixed point $\mathbf{r}^m = \mathrm{col}\left[m/\alpha^2, 0\right]$ must be in band m, thus

$$\frac{|m|}{P} \leq \frac{|m|}{\alpha^2} < \frac{(|m|+1)}{P}.$$

If the fixed point crosses the switching line, it becomes virtual via a border collision bifurcation. Since all the fixed points are saddles, strange attractors or repellers can be formed between neighbouring fixed points (and between a fixed point and a virtual fixed point). These primary structures can merge with each other, forming new, secondary strange attractors. The micro-chaos map is similar to a series of Baker maps in a large portion of the domain of stability. In this case

- The attractor farthest from the origin is close to the $m_{\max}$th switching line, at position $y^{SW}_{\max}$, where $m_{\max}$ is the index of the virtual fixed point that is closest to the origin:

$$m_{\max} = \mathrm{Int}\left(\frac{P}{P-\alpha^2}\right), \quad y^{SW}_{\max} = \frac{m_{\max}}{P}.$$

- If the size of the primary attractors is small (i.e., the trajectory cannot jump over a whole band and cannot escape from the structure at the virtual fixed point towards larger displacements), $y^{SW}_{\max}$ provides an estimation for the maximal control error.

Another estimation method is based on the fact that the micro-chaos map (4) can be rewritten to the following form:

$$\mathbf{y_{j+1}} = \mathbf{S}^j\mathbf{y_0} - \sum_{k=0}^{j-1}\mathbf{S}^k\mathbf{c}\chi_k, \quad \chi_k \in (-1,1), \tag{6}$$

where $\mathbf{S}$ would be the coeffcient matrix of the continuous, controlled system and χ_k corresponds to the fractional part of the control force. Because $\lim_{j\to\infty}\mathbf{S}^j\mathbf{y_0} = \mathbf{0}$, the maximal possible control error can be estimated by the proper choice of the numbers χ_k. If the eigenvalues of $\mathbf{S}$ are positive real numbers, $\chi_k = 1, \forall k$ provides the maximal error. In this case,

$$-\sum_{k=0}^{\infty}\mathbf{S}^k\mathbf{c} = -\mathbf{T}\begin{bmatrix}\frac{c_1}{1-\mu_1}\\ \frac{c_2}{1-\mu_2}\end{bmatrix} = \begin{bmatrix}\frac{1}{P-\alpha^2}\\ 0\end{bmatrix}.$$

μ_i, $i = 1,2$ denote the eigenvalues of $\mathbf{S}$, matrix $\mathbf{T}$ transforms $\mathbf{S}$ to diagonal form and c_i are the elements of vector $\mathbf{c}$ from (6), transformed with $\mathbf{T}$. Thus, the maximal distance from the origin is $y_\infty = 1/(P-\alpha^2)$.

As Figure 1 shows, the two methods provide quite similar results. Certainly, the control error will be less than these estimates, especially if the farthest Baker map-like structure turns to a repeller. This favourable situation can occur quite frequently in the practice, as it is also illustrated in the figure. On the other hand, the choice $\chi_k = 1, \forall k$ does not provide the maximal control error if the eigenvalues of $\mathbf{S}$ are complex or negative real numbers. Our goal is to give a more general estimate for the control error and to describe the bifurcations that lead to the merging of primary attractors.

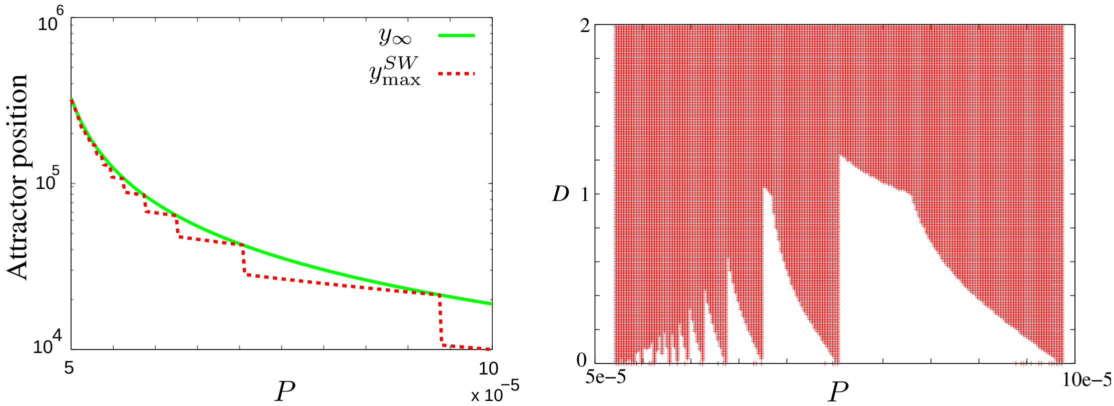

Figure 1: The estimated control error at realistic parameters $\alpha = 6.85e-3$, $\beta = 0$ and $D = 2.566e-3$ (left panel) and the parameters where the farthest attractor disappears or merges with other attractors (red on the right panel).

Conclusions

We showed that several disconnected attractors may coexist in a simple model of digitally controlled systems. The position of the attractor, farthest from the origin, was estimated by two different methods that led to similar results. These methods can be used for the estimation of the largest possible position error of the control system in a broad parameter domain.

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