STATISTICAL PROBLEMS OF THE ELEMENTARY GAUSSIAN PROCESSES

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Part I

STOCHASTIC PROCESSES

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Contents:

Chapter 1:

Basic concepts and definitions

In this book we shall be concerned primarily with the statistical problems of certain types of stochastic processes, or random functions of a variable, which in most practical cases, will mean time.

In the first part of the book we begin with some preliminary materials on stochastic processes. The standard reference will be Gikhman-Skorokhod's book [1] where the reader may find the proofs which are not given here and which are far from the aims of this book.

A stochastic process is a parametrized family of random variables, where the range of random variables is a finite - dimensional Euclidean space, denoted by R^k in the k-dimensional case.

Let be given the parameter or index space T and $t \in T$ denoting the parameter, where in most cases t means the time. The vector random variables $\underline{\xi}^*(t)=(\xi_0(t),\ldots,\xi_{k-1}(t))$ depending on parameter t, where * means the "transpose" of a vector (matrix), form a stochastic process if for any values $t_1, t_2, \ldots, t_n$ ($t_i \in T$, $i=1,2,\ldots,n$) there is given the common probability distribution function of $\underline{\xi}(t_1),\ldots,\xi(t_n)$. That is, for any sets $E_1,\ldots, E_n$ of the k-dimensional Euclidean space R^k

$$P(E_1,\ldots,E_n)=P\{\underline{\xi}(t_1)\in E_2,\ldots,\underline{\xi}(t_n)\in E_n\}$$

is given. $\xi_j(t)$ $(j=0,1,\ldots,k-1)$ are called the components of the process. This gives the direct definition of a stochastic vector process $\underline{\xi}(t)$.

The probabilistic properties of the parametrized set of random variables are uniquely determined by the corresponding finite - dimensional distributions. That this is so is a consequence of the extension theorem of Kolmogorov (see [1] , or Gikhman-Skorokhod [1]). This theorem of Kolmogorov may be applied when T is an interval (in the continuous case) , but the situation is more complicated than in the discrete case.

Generally we say, that on the probability space $(\Omega, \mathcal{F}, P)$ there is given the stochastic process $\xi(t, \omega)$ (the space is Ω and $\omega \in \Omega$ denoting the elements, $\mathcal{F}$ is a σ - algebra with elements $A \in \mathcal{F}$, P is the probability measure), if for every $t \in T$ $\underline{\xi}(t)$ is a random vector variable.

Note that if we have a directly defined stochastic process we can determine the basic probability space in several way.

Supposing a family of random vector variables whose finite dimensional distributions coincide with the given distributions (see Gikhman-Skorokhod [1]), if we take simply the function - value at each t then we get the sample space as the function space X and the process $\underline{\xi}(t, \omega)$ is a function space process, where the mapping $\omega \to \underline{\xi}(t, \omega)$ must be a measurable mapping of Ω into X .

In the whole book, when T is the real line or an interval of it, for simplicity we assume that Ω, or the sample space X , consists of the componentwise continuous vector functions. So we aviod the question of seperability.

We say that $\underline{\xi}(t)$ is continuous with probability one when $\underline{\xi}(t,\omega)$ is continuous in t for almost all ω. In the book we shall be concerned with processes continuous with probability one. In such a case it is natural that we confine ourselfes to a smaller space, the space of continuous functions.

We say that the process $\underline{\xi}(t,\omega)$ is separable if we can find a countable dense set $\{t_i\}$ in T and a set $\mathcal{N}\in\mathcal{F}$ with measure 0 such that for any open set G in T and any arbitrary closed set $E\in R^k$, the set

$$\{\omega:\underline{\xi}(t,\omega)\in E \quad \text{for all}$$

differs from the set

$$\{\omega:\underline{\xi}(t_i,\omega)\in E \text{ for all } t_i\in G\}$$

by a subset of $\mathcal{N}$. Doob has shown (see Gikhman-Skorohod [1]) that for any process (with range in a locally compact space) there exists an equivalent separable process.

We say that two processes $\underline{\xi}(t,\omega)$ and $\underline{\xi}'(t,\omega)$ are <u>equivalent</u> if.

$$P\{\xi(t,\omega)=\xi'(t,\omega)\}=1 \qquad , \text{ for every } t\in T .$$

Later we shall see examples where we choose that process from the class of equivalent processes, which has the best qualities, for example continuity, differentiability etc. (see e.g.

the § on stochastic integrals).

In most cases we do not exhibit the variable ω in $\underline{\xi}(t,\omega)$ even if an integration is according to $P(d\omega)$.

If $\underline{\xi}(t)$ is given in the interval $[a,b]$ we say that there is given a realization on $[a,b]$ of the process, the "sample function", the "trajectory" or "history" of the process. The process is given directly if the space consists of the realizations X.

In the case when T consists of the integer numbers we speak on a stochastic process with discrete time /or a "time series", or a "random sequence"/. The process $\underline{\xi}(t)$ is continuous parameter stochastic process when T is the real line, or a part of it.

The first moment, or expectation, of the process $\underline{\xi}(t)$ is denoted by

$$E\,\underline{\xi}(t) = \underline{m}(t) = (m_0(t),\dots,m_{k-1}(t))^*$$

and it is called the expected /or mean/ value function. By definition $E\underline{\xi}(t) = \int_\Omega \underline{\xi}(t,\omega)\,P(d\omega)$. We always assume, that the second moments

$$E(\xi_j(t) - m_j(t))(\xi_i(s) - m_i(s)) = \sigma_{ij}(t,s)$$

exist. If we arrange them into a matrix $B(t,s) = (\sigma_{ij}(t,s)) = E(\xi(t) - m(t))(\xi(s) - m(s))^*$ which is symmetrical, than we refer to it as the covariance matrix.

We say that the sequence of random variables $\underline{\xi}_n$ tends to the random variable $\underline{\xi}$ in mean square, which will be denoted by l.i.m. $\underline{\xi}_n = \underline{\xi}$, if

$$E|\underline{\xi}_n - \underline{\xi}|^2 = E(\underline{\xi}_n - \underline{\xi})(\underline{\xi}_n - \underline{\xi})^* \rightarrow 0.$$

The stochastic process $\underline{\xi}(t)$ is a called stationary in the wide sense /or second-order stationary/ when

$$m(t) = E\,\underline{\xi}(t) = \text{const},$$

$$B(t,s) = B(t-s)$$

that is, the covariance matrix depends only on the difference $t-s$.

By a strictly stationary vector process $\underline{\xi}(t)$ we mean one for which, for all n, t_1, t_2,..., t_n and h the distributions of $\underline{\xi}(t_1), \dots \underline{\xi}(t_n)$ and $\underline{\xi}(t_1+h), \dots, \underline{\xi}(t_n+h)$ are the same. If process $\underline{\xi}(t)$ has finite mean square, this means that

$$E(\underline{\xi}(t) - \underline{m}(t))(\underline{\xi}(s) - \underline{m}(s))^* = E(\xi(t-s) - \underline{m})(\underline{\xi}(0) - m)^* =$$

$$= B(t-s,0) = B(t-s)$$

i.e. it is stationary in the wide sense.

By a Markov vector process $\underline{\xi}(t)$ we mean one for which, for all n $t_1 < t_2 < \dots < t_n$, $t > t_n$ and arbitrary Borel set and $\underline{X}_1, \dots, \underline{X}_n \in R^h$

$$P(\underline{\xi}(t) \in E \,|\, \underline{\xi}(t_1) = \underline{X}_1, \dots, \underline{\xi}(t_n) = \underline{X}_n) = P(\underline{\xi}(t) \in E \,|\, \xi(t_n) = \underline{X}_n\}$$

holds with probability 1.

A Markov process can be given by the transition probabilities

$$P(\xi(t)\in E \mid \xi(s)=\underline{x}) = P(\underline{x}, s, E, t)$$

and for them the Kolmogorov-Chapman equation

$$(1.1)\quad P(\underline{x}, s, E, t) = \int_{-\infty}^{\infty} P(\underline{y}, \tau, E, t) P(\underline{x}, s, dy, \tau)$$

is valid often P is given by the probability density function

$$P(\underline{x}, s, E, t) = \int_E P(\underline{x}, s, y, t)\, dy$$

The Markov process $\xi(t)$ is a diffusion type one, when the following conditions are satisfied:

a/ for any $\varepsilon > 0$ and $t \geqq 0$ $\quad -\infty < t < \infty$

$$(1.2)\qquad \lim_{\Delta\to 0} \frac{1}{\Delta} \int_{/x-y/>\varepsilon} P(t, x, t+\Delta, dy) = 0,$$

b/ there exist functions $a(t,x)$, $b(t,x)$ such that for any $\varepsilon \geqq 0$, $t \geqq 0$, $-\infty < x < \infty$ the relations

$$(1.3)\quad \lim_{\Delta\to 0} \frac{1}{\Delta} \int_{(x-y)<\varepsilon} (y-x) P(t, x, t+\Delta, dy) = a(t,x),$$

$$(1.4)\quad \lim_{\Delta\to 0} \frac{1}{\Delta} \int_{(x-y)<\varepsilon} (y-x)^2 P(t, x, t+\Delta, dy) = b(t,x),$$

hold.

The functions $a(t,x)$ and $b(t,x)$ are called the coefficients of transition resp. diffusion /or local mean and local dispersion, see later ch.9. the definition of stochastic differential/.

The name "diffusion process" corresponds the fact that the move of a particle in liquor or in gas can be described by this process under very general assumptions. The function $a(t,x)$ describes the trend of the particle in the sense that during a time period of length Δ the particle moves with the distance $a(t,x)\Delta\delta\xi+0(\Delta)$ where $\delta\xi$ is a random variable with mean Δ and dispersion $b(t,x)\Delta+0(\Delta)$

Conditions a/ and b/ are hardly varificable. We give below stricter, but easier conditions for a diffusion process. For $\xi(t)$ to be a diffusion Markov process it is sufficient to have the properties

a*/ for some $\delta>0$

$$1.5 \quad \lim_{\Delta\to 0} \frac{1}{\Delta} \int (y-x)^{2+\delta} P(t,x,t+\Delta,dy)=0,$$

b*/ there exist functions $a(t,x)$ and $b(t,x)$ such, that for all t, x

$$(1.6) \quad \lim_{\Delta\to 0} \frac{1}{\Delta} \int (y-x) P(t,x,t+\Delta,dy)=a(t,x),$$

and

$$(1.7) \quad \lim_{\Delta\to 0} \frac{1}{\Delta} \int (y-x)^2 P(t,x,t+\Delta,dy)=b(t,x).$$

Indeed in this case

$$\int_{|y-x|>\varepsilon} P(t,x,t+\Delta,dy) \leqq \frac{1}{\varepsilon^{2+\delta}} \int |y-x|^{2+\delta} P(t,x,t+\Delta,dy) = o(\Delta)$$

and

$$\int_{|y-x|>\varepsilon}(y-x)P(t,x,t+\Delta,dy)\leq\frac{1}{\varepsilon^{1+\delta}}\int|y-x|^{2+\delta}P(t,x,t+\Delta,dy)=\gamma(\Delta)$$

$$\int_{|y-x|>\varepsilon}(y-x)^2P(t,x,t+\Delta,dy)\leq\frac{1}{\varepsilon^{\delta}}\int|y-x|^{2+\delta}P(t,x,t+\Delta,dy)=\gamma(\Delta).$$

It can be proved /see e.g. Gikhman-Skorokhod [2] p. 65/ that if $\xi(t)$ is a diffusion process and $g(t,x)$ two times continuously differentiable function of x and a continuous function of t then $g(t,\xi(t))$ is also a diffusion process with the coefficients $\bar{a}(t,x)$, $\bar{b}(t,x)$ where

$$(1.8)\quad \bar{a}(t,x)=\frac{\partial}{\partial t}g(t,g^{-1}(t,x))+a(t,g^{-1}(t,x))\frac{\partial}{\partial x}g(t,g^{-1}(t,x))+$$

$$+\frac{1}{2}b(t,g^{-1}(t,x))\frac{\partial^2}{\partial x^2}g(t,g^{-1}(t,x))$$

$$(1.9)\quad \bar{b}(t,x)=b(t,g^{-1}(t,x))\left[\frac{\partial}{\partial x}g(t,g^{-1}(t,x))\right]^2.$$

The reader may compare (1.8) - (1.9) with the Ito formula (see § 9.).

Let there be given a stochastic process $\xi(t)$, $t\geq 0$, and a family of σ-algebras $\mathfrak{F}_t$ with the property $\mathfrak{F}_{t_1}\subseteq\mathfrak{F}_t$ if $t_1\leq t$ and such that $\xi(t)$ is measurable with respect to $\mathfrak{F}_t$. We say that the paire $(\xi(t),\mathfrak{F}_t)$ forms a <u>martingale</u> if $E|\xi(t)|<\infty$, $t\geq 0$, and $E(\xi(t)|\mathfrak{F}_s)=\xi(s)$, $0\leq s\leq t$, with probability 1. We say that the random variable ξ is normally distributed if the characteristic function of it equals

$$E e^{iu\xi} = e^{ium - \frac{1}{2}\sigma^2 u^2}$$

where $m = E\xi$, $\sigma^2 = E(\xi - E\xi)^2$. In the case $\sigma \neq 0$ the random variable ξ has density function

$$f_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

The random vector $\underline{\xi}^* = (\xi_1, \dots, \xi_n)$ is normally distributed when the characteristic function has the form

$$(1.10)\quad E e^{i(\underline{u}, \underline{\xi})} = \exp\left\{ i(\underline{u}, \underline{m}^*) - \frac{1}{2}\underline{u} R \underline{u}^* \right\} = \exp\left\{ i\sum_1^n u_j m_j - \right.$$

$$\left. - \frac{1}{2}\sum_{j,k} u_j u_k \sigma_{jk} \right\},$$

where $m_j = E\xi_j$, $\sigma_{jk} = E(\xi_j - m_j)(\xi_k - m_k)$ and $R = (\sigma_{jk})$ is a symmetric pozitiv semidefinite matrix. If R has rank n the n-dimensional density function of $\underline{\xi}$ is

$$(1.11)\quad f_{\underline{\xi}}(\underline{x}) = (2\pi)^{-n/2} |R|^{-1/2} \exp\{-\tfrac{1}{2}(\underline{x} - \underline{m}) R^{-1} (\underline{x} - \underline{m})^*\}.$$

It is well known that if $\underline{\xi}$ is a Gaussian random vector and $A = (a_{ij})(i = 1, 2, \dots, n;\ \ j = 1, 2, \dots, m)$ is a matrix then

$$\underline{\eta} = A\underline{\xi}$$

is normally distributed with parameters $\overline{\underline{m}} = A\underline{m}$, $\overline{R} = A'RA$. If the joint distribution of $\underline{\xi}$ and $\underline{\eta}$ is normal and they are uncorrelated ($E\xi_i\eta_j = 0$ for $i = 1, \dots, n$; $j = 1, \dots, m$) then they are independent.

We assume that the reader is acquainted with the elementary facts with respect to the normal distribution /the conditional distribution, expected value e.t.c/.

We remind some fact /see e.g. Rao [1] /

1. If $\xi_1, \xi_2, \xi_3, \xi_4$ are normally distributed $(E\xi_i=0)$ then

$$(1.12)\quad E\xi_1\xi_2\xi_3\xi_4 = E\xi_1\xi_2\, E\xi_3\xi_4 + E\xi_1\xi_3 E\xi_2\xi_4 + E\xi_1\xi_4 E\xi_2\xi_3 .$$

2. If $E(\xi_n-\xi)^2 \to 0$ and ξ_n are normally distributed then ξ has also normal distribution.
3. A necessary and sufficient condition for normally distributed random vectors to converge in distribution is that

$$E\underline{\xi}_n = \underline{m}_n \longrightarrow \underline{m} \text{ and } E(\underline{\xi}_n-\underline{m}_n)(\underline{\xi}_n-m_n)^* \longrightarrow R.$$

4. The random vector $\underline{\xi}$ in R^n in normally distributed if and only if when $(\underline{\xi}, \underline{u})$/the scalar product of wo vectors/ is a random variable with normal distribution for every $u \in R^n$.

The process $\underline{\xi}(t)$ is a Gaussian one /or normal process/ if its all finite dimensional distributions are Gaussian.

The measure $\underline{P}_\xi$ generated by the variables $\underline{\xi}(t)$ is called a Gaussian /or normal/ measure.

A Gaussian process $\underline{\xi}(t)$ is determined by the mean value function $\underline{m}(t) = E\underline{\xi}(t)$ and by the covariance function $B(s,t) = E(\underline{\xi}(s)-m(s))(\underline{\xi}(t)-\underline{m}(t))^*$. $\underline{m}(t)$ is an arbitrary function but $B(s,t)$ must be nonnegative definite, i.e. for arbitrary real numbers c_{t_i} and integer n

$$(B\underline{c}, \underline{c}) - \sum_{i,j=1} c_i\; c_j\; B_{ij}(s,t) \geq 0 .$$

Exercises

1. Prove that the process $\xi(t)$ is Gaussian if and only if every linear combination

$$c_{t_1}\xi(t_1)+\dots c_{t_n}\xi(t_n),$$

$(n<\infty;\ t_1,\dots,t_n$ arbitrary, c_{t_i} real arbitrary numbers,$)$ is a Gaussian random variable.

2. On the basis of Kolmogorov's theorem prove that for every $m(t)$ function and positive definite function $B(s,t)$ (i.e. $\sum_{i,j=1}^{n} c_{t_i} c_{t_j} B(t_i,t_j)\geq 0$, where n is an integer, c_{t_i}, c_{t_j} are arbitrary real$)$ there exist a probability space $(\Omega,\mathcal{F},P)$ and stochastic process $\xi(t)$ that $E\,\xi(t)=m(t)$ and cov $(\xi(s),\xi(t))=B(s,t)$.

3. Let $\xi_1,\xi_2,\dots,\xi_n$ be a Gaussian system with $E\xi_i=0$ and covariance matrix B where rank of B is $r\leq n$. It is known that there exists an orthogonal transformation C $(CC^*=I)$ for which $C^*R\,C$ has diagonal form. Prove that there exist r independent Gaussian random variables $\eta_1,\dots,\eta_r$ such that ξ_i (for every i) is a linear combination of them. Further, if $r<n$ prove that there exist exactly $n-r$ linear relations between the variables $\xi_1,\dots,\xi_n$.

4. Let $\xi_1,\xi_2,\dots$ be a Gaussian independent sequence with $E\,\xi_i=0$. Prove that $\sum_1^\infty \xi_i^2<\infty$ with probability one if and only if $\sum_1^\infty E\xi_i^2<\infty$.

5. Let $\xi_1,\xi_2,\dots$ finite or infinite sequence of random variables with $E\,\xi_i=0$, $E\xi_i^2<\infty$. We suppose that they are linearly independent. If $(\xi_i,\xi_k)=E\,\xi_i\,\xi_k$ and

$$\tilde{\eta}_i = \begin{vmatrix} \xi_i & \xi_{i-1} \dots & \xi_1 \\ (\xi_i,\xi_1) & (\xi_{i-1},\xi_1)\dots & (\xi_1,\xi_1) \\ \vdots & & \\ (\xi_i,\xi_{i-1}) & (\xi_{i-1},\xi_{i-1})\dots & (\xi_{i-1},\xi_1) \end{vmatrix}$$

then $\eta_i = \tilde{\eta}_i / E\tilde{\eta}_i^2$ form an orthonormal sequence of random variables, i.e. $E\eta_i\eta_j = \begin{cases} 1 & i=j, \\ 0 & i\neq j. \end{cases}$

6. Let $\Theta = \{\Theta_1, \dots, \Theta_k\}$ and $\underline{\xi} = \{\xi_1, \dots, \xi_k\}$ be two random vectors, and the common distribution of $\underline{\Theta}$ and $\underline{\xi}$ be normal; if, moreover, the matrix cov $(\underline{\xi},\underline{\xi})$ has an inverse $(\operatorname{cov}^{-1}(\xi,\xi))$ then

$$E(\underline{\Theta}|\underline{\xi}) = E\underline{\Theta} + \operatorname{cov}(\underline{\Theta},\underline{\xi})\ \operatorname{cov}(\underline{\xi},\underline{\xi})(\underline{\xi} - E\underline{\xi})$$

$$\operatorname{cov}(\underline{\Theta}|\underline{\xi}) = \operatorname{cov}(\underline{\Theta},\underline{\Theta}) - \operatorname{cov}(\underline{\Theta},\underline{\xi})\operatorname{cov}^{-1}(\underline{\xi},\underline{\xi})\operatorname{cov}^{*}(\underline{\Theta},\underline{\xi})$$

7. Prove that the definition of the Markov processes may be replaced by anyone of the following

a/ There are families of σ-algebras $\mathfrak{F}_t$ and $\mathcal{G}_t$ such that:

α.) $\mathfrak{F}_t \subset \mathfrak{F}_s$, $\mathcal{G}_t \supset \mathcal{G}_s$ if $t < s$;

β.) $\xi(t)$ is measurable with respect to both $\mathfrak{F}_t$ and $\mathcal{G}_t$

γ.) the sets of $\mathfrak{F}_t$ and $\mathcal{G}_t$ are independent under the condition of $\mathfrak{F}_t \cap \mathcal{G}_t$ with probability 1., i.e. if $A \in \mathfrak{F}_t$ $B \in \mathcal{G}_t$, then

$$(*) \quad P(A\cap B|\mathfrak{F}_t \cap \mathcal{G}_t) = P(A|\mathfrak{F}_t\cap\mathcal{G}_t)P(B|\mathfrak{F}_t\cap\mathcal{G}_t).$$

(The σ-algebras $\sigma(\xi(s): s \leq t)$ and $\sigma(\xi(\tau): \tau \geq t)$ may be chosen for $\mathcal{F}_t$ resp. $\mathcal{G}_t$).

b/ For any t and any bounded, $\mathcal{G}_t$ - measurable random variable η we have

$$(**) \quad E(\eta \mid \mathcal{F}_t) = E(\eta \mid \mathcal{F}_t \cap \mathcal{G}_t) \qquad \text{a.s.}$$

c/ for $s \geq t$ and any bounded $\varphi = \varphi(x) \qquad (x \in R^{1})$

$$(***) \; E(\varphi(\xi(s)) \mid \mathcal{F}_t) = E(\varphi(\xi(s)) \mid \mathcal{F}_t \cap \mathcal{G}_t) \quad \text{a.s.}$$

(Hint: a./ It is enough to prove $(*)$ for any finite dimensional A and B.

b./ Prove $(**)$ first for characteristic functions of sets from $\mathcal{G}_t$.

c./ Obviously b.) $\Longrightarrow$ c.). Prove $(**)$ from $(***)$ using the hint for b.

8. Prove that for any diffusion process $\xi(t)$ with continuous coefficient of transition $a(t,x)$ and coefficient of dispersion $b^2(t,x)$ and any continuous bounded function $\varphi(x)$ such that the function

$$u(s,x) = E\left[\varphi(\xi(t)) \mid \xi(s)\right]_{\xi(s)=x} \quad , \qquad (s \leq t)$$

has bounded derivatives of first and second order with respect to x the function $u(s,x)$ has the derivative $\dfrac{\partial u(s,x)}{\partial s}$ and the equation

$$(*) \qquad -\frac{\partial u}{\partial s} = a(s,x)\frac{\partial u}{\partial x} + \frac{1}{2} b^2(s,x)\frac{\partial^2 u}{\partial^2 x}$$

is satisfied in the region $s\in(0,t)$, $x\in R$, and the boundary condition

$$\lim_{s\nearrow t} u(s,x)=\varphi(x)$$

holds.

(Hint: The boundary condition is a direct consequence of the boundedness and continuity of $\varphi(x)$. To prove (*) show first that for any $0<s_1<s_2<t$

$$u(s_1,x)=\int u(s_2 z)P(s_1,x;\ s_2,\ dz).$$

Then expanding $u(s_1,x)$ into Taylor series with respect to x take the limit $s_2-s_1\to 0$ in

$$\frac{u(s_1,x)-u(s_2,x)}{s_2-s_1}\ )$$

9. Prove that if for a diffusion process $\xi(t)$ the conditions a/ and b/ are satisfied uniformly in x and the partial derivatives

$$\frac{\partial p(s,x;t,y)}{\partial t},\ \frac{\partial}{\partial y}\Big(a(t,y)\,p(s,x;t,y),\ \frac{\partial^2}{\partial y^2}\,(b^2(t,y)p(s,x;t,y))\Big)$$

exist, then $p(s,x,t,y)$ satisfies the equation

$$\frac{\partial}{\partial t}p=-\frac{\partial}{\partial y}[a\cdot p]+\frac{1}{2}\frac{\partial^2}{\partial y^2}\ [b^2\cdot p]$$

/Hint: prove that for any twice continuously differentiable function $g(y)$ disappearing outside a finite interval we have

$$\int\frac{\partial}{\partial t}p(s,x;t,y)g(y)dy=$$

$$\int\left\{-\frac{\partial}{\partial y}\left[a(t,y)p(s,x;t,y)\right]+\frac{1}{2}\frac{\partial}{\partial y^2}\left[b^2(t,y)p(s,x;t,y)\right]\right\}g(y)dy,$$

for this prove first that

$$\lim_{h\to 0}\frac{1}{h}\left[\int g(y)p(s,x;s+h,y)dy-g(x)\right]=$$

$$=a(s,x)g'(x)+\frac{1}{2}b^2(s,x)g''(x),$$

then use the Markov equation

$$p(s,x;t+h,y)=\int p(s,x;t,z)\,p(t,z,t+h,y)\,dz$$

and integrate by parts in the expression for

$$\frac{\partial}{\partial t}\left[\int p(s,x;t,y)\,g(y)dy\right].$$

A random element ξ in a Hilbert space H is called Gaussian if for every $u\in H$ the scalar product (ξ, u) is a normally distributed random variable /see remark 4. for random vectors/.

Let us consider a set of random variables $\{\xi\}$ and assume that for every ξ (for simplicity $M\underline{\xi}=\underline{0}$) $M|\xi|^2<\infty$. The linear space generated by the scalar product /the "inner product" $(\underline{\xi}, \underline{\eta})=M\underline{\xi}\,\underline{\eta}^*$ can be extended to a Hilbert space. This Hilbert space is generated by $\{\xi\}$ and we denote it H_ξ. In our case the Hilbert space is called a vector Hilbert space.

If $\{\xi\}\subset\{\eta\}$ then $H_\xi\subset H_\eta$. Let $\underline{\xi}(t)$ be a stationary process then for the Hilbert space H_ξ^t generated by the random variables $\underline{\xi}(s)$, $s\leqq t$, $H_\xi^t\leqq H_\xi^{t+1}$.

Chapter 2:

Regularity and singularity

Let us denote by H_ξ^t the subspace generated by $\xi(s), s \leq t$, and let

$$H_\xi^{-\infty} = \bigcap_t H_\xi^t \quad , \quad H_\xi^{\infty} = \bigcup_t H_\xi^t$$

i.e. H_ξ^∞ is the Hilbert space generated by the process $\xi(t)$. If $H_\xi^{-\infty}$ consists only of the element 0 we say that the process is linearly regular /purely non deterministic/.

When $H_\xi^\infty = H_\xi^{-\infty}$ we say that the process is linearly singular /purely deterministic/.

Regularity means, that the future always contains new information which is uncorrelated with the past.

When $\xi(t)$ is linearly regular there exists a sequence C_k such that $\xi(t) = \sum_{k=0}^{\infty} C_k \varepsilon(t-k)$ with uncorrelated $\varepsilon(t)$. This is the so called Wold expansion.

Example 1. For $|\varrho| < 1$ the process

$$(2.1) \qquad \xi(t) = \sum_{n=0}^{\infty} \varrho^n \varepsilon(t-n),$$

where $\varepsilon(n)$ is a sequence of independent indentically distributed random variables $(M\varepsilon(t) = 0, M\varepsilon^2(n) = 1)$, is stationary and regular.

Example 2. If $|\varrho| > 1$ the process

$$\xi(t) = -\sum_{n=1}^{\infty} \varrho^{-n} \, \varepsilon(t+n)$$

is stationary and regular, where $\varepsilon(t)$ is the same as in Example 1.

It is remarkable that the processes of examples 1 and 2 satisfy the equation /a stochastic difference equation/

$$\xi(t) = \varrho\, \xi(t-1) + \varepsilon(t). \tag{2.2}$$

In example 1 the process is a Markov one depending on the past, but in example 2 it has the Markov property depending on the future. In example 2. $\varepsilon(t)$ and $\xi(t-1)$ are not independent as in example 1.

It is a well known fact that if we have a series of Hilbert spaces with the property $H_{t+1} \geqq H_t$ and for any element $\xi \in H_0$ then the projection of ξ to H_t tends to 0 in norm if $t \to -\infty$ then $\wedge H_t$ reduces to the element 0.

Using this fact we can show the regularity of both processes. In example 1 the projection of $\xi(0)$ to H_t is $\sum_{n=t}^{\infty} \varrho^n \varepsilon(-n)$ and this $\|\sum_{n=t}^{\infty} \varrho^n \varepsilon(-n)\| \to 0$ if $t \to -\infty$. In example 2 it follows from /x/ that $M\, \xi(t)\, \xi(0) = -\varrho^{+t^{-1}}$ $M\, \xi^2(0)\ (t<0)$. From this fact we get that if $\tilde{\varrho}(0)$ is the projection of $\xi(0)$ on $H_t\ (t<0)$ than $M|\tilde{\xi}(0)|^2 \leqq c_1 \varrho^t \to 0$ when $t \to -\infty$.

<u>Example 3</u>. Let ξ_0, ξ_1 be independent random variables

$(M\xi_i=0,\ D^2\xi_i=1)$. $\quad \xi(t)=\xi_0\sin t+\xi_1\cos t$

is stationary and singular. In this case it is trivial that

$$H_\xi^t=H_\xi^{t+1}=H_\xi^{-\infty}=H_\xi^{\infty}.$$

Example 4. Pinsker gave an interesting example for a two dimensional process $\underline{\xi}(t)^*=(\xi_1(t),\ \xi_2(t))$ which is regular, but the process $\underline{\eta}(t)=\underline{\xi}(-t)$ is singular. /It may be proved that in the one dimensional case if $\xi(t)$ is regular than $\eta(t)=\xi(-t)$ has the same property./

Let $\xi_1(t)$ be an independent stationary process with $M\xi_1=0$, $M\xi_1^2=\varrho$ and $\xi_2(t)=\sum\limits_{k=0}^{\infty}c_k\xi_1(t-k)$ $(\sum c_k^2<\infty)$, $\underline{\xi}_2(t)$ is obviously regular.

If

(2.3) $\quad \lim\limits_{t\to\infty}\frac{1}{c_t^2}\sum\limits_{k>t}\longrightarrow 0$

then the process $\underline{\eta}(t)=\underline{\xi}(-t)$ is singular. It is sufficient to prove $\underline{\eta}(0)\in H_\eta^{-1}$ because of stationarity. Indeed H_η^{-1} contains the elements $\xi_1(1),\ \xi_1(2),\ \dots$

and $\xi_2(1)=\sum\limits_{k=0}^{\infty}c_k\,\xi_1(1-k)$, $\ \xi_2(2)=\sum\limits_{k=0}^{\infty}c_k\,\xi_1(2-k)$, $\dots$

Hence H_η^{-1} contains the elements $\frac{1}{c_n}\sum\limits_{k=n}^{\infty}c_k\,\xi_1(n-k)$ $(n=1,2,\dots)$. Further

$$\|\xi_1(0)-\frac{1}{c_n}\sum_{k=n}^{\infty}c_k\,\xi_1(n-k)\|^2=\frac{1}{c_n^2}\sum_{k>n}c_k^2$$

which tends to 0 by (2.3).

Let we denote by A_s^t the σ-algebra generated by the random variables $\xi(u), s \leq u \leq t$, i.e. by the sets of type $\omega : \xi_{k_1}(t_1) \in E_1, \ldots, \xi_{k_n}(t_n) \in E_n$, where $s \leq t_k \leq t$ for every $k = 1, 2, \ldots, n$ and let

$$A^- = \lim_{t \to -\infty} A^t_{-\infty} ,$$
$$A^+ = \lim_{t \to \infty} A^\infty_t .$$

We say that the stationary process $\xi(t)$ is regular, if the σ-algebra A^- is a trivial one, this means that it contains only sets of probability 1 or 0.

From the $0-1$ law of Kolmogorov it follows that an independent sequence is always regular.

Let denote $H^{(s,t)}$ the Hilbert space generated by the random variables $\eta (E\eta = 0)$, which are measurable with respect to A_s^t and integrable with their square. Regularity means that

$$(2.4) \qquad \bigcap_t H^{(-\infty, t)} = 0 \quad .$$

That the regularity follows from /x/ can be easily seen, because $\chi_A \in H^{(s,t)}$ when $A \in A_s^t$. On the other hand for any $\eta \in H^{(s,t)}$ there exists $\sum_1^N c_k \chi_{A_k}$ for which $\|\sum_1^N c_k \chi_{A_k} - \eta\| < \varepsilon$ and $A_k \in A_s^t$ $(k = \overline{1, N})$ and from regularity follows (2.4).

Let T denote the shift operator $\xi(Tt) = \xi(t+1)$, then from stationarity follows that the operator $U\xi(t) = \xi(Tt)$ is isometric and it can be proved that U may be extended to a unitary operator on H_ξ /see Rozanov [1] p.72./

From /x/ it follows, that if $\xi(t)$ is regular then for every $\eta \in H_\xi$ the stationary process $\eta(t) = U^t_\eta$ is linearly regular.

<u>Theorem 1</u>. For Gaussian processes regularity and linear regularity are equivalent.

<u>Proof</u>. From the Wold decomposition it follows that for the Gaussian processes $\xi(t)$ there exists a sequence of independent. Gaussian sequence of random variables $\xi(t)$ so that $A_s^t(\xi) = A_s^t(\xi)$. But for $\xi(t)$ the zero-one law is true and hence it is true for $\xi(t)$ too.

<u>Theorem 2</u>. Sufficient and necessary condition for regularity is the following

$$(2.5) \qquad \sup_{B \in A_{-\infty}^t} |P(AB) - P(A)P(B)| \to 0$$

when $t \to -\infty$, for any $A \in A_{-\infty}^{\infty}$.

<u>Proof</u>. Sufficiency. Let $A \in A_{-\infty}^{-\infty}$ and $B = A$, then from (2.5) follows that $P(A) = P^2(A)$ i.e. $P(A) = 0$ or 1.

<u>Necessity</u>. If $\xi(t)$ is regular, then it is linearly regular and for every $\eta \in H_\xi$ the projection of η on H_ξ^t, $\eta^{(t)}$, has the property $\|\eta(t)\| \to 0$.

If $\xi \in H_\xi^t$ then for any $\eta \in H_\xi$

$$(2.6) \quad (\eta, \xi) = (\eta(t), \xi) \text{ and } |(\eta, \xi)| \leq \|\eta(t)\| \, \|\xi\|.$$

Let $A \in A_{-\infty}^{\infty}$ and $B \in A_{-\infty}^{t}$ then from (2.6) follows for $\eta = \chi_A - P(A)$, $\xi = \chi_B - P(B)$ that $|(\eta,\xi)| P(AB) - P(A)P(B)| \leq \|\eta(t)\|_{\to 0}$ when $t \to -\infty$ which does not depend on B.

A stronger condition than (2.5) is the uniform mixing condition which we define in the following way. If

$$\sup_{\substack{A\in A_{-\infty}^{t} \\ B\in A_{t+\tau}^{\infty}}} |P(AB)-P(A)P(B)| \longrightarrow 0 \qquad (2.7)$$

when $\tau \longrightarrow \infty$ then $\xi(t)$ is said to satisfy the uniform mixing condition. It was introduced by M. Rosenblatt.

Exercises

For the process $\xi(t)$ let us denote by $P(\xi(t))$ the Hilbert-space generated by polinomials $\sum c_{t_1,\dots,t_n}\, \xi_{(t_1)}^{\alpha_1},\dots,\xi_{(t_n)}^{\alpha_n}$ in mean square norm, and by $M(\xi(t))$ the Hilbert-space of random variables with finite second moment and measurable with respect to the σ-field $A_{-\infty}^{\infty}$

1. Prove, that $P(\xi(t)) = M(\xi(t))$ under the following condition: there exists a $c(t)>0$ such, that $E\, e^{c(t)|\xi(t)|} < \infty$ /Notice, that this condition is sufficient for the solvability of the problem of moments for the individual distributions/ $F_t(x)=P(\xi(t)<x)$.) (Hint: It is sufficient to prove, that finite, bounded, continuous functions of n variables $f(\xi(t_1),\dots,\xi(t_n))$ may be approximated by polinomials in L^2 norm. For this purpose prove, that finite, bounded, continuous functions of one variable $g(\xi(t))$ may be approximated by polinomials in L^{2n} norm. Approximate then at first by periodic functions, then use the second Weierstrass approximation theorem, and the power series expansion of trigonometric functions.)

2. Suppose, that $\xi(t)$ is a Gaussian process, and $\{\eta_\lambda\}$ is a complete orthonormal system in H_ξ^∞. Denote by $h_n(x)$ the n -th Hermite polinomial. The polinomials

$$\Psi_{p_1,\ldots,p_k;\ \lambda_1,\ldots,\lambda_k} = h_{p_1}(\lambda_1),\ldots,h_{p_k}(\lambda_k)$$

$(p_1+,\ldots,+p_k=n,\ \lambda_1,\ldots,\lambda_k$ are different) from a complete orthonormal system in the space $P_n=\hat{P}_n \ominus \hat{P}_{n-1}$ where $\hat{P}_n$ is the span of the polinomials of degree at most .
(Hint: Recall, that the Hermite polinomials are orthonormal with respect to the weight function $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$).
Consequence: any $\eta\in M(\xi(t))$ has the representation

$$\eta = \eta_0 + \sum_{(n)} \sum_{(\lambda)(p)} \eta \begin{pmatrix} \lambda_1,\ldots,\lambda_k \\ p_1,\ldots,p_k \end{pmatrix} \Psi_{p_1,\ldots,p_k;\ \lambda_1,\ldots,\lambda_k}$$

where the coefficients $a\begin{pmatrix} \lambda_1,\ldots,\lambda_k \\ p_1,\ldots,p_k \end{pmatrix}$ are uniquelly determined by the formulas

$$\eta \begin{pmatrix} \lambda_1,\ldots,\lambda_k \\ p_1,\ldots,p_k \end{pmatrix} = E\,\eta\,\Psi_{p_1,\ldots,p_k;\ \lambda_1,\ldots,\lambda_k}$$

/Cameron-Martin expansion/.

3. Let the sequence $\{\eta, \xi_1,\ldots,\xi_n,\ldots\}$ of random variables have jointly normal distribution. The optimal approximation /in L^2 norm/ of the random variable η^n by elements from $M(\xi(t))$ belongs to $P_n(\xi)$
(Hint: Use the uniqueness of the Cameron-Martin expansion.)

4. Prove the Wold expansion.

Chapter 3:

The Brownian motion process /Wiener Process/

The process $w(t)$ (for $t \geq 0$) is called a Brownian motion process /or Wiener Process/ if it is

a/ Homogeneous, i.e. the distribution of $w(t+h)-w(t)$ does not depend on t.

b/ A process with independent increments, i.e. for every $t_1 < t_2 < \ldots < t_n$ and n the random variables are
$\eta_n = w(t_n) - w(t_{n-1}), \ldots, \eta_2 = w(t_2) - w(t_1), \eta_1 = w(t_1)$
independent.

c/ A Gaussian process, for which $w(0) \equiv 0$, $M w(t) = 0$, $M w^2(t) = \sigma^2 t$.

We shall investigate only continuous Brownian motion processes.

From the definition it follows that

$$\underline{P}\{a < w(k) < b\} = \underline{P}\{a < w(t+k) - w(t) < b\} = \frac{1}{\sqrt{2\pi k}} \int_a^b e^{-\frac{u^2}{2k}}\, du,$$

and the characteristic function of $w(h)$ is given by

$$M e^{izw(k)} = e^{-\frac{z^2 k}{2}}$$

It is trivial that a sufficient and necessary condition for the process $w(t)$ to be a Brownian motion process is the following: for every $0 = t_0 < t_1 <, \ldots, < t_n$, n and $z_0, z_1, \ldots, z_n$ the relation

$$E\exp\left\{i \sum_{k=1}^{n} z_k \left[w(t_k) - w(t_{k-1})\right] + i z_0 w(t_0)\right\} = \exp\left\{\frac{1}{2} \sum z_k^2 (t_k - t_{k-1})\right\}$$

holds. This formula will be used to verify is a process a Brownian motion one or not.

We shall prove some theorems concerning Brownian motion processes.

<u>Theorem 1.</u> If $m(t)$ is a differentiable function with $\int_0^T |m'(t)|\,dt < \infty$ and $\eta(t) = m(t) + w(t)$ then the variable

$$\xi_n = \frac{f_\eta(t_1, \ldots, t_n)}{f_w(t_1, \ldots, t_n)}$$

tends, with probability 1 /when max/ $t_i - t_{i-1}$ /$\longrightarrow 0$ /, to a random variable

$$\xi = \exp\left\{ -\frac{1}{2}\int_0^T [m'(t)]^2\,dt + \int_0^T m'(t)\,d\eta(t)\right\}. \tag{3.1}$$

<u>Proof.</u> It is easy to calculate

$$f_\eta(t_1, \ldots, t_n) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} (t_i - t_{i-1})^{-1/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{1}{t_i - t_{i-1}} (\eta_i - \eta_{i-1} - m(t_i) + m(t_{i-1}))^2\right\}$$

$$f_w(t_1, \ldots, t_n) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} (t_i - t_{i-1})^{-1/2} \left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{1}{t_i - t_{i-1}} (w_i - w_{i-1})^2\right\}$$

where $\eta_i = \eta(t_i)$, $w_i = w(t_i)$. Here we get

$$\xi_n = \frac{f_\eta(t_1, \ldots, t_n)}{f_w(t_1, \ldots, t_n)} (\eta(t)) = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{1}{t_i - t_{i-1}} \Big[(m(t_i) - m(t_{i-1}))^2 - \right.$$

$$\left. - 2\frac{(m(t_i) - m(t_{i-1}))}{t_i - t_{i-1}} (\eta_i - \eta_{i-1})\Big]\right\} =$$

$$= \exp\left\{ -\frac{1}{2}\sum_{i=1}^{n}\left[\frac{m(t_i)-m(t_{i-1})}{t_i-t_{i-1}}\right]^2 (t_i-t_{i-1}) + \right.$$

$$\left. + \sum_{i=1}^{n}\frac{(m(t_i)-m(t_{i-1}))}{t_i-t_{i-1}}(\eta(t_i)-\eta(t_{i-1}))\right\}.$$

Under the assumptions of the theorem the first sum tends to $\frac{1}{2}\int[m'(t)]^2\,dt$ and the second tends in mean square to $\int m'(t)\,d\eta(t)$. We may choose such a subsequence for which the second sum is convergent with probability 1.

__Theorem 2.__ If $\max(t_i-t_{i-1})\to 0$, $(0=t_0<t_1<,\dots<t_{2^n}=\tau)$, then

$$(3.2) \qquad \xi_n=\sum_{i=1}^{2^n}(w(t_i)-w(t_{i-1}))^2 \to \sigma^2 T$$

with probability 1.

__Proof.__ The random variable

$$\xi_n=\sum_{i=1}^{2^n}(w(t_i)-w(t_{i-1}))^2$$

has a χ^2 distribution with 2^n degrees of freedom, and we have

$$E\,\xi_n=\sigma^2\sum_{i=1}^{2^n}(t_i-t_{i-1})=\sigma^2\,T$$

$$E\,\xi_n^2=\sum_{i,j}E(w(t_i)-w(t_{i-1}))^2(w(t_j)-w(t_{j-1}))^2=$$

$$=\sum_{i\neq j}E(w(t_i)-w(t_{i-1}))^2(w(t_j)-w(t_{j-1}))^2+2\sum_{i=1}^{2^n}E(w(t_i)-w(t_{i-1}))^4=$$

$$= \sigma^4 \tau^2 + \frac{\sigma^4 \tau^2}{2^{n-1}}$$

So we get

$$D^2 \xi_n = E\xi_n^2 - (E\xi_n)^2 = \frac{\sigma^4 T}{2^{n-1}}$$

for the variance

/Here we used the relation (1.12)/

From the Chebishev-inequality

$$P\{|\xi_n - \sigma^2 T| > \varepsilon\} \leq \frac{\sigma^4 T^2}{2^{n-1}\varepsilon^2},$$

and we get at once that ξ_n tends to $\sigma^2 T$ stochastically. As $\sum_{n=1}^{\infty} \frac{c}{2^n}$ is convergent, we deduce from the Borel-Cantelli lemma the convergence with probability 1.

Brownian motions are often considered together with a family of σ-algebras $\{\mathfrak{F}_t\}$ for which $\mathfrak{F}_{t_1} \subseteq \mathfrak{F}_{t_2}$ $(\subseteq A)$, $w(t)$ is measurable with respect to $\mathfrak{F}_t$ and $w(t+h) - w(t)$ is independent of $\mathfrak{F}_t$ (i.e. of the events $B \in \mathfrak{F}_t$). It is possible that $\mathfrak{F}_t = A_0^t$, and always $A_0^t \subseteq \mathfrak{F}_t$.

<u>Theorem 3.</u> /The Markov property of the Brownian motion process./ The process $\eta(t) = w(T+t) - w(T)$ with fixed T, is a Brownian motion process, independent of A_0^T.

<u>Proof</u>. The Brownian motion character of the process $\eta(t)$ follows directly from the fact, that $\eta(t)$ is Gaussian, with independent increments with the same mean and correlation functions as the Brownian motion process. On the other hand for every $0 = t_0 \leq t_1 \leq t_2 \leq, \ldots, \leq t_n = T$

$$\sigma(w(t_0), w(t_1), \ldots, w(t_n)) = \sigma(w(t_0), w(t_1)-w(t_0), \ldots, w(t_n)-w(t_{n-1}))$$

and $\eta(t)$ is independent of the variables on the right hand side.

The question is, that if we replace T by a random variable will this theorem remain true? It turns out that this is the case for a wide class of random variables.

Definition 1. The random variable $\tau(\omega)$ is called a Markov moment (Markov point, or stopping time) with respect to the family of σ-algebras $\{\mathcal{F}_t\}$ if for every

$$\{\omega : \tau(\omega) < t\} \in \mathcal{F}_t$$

For example $\tau = T_0$ /constant/ is a Markov moment. It is easy to see that the first upcrossing time of the level a that is the random variable $\tau_a = \{\min t : w(t) \geqq a\}$ is a Markov moment. Indeed from the continuity of $w(t)$

$$\{\tau_a < t\} = \bigcap_{n=1} \bigcup_{r<t} \{w(r,\omega) > a - \tfrac{1}{n}\} \qquad /r \text{ is rational/.}$$

The random variable ξ which denotes the last moment of crossing the 0 level before reaching the level a is not a Markov moment, as it depends on the events occuring in the "future".

Definition 2. Let τ be a Markov moment with respect the σ-algebras $\mathcal{F}_t$ then we say that $A \in \mathcal{F}_\tau$ if for every $t \geqq 0$

$$A \wedge \{\tau \leq t\} \in \mathcal{F}_t \,.$$

It may be proved that $\mathcal{F}_\tau$ is a σ-algebra.

As we shall see $w(\tau)$ is measurable with respect to $\mathcal{F}_\tau$ for any stopping time τ. Indeed if $\{v_i\} = \left\{\frac{a_i}{b_i}\right\}$ is the set of the rational numbers then, using the fact that $w(t)$ is continuous with probability one, the set $\{\omega : w(\tau) < c \bigcap \{\tau \leq t\}$ can be written in the form

$$\bigcap_N \bigcup_{b_i \geq N} \bigcup_n \bigcup_{\substack{|\tau - v_i| < b_i \\ v_i \leq t}} \left\{ w(v_i) < c - \frac{1}{n} \right\}$$

So it is measurable with respect to $\mathcal{F}_t$.

<u>Theorem 4.</u> /The strong Markov property./ Let $\tau(\omega)$ be with probability 1 finite Markov moment. The process $\eta(t) = w(t+\tau) - w(\tau)$ is a Brownian motion process, independent of $\mathcal{F}_\tau$. The attached family of σ-algebras is $\mathcal{F}_{\tau+t}$.

<u>Proof.</u> We introduce the sequence of random variables $\tau_n(\omega) = \frac{k}{n}$, $\tau(\omega) \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$. Obviously $\tau_n \downarrow \tau$ and is a Markov moment. Let us consider some event $B \in \mathcal{F}_\tau$ and we shall prove that it is independent of $\eta(t_1), \ldots, \eta(t_m)$ where $0 < t_1 <, \ldots, < t_m$ and $\eta(t_i) = w(\tau + t_i) - w(\tau)$ is enough to verify that

$$E\ \chi_B \cdot f(\eta(t_1), \ldots, \eta(t_m)) = P(B)\, E f(\eta(t_1), \ldots, \eta(t_m))$$

for a family of functions f , wide enough. For example we suppose $f \in C_{R^m}$ and $\| f \| = \sup |f| < \infty$

Let

$$\xi = f(w(\tau+t_1)-w(\tau),\dots,w(\tau+t_m)-w(\tau+t_{m-1})),$$

and

$$\xi_n = f(w(\tau_n+t_1)-w(\tau_n),\dots,w(\tau_n+t_m)-w(\tau_n+t_{m-1})).$$

As f and $w(t)$ are continuous $\xi_n \longrightarrow \xi$ with probability 1. From Lebesgue theorem and the fact that $\|\xi_n\| \leq \|f\|$

$$E\,\chi_B\,\xi = \lim_{n\to\infty} E\,\chi_B\,\xi_n \;.$$

But

$$E\chi_B\,\xi_n = E \sum_{k=1}^{\infty} \chi_B \cdot \chi_{\{\tau_n=\frac{k}{2^n}\}} \;\xi_n = \sum E(\chi_B \cdot \chi_{\{\tau_n=\frac{k}{2^n}\}}) \cdot \xi_n =$$

$$= \sum_{k=1} E(\chi_{B\cap\{\tau_n=\frac{k}{2^n}\}} \cdot \xi_n)\,.$$

Now using the Markov property $B \cap \{\tau_n = \frac{k}{2^n}\} \in A_0^{\frac{k}{2^n}}$ and

$$E\,[\chi_{B\cap\{\tau_n=\frac{k}{2^n}\}} f(w(\tfrac{k}{2^n}+t_1)-w(\tfrac{k}{2^n}),\dots,w(\tfrac{k}{2^n}+t_m)-$$

$$-w(\tfrac{k}{2^m}+t_{m-1}))] =$$

$$= E\left[\chi_{B\cap\{\tau_n=\frac{k}{2^n}\}}\right] E f(w(\tfrac{k}{2^n}+t_1)-w(\tfrac{k}{2^n}),\dots,w(\tfrac{k}{2^n}+t_m)-w(\tfrac{k}{2^n}+t_{m-1})) =$$

$$= P(B\cap\{\tau_n=\tfrac{k}{2^n}\})\,E f(\tilde{\eta}(t_1),\dots,\tilde{\eta}(t_m)),$$

Where $\tilde{\eta}(t_1),\dots,\tilde{\eta}(t_m)$ is a Brownian motion process. We get

$$E\chi_B\xi_n=\sum_{k=1} P(B\cap\{\tau_n=\tfrac{k}{2^n}\})Ef(\tilde{\eta}(t_1),\dots,\tilde{\eta}(t_m))=$$

$$=P(B)Ef(\tilde{\eta}(t_1),\dots,\tilde{\eta}(t_m))\to P(B)Ef(\eta(t_1),\dots,\eta(t_m)),$$

i.e. the process $\eta(t)$ is independent of $\mathcal{F}_\tau$. Replacing $B=\Omega$ and using again the Lebesgue theorem $\eta(t)$ is a Brownian motion process, as $\eta(t)$ is measurable with respect to $\mathcal{F}_{\tau+t}$. By using the strong Markov property of the Brownian motion process we can prove the so called reflection principle /Desire André/.

<u>**Theorem 5.**</u> For $a>0$

$$(3.3)\quad P\{\sup_{0\le t\le T} w(t)>a\}=2\underline{P}\{w(\tau)>a\}=\sqrt{\frac{2}{\pi T}}\int_a^\infty e^{-\frac{u^2}{2T}}\,du.$$

<u>**Proof.**</u> Let τ_a the moment of reaching the level $a(>0)$. We consider

$$\int_0^\infty e^{-\lambda t}\underline{P}\{w(t)>a\}\,dt=E\int_0^\infty e^{-\lambda t}\chi_{(a,\infty)}(w(t))\,dt=$$

$$=E\int_{\tau_a}^\infty e^{-\lambda t}\chi_{(a,\infty)}(w(t))\,dt=E\int_0^\infty e^{-\lambda(\tau_a+s)}\chi_{(a,\infty)}(w(\tau_a+s))\,ds=$$

$$=Ee^{-\lambda\tau_a}\int_0^\infty e^{-\lambda s}\chi_{(a,\infty)}(w(\tau_a+s)-w(\tau_a)+w(\tau_a))\,ds=$$

$$= E e^{-\lambda\tau_a} \int_0^\infty e^{-\lambda s} \chi_{(a,\infty)}(w(s)+a)\, ds$$

where we used the strong Markov property of $w(t)$ and that $w(\tau_a) = a$. Further

$$= E\, e^{-\lambda\tau_a} \int_0^\infty e^{-\lambda s} \chi_{(0,\infty)}(\tilde{w}(s))\, ds = E e^{-\lambda\tau_a} E \int_0^\infty e^{-\lambda s} \chi_{(0,\infty)}(\tilde{w}(s))\, ds =$$

$$= E\, e^{-\lambda\tau_a} \int e^{-\lambda s} \tfrac{1}{2}\, ds = \frac{1}{2\lambda} E\, e^{-\lambda\tau_a}.$$

In such a way we get

$$\int_0^\infty e^{-\lambda t} P\,\{w(t) > a\}\; dt = \frac{1}{2\lambda} E e^{-\lambda\tau_a} = \int_0^\infty \frac{e^{-\lambda t}}{2\lambda} \underline{P}\,\{\tau_a \in dt\} =$$

$$= \frac{1}{2} \int_0^\infty e^{-\lambda t} \underline{P}\,\{\tau_a < t\}\, dt,$$

and from the uniqueness of Laplace transform

$$(3.4) \quad P\{w(t) > a\} = \frac{1}{2} P\{\tau_a < t\}.$$

The last equation is equivalent to the reflexion principle and our theorem is proved.

<u>Remark 1.</u> The distribution of τ_a is called the Wald distribution and we get for it

$$(3.5)\quad P\{\tau_a < t\} = \sqrt{\frac{2}{\pi t}} \int_a^\infty e^{-\frac{u^2}{2t}}\,du = \sqrt{\frac{2}{\pi}} \int_{a/\sqrt{t}}^\infty e^{-\frac{u^2}{2}}\,du .$$

The density function

$$(3.6)\quad p_{\tau_a}(t) = \frac{a}{\sqrt{2\pi}} \cdot \frac{e^{-a^2/2t}}{t^{3/2}} .$$

It is interesting to note that

$$p_{\tau_a}(t) \sim \frac{c}{t^{3/2}}, \qquad \text{as} \qquad t \longrightarrow \infty,$$

and so

$$E\,\tau_a = \infty,$$

though it is well known /the reader may prove/, that

$$P\{\tau_a < \infty\} = 1$$

<u>Remark 2.</u> The proof of the theorem may be done in the following intiutive way. Let τ_a denote the first upcrossing moment of level a, where $w(\tau_a) = a$. From this moment let us reflect the trajectories for the line $y = a$. It is obvious that

$$P\{\sup_{0\le t\le T} w(t) > a,\ w(\tau) > a\} = P\{w(T) \ge a\} .$$

On the other hand from strong Markov property the behaviour of the process $w(t) - w(\tau)$ for $t > \tau$ is independent of A_0^τ and $w(t) - w(\tau)$ is symmetrically distributed, this means

$$\mathcal{P}\{\sup_{0\leq t\leq T} w(t)>a, w(T)<a\}=\mathcal{P}\{\sup w(t)>a, w(T)\geq a\}=\mathcal{P}\{w(T)\geq a\}.$$

From this two equations we get the desired equality.

Multidimensional Brownian Motion. A process $\underline{\xi}(t)$, taking values from R^m is called an m - dimensional Brownian motion, if $\underline{\xi}(t)$ is homogeneous, $\underline{\xi}(0)\equiv 0$, continuous with probability 1, having independent increments, for which the scalar process $(\underline{z}, \underline{\xi}(t))$ is a Brownian motion process for each $\underline{z}\in R^m$ with $|z|=1$, and there is a family of σ-algebras $\{\mathcal{F}_t\}$ in Ω for which $\mathcal{F}_{t_1}\subseteq\mathcal{F}_{t_2}$ $(\subseteq A_0^t)$, if $t_1\leq t_2$ and $\underline{\xi}(t)$ is measurable with respect to $\mathcal{F}_t$. For such a process we have the relations $E(\underline{z}, \underline{\xi}(t))=0$, $D^2(\underline{z}, \underline{\xi}(t))=t$. The distribution of $\underline{\xi}(t)$ is determined by the density function

$$(1)\qquad p_t(\underline{x})=(2\pi t)^{-\frac{m}{2}} \exp\left\{-\frac{1}{2t}|x|^2\right\},$$

so that for any Borel set $A\in R^m$

$$(2)\quad \mathcal{P}\{\underline{\xi}(t)\in A\}=(2\pi t)^{-\frac{m}{2}}\int_A \exp\left\{-\frac{1}{2t}|x|^2\right\}\mu_m(d\underline{c}),$$

where μ_m is the Lebesgue measure in R^m.

Obviously if U is an orthonormal transformation of R^m and $\xi(t)$ is a Brownian motion in R^m then $U\underline{\xi}(t)$ is a Brownian motion in R^m too.

It easily follows now that if S_ϱ is a ball with radius ϱ with its center in the beginning of the system of coordinates, and τ_ϱ the first exit time of $\underline{\xi}(t)$ from S_ϱ then $\xi(\tau_\varrho)$ is uniformly distributed on the surface of S_ϱ. Strong Markovity for multivariate Brownian motion follows easily from the fact that its coordinates are independent one dimensional Brownian motions.

Theorem 1. For any $c>0, T>0$ the Brownian motion $\underline{\xi}(t)$ has the property

$$\underline{P}\left(\sup_{0\leq t\leq T}|\underline{\xi}(t)|>c\right)\leq 2\underline{P}(|\underline{\xi}(T)|>c).$$

Proof. Let τ be the first exit time from the ball S_c. Then the process $\underline{\xi}(t+\tau)-\underline{\xi}(\tau)$ is an m-dimensional Brownian motion too. Hence

$$\underline{P}\{|\underline{\xi}(T)|>c\}=\underline{P}(\tau<T, |\underline{\xi}(T)-\underline{\xi}(\tau)|>c) =$$

$$=\int_0^T \underline{P}\{|\underline{\xi}(T)-\underline{\xi}(t)+\underline{\xi}(t)|>c|\tau=t\}\underline{P}(\tau\in dt) =$$

$$=\int_0^T \underline{P}(|\underline{\xi}(T)-\underline{\xi}(t)+\underline{z}|>c)\underline{P}(\tau\in dt),$$

where z is any vektor for which $|z|=c$. But

$$\underline{P}\{|\underline{\xi}(T)-\underline{\xi}(t)+\underline{z}|>c\}\geq\underline{P}\{(\underline{\xi}(T)-\underline{\xi}(t),\underline{z})\geq 0\}=1/2,$$

so that

$$\underline{P}(|\underline{\xi}(T)|\geq c)\geq\frac{1}{2}\int_0^T \underline{P}(\tau\in dt)=\frac{1}{2}\underline{P}\{\sup_{0\leq t\leq T}|\underline{\xi}(t)|>c\}$$

proving the theorem.

Exercises

1. /The Wiener-Representation of the Brownian motion process./
   Let $\{H_k(t)\}$ $(0\leq t\leq 1)$ be the Haar's system, i.e.

$H_0(t) = 1$

and if $2^n \leq k < 2^{n+1}$ then

$$H_k(t) = \begin{cases} 2^{\frac{n}{2}} & \text{if } \frac{k-2^n}{2^n} \leq t < \frac{k-2^n+1/2}{2^n} \\ -2^{\frac{n}{2}} & \text{if } \frac{k-2^n+1/2}{2} \leq t < \frac{k-2^n+1}{2^n} \\ 0 & \text{other wise} \end{cases}$$

Furthermore let η_n be independent standard Gaussian random variables. The series $\sum_{n=0}^{\infty} \eta_n \int_0^t H_n(\tau)\, dt$ uniformly converges and represents the Brownian motion process.

(Hint: At first prove, that for deterministic coefficients a_k the series $\sum_{n=0}^{\infty} a_k \int_0^t H_k(\tau) \cdot (\tau)$ uniformly converges under the condition $|a_k| \subset O(k^{\varepsilon})$ $(0 < \varepsilon < 1/2)$ Then verify, that this condition fullfills with probability 1 for the random coefficients η_n. The characteristic functions of the desired distributions can be computed directly.)

2. /The interated logarithm theorem./ If $w(t)$ is the standard Brownian motion process, then

$$P\left(\lim_{t \to \infty} \frac{w_t}{\sqrt{2t \ln|\ln t|}} = 1\right) = 1$$

(Hint: Use the iterated logarithm theorem for the sequence of i.i.d. random variables $w(n) - w(n-1)$ and prove - by means of André's reflection principle and Borel-Cantelli's lemma - that the defect $\sup\limits_{n-1 \leq t < n} (w(t) - w(n-1))$ has order $O(\sqrt{2n \ln \ln n})$ with probability 1.)

3. Prove the local iterated logarithm theorem: If $w(t)$ is the standard Brownian motion process, then

$$P\left\{ \overline{\lim_{t \to 0}} \frac{w(t)}{\sqrt{2t \ln|\ln t|}} = 1 \right\} = 1$$

(Hint: Introduce the new process $\tilde{w}(t) = t\, w(\frac{1}{t})$, show that it is also a standard Brownian motion process, and apply to it the global iterated logarithm theorem.)

4. The local iterated logarithm theorem remains valid for the elementary Gaussian processes $\xi(t)$ too.
(Hint: The difference $\xi(t) - w(t) = a \int_0^t \xi(t)\, dt$ satisfies the relation $P\ \left(\overline{\lim_{t \to 0}} \frac{\xi(t) - w(t)}{t^{\alpha}} < 1 \right) = 1$ for every $0 < \alpha < 1$)

5. With probability 1 the trajectories of the Wiener process $w(t)$ are nowhere differentiable.
Hint: Suppose that the trajectory $w(t)$ has a derivative less than ℓ at a point s.
Then $|w(\frac{j}{n}) - w(\frac{j-1}{n})| < \frac{7\ell}{n}$ for $i = [ns]+1,\ i < j \leq n+3$ and sufficiently large n. Therefore the event " $w(t)$ anywhere differentiable" involves the event

$$B = \bigcup_{\ell \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} \bigcup_{0 \leq i \leq n+1} \bigcap_{i < j \leq i+3} \left\{ |w(\tfrac{j}{n}) - w(\tfrac{j-1}{n})| < \tfrac{7\ell}{n} \right\}$$

Prove that

$$P(B) = 0$$

6. Prove, that for every $\varepsilon > 0$ there exists a compact subset of Wiener trajectories on the interval $[0,1]$ of probability $1-\varepsilon$ (in the sense of the uniform topology).

(Hint. Recall, that the compact subsets of the space of continuous functions are exactly the subsets of uniformly bounded equicontinuous functions. Using the iterated logarithm theorem prove, that for a suitable choose of the constants N_n and δ_n:

$$P\Big(\bigcup_{0\leq t\leq 1}\big((|\xi(t)|>N_n)\bigcup_{-N_n\leq k\leq N_n}((\xi(t)<\tfrac{k}{n})\cap(\xi(t+\delta_n)>\tfrac{k+1}{n})\cup(\xi(t)>\tfrac{k}{n})\cap$$

$$(\xi(t+\delta)<\tfrac{k-1}{n}))\Big)<\frac{\varepsilon}{2^n}$$

Remark: A theorem of Lévy gives the exact estimation of the modulus of the continuity of Wiener trajectories.

$$P\Big[\overline{\lim_{\substack{0\leq t_1-t_2\leq 1\\ t=t_2-t_1\downarrow 0}}}\frac{|w(t_1)-w(t_2)|}{\sqrt{2t\ln(1/t)}}=1\Big]=1$$

The proof of this tehorem is complicated and we need only the above rougher assertion.)

7. Let $\underline{w}^*(t)=\{w_1(t),\dots,w_n(t)\}$ an n-dimensional Brownian motion process $E\underline{w}(t)=0$, $E\underline{w}(t)\underline{w}^*(t)=B_w.t$ where B_w is the local covariance matrix (it is positiv semidefinite). We say that $\underline{w}(t)$ is an n-dimensional Brownian motion process if it is homogeneous, with independent increments, Gaussian and continuous with probability 1.

Prove that if $\underline{w}(t)$ is an n-dimensional Brownian motion process then there exist a matrix C such that

$$C\,\underline{w}(t)=w'(t)$$

and $w'(t)$ is a Brownian motion process with independent components.

Chapter 4:

Differentiation and integration

In the sequel we shall need the following.

Lemma 1. The random variables ξ_h tend to the random variable ξ in mean square when $h \to 0$ if and only if the limit

$$\lim_{h, h' \to 0} E\,\xi_h \xi_{h'}^* = \lim_{h, h' \to 0} (\xi_h, \xi_{h'}^*) = a$$

exists, independently of the choice of h, h'.

Proof. Necessity follows from the inequality

$$|(\xi_h, \xi_{h'}) - (\xi, \xi)| = |(\xi_h, \xi_{h'}) - (\xi_h, \xi) + (\xi_h, \xi) - (\xi, \xi)| \leq$$

$$\leq |(\xi_h, \xi_{h'} - \xi)| + |(\xi_h - \xi, \xi)| \leq \|\xi_h\| \, \|\xi_{h'} - \xi_h\| + \|\xi\| \, \|\xi_h - \xi\| .$$

Sufficiency is a consequence of the relation

$$(4.1) \qquad \|\xi_h - \xi_{h'}\|^2 = (\xi_h - \xi_{h'}, \xi_h - \xi_{h'}) =$$

$$= (\xi_h, \xi_h) - 2(\xi_h, \xi_{h'}) + (\xi_{h'}, \xi_{h'})$$

As the right-hand side of (4.1) tends to 0 as $h, h' \to 0$ so Cauchy's convergence criterion is satisfied. As the Hilbert space of the square-integrable functions is complete, so there exists ξ such, that $\mathrm{l.i.m.}\, \xi_h = \xi$. A consequence of this lemma is that if $\xi_h \to \xi$, $\eta_h \to \eta$ then $E\,\xi_h \eta_h^* = E\,\xi \eta^*$. Another consequence is that a necessary and sufficient condition for the process $\xi(t)$ to be continuous at the point t_0 is the continuity of the trace of the covariance function

$B(t,s)-E\xi(t)\xi^*(s)$ at the point (t_0, t_0) and if $\xi(t_0)\to\xi(t_0)$ $(t \to t_0)$ then

$$E|\xi(t_0)|^2 = \lim_{t\to t_0,\ s\to t_0} B(t,s)$$

The proof of the sufficiency follows directly from the relation

$$E|\xi(t)-\xi(t_0)|^2 = B(t,t) - 2B(t,t_0) + B(t_0,t_0)$$

while the necessity is a consequence of the lemma 1, if we put $\xi_h = \xi(t_0+h)$. We say that the process $\xi(t)$ is differentiable at the point to if the limit

$$\text{l.i.m.}\ \frac{\xi(t_0+h)-\xi(t_0)}{h} = \xi'(t_0)$$

exists.

As

$$E\,\frac{(\xi(t_0+h)-\xi(t_0))\,(\xi(t_0+h')-\xi(t_0))}{h\,h'} =$$

$$= \frac{1}{hh'}\left\{B(t_0+h,\ t_0+h') - B(t_0,t_0+h') - B(t_0+h,t_0) - B(t_0,t_0)\right\}$$

it follows that a necessary and sufficient condition for the differentiability of the process is the existence of the derivative $\left.\frac{\partial^2 B(t,s)}{\partial t\,\partial s}\right|_{t=s=t_0}$. It is easy to show that the expectation of $\xi'(t)$ exists and

$$E\,\xi'(t) = \frac{d}{dt}E\,\xi(t).$$

If $\xi(t)$ is differentiable at every point t of $(0, T)$, then $\xi'(t)$ is a process of finite variance too. We shall show that

$$(4.2) \qquad E\,\xi'(t)\ \xi'(s)^* = \frac{\partial^2 B(t,s)}{\partial t\,\partial s}\ ,$$

$$(4.3) \qquad E\,\xi'(t)\ \xi(s)^* = \frac{\partial B(t,s)}{\partial t}\ ,$$

if for every $t \in (0,T)$ the derivative $\left.\frac{\partial^2 B(t,s)}{\partial t\,\partial s}\right|_{s=t}$ exists. Namely the existence of the limits

$$E\,\xi'(t)\,\xi(s)^* = \lim_{h\to 0} E\,\frac{\xi(t+h)-\xi(t)}{h}\,\xi(s)^* =$$

$$= \lim_{h\to 0} \frac{B(t+h,s)-B(t,s)}{h} \qquad \text{and}$$

$$E\,\xi'(t)\,\xi'(s)^* = \lim_{h,h'\to 0} E\,\frac{\xi(t+h)-\xi(t)}{h} - \frac{\xi(s+h')-\xi(s)}{h'}$$

$$= \underset{h,h'\to 0}{} \frac{B(t+h,\ s+h')-B(t,s+h')-B(t+h,s)-B(t,s)}{h\ h'}$$

follows from the differentiability and the lemma. (So from the differentiability of $B(t,s)$ along the line $t=s$ its differentiability follows for every $t,s \in (0,T)$).

As a consequence we get that the stationary process $\xi(t)$ is differentiable if and only if its covariance function $B(\tau)$ is twice differentiable at the point $\tau = 0$. Then $\frac{d^2 B(\tau)}{d\tau^2}$ exists for any τ and

$$E\,\xi'(t)\ \xi'(t+\tau)^* = \frac{d^2 B(\tau)}{d\tau^2}\ ,$$

$$E\,\xi(t)\,\xi'(t+\tau)^* = \frac{dB(\tau)}{d\tau}\ .$$

Similar relations are true for derivatives of higher order. By the integral

$$(4.4)\qquad \int_a^b \xi(t)\,dt$$

of the process $\xi(t)$ over the interval (a,b) we mean the limit in mean square of the sum

$$\sum \xi(t'_{n_k})(t_{n_k}-t_{n_{k-1}}), \quad a=t_{n_0}<t_{n_1}<\dots<t_{n_N}=b,\; t'_{n_k}\in(t_{n_{k-1}},t_{n_k})$$

where $\max(t_{n_k}-t_{n_{k-1}}) \longrightarrow 0$ as $n \longrightarrow \infty$. By the lemma the integral exists if and only if the limit of the sum

$$E \sum_k \xi(t'_{n_k})(t_{n_k}-t_{n_{k-1}}) \sum_j \xi(t'_{n_j})^*(t_{n_j}-t_{n_{j-1}}) =$$

$$= \sum_k \sum_j B(t'_{n_k}, t'_{n_j})(t_{n_k}-t_{n_{k-1}})(t_{n_j}-t_{n_{j-1}})$$

exists, that is, the function $B(t,s)$ is Riemann integrable over the region $a \leqq t,\; s \leqq b$.

<u>Remark 1.</u> The integral of the process $\xi(t)$ can be defined also in another way. Let us suppose that the process $\xi(t,\omega)$ as the function of the two wariables t,ω is measurable and

$$(4.5)\qquad \int_a^b E|\xi(t)|^2\,dt<\infty$$

Then as we know from the theorem of Fubini, the function $|\xi(t,\omega)|^2$ is integrable over the space $\{[a,b]\times\Omega$ $B\times A,\ \mu\times P\}$, where B is the σ-algebra of Borel sets of the interval $[a,b]$ and μ the Lebesgue measure, and we have

$$E\int_a^b |\xi(t)|^2\,dt = \int_{[a,b]\times\Omega} |\xi(t,\omega)|^2\,d(\mu\times P) =$$

$$= \int_a^b E|\xi(t)|^2\,dt.$$

So the integral $\int_a^b |\xi(t)|^2\,dt$ exists with probability 1 together with the integral $\int_a^b \xi(t)\,dt$. If the functions $f_i(t)$ are square Lebesgue integrable on the interval [a,b] then the integrals $\int_a^b f_i(t)\,\xi(t)\,dt$ also exist and using again the theorem of Fubini we get

$$(4.6)\quad E\int_a^b f_1(t)\;\xi(t)\,dt\left(\int_a^b f_2(t)\,\xi(t)\,dt\right)^* =$$

$$= E\int_a^b\int_a^b f_1(t)\,f_2(\tau)\,\xi(t)\,\xi^*(\tau)\,dt\,d\tau =$$

$$= \int_a^b\int_a^b f_1(t)\,B(t,\tau)\,f_2(\tau)\,dt\,d\tau.$$

As

$$\frac{1}{T^2}\int_a^{a+T}\int_a^{a+T} B(t-s)\,dt\,ds = \frac{1}{T^2}\int_0^T\int_0^T B(t-s)\,dt\,ds =$$

$$= \frac{1}{T^2}\int_0^T\int_{t-T}^t B(u)\,du\;\;dt = \frac{1}{T}\int_{-T}^T B(u)\left(1-\frac{|u|}{T}\right)du.$$

That is if $\xi(t)$ is stationary then the limit

$$\frac{1}{T}\int_a^{a+T} \xi(t)\;dt \longrightarrow m = E\,\xi(t)$$

is true if and only if

$$(4.7)\quad \frac{1}{T}\int_{-T}^{T} B(u)\left(1-\frac{|u|}{T}\right)du \to 0.$$

(4.7) will be satisfied if

$$\frac{1}{2T}\int_{-T}^{T} B(u)\,du \to 0.$$

Remark 2. Let $\xi(t)$ be a measurable process for which $E|\xi(t)|^2<\infty\ (-\infty<t<\infty)$ and condition (4.6) is satisfied for every finite a,b. We may ask when the limit in mean square exists

$$(4.8)\quad \underset{T\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{T}\int_a^{a+T} \xi(t)\,dt.$$

From Lemma 1 it follows that for the existence of the limit in mean square (4.8) a necessary and sufficient condition is the existence

$$\lim_{T,T'\to\infty} E\left(\frac{1}{T}\int_a^{a+T} \xi(t)\,dt\right)\left(\frac{1}{T'}\int_a^{a+T'} \xi(t)\,dt\right)^* =$$

$$= \lim_{T,T'\to\infty} \frac{1}{TT'}\int_a^{a+T}\int_a^{a+T'} \mathrm{tr}\, B(t,\tau)\,dt\,d\tau.$$

Moreover, for the limit

$$\mathrm{l.i.m.}\left\{\frac{1}{T}\int_a^{a+T} \xi(t)\,dt-\frac{1}{T}\int_a^{a+T} E\,\xi(t)\,dt\right\}=0$$

it is necessary and sufficient that

$$\lim_{T\to\infty}\frac{1}{T^2}\int_a^{a+T}\int_a^{a+T} \mathrm{tr}\, B(t,\tau)\,dt\,d\tau=0\,.$$

For processes stationary in wide sense we have $B(t,\tau)=B(t-\tau)$, $E\,\xi(t)=0$.

Chapter 5:

Stochastic measures and integrals

We often need integrals according to a process $\xi(t)$

$$(5.1) \quad \int_a^b f(t)\, d\xi(t)$$

Such functionals can not be understand as Stieltjes /or Lebesgue-Stieltjes/ integrals as the realisation of the process $\xi(t)$ has infinite variation in most cases. In spite of this we can define the integrale (5.1) so that it will be very useful in the sequel and is suitable for the practical purposes too. A complex of random variables $\Phi(\Delta, \omega)$ where Δ is any measurable set of the interval $[a,b]$ and $\omega \in \Omega$ is called a stochastic measure of the probability space $(\Omega, \mathcal{F}, P)$ if the following conditions are satisfied

1./ $\Phi(\Delta)$ is additive with probability 1, i.e. if

$\Delta_1 \cap \Delta_2 = \emptyset$ then $\Phi(\Delta_1 \cup \Delta_2) = \Phi(\Delta_1) + \Phi(\Delta_2)$,

2./ $E|\phi(\Delta)|^2 = F(\Delta) < \infty$,

3./ $E\Phi(\Delta_1)\Phi(\Delta_2)^* = 0$ if $\Delta_1 \cap \Delta_2 = \emptyset$.

From this property it follows that

$$F(\Delta_1 \cup \Delta_2) = F(\Delta_1) + F(\Delta_2) \quad \text{if} \quad \Delta_1 \cap \Delta_2 = \emptyset$$

4./ $\Phi(\Delta)$ is σ-additive in mean square, i.e.

if $\Delta = \bigcup_{i=1}^{\infty} \Delta_i$, $\Delta_i \cap \Delta_k = \emptyset$ if $i \neq k$, then $\Phi(\bigcup_1^n \Delta_i) = \sum_1^n \Phi(\Delta_i) \longrightarrow \Phi(\Delta)$ in mean square. From this property it follows that F is σ-additive, i.e it is a measure.

It can be shown that if a random set function determined on a semiring satisfies the axioms 1./-4./ then it can be extended to a random measure on the σ-algebra generated by the semiring.

<u>Example.</u> We can determine a stochastic measure on the Borel sets of the intervall /0,1/ with the help of a Brownian motion $\omega(t)$ as follows. If $\Delta=[t_1,t_2)$ is an interval closed from the left hand side and open from the right-hand side, then let $\Phi(\Delta)=\omega(t_2)-\omega(t_1)$. The intervals of such type form a semiring and we see at once that $\Phi(\Delta)$ satisfies the axioms 1./-4./, so it can be extended to a random measure on the Borel sets. $F(\Delta)$ will be the Lebesgue measure multiplied by a constant.

Let us define the integral according to the random measure $\Phi(\Delta)$ first for simple functions. Let the integral of the function

$$f(t)=\sum_{k=1}^{n} c_k \chi_{\Delta_k}(t), \quad \Delta_i\cap\Delta_j=\emptyset \quad \text{if} \quad i\neq j,$$

be by definition

$$(5.2) \qquad \int_a^b f(t)\,\Phi(dt)=\sum_1^n c_k\,\Phi(\Delta_k)$$

If

$$g(t)=\sum_{j=1}^{m} d_j\,\chi_{\Delta_j'}(t), \quad \Delta_i'\cap\Delta_j'=\emptyset \quad \text{if} \quad i\neq j,$$

then we get by simple transformations that

$$(5.3) \qquad E\int_a^b f(t)\,\Phi(dt)\Big(\int_a^b g(s)\,\phi(ds)\Big)^*=E\sum_k c_k\phi(\Delta)\Big(\sum d_j\phi(\Delta_j)\Big)^*=$$

$$=\sum_{i,j} c_k d_j^*\,E|\Phi(\Delta_k\cap\Delta_j)|^2=\int_a^b f(t)g(t)^*F(dt)$$

and

$$(5.3') \quad E\left|\int_a^b f(t)\,\Phi(dt)\right|^2 = E\left(\sum c_k \phi(\Delta_k)\right)\left(\sum c_k \phi(\Delta_k)\right)^* =$$

$$= \sum |c_k|^2 F(\Delta_k) = \int_a^b |f(t)|^2 F(dt).$$

The integral of limits of simple functions exists if and only if

$$\int_a^b |f(t)|^2 F(dt) = \sum_{k=1}^{\infty} |c_k|^2 F(\Delta_k) < \infty.$$

For simple functions it is true that

$$\int_a^b [\alpha f(t) + \beta g(t)]\,\phi(dt) = \alpha \int_a^b f(t)\,\Phi(dt) + \beta \int_a^b g(t)\,\Phi(dt).$$

If $f(t)$ is a limit in mean square of the functions $f_n(t)$ i.e

$$\int_a^b |f(t) - f_n(t)|^2 \quad F(dt) \longrightarrow 0 \qquad \text{if} \qquad n \longrightarrow \infty$$

then by (5.3)

$$E\left|\int_a^b f_n(t)\,\Phi(dt) - \int_a^b f_m(t)\,\Phi(dt)\right|^2 = \int_a^b |f_n(t) - f_m(t)|^2 F(dt) \longrightarrow 0, \text{ if } n, m \longrightarrow \infty$$

consequently the random variables $\int_a^b f_n(t)\,\Phi(dt)$ will have a limit in mean square and this random variables will be called the stochastic integral of $f(t)$

$$\int_a^b f(t)\,\Phi(dt) = \text{l.i.m.} \int_a^b f_n(t)\,\Phi(dt).$$

The value of this integral will not depend on the choise of the sequence $f_n(t)$. It can be defined for every function satisfying the condition

(5.4) $\int_a^b |f(t)|^2 F(dt) < \infty$

and the relation(5.3)will be satisfied too.

Let us examine now some properties of the stochastic integral. Let $\Phi(\Delta)$ a stochastic measure $E\Phi(\Delta)\Phi^*(\Delta)=F(\Delta)$, where $F(\Delta)$is a totally additive positive definite matrix function. Let $g(t)\in L_2(F)$

i.e

$$\int_a^b |g(t)|^2 F(dt) < \infty$$

A random set function $\psi(\Delta)$ will be defined as follows

$$\psi(\Delta) = \int_a^b \chi_\Delta(t) g(t)\,\Phi(dt).$$

$\psi(\Delta)$ is obviously a random measure and on the basis of the property (3')

$$E|\Psi(\Delta)|^2 = \int_\Delta |g(t)|^2 F(dt) = G(\Delta),$$

while on the basis of (3)

(5.5) $E\,\Psi(\Delta_1)\Psi(\Delta_2)^* = \int \chi_{\Delta_1}(t)\ \chi_{\Delta_2}(t)|g(t)|^2 F(dt) =$

$$= \int_{\Delta_1 \cap \Delta_2} |g(t)|^2\ F(dt).$$

__Theorem 1.__ If $f(t)\in L_2(G)$ then $f(t)\,g(t)\in L_2(F)$ and

$$\int_a^b f(t)\Psi(dt) = \int_a^b f(t)\,g(t)\Phi(dt).$$

__Proof.__ For simple functions $(f(t)=\sum_1^n c_k\chi_{\Delta_k}(t))$ the statement is obvious

$$\int_a^b f(t)\psi(dt) = \sum_1^n c_k\Psi(\Delta) = \int_a^b \sum_1^n c_k\,\chi_{\Delta_k}(t)\,g(t)\Phi(dt) =$$

$$= \int_a^b f(t)\,g(t)\Phi(dt).$$

Let $f_n(t)$ be a fundamental sequence of simple functions in $L_2(G)$ then

$$E\left|\int f_n(t)\Psi(dt)-\int f_m(t)\Psi(dt)\right|^2=\int |f_n-f_m|^2\, G(dt)=$$

$$=\int |f_n-f_m|^2 (g(t))^2\, F(dt),$$

i.e. $f_n(t)\, g(t)$ is fundamental in $L_2(F)$ and the statement of the theorem is obvious.

<u>Theorem 2</u>. If $\Psi(\Delta)=\int \chi_\Delta(t)\, g(t)\Phi(dt)$, $g(t)\in L_2(F)$, then

$$\Phi(\Delta)=\int \frac{1}{g(t)}\chi_\Delta(t)\,\Psi(dt).$$

<u>Proof</u>. The function $g(t)$ can be equal to o only on a set of measure o (mod G) so $\frac{1}{g(t)}\neq\infty$. Moreover

$$\int_a^b \frac{1}{|g(t)|^2}\chi_\Delta(t)\, G(dt)=\int_\Delta \frac{1}{|g(t)|^2}|g(t)|^2 F(dt)=F(\Delta)<\infty$$

and so by Theorem 1.

$$\int \frac{1}{g(t)}\chi_\Delta(t)\Psi(dt)=\int \frac{1}{g(t)}\chi_\Delta(t)\, g(t)\,\Phi(dt)=\Phi(\Delta).$$

<u>Example</u>. Let $\xi(\Delta)$ be a random measure over the interval $-\infty<t<\infty$

$$E\,\xi(\Delta)=0,$$
$$E\,\xi(\Delta)\,\xi(\Delta)^*=B(0)\cdot|\Delta|,$$

where $B(0)$ is a positiv definite matrix. If $c(t)$ is square integrable i.e.

$$\int |c(t)|^2\, dt<\infty,$$

then for any t the integral

$$\eta(t)=\int_{-\infty}^{\infty} c(t-s)\,\xi(ds) \tag{5.6}$$

exists and

$$(5.7) \qquad \begin{aligned} &E\,\eta(t) = 0, \\ &E\,\eta(t)\,\eta(s)^* = B(0)\int_{-\infty}^{\infty} c(t-s+u)\,c^*(u)\,du. \end{aligned}$$

i.e. the process $\eta(t)$ is stationary.

Let we have now a function with two variables $f(t,\lambda)$ $(a \leqq \lambda \leqq b,\ c \leqq t = d)$ measurable with respect to the two variables t, λ and $\Phi(d\lambda)$ a stochastic measure such, that

$$\int_a^b |f(t,\lambda)|^2 \ \Phi(d\lambda) \infty$$

for almost every t $(F(d\lambda) = E\,\Phi(d\lambda)\,\Phi^*(d\lambda))$.

For these t the integral

$$\xi(t) = \int_a^b f(t,\lambda)\,\Phi(d\lambda)$$

exists and the process $\xi(t)$ has finite variance. If $\xi(t,\omega)$ is measurable as a function of two variables and

$$\int_c^d \Big[\int_a^b |f(t,\lambda)|^2\, F(d\lambda) \Big]^{1/2} dt < \infty$$

then the integral of $\xi(t)$ exists and

$$\int_a^b \xi(t)\,dt = \int_c^d \int_a^b f(t,\lambda)\ \Phi\,(d\lambda)\,dt.$$

Chapter 6:

Integral representation of stochastic processes

With the help of stochastic integrals we can get different representations of stochastic processes. If a process $\xi(t)$ can be represented in the form

$$(6.1) \qquad \xi(t) = \int f(t,\lambda)\, \Phi(d\lambda), (E\Phi(d\lambda)\Phi^*(d\lambda) = F(d\lambda)),$$

then its covariance function has the form

$$(6.2) \qquad E\,\xi(t)\,\xi(s)^* = \int f(t,\lambda) f(s,\lambda) F(d\lambda),$$

on the basis of the property (5.3).

Let $L_2\{f(t)\}$ denote the set of the linear combinations of the functions $f(t,\lambda)$ and their limit in mean square according to the measure $F(d\lambda)$. If $L_2\{f(t)\}$ coincides with $L_2(F)$ the system $f(t,\lambda)$ will be called complete.

Theorem 1. If the covariance matrix of the process $\xi(t)$ can be represented in the form (6.2) where $f(t,\lambda) \in L_2(F)$ then there exists a stochastic measure $\Phi(d\lambda)$ such that $E\Phi(d\lambda)\,\Phi(d\lambda)^* = F(d\lambda)$ and the relation (6.1) is satisfied with probability 1.

Proof. Let us bring the linear combinations of the functions $f(t,\lambda)$

$$(6.3) \qquad g(\lambda) = \sum_{k=1}^{n} \alpha_k\, f(t_k,\lambda)$$

into correspondence with the random variables

$$(6.4) \qquad \eta = \sum_{k=1}^{n} \alpha_k \, \xi(t_k)$$

Let $\{f\}$ resp. $\{\xi\}$ denote the mainfold of the functions of the form (6.3) resp. (6.4) . Let us define the scalar product with the integral

$$(6.5) \qquad (g_1, g_2) = \int g_1(\lambda)\, g_2(\lambda)\, \mathrm{tr}\, F(d\lambda).$$

On the basis of the relation $(\eta_1, \eta_2) = E\eta_1\eta_2^* = (g_1, g_2)$ this correspondence is isometric. This correspondence can be extended to the Hilbert space $L_2\{f(t)\}$ resp. $L_2\{\xi\}$ keeping isometricity. Namely let $g(t,\lambda) \in L_2\{f(t)\}$ then we can find $g_n(t,\lambda) \in M\{f\}$ such that $\|g_n(t,\lambda) - g(t,\lambda)\| \to 0$ if $n \to \infty$. If the functions $g_n(t,\lambda)$ correspond to the random variables $\eta_n, \eta_n \in M\{\xi\}$, then from the isometrical correspondence

$$\|\eta_n - \eta_m\| = \|g_n - g_m\| \longrightarrow 0 \quad \text{, if} \quad n, m \longrightarrow \infty,$$

that is there exists a limit η . $\in L_2\{\xi\}$. To prove uniqueness let $g_n^* \to g$. For the variables η_n^* corresponding to g_n^* we have $\|\eta_n^* - \eta_0^*\| \to 0$. Let moreover $\tilde{g}_{2n} = g_n^*$, $\tilde{g}_{2n-1} = g_n$ then we have $\|\tilde{g}_n - g\| \to 0$, and for some $\tilde{\eta}_0$, $\|\tilde{\eta}_0 - \tilde{\eta}_0\| \to 0$, so we must have $\eta_0 = \eta_0^* = \tilde{\eta}_0$ with probability 1. So we have a one to one correspondence between the spaces $L_2\{f(t)\}$ and $L_2\{\xi\}$ which preserves the scalar product (6.3).

Let us suppose that the system $f(t,\lambda)$ is complete in $L_2(F)$. Let Δ be a Borel measurable set, then $\chi_\Delta(\lambda) \in L_2(F) = L_2\{f\}$ and let $\Phi(\Delta)$ denote the random variable corresponding to $\chi_\Delta(\lambda)$. $\phi(\Delta)$ is a stochastic measure for which

$$E\,\Phi(\Delta_1)\,\Phi(\Delta_2)^* = \int \chi_{\Delta_1}(\lambda)\,\chi_{\Delta_2}(\lambda)\,F(d\lambda) = F(\Delta_1 \cap \Delta_2).$$

The process $\eta(t)$ defined by the stochastic integral

$$\eta(t) = \int f(t,\lambda)\,\Phi(d\lambda)$$

coincides with $\xi(t)$ as

$$E\,\xi(t)\,\eta(t)^* = E\,\xi(t)\left(\int f(t,\lambda)\,\phi(d\lambda)\right)^* = \int f(t,\lambda)^2\,F(d\lambda),$$

namely

$$E\,\xi(t)\,\Phi(\Delta)^* = (f(t,\lambda),\ \chi_\Delta(\lambda)) = \int f(t,\lambda)\,\chi_\Delta(\lambda)\,\mathrm{tr}\,F(d\lambda),$$

and from this we get

$$E|\eta(t) - \xi(t)|^2 = E(\xi(t) - \eta(t))(\xi(t) - \eta(t))^* =$$

$$= E|\xi(t)|^* - E\,\xi(t)\,\eta(t)^* - E\,\eta(t) + E|\eta(t)|^2 = 0.$$

If the system $f(t,\lambda)$ is not complete in $L_2(F)$ then let us chose $h(t,\lambda)$ $(t \in \tilde{T},\ \tilde{T} \cap T = \emptyset)$ to be complete over the Hilbert space $L_2(F) \ominus L_2\ \{f(t)\}$. Let the Gaussian process $\tilde{\xi}(t)$, $t \in \tilde{T}$ be independent of $\xi(t)$ and let $E\,\tilde{\xi}(t) = 0$

$$E\,\tilde{\xi}(t_1)\,\tilde{\xi}(t_2)^* = \int h(t_1,\lambda)\,h(t_2,\lambda)\,F(d\lambda).$$

We may apply the previous considerations to the process $\eta(t) = \xi(t)$ if $t \in T$ and $\eta(t) = \tilde{\xi}(t)$ if $t \in \tilde{T}$, and complete the proof of the theorem.

If the system $\{f(t,\lambda)\}$ is complete in $L_2(F)$ then the stochastic measure $\Phi(\Delta)$ is an element of the Hilbert space $L_2\{\xi\}$ i.e. it can be determined from the realizations of the process $\xi(t)$. In such cases we say that Φ is subordinated to $\xi(t)$.

Exercises

1. Let $\xi(t)$ be a continuous stochastic process on $[0,T]$ with mean zero and covariance function $R(s,t)$. The mapping $L_2(T)$ into $L_2(T)$, defined by

$$R: \quad g(t) = \int_0^T R(s,t) f(s)\,ds \qquad , \quad t \in [0,T]$$

has positive eigenvalues λ_n and the corresponding eigen functions $\phi_n(t)$. Prove the Karhunen-Loeve expansion theorem:

$$\underset{N \to \infty}{\mathrm{l.i.m.}} \left(\xi(t) - \sum_1^N \xi_n \phi_n(t)\right) = 0 ,$$

where

$$\xi_n = \int_0^T \xi(t) \phi_n(t)\,dt.$$

2. If, in addition, the process $\xi(t)$ is Gaussian prove by virtue of the Kolmogorov inequality, that the series

$$\sum \xi_n \phi_n(t)$$

converges also with probability one.

3. Let $w(t)$ the Brownian-motion process with mean zero and covariance

$$R(s,t) = \min(s,t),$$

and $T = 1$. The corresponding eigenvalues and eigenfunctions are well-known:

$$\phi_k(t) = \sqrt{2}\ \sin(2k+1)\pi t/2 ,$$

$$E\,\xi_k^2 = \frac{4}{(2k+1)^2 \pi^2}$$

So we have

$$w(t) = \sum_{1}^{\infty} \xi_k \phi_k(t),$$

where the series converges in the mean square and with probability one.

Chapter 7:

Stochastic integrals

In this section we define the stochastic integral for a stochastic process

$$(7.1) \qquad \int_0^T f(t,\omega)\, w(dt) = \int_0^T f(t,\omega)\; dw(t)$$

where $w(t)$ is the standard Brownian motion process.
In the case whene $f(t)$ was a function /and not a process/ we saw that the integral cannot be defined as a Stieltjes or Lebesgue-Stieltjes one.

On the basis of the fact that $\sum_k (w(t_k) - w(t_{k-1}))^2 \to T$ /see theorem 3.2 for Brownian motion process, where for simplicity $M w(t)^2 = t$ / we see that the definition of the integral /1/ is not an obvious one from the following examples:

$$\sum \left[w(t_{k+1}) - w(t_k)\right]\; w(t_k) \to \tfrac{1}{2}(w(T)^2 - T),$$

$$\sum \left[w(t_{k+1}) - w(t_k)\right]\; w(t_{k+1}) \to \tfrac{1}{2}(w(T)^2 + T),$$

$$\sum \left[w(t_{k+1}) - w(t_k)\right]\; \frac{w(t_{k+1}) - w(t_k)}{2} \to \tfrac{1}{2}\, w(T)^2,$$

where the convergence is true with probability 1 and in mean square too.

To prove 1. we know

$$\sum_{k=0}^{N} \left[w(t_{k+1}) - w(t_k)\right]^2 \to T$$

and from there

$$\sum [w(t_{k+1}) - w(t_k)]^2 = \sum [w(t_{k+1}) - w(t_k)]\, w(t_{k+1}) -$$

$$- \sum [w(t_{k+1}) - w(t_k)]\, w(t_k) = w(T)^2 - 2\sum [w(t_{k+1}) - w(t_k)]\, w(t_k),$$

which proves the statement.

The way as we defined the stochastic integral in the preceding paragraph for a deterministic function $f(t)$, proposed in the thirties by Paley, cannot be extended to the case of random function /stochastic process/. It was K. Ito who proposed a much more general way of constructing stochastic integrals, applicable in the case of a wide class of random functions.

In the following let $\mathcal{F}_t (\subset A)$ denote σ-algebras for which $\mathcal{F}_{t_1} \subseteq \mathcal{F}_t$ if $t_1 \leq t$ and $\mathcal{F}_t$ be independent of the future of Brownian motion process $w(t)$, that is the events $B \in \mathcal{F}_t$ and $\{w(t+h) - w(t) < x\}$ must be independent for every B and $h > 0$, x. In this case we say that $\{w(t), \mathcal{F}_t\}$ forms a Brownian motion process. It may happen that $\mathcal{F}_t$ is the σ-algebra A_t generated by $\omega(s)$, $0 \leqq s \leqq t$.

Definition 1. Let $\{w(t), \mathcal{F}_t\}$ be a Brownian motion process on the probability space $(\Omega, \mathcal{A}, P)$. We shall say that the stochastic process $f(t,\omega)$ does not depend on the future if it is measurable in (t,ω) (with respect to $\mathcal{B}_{[0,T]} \times \mathcal{A}$) and for any $t \geqq 0$ $f(t,\omega)$ is measurable according to $\mathcal{F}_t$. The class of such processes will be denoted by $\mathfrak{M}$.

Let us denote $L^2_{\Omega \times [0,T]}$ the class of the functions $f(t,\omega)$ for which

$$(7.2)\qquad \int_{\Omega}\int_{0}^{T} f^2(t,\omega)\,dt\,P(d\omega) = E\int_0^T f^2(t)\,dt < \infty \Leftrightarrow f(t,\omega)\in L^2_{\Omega\times[0,T]}$$

and

$$\mathfrak{M}^2 = \mathfrak{M} \cap L^2_{\Omega\times[0,T]}$$

Obviously $\mathfrak{M}^2$ is a closed subspace of $L^2_{\Omega\times[0,T]}$.

<u>Definition 2.</u> $f(t,\omega)\in L^2_{\Omega\times[0,T]}$ is a simple function if

$$f(t,\omega) = f_k(\omega), \quad \text{if} \quad t\in[t_k, t_{k+1}); \quad k = 0, 1, \ldots, n-1,$$

where $(0 = t_0 < t_1 < \ldots < t_n = T)$ is a decomposition of the interval $[0,T]$.

The stochastic integral of a simple function $f(t,\omega)$ is defined by the formula

$$\int_0^t f(s,\omega)\,dw(s) = \sum_{k=0}^{m:t_{m+1}<t} f_k(\omega)\left[w(t_{k+1}) - w(t_k)\right] + f_{m+1}(\omega)(w(t) - w(t_{m+1})).$$

The basic properties of the stochastic integral of simple functions are:

a/ $\int_0^t (\alpha f(s) + \beta g(s))\,dw(s) = \alpha\int_0^t f(s)\,dw(s) + \beta\int_0^t g(s)\,dw(s)$,

b/ $\int_0^t f(s)\,dw(s) = \int_0^{t_1} f(s)\,dw(s) + \int_{t_1}^t f(s)\,dw(s)$, $0 \leq t_1 \leq t$,

c/ the integral is a continuous function of the upper bound;

d/ $E(\int_0^t f(s)\,dw(s)/\mathcal{F}_u) = \int_0^u f(s)\,dw(s)$, for $0 < u < t$ with probability 1, especially
$E(\int_0^t f(s)\,dw(s)) = 0$

e/ $E(\int_0^t f(s)\,dw(s))(\int_0^t g(s)\,dw(s) = E\int_0^t f(s)\,g(s)\,ds.$

Properties a/ and b/ are obvious. Property c/ follows from the continuity of Brownian motion process. To prove property d/ it is enough to note that if $t_k > u$

$$E(f_k(\omega)[w(t_{k+1})-\omega(t_k)]/\mathcal{F}_u) = E(E(f_k[w(t_{k+1})-w(t_k)]/\mathcal{F}_{t_k})/\mathcal{F}_u) =$$

$$= E(f_k(E[w(t_{k+1})-w(t_k)]/\mathcal{F}_{t_k})\ \mathcal{F}_u) = 0$$

In the same way can be proved e/. Indeed, without restriction of generality we may suppose that $f(t)$ and $g(t)$ are piecewise constant on the same intervals. Let $t_n \geqq t_{k+1}$ then

$$E\ f_k(w(t_{k+1})-w(t_2))\ g_n(w(t_{n+1})-w(t_n)) =$$

$$= E\ f_k(w(t_{k+1})-w(t_k))\ g_n E[(w(t_{n+1})-w(t_n))/\mathcal{F}_{t_n}] = 0,$$

and

$$E\ f_k(w(t_{k+1})-w(t_k))\,g_k(w(t_{k+1})-w(t_k)) = E\ f_k\cdot g_k(t_{k+1}-t_k),$$

and from here we get e/.

The properties d/ and e/ mean that the transformation of a stochastic simple process $f(t,\omega) \in \mathfrak{M}^2$ determined by the stochastic integral $\int f\,dw$ into $L^2_{\Omega\times[0,T]}$ is isometric.

For the definition of the stochastic integral $\int f\,dw$ of any process $f(t,\omega)\in\mathfrak{M}^2$ it remains to prove that the set of simple processes is everywhere dense in $\mathfrak{M}^2$. The proof of this fact is the following.

Let $f(t,\omega)\in\mathfrak{M}^2$, $|f(t,\omega)| < C$ and continuous with probability 1 according to t. Then for simple functions

$$f_n(t,\omega)=f\left(\frac{kT}{n}\right), \quad \text{if} \quad t\in\left[\frac{kt}{n}, \frac{(k+1)T}{n}\right),$$

we have $f_n \in \mathfrak{M}^2$ and $f_n \to f$ with probability 1, and

$$E\int_0^T (f-f_n)^2\, dt \to 0, \quad \text{if} \quad n\to\infty.$$

Further now we assume that $|f(t,\omega)|< C$. The functions

$$f_n(t,\omega)=n\int_{\{\max(t-\frac{1}{n}),0\}}^{t} f(s,\omega)\, ds$$

are continuous and are contained in $\mathfrak{M}^2$. The sequence $f_n(t,\omega)$ converges to $f(t,\omega)$ with probability one and as $|f_n| < C$ in mean square too, i.e.

$$E\int_0^T (f-f_n)^2\, dt \to 0, \quad \text{if} \quad n\to\infty.$$

Finally let $f(t,\omega)$ arbitrary in $\mathfrak{M}^2$. It may be approximated in mean square by the bounded functions

$$f_n(t,\omega)=f(t,\omega)\ \chi_{[-n,n]}(t).$$

So we have defined the stochastic integral all over $\mathfrak{M}^2$. Now we shall prove that the properties a/-e/ are valid for any process $f(t,\omega)\in\mathfrak{M}^2$.

To prove a/ let us choose two sequences of simple functions $f_n(t,\omega), g_n(t,\omega)$ converging in mean square to $f(t,\omega)$ resp. $g(t,\omega)$.

Then a/ is true for f_n and g_n and it remains true for the limit too.

In the same way we can prove b/. But we can prove it using a/ for arbitrary $f(t,\omega)\in\mathfrak{M}^2$. Indeed

$$\int_0^{t_1} f(t)\,dw(t) = \int_0^{t} f(s)\,\chi_{[0,t_1]}(s)\,dw(t).$$

It is well known and may be easily proved, that if $E(\xi_n-\xi)^2 \to 0$ $(n \to \infty)$ and $\mathcal{F}$ is a σ-algebra, then

$$E|E(\xi_n/\mathcal{F})-E(\xi/\mathcal{F})|^2 \to 0, \quad \text{if} \quad n \to \infty.$$

Using this fact we shall prove property d/. Let $f_n(t)\in \mathfrak{M}^2$ a simple process such, that $E\int_0^T (f_n(t)-f(t))^2\,dt \to 0$ if $n \to \infty$. As

$$E\left[\int_0^T f_n(t)\,dw(t)/\mathcal{F}_{T_1}\right] = \int_0^{T_1} f_n(t)\,dw(t)$$

and

$$E\left(\int_0^{T_1} f_n(t)\,dw(t) - \int_0^T \chi_{[0,T_1]}(t)\,f(t)\,dw(t)\right)^2 \to 0, \quad \text{if} \quad n \to \infty,$$

$$E\left(\int_0^T f_n(t)\,dw(t) - \int_0^T f(t)\,dw(t)\right)^2 \to 0, \quad \text{if} \quad n \to \infty,$$

we get the required result.

Property e/ is a trivial consequence of the definition of stochastic integral, where there exists an isometry between $f(t,\omega)$ and its integral

$$f(t,\omega) \xrightarrow{y} \int_0^t f(s,\omega)\,dw(s).$$

Let us turn to property c/, which states that the stochastic integral is a continuous function of its upper bound. Here we prove the following theorem.

<u>Theorem 1</u>. The process $\xi(t)= \quad f(s,\omega)\,dw(s)$ is equivalent to a process with continuous trajectories.

First we shall prove the Kolmogorov inequality for martingales with continuous time parameter.

<u>Lemma 1</u>. If $(\xi(t), \mathcal{F}_t)$ is a martingale, where $\xi(t)$ is continuous with probability 1, then

$$(7.3) \qquad P\{\sup_{0\leq t\leq T} |\xi(t)| \geq c\} \leq \frac{E|\xi(T)|^2}{c^2}$$

The proof of this lemma may be carried out in two steps. First the Kolmogorov inequality can be proved for the discrete time martingale

$$\eta_k^{(n)} = \xi\left(\frac{kT}{n}\right), \qquad k = 0, 1, 2, \dots, n,$$

in the same way as in the case of sums of independent random variables /with zero mean/.

In the second step we should take the limit from the martingales

$$\eta_k^{(n)} \text{ to } \xi(t) \text{ as } n \longrightarrow \infty .$$

Let

$$B_k = \{\omega : |\eta_j| < c,\ j = 1, 2, \dots, k-1,\ |\eta_k| \geq c\} \in \mathcal{F}_{\frac{kt}{n}} = \mathcal{F}_k,$$

where $B_k \cap B_\ell = \emptyset$ if $k \neq \ell$, and

$$B = \{\omega : \max_{k\leq n} |\eta_k| \geq c\} = \bigcup_{k\leq n} B_k .$$

The following inequality proves the theorem in the discrete time case:

$$E\eta_n^2 \geq E(\eta_n^2 \cdot \chi_B) = \sum_{k\leq n} E(\eta_n^2 \chi_{B_k}) =$$

$$= \sum_{k\leq n} E[\chi_{B_k} E(\eta_n^2/\mathcal{F}_k)] \geq \sum_{k\leq n} E(\chi_{B_k}(E(\eta_n/\mathcal{F}_k))^2] =$$

$$= \sum_{k\leq n} E[\chi_{B_k} \eta_k^2] \geq c^2 \sum_k E[\chi_{B_k}] = c^2 P(B),$$

where we used the Jensen inequality and the martingal equality.

In continuous time case we have the equality for continuous processes:

$\{\sup_t \xi(t) \geq c\} = \{\sup_r \xi(r) \geq c\}$ where $\{r\}$ is the set of rationals. Moreover

$$\{\sup_r \xi(r) \geq c\} = \lim_{n\to\infty} \{\max_{k \leq 2^n} \xi\left(\frac{kT}{2^n}\right) \geq c\}, \text{ with integer } k,n.$$

From here using the Kolmogorov inequality in discrete time case we get

$$\mathcal{P}\{\sup_{0 \leq t \leq T} |\xi(t)| \geq c\} = \lim_{n\to\infty} \mathcal{P}\{\max_{k \leq 2^n} . \xi\left(\frac{kT}{2^n}\right) \geq c\} \leq \frac{E\xi(T)^2}{c^2}$$

Proof of the theorem. For simple processes $f_n(t,\omega)$ the paire $\{\eta^{(n)}(t) = \int_0^t f_n(s)\, dw(s), \mathcal{F}_t\}$ forms a continuous martingale. From the Kolmogorov inequality we get

$$\mathcal{P}\left\{\sup_{t\leq T} \left|\int_0^t f_n(s)\, dw(s) - \int_0^t f_m(s)\, dw(s) > c\right.\right\} \leq \frac{E[\int_0^T (f_n(s)-f_m(s))\, dw(s)]^2}{c^2} =$$

$$= \frac{1}{c^2} E\int_0^T (f_n(s) - f_m(s))^2\, ds.$$

If f_n tends to f so quickly, that

$$E\int_0^T (f_m(s) - f_{m+1}(s))^2\, ds \leq \frac{1}{2^m}$$

then $\int_0^t f\, dw$ is the sum of continuous functions so that the convergence is uniform with probability 1, and this means that $\int_0^t f\, dw$ is also continuous. Indeed

$$\int_0^t f\, dw = \int_0^t f_1\, dw + \int_0^t (f_2 - f_1)\, dw + \dots$$

where the convergence is in mean square. The members of this sum are continuous /with probability 1/, and the uniform convergence with probability 1 follows from the inequality

$$\sum_{n=0}^{\infty} P\left\{ \sup_t \Big| \int_0^t (f_{n+1} - f_n)\, ds \Big|^2 > \frac{1}{n^2} \right\} \leq \sum_{n=1}^{\infty} \frac{n^2}{2^n} < \infty$$

and from the Borel-Cantelli lemma.

If the processes a, b $\in \mathfrak{M}^2\,[0,T]$ and $\xi(t)$ is defined by the equation

$$(7.4) \quad \xi(t) = \xi(0) + \int_0^t a(t,\omega)\, dt + \int_0^t b(t,\omega)\, dw(t)$$

we say that the process $\xi(t)$ has a stochastic differential

$$(7.4') \qquad d\xi(t) = a(t,\omega)\, dt + b(t,\omega)\, dw(t)$$

The last expression has not meaning in itself, it is only a short writing of the integral expression.

It is possible to extend the definition of the stochastic integral to the case where $f = f(t,\omega) \in \mathfrak{M}$, i.e. it is measurable with respect to $\mathcal{B}_{[0,T]} \times A$ $(\mathcal{B}_{[0,T]}$ is the σ-algebra of Borel sets/, and $f(t,\omega)$ is $\mathcal{F}_t$ measurable for every fixed t where $(w(t), \mathcal{F}_t)$ is a Brownian motion process, and finally we suppose only that

$$(7.5) \qquad P\{ \int_0^T f^2(t,\omega)\, dt < \infty \} = 1.$$

The last condition is weaker than $E \int_0^T f^2(t,\omega)\, dt < \infty$.
The definition of the stochastic integral for simple functions is the same as in the discussed case.

It is obvious that for simple functions the integral has the properties a/-c/.

Before studying further properties of stochastic integrals we prove the following lemma.

<u>Lemma 2</u>. If $f_n(t,\omega) \in \mathfrak{M}$ is a simple function then for the process

$$(7.6) \qquad \xi(t) = \exp\{\int_0^t f_n(s,\omega)\,dw(s) - \frac{1}{2}\int_0^t f_n^2(s,\omega)\,ds\}$$

$(\xi(t), \mathcal{F}_t)$ is a martingale, for which

$$E[\xi(t)/\xi(s)] = 1, \qquad s \leq t,$$

i.e.

$$(7.7) \qquad E\exp\{\int_s^t f_n(s)\,dw(s) - \frac{1}{2}\int_s^t f_n^2(s)\,ds\} = 1 \text{ for every } 0 \leq s \leq t \leq T.$$

Proof. We have that

$$(7.8) \quad E(\xi(t)/\mathcal{F}_s) = \exp\{\int_0^s f_n(u)\,dw(u) - \frac{1}{2}\int_0^s f_n^2(u)\,du\}\,E\{\exp[\int_s^t f_n\,dw - \frac{1}{2}\int_s^t f_n^2\,du]/\mathcal{F}_s\}$$

Let for simplicity

$$f_n(t,\omega) = f_k(\omega), \qquad \text{if } t_k \leq t < t_{k+1}, \quad k = 1,2,\dots,m.$$

$$s = t_i, \qquad t = t_j.$$

Then

$$E\{\exp[\int_s^t f_n\,dw - \frac{1}{2}\int_s^t f_n^2\,du]/\mathcal{F}_s\} = E\{\exp[\int_s^{t_{j-1}} f_n\,dw - \frac{1}{2}\int_s^{t_{j-1}} f_n^2\,du].$$

$$E[\exp[\int_{t_{j-1}}^{t_j} f_n\,dw - \frac{1}{2}\int_{t_{j-1}}^{t_j} f_n^2\,du]/\mathcal{F}_{t_{j-1}}]/\mathcal{F}_s\}.$$

As the conditional distribution of $f_n(t_{j-1})(w(t_j)-w(t_{j-1})) =$ $= c_{j-1}(w(t_j)-w(t_{j-1}))$ under the condition $\mathcal{F}_{t_{j-1}}$ is normal with parameters $(0,\ c_{j-1}^2(t_j - t_{j-1}))$ we have

$$E\Big[\exp\Big[\int_{t_{j-1}}^{t_j} c_{j-1}\, dw - \frac{1}{2}\int_{t_{j-1}}^{t_j} c_{j-1}^2\, du\Big] / \mathcal{F}_{t_{j-1}}\Big] = 1 .$$

Applying this relation repeatedly we get (7.7) and from (7.8) that $\xi(t)$ is a martingale.

Here we shall not give the extension of the stochastic integral for arbitrary $f \in \mathfrak{M}$ and $P(\int_0^T f^2(t)\, dt < \infty) = 1$,the reader can find it in Gikhman-Skorokhod [2] , or in Shiryayev [1].

The definition of stochastic integral can be extended to the case $T = \infty$ if

$$P\{\int_0^\infty f^2(t,\omega)\, dt < \infty\} = 1 .$$

The definition may be given so, that

$$P\{\lim_{T\nearrow\infty} \int_0^T f(t,\omega)\, dw(t) = \int_0^\infty f(t,\omega)\, dw(t)\} = 1$$

The generalized stochastic integrals have the properties a/-c/ and

d' If $E\Big[\int_0^\infty f^2(s,\omega)\, ds\Big]^{1/2} < \infty$ then $E\int_0^\infty f(s,\omega)\, dw(s) = 0,$

e' If $E\int_0^\infty f^2(s,\omega)\, ds < \infty$ then

$$E\Big[\int_0^\infty f(s,\omega)\, dw(s)\Big]^2 = E\int_0^\infty f^2(s,\omega)\, ds.$$

Let be τ a Markov moment with respect $\mathcal{F}_t$, and $P(\tau<\infty)=1$. Further $\int_0^\tau f(s,\omega)dw(s)=\int_0^t f(s,\omega)dw(s)$ on $\{\omega:\tau(\omega)=t\}$. Then

$$\int_0^\tau f(s,\omega)\,dw(s)=\int_0^\infty \chi(\tau\geqq s)\,f(s,\omega)\,dw(s)$$

g) If for $\delta>0$ $E\exp\{(\delta+1/2)\int_0^\infty f^2(s,\omega)\,ds\}<\infty$ then $\{\xi(t),\ \mathcal{F}_t\}$ is a martingale with the property

$$E\,\xi(t)=1$$

where

$$\xi(t)=\exp\{\int_0^t f(s,\omega)\,dw(s)-1/2\int_0^t f^2(s,\omega)\,ds\}$$

<u>Remark 1.</u> Recently A. Novikov [1] has shown that g) is true under the assumption

$$E\exp\{1/2\int_0^\infty f^2(s,\omega)\,ds\}<\infty .$$

<u>Remark 2.</u> For stopping time /Markov-moment/ of a Brownian motion process $w(t)$ we have

$$E\,w(\tau)=0,\qquad E\,w^2(\tau)=E\tau$$

(if $E\,w^2(t)=t$), which is known as the Wald indentity for Brownian motion process.

<u>Remark 3.</u> (7.7) is the generalization of the so-called "fundametal identity" of sequential procedure.

Chapter 8:

A theorem of Levy

Theorem. Let there be given the continuous process $\xi(t)$, $t \geqq 0$, $\xi(0)=0$, and for every $t \geqq 0$ there are given the σ-algebras $\mathcal{F}_t$, with the property $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$, when $t_1 \leqq t_2$. If for process $\xi(t)$ the conditions

a/ for all $t \geqq 0$ the random variable $\xi(t)$ is $\mathcal{F}_t$ measurable

b/ for every $t \geqq 0$ and $h>0$ with probability 1

$$E([\xi(t+h)-\xi(t)]/\mathcal{F}_t)=0$$

c/ for every $t \geqq 0$ and $h>0$ with probability 1

$$E([\xi(t+h)-\xi(t)]^2/\mathcal{F}_t)=h$$

are fulfilled, then $\xi(t)$ is a Brownian motion process. The theorem is due to P. Levy.

Proof. We want to prove that for any decomposition $t=t_0<t_1<\ldots<t_n=t+h$ of the time interval $[t,t+h]$ we have

$$E\exp\{i\sum_{k=1}^{n} z_k[\xi(t_k)-\xi(t_{k-1})]\}=\exp\{-\frac{1}{2}\sum_{k=1}^{n} z_k^2(t_k-t_{k-1})\},$$

i.e. the increments are independent and normally distributed. We first compute the conditional characteristic function of the increment $\xi(t+h)-\xi(t)$ under the condition $\mathcal{F}_t$:

$$E[e^{i(\xi(t+h)-\xi(t))z}/\mathcal{F}_t].$$

Let $\eta_k^n = \xi(t+\frac{k}{n}h) - \xi(t+\frac{k-1}{n}h)$, then

$$\exp a[\xi(t+h)-\xi(t)] = \exp a\left(\sum_{k=1}^{n}\eta_k^n\right) = \sum_{r=0}^{n-1}\left[e^{a\sum_{k=1}^{r+1}\eta_k^n + \frac{n-r-1}{2n}ha^2} - \right.$$

$$\left. - e^{a\sum_{k=1}^{r}\eta_k^n + \frac{n-r}{2n}ha^2}\right] + e^{\frac{ha^2}{2}}.$$

So we have for the conditional expectation

$$E\{\exp iz[\xi(t+h)-\xi(t)]/\mathcal{F}_t\} = e^{-\frac{hz^2}{2}} + \sum_{r=0}^{n-1} E\left\{\exp[iz\sum_{k=1}^{r}\eta_k^n](e^{iz\eta_{r+1}^n} - \right.$$

$$\left. - e^{-\frac{hz^2}{2n}})/\mathcal{F}_t\right\} e^{-\frac{n-r-1}{2n}hz^2}.$$

We want to estimate the sum on the right hand side. As

$$E\left\{\exp[iz\sum_{k=1}^{r}\eta_k^n](e^{iz\eta_{r+1}^n} - e^{-\frac{hz^2}{2n}})/\mathcal{F}_t\right\} = E\ \exp[iz\sum_{k=1}^{r}\eta_k^n]\ \cdot$$

$$\cdot E(e^{iz\eta_{r+1}^n} - e^{-\frac{hz^2}{2n}}/\mathcal{F}_{t+\frac{r}{n}h})/\mathcal{F}_t\} =$$

$$= E\left\{\exp[iz\sum_{k=1}^{r}\eta_k^n]\,E(e^{iz\eta_{r+1}^n} - 1 - iz\eta_{r+1}^n + \frac{z^2}{2}(\eta_{r+1}^n)^2/\mathcal{F}_{t+\frac{r}{n}h})/\mathcal{F}_t\right\} +$$

$$+ E\left\{\exp[iz\sum_{k=1}^{r}\eta_k^n](1-\frac{z^2}{2n}h - e^{-\frac{hz^2}{2n}})/\mathcal{F}_t\right\}$$

and using the relation $e^{-x} - 1 - x = o(x^{2-\varepsilon})$, where $0<\varepsilon<1$, we get

$$(8.1)\qquad |E(\exp[iz(\xi(t+h)-\xi(t))]/\mathcal{F}_t - e^{-\frac{hz^2}{2}}| \leq o\left(\frac{h^{2-\varepsilon}}{n^{1-\varepsilon/2}}\right) +$$

$$+ \sum_{r=0}^{n-1} E\left\{|E(e^{iz\eta_{r+1}^n} - 1 - iz\eta_{r+1}^n + \frac{z^2}{2}(\eta_{r+1}^n)^2/\mathcal{F}_t\right\}.$$

To estimate the second term on the right hand side we have to estimate the sum of the form $\sum_r E|\eta_r^n|^\delta$, where $\delta > 2$.
From condition c/ we get that $E\sum_{k=1}^{n}(\eta_k^n)^2 = h$, i.e. the sum $\sum_{k=1}^{n}(\eta_k^n)^2$ is bounded in probability, (this follows from Markov inequality) . From here and from continuity of $\xi(t)$ we get that

$$\sum_{k=1}^{n}(\eta_k^n)^\delta \leq \max_k (\eta_k^n)^{\delta-2} \sum_{k=1}^{n}(\eta_k^n)^2 \to 0 \quad , \text{ if } \delta > 2,$$

in probability.

From b/-c/ by repeated applications of conditional expectation we get

$$E(\eta_k^n \eta_j^n \eta_\ell^n \eta_r^n / \mathcal{F}_t) = E(\eta_r^n(E(\eta_\ell^n(E(\eta_j^n(E(\eta_j^n(E(\eta_k^n/\mathcal{F}_{t+\frac{kh}{n}})/$$

$$\mathcal{F}_{\frac{jh}{n}+t})/\mathcal{F}_{\frac{\ell h}{n}+t})/\mathcal{F}_{\frac{rh}{n}+t})/\mathcal{F}_t) = 0, \quad \text{if} \quad 0 < r \leqq \ell \leqq j \leqq k$$

and

with at least one definite inequality among r,l,j,k.
Further we have the following limit relation in probability

$$|\xi(t+h)-\xi(t)|^4 = \lim_{n\to\infty} \left[(\sum_{k=1}^{n} \eta_k^n)^4 + 3\sum_{k=1}^{n}(\eta_k^n)^4 - 4\sum_{k=1}^{n}(\eta_k^n)^3 \sum_{k=1}^{n} \eta_k^n\right].$$

So by the Fatou lemma and from the above relations

$$E[(\xi(t+h)-\xi(t))^4/\mathcal{F}_t] \leqq \overline{\lim_{n\to\infty}}\, E([(\sum_k \eta_k^n)^4 + 3\sum_k(\eta_k^n)^4 -$$

$$-4\sum_k(\eta_k^n)^3 \sum_\ell \eta_\ell^n]/\mathcal{F}_t) = 6 \overline{\lim_{n\to\infty}}\, E(\sum_{k<j}(\eta_k^n)^2(\eta_j^n)^2/\mathcal{F}_t) =$$

$$= 6 \overline{\lim_{n\to\infty}} E\Big[\sum_{k<j} (\eta_k^n)^2 \cdot E((\eta_j^n)^2 | \mathcal{F}_{t+\frac{j-1}{n}h}) | \mathcal{F}_t\Big] = 6 \lim \frac{n(n-1)}{2} \cdot \frac{h^2}{n^2} \leq 3h^2 .$$

Now we are able to estimate $E(|\xi(t+h)-\xi(t)|^3 | \mathcal{F}_t$; indeed

$$E(|\xi(t+h)-\xi(t)|^3 | \mathcal{F}_t) \leq$$

$$\leq \{ E(|\xi(t+h)-\xi(t)|^2 / \mathcal{F}_t) E(|\xi(t+h)-\xi(t)|^4 | \mathcal{F}_t)\}^{1/2} \leq \sqrt{3}\, h^{3/2} .$$

Applying this to η_k^n

$$E(|\eta_k^n|^3 | \mathcal{F}_{t+\frac{k-1}{n}h}) \leq c.\left(\frac{h}{n}\right)^{3/2}$$

As $\left|e^{izx} - 1 - izx - \frac{z^2x^2}{2}\right| \leq \frac{|zx|^3}{6}$ we have the estimate for the second term in (8.1)

$$\sum_{r=0}^{n-1} E\{| e^{iz\eta_{r+1}^n} - 1 - iz\eta_{r+1}^n + \frac{z^2}{2}(\eta_{r+1}^n)^2 | | \mathcal{F}_t\} \leq cznh.\left(\frac{h}{n}\right)^{3/2} =$$

$$= o\left(\frac{1}{n^{1/2}-\varepsilon}\right).$$

So we showed that

$$E[\exp iz(\xi(t+h)-\xi(t)) | \mathcal{F}_t] = e^{-\frac{z^2h^2}{2}} .$$

From this and the definition of conditional expectation for every decomposition $0 = t_0 < t_1 <, \dots, < t_n = t$ it follows that

$$E \exp \{ i \sum_{k=1}^{n} z_k(\xi(t_k) - \xi(t_{k-1}))\} = E \exp \{ i \sum_{k=1}^{n-1} z_k(\xi(t_k)-\xi(t_{k-1}))\} .$$

$$\cdot E[\exp i z_n(\xi(t_n)-\xi(t_{n-1}))/\mathcal{F}_{t_{n-1}}] = E\exp\{i\sum_{k=1}^{n-1} z_k(\xi(t_k)-\xi(t_{k-1}))\}\cdot$$

$$\cdot\exp\left\{-\frac{z_n^2(t_n-t_{n-1})}{2}\right\}.$$

By induction the desired result follows and the theorem is proved.

The statment of the theorem may be deduced from the following general lemma.

<u>Lemma.</u> Let $(\xi(t), \mathcal{F}_t)$ satisfy the conditions of the theorem and $f(x)$ be a bounded twice continuously differentiable function where f' and f'' are bounded, then

$$E[f(\xi(t))/\mathcal{F}_s] - f(\xi(s)) = \frac{1}{2}\int_s^t E[f''(\xi(u))/\mathcal{F}_s]du, \quad t \geqq s.$$

We shall not prove this lemma but we show how can we get the theorem from it.

Let $f(x) = e^{i\lambda x}$ and we can apply the lemma; so we get

$$E[e^{i\lambda\xi(t)}/\mathcal{F}_s] - e^{i\lambda\xi(s)} = -\frac{\lambda^2}{2}\int_s^t E[e^{i\lambda\xi(u)}/\mathcal{F}_s]du, \quad t \geqq s,$$

or

$$E[e^{i\lambda(\xi(t)-\xi(s))}/\mathcal{F}_s] - 1 = -\frac{\lambda^2}{2}\int_s^t E[e^{i\lambda(\xi(u)-\xi(s))}/\mathcal{F}_s]\,du.$$

using the notation

$$\psi(t,s) = E[e^{i\lambda(\xi(t)-\xi(s))}/\mathcal{F}_s]$$

we have

$$\psi(t,s) - 1 = -\frac{\lambda^2}{2}\int_s^t \psi(u,s)\,du.$$

By differentiation

$$\frac{\partial \psi(t,s)}{\partial t} = -\frac{\lambda^2}{2}\,\psi(t,s), \quad t \geq s, \; \psi(s,s) = 1 .$$

It is well known that the only continuous solution of the above equation with the given boundary condition is

$$\psi(t,s) = e^{-\frac{\lambda^2}{2}(t-s)}$$

which proves the theorem.

Remark 1. In other words the theorem states that if $\xi(t)$ is a continuous martingale and $\xi^2(t)-t$ is a martingale too then $\xi(t)$ is a Brownian motion process /see e.g. Doob theorem 11.9/.

Remark 2. The statement of the theorem remains true under a bit weaker condition. Namely if the following conditions are satisfied: $\xi(t)$ is continuous with probability 1, $\xi(0)=0$, there exist such random variables $\eta_1, \eta_2 > 0$ with $M\eta_i < \infty$, $(i = 1,2)$, that

$$\frac{1}{\Delta}\left|E[(\xi(t+h)-\xi(t))/\mathcal{F}_t]\right| \leq \eta_1, \quad \frac{1}{\Delta}E[|\xi(t+h)-\xi(t)|^2/\mathcal{F}_t] \leq \eta_2,$$

and with probability 1 the following limits exist

$$\lim_{\Delta \to 0}\frac{1}{\Delta}E[\xi(t+\Delta)-\xi(t)/\mathcal{F}_t] = 0,$$

$$\lim_{\Delta \to 0}\frac{1}{\Delta}E[(\xi(t+\Delta)-\xi(t))^2/\mathcal{F}_t] = 1,$$

then the process $\xi(t)$ is a Brownian motion process.

Proof of the latest statement: Let us denote in the case $t > s$

$$\varphi(t) = E[(\xi(t) - \xi(s))/\mathcal{F}_s].$$

Then from our conditions, using the Lebesgue theorem,

$$\lim_{\Delta \downarrow 0} \frac{\varphi(t+\Delta) - \varphi(t)}{\Delta} = \lim_{\Delta \downarrow 0} E\left(\frac{\xi(t+\Delta) - \xi(t)}{\Delta} / \mathcal{F}_s\right) =$$

$$= \lim_{\Delta \downarrow 0} E\left(E\left\{\frac{\xi(t+\Delta) - \xi(t)}{\Delta} / \mathcal{F}_t\right\} / \mathcal{F}_s\right) = 0.$$

<u>Exercise.</u> Generalize the theorem of Levy to the multidimensional case, that is prove the following statement: If the process $\underline{\xi}(t)$ is continuous with probability 1, $\underline{\xi}(0) = \underline{0}$ and

a/ $\underline{\xi}(t)$ is $\mathcal{F}_t$ measurable for all $t \geqq 0$,

b/ $E(\underline{\xi}(t+h) - \underline{\xi}(t)/\mathcal{F}_t) = \underline{0}$ for every $t \geqq 0$ and $h > 0$ with probability 1,

c/ there exists a positiv semidefinite matrix B such that

$$E((\underline{\xi}(t+h) - \underline{\xi}(t))(\underline{\xi}(t+h) - \underline{\xi}(t))^* / \mathcal{F}_t) = h.B$$

for every $t \geqq$ and $h > 0$ with probability 1

then $\underline{\xi}(t)$ is a multidimensional Brownian motion process. The same statement is true under the weaker conditions of remark 2 /p.61/.

It may be easily proved that $\varphi(t)$ is continuous with probability 1 and so $\varphi(t)$ must be constant. As $\varphi(s) = 0$, $\varphi(t) = 0$ for $t \geqq s$. So we have proved that condition b/ in the theorem is satisfied. Similar argument shows that for the function

$$\psi(t) = E\left[(\xi(t) - \xi(s))^2 / \mathcal{F}_s\right]$$

$\lim\limits_{\Delta \downarrow 0} \dfrac{\psi(t+\Delta) - \psi(t)}{\Delta} = 1$ is satisfied, if $t \geqq s$, and so $\psi(t) = t - s$. Which proves the statement.

Chapter 9:

Stochastic differentials and a theorem of Ito

If the process $\xi(t)$ may be represented in the form

$$(9.1)\qquad \xi(t)=\xi(0)+\int_0^t a(t,\omega)\,dt+\int_0^t b(t,\omega)\,dw(t)$$

where the processes $a(t,\omega)$, $b(t,\omega)$ belong to $\mathfrak{M}^2[0,T]$ then we say that it satisfies the stochastic differential equation /or it has the stochastic differential/

$$d\xi(t)=a(t,\omega)\,dt+b(t,\omega)\,dw(t).$$

We have to remark that the termin stochastic differential has only meaning in the sense of 9.1 , but we shall use this termin for brevity. $a(t,\omega)$ and $b(t,\omega)$ we shall call the stream coefficient /local expectation/ resp. diffusion coefficient /local variance/ of the process $\xi(t)$.

In the general case we suppose that $a(s,\omega)$ and $b(s,\omega)$ are measurable with respect to $\mathfrak{F}_s$ for every fixed s and

$$P\{\int_0^T |a(s,\omega)|\,ds<\infty\}=1,$$

$$P\{\int_0^T b^2(s,\omega)\,ds<\infty\}=1,$$

where $\{w(t),\mathfrak{F}_t,P\}$ form a martingal in $0\leq t\leq T$ (T may be ∞).

In the sequel when we say $\xi(t)$ has stochastic differential /or satisfies stochastic differential equation/ we mean that a, b satisfy the above conditions.

<u>Theorem</u> (Ito). Let $u = u(t, x_1, \dots, x_n)$ a function on $[0,\infty] \times R^n$, where the functions

$$u_0 = \frac{\partial u}{\partial t}, \quad u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

are continuous. Let us suppose that the processes $\xi_i(t)$ $i = 1, 2, \dots, n$ satisfy the stochastic differential equations

$$d\,\xi_i(t) = a_i(t,\omega)\,dt + b_i(t,\omega)\,dw(t).$$

Then the process $\eta(t) = u(t, \xi_1(t), \dots, \xi_n(t))$ has the stochastic differential

$$d\eta(t) = \Big[u_0(t, \xi_1, \dots, \xi_n) + \frac{1}{2}\sum_{i,j=1}^{n} b_i b_j u_{ij}\Big]dt + \sum_{i=1}^{n} u_i\, d\xi_i(t) =$$

$$= \Big[u_0 + \frac{1}{2}\sum_{i,j=1}^{n} b_i b_j u_{ij}\Big]dt + \sum_{i=1}^{n} u_i\,\big[a_i\,dt + b_i\,dw(t)\big]$$

Before the proof of this theorem let us consider some examples. In the case $n = 1$ Ito's formula is the following

$$d\eta(t) = \Big[\frac{\partial u(t,\xi)}{\partial t} + a(t,\omega)\frac{\partial u(t,\xi)}{\partial x}\Big]dt + \frac{1}{2}b^2 \cdot \frac{\partial^2 u(t,\xi)}{\partial x^2}\,dt +$$

$$+ b\,\frac{\partial u(t,\xi)}{\partial x}\cdot dw(t).$$

The difference between the ordinary and stochastic differentials is expressed by the term $\frac{1}{2}b^2 \cdot \frac{\partial^2 u}{\partial x^2}\,dt$. Its appearance may be explained by the known properties of the Brownian motion process, for which

$$dt \cdot dw(t) = 0, \qquad (dw(t))^2 = dt$$

at the same time $(dt)^2=0$. The formula $(dw)^2=dt$ is an equivalent form of the relation

$$\int_0^T (dw(t))^2 = T.$$

Let $u(t,x) = x^2$ and $\xi(t) = w(t)$. In this case $a\equiv 0$ $b\equiv 1$. From Ito's formula we get for

(9.2) $\qquad d\eta(t) = dt + 2w(t)\,dw(t).$

This result is nothing else as the differential form of the known relation for Brownian motion processes

$$\int_0^T w(t)\,dw(t) = \frac{1}{2}[w^2(T) - T].$$

Heuristically (9.2) may have the following explanation:

$$\Delta\eta(t) = w^2(t+\Delta) - w^2(t) = (w(t+\Delta) - w(t))(w(t+\Delta) + w(t)) =$$

$$= (w(t+\Delta) - w(t))(w(t+\Delta) - w(t) + 2w(t)) =$$

$$= (\Delta w(t))^2 + 2w(t)\,\Delta w(t),$$

where $(\Delta w(t))^2 \sim \Delta t$.

This gives (9.2) .

As a next example let $f(t) = f(t,\omega)$ measurable with respect to $\mathcal{F}_t$, $P\{\int_0^1 f^2(t,\omega)\,dt < \infty\} = 1$,

$$\zeta(t) = \exp\{\int_0^t f(s,\omega)\,dw(s) - \frac{1}{2}\int_0^t f^2(s,\omega)\,ds\},$$

that is

$$\zeta(t) = \exp\{\xi(t)\},$$

where

$$d\xi(t) = -\frac{1}{2} f^2(t,\omega)\,dt + f(t,\omega)\,dw(t).$$

From Ito's formula we get

$$d\xi(t) = \xi(t)\, f(t)\, dw(t),$$

$$d\left(\frac{1}{\xi(t)}\right) = \frac{f^2(t)}{\xi(t)}\, dt - \frac{f(t)}{\xi(t)}\, dw(t) .$$

This means that the process $\xi(t)$ is the solution of the following stochastic integral equation

$$(9.3) \qquad \eta(t) = 1 + \int_0^t \eta(s)\, f(s)\, dw(s).$$

We prove with the help of Ito's formula that the only continous solution of (9.3) is given by

$$\xi(t) = \exp\{ \int_0^t f(s)\, dw(s) - \int_0^t f^2(s)\, ds \} .$$

Let $\eta(t)$ a continuous solution of (9.3) . As we have seen $\xi(t)$ is a solution too. Applying Ito's formula to the process $\eta(t) \cdot \frac{1}{\xi(t)}$ we get if we put $u(t, x_1, x_2) = x_1 \cdot x_2$)

$$d\left(\frac{\eta(t)}{\xi(t)}\right) = \frac{1}{\xi(t)}\, d\eta(t) + \eta(t)\, d\left(\frac{1}{\xi(t)}\right) - \frac{f(t)}{\xi(t)}\, \eta(t)\, f(t)\, dt =$$

$$= \frac{1}{\xi(t)}\left[\eta(t) f(t)\, dw(t)\right] + \eta(t)\left[\frac{f^2(t)}{\xi(t)}\, dt - \frac{f(t)}{\xi(t)}\, dw(t)\right] -$$

$$- \frac{f^2(t)}{\xi(t)}\, \eta(t)\, dt \equiv 0,$$

this means that $\eta(t) = \xi(t)$ with probability 1.

Proof of the theorem. It is enough to prove the theorem for the case when $a_i(t, \omega)$ and $b_i(t, \omega)$ are simple functions, that is they are constantson some intervals of t. This means that

$$\xi_i(t) = a_i t + b_i(w(t) - w(t_k)), \quad \text{on} \quad t_k < t < t_{k+1},$$

where a_i and b_i are constants. In this case there exists a smooth function v such that

$$u(t, \xi_1, \ldots, \xi_n) = v\,(t, w(t)).$$

So we prove Ito's formula for functions of the form

$$\xi(t) = v(t, w(t)), \qquad 0 \leqq t \leqq 1.$$

Let $\ell = [2^n \cdot t]$ and $\Delta w_k = w(k \cdot 2^n) - w((k-1)2^{-n})$; $k = 1, 2, \ldots, \ell$, then

$$v(t, w(t)) - v(0,0) = \sum_{k \leqq \ell} \{v(k.2^{-n}, w(k.2^{-n})) - v((k-1)2^{-n}, w(k.2^{-n}))\}$$

$$+ \sum_{k \leqq \ell} \{v((k-1)2^{-n}, w(k2^{-n})) - v((k-1)2^{-n}, w((k-1)2^{-n}))\} +$$

$$+ \{v(t, w(t)) - v(\ell.2^{-n}, w(\ell 2^{-n}))\},$$

from our assumptions of differentiability we have, with the notations

$$v_0 = \frac{\partial v}{\partial t}, \qquad v_i = \frac{\partial v}{\partial x_i}, \qquad v_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j},$$

$$\sum_{k \leqq \ell} \{v(k.2^{-n}, w(k.2^{-n})) - v((k-1)2^{-n}, w(k.2^{-n}))\} =$$

$$= \sum_{k \leqq \ell} \{v_0((k-1)2^{-n}, w(k.2^{-n})).2^{-n} + o(2^{-n})\}$$

$$\sum_{k \leqq \ell} \{v((k-1)2^{-n}, w(k.2^{-n})) - v((k-1)2^{-n}, w((k-1)2^{-n}))\} =$$

$$=\sum_{k\leq \ell}\{v_1((k-1)2^{-n},\ w((k-1)2^{-n}))\Delta w_k+\frac{1}{2}v_{11}((k-1)2^{-n},w((k-1)2^{-n}))\cdot$$

$$\cdot(\Delta w_k)^2+\ o(\Delta w_k)^2\}$$

and

$$v(t,w(t))-v(\ell.2^{-n},\ w(\ell.2^{-n}))=o(1).$$

From the definition of stochastic integrals we get

$$\sum_{k\leq\ell}\{v_0((k-1)2^{-n},\ w(k.2^{-n})).2^{-n}+o(2^{-n})\}=\int_0^t v_0(s,w(s))ds+o(1),$$

$$\sum_{k\leq\ell}\{v_1((k-1)2^{-n},\ w((k-1)2^{-n}))\Delta w_k+\frac{1}{2}v_{11}((k-1)2^{-n},w((k-1)2^{-n}))(\Delta w_k)^2$$

$$+\ o(\Delta w_k)^2\}=\int_0^t v_1(s,w(s))\,dw(s)+\frac{1}{2}\int_0^t v_{11}(s,w(s))\,ds+$$

$$+\frac{1}{2}\sum_{k\leq\ell}v_{11}((k-1)2^{-n},\ w((k-1)2^{-n}))\,[(\Delta w_k)^2-2^{-n}]+o(1).$$

From the last relations we see that to complete the proof for the special case $v(t,w(t))$ it is enough to show that

$$\sum v_{11}((k-1)2^{-n},\ w((k-1)2^{-n})).\,[(\Delta w_k)^2-2^{-n}]\longrightarrow 0$$

in probability if $n\to\infty$.

Let

$$\varepsilon_k=[(\Delta w_k)^2-2^{-n}],$$

and for fixed N

$$\chi_k^N=\chi\ \{\max_{i\leq k}|w_{i.2^{-n}}|\leq N\}.$$

We have

$$P\{\sum_{k\leq\ell}v_{11}((k-1)2^{-n},\ w((k-1)2^{-n}))\,\varepsilon_k\cdot[1-\chi_{k-1}^N]\neq 0\}\leq$$

$$\leq P\{\sup_{0\leq t\leq 1} |w(t)|>N\} \to 0, \quad \text{if } N\to\infty.$$

On the other hand

$$E\sum_{k\leq \ell} v_{11}((k-1)2^{-n},\ w((k-1)2^{-n}))\,\varepsilon_k = 0,$$

and (using $E\ \varepsilon_k^2 = 2.2^{-2n}$)

$$E\Big(\sum_{k\leq\ell} v_{11}((k-1)2^{-n},\ w((k-1)2^{-n}))\,\varepsilon_k\ \chi_{k-1}^N\Big)^2 =$$

$$=\sum_{k\leq\ell} E\ v_{11}^2((k-1)2^{-n},\ w((k-1)2^{-n}))\,\chi_{k-1}^N\ .\ \varepsilon_k^2 \leq$$

$$\leq \sup_{\substack{0\leq t\leq 1\\ |x|\leq N}} v_{11}^2(t,x)\sum_{k\leq\ell} E\,\varepsilon_k^2 = 2\sup_{\substack{0\leq t\leq 1\\ |x|\leq N}} v_{11}^2(t,x)\sum_{k\leq\ell}(2^{-n})^2 \to 0,$$

if $n\to\infty$. So the proof is completed.

Exercises

1. Prove that for natural $m\geq 2$

$$d(w(t))^m = m(w(t))^{m-1}\,dw(t) + \frac{m(m-1)}{2}(w(t))^{m-2}\,dt.$$

2. Prove, if $f(x)$ is twice differentiable and $f''(x)$ is continuous then

$$df(w(t)) = f'(w(t))\,dw(t) + \frac{1}{2}f''(w(t))\,dt.$$

3. Let $b_i(t,\omega)$ $(i=1,2)$ measurable with respect $\mathcal{F}_t$, $P\{\int b_i^2(t)\,dt<\infty\} = 1$, $\int E\,b_i^2(t)\,dt<\infty$, then

$$E\int b_1(t)\,dw(t)\int b_2(t)\,dw(t) = \int E\,b_1(t)\,b_2(t)\,dt.$$

4. Let $b(t,\omega)$ has the properties of the preceding exercise and $\int E b^{2m}(t)\,dt<\infty$ (m - natural), then

$$E\Big[\int_0^T b(t)\,dw(t)\Big]^{2m} \leq [m(2m-1)]^{m-1}\,T^{m-1}\int_0^T E b^{2m}(t)\,dt.$$

Chapter 10:

Martingales, semi martingales

Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a probability space, $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \dots$ - algebras $/ n = 1, 2 \dots /$.

Definition We call the sequence $\{\xi_n, \mathcal{F}_n\}$ a martingale, if

a/ ξ_n is $\mathcal{F}_n$ measurable,

b/ $E|\xi_n| < \infty$,

c/ $E(\xi_{n+1}/\mathcal{F}_n) = \xi_n$.

If we have the inequality

c'/ $E(\xi_{n+1}/\mathcal{F}_n) \leq \xi_n$ resp.

c"/ $E(\xi_{n+1}/\mathcal{F}_n) \geq \xi_n$ instead of

the equality c/ we call the sequence a super- /resp. sub-/ martingale.

Examples

1./ Let $E|\xi| < \infty$ and $\xi_n = E(\xi/\mathcal{F}_n)$; then $\{\xi_n, \mathcal{F}_n\}$ is a martingale.

2./ Let $\xi_1, \xi_2, \dots$ be a sequence of independent identically distributed random variables $(E\xi_i = 0)$. If $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ and $\eta = \xi_1 + \dots + \xi_n$ then $\{\eta_n, \mathcal{F}_n\}$ is a martingale.

3./ Let $\mathcal{P} \ll Q$ be two measures on $\mathcal{F}$, $\mathcal{P}_n$ and Q_n the restrictions of $\mathcal{P}$ and Q to $\mathcal{F}_n (\subseteq \mathcal{F}_{n+1})$. Then obviously $\mathcal{P}_n \ll Q_n$. Let $\xi_n = \frac{d\mathcal{P}_n}{dQ_n}$ the Radon-Nikodym derivative, i.e. for any $A \in \mathcal{F}_n$ $\mathcal{P}_n(A) = \int_A \xi_n Q_n(d\omega)$.
Then $\{\xi_n, \mathcal{F}_n, Q\}$ is a martingale. Indeed, ξ_n is a

martingale if and only if for any $A \in \mathcal{F}_m$, $m < n$

$$\int_A \xi_m(\omega)\, Q(d\omega) = \int_A \xi_n(\omega)\, Q(d\omega).$$

4./ Let $\xi_1, \xi_2, \dots$ be independent, identically distributed random variables.

$\mathcal{F}_n = \sigma(\xi_1, \xi_2, \dots, \xi_n)$,

$\mathcal{F}_\infty = \sigma(\xi_1, \xi_2, \dots\dots)$.

Let us have two hypotheses for the distribution of ξ_n

H_p: the probability density function is $p(x)$

H_q: the probability density function is $q(x)$

Let P and Q be the corresponding two measures generated by $(\xi_1, \xi_2, \dots)$ on $(\Omega, \mathcal{F}_\infty)$. If for any Borel measurable A from $\int_A q(x)\,dx = 0$ follows $\int_A p(x)\,dx = 0$, then Q a. e. $\longrightarrow$

$$\frac{dP_n}{dQ_n} = \prod_{i=1}^{n} \frac{p(\xi_i(\omega))}{q(\xi_i(\omega))} = \zeta_n \, .$$

The sequence $\{\zeta_n, \mathcal{F}_n, Q\}$ is a martingale.

5./ Let $\xi_1, \xi_2, \dots$ be random variables $E|\xi_i| < \infty$.

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$

and

$$\eta_n = \sum_{k=1}^{n} (\xi_n - E(\xi_k | \mathcal{F}_{k-1})).$$

The sequence $\{\eta_n, \mathcal{F}_n\}$ is a martingale.

6./ Let $\{\xi_n, \mathcal{F}_n\}$ be a submartingale $/n = 1, 2, \dots/$.

Let $\xi_0 = 0, \dots,$ $\zeta_n = \xi_n - \xi_{n-1}$, for $n \geq 1$.

Then

$$\xi_n = \sum_{k=1}^{n} \zeta_k = \sum_{k=1}^{n} [\zeta - E(\zeta_k | \mathcal{F}_{k-1}] + \sum_{k=1}^{n} E(\zeta_k | \mathcal{F}_{k-1}),$$

where

$$E(\xi_k|\mathcal{F}_{k-1}) = E(\xi_k - \xi_{k-1}|\mathcal{F}_{k-1}) \geq 0.$$

So we get $\xi_n = \eta_n + \alpha_n$, where

$$\eta_n = \sum_{k=1}^{n} [\xi_k - E(\xi_k|\mathcal{F}_{k-1})]$$

is a martingale and $\alpha_n = \sum_{1}^{n} E(\xi_k|\mathcal{F}_{k-1})$ is a non decreasing sequence.

In the sequel we recite some well known theorems from Lebesgue's integral theory.

Theorem 1/monotone convergence/. If $f_n \to f$ with probability 1, then from $f_n \uparrow f$ and $E f_1^- < \infty$ /resp. $f_n \downarrow f$ and $E f_1^+ < \infty$ /follows $E f_n \uparrow E f$ / resp. $E f_n \downarrow E f$ /

Definition. The sequence $\{f_n\}$ of integrable functions is uniformly integrable if

$$\lim_{a\to\infty} \sup_n \int_{\{|f_n|>a\}} |f_n| \, dP = 0.$$

This condition is equivalent to the following two conditions:

(i) $\sup_n E|f_n| < \infty$,

(ii) $\lim_{P(A)\to 0} \sup_n \int_A |f_n| \, dP = 0.$

Theorem 2.(Fatou's lemma):

If the sequence $\{f_n^+\}$ is uniformly integrable and $|E \limsup_{n\to\infty} f_n| < \infty$, then $E(\limsup_{n\to\infty} f_n) \geq \limsup_n E f_n$

Theorem 3. If $f_n \geq 0$, $f_n \to f$ a.e. and $E f_n < \infty$, then the convergence $E f_n \to E f$ is equivalent to the uniform integrability of the sequence $\{f_n\}$.

<u>Theorem</u> 4.(majorated convergence). If $f_n \to f$ a.e. and there exists a function g, for which $|f_n| \leqq g$ and $Eg < \infty$, then $E|f_n - f| \to 0$.

<u>Theorem</u> 5.(Lévy). Let $\mathcal{F}_n$ be a non decreasing sequence of σ-subalgebras of $\mathcal{F}$, and denote $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ by $\mathcal{F}_\infty$. For any $\mathcal{F}$ measurable random variable ξ with $E|\xi| < \infty$, the following relation holds a.e.:

$$\lim_{n\to\infty} E(\xi|\mathcal{F}_n) = E(\xi|\mathcal{F}_\infty).$$

<u>Proof</u>.

Without loss of generality we may assume that $\mathcal{F}_\infty = \mathcal{F}$. We shall use the following lemma:

Let $(\Omega, \mathcal{F}, P)$ a probability space and G an algebra, which generates the σ-algebra $\mathcal{F}$. For arbitrary $\varepsilon > 0$ and $A \in \mathcal{F}$ there exists a $B \in G$, such that

$$(10.1) \qquad P(A \setminus B) + P(B \setminus A) < \varepsilon$$

(Hint for the proof. Let $\mathfrak{U}$ be the class of the sets of the above property. $\mathfrak{U} \supset G$ and $\mathfrak{U}$ is a σ-algebra, so $\mathfrak{U} \supseteq \mathcal{F}$.). Obviously (1) means, that

$$(10.2) \qquad E|\chi_A - \chi_B| < \varepsilon.$$

From (2) and the $\mathcal{F}_\infty$ measurability of ξ follows that for any given $\varepsilon > 0$ there exist a natural $n_0 = n_0(\varepsilon)$ and an $\mathcal{F}_{n_0}$ - measurable random variable ξ_{n_0}, of property

$$E|\xi - \xi_{n_0}| < \varepsilon^2/2.$$

Let $\eta = |\xi - \xi_{n_0}|$ and

$\tau = \inf\{n \geq n_0 : E(\eta|\mathcal{F}_n) > \varepsilon\}$

(if there is no such n, then $\tau = \infty$). As for every $n \geqq n_0$

$\{\tau = n\} \in \mathcal{F}_n$, so

$$\mathcal{P}\{E(\eta|\mathcal{F}_n) > \varepsilon \quad \text{for some} \quad n \geqq n_0\} =$$

$$= \sum_{n=n_0}^{\infty} \mathcal{P}\{\tau = n\} = \sum_{n=n_0}^{\infty} \int_{\{\tau = n\}} 1 .\, dP \leqq \sum_{n=n_0}^{\infty} \frac{1}{\varepsilon} \int_{\{\tau \leqq n\}} E(\eta|\mathcal{F}_n)\, dP =$$

$$= \sum_{n=n_0} \frac{1}{\varepsilon} \int_{\{\tau=n\}} \eta \, dP \leqq \frac{1}{\varepsilon} E\,\eta = \frac{\varepsilon}{2} .$$

Furthermore for every $n \geqq n_0$

$$|E(\xi|\mathcal{F}_n) - \xi| = |E(\xi - \xi_{n_0}|\mathcal{F}_n) + (\xi_{n_0} - \xi)| \leqq |E(\xi - \xi_{n_0}|\mathcal{F}_n)| +$$

$$+ |\xi_{n_0} - \xi| \leqq E(\eta|\mathcal{F}_n) + \eta .$$

Hence

$$\mathcal{P}\{|E(\xi|\mathcal{F}_n) - \xi| > 2\varepsilon \quad \text{for some} \quad n \geqq n_0\} \leqq$$

$$\mathcal{P}\{E(\eta|\mathcal{F}_n) > \varepsilon \quad \text{for some} \quad n \geqq n_0\} +$$

$$\mathcal{P}\{\eta > \varepsilon\} \leqq \varepsilon/2 + 1/\varepsilon\, E\,\eta \leqq \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

This proves Lévy's theorem.

The following theorem includes the theorems of Lebesgue and Lévy too.

<u>Theorem</u> 6. If $\xi_n \to \xi$ a.e., $|\xi_n| < \eta$, $E\,\eta < \infty$, $\{\mathcal{F}_n \subseteq \mathcal{F}\}$ is a non-decreasing sequence of σ-algebras and $\mathcal{F} = \sigma(\cup \mathcal{F}_n)$, then a.e. $\lim\limits_{n,m} E(\xi_n|\mathcal{F}_m) = E(\xi|\mathcal{F}_\infty)$.

Proof. Let $\overline{\alpha} = \lim\limits_{N\to\infty} \sup\limits_{n \geqq N} E(\xi_n|\mathcal{F}_m)$,

$$\underline{\alpha} = \lim_{N\to\infty} \inf_{\substack{n \geqq N \\ m \geqq N}} E(\xi_n|\mathcal{F}_m).$$

We shall prove, that with probability 1

$$\overline{\alpha} \leqq E(\xi|\mathcal{F}_\infty) \leqq \underline{\alpha} .$$

We set for fixed K

$$\eta_K = \sup_{n \geq K} \xi_n ,$$

Then for $n \geq K$:

$$\eta_K \geq \xi_n , \quad \text{and}$$

$$\bar{\alpha} = \lim_{N} \sup_{\substack{n \geq N \\ m \geq N}} E(\xi_n | \mathcal{F}_m) \leq \lim_{N \to \infty} \sup_{m \geq N} E(\eta_K | \mathcal{F}_m)$$

It is clear, that

$$E|\eta_K| = E[|\sup_{n \geq K} \xi_n|] \leq E\eta < \infty ,$$

by Lévy's theorem

$$\lim_{N \to \infty} \sup_{m \geq N} E(\eta_K | \mathcal{F}_m) = \lim_{N \to \infty} E(\eta_K | \mathcal{F}_n) = E(\eta_K | \mathcal{F}_\infty) .$$

So for any K we get $\bar{\alpha} \leq E(\eta_K | \mathcal{F}_\infty)$ and as $\eta_K \downarrow \lim_{K \to \infty} \sup_{n \geq K} \xi_n = \lim_{n \to \infty} \xi_n = \xi$, furthermore $E|\sup_n \xi_n| \leq E\eta < \infty$, by the monotone convergence-theorem (which is valid for conditional expectations too):

$$\bar{\alpha} \leq \lim_K E(\eta_K | \mathcal{F}_\infty) = E(\xi | \mathcal{F}_\infty) .$$

Similarly we can prove, that

$$E(\xi | \mathcal{F}_\infty) \leq \underline{\alpha} .$$

As $\bar{\alpha} \geq \underline{\alpha}$, we get the desired relation

$$\lim_{n, m \to \infty} E(\xi_n | \mathcal{F}_\infty) = E(\xi | \mathcal{F}_\infty) .$$

<u>Semi martingale convergence</u>

<u>Theorem 7.</u> If $\xi_n, \mathcal{F}_n$ is a submartingale and $\sup E\xi_n^+ < \infty$ then there exists /the limit/ $\lim_{n \to \infty} \xi_n \; (= \xi_\infty)$ with probability 1.

Theorem 8. If $\{\xi_n, \mathcal{F}_n\}$ is a supermartingale and $\sup E\xi_n^- < \infty$ then there exists /the limit/ $\lim\limits_{n\to\infty} \xi_n (= \xi_\infty)$

Obviously it is sufficient to prove theorem 7. For this purpose we recite an inequality of Doob:

Let $y_1, y_2, \dots, y_n$ be a sequence of real numbers, and $a < b$.

Set $y, t_0 = 0,$

$$t_1 = \min\{k;\ 1 \le k \le n,\ y_k \le a\}$$
$$t_2 = \min\{k:\ t_1 \le k \le n,\ y_k \ge b\}$$
$$\vdots$$
$$t_{2m-1} = \min\{k:\ t_{2m-2} < k \le n,\ y_k \le a\}$$
$$t_{2m} = \min\{k:\ t_{2m-1} < k \le n,\ y_k \ge b\}$$

If one of above sets is empty, then the corresponding t is equal to ∞.

Let us denote $\max\{m: t_{2m} < \infty\}$ i.e. the number of intersections from below by $\beta(a,b)$ and the number of intersections from above by $\alpha(a,b)$.

If we replace $y_1, \dots, y_n$ by random variables $\eta_1, \dots, \eta_n$ then $t_2, \dots, t_n$ $\alpha(a,b)$ and $\beta(a,b)$ also become random variables. Doob's lemma asserts the following:

Let $\{\xi_n, \mathcal{F}_n\}$ be a submartingale and $n \le N$. Then

$$E\,\beta(a,b) \le \frac{E(\xi_n - a)^+}{b-a} \le \frac{E(\xi_n^+ + |a|)}{b-a}, \tag{103}$$
$$E\,\alpha(a,b) \le \frac{E(\xi_n - b)^+}{b-a} \le \frac{E(\xi_n^+ + |b|)}{b-a}$$

Proof: Let first $\{\xi_n, \mathcal{F}_n\}$ be a nonnegative submartingale and $a=0$; we show that

$$E\beta(0,b) \leq \frac{E\xi_n}{b}$$

Let $\xi_0 = 0$, $u_0 = 1$ and

$$1 \quad \text{if} \quad t_m < i \leq t_{m+1}, \quad (m \text{ is odd})$$

$$0 \quad \text{if} \quad t_m < i \leq t_{m+1}, \quad (m \text{ is even})$$

We suppose that $t_1 = N$ (instead of $t_1 = \infty$) if the corresponding set is empty.

It is easy to see that

$$(10.4) \qquad u_0\xi_0 + \sum_{i=1}^{N} u_i(\xi_i - \xi_{i-1}) \geq b\beta(0,b)$$

and so

$$b\,E\beta(0,b) \leq E\sum_{i=0}^{N} u_i[\xi_i - \xi_{i-1}].$$

As

$$\{u_i = 1\} = \bigcup_{m \text{ odd}} \{t_m < i \leq t_{m+1}\} =$$

$$= \bigcup_{m \text{ odd}} \Big\{ \{t_m < i\} - \{t_{m+1} < i\} \Big\} \in \mathcal{F}_{i-1},$$

denoting $\xi_i - \xi_{i-1}$ by η_i we get

$$(10.5) \qquad E\xi_0 + E\sum_i u_i(\xi_i - \xi_{i-1}) = E\xi_0 + E\sum u_i\eta_i =$$

$$= E\xi_0 + E\sum_{i=1}^{N} u_i E(\eta_i | \mathcal{F}_{i-1}) \leq E\xi_0 + E\sum_{i=1}^{N} E(\eta_i | \mathcal{F}_{i-1}) =$$

$$= E\xi_0 + E\sum_{1}^{N} \eta_i = E\xi_N$$

If $\{\xi_i, \mathcal{F}_i\}$ is a nonnegative submartingale then from the relations (4) and (5) we get

$$E\beta(0,b) \leq \frac{E\xi_N}{b}.$$

To complete the proof of the lemma it is sufficient to show that the number of intersections $\beta(a,b)$ is equal to the number of intersections $\beta(0,\ b-a)$ for the submartingale $\{(\xi_i - a)^+,\ \mathcal{F}_i\}$

__Remark.__ If for the nonnegativ submartingale $\{\xi_n, \mathcal{F}_n\}$ u denotes the indicator function of the set $\{\max\limits_{i \leqq N} \xi_i > b\}$ then we get

$$\mathcal{P}\{\max_{i \leqq N} \xi_i \geqq b\} \leqq \frac{E u \xi_N}{b}.$$

__Proof.__ $u_i \leqq u$, so in (5) we have

$$E\xi_0 + E\sum_{i=1}^{N} u_i E(\eta_i | \mathcal{F}_{i-1}) \leqq E\xi_0 + E(u \sum_{i=1}^{N} E(\eta_i | \mathcal{F}_{i-1})) =$$

$$= E\xi_0 + E u \sum_{i=1}^{N} \eta_i = E u \xi_N \Big(= \int_{\{\max\limits_{i \leqq N} \xi_i > b\}} \xi_N \, d\mathcal{P}\Big).$$

__The proof of Theorem 7.__

Let

$$\xi^* = \lim_n \sup \xi_n,$$
$$\xi_* = \lim \inf \xi_n,$$

and suppose that $\{\mathcal{P}\ \xi^* > \xi_*\} > 0$.

From the identity

$$\{\xi^* > \xi_*\} = \{\bigcup_{\substack{a<b \\ a,b \text{ rationals}}} \xi^* > a > b > \xi_*\}$$

we get then, that there are rationals a,b such that

$$\mathcal{P}\{\xi^* > a > b > \xi_*\} > 0$$

From this if follows that

$$\text{(1o.6)} \qquad \mathcal{P}\{\beta(a,b) = \infty\} > 0 .$$

But according to the inequality e we have

$$E\,\beta(a,b) \leqq \frac{\sup_n \xi_n^+ + |a|}{b-a} < \infty ,$$

contradicting (6).

So the first statement of Theorem 7. is proved. The second statement is a consequence of the first.

Two examples for the use of the supermartingale convergence theorem.

1. Let

$$\xi_n = E(\xi/\mathcal{F}_n), \quad E|\xi| < \infty$$

be a martingale. On the basis of the supermartingale convergence theorem there exists the limit $\xi_\infty = \lim E(\xi|\mathcal{F}_n)$ and it is $\mathcal{F}_\infty$ measurable.

Show, that $\lim_n E(\xi/\mathcal{F}_n) = E(\xi/\mathcal{F}_\infty)$.

2. The Kolmogorov's $0-1$ law.

Let $\eta_1, \eta_2, \ldots$ be independent random variables, and $\mathcal{F}_n = \sigma(\eta_1, \eta_2, \ldots, \eta_n)$. Suppose that $A \in \sigma(\eta_{n+1}, \ldots)$ for every n. (E.g. the sets of the form $\{\sup_n \eta_n < \infty\}$ or $\{$there exists $\lim_{n\to\infty} \eta_n\}$).

The fields $\mathcal{F}_n$ and $\sigma(\eta_{n+1}, \eta_{n+2}, \ldots)$ are independent, and if $A \in \sigma(\eta_{n+1}, \ldots)$ then $\mathcal{P}(A|\mathcal{F}_n = \mathcal{P}(A)$.

On the other hand, from 1.

$$\mathcal{P}(A|\mathcal{F}_n) = E(\chi_A|\mathcal{F}_n) \to E(\chi_A|\mathcal{F}_\infty) = \chi_A ,$$

therefore

$$\mathcal{P}(A) = \chi_A , \quad \text{i.e. } \mathcal{P}(A) = 0 \text{ or } 1 .$$

Martingal Inequalities

From Doob's Lemma it follows that if $\{\xi_n, \mathcal{F}_n\}$ is a submartingal then

$$(10.7) \quad \mathcal{P}\{\max_{k\leqq n} \xi_n \geqq c\} \leqq \frac{E\xi_n^+}{c} .$$

This is the so called Kolmogorov's inequality. We can easily see that even the stronger inequality

$$\mathcal{P}\{\max_{k\leqq n} \xi_n \geqq c\} \leqq \frac{1}{c} \int\limits_{\{\max_{k\leqq n} \xi_k \geqq c\}} \xi_n^+ \, d\mathcal{P}$$

holds.

Leter for stochastic integrals we want to prove the inequality

$$E \sup_{0\leqq t\leqq T} \Big(\int_0^t f(s,\omega)\, dw(s)\Big)^2 \leqq 4E \int_0^T f^2(s,\omega)\, ds.$$

In order to do this we need the following inequality.

Lemma 1. Let $\alpha = 1$ and the random variables $\xi_1, \ldots, \xi_n$ satisfy the conditions: $E|\xi_k|^\alpha < \infty$ and for each k

$$E(\xi_n - \xi_k | \xi_1, \ldots, \xi_k) = 0 .$$

Then denoting $\xi = \sup(0, \xi_1, \dots, \xi_n)$, $\xi_n^+ = \sup(0, \xi_n)$

we have

$$E\,\xi^{\alpha} \leq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha} E(\xi_n^+)^{\alpha}.$$

<u>Proof</u> Let $a>0$ and $\chi_k(a)=1$ if $\xi_1<a, \dots, \xi_{k-1}<a, \xi_k \geq a$ else $\chi_k(a)=0$. As $\sum_{k=1}^{n} \chi_k(a)=1$ if $a \leq \xi$ and $\sum \chi_k(a) = 0$ if $\xi < a$ we have for

$$\xi^{\alpha} = \alpha \int_0^{\infty} a^{\alpha-1} \sum_{k=1}^{n} \chi_k(a)\, da.$$

From the definition of $\chi_k(a)$ we may deduce the inequality a $\chi_k(a) \leq \xi_k\, \chi_k(a)$. So we have

$$a \sum_{k=1}^{n} \chi_k(a) \leq \sum_{k=1}^{n} \xi_k\, \chi_k(a)$$

and

$$a^{\alpha-1} \sum_{k=1}^{n} \chi_k(a) \leq \sum_{k=1}^{n} \xi_k\, a^{\alpha-2}\, \chi_k(a).$$

$\chi_k(a)$ are measurable functions of the variables $\xi_1, \dots, \xi_k$ consequently

$$E(\xi_n - \xi_k)\chi_k(a) = E\,\chi_k(a) E(\xi_n - \xi_k / \xi_1, \dots, \xi_k) = 0$$

and

$$E\, a^{\alpha-1} \sum_{k=1}^{n} \chi_k(a) \leq E \sum_{k=1}^{n} a^{\alpha-2} \chi_k(a)\xi_n \leq E \sum_{k=1}^{n} a^{\alpha-2} \chi_k \cdot \xi_n^+$$

Taking the integral of both sides of this relation with respect to a from 0 to infinity we get

$$E\,\frac{1}{2}\,\xi^{\alpha} \leq E\,\frac{1}{\alpha-1}\,\xi^{\alpha-1}\,\xi_n^+$$

Using Hölder's inequality we shall have

$$E\xi^{\alpha} \leq \frac{\alpha}{\alpha-1} E\xi^{\alpha-1}\xi_n^{+} \leq \frac{\alpha}{\alpha-1}[E\xi^{\alpha}]^{\frac{\alpha-1}{\alpha}}[E(\xi_n^{+})]^{1/\alpha}$$

which is equivalent with the statement of the lemma.

Remark 1. Let us notice that

$$P\{\xi > a\} = \sum_{k=1}^{n} E\chi_k(a)$$

and so from the relation

$$a^r \sum_{k=1}^{n} \chi_k(a) \leq \sum_{k=1}^{n} (\xi_k^{+})^r \chi_k(a) \qquad (r = 1,2)$$

we get the inequalities

$$P(\xi > a) \geq \frac{E\xi_n^{+}}{a}, \qquad P\{\xi > a\} \leq \frac{E(\xi_n^{+})^2}{a^2}$$

Corollary. Applying the lemma to the variables $-\xi_1, -\xi_2, -\xi_3, \dots, -\xi_n$, we get

$$E\xi_-^{\alpha} \leq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha} E(\xi_n^{-})^{\alpha},$$

where $\xi_- = \max(0, -\xi_1, \dots, -\xi_n)$, $\xi_n^{-} = \max(0, -\xi_n)$. Hence, using equalities $\max|\xi_k| = \max_k(\xi, \xi_-)$, $|\xi_n|^{\alpha} = (\xi_n^{+})^{\alpha} + (\xi_n^{-})^{\alpha}$ we get

$$E(\max_k|\xi_k|)^{\alpha} \leq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha} E|\xi_n|^{\alpha}.$$

Theorem 9. Let $1 < \alpha < \infty$ and $(\xi_n, \mathcal{F}_n)$ a non-negative submartingal for which

$$\sup_n E\xi_n^{\alpha} < \infty.$$

Then

$$(10.8)\quad E\,(\sup_n \xi_n)^{\alpha} < \infty$$

$$(10.9)\quad E\,(\sup_n \xi_n)^{\alpha} < \left(\frac{\alpha}{\alpha-1}\right)^{\alpha} \qquad \sup_n \xi_n^{\alpha}\,.$$

__Proof.__ The submartingal convergence theorem ensures the existence of lim $\xi_n = \xi$. By Fatou's lemma

$$E\,\xi \leq \underline{\lim}\, E\,\xi_n < \infty$$

Let $\eta_N = \sup\limits_{n \leq N} \xi_n$. For any $\lambda > 0$ we get from Kolmogorov's inequality

$$\lambda P\{\eta_N \geq \lambda\} \leq \int\limits_{\{\eta_N \geq \lambda\}} \xi_N \, dP\,.$$

Set $F(\lambda) = P(\eta_N > \lambda)$. Then

$$E\big[\sup_{n\leq N} \xi_n\big]^{\alpha} = E\,\eta_N^{\alpha} = -\int_0^{\infty} \lambda^{\alpha}\, dF(\lambda) = \int_0^{\infty} F(\lambda)\, d(\lambda^{\alpha}) -$$

$$-\lim_{h\to\infty}\big[\lambda^{\alpha}F(\lambda)\big]_0^h \leq \int_0^{\infty} F(\lambda)\, d(\lambda^{\alpha}) \leq \int_0^{\infty} \frac{1}{\lambda}\Big(\int\limits_{\{\eta_n>\lambda\}} \xi_N\, dP\Big)\, d(\lambda^{\alpha}) =$$

$$= \int\limits_{\Omega} \xi_N \Big(\int_0^{\eta_N} \frac{d(\lambda^{\alpha})}{\lambda}\Big)\, dP = \frac{\alpha}{\alpha-1}\, E\,\xi_N\, \eta_N^{\alpha-1}\,.$$

Using Hölder's inequality

$$E\,\xi_N\, \eta_N^{\alpha-1} \leq (E\,\xi_N^{\alpha})^{1/\alpha}(E\,\eta_N^{(\alpha-1)q})^{1/q} =$$

$$= (E\,\xi_N^{\alpha})^{1/\alpha}(E\,\eta_N^{\alpha})^{1/q}\,, \qquad \Big(q = \frac{\alpha}{1-\alpha}\Big).$$

Now if $E\,\eta_N^{\alpha} < \infty$, then the statement follows easily. Otherwise let $\eta_N^{(n)} = \min\{\eta_N, n\}$; then the inequality

$$\lambda P\{\eta_N^{(n)} \geq \lambda\} \leq \int_{\{\eta_N^{(n)} \geq \lambda\}} \xi_N \, dP$$

holds, and applying the above result we get

$$E(\eta_N^{(n)}) \leq q^{\alpha} E \xi_N^{\alpha} .$$

As $\eta_N^{(n)} \uparrow \eta_N$ it follows by Fatou's lemma that

$$E(\eta_N^{(n)})^{\alpha} \leq \underline{\lim}_{n} E(\eta_N^{(n)})^{\alpha} \leq q^{\alpha} E \xi_N^{\alpha} ,$$

which proves the theorem.

Remark. Lemma 1 is a special case of theorem 1.

<u>Corollary 1</u>. Let $(\xi_n, \mathcal{F}_n)$ a martingal, $\alpha > 1$, $\sup_n E|\xi_n|^{\alpha} < \infty$ then, as $\{|\xi_n|, \mathcal{F}_n\}$ is a submartingal

$$E(\sup_n |\xi_n|)^{\alpha} < \infty,$$

$$E(\sup_n |\xi_n|)^{\alpha} \leq \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} \sup E|\xi_n|^{\alpha}.$$

<u>Corollary 2</u>. Let the martingal $(\xi_n, \mathcal{F}_n)$ be square integrable, for which $\sup_n E\xi_n^2 < \infty$. Then

$$E(\sup \xi_n^2) < \infty,$$

$$E(\sup_n \xi_n^2) \leq 4 \sup_n E\xi_n^2 .$$

<u>Remark</u>. If $\alpha = 1$, the theorem is not true, so the following theorem is useful.

<u>Theorem lo</u>. Let $(\xi_n, \mathcal{F}_n)$ be a martingal, for which

$$\sup_n E(|\xi_n| \quad \log^+ |\xi_n|) < \infty,$$

then

$$E(\sup_n |\xi_n|) \leq \frac{e}{e-1}(1+\sup_n E(|\xi_n| \log^+ |\xi_n|) < \infty .$$

Proof. Let $a, b > 0$; then

$$a \log^+ b \leq a \log^+ a + \frac{b}{e}$$

/This can be seen as follows:

$$\log b \leq \frac{b}{e},$$

from where

$$a \log \frac{b}{a} \leq a \frac{b}{a} \cdot \frac{1}{e} = \frac{b}{e},$$

$$a \log b \leq a \log a + \frac{b}{e} \leq a \log^+ a + \frac{b}{e}$$

and

$$a \log^+ b \leq a \log^+ a + \frac{b}{e}.)$$

Integrating the above mentioned inequality

$$P\{\sup_{m\leq n} |\xi_m| \geq a\} \leq \frac{1}{a} \int_{\{\sup_{m\leq n} |\xi_m| > a\}} |\xi_m| \, dP,$$

according to a in $(1, \infty)$ we get

$$(10.10) \quad \int_1^\infty P(\sup |\xi_m| > a)\, da \leq \int_1^\infty \frac{da}{a} \int_{\{\sup_{m\leq n} |\xi_m| \geq a\}} |\xi_n| \, dP =$$

$$= E(|\xi_n| \log^+ (\sup_{m\leq n} |\xi_n|) \leq E(|\xi_n| \log^+ \xi_n) + \frac{1}{e} E(\sup_{m\leq n} \xi_m).$$

Furthermore

$$(10.11) \quad E(\sup_{m\leq n} |\xi_m|) = \int_0^\infty P\{\sup_{m\leq n} |\xi_m| > a\}\, da \leq$$

$$\leq 1 + \int_1^\infty P\{\sup_{m\leq n} |\xi_m| > a\}\, da$$

consequently from (10)

$$\int_1^\infty P\{\sup_{m\leq n}|\xi_m|>a\}\,da \leq \frac{1}{e}+E[|\xi_n|\log^+|\xi_n|]+\frac{1}{e}\int_1^\infty P\{\sup_{m\leq n}|\xi_m|>a\}\,da.$$

Supposing that $E[\sup_{m\leq n}|\xi_m|]<\infty$ we get

$$\left(1-\frac{1}{e}\right)\int_1^\infty P\{\sup|\xi_m|>a\}\,da \leq \frac{1}{e}+E[|\xi_n|\log^+|\xi_n|]$$

and from (11)

$$E[\sup_{m\leq n}|\xi_m|]\leq 1+\frac{e}{e-1}[1+E[|\xi_n|\log^+|\xi_n|]\leq$$

$$\leq \frac{e}{e-1}[1+\sup_n E[|\xi_n|\log^+|\xi_n|]].$$

To see that $E[\sup_{m\leq n}|\xi_m|]<\infty$ is always true we can use the "truncating" method, and get the above inequality from where the theorem follows directy.

Martingales and semi martingales with random time.

Theorem 11. Let $\{\xi_n, \mathcal{F}_n\}$ be a non-negative supermartingale /that is $E(\xi_{n+1}/\mathcal{F}_n)\leq\xi_n$) τ and σ two stopping times/ according to $\{\mathcal{F}_n\}$).

Then ξ_τ and ξ_σ are integrable and on the set $\tau>\sigma$ the relation

$$\xi_\sigma \geq E(\xi_\tau/\mathcal{F}_\sigma)$$

holds with probability 1.

Proof. The limit $\xi_\infty=\lim_{n\to\infty}\xi_n$ exists and by Fatou's lemma

$$E\,\xi_\infty = E\,\underline{\lim}\,\xi_n \leq \underline{\lim}\,E\,\xi_n \leq E\,\xi_1<\infty.$$

Similarly, if $\mathbb{P}(\tau<\infty)=1$

$$E\xi_\tau = E\lim_{n\to\infty} \xi_{\tau\wedge n} \leqq \underline{\lim}_{n\to\infty} E\xi_{\tau\wedge n} \leqq E\xi_1 < \infty .$$

So we have for any

$$E\xi_\tau = E\chi_{\{\tau=\infty\}}\xi_\infty + E\chi_{\{\tau<\infty\}}\xi_\tau < \infty .$$

That is ξ_τ is integrable /and ξ_0 too/.

$\{\xi_{\tau\wedge n}, \mathcal{F}_n\}$ is a supermatingale, as

$$\xi_{\tau\wedge n} = \sum_{m<n} \xi_m \chi_{\{\tau=m\}} + \xi_n \chi_{\{\tau>n\}}$$

so $\xi_{\tau\wedge n}$ is $\mathcal{F}_n$ measurable, and

$$E\{\xi_{\tau\wedge n}/\mathcal{F}_{n-1}\} = \sum_{m<n} \xi_m \chi_{\{\tau=m\}} + E(\xi_n/\mathcal{F}_{n-1})\chi_{\{\tau\geqq n\}} \leqq$$

$$\leqq \sum_{m<n-1} \xi_m \chi_{\{\tau=m\}} + \xi_{n-1}\chi_{\{\tau=n-1\}} + \xi_{n-1}\chi_{\{\tau\geqq n\}} = \xi_{\tau\wedge(n-1)}$$

Let now $E|\Theta|<\infty$. Then the equality $E(\Theta/\mathcal{F}_\sigma)=E(\Theta/\mathcal{F}_n)$ will be satisfied on the set $\{\sigma=n\}$. To see this let us define $\eta(\omega)$ on the set $\{\sigma=n\}$ by

$$\eta(\omega) = E(\Theta/\mathcal{F}_n).$$

As

$$\{\omega:\eta(\omega)\leqq c\}\cap\{\sigma=n\} = \{\omega: E(\Theta/\mathcal{F}_n)\leqq c\}\cap\{\sigma=n\}\in\mathcal{F}_n$$

furthermore for any $A\in\mathcal{F}_\sigma$

$$\int_A \eta(\omega)\,dP = \sum_n \int_{A\cap\{\sigma=n\}} E(\Theta/\mathcal{F}_n)\,dP = \sum_n \int_{A\cap\{\sigma=n\}} \Theta\,dP =$$

$$= \int_A \Theta\,dP = \int_A E(\Theta/\mathcal{F}_\sigma)\,dP .$$

That is η is $\mathcal{F}_\sigma$ measurable.

Now on the set $\{\sigma=n\}$ we have $E\{\xi_\tau/\mathcal{F}_\sigma\}=E\{\xi_\tau/\mathcal{F}_n\}$ and it is enough to show that

$$\xi_n \geqq E\{\xi_\tau|\mathcal{F}_n\} \ . \tag{10.12}$$

As $\{\xi_{\tau\wedge n}, \mathcal{F}_n\}$ is a super-martingale, we get, using Fatou's lemma

$$\xi_{\tau\wedge n} \geqq E(\xi_{\tau\wedge\infty}|\mathcal{F}_n) = E(\xi_\tau|\mathcal{F}_n).$$

So we have (12) on the set $\{\tau \geq n\}$ and the theorem is proved.

<u>Theorem12</u>. Let $\xi_n = E(\eta|\mathcal{F}_n)$, where $E|\eta| < \infty$. Then for any two stopping times σ, τ with $P\{\tau<\infty\}=1$ we have $(\sigma \leqq \tau)$

$$\xi_\sigma = E(\xi_\tau|\mathcal{F}_\sigma).$$

<u>Proof</u>. We get, as above that $\xi_\tau = \xi_n$ on the set $\{\tau=n\}$ and

$$E(\eta|\mathcal{F}_\tau) = E(\eta|\mathcal{F}_n)$$

on the same set. That means

$$\xi_\tau = E(\eta|\mathcal{F}_\tau).$$

Furthermore $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ as we can easily see and

$$E(\xi_\tau|\mathcal{F}_\sigma) = E(E(\eta|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = E(\eta|\mathcal{F}_\sigma) = \xi_\sigma$$

<u>Theorem13</u>. Let $\{\xi_n, \mathcal{F}_n\}$ be a super-martingal, such that $\xi_n \geqq E(\eta|\mathcal{F}_n)$, where $E|\eta| < \infty$. Then for any τ, σ with $\tau \geqq \sigma$

$$\xi_\sigma \geq E(\xi_\tau / \mathcal{F}_\sigma).$$

Proof. Follows from the theorems 1 and 2 and from the identity

$$\xi_n = E(\eta/\mathcal{F}_n) + (\xi_n - E(\eta/\mathcal{F}_n))$$

using the fact that $\xi_n - E(\eta/\mathcal{F}_n)$ is a non negative super-martingale.

Applying Theorem 2 with $\sigma=1$ we get the very useful

(10.13) $\quad E\xi_\tau = E\xi_1$

relation. (13) is true under different sufficient conditions too.

Theorem 14. Let $\{\xi_n, \mathcal{F}_n\}$ be a martingale, τ stopping time with $P(\tau<\infty)=1$, $E|\xi_\tau|<\infty$ and $\lim\limits_{n\to\infty} \int\limits_{\{\tau>n\}} \xi_n \, dP = 0$ then

$$E\xi_\tau = E\xi_1$$

Proof. For any $n>0$ we can have the formulae

$$E\xi_\tau = \sum_{k=1}^{n} E(\xi_k/\tau=k)P(\tau=k) + E(\xi_\tau/\tau>n)P(\tau>n) =$$

$$= \sum_{k=1}^{n} E(E(\xi_n/\mathcal{F}_k)/\tau=k)P(\tau=k) + E(\xi_\tau/\tau>h)P(\tau>h) =$$

$$= \sum E(\xi_n/\tau=k)P(\tau=k) + E(\xi_\tau/\tau>h)P(\tau\geq n) =$$

$$= E(\xi_n/\tau\leq)P(\tau\leq n) + E(\xi_\tau/\tau>n)P(\tau>n),$$

$$E\xi_\tau = E(\xi_\tau|\tau\leq n)P(\tau\leq n)+E(\xi_\tau|\tau>n)P(\tau>n) =$$

$$= E\xi_n - E(\xi_n|\tau>n)P(\tau>n)+E(\xi_\tau|\tau>n)P(\tau>n).$$

As $E\xi_n = E\xi_1$ and the second and third terms in the right hand side of the above equation tend to 0 as $n\to\infty$, we have the statement of the theorem.

<u>Corollary</u>. If $E(\xi_n)^2 < K < \infty$, then $E\xi_\tau = E\xi_1$. Indeed $E(\xi_\tau)^2 \leq K < \infty$, and

$$\Big|\int_{\{\tau>n\}} \xi_n \, dP\Big| \leq \int_{\{\tau>n\}} |\xi_n| \, dP \leq \Big(\int_\Omega \xi_n^2 \, dP\Big)^{1/2} (P(\tau>n))^{1/2} \leq$$

$$\leq K^{1/2}(P(\tau>n))^{1/2} \to 0 .$$

<u>Example</u>. Let $\eta_1, \eta_2, \ldots$ be a sequence of i.i.d random variables with $P(\eta_i = 1) = P(\eta_i = -1) = 1/2$, and

$$\xi_n = \eta_1 + \ldots + \eta_n$$

$$\tau = \inf\{n : \xi_n = M, \text{ or } \xi_n = -N\}, \ (M, N \text{ naturals})$$

Let $p = P(\xi_\tau = M)$, $q = P(\xi_\tau = -N)$.
/Obviously $P(\xi_\tau < \infty) = 1$). Then $p + q = 1$,
$E|\xi_\tau| \leq \max(M, N) < \infty$. Moreover $|\xi_n| \leq \max(M, N)$ on the set $\{\tau > n\}$ and

$$\int_{\{\tau>n\}} |\xi_n| \, dP \leq \max(M,N)P(\tau>n) \to 0, \qquad (n\to\infty).$$

So according to Theorem14

$$0 = E\xi_1 = E\xi_\tau = pM + q(-N) = pM + (1-p)(-N) \Rightarrow p = \frac{N}{M+N}, \qquad q = \frac{M}{M+N} .$$

Based on Theorem14 we can prove the so called Wald's identity.

Theorem15. Let $\eta_1, \eta_2, \ldots$ be i.i.d random variables, $E|\eta_i| < \infty$, τ- stopping time for $\mathcal{F}_n = \sigma\{\eta_1, \ldots, \eta_n\}$, then with $E\tau < \infty$

$$E(\eta_1 + \ldots + \eta_\tau) = E\eta E\tau.$$

Proof. Let $\tau_N = \min(\tau, N)$, $N < \infty$

$$\xi_n = (\eta_1 + \ldots + \eta_n) - nE\eta_1$$

$\{\xi_n, \mathcal{F}_n\}$ is a martingale. According to Theorem14 /or Theorem12/

$$E\xi_{\tau_N} = E\xi_1 = 0.$$

Applying this result to $/\eta_i/$ we get

$$E\{|\eta_1| + \ldots + |\eta_{\tau_N}|\} = E\tau_N \cdot E|\eta_1| \leq E\tau E|\eta_1| < \infty$$

and $\tau_N \uparrow \tau$, $\sum_{i=1}^{N} |\eta_i| \uparrow \sum_{i=1}^{\tau} |\eta_i|$, so by Fatou's lemma whence

$$E|\xi_\tau| \leq E\tau E|\eta_1| + E\{|\eta_1| + \ldots + |\eta_\tau|\} \leq 2E\tau E|\eta_1| < \infty.$$

Now we show that $\lim_{n\to\infty} \int_{\{\tau > n\}} |\xi_n| \, dP = 0$.

Obviously

$$|\xi_n| \leq |\eta_1 - E\eta_1| + \ldots + |\eta_n - E\eta_n| \leq |\eta_1| + \ldots + |\eta_n| + nE\eta_1, \quad \{\tau > n\}$$

and on the set

$$|\xi_n| \leq |\eta_1| + \ldots + |\eta_\tau| + \tau E|\eta_1|.$$

So $\int_{\{\tau > n\}} |\xi_n| \, dP \leq \int_{\{\tau > n\}} \{|\eta_1| + \ldots + |\eta_\tau| + \tau E|\eta_1|\} \, dP \to 0$

as $\{E|\eta_1|+\ldots+|\eta_\tau|+\tau E|\eta_1|\}<\infty$ and $P\{\tau<\infty\}=1$. That is the conditions of Theorem14. are satisfied.

Exercise 1. Prove that if $E(\eta_1)^2<\infty$ and $E\tau<\infty$ then

$$D^2(\eta_1+\ldots+\eta_\tau)=D^2\eta E\tau$$

Exercise 2. Prove that in the example after Theorem 4 .

$$E\tau=M.N.$$

Chapter 11:

Some properties of the stochastic integrals as functions of the upper bound

Theorem 1. If $f(t,\omega)\in \mathfrak{M}$ and $\int_0^T Ef^2(t)\,dt<\infty$ then the continuous process $\xi(t)=\int_0^t f(s)\,dw(s)$ satisfies the following inequalities

$$P\{\sup_{0\leq t\leq T}\left|\int_0^t f(s)\,dw(s)\right|>a\}\leq \frac{1}{a^2}\int_0^T Ef^2(t)\,dt,$$

$$E\sup_{0\leq t\leq T}\left|\int_0^t f(s)\,dw(s)\right|^2\leq 4\int_0^T Ef^2(t)\,dt.$$

Proof of the theorem. Let us first suppose that $f(t)$ is a piecewise constant function from $\mathfrak{M}$. Let $\Lambda_n=\{t_{n_k}\}$ a sequence of decompositions of the intervall $[0,T]$ such that $\Lambda_n\subset\Lambda_{n+1}$. $\bigcup_n\Lambda_n$ is a set everywhere dense in $(0,T)$ and $f(t)$ is constant on $(t_{ni},\ t_{ni+1})$. Then

$$\xi=\sup_{0<t<T}\left|\int_0^t f(s)\,dw(s)\right|=\lim_{n\to\infty}\xi_n$$

with probability 1 where

$$\xi=\sup_{t_{n_k}}\left|\int_0^{t_{n_k}} f(t)\,dw(t)\right|$$

As the variables $\int_0^{t_{n_k}} f(t)\,dw(t)$ are measurable with respect to $\mathcal{F}_{t_{n_k}}$ we have

$$E\left(\int_0^{t_{nn}} f(t)\,dw(t)-\int_0^{t_{nj}} f(t)\,dw(t)\,\middle|\,\int_0^{t_{n1}} f(t)\,dw(t),\ldots,\int_0^{t_{nj}} f(t)\,dw(t)\right)=0$$

and the conditions of Lemma 1. are satisfied for the variables $\int_0^{t_{nj}} f(t)\,dw(t)$. Using Remark 1 and Corollary 1 we can write

$$\mathrm{P}\{\xi_n > a\} \leq \frac{1}{a^2}\int_0^T E f^2(t)\,dt,$$

$$E\xi_n^2 \leq 4\int_0^T E f^2(t)\,dt.$$

Taking the limits in these relations we get the proof of the theorem in the case of piecewise constant functions.

Let us now consider the general case i.e. when $f(t) \in \mathfrak{M}$ and $\int E f^2(t)\,dt < \infty$. Then we can choose a sequence of stepwise constant functions $f_n(t)$ so that

$$\lim_{n\to\infty} \int_0^T E(f(t) - f_n(t))^2\,dt = 0.$$

Let us choose $f_n(t)$ so, that

$$\int_0^T E(f(t) - f_n(t))^2 \leq \frac{1}{2^n}$$

be satisfied. Then

$$\int_0^T E(f_{n+1}(t) - f_n(t))^2\,dt \leq$$

$$\leq 2\int_0^T E(f_{n+1}(t) - f(t))^2\,dt + 2\int_0^T (f_n(t) - f(t))^2\,dt \leq \frac{3}{2^n}.$$

The function $f_{n+1}(t) - f_n(t)$ is piecewise constant so

$$\mathrm{P}\{\sup_{0\leq t\leq T} \left|\int_0^t f_{n+1}(s)\,dw(s) - \int_0^t f_n(s)\,dw(s)\right| \geq \frac{1}{n^2}\} \leq$$

$$\leq n^4\int_0^T E(f_{n+1}(t) - f_n(t))^2\,dt \leq \frac{3n^4}{2^n}$$

As the series $\sum \frac{3n^4}{2^n}$ is convergent we can use the lemma of Borel-Cantelli, and see that there exists a /random/ integer n_0, finite with probability one, so that if $n > n_0$ then

$$\sup_{0 \leq t \leq T} \left| \int f_{n+1}(s)\,dw(s) - \int f_n(s)\,dw(s) \right| \leq \frac{1}{n^2}$$

Hence the series

$$\int_0^t f_1(s)\,dw(s) + \sum_{n=1}^{\infty} \left(\int_0^t f_{n+1}(s)\,dw(s) - \int_0^t f_n(s)\,dw(s) \right)$$

converges uniformly with probability one. So their sum will be continuous with probability one.

We can complete the proof of the theorem in the same way as we did in the case of piecewise constant functions.

Theorem 2. If $f \in \mathfrak{M}$ the process $\int_0^t f(s)\,dw(s)$ continuous with probability 1 and

$$P\{\sup_{0 \leq t \leq T} \left| \int_0^t f(s)\,dw(s) \right| > c\} \leq P\{\int_0^T f^2(t)\,dt > N\} + \frac{N}{c^2}$$

Let ξ_1, ξ_2 be two stopping times with respect to the σ-algebras $\mathcal{F}_t\ (0 \leq t \leq T)$ such that

$$P(0 \leq \xi_1 \leq \xi_2 \leq T) = 1$$

and $\mathcal{F}_{\xi_1}$ the σ-algebra belonging to ξ_1 as defined in definition 2 § 1. $\mathcal{F}_{\xi_1}$ is the σ-algebra generated by the sets of the form $B \cap (\xi > t)$, $t > 0$, $B \in \mathcal{F}_t$. (see definition 2 p. 19)

Theorem 3. If $f(t) \in \mathfrak{M}\,(0, T)$ and $Ef^2(t)\,dt < \infty$ then

$$E\left(\int_{\xi_1}^{\xi_2} f(t)\,dw(t) / \mathcal{F}_{\xi_1}\right) = 0$$

$$E\left(\left[\int_{\xi_1}^{\xi_2} f(t)\,dw(t)\right]^2 / \mathcal{F}_{\xi_1}\right) = E\left(\int_{\xi_1}^{\xi_2} f^2(t)\,dt / \mathcal{F}_{\xi_1}\right)$$

Proof. Let η be any random variable measurable with respect to $\mathcal{F}_{\xi_1}$ and $\chi_i(t)$ $(i = 1,2)$ be two processes defined by $\chi_i(t) = 1$ if $\xi_i \leqq t$ and $\chi_i(t) = 0$ if $t < \xi_i$. Then $\chi_i(t)$ are measurable with respect to $\mathcal{F}_t$.
Furthermore, as we shall see the process $\eta(\chi_2(t) - \chi_1(t))$ is measurable with respect to $\mathcal{F}_t$ too. Indeed, let us first assume, that η is an indicator function of a set of the form $A \cap (\xi_1 \leqq s)$, where $A \in \mathcal{F}_{\xi_1}$. Then if $s \leqq t$ then both coefficients of the product are measurable with respect to $\mathcal{F}_t$ and if $s > t$ then

$$\chi_{A\cap(\xi_1 \leqq s)}(\chi_2(t) - \chi_1(t)) \leqq$$

$$\leqq \chi_{A\cap(\xi_1 \leqq t)}(\chi_2(t) - \chi_1(t)) \leqq$$

$$\leqq \chi_{A\cap(\xi_1 \leqq t)}(1 - \chi_1(t)) = \chi_A \cdot \chi_1(t)(1 - \chi_1(t)) = 0$$

Now we see that $\eta(\chi_2(t) - \chi_1(t))$ is measurable with respect to $\mathcal{F}_t$ for any indicator function of some $A \in \mathcal{F}_{\xi_1}$
As any variable η measurable with respect to $\mathcal{F}_{\xi_2}$ can be represented as an almost everywhere convergent limit of sums of the form $\sum c_k \chi_{A_k}$, $A_k \in \mathcal{F}_{\xi_1}$ we proved that $\eta(\chi_2(t) - \chi_1(t))$ has the desired measurability. Now

$$\int_{\xi_1}^{\xi_2} \eta f(t)\, dw(t) = \int_0^{\xi_2} \eta f(t)\, dw(t) - \int_0^{\xi_1} \eta f(t)\, dw(t) =$$

$$= \int_0^T \eta f(t)\, \chi_2(t)\, dw(t) - \int_0^T \eta f(t)\, \chi_1(t)\, dw(t) =$$

$$= \int_0^T \eta f(t)\, (\chi_2(t) - \chi_1(t))\, dw(t).$$

Obviously

$$\int_0^T E\, \eta^2 (\chi_2(t) - \chi_1(t))^2 f^2(t)\, dt \leqq 4 \int_0^T E\, \eta^2 f^2(t)\, dt < \infty.$$

Consequently

$$E\eta \int_{\xi_1}^{\xi_2} f(t)\, dw(t) = E\int_0^T \eta f(t)(\chi_2(t) - \chi_1(t))\, dw(t) = 0$$

$$E\Big[\eta \int_{\xi_1}^{\xi_2} f(t)\, dw(t)\Big]^2 = E\Big\{\int_0^T \eta f(t)(\chi_2(t) - \chi_1(t))\, dw(t)\Big\}^2 =$$

$$= \int_0^T E\, \eta^2 f^2(t)(\chi_2(t) - \chi_1(t))^2\, dt = E\int_0^T \eta^2 f^2(t)(\chi_2(t) - \chi_1(t))\, dt =$$

$$= E\eta \int_{\xi_1}^{\xi_2} f^2(t)\, dt$$

That means that the proof of Theorem 11 is completed.

<u>Remark</u> Taking the expectations in the two equations of the theorem we see that

$$E\Big(\int_{\xi_1}^{\xi_2} f(t)\, dw(t)\Big) = 0$$

$$E\Big(\int_{\xi_1}^{\xi_2} f(t)\, dw(t)\Big)^2 = E\int_{\xi_1}^{\xi_2} f^2(t)\, dt.$$

Theorem 4. Let $f(t) \geqq 0$, $t \geqq 0$, $f(t) \in \mathfrak{M}[0,T]$ for every $T > 0$ Let us assume that $P\{\int_0^\infty f^2(t)\,dt < \infty\} = 1$, and let $\tau_k = \text{if}\{t : \int_0^t f^2(s)\,ds \geqq k\}$. The process

$$\xi(t) = \int_0^{\tau_t} f(s)\,dw(s)$$

is a Brownian motion process.

<u>Proof</u> As we have seen τ_k is a stopping time and $\tau_{t_1} < \tau_{t_2}$ if $t_1 < t_2$. In the previous theorem we have proved that

$$E\left(\int_{\tau_{t_1}}^{\tau_{t_2}} f(s)\,dw(s) \,/\, \mathcal{F}_{\tau_{t_1}}\right) = 0$$

$$E\left(\left[\int_{\tau_{t_1}}^{\tau_{t_2}} f(s)\,dw(s)\right]^2 \,/\, \mathcal{F}_{\tau_{t_1}}\right) =$$

$$= E\left(\int_{\tau_1}^{\tau_2} f^2(s)\,ds \,/\, \mathcal{F}_{t_1}\right) = t_2 - t_1$$

So to use Levy's theorem we have to prove that the process $\xi(t) = \int_0^{\tau_t} f(s)\,dw(s)$ is continuous with probability 1. As τ_t is monotonic with probability 1, its only discontinuities are jumps. So the only discontinuities of ξ_t are jumps and they are placed at the jumps of the process τ_t. Let us suppose now for some t $\tau_{t-0} < \tau_{t+0}$, then as

$$E\int_{t-\varepsilon}^{t+\varepsilon} f^2(s)\,ds = 2\varepsilon \to 0 \quad \text{if} \quad \varepsilon \to 0,$$

using Chebyshev's inequality we get that

$$\int_{\tau_t-\varepsilon}^{\tau_t+\varepsilon} f(s)\,dw(s) \to 0$$

with probability 1.

Remark If the function $f(t)$ is determined only for $t \in (0, T]$ then we can apply Theorem 12 by putting $f(t) = 1$, $t \geqq T$. Then ξ_t will be a Brownian motion process with lifetime

$$\Theta - \int_0^T f^2(t)\, dt.$$

Chapter 12:

Solutions of stochastic differential equations

In the following we want to explain what do we mean by a solution of a stochastic differential equation. In the dis crete time case the solution of the equation

$$\xi_n = a(\xi_{n-1}, n) + b(\xi_{n-1}, n) \cdot \varepsilon_n ,$$

where ε_n is an independent $(E\xi_n = 0)$ sequence of random variables can be defined in the following way. At first from the definition we see that ξ_n is measurable with respect to the σ-algebra $A^n_{\varepsilon,0}$ generated by the random variables $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$. ξ_0 may be any random variable, $\xi_0 = \varepsilon_0$ and it is independent of $A^n_{\varepsilon,1}$. The recursion (11.1), with the functions a,b and sequence ε_n defines the new process ξ_n which is called the solution of the difference equation. The properties of the process ξ_n depend on the choice of the functions a and b. From (11.1)

$$E(\xi_n / A^{n-1}_{\varepsilon,0}) = a(\xi_{n-1}, n),$$

$$E[(\xi_n - a(\xi_{n-1}, n))^2 / A^{n-1}_{\varepsilon,0}] = E[b^2(\xi_{n-1}, n)\, \varepsilon_n^2 / A^{n-1}_{\varepsilon,0}] =$$

$$= b^2(\xi_{n-1}, n) \cdot \sigma^2_{\varepsilon_n}$$

where $\sigma^2_{\varepsilon_n} = E\,\varepsilon_n^2$. The distribution of ξ_n in many cases, under condition $\xi_0, \xi_1, \dots, \xi_{n-1}$, depends only on ξ_{n-1} which means that the process ξ_n is Markovian.

Now let us consider the continuous time case when we have the differential equation

$$(11.2) \qquad d\xi(t) = a(t,\xi(t))\,dt + b(t,\xi(t))\,dw(t),$$

where $(w(t), \mathcal{F}_t)$ is a Brownian motion process, $\mathcal{F}_t$ are σ-algebras $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ when $t_1 \leq t_2$ and $w(t+h) - w(t)$ is independent of $\mathcal{F}_t$ /for every t/. The functions $a(t,x)$ and $b(t,x)$ are measurable in (t,x).

We say that the process $\xi(t)$ is a solution of the equation (12.2) in the interval $0 \leq t \leq T$ if the following conditions are satisfied

a/ $\xi(0)$ is measurable with respect to $\mathcal{F}_0$, $\xi(0)$ is the initial value of the process;

b/ $\xi(t,\omega)$ is measurable in (t,ω) ;

c/ $\xi(t,\omega)$ is $\mathcal{F}_t$ -measurable for every $0<t \leq T$;

d/ The integrals $\int_0^T |a(t,\xi(t))|\,dt$, $\int_0^T b^2(t,\xi(t))\,dt$ exist and are finite with probability 1;

e/ With probability 1 the equation

$$\xi(t) - \xi(0) = \int_0^t a(s,\xi(s))\,ds + \int_0^t b(s,\xi(s))\,dw(s)$$

holds for every $0 \leq t \leq T$.

<u>Theorem 1</u>. Let the functions $a(t,x)$, $b(t,x)$ satisfy the following conditions

$$|a(s,x) - a(s,y)| + |b(s,x) - b(s,y)| \leq K|x-y|,$$

$$|a(s,x)|^2 + |b(s,x)|^2 \leq K(1+x^2),$$

for a fixed K and for every $0 \leqq s \leqq T$ and $-\infty < x, y < \infty$.
Further let us have for the initial value $\xi(0)$ measurable with respect to $\mathcal{F}_0$ and satisfying $E|\xi(0)|^2 < \infty$, then

1. There exists a solution $\xi(t)$, continuous with probability 1 with the initial value $\xi(0)$;
2. $\sup_t E|\xi(t)|^2 < \infty$;
3. If $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ are solutions with properties 1. and 2. then

$$\mathcal{P}\{\sup_{0\leqq t\leqq T} |\xi^{(1)}(t) - \xi^{(2)}(t)| = 0\} = 1.$$

__Proof.__ The uniqueness follows from the following. If $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ satisfy

$$(12.2') \qquad \xi(t) = \xi(0) + \int_0^t a(s,\xi(s))\,ds + \int_0^t b(s,\xi(s))\,dw(s),$$

then

$$E[\xi^{(1)}(t) - \xi^{(2)}(t)]^2 = E\{\int_0^t [a(s,\xi^{(1)}(s)) - a(s,\xi^{(2)}(s))]\,ds +$$

$$+ \int_0^t [b(s,\xi^{(1)}(s)) - b(s,\xi^{(2)}(s))]\,dw(s)\}^2 \leqq$$

$$\leqq 2E[\int_0^t (a(s,\xi^{(1)}(s)) - a(s,\xi^{(2)}(s)))\,ds]^2 + 2E[\int_0^t (b(s,\xi^{(1)}(s)) - b(s,\xi^{(2)}(s)))\,dw(s)]^2 \leqq$$

$$\leqq 2tE\int_0^t [a(s,\xi^{(1)}(s)) - a(s,\xi^{(2)}(s))]^2\,ds + 2E\int_0^t (b(s,\xi^{(1)}(s)) - b(s,\xi^{(2)}(s)))^2\,ds \leqq$$

$$\leqq 2Kt\int_0^t E(\xi^{(1)}(s) - \xi^{(2)}(s))^2\,ds + 2K^2\int_0^t E(\xi^{(1)}(s) - \xi^{(2)}(s))^2\,ds \leqq$$

$$\leqq L.\int_0^t E(\xi^{(1)}(s) - \xi^{(2)}(s))^2\,ds.$$

From this inequality it follows that the function $u(t) = E(\xi^{(1)}(t)-\xi^{(2)}(t))^2$ is 0, which means $P\{\xi^{(1)}(t)=\xi^{(2)}(t)\}=1$. The fact that $u(t)=0$ is a consequence of the following lemma.

Lemma 1. If $c_1>0$, $u(t)\geq 0$, $v(t)\geq 0$ then from

$$u(t)\leq c_1+\int_0^t u(s)v(s)\,ds$$

follows

$$u(t)\leq c_1 \exp\{\int v(s)\,ds\}.$$

Proof of the lemma. We have

$$\frac{u(t)\,v(t)}{c_1+\int_0^t u(s)v(s)\,ds}\leq v(t)$$

By integration

$$\ln[c_1+\int_0^t u(s)v(s)\,ds]-\ln c_1\leq \int_0^t v(s)\,ds,$$

or

$$u(t)\leq c_1+\int_0^t u(s)v(s)\,ds\leq c_1\exp\{\int_0^t v(s)\,ds\},$$

which gives the desired result. The case $c_1=0$ we may get from here by limiting $(c_1\downarrow 0)$.

As $P\{\xi^{(1)}(t)=\xi^{(2)}(t)\}=1$ and the processes $\xi^{(1)}(t)$, $\xi^{(2)}(t)$ are continuous we have

$$P\{\sup_t|\xi^{(1)}(t)-\xi^{(2)}(t)|=0\}=1$$

Indeed if R is the set of rationals

$$P\{\sup_{\substack{t\in R\\ 0\leq t\leq T}}|\xi^{(1)}(t)-\xi^{(2)}(t)|=0\}=1,$$

but R is dense in $[0,T]$ and from the continuity of the processes we have

$$\mathbb{P}\{ \sup_{\substack{t\in R \\ 0\leqq t\leqq T}} |\xi^{(1)}(t)-\xi^{(2)}(t)|=0\} = \mathbb{P}\{\sup_t |\xi^{(1)}(t)-\xi^{(2)}(t)|=0\}=1,$$

which proves the uniqueness.

To prove the existence we shall apply the usual iterational procedure.

Let the first approximation be,

$$\xi(t) \equiv \xi(0)$$

and

$$\xi^n(t) = \xi(0) + \int_0^t a(s, \xi^{n-1}(s))\, ds + \int_0^t b(s, \xi^{n-1}(s))\, ds.$$

With a similar argument as we did in the proof of uniqueness we get

$$E|\xi^{n+1}(t)-\xi^n(t)|^2 \leq L \int_0^t E|\xi^n(s)-\xi^{n-1}(s)|^2 ds.$$

As

$$E|\xi^1(t)-\xi^0(t)|^2 = E\{\int_0^t a(s,\xi(0))\, ds + \int_0^t b(s,\xi(0))\, dw(s)\}^2 \leq$$

$$\leq L.T.K.(1+E(\xi(0))^2).$$

From the last two inequalities for suitably choosen C we get

$$(12.3)\quad E|\xi^{n+1}(t)-\xi^n(t)|^2 \leq C\frac{(L.T.)^n}{n!} .$$

Further

$$\sup_{0\leq t\leqq T} |\xi^{n+1}(t)-\xi^n(t)| \leqq \sup_{0\leqq t\leqq T} \int_0^t |a(s,\xi^n(s))-a(s,\xi^{n-1}(s))|\, ds +$$

$$+ \sup_{0\leq t\leqq T} \int_0^t |b(s,\xi^n(s))-b(s,\xi^{n-1}(s))|\, dw(s).$$

Now we use the theorem 1. of the preceeding 1o. paragraph, namely

$$E \sup_{0 \leq t \leq T} \left| \int_0^t f(s,\omega)\, dw(s) \right|^2 \leq 4 \int_0^T E f^2(s,\omega)\, ds.$$

Using the Lipschitz condition for the functions a and b we get

$$E \sup_t |\xi^{n+1}(t) - \xi^n(t)|^2 \leq 2TK^2 \int_0^T E|\xi^n(t) - \xi^{n-1}(t)|^2\, dt +$$

$$+ 8K^2 \int_0^T E|\xi^n(t) - \xi^{n-1}(t)|^2\, dt = c_2 \int_0^T E|\xi^n(t) - \xi^{n-1}(t)|^2\, dt,$$

and from (12.3)

$$E \sup_t |\xi^{n+1}(t) - \xi^n(t)|^2 \leq c_2 \,.\, c \,.\, T \,\frac{(LT)^{n-1}}{(n-1)!}$$

Using the Chebisev inequality

$$\sum_{n=1}^{\infty} P \left(\sup_{0 \leq t \leq T} |\xi^{n+1}(t) - \xi^n(t)| > \frac{1}{n^2} \right) \leq \sum_{n=1}^{\infty} \frac{c_3 (LT)^{n-1}}{(n-1)!}\, n^4 < \infty$$

So the series

$$\xi(0) + \sum_{n=1}^{\infty} |\xi^{n+1}(t) - \xi^n(t)|$$

converges uniformly with probability 1. I.e $\xi^n(t)$ tends to a certain process, let us denote it by $\xi(t)$, which is continuous with probability 1.

Taking the limit in the equation

$$\xi^n(t) = \xi(0) + \int_0^t a(s, \xi^{n-1}(s))\, ds + \int_0^t b(s, \xi^{n-1}(s))\, dw(s)$$

we find that the process $\xi(t,\omega)$ is measurable in (t,ω)

and satisfies the relation (11.2')

Further for any fixed the random variables $\xi(t)$ are $\mathcal{F}_t$ measurable.

Finally

$$E(\xi^n(t))^2 \leq 3\{E(\xi(0))^2 + E[\int_0^t a(s,\xi^{n-1}(s))\,ds]^2 +$$

$$+ E[\int_0^t b(s,\xi^{n-1}(s))\,dw(s)]^2\} \leq 3E(\xi(0))^2 + 3L\int_0^t E(\xi^{n-1}(s))^2\,ds,$$

and by iteration

$$E(\xi^n(t))^2 \leq 3E(\xi(0))^2 + 3E(\xi(0))^2 \cdot 3Lt + (3L)^2\int_0^t (t-s)E(\xi^{n-2}(s))^2\,ds \leq$$

$$\leq 3E(\xi(0))^2 + 3Lt\,3E(\xi(0))^2 + 3E(\xi(0))^2\frac{(3Lt)^2}{2} + \ldots \leq 3E(\xi(0))^2 e^{LT}.$$

That means

$$\sup_{0\leq t\leq T} E(\xi(t))^2 \leq 3E(\xi(0))^2 e^{3LT},$$

that is the theorem is proved.

<u>Exercises</u>

1. Let $(\Omega, \mathcal{F}, P)$ a probability space and $\mathcal{F}_t \subseteq A$, $0 \leq t \leq T$ a family of nondecreasing σ-algebras and $(w(t), \mathcal{F}_t, P)$ a Brownian motion process.

   Let the functions $\varphi(t,\omega)$, $a(t,x,\omega)$, $b(t,x,\omega)$ have the properties

   1. they are measurable in t, x, ω
   2. for fixed t and x they are $\mathcal{F}_t$ measurable.

   We say that the process $\xi(t) = \{\xi(t,\omega),\ 0 \leq t \leq T\}$ is a solution of the equation

$$(*) \quad \xi(t) = \varphi(t,\omega) + \int_0^t a(s,\xi(s),\omega)\,ds + \int_0^t b(s,\xi(s),\omega)\,dw(s)$$

if the following conditions are satisfied:

a/ $\xi(t,\omega)$ is (t,ω) measurable;

b/ for any $0 \leqq t \leqq T$, $\xi(t,\omega)$ is $\mathcal{F}_t$ -measurable,

c/ the integrals in $(*)$ exist

d/ equation $(*)$ is satisfied for every t with probability 1.

Prove that if $\sup\limits_{0\leqq t\leqq T} E\varphi^2(t,\omega)<\infty$ and there exists a constant K, that the inequalities

$$|a(t,x,\omega)|^2 + |b(t,x,\omega)|^2 \leqq K^2(1+x^2),$$

$$|a(t,x,\omega) - a(t,y,\omega)| + |b(t,x,\omega) - b(t,y,\omega)| \leqq K|x-y|$$

hold with probability 1, then the equation $(*)$ has a solution, for which

$$\sup_{0\leqq t\leqq T} E\xi^2(t)<\infty$$

and if $\xi_1(t)$ and $\xi_2(t)$ are any two solutions then they are stochastically equivalent, i.e.

2. Prove that under the conditions of Theorem 1. the solution $\xi(t)$ of the stochastic differential equation (12.2) is a Markov process.

Hint: it is enough to prove that for any A_t measurable random variable $\alpha(\omega)$ and bounded continuous function $\beta(x)$ we have

$$E(\alpha\beta(\xi(s))/A_t) = E(\alpha E(\beta(\xi(s))|\xi(t))).$$

Using the unique solvability of the equation (11.2) and denoting its solution in the interval (t,s) by $\xi_{t,y}(u)$ we get that

$$\xi(s) = \xi_{t,\xi(t)}(s) \qquad \text{a.s.}$$

The function

$$B(\xi(t),\omega) = \beta(\xi_{t,\xi(t)}(s))$$

depends on only through the increments $w(u)-w(t)$ $(t \leqq u < s)$. Approximate $B(x,\omega)$ by functions of the form

$$\sum f_k(x)\, g_k(\omega))$$

3. Let the functions $a(t,x)$ and $b(t,x)$ be continuous with respect to the pair (t,x) and satisfy the conditions of Theorem 1. Then the process $\xi(t)$, which is the solution of the equation (11.2) is a diffusion process.
 Hint: use Exercise 4. to prove that $E|\xi_{s,x}(t)-x|^4 =$ $= o(t-s)$, then use this, Lipschitz condition and Hölder's inequality to prove that $E(\xi_{s,x}(t)-x) = a(s,x)(t-s) + o(t-s)$; similar estimations lead to $E[\xi_{s,x}(t)-x]^2 = b^2(s,x)(t-s)+o(t-s)$.)

4. Prove that if for a diffusion process $\xi(t)$
 A/ the coefficient of transmission $a(s,x)$ is continuous with respect to the pair (s,x), and

$$/a(s,x)/ \leqq K(1+(x))$$

for some $K > 0$,

B/ There exists a function $\varphi(x)$ independent of s and $\Delta > 0$, such that $\varphi(x) > 1 + |x|$, $\sup\limits_{0 \leq s \leq T} E\varphi(\xi_s) < \infty$ and

$$|E(\xi_{s+\Delta} - \xi_s \mid \xi_s = x)| + E((\xi_{s+\Delta} - \xi_s)^2 \mid \xi_s = x) \leq \varphi(x)\Delta$$

$$E\{|\xi_{s+\Delta}| + |\xi_{s+\Delta}|^2 \mid \xi_s = x\} \leq \varphi(x)$$

Then there exists a Brownian motion $w(s)$, measurable with respect to $A^s_{-\infty}$ such that ξ_s satisfies the stochastic differential equation

$$d\xi_s = a(\xi_s, s)\, ds + dw(s)$$

(Hint: prove first the relations(with $\eta_s = \xi_s - \int_0^s a(\xi_u, u)\, du$)

$$|E(\eta_{s+\Delta} - \eta_s \mid A^s_{-\infty})| \leq K_1 \varphi(x_s)\Delta$$

$$E((\eta_{s+\Delta} - \eta_s)^2 \mid A^s_{-\infty}) = K_2 \varphi(x_s)\Delta$$

$$\lim_{\Delta \to 0} \frac{1}{\Delta} E(\eta_{s+\Delta} - \eta_s \mid A^s_{-\infty}) = 0$$

$$\lim_{\Delta \to 0} \frac{1}{\Delta} E(\eta_{s+\Delta} - \eta_s)^2 \mid A^s_{-\infty}) = 0$$

then, using Levy's theorem prove that η_s is a Brownian motion, considering the expectation

$$E(\eta_t - \eta_s \mid A^s_{-\infty}) = E(\Sigma E(\eta_{(k+1)\Delta} - \eta_{k\Delta} \mid A^{k\Delta}_{-\infty}) \mid A^s_{-\infty})$$

and using Lebesque's theorem about majorated convergence.

Chapter 13:

Stochastic integrals and differential equations in multidimensional case.

Let $\{ w_k(t), \mathcal{F}_t,\ k = 1, 2, \ldots, m \}$ independent Brownian motion processes with the same family of non-decreasing σ-algebras $\mathcal{F}_t$. Let $\underline{f}^*(t) = (f_1(t)\omega), \ldots, f_n(t, \omega))$ a vector process, where $f_i(t)$ are $\mathcal{F}_t$ measurable for every i and any fixed t. We denote $\underline{f} \in L^2_{[0,T] \times \Omega}$ if $|f|^2 = f_1^2 + f_2^2 + \ldots + f_n^2$ is integrable on $[0, T] \times \Omega$ with respect $dt\ dP$. The class of vector functions with the above two properties will be denoted by $\mathfrak{M}^2$.

The stochastic vector integral with n components $\int_0^T \underline{f}(t,\omega)\, dw_i(t)$ is determined as the vector $\{ \int_0^T f_1(t,\omega)\, dw_i(t), \int_0^T f_2(t,\omega)\, dw_i(t), \ldots, \int_0^T f_n(t,\omega)\, dw_i(t) \}$. All properties of the stochastic integrals a/- e/ in one dimensional case remain true. We must substitute $|f| = \sqrt{f_1^2 + \ldots + f_n^2}$.

Now let us take m stochastic vector processes $\underline{f}^1(t, \omega), \ldots, \underline{f}^m(t, \omega) \in \mathfrak{M}^2$, then for every k the integral $\int_0^T \underline{f}^k(t,\omega)\, dw_k(t)$ is defined. Let $\underline{a}(t, \omega)$ be an n-dimensional vector function, of the real variable t.

As in the one dimensional case we may define the stochastic differential

$$(12.1) \qquad d\underline{\xi}(t) = \underline{a}(t)\, dt + \sum_{k=1}^{m} \underline{f}^k(t)\, dw_k(t)$$

if

$$\underline{\xi}(t)-\underline{\xi}(t_0)=\int_{t_0}^{t}\underline{a}(t)\,dt+\sum_{k=1}^{m}\int_{t_0}^{t}\underline{f}^k(t)\,dw_k(t).$$

Let us have a vector function $\underline{u}(t,\underline{x})$ in $R_1(n)$ where $\underline{x}\in R^{(n)}$. We suppose that all the functions $\underline{u}(t,\underline{x}),\ \frac{\partial \underline{u}}{\partial t},\ \frac{\partial}{\partial x_i}\underline{u},\ \frac{\partial^2}{\partial x_i\,\partial x_j}\underline{u}(t,\underline{x})\ (i,j=1,2,\dots,n)$ are continuous. If the stochastic process $\xi^*(t)=(\xi_1(t),\dots,\xi_n(t))$ has a differential (12.1) then the process $\underline{\eta}(t)=\underline{u}(t,\underline{\xi}(t))$ has a differential too and

$$d\underline{\eta}(t)=\Big[\frac{\partial \underline{u}}{\partial t}+\sum_i a_i(t)\frac{\partial}{\partial x_i}\underline{u}+\frac{1}{2}\sum_{i,j,l}\frac{\partial^2}{\partial x_i\partial x_j}\underline{u}\cdot f_i^l f_j^l\Big]dt+$$

$$+\sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\frac{\partial \underline{u}}{\partial x_j}f_j^i\,dw_i\Big).$$

(Ito's formula in multidimensional case).

Let $\underline{a}(t,\underline{x})$ and $\underline{b}_k(t,\underline{x})$ $(k=1,2,\dots,m)$ be vector valued measurable functions with values in $R^{(n)}$, $x\in R^{(n)}$. We shall examine the solution of the stochastic differential equation

$$(12.2)\qquad d\underline{\xi}(t)=\underline{a}(t,\underline{\xi}(t))\,dt+B(t,\xi(t))\,d\underline{w}(t)=$$

$$=\underline{a}(t,\underline{\xi}(t))\,dt+\sum_{k=1}^{m}b_k(t,\xi(t))\,dw_k(t)$$

or in the equivalent form

$$(12.2')\quad \underline{\xi}(t)-\underline{\xi}(t_0)=\int_{t_0}^{t}a(s,\underline{\xi}(s))\,ds+\sum_{k=1}^{m}\int_{t_0}^{t}\underline{b}_k(s,\underline{\xi}(s))\,dw_k(s)$$

$\underline{\xi}(t_0)$ does not depend on $\underline{w}(t) - \underline{w}(t_0)$ for every $t > t_0$
We say $\underline{\xi}(t)$ is the solution of above differential equation if the integrals in (12.2) exist and the equation (12.2) is satisfied with probability 1 for every $t\,(0 \leqq t \leqq T)$.

The existence and the uniqueness of the solution of equation (12.2) can be proved under similar conditions and on the same way as in the one dimensional case.

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