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Rainbow matchings and transversals*

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Abstract

We show that there exists a bipartite graph containing n matchings of sizes $m_i \geqslant n$ satisfying $\sum_i m_i = n^2 + \lfloor n/2 \rfloor - 1$, such that the matchings have no rainbow matching. This answers a question posed by Aharoni, Charbit and Howard.

We also exhibit $(n-1) \times n$ latin rectangles that cannot be decomposed into transversals, and some related constructions. In the process we answer a question posed by Häggkvist and Johansson.

Finally, we propose a Hall-type condition for the existence of a rainbow matching.

Two edges in a graph are *independent* if they do not share an endpoint. A *matching* is a set of edges that are pairwise independent. If $M_1, M_2, \ldots, M_n$ are matchings and there exist edges $e_1 \in M_1, e_2 \in M_2, \ldots, e_n \in M_n$ that form a matching then we say that $\{M_1, M_2, \ldots, M_n\}$ possesses a *rainbow matching*. Note that we do *not* require the matchings M_i to be disjoint, so it may even be the case that $e_j \in M_i$ for some $j \neq i$.

A $k \times n$ matrix R, containing symbols from an alphabet Λ, can be viewed as listing edges in a bipartite graph G_R. The two parts of G_R correspond respectively to the columns of R, and to Λ. Each cell in R corresponds to an edge in G_R that links the column and the symbol in the cell. A *partial transversal of length ℓ* in R is a selection of ℓ cells of R that contain ℓ different symbols and come from ℓ

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different rows and ℓ different columns. A partial transversal of length $\min(k, n)$ is a *transversal* of R. The study of transversals of this kind has a rich history [10]. Of particular interest are cases where further structure is imposed on the matrix. We say that R is

- *generalised row-latin* if no symbol in Λ is repeated within any row,
- *row-latin* if it is generalised row-latin and $n = |\Lambda|$,
- *generalised latin* if no symbol is repeated in any row or in any column,
- *latin* if it is generalised latin and $n = |\Lambda|$.

In a generalised row-latin rectangle R, each row specifies a matching in G_R. If $k \leqslant n$ then transversals of R are in bijective correspondence with rainbow matchings for the set of matchings defined by the rows of R. For this reason, rainbow matchings and transversals are intimately connected. We will change back and forth between the two concepts freely.

In any instance where $k = n$ we refer to R as a square. The following conjecture, attributed variously to Brualdi and Stein, is one of the most prominent questions in the area [10].

Conjecture 1 *Every $n \times n$ latin square contains a partial transversal of length $(n-1)$.*

The following conjecture from [1] implies Conjecture 1.

Conjecture 2 *Any k matchings of size $k+1$ in a bipartite graph have a rainbow matching.*

The following theorem is due to Drisko [3], although the proof we offer is in the spirit of the Delta Lemma, which has proved so useful in studying transversals [10].

Theorem 3 *Suppose $m \geqslant 2$. Let R be the $(2m-2)\times m$ row-latin rectangle consisting of $m-1$ rows that are equal to $[1, 2, 3, \dots, m]$ followed by $m-1$ rows that are equal to $[2, 3, \dots, m, 1]$. Then R has no transversal.*

Proof: Define a function $\Delta : R \mapsto \mathbb{Z}_m$ on the cells in R by $\Delta(r, c) \equiv R_{rc} - c \mod m$. Suppose T is a transversal of R. Now T uses every symbol and every column exactly once so the sum, S, of Δ over the entries of T must be $\sum_{i\in\mathbb{Z}_m} i - \sum_{i\in\mathbb{Z}_m} i = 0$. However, Δ is zero on every entry in the first $m-1$ rows, and one on every entry thereafter. As T uses m different rows, we see that S cannot be zero. This contradiction proves the result. □

In formulating Conjecture 2, Aharoni et al. [1] pondered how much it may be able to be strengthened. Our next result shows there is not as much slack as might have been suspected. For example, it shows that there is a set of $\lfloor k/2\rfloor - 1$ matchings of size $k+1$ and $k - \lfloor k/2\rfloor + 1$ matchings of size k, that together have no rainbow matching.

Theorem 4 *Let n be a positive integer and $m_1, m_2, \ldots, m_n$ any integers satisfying $m_i \geqslant n$ for each i, and $\sum_i m_i \leqslant n^2 + \lfloor n/2 \rfloor - 1$. Then there exists a bipartite graph containing matchings $M_1, M_2, \ldots, M_n$ of respective sizes $m_1, m_2, \ldots, m_n$ that have no rainbow matching.*

Proof: The two parts of our bipartite graph will have vertices $\{u_i\}_{i \geqslant 1}$ and $\{v_i\}_{i \geqslant 1}$ respectively. We start by considering matchings $N_1, \ldots, N_n$, each of size n, constructed as follows. Let $h = \lfloor n/2 \rfloor$. Let $N_1, \ldots, N_h$ each consist of the edges $u_1v_1, u_2v_2, \ldots, u_nv_n$. Let $N_{h+1}, \ldots, N_n$ each consist of the edges $u_1v_2, u_2v_3, \ldots, u_{n-h}v_{n-h+1}u_{n-h+1}v_1$ together with the edges $u_{n-h+2}v_{n-h+2} \ldots, u_nv_n$. Next we build matchings M_i by modifying N_i according to the following algorithm.

1: $t := n - h + 1$
2: **for** i from 1 to n **do**
3:   **for** j from 1 to $m_i - n$ **do**
4:     Increment t
5:     Remove edge u_tv_t from N_i and add edges u_tv_{h+t-1} and $u_{h+t-1}v_t$
6:   **end for**
7:   $M_i := N_i$
8: **end for**

The assumption that $\sum_i m_i \leqslant n^2 + \lfloor n/2 \rfloor - 1$ ensures that $n - h + 2 \leqslant t \leqslant n$ whenever Step 5 is reached. Each time Step 5 is performed it increases the size of N_i by 1, which means that M_i ends up having size m_i, as desired.

Suppose that $\{M_1, M_2, \ldots, M_n\}$ has a rainbow matching $\mathscr{M}$. For each t in the range $n - h + 2 \leqslant t \leqslant n$ there can be at most one of the edges u_tv_t, u_tv_{h+t-1} and $u_{h+t-1}v_t$ in $\mathscr{M}$. Hence $\mathscr{M}$ includes $h + 1$ edges between $u_1, u_2, \ldots, u_{h+1}$ and $v_1, v_2, \ldots, v_{h+1}$. However, this is impossible, by Theorem 3. This contradiction finishes the proof. $\square$

The hypothesis that $m_i \geqslant n$ for each i cannot be omitted from Theorem 4. For example, suppose that $n = 4$, $m_1 = 2$, $m_2 = 3$, $m_3 = 5$ and $m_4 = 7$. Any four matchings of sizes $m_1, \ldots, m_4$ have a rainbow matching that can be found by a simple greedy algorithm. Make any choice of an edge from each matching in turn, subject only to the constraint that the chosen edges form a matching. This process is bound to succeed, since each chosen edge eliminates at most two choices from each subsequent matching and $m_i > 2(i - 1)$ for each i.

Although there are important exceptions [4, 5, 7, 11], it seems to often be the case that the existence of one rainbow matching is enough to ensure that a set of matchings of equal size can be *partitioned* into rainbow matchings. Häggkvist and Johansson [8] asked for an $(n-1) \times n$ latin rectangle without a decomposition into transversals. We searched catalogues of small candidates and found a number of examples with $n \in \{5, 6, 7\}$. Here we give one example for each of those orders.

1*	2*	3	4	5
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	1	3	2
5	6	4	2	1	3

1	2	3	6	7	4	5
2	3	1	7	6	5	4
3	1	2	4	5	7	6
6	7	5	2	4	3	1
7	6	4	5	2	1	3
4	5	7	1	3	6	2

It is easy to confirm the validity of the example for $n = 5$ by hand. It suffices to check that the only transversals that include either of the asterisked entries include all of the shaded entries. In the $n = 6$ example it is not possible to cover more than two of the entries in any of the four shaded blocks of three entries. Hence it is not even possible to find $n - 1$ disjoint transversals. For $n \neq 6$ we found $n - 1$ disjoint transversals in all the candidates that we tested.

Moreover, we checked exhaustively and found that every 7×8 latin rectangle can be decomposed into transversals. We also tried to generalise the structure of some small examples to $n > 8$, but without success. Thus it remains an interesting open question as to whether there are only finitely many examples of the type sought by Häggkvist and Johansson.

If we relax the conditions slightly it becomes easy to build infinite families of examples, as our next two results show.

Theorem 5 *For any $n \geqslant 3$ there exists an $(n-1) \times n$ row-latin rectangle that cannot be decomposed into transversals.*

Proof: Let R be the row-latin rectangle consisting of $n - 2$ rows that are copies of $[1, 2, \dots, n]$, followed by a single row $[2, 3, \dots, n-2, n-1, 1, n]$. Now consider any $x \in \{1, 2, \dots, n-1\}$. A transversal that includes symbol x from the last row cannot use any entry in column x, so it has to use an entry in the last column. However there are only $n - 2$ entries in the last column outside the last row, and these have to be paired with the $n - 1$ different values of x in any decomposition of R into transversals. Hence R cannot be decomposed into transversals. □

An interesting counterpoint to Theorem 5 is the unresolved conjecture by Stein ([9], see also [10]) that any $(n-1) \times n$ row-latin rectangle has a transversal.

Theorem 6 *For any odd $n \geqslant 3$ there exists an $(n-1) \times n$ generalized latin rectangle that cannot be decomposed into transversals.*

Proof: Since $n - 1$ is even, there exists a latin square L of order $n - 1$ that contains no transversal [10]. Adjoin to L a new column consisting of symbols that are distinct from each other and from the symbols in L. This produces an $(n-1) \times n$ generalized latin rectangle. It cannot be decomposed into transversals, since any such decomposition would have to include one transversal of L. □

Akbari and Alireza [2] define $l(n)$ to be the least number of symbols in an $n \times n$ generalised latin rectangle that guarantees a transversal. Erdős and Spencer [6] showed that if no symbol occurs more than $n/16$ times in an $n \times n$ matrix then the matrix has a transversal. Hence, if all symbols occur roughly equally frequently then there is a linear bound on the number of different symbols needed to guarantee a transversal. Akbari and Alireza conjecture that $l(n) \leqslant n^2/2$ and that there is no constant c such that $l(n) \leqslant n + c$ for all n. The first part of this claim seems at first sight to be highly plausible, maybe even generous, but the following related result is worth bearing in mind.

Theorem 7 *For any $n > 1$ there exists an $n \times n$ generalised row-latin rectangle that contains $n^2 - (n-1)(\lceil n/2 \rceil + 1)$ different symbols and has no transversal.*

Proof: Theorem 3 gives a construction for an $n \times (\lceil n/2 \rceil + 1)$ row-latin rectangle that has no transversal (use $m = \lceil n/2 \rceil + 1$ and omit one row if n is odd). Appending $n - \lceil n/2 \rceil - 1$ columns in which each cell contains a new symbol gives a generalised row-latin square that has no transversal. It contains $\lceil n/2 \rceil + 1 + (n - \lceil n/2 \rceil - 1)n$ different symbols. $\square$

In other words, if we allow repeats in columns but not in rows then there are $n \times n$ matrices with $n^2/2 - O(n)$ different symbols and no transversal. Of course, if you abandon any latin constraint and look at arbitrary matrices, it is easy to have $n^2 - 2n + 1$ different symbols and no transversal. Simply take two entire rows containing just one symbol, and have all other symbols distinct.

We finish by considering the following Hall-type condition for a set M of matchings in a bipartite graph:

$$\begin{array}{l}\text{For every subset } S \subseteq M \text{ with } |S| = k,\\ \text{the total size of } S \text{ is at least } h(k).\end{array} \tag{$*$}$$

Here, the total size of S is the sum of the sizes of the matchings in S. We want to find a function h, as small as possible, such that M must have a rainbow matching provided it satisfies $(*)$ for all k. We choose to define h inductively. Suppose that we have decided the value of $h(i)$ for $i < k$. We know that $h(k) \leqslant 2k^2 - 2k + 1$ since among k matchings of total size $2k^2 - 2k + 1$ there is one, say M_0, of size at least $2k - 1$. By the induction hypothesis, we can find a rainbow matching $\mathscr{M}$ in the matchings other than M_0. Each of the $k - 1$ edges in $\mathscr{M}$ meets at most two edges of M_0. So there is an edge of M_0 disjoint from all the edges in $\mathscr{M}$, justifying our bound on $h(k)$. Hence, up to isomorphism there are only finitely many relevant examples to consider when deciding $h(k)$. We simply choose $h(k)$ to be the smallest value that works for this finite list of examples.

It may be that $h(k) = k^2 + k - 1$ for all k. The following example with no rainbow matching shows that it cannot be any smaller:

$$\begin{aligned} M_1 &= \{u_1v_2, u_2v_1\} \\ M_2 &= \{u_1v_1, u_2v_2, u_3v_4, u_4v_3\} \\ &\vdots \\ M_i &= \{u_1v_1, \dots, u_{2i-2}v_{2i-2}, u_{2i-1}v_{2i}, u_{2i}v_{2i-1}\} \\ &\vdots \\ M_{k-1} &= \{u_1v_1, \dots, u_{2k-4}v_{2k-4}, u_{2k-3}v_{2k-2}, u_{2k-2}v_{2k-3}\} \\ M_k &= \{u_1v_1, \dots, u_{2k-2}v_{2k-2}\}. \end{aligned}$$

Try to form a rainbow matching by choosing an edge from each matching in turn. By induction for $j < k$, the edge chosen from M_j blocks two edges from all subsequent matchings. Hence there are no choices available when we reach M_k. There are

$$2 + 4 + 6 + \cdots + (2k-2) + (2k-2) = k^2 + k - 2$$

edges in total among the matchings.

In the above it would also be possible to remove the constraint that the underlying graph is bipartite. It is not clear how much this changes the value of $h(k)$. However, $\{01, 23, 45, 67\}$, $\{02, 13, 46, 57\}$, $\{03, 12, 47, 56\}$ are 3 matchings of total size 12 without a rainbow matching.

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References

[1] R. Aharoni, P. Charbit and D. Howard. On a generalization of the Ryser-Brualdi-Stein conjecture, *J. Graph Theory* (to appear).

[2] S. Akbari and A. Alireza, Transversals and multicolored matchings, *J. Combin. Des.* **12** (2004), 325–332.

[3] A. A. Drisko, Transversals in row-latin rectangles, *J. Combin. Theory Ser. A* **84** (1998), 181–195.

[4] J. Egan, Bachelor latin squares with large indivisible plexes, *J. Combin. Des.* **19** (2011), 304–312.

[5] J. Egan and I. M. Wanless, Latin squares with restricted transversals, *J. Combin. Designs* **20** (2012), 344–361.

[6] P. Erdős and J. Spencer, Lopsided Lovász local lemma and latin transversals, *Discrete Appl. Math.* **30** (1991), 151–154.

[7] A. B. Evans, Latin squares without orthogonal mates, *Des. Codes Cryptogr.* **40** (2006), 121–130.

[8] R. Häggkvist and A. Johansson. Orthogonal latin rectangles. *Combin. Probab. Comput.* **17** (2008), no. 4, 519–536.

[9] S. K. Stein. Transversals of latin squares and their generalizations. *Pacific J. Math.* **59** (1975), 567–575.

[10] I. M. Wanless, "Transversals in latin squares: A survey", in R. Chapman (ed.), *Surveys in Combinatorics 2011*, London Math. Soc. Lecture Note Series **392**, Cambridge University Press, 2011, pp. 403–437.

[11] I. M. Wanless and B. S. Webb, The existence of latin squares without orthogonal mates, *Des. Codes Cryptogr.* **40** (2006), 131–135.

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