Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2010-10. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

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November 2010

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Abstract

We prove two results on packing common bases of two matroids. First, we show that the computational problem of common base packing reduces to the special case where one of the matroids is a partition matroid. Second, we give a counterexample to a conjecture of Chow, which proposed a sufficient condition for the existence of a common base packing. Chow's conjecture is a generalization of Rota's basis conjecture.

1 Introduction

Let $\mathbf{M}_1 = (S, \mathcal{I}_1)$ and $\mathbf{M}_2 = (S, \mathcal{I}_2)$ be matroids on ground set S, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the respective families of independent sets. A set $B \subseteq S$ that is both a base of $\mathbf{M}_1$ and of $\mathbf{M}_2$ is called a ***common base***. The problem PartitionIntoCommonBases is to decide if S can be partitioned into common bases. Two well-studied special cases of this problem include edge-coloring bipartite graphs and packing arborescences in digraphs.

The computational complexity of PartitionIntoCommonBases is unclear. In particular, the answers to the following questions are unknown.

- If each matroid is given by an oracle which tests independence in the matroid, is there an algorithm which solves the problem using a number of queries which is polynomial in $|S|$?
- If each matroid is linear and given by an explicit matrix representing the matroid, is there an algorithm which solves the problem using a number of steps which is polynomial in the size of this matrix?

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Supported by an NSERC Discovery Grant.
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For the two special cases mentioned above, both of these questions have a positive answer; this follows from results of Kőnig [11], Tarjan [16] and Lovász [12]. The latter two results give an efficient, constructive proof of a min-max relation originally proved by Edmonds [6].

Another problem related to packing matroid bases is Rota's basis conjecture.

Conjecture 1.1 (Rota, 1989)**.** *Let* $\mathbf{M} = (T, \mathcal{I})$ *be a matroid of rank* n*. Let* $A_1, \ldots, A_n$ *be a partition of* T *into bases of* $\mathbf{M}$*. Then there are disjoint bases* $B_1, \ldots, B_n$ *such that* $|A_i \cap B_j| = 1$ *for every* $i = 1, \ldots, n$ *and* $j = 1, \ldots, n$*.*

Rota's conjecture remains open. It is explicitly stated in the work of Huang and Rota [10, Conjecture 4]. An equivalent statement is as follows. Let $\mathbf{M}_1$ be a matroid of rank n and let $A_1, \ldots, A_n$ be disjoint bases of $\mathbf{M}_1$. Let $\mathbf{M}_2$ be the partition matroid with parts $A_1, \ldots, A_n$. Then the solution to PartitionIntoCommonBases for $\mathbf{M}_1$ and $\mathbf{M}_2$ is "yes".

Recently, Chow [1] proposed the following generalization of Rota's conjecture.

Conjecture 1.2. *Let* $\mathbf{M} = (T, \mathcal{I})$ *be a matroid of rank* n *with the property that* T *can be partitioned into* b *bases, where* $3 \leq b \leq n$*. Let* $I_1, \ldots, I_n \in \mathcal{I}$ *be disjoint independent sets, each of size* b*. Then there are disjoint bases* $B_1, \ldots, B_b$ *such that* $|I_i \cap B_j| = 1$ *for every* $i = 1, \ldots, n$ *and* $j = 1, \ldots, b$*.*

Obviously Chow's conjecture implies Rota's conjecture, by setting $b = n$. A stronger statement is also true: Chow [1] proved that, for *every* value of b, his conjecture implies Rota's conjecture. In particular, this suggests an interesting approach to proving Rota's conjecture, which is to prove Chow's conjecture in the special case $b = 3$.

Chow had originally proposed a stronger conjecture [1], which unfortunately turned out to be false. This stronger conjecture is as follows.

Conjecture 1.3. *Let* $\mathbf{M} = (S, \mathcal{I})$ *be a matroid of rank* m *with the property that* S *can be partitioned into* b *bases, where* $3 \leq b \leq m$*. Let* $A_1, \ldots, A_m$ *be disjoint sets, each of size* b*. Then there are disjoint bases* $B_1, \ldots, B_b$ *such that* $|A_i \cap B_j| = 1$ *for every* $i = 1, \ldots, m$ *and* $j = 1, \ldots, b$*.*

The reason that Conjecture 1.2 and Conjecture 1.3 require $b \geq 3$ is that they are false if $b = 2$, as is shown by a well-known instance based on the graphic matroid of the complete graph K_4. See, e.g., [5], [13, Exercise 12.3.11(ii)] or [14, Section 42.6c]. A slightly modified example due to Colin McDiarmid gives a counterexample to Conjecture 1.3 for all $b \geq 3$; this is described in [1] and in Appendix A.

1.1 Our Results

This paper contains two related results. First, we give a reduction from PartitionIntoCommonBases for arbitrary $\mathbf{M}_1$ and $\mathbf{M}_2$ to the special case of the problem in which $\mathbf{M}_2$ is a partition matroid. This reduction is efficiently computable, implying the following statement.

Theorem 1.4. *The general problem* PartitionIntoCommonBases *can be solved in polynomial time if and only if this is true under the additional assumption that* $\mathbf{M}_2$ *is a partition matroid.*

This shows that the computational difficulty of PartitionIntoCommonBases does not stem from the interaction of two potentially complicated matroids — the problem is equally difficult when one of the matroids is very simple.

Our second result disproves Chow's conjecture.

Theorem 1.5. *Conjecture 1.2 is false for every* b *such that* $2 \le b \le n/3$.

In fact, we give two proofs of Theorem 1.5. The first proof, given in Section 3, shows that Conjecture 1.3 and Conjecture 1.2 are actually equivalent: given any counterexample to Conjecture 1.3 with parameters b, m and ground set size $|S|$, we obtain a counterexample to Conjecture 1.2 with parameters b, $n = |S|$, and ground set size $|T| = b \cdot |S|$. The second proof, given in Section 4, uses a connection between packing common bases and packing dijoins. Chow's conjecture remains open when $b > n/3$; in particular, Rota's conjecture remains open.

By combining our two results, we obtain the following refinement.

Corollary 1.6. *The problem* PartitionIntoCommonBases *can be solved in polynomial time if and only if this is true under the additional assumption that* $\mathbf{M}_2$ *is a partition matroid whose parts are each independent in* $\mathbf{M}_1$.

2 Packing common bases and partition matroids

In this section we prove Theorem 1.4. Suppose we are given two matroids $\mathbf{M}_1$ and $\mathbf{M}_2$ on a ground set S. We will show how to construct a matroid $\mathbf{M}$ and a partition matroid $\mathbf{M}_P$ on a ground set $S \cup \bar{S}$ such that S can be partitioned into common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$ if and only if $S \cup \bar{S}$ can be partitioned into common bases of $\mathbf{M}$ and $\mathbf{M}_P$.

We may assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ contain no loops, their rank is the same number r, they have at least one common base, and $|S|$ is a multiple of r, say $|S| = (k+1) \cdot r$. These assumptions can easily be tested in polynomial time, and if they do not hold then the solution to PartitionIntoCommonBases is "no".

The new matroids $\mathbf{M}$ and $\mathbf{M}_P$ will have a larger ground set, which we now begin to define. Let $[k]$ denote the set $\{1, \ldots, k\}$. We define a new set $\bar{S} = S \times [k]$ where $\times$ denotes Cartesian product. Any subset $A \subseteq S$ is extended to a subset $\bar{A} \subseteq \bar{S}$ by taking $\bar{A} = A \times [k]$. Similarly, for any $s \in S$, let $\bar{s} = \{s\} \times [k]$. Conversely, the projection onto S of any subset $A \subseteq \bar{S}$ is

$$\pi(A) \;=\; \{\, s \in S \,:\, \exists x \in [k] \;\text{ s.t. }\; (s,x) \in A \,\}.$$

The dual matroid of $\mathbf{M}_2$ is denoted $\mathbf{M}_2^* = (S, \mathcal{I}_2^*)$. We define a new matroid $\mathbf{M}' = (\bar{S}, \mathcal{I}')$ as the matroid obtained from $\mathbf{M}_2^*$ by replacing each element by k parallel elements, i.e.

$$\mathcal{I}' \;=\; \left\{\, I \subseteq \bar{S} \,:\, \pi(I) \in \mathcal{I}_2^*,\; |I \cap \bar{s}| \le 1 \;\; \forall s \in S \,\right\}.$$

Note that the rank of $\mathbf{M}'$ is the same as the rank of $\mathbf{M}_2^*$, which is $|S| - r$.

Finally we can define $\mathbf{M}$ and $\mathbf{M}_P$. We let $\mathbf{M} = (S \cup \bar{S}, \mathcal{I})$ be the direct sum of $\mathbf{M}_1$ and $\mathbf{M}'$. For each $s \in S$, define $\hat{s} = \{s\} \cup \bar{s}$. Then $\mathbf{M}_P = (S \cup \bar{S}, \mathcal{I}_P)$ is the partition matroid whose parts are the sets $\hat{s}$, i.e.,

$$\mathcal{I}_P \;=\; \left\{\; I \subseteq S \cup \bar{S} \,:\, |I \cap \hat{s}| \leq 1 \;\; \forall s \in S \;\right\}.$$

Note that $\mathbf{M}$ and $\mathbf{M}_P$ both have rank $|S|$.

Claim 2.1. *The common bases of* $\mathbf{M}$ *and* $\mathbf{M}_P$ *are precisely the subsets* $B \subseteq S \cup \bar{S}$ *satisfying*

$$|B \cap \hat{s}| = 1 \;\; \forall s \in S \qquad \textit{and} \qquad B \cap S \textit{ is a common base of } \mathbf{M}_1 \textit{ and } \mathbf{M}_2. \tag{1}$$

Proof. Recall that r is the rank of both $\mathbf{M}_1$ and $\mathbf{M}_2$. Clearly the bases of $\mathbf{M}_P$ are the subsets $B \subseteq S \cup \bar{S}$ for which $|B \cap \hat{s}| = 1$ for every $s \in S$. Of these subsets, the bases of $\mathbf{M}$ are exactly those for which $B \cap S \in \mathcal{I}_1$ and $\pi(B \cap \bar{S}) \in \mathcal{I}_2^*$. Note that $\pi(B \cap \bar{S}) = S \setminus B$. The condition $S \setminus B \in \mathcal{I}_2^*$ is equivalent to $B \cap S$ containing a base of $\mathbf{M}_2$, which implies $|B \cap S| \geq r$. But $B \cap S \in \mathcal{I}_1$ implies $|B \cap S| \leq r$, so $|B \cap S| = r$ and $B \cap S$ is a common base of $\mathbf{M}_1$ and $\mathbf{M}_2$. Thus we have argued that the common bases of $\mathbf{M}$ and $\mathbf{M}_P$ are exactly those bases of $\mathbf{M}_P$ for which $B \cap S$ is a common base of $\mathbf{M}_1$ and $\mathbf{M}_2$. ∎

Corollary 2.2. *If* $B_1, \ldots, B_{k+1}$ *is a partition of* $S \cup \bar{S}$ *into common bases of* $\mathbf{M}$ *and* $\mathbf{M}_P$*, then* $B_1 \cap S, \ldots, B_{k+1} \cap S$ *is a partition of* S *into common bases of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$*.*

Claim 2.3. *Given a partition* $B_1, \ldots, B_{k+1}$ *of* S *into common bases of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$*, we can construct a partition* $B_1', \ldots, B_{k+1}'$ *of* $\bar{S}$ *into common bases of* $\mathbf{M}$ *and* $\mathbf{M}_P$*.*

Proof. We will partition $\bar{S}$ into $C_1, \ldots, C_{k+1}$ such that the following properties are satisfied.

$$\pi(C_j) = S \setminus B_j \qquad \text{and} \qquad |C_j \cap \bar{s}| \leq 1 \;\; \forall s \in S.$$

Then we will set $B_j' = B_j \cup C_j$. The resulting sets B_j' will satisfy (1), so by Claim 2.1 they are common bases of $\mathbf{M}$ and $\mathbf{M}_P$. The construction of the sets C_j is by a simple greedy approach that proceeds by sequentially constructing C_1, then C_2, etc. To construct C_j, for each element $s \in S \setminus B_j$ we add to C_j an arbitrary element in $\bar{s} \setminus \bigcup_{\ell < j} C_\ell$. Such an element exists because the sets B_j are a partition of S, so for every $s \in S$, we have $|\{\, j \,:\, s \notin B_j \,\}| = k = |\bar{s}|$. ∎

Claim 2.1 and Claim 2.3 together imply Theorem 1.4.

3 A counterexample to Chow's Conjecture

In this section, we show that Conjecture 1.2 and Conjecture 1.3 are equivalent. Since Conjecture 1.3 is false, this yields a counterexample to Conjecture 1.2. In fact, we will prove the following more general statement.

Theorem 3.1. *Let* $\mathbf{M}_1 = (S, \mathcal{I}_1)$ *be a matroid with rank* m*, no loops and* $|S| = (k+1)\cdot m$*. Let* $\mathbf{M}_2 = (S, \mathcal{I}_2)$ *be a partition matroid defined by*

$$\mathcal{I}_2 \;=\; \{\, I \,:\, |I \cap A_i| \leq 1 \;\; \forall i \in [m] \,\},$$

where $A_1, \ldots, A_m$ *is a partition of* S *and each* $|A_i| = k+1$*. Note that* $\mathbf{M}_2$ *also has rank* m*. Then there exist matroids* $\mathbf{M}$ *and* $\mathbf{M}_P$ *on a common ground set such that*

$$\mathbf{M} \text{ and } \mathbf{M}_P \text{ have } k+1 \text{ disjoint common bases if and only if } \mathbf{M}_1 \text{ and } \mathbf{M}_2 \text{ do,} \tag{2a}$$

$$\text{the parts in } \mathbf{M}_P \text{ are each independent in } \mathbf{M}, \text{ and} \tag{2b}$$

$$\text{if } \mathbf{M}_1 \text{ has } k+1 \text{ disjoint bases then } \mathbf{M} \text{ has } k+1 \text{ disjoint bases.} \tag{2c}$$

Proof. We use the same notation as Section 2, e.g., $\bar{S} = S \times [k]$. Now define the matroid $\mathbf{M}' = (\bar{S}, \mathcal{I}')$ where

$$\mathcal{I}' \;=\; \left\{\, I \subseteq \bar{S} \,:\, |I \cap \bar{A}_i| \leq |A_i| - 1 \;\; \forall i \in [m] \,\right\}.$$

Note that the rank of $\mathbf{M}'$ is $\sum_{i=1}^{m}(|A_i| - 1) = |S| - m$. Let $\mathbf{M} = (S \cup \bar{S}, \mathcal{I})$ be the direct sum of $\mathbf{M}_1$ and $\mathbf{M}'$. Let $\mathbf{M}_P = (S \cup \bar{S}, \mathcal{I}_P)$ be the partition matroid whose parts are the sets $\hat{s}$, i.e., $\mathcal{I}_P \;=\; \left\{\, I \subseteq S \cup \bar{S} \,:\, |I \cap \hat{s}| \leq 1 \;\; \forall s \in S \,\right\}$. Note that $\mathbf{M}$ and $\mathbf{M}_P$ both have rank $|S|$.

To prove (2b), we must show that $\hat{s} \in \mathcal{I}$ for every $s \in S$. Since $\mathbf{M}$ is a direct sum of $\mathbf{M}_1$ and $\mathbf{M}'$, it suffices to show that $\hat{s} \cap S \in \mathcal{I}_1$ and $\hat{s} \cap \bar{S} \in \mathcal{I}'$. Since $\mathbf{M}_1$ has no loops, $\hat{s} \cap S = \{s\} \in \mathcal{I}_1$ for every $s \in S$. On the other hand, for every i, $|\hat{s} \cap \bar{A}_i| = k = |A_i| - 1$, so $\hat{s} \cap \bar{S} \in \mathcal{I}'$.

Now we prove (2c). Since $\mathbf{M}$ is a direct sum of $\mathbf{M}_1$ and $\mathbf{M}'$, it suffices to show that S can be partitioned into $k+1$ bases of $\mathbf{M}_1$ and that $\bar{S}$ can be partitioned into $k+1$ bases of $\mathbf{M}'$. The first condition holds by assumption, so we consider the second condition. Since $|A_i| = k+1$ for every i, there exists a partition of S into bases $B_1, \ldots, B_{k+1}$ of $\mathbf{M}_2$. We will greedily construct a partition of $\bar{S}$ into $C_1, \ldots, C_{k+1}$ such that $|C_j \cap \bar{A}_i| = |A_i| - 1$ for each i and each j, implying that each C_j is a base of $\mathbf{M}'$. To construct C_j, for each element $s \in S \setminus B_j$ we add to C_j an arbitrary element in $\bar{s} \setminus \bigcup_{\ell < j} C_\ell$. Such an element exists because the sets B_j are a partition of S, so for every $s \in S$, we have $|\{\, j \,:\, s \notin B_j \,\}| = k = |\bar{s}|$.

To prove (2a) we require the following claim, which is similar to Claim 2.1.

Claim 3.2. *The common bases of* $\mathbf{M}$ *and* $\mathbf{M}_P$ *are precisely the subsets* $B \subseteq S \cup \bar{S}$ *satisfying*

$$|B \cap \hat{s}| = 1 \;\; \forall s \in S \qquad \textit{and} \qquad B \cap S \textit{ is a common base of } \mathbf{M}_1 \textit{ and } \mathbf{M}_2.$$

Proof. The common bases of $\mathbf{M}$ and $\mathbf{M}_P$ are the subsets $B \subseteq S \cup \bar{S}$ satisfying

$$|B \cap \hat{s}| \;=\; 1 \;\; \forall s \in S \tag{3a}$$

$$B \cap S \text{ is a base of } \mathbf{M}_1 \tag{3b}$$

$$|B \cap \bar{A}_i| \;=\; |A_i| - 1 \;\; \forall i. \tag{3c}$$

The main point is that, under the assumption that (3a) holds, (3c) is equivalent to

$$|B \cap A_i| \;=\; 1 \;\; \forall i.$$

This last condition is equivalent to $B \cap S$ being a base of $\mathbf{M}_2$. □

Now we prove (2a). If $B_1, \ldots, B_{k+1}$ are disjoint common bases of $\mathbf{M}$ and $\mathbf{M}_P$ then by Claim 3.2 $B_1 \cap S, \ldots, B_{k+1} \cap S$ are disjoint common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$. Conversely, suppose that $B_1, \ldots, B_{k+1}$ are disjoint common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$. Then the argument of Claim 2.3 shows that we can construct $k+1$ disjoint common bases of $\mathbf{M}$ and $\mathbf{M}_P$. ■

Proof (of Theorem 1.5). Clearly Conjecture 1.3 is stronger than Conjecture 1.2. We show the converse. Let $k = b - 1$. Suppose that $\mathbf{M}_1$ is a counterexample to Conjecture 1.3. Let $\mathbf{M}_2$ be the partition matroid whose parts are the sets $A_1, \ldots, A_m$, each of which has $|A_i| = k + 1$. Then $\mathbf{M}_1$ and $\mathbf{M}_2$ can both be partitioned into $b = k+1$ bases, but S cannot be partitioned into $k+1$ common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$.

Now let $\mathbf{M}$ and $\mathbf{M}_P$ be the matroids constructed in Theorem 3.1. Then $\mathbf{M}$ has rank $|S|$ and it can be partitioned into b bases. Furthermore $\mathbf{M}_P$ is a partition matroid whose parts each have size b and each is independent in $\mathbf{M}$. Since $\mathbf{M}_1$ and $\mathbf{M}_2$ do not have $k+1$ disjoint bases, neither do $\mathbf{M}$ and $\mathbf{M}_P$. Thus $\mathbf{M}$ and $\mathbf{M}_P$ give a counterexample to Conjecture 1.2.

McDiarmid's example (briefly described in Appendix A) shows that, for any $b \geq 2$, Conjecture 1.3 is false with $m = 3$ and $|S| = 3b$. Thus our construction shows that Conjecture 1.2 is false for any $b \geq 2$ and $n = 3b$. By Chow's theorem [1], this implies that Conjecture 1.2 is false whenever $2 \leq b \leq n/3$. ■

3.1 A refinement of Theorem 1.4

Theorem 1.4 describes a polynomial-time reduction from an arbitrary instance of PartitionIntoCommonBases to an instance in which one of the matroids is a partition matroid. Theorem 3.1 describes a polynomial-time reduction from an instance of PartitionIntoCommonBases in which one of the matroids is a partition matroid to an instance in which one of the matroids is a partition matroid whose parts are independent in the other matroid. Composing these two reductions proves Corollary 1.6.

4 Chow's Conjecture, Clutters and Dijoins

4.1 Clutters

We now interpret Chow's conjecture as a statement about clutters. For an introduction to this topic, see Cornuéjols [2] or Cornuéjols and Guenin [3].

Definition 4.1. A ***clutter*** $\mathcal{C}$ is a pair $(V(\mathcal{C}), E(\mathcal{C}))$, where $V(\mathcal{C})$ is a finite set and $E(\mathcal{C}) = \{E_1, E_2, \ldots\}$ is a family of distinct subsets of $V(\mathcal{C})$ such that $E_i \subseteq E_j$ implies

$i = j$. The elements of $V(\mathcal{C})$ are called ***vertices*** and the elements of $E(\mathcal{C})$ are called ***edges***.

Definition 4.2. A ***transversal*** is a subset of the vertices that intersects all edges. Let $\tau(\mathcal{C})$ denote the minimum cardinality of any transversal. We say that a clutter ***packs*** if there exist $\tau(\mathcal{C})$ pairwise disjoint edges.

As in Conjecture 1.2, let $\mathbf{M} = (T, \mathcal{I})$ be a matroid of rank n with the property that T can be partitioned into b bases, where $3 \leq b \leq n$. Let $I_1, \ldots, I_n \in \mathcal{I}$ be disjoint independent sets, each of size b. Consider the clutter $\mathcal{C}$ with $V(\mathcal{C}) = T$ and

$$E(\mathcal{C}) \;=\; \{\, B \,:\, B \in \mathcal{I} \,\wedge\, |I_i \cap B| = 1 \;\; \forall i \in [n] \,\}. \tag{4}$$

Note that every $B \in E(\mathcal{C})$ is a base of $\mathbf{M}$.

An equivalent statement of Chow's conjecture (and Rota's conjecture) is that the clutter $\mathcal{C}$ packs, since we show in Claim 4.3 that $\tau(\mathcal{C}) = b$. Therefore we can obtain a deeper understanding of these conjectures by analyzing the clutter $\mathcal{C}$. In particular, any counterexample to Conjecture 1.2 necessarily involves a clutter which does not pack. Characterizing clutters which do not pack seems difficult, although there has been significant work on identifying the minimal such clutters [4].

Our counterexample to Conjecture 1.2 given in Section 3 is based on a well-known clutter Q_6 which does not pack, and which underlies the K_4 counterexample described in Appendix A. To further our understanding of this conjecture, we would like to understand which other minimal non-packing clutters can arise in $\mathcal{C}$. In particular, finding additional counterexamples might allow one to strengthen the parameters in Theorem 1.5. The following section shows that another famous such clutter, known as $Q_6 \otimes \{1, 3, 5\}$ yields a counterexample to Conjecture 1.2. This clutter was developed by Schrijver [15] to disprove a conjecture of Edmonds and Giles [7] on packing dijoins.

Claim 4.3. *$\tau(\mathcal{C}) = b$.*

Proof (of Claim 4.3). Obviously $\tau(\mathcal{C}) \leq b$ as any set I_i is a transversal. So suppose there exists a transversal $D \subseteq T$ with $|D| < b$. Let $\mathbf{M}_P = (T, \mathcal{I}_P)$ be the partition matroid with

$$\mathcal{I}_P \;=\; \{\, I \subseteq T \,:\, |I \cap I_i| \leq 1 \;\; \forall i \in [n] \,\}.$$

We wish to show that there is an edge with does not intersect D, which is equivalent to showing that $\mathbf{M} \backslash D$ and $\mathbf{M}_P \backslash D$ have a common base. Let $r_{\mathbf{M}}$ and $r_{\mathbf{M}_P}$ respectively be the rank function of $\mathbf{M}$ and $\mathbf{M}_P$. By the matroid intersection theorem [14, Theorem 41.1], it suffices to show that

$$r_{\mathbf{M}}(A) + r_{\mathbf{M}_P}(T \setminus (D \cup A)) \;\geq\; n \quad \forall A \subseteq T \setminus D. \tag{5}$$

By Edmonds' matroid base covering theorem [14, Corollary 42.1c], for any set A we have $r_{\mathbf{M}}(A) \geq \lceil |A|/b \rceil$ and $r_{\mathbf{M}_P}(A) \geq \lceil |A|/b \rceil$. Thus

$$\begin{aligned} r_{\mathbf{M}}(A) + r_{\mathbf{M}_P}(T \setminus (D \cup A)) \;&\geq\; \left\lceil \frac{|A|}{b} \right\rceil + \left\lceil \frac{|T \setminus (D \cup A)|}{b} \right\rceil \\ &=\; \left\lceil \frac{|A|}{b} \right\rceil + \left\lceil \frac{|T \setminus A|}{b} \right\rceil - \epsilon \;\geq\; n - \epsilon, \end{aligned}$$

where $\epsilon \in \{0,1\}$, since $|D| < b$. If the last inequality is strict, then (5) must be satisfied. If last inequality holds with equality then $|A|/b$ and $|T \setminus A|/b$ are both integers, which implies that $\lceil |T \setminus (D \cup A)|/b \rceil = |T \setminus A|/b$, since $|D| < b$. Thus $\epsilon = 0$ and so (5) is satisfied. ■

4.2 Dijoins

In this section we give an alternative proof of Theorem 1.5. The proof is based on a connection between dijoins and common matroid bases, due to Frank and Tardos [8], and Schrijver's counterexample on packing dijoins [15] which is based on the clutter $Q_6 \otimes \{1,3,5\}$.

Given a directed graph $D = (V, A)$, a ***k-dijoin*** is an arc set $F \subseteq A$ that contains at least k arcs from each directed cut of D. A 1-dijoin is called simply a ***dijoin***. Schrijver's showed the existence of a digraph and a 2-dijoin that cannot be partitioned into two disjoint dijoins. By adding three arcs x', y', z' to Schrijver's example, we can obtain a 3-dijoin that cannot be decomposed into three dijoins. The resulting example is shown in Figure 1 and is denoted $D = (V, A)$. Let F be the set of red arcs in this example.

Claim 4.4. *The arc set F cannot be decomposed into 3 dijoins.*

Proof. First, we can observe that if a decomposition exists, none of the dijoins can be $\{x', y', z'\}$, since the rest is Schrijver's counterexample which cannot be decomposed into two dijoins. Furthermore, all other dijoins contain at least 4 arcs because of the nodes of in-degree and out-degree 0 represented on Figure 2(a), and a dijoin of 4 arcs must contain two nonparallel arcs from $\{x, x', y, y', z, z'\}$. Since F has 12 elements, each of the 3 disjoint dijoins must have exactly 4 arcs and each must contain two nonparallel arcs from $\{x, x', y, y', z, z'\}$. But Figure 2(b) shows that such an arc set cannot be a dijoin: there is a set of out-degree 0 in D that it does not enter. ■

We define an arc set F' (which is not a subset of A) by taking F and adding two reversed arcs for each arc of F. For $a \in F$, these reversed arcs will be denoted by a_1^{-1} and a_2^{-1}. Our counterexample for Chow's conjecture is a matroid with ground set F'. We define the matroid by its bases: a set $B \subseteq F'$ is a base if and only if $|B| = |F|$ and

$$\sum_{v \in X} d_B^{in}(v) \geq i_F(X) + 1 \text{ for every } \emptyset \neq X \subsetneq V \text{ with } d_A^{out}(X) = 0, \tag{6}$$

where $i_F(X)$ is the number of arc of F induced by X. It was shown by Frank and Tardos [8] [14, Section 55.5] that this construction gives a matroid. Moreover, since F is a 3-dijoin and $i_{F'}(X) = 3i_F(X)$ for every $X \subseteq V$, we have

$$\sum_{v \in X} \frac{d_{F'}^{in}(v)}{3} \geq i_F(X) + 1 \text{ for every } \emptyset \neq X \subsetneq V \text{ with } d_A^{out}(X) = 0,$$

which implies by the matroid union theorem that F' can be partitioned into three bases.

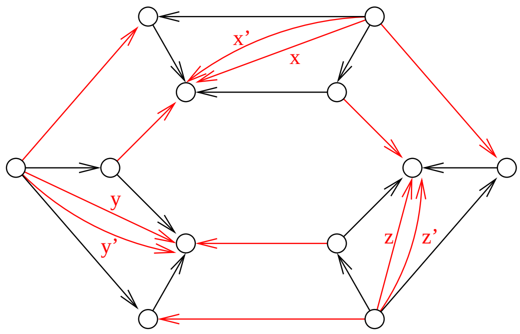

Figure 1: Schrijver's example, augmented with additional arcs x', y' and z'.

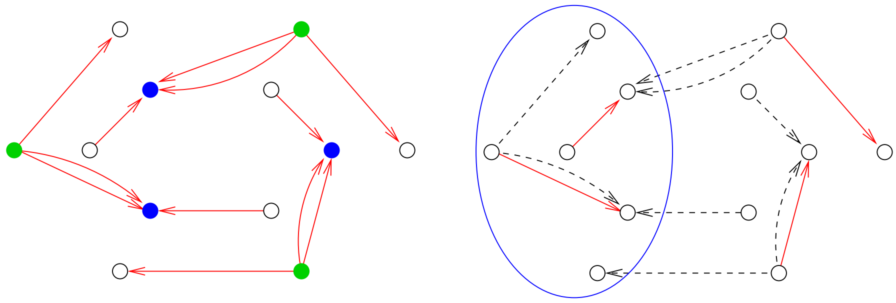

Figure 2: (a) Nodes of out-degree 0 (blue) and in-degree 0 (green) in D. (b) The solid (red) arcs do not form a dijoin because of the set of out-degree 0, indicated by the oval (blue).

Let us define the sets $I_a = \{a, a_1^{-1}, a_2^{-1}\}$ $(a \in F)$. These triplets are independent sets: $(F \setminus \{a, b, c\}) \cup I_a$ is a base for arbitrary distinct arcs $a, b, c \in F$ because F satisfies inequality (6) with $i_F(X) + 3$ instead of $i_F(X) + 1$.

Conjecture 1.2 would imply that F' can be decomposed into three bases B_1, B_2, B_3 such that $|B_j \cap I_a| = 1$ for any $j \in \{1, 2, 3\}$ and $a \in F$. Suppose that this is possible; then $i_{B_j}(X) = i_F(X)$ for every $X \subseteq V$, so $\sum_{v \in X} d^{in}_{B_j}(v) \geq i_F(X) + 1$ implies that

$$d^{in}_{B_j}(X) \geq 1 \text{ for every } \emptyset \neq X \subsetneq V \text{ with } d^{out}_A(X) = 0.$$

In other words, B_j has at least one arc in every directed cut of D. However, the only arcs that are in directed cuts of D are the arcs of F. Thus the conjecture would imply that F can be decomposed into three dijoins, but by Claim 4.4 this is impossible.

This proves Theorem 1.5 for the case $b = 3$. By adding additional arcs parallel to x', y', z' one can extend this argument to obtain a counterexample to Conjecture 1.2 for all $3 \leq b \leq n/3 - 1$. This proves Theorem 1.5, with slightly weaker parameters.

Concluding Remarks

Several basic questions on disjoint common bases of two matroids remain open. One question is to determine the computational complexity of PartitionIntoCommon-

Bases. As we have shown, it suffices to consider the case when one of the matroids is a partition matroid. Even when the other matroid is a graphic matroid, the computational complexity is still unknown. Another question is to find a sufficient condition that guarantees the existence of k disjoint common bases. Geelen and Webb [9] showed that there are $\sqrt{n}$ disjoint common bases under the setting in Rota's conjecture.

Finding further counterexamples to Chow's conjecture may lead to an improvement of the parameters in Theorem 1.5, and perhaps a better understanding of Rota's conjecture. One can show that the clutter $\mathcal{C}$ defined in (4) is not necessarily ideal: there is a laminar matroid on nine elements such that C_3^2 is a minor of $\mathcal{C}$. On the other hand, our two counterexamples are based on Q_6 and $Q_6 \otimes \{1,3,5\}$, which are both ideal. Is there a counterexample based on a non-ideal, non-packing clutter?

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A The K_4 counterexample

Consider K_4, the complete graph on 4 vertices. As shown in Figure 3(a), we write its edges as $S = \{a_0, a_1, b_0, b_1, c_0, c_1\}$, where $a_0 \cap a_1 = \emptyset$, $b_0 \cap b_1 = \emptyset$, $c_0 \cap c_1 = \emptyset$, and $\{a_1, b_1, c_1\}$ forms a spanning star. Let $\mathbf{M}_1 = (S, \mathcal{I}_1)$ be the graphic matroid of K_4. Let $\mathbf{M}_2 = (S, \mathcal{I}_2)$ be the partition matroid on the same ground set, whose parts are $\{a_0, a_1\}$, $\{b_0, b_1\}$ and $\{c_0, c_1\}$. It is well-known [5] that both $\mathbf{M}_1$ and $\mathbf{M}_2$ have two disjoint bases, but they do not have two disjoint common bases. The common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$ are precisely the spanning stars in K_4.

McDiarmid [1] showed how to extend this example to obtain, for any $k \geq 1$, two matroids $\mathbf{M}_1 = (S, \mathcal{I}_1)$ and $\mathbf{M}_2 = (S, \mathcal{I}_2)$ such that

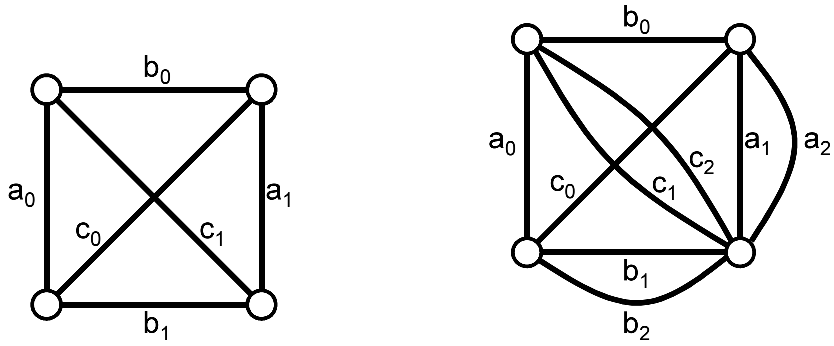
- S can be partitioned into $k+1$ bases of $\mathbf{M}_1$,
- S can be partitioned into $k+1$ bases of $\mathbf{M}_2$, and
- S cannot be partitioned into $k+1$ common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$.

We now describe this extension. The example is based on the graph G_k, which is constructed from K_4 by adding new edges:

- $a_2, \ldots, a_k$ parallel to a_1,
- $b_2, \ldots, b_k$ parallel to b_1, and
- $c_2, \ldots, c_k$ parallel to c_1.

The graph G_2 is shown in Figure 3(b). Define

$$\begin{aligned} E_a &= \{a_1, \ldots, a_k\} & E_b &= \{b_1, \ldots, b_k\} & E_c &= \{c_1, \ldots, c_k\} \\ F_a &= \{a_0, \ldots, a_k\} & F_b &= \{b_0, \ldots, b_k\} & F_c &= \{c_0, \ldots, c_k\}. \end{aligned}$$



(a) (b)

Figure 3: (a) The graph K_4 with our chosen edge labeling. (b) The graph G_2 is obtained by letting a_2, b_2 and c_2 be parallel copies of a_1, b_1 and c_1, respectively.

Let $\mathbf{M}_1$ be the graphic matroid of G_k. Let $\mathbf{M}_2$ be the partition matroid whose parts are F_a, F_b and F_c. It is easy to see that the edges can be partitioned into $k+1$ bases of $\mathbf{M}_1$, or into $k+1$ bases of $\mathbf{M}_2$.

Claim A.1. *The edges cannot be partitioned into $k+1$ common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$.*

Proof. As remarked above, the common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$ are precisely the spanning stars in G_k.

Suppose $k \geq 3$. Since there are only 3 edges not in $E_a \cup E_b \cup E_c$, at least one of the $k+1$ common bases is contained in $E_a \cup E_b \cup E_c$. Removing this common base, the resulting graph is G_{k-1}. By induction, this instance cannot be partitioned into k common bases.

Suppose $k = 2$. Note that there is no spanning star using exactly two edges from $E_a \cup E_b \cup E_c$. So two of the common bases use three of those edges, and the other common base uses none. But the complement of $E_a \cup E_b \cup E_c$ is not a spanning star. ■

The matroids $\mathbf{M}_1$ and $\mathbf{M}_2$ give a counterexample to Conjecture 1.3 for $m = 3$ and arbitrary $b \geq 2$: take $k = b-1$, and define the sets A_1, A_2, A_3 to be the partitions of the matroid $\mathbf{M}_2$. However, this does not directly yield a counterexample to Conjecture 1.2 for $b \geq 3$ since the sets A_1, A_2, A_3 are not independent in $\mathbf{M}_1$.