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SOME NEW RESULTS IN THE STATISTICAL INVESTIGATION OF ELEMENTARY GAUSSIAN PROCESSES

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The subject of our paper is the statistical analysis of some problems connected with the elementary Gaussian processes, with discrete and continuous time parameter. The k-dimensional stochastic process $\xi^*(t) = (\xi_0(t), \xi_1(t), \dots, \xi_{k-1}(t))$, where the $*$ denotes the transpose of a matrix or vector, is called an elementary Gaussian one if it is stationary Markovian and Gaussian.

It is well known that, if $\mathsf{M}\xi(t) = 0$, the elementary Gaussian process satisfies the stochastic differential equation

$$d\xi(t) = A\xi(t)dt + dw(t) \tag{1}$$

with a continuous time parameter, where the Brownian motion process $w(t)$ has the parameters $\mathsf{M}dw = 0$, $\mathsf{M}dwdw^* = B_w dt$, and the real parts of the characteristic roots of the matrix A are negative. The covariance function is of the form

$$B(t) = \mathsf{M}\xi(t+s)\xi^*(s) = e^{A|t|}B(0)\ ,$$

where

$$(2) \qquad AB(0) + B(0)A^* = -B_w$$

(see Arató [8]). In the case of discrete time parameter the process $\xi_n = \xi(n\Delta)$ $(\Delta > 0)$ fulfils the difference equation

$$(3) \qquad \xi_n = Q\xi_{n-1} + \epsilon_n$$

where $Q = e^{A\Delta}$, $\mathrm{M}\epsilon_n = 0$, and $B_\epsilon = \mathrm{M}\epsilon_n \epsilon_n^*$ may be calculated from

$$(4) \qquad B(0) = QB(0)Q^* + B_\epsilon .$$

In our investigations we are interested in the calculations of exact distributions of estimators of the unknown parameters of A (resp. Q), if we have a realization of $\xi(t)$ on the interval $0 \leq t \leq T$. First, for the sake of simplicity we assume that the matrix A is given in the canonical form, and the components of the Brownian motion process $w(t)$ are independent. It is interesting to study the case in it's all generality.

There are many results which gave the asymptotic distributions of estimators (see Mann and Wald [16], Anderson [1], Hannan [14], Walker [26], Whittle [30], Grenander and Rosenblatt [13]) and the approximate distributions of the estimators (White [28], [29]) in the case of discrete time. For the continuous time there is much less literature (we refer to Arató [9], Pisarenko [22], Širjaev [23], Novikov [17], [18]). The first attempt at the calculation of exact distributions was given by Pisarenko [21], Arató [3], where they gave the characteristic function of the sufficient statistics of autoregressive scheme. In an early paper of Arató, Kolmogorov, Sinay [12], an interesting theorem was given for the exact distribution of a statistics in the two dimensional case (see also Novikov [19]). For exact distribution see Arató [7], Arató, Benczúr [10]. In our paper we show, that in the case of continuous time parameter it is possible to get exact distributions and they may be used to the approximation of the case of discrete time parameter, and on the other hand, we show that the normal approximation for the distributions of estimators cannot be used in many practical cases.

At last we want to show the usefulness of computers in our investigations, for example in the case when the realization is

$$\eta(t) = \xi(t) + m$$

where $\xi(t)$ is an elementary Gaussian process and m is an unknown constant. In other aspect the problem was investigated in Orcutt and Winokur [20].

The idea of investigating the process $\xi(t)$ with continuous time parameter was given by A. N. Kolmogorov in 1948 at the meeting of probability theory in Jerevan. He also asked how to get an approximation for the discrete time case from the continuous one.

1.

In this paragraph we assume that $\xi(t)$ is one dimensional and $\mathsf{M}\xi(t) = 0$. Further, we suppose that we know a realization on $0 \leq t \leq 1$ and $B(t) = \mathsf{M}\xi(t+s)\xi(s) = \frac{1}{2\lambda}e^{-\lambda|t|}$. The sufficient statistics can be deduced from the likelihood function

$$\frac{d\mathsf{P}_\lambda}{d\mathsf{V}} = \sqrt{\frac{\lambda}{\pi}} \exp\left\{-\lambda[s_1^2 - \frac{1}{2} + \frac{1}{2}\lambda s_2^2]\right\} \tag{1.1}$$

where $\mathsf{V} = \mathsf{L} \times \mathsf{W}$, and here L denotes the Lebesgue measure and W the conditional Wiener measure (see Arató [9], Striebel [24]), further

$$s_1^2 = \frac{1}{2}(\xi^2(0) + \xi^2(1)), \qquad s_2^2 = \int_0^1 \xi^2(t)dt\,. \tag{1.2}$$

The characteristic funttion of statistics s_1^2, s_2^2 has the following form

$$\begin{aligned} \varphi(\alpha_1, \alpha_2) &= \mathsf{M}\exp\{i\alpha_1 s_1^2 + i\alpha_2 s_2^2\} = \\ &= 2\sqrt{\frac{\lambda e^\lambda \sqrt{\Lambda}}{(\lambda - i\alpha_1 + \Lambda)^2 e^\Lambda - (\lambda - i\alpha_1 - \Lambda)^2 e^{\Lambda}}} \end{aligned} \tag{1.3}$$

where

$$\Lambda = \sqrt{\lambda^2 - 2i\alpha_2}\,.$$

As to the proof see Arató [5], Arató, Benczúr [10] or Novikov [19].

With the notations $z = \lambda x$, $\eta_\lambda = \lambda^2 x^2 s_2^2 + \lambda x s_1^2$ we have the following relation for the maximum likelihood estimator

$$\mathsf{P}_\lambda \{\hat{\lambda} > z\} = \mathsf{P}_\lambda \left(\eta_\lambda < \frac{1}{2} + \frac{1}{2^2} z \right) .$$

From (1.3) we may get the characteristic function of η_λ. The distribution function of the maximum likelihood estimator $\hat{\lambda}$ can be calculated only numerically, and the computing time is very large. A table for the quantiles of this distribution is given in our earlier paper (Arató, Benczúr [10]).

Here we present this table at the end of our paper (Table 1). To win confidence limits for λ we can use this table. The lower confidence limit is always greater than 0.

From the form of characteristic function we can see that $\hat{\lambda}$ is assymptotically normally distributed, when $\lambda \to \infty$, with mean λ and variance λ. From Table 1 it is obvious that the rate of convergence is very slow. Normality is true if $\lambda \approx 1000.000$.

In the discrete time case, when (with the notations $\xi_k = \xi\left(k \cdot \frac{T}{N}\right)$ and $\rho = e^{-\lambda \frac{T}{N}}$, $\sigma_\epsilon^2 = (1 - \rho^2)\sigma_\xi^2 = \frac{\left(1 - e^{-2\frac{T}{N}}\right)}{2\lambda} \sigma_w^2$)

$$\xi_k = \rho \xi_{k-1} + \epsilon_k ,$$

the unknown parameter ρ may be estimated by several methods. We take the following estimators

$$\hat{\rho}_1 = \frac{\sum_{i=1}^{N} \xi_i \xi_{i-1}}{\sum_{i=1}^{N} \xi_i^2} ,$$

$$\hat{\lambda}_2 = \frac{N+1}{2\sum\limits_{0}^{N} \xi_i^2} \left(\sim \frac{1}{2\int\limits_0^1 \xi^2(t)dt} \right),$$

$\hat{\rho}_3$ the maximum likelihood estimator in the case of discrete time. In Tables 2-4 we give the quantiles of empirical distributions of the estimators $\hat{\lambda}_i$. The number of observations in the time interval $[0, 1]$ is 100. The size (n) of the samples is varying from 1000 to 2000 (depending on λ). The program of simulation was carried out on CDC 3300.

It is remarkable that $\hat{\lambda}_1 = -N \log \hat{\rho}_1$ behaves very badly for $\lambda \ll 1$. At the same time the distributions of $\hat{\lambda}_2$ and $\hat{\lambda}_3 = -N \log \hat{\rho}_3$ agree very well with the values of Table 1.

Further detailes and the behaviour of the other estimators can be found in Arató, Benczúr [11]. Results are given here when the number of observations in $[0, 1]$ is 20, 60, 500, 1000. We found that for the good approximation of the continuous time case 60-100 observations are enough in the parameter interval $\lambda \leqslant 10$.

As we have showed (see Arató, Benczúr [10]) the approximation with t-distribution, proposed by White [28] cannot be used when $\lambda \ll 1$.

2.

In this section we suppose that $\xi(t)$ is one dimensional, but $\mathsf{M}\xi(t) = m$ and λ are both unknown.

One of the authors (Arató [5]) have calculated the characteristic function of the sufficient statistics, but this cannot be used for the evaluation of the distribution of elementary estimators.

On the other hand, it was proved (see Arató [4]) that every estimator $\tilde{\lambda}$ of λ has the property that for any fixed p there exists an x_p (independent of λ) such that

(2.1) $\mathsf{P}_\lambda \{\tilde{\lambda} > x_p\} \geqslant 1 - p$.

Table 1.

In the table the quantiles $z_p(\lambda)$ of the maximum likelihood estimator $\hat{\lambda}$ are given, i.e. $P_\lambda\{\hat{\lambda} > z_p\} = p$.
In brackets the values $x = \frac{z}{\lambda}$ are given.

λ \ p	0,001	0,01	0,025	0,05	0,1	0,9	0,95	0,975	0,99	0,999
0	0 (637000)	0 (6370)	0 (1020)	0 (255,0)	0 (63,60)	0 (0,369)	0 (0,260)	0 (0,199)	0 (0,151)	0 (0,092)
0,01	10,60 (1060)	4,232 (423,2)	2,274 (227,4)	1,170 (117,0)	0,4734 (47,34)					
0,05	11,195 (263,9)	6,330 (126,6)	4,0065 (80,13)	2,5130 (50,26)	1,3375 (26,75)					
0,1	14,38 (143,8)	7,344 (73,44)	4,879 (48,79)	3,268 (32,68)	1,908 (19,08)					
0,2	15,664 (78,32)	8,468 (42,34)	5,902 (29,51)	4,154 (20,77)	2,624 (13,12)					
0,3	16,488 (54,96)	9,207 (30,69)	6,561 (21,87)	4,746 (15,82)	3,120 (10,40)					
0,4	17,080 (42,70)	9,756 (24,39)	7,080 (17,70)	5,208 (13,02)	3,517 (8,793)					
0,5	17,670 (35,34)	10,230 (20,46)	7,515 (15,03)	5,605 (11,21)	3,8610 (7,722)	0,2085 (0,417)	0,1510 (0,302)			
0,6	18,108 (30,18)	10,638 (17,73)	7,896 (13,16)	5,9532 (9,922)	4,1676 (6,946)					
0,7	18,522 (26,46)	11,011 (15,73)	8,239 (11,77)	6,2713 (8,959)	4,4471 (6,353)					
0,8	18,896 (23,62)	11,360 (14,20)	8,560 (10,70)	6,5648 (8,206)	4,7103 (5,887)					
0,9	19,260 (21,40)	11,682 (12,98)	8,8587 (9,843)	6,8409 (7,601)	4,9446 (5,494)					
1	19,60 (19,60)	11,98 (11,98)	9,140 (9,140)	7,103 (7,103)	5,188 (5,188)	0,445 (0,445)	0,332 (0,332)	0,269 (0,269)	0,205 (0,205)	0,130 (0,130)
1,5	21,060 (14,04)	13,3080 (8,872)	10,3845 (6,923)	8,2590 (5,506)	6,2325 (4,155)	0,7005 (0,467)	0,5325 (0,355)	0,4275 (0,285)	0,3360 (0,224)	0,2145 (0,143)
2	22,32 (11,16)	14,462 (7,231)	11,426 (5,713)	9,272 (4,636)	7,156 (3,278)	0,972 (0,486)	0,750 (0,375)	0,606 (0,303)	0,480 (0,240)	0,310 (0,155)
2,5	23,4750 (9,390)	15,515 (6,206)	12,4575 (4,983)	10,2050 (4,082)	8,0100 (3,204)	1,2575 (0,503)	0,9850 (0,394)	0,8000 (0,320)	0,6500 (0,256)	0,4125 (0,165)
3,	24,342 (8,114)	16,506 (5,502)	13,389 (4,463)	11,082 (3,694)	8,811 (2,937)	1,557 (0,519)	1,233 (0,411)	1,111 (0,337)	0,807 (0,269)	0,525 (0,175)
3,5	25,5850 (7,310)	17,4440 (4,984)	14,2800 (4,080)	11,9105 (3,403)	9,5970 (2,742)	1,8655 (0,533)	1,4910 (0,426)	1,2355 (0,353)	0,9975 (0,285)	0,6405 (0,183)
4	26,576 (6,644)	18,352 (4,588)	15,136 (3,784)	12,732 (3,183)	10,352 (2,588)	2,180 (0,545)	1,760 (0,440)	1,468 (0,367)	1,192 (0,298)	0,776 (0,194)
4,5	27,5355 (6,119)	19,2285 (4,273)	15,9705 (3,549)	13,5225 (3,005)	11,0835 (2,463)	2,5020 (0,556)	2,0430 (0,454)	1,7100 (0,380)	1,3905 (0,309)	0,9135 (0,203)

λ \ p	0,001	0,01	0,025	0,05	0,1	0,9	0,95	0,975	0,99	0,999
5	28,470 (5,694)	20,090 (4,018)	16,795 3,359)	14,395 (2,879)	11,755 (2,351)	2,835 (0,567)	2,325 (0,0465)	1,965 (0,393)	1,615 (0,323)	1,070 (0,214)
5,5	29,3865 (5,343)	20,9275 (3,805)	17,5835 (3,197)	15,0535 (2,737)	12,5125 (2,275)	3,1680 (0,576)	2,6235 (0,477)	2,4640 (0,448)	1,8315 (0,333)	1,2265 (0,223)
6	30,318 (5,053)	21,750 (3,625)	18,366 (3,061)	15,792 (2,632)	13,282 (2,212)	3,510 (0,585)	2,922 (0,487)	2,490 (0,415)	2,061 (0,347)	1,392 (0,232)
6,5	31,1610 (4,794)	22,5615 (3,471)	19,1295 (2,943)	16,5295 (2,543)	13,8905 (2,137)	3,8545 (0,593)	3,2305 (0,497)	2,7690 (0,426)	2,3140 (0,356)	1,5730 (0,242)
7	32,025 (4,575)	23,359 (3,337)	19,894 (2,842)	17,248 (2,464)	14,574 (2,082)	4,207 (0,601)	3,542 (0,506)	3,052 (0,436)	2,555 (0,365)	1,764 (0,252)
7,5	32,8882 (4,385)	24,1500 (3,220)	20,6475 (2,753)	17,9550 (2,394)	15,2475 (2,033)	4,5600 (0,608)	3,8550 (0,514)	3,3375 (0,445)	2,8200 (0,376)	1,9575 (0,261)
8	33,728 (4,216)	24,920 (3,115)	21,384 (2,673)	18,672 (2,334)	15,912 (1,989)	4,920 (0,615)	4,176 (0,522)	3,632 (0,454)	3,088 (0,386)	2,168 (0,271)
8,5	34,5610 (4,066)	25,6870 (3,022)	22,1170 (2,602)	19,3715 (2,279)	16,5750 (2,950)	5,2700 (0,620)	4,5050 (0,530)	3,9270 (0,462)	3,3490 (0,394)	2,3630 (0,278)
9	35,388 (3,932)	26,451 (2,939)	22,689 (2,521)	20,070 (2,230)	17,226 (1,914)	5,643 (0,627)	4,842 (0,538)	4,230 (0,470)	3,618 (0,402)	2,592 (0,288)
9,5	36,1855 (3,809)	27,1700 (2,860)	23,5790 (2,482)	20,7480 (2,184)	17,8790 (1,882)	6,0135 (0,633)	5,1870 (0,546)	4,5315 (0,477)	3,8950 (0,410)	2,812 (0,296)
10	37,04 (3,704)	27,55 (2,755)	24,28 (2,428)	21,47 (2,147)	18,53 (1,853)	6,38 (0,638)	5,50 (0,550)	4,84 (0,484)	4,20 (0,420)	3,04 (0,304)
20	52,200 (2,610)	42,040 (2,102)	37,800 (1,890)	34,4360 (1,7218)	30,8960 (1,5448)	14,1780 (0,7089)	12,7140 (0,6357)	11,580 (0,579)	10,380 (0,519)	8,320 (0,416)
30	66,270 (2,209)	55,230 (1,841)	50,520 (1,684)	46,7370 (1,5579)	42,7080 (1,4236)	22,4310 (0,7477)	20,4960 (0,6832)	18,960 (0,632)	17,340 (0,578)	14,400 (0,480)
40	79,800 (1,995)	67,920 (1,698)	62,800 (1,570)	58,6800 (1,4670)	54,2320 (1,3558)	30,9400 (0,7735)	28,5920 (0,7148)	26,720 (0,668)	24,680 (0,617)	21,000 (0,525)
50	92,900 (1,858)	80,3200 (1,6064)	74,8400 (1,4968)	70,3950 (1,4079)	65,5800 (1,3116)	39,6150 (0,7923)	36,9000 (0,7380)	34,7100 (0,6972)	32,350 (0,647)	27,950 (0,559)
60	105,780 (1,763)	92,520 (1,542)	86,6820 (1,4447)	81,9540 (1,3659)	76,7940 (1,2799)	48,4140 (0,8069)	45,3660 (0,7561)	42,8880 (0,7148)	40,200 (0,670)	30,300 (0,585)
70	118,370 (1,691)	104,510 (1,493)	98,3770 (1,4054)	93,3800 (1,3340)	87,9130 (1,2559)	57,3020 (0,8186)	53,9560 (0,7708)	51,2120 (0,7316)	48,2300 (0,689)	42,490 (0,607)
80.	130,800 (1,635)	116,40 (1,455)	109,960 (1,3745)	104,7120 (1,3089)	98,9600 (1,2370)	66,2722 (0,8284)	62,6240 (0,7828)	59,3320 (0,7454)	56,400 (0,0705)	50,080 (0,326)
90	143,100 (1,590)	128,160 (1,424)	121,4550 (1,3495)	115,6950 (1,2885)	109,9350 (1,2215)	75,2940 (0,8366)	71,3880 (0,7932)	68,1570 (0,7573)	64,620 (0,718)	57,870 (0,643)
100	155,32 (1,5532)	139,79 (1,3979)	132,86 (1,3286)	127,15 (1,2715)	120,86 (1,2086)	84,37 (0,8737)	80,21 (0,8021)	76,8 (0,768)	73,0 (0,730)	65,7 (0,657)
500	609,000 (1,218)	580,000 (1,160)	567,0500 (1,1341)	555,800 (1,1116)	543,15 (1,0833)	462,05 (0,9241)	451,4500 (0,9029)	442,600 (0,8852)	432,600 (0,8652)	412,00 (0,824)
1000	1149 (1,149)	1110,6 (1,1106)	1092,6 (1,0926)	1077,3 (1,0773)	1060,00 (1,0600)	945,3 (0,9453)	929,9 (0,9299)	917,00 (0,9170)	902,3 (0,9023)	872,00 (0,872)
10 000	10477 (1,0477)	10336 (1,03360)	10282,1 (1,02821)	10236,3 (1,02333)	10183,9 (1,01839)	9821,4 (0,98214)	9771,1 (0,97711)	9727,6 (0,97276)	9377,5 (0,93775)	9274 (0,9574)

Table 2.

In the table the values z_p are given for which $P_\lambda^{(n)}\{\hat{\lambda}_1 > z_p\} = p$.

λ \ p	0,05	0,1	0,9	0,95	Empirical mean
0,00001	0,04	0,01	−0,02	−0,04	$0,5\cdot10^{-2}$
0,0001	0,15	0,14	−0,04	−0,08	0,03
0,01	1,32	0,67	−0,25	−0,41	0,22
0,1	4,13	2,45	−0,39	−0,64	0,84
1	7,69	5,69	−0,09	−0,44	2,48
2	9,57	7,27	0,67	0,36	3,59
5	15,16	12,64	2,50	1,97	6,88
10	22,75	18,04	5,71	5,04	11,97

Table 3.

In the table the values z_p are given for which $P_\lambda^{(n)}\{\hat{\lambda}_2 > z_p\} = p$.

λ \ p	0,05	0,1	0,9	0,95	Empirical mean
0,00001	$0,73\cdot10^{-2}$	0,0007	$0,33\cdot10^{-5}$	$0,24\cdot10^{-5}$	$0,5\cdot10^{-2}$
0,0001	0,03	0,009	$0,37\cdot10^{-4}$	$0,27\cdot10^{-4}$	0,03
0,01	0,92	0,42	$0,37\cdot10^{-2}$	$0,27\cdot10^{-2}$	0,21
0,1	3,96	2,21	0,04	0,03	0,82
1	6,86	5,21	0,42	0,3	2,44
2	8,64	6,86	0,91	0,74	3,44
5	13,99	11,08	2,65	2,23	6,67
10	20,78	18,45	5,87	5,09	11,69

Table 4.

In the table the values z_p are given for which $P_\lambda^{(n)}\{\hat{\lambda}_3 > z_p\} = p$.

λ \ p	0,05	0,1	0,9	0,95	Empirical mean
0,00001	0,004	0,0004	$0,18\cdot10^{-5}$	$0,13\cdot10^{-5}$	$0,3\cdot10^{-2}$
0,0001	0,02	0,005	$0,2\cdot10^{-4}$	$0,14\cdot10^{-4}$	0,02
0,01	0,75	0,37	$0,3\cdot10^{-2}$	$0,2\cdot10^{-2}$	0,25
0,1	3,67	1,72	0,03	0,02	0,74
1	6,81	4,98	0,36	0,25	2,21
2	8,34	6,55	1,01	0,71	3,30
5	14,36	11,78	2,40	1,91	6,50
10	21,53	18,16	6,23	5,07	11,47

From this fact it follows that it is impossible to construct a lower confidence limit for λ which differs from 0.

We could not calculate the x_p values analitically for given p. The results of simulation for the estimators

$$\tilde{m}_1 = \frac{1}{N}\sum \xi_i , \quad \tilde{\lambda}_1 = \frac{1}{2\left[\frac{1}{N}\sum \xi_i^2 - \frac{1}{N}\sum \xi\right]^2} ,$$

the maximum likelihood estimators with discrete time $\tilde{m}_2, \tilde{\lambda}_2$, and

$$\tilde{m}_3 = \frac{\xi_0 + \xi_N}{2} , \qquad \tilde{\lambda}_3 = \frac{2}{(\xi_N - \xi_0)^2}$$

are given in Tables 5-7.

It is well known (see for example Širjaev [23], Novikov [19]) that the estimators of λ are biased. But in the stationary case we must have nonnegative estimators, which requirement is not fulfilled for estimators recommended by Širjaev and Novikov. Simulation was completed for these type of estimation too. They behave like the estimator $\hat{\lambda}_1$ in the one parameter case (see Section 1). Estimators $\tilde{m}_i$ $(i = 1, 2, 3)$ are nearly normally distributed with parameters m and $\frac{1}{2\lambda}$ but not $\frac{1}{2\tilde{\lambda}}$. The tables of this distribution is given in our paper (Arató, Benczúr [11]).

3.

In this paragraph we study the two dimensional process $\xi(t) = (\xi_1(t), \xi_2(t))$ which satisfies the system of differential equations

$$d\xi_1(t) = (-\lambda\xi_1 - \omega\xi_2)dt + dw_1$$

$$d\xi_2(t) = (-\lambda\xi_2 + \omega\xi_1)dt + dw_2$$

where w_1, w_2 are independent Brownian motion processes. We assume that $M\xi(t) = 0$.

If ω is known the maximum likelihood estimator $\hat{\lambda}$ of unknown para-

Table 5.

In the table the values z_p are given for which $P^{(n)}_{\lambda;m}\{\tilde{\lambda}_1 > z_p\} = p$. m, λ are unknown and estimated.

λ \ p	0,05	0,1	0,9	0,95	Empirical mean
0,00001	12,58	10,04	1,15	0,80	5,09
0,0001	12,98	10,49	1,39	0,88	5,18
0,01	13,40	10,74	1,47	1,00	5,30
0,1	14,62	10,76	1,50	1,00	5,41
1	13,47	11,35	1,74	1,33	5,72
2	13,87	11,87	2,13	1,78	6,51
5	19,14	16,43	3,80	2,97	9,46
10	23,18	22,42	7,22	5,95	14,05

Table 6.

In the table the values z_p are given for which $P^{(n)}_{\lambda,m}\{\tilde{\lambda}_2 > z_p\} = p$. m, λ are unknown and estimated.

λ \ p	0,05	0,1	0,9	0,95	Empirical mean
0,00001	12,68	9,77	0,87	0,55	4,75
0,0001	13,25	10,24	0,94	0,59	4,86
0,01	13,85	10,41	0,96	0,61	5,00
0,1	13,95	10,87	0,99	0,68	5,12
1	14,33	11,34	1,40	1,01	5,63
2	14,78	12,24	1,86	1,38	6,54
5	19,94	17,36	3,61	2,89	9,56
10	26,74	23,61	7,40	6,38	14,74

Table 7.

In the table the values z_p are given for which $P^{(n)}_{\lambda,m}\{\tilde{\lambda}_3 > z_p\} = p$. m, λ are unknown and estimated.

λ \ p	0,05	0,1	0,9	0,95	Empirical mean
0,00001	398	116	0,69	0,49	815
0,0001	423	128	0,71	0,50	3100
0,01	423	150	0,75	0,51	3200
0,1	710	179	0,85	0,57	5106
1	1168	326	1,11	0,79	10772
2	2039	426	1,58	1,23	–
5	3531	602	3,37	2,26	–
10	3950	1331	5,71	4,92	–

meter λ has a known characteristic function (see A r a t ó [5] III, or [6]). The characteristic function of the sufficient statistics

$$s_1^2 = \frac{1}{2}[\xi_1^2(0) + \xi_1^2(1) + \xi_2^2(0) + \xi_2^2(1)] ,$$

$$s_2^2 = \int_0^1 [\xi_1^2(t) + \xi_2^2(t)]dt ,$$

is the square of the function $\varphi(\alpha_1, \alpha_2)$ given in (1.3). The calculations of the distribution function of the maximum likelihood estimator were carried out on a computer. The table of quantiles is given in the paper of A r a t ó [6]. As to other estimators and approximations see A r a t ó [7]. Here we do not intend to present the tables, which can be used also to the approximation of the discrete time case.

The normal approximation of the distribution function may be used only if $\lambda > 100$. The tables may be applied to construct confidence limits for λ. This cannot be done with the help of normal approximation if $\lambda < 10$.

If λ is known and ω is unknown the sufficient statistics are

$$s_2^2 = \int_0^1 [\xi_1^2(t) + \xi_2^2(t)]dt , \quad r = \int_0^1 [\xi_1 d\xi_2 - \xi_2 d\xi_1] = \int_0^1 |\zeta|^2 d\Theta$$

where $\zeta = \xi_1 + i\xi_2 = |\zeta| e^{i\theta(t)}$.

We do not know the joint distribution of these two statistics, but it is a very interesting fact, found by K o l m o g o r o v, (see A r a t ó – K o l m o g o r o v – S i n a y [12]) that the statistic

$$(\hat{\omega}_t - \omega)\left(\int_0^t |\zeta(s)|^2 ds\right)^{\frac{1}{2}} ,$$

where $\hat{\omega}_t = \frac{r}{s_2^2}$ $(t > 0)$, is exactly normally distributed with parameters (0, 1). An elegant proof for this theorem can be found in N o v i k o v [19]. The proof is based on deep results of the stochastic differential equations. From these results it follows that the joint distribution of estimators of parameters λ, ω is known, but it would be for us useful to find the same for the case when the observed process is $\zeta(t) + m$ and $\lambda, \omega,\ m = (m_1, m_2)$ are unknown parameters. In the latter case the situation seems to be quite different than in the one dimensional case, the parameters might be estimated and we would not have such an irregular situation like that in the one dimensional case, formulated in (2.1).

4.

On the basis of results of the preceeding paragraphs we can go over to the multidimensional case, too. Indeed, if the matrix A is given in the standard Jordan form and the characteristic numbers have multiplicity 1, the problem of estimation is reduced to the case investigated in 1 and 3.

We do not know enough if the roots have multiplicity greater than 1. The same difficulties arise if the matrix A is not given in the Jordan form.

The tables of distribution functions, exact ones and those got by simulation method, are all reserved in magnetic tapes. So we are able to determine confidence limits, quantiles and other characteristics by using short programs.

The sequential estimation procedures, proposed by Lipcer and Širjaev, (see their work [15] Širjaev [23] or Novikov [17]) are investigated in the case of one unknown parameter (linear case). It seems to be natural to formulate this procedure in the case of Paragraph 3, when we no longer have a linear problem, indeed

$$d\xi(t) = -\lambda(\xi(t) - m)dt + dw(t) = -\lambda\xi(t)dt + \lambda m dt + dw(t)\,.$$

We do not know results of this type.

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