Optimal packages: binding regular polyhedra

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ABSTRACT: Strings on the surface of gift boxes can be modelled as a special kind of cable-and-joint structure. This paper deals with systems composed of idealised (frictionless) closed loops of strings that provide stable binding to the underlying convex polyhedron ('package'). Optima are searched in both the sense of topology and geometry in finding minimal number of closed loops as well as the minimal (total) length of cables to ensure such a stable binding for simple cases of polyhedra.

1 INTRODUCTION

1.1 *Loops on a polyhedral surface*

There are some practical occurrences of closed strings on polyhedral surfaces. For example, mailed packages are traditionally bound by some pieces of string. Another example is from basketry, when a woven pattern over a polyhedral surface is also formed by some (mainly closed) strings (Tarnai, Kovács, Fowler, & Guest 2012, Kovács 2014). In both cases, strings are driven along geodesics of the underlying surface (Fuchs & Fuchs 2007) in order to ensure that fastening of the network should not modify the geometry of strings.

Figure 1 shows two typical bindings of gift boxes: the picture to the left with horizontal and vertical pieces is more common but the skew string to the right is also applied (mainly in Japan). Although both solutions are well enough for secure wrapping, an idealisation assuming no knotted strings and no friction makes possible to free the package from both bindings without any damage or even extension of strings, just by sliding them along the surface of the package: these loops cannot provide stable equilibrium by prestressing. It is therefore a natural question: which are the suitable topological and geometrical conditions for a set of such loops to keep the package tightened in the above sense?

For further purposes, let us introduce the term *simple loop* for a closed string that (i) is interlaced with each its neighbour exactly once and (ii) has no self-intersection; otherwise the loop is termed *complex*. In another aspect, a loop will be referred to as *convex* if it has no change of direction in different senses (left-right) when followed along the underlying surface, and *concave* otherwise (for example, 'zigzag' loops with alternating turns are concave). In these terms, the loop on the left hand side of Fig. 1 is complex due to the self-intersection at the top and concave because of the two rectangular turns with different handedness at the bottom (such connections will be called *self-binding* from now on); whereas the skew string to the right is simple and convex (by having no turns within the polyhedral surface at all).

1.2 *Mechanical model*

Cable-and-joint networks are most commonly modelled as trusses, i.e., with geometric constraint functions on the distances of connected nodes. There are well-known methods to detect inextensional mechanisms as well as states of self-stress as some left and right singular vectors of the compatibility matrix of the assembly, see e.g. Pellegrino (1993). Even in the presence of mechanisms and a state of self-stress, it may occur that this latter one results in an increasing potential energy when mobilising the structure in any possible directions of mechanisms. It means that none of such mechanisms can be higher than second-order infinitesimal, and pre-stressing yields a stable equilibrium of the assembly (Connelly & Whiteley 1996).

For frictionless loops on polyhedral surfaces it is necessary to insert extra nodes into the described model at edge crossings and turn points of loops (i.e., where a loop changes its direction either together with or within the polyhedral surface). At the same time, geometric constraint should be written for the full length of the loop instead of all straight segments ('bar-length constraints'), which means that all rows in the compatibility matrix **C** pertaining to segments of a loop should be summed up instead in a single

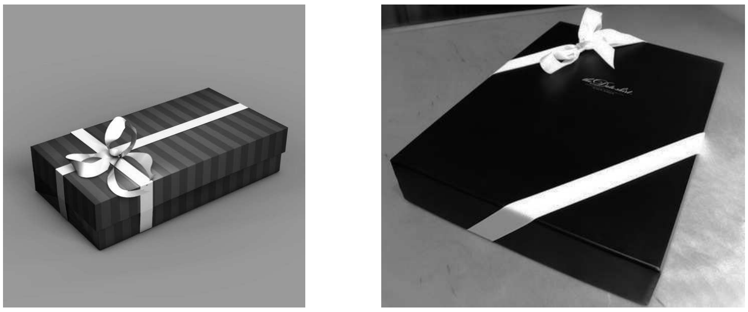

Figure 1: Typical wrappings (bindings) of packages with vertical-horizontal strings (left) and with a skew string (right).

row. Likewise, when testing the structure for second-order rigidity, second derivatives of such a summarized loop-length constraint should be calculated as the sum of second derivatives of each bar-length constraint F_i times the member force S_i involved (it is quite straightforward to see that member forces in each segment of a loop are the same without friction). Such derivatives form the Hessian matrix $\mathbf{H}$ of the expression $\sum S_i F_i$. If the column vector space of infinitesimal mechanisms is denoted by $\mathbf{v}$, then sign definiteness of $\mathbf{W} = \mathbf{v}^{\mathrm{T}}\mathbf{H}\mathbf{v}$ is a sufficient condition for the existence of second-order stiffness (imparted by pre-stressing) against any mechanism (Tarnai & Szabó 2002).

If similar questions arise on strings on a spherical rather than polyhedral surface, note that an analogous method for spherical trusses has been developed in Kovács & Tarnai (2009). For nonsmooth (polyhedral) surfaces we adopt two assumptions: (a) no strings can pass any vertex of the polyhedron, (b) no connections of strings can appear on any edge of the polyhedron (in this sense, 'connection' can mean either a simple under/overcrossing of strings or a point when two pieces of string change their directions by twisting upon each other; this latter kind of point will be termed *binding point*). Before presenting numeric tests for loops, some statements will be proved in the following chapter.

2 GENERAL CONVEX POLYHEDRA WITH A SINGLE LOOP

Consider a rectangular block bound by a simple loop as shown in Fig. 2 in dashed lines. If such a binding is in equilibrium along the surface of the block, any angle between the string and a crossed edge must be the same before and after that crossing; therefore any simple loop on any convex surface can obviously be developed into a straight line between two parallel copies of an edge. Let us now translate that development along the respective edge until the loop reaches a vertex (the sum of angles of polygons or simply vertex angles at any vertex must be less than 2π due to convexity): any further translation resul ts in a definitive loosening of the loop (shown in solid lines). From this observation we can formulate

Theorem 1. *Simple closed loops can provide no stable binding on any convex polyhedron.*

This theorem can easily be extended to any single loop with arbitrary number of self-bindings with the following argument. Consider a point of self-binding as a joint of our generalized truss: because of the constant force in all four segments adjacent to our point, those four forces must appear with pairwise coincident lines of action. This makes possible to change connectivity from angulated to straight without modifying the equilibrium of such a joint, yielding thus

Theorem 2. *A complex closed loop can provide no stable binding on any convex polyhedron.*

3 RECTANGULAR BLOCKS WITH MULTIPLE LOOPS

If at least two loops are used in a binding, they must meet at some binding points. It results then in several changes in directions of strings, making impossible to follow a single development of the polyhedron as seen in Fig. 2: geometry of loops in equilibrium turns to be highly sensitive to the geometry and topology of the underlying polyhedron. Considering only rectangular blocks (with edges a, b, c) bound by at least two loops from now on, let us depart from the traditional (rectangular) but obviously neutral binding pattern shown in Fig. 3a. The first idea can be to bind two simple loops together following the sample of Fig. 3b, but it is easy to prove that such a pair of loops is still insufficient: consider a synchronised finite translation of the binding point along the bisector of right angles marked by the arrow. If one departs from three mutually perpendicular simple loops then all simple crossings are replaced by binding points, configurations of Fig. 3c-d can also be obtained (the term *completely bound* is used to show that no simple crossings remained).

Numeric tests filtered 21 independent infinitesimal mechanisms for the case c) with $\mathbf{W}$ being sign-definite. It is easy to proof even by inspection that no piece of strings can be transformed into an oblique position since it would result in a total length of strings larger than $4(a+b+c)$. Consequently, any given displacement of a binding point determines uniquely all other four binding points on the adjacent two loops

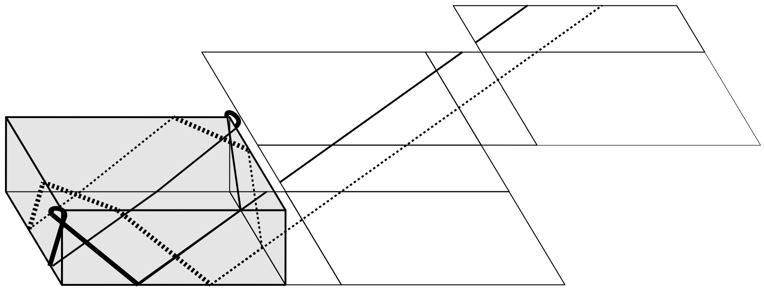

Figure 2: Binding a rectangular block: simple loop corresponding to a simple (non-intersecting) geodesic and its development is shown in dashed lines. The development in solid line is obtained by translation, notice its missing segments and corresponding loose parts of the string on the block.

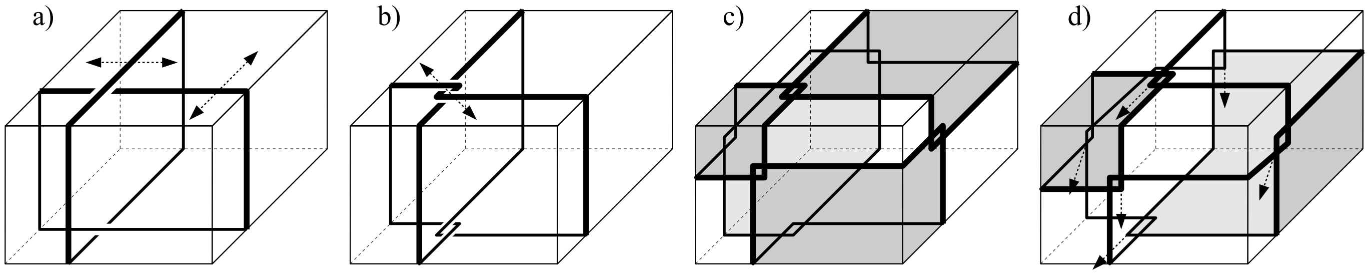

Figure 3: Different bindings on a rectangular box: unbound, neutral double loop (a), doubly bound, neutral double loop (b), completely bound, stable quadruple loop (c), completely bound triple loop with a zigzag (d); convex loops in cases c-d are highlighted; six small arrows mean concerted displacements with a single degree of freedom.

etc. that leads to contradictory displacement vectors just in the next step.

It is quite straightforward to see again that if there are only three loops in such a rectangular arrangement, they either correspond to a set of three mutually bound loops (as if two bound ones in the case (c) were transformed into a single one by self-binding) or two loops are disjoint and the topology of case (d) is obtained. Even without a computational proof, one can see here that finite displacements of binding points according to the figure (oblique motions occur along bisectors) results in no change either of joint equilibria or loop lengths: it is a neutral configuration. Interestingly, however, if the rest-length of two convex loops decreases but the zigzag loop is elongated accordingly, a stable equilibrium with oblique segments can be found by any iterative form-finding procedure (such geometries, except those having D_{3d} metric symmetry on a cube, are not obvious to get analytically).

In summary, for configurations where each face of the box contains exactly one binding point and each edge of the box is crossed by exactly one string, we can state

Theorem 3. *The minimal total length of strings that can be achieved by four loops is* $4(a+b+c)$*, and the same optimal value can be arbitrarily approximated (but never achieved) by three loops with one being a zigzag loop in any stable configuration.*

4 SIMPLE SOLIDS WITH TWO LOOPS

As far as Section 3 dealt with bindings derived from some simple closed geodesics, now a different concept will be introduced. Consider a point at random on the polyhedral surface and start drawing a geodesic from there at random again. The geodesic (typically *not* a simple closed one) will sooner or later intersect itself, let us denote that point of intersection by A. Consider now the segment AA as a closed loop with a turn at A: the internal angle $0 < \gamma < \pi$ measured here is determined by the total angular defect of vertices inside the loop according to the Gauss-Bonnet formula for polyhedral surfaces:

$$\sum D_i + \sum (\pi - \gamma_j) = 2\chi\pi, \qquad (1)$$

where D_i stands for the angular defect (2π minus the sum of vertex angles) at the ith vertex; χ is the Euler characteristic of the surface ($\chi = 1$ for any polyhedral domain with a single continuous boundary); and γ_j measures the (oriented) angular deviation at the jth turn within the surface.

4.1 *Rectangular blocks*

A rectangular block has $D_i = \pi/2$ at any vertex, so the turn at A can only be of $\gamma = \pi/2$, as well as the number of vertices inside the loop is exactly three. Such a loop with only one turn cannot be in equilibrium in itself but may be balanced by another one if bisectors of both turn angles γ coincide. For the existence of a two-loop system in equilibrium for rectangular blocks it is therefore a necessary and sufficient condition that string segments should follow a closed geodesic with *at least* one self-crossing (where the loops are then split). Fig. 4 shows a 3D image and the development of such a loop with point A located at a face centre, which results in a C_2 symmetry for

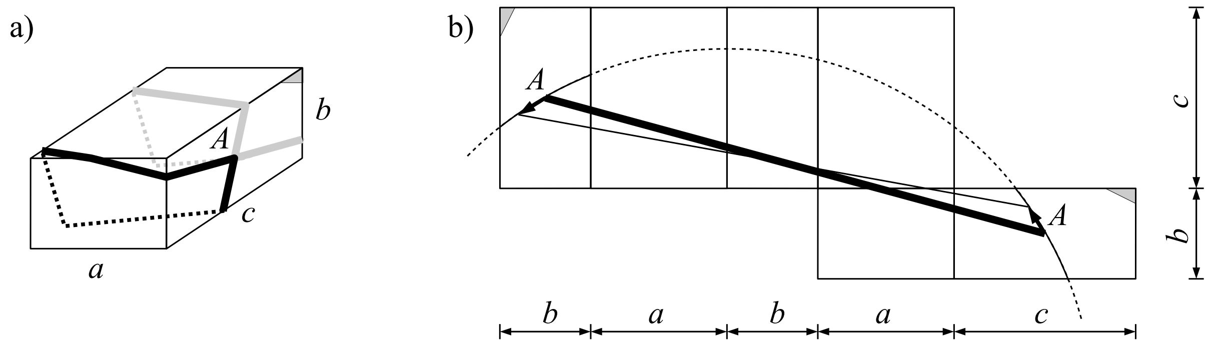

Figure 4: Stable binding of a rectangular block by two simple convex loops. Each loop encompasses three vertices and has therefore a rectangular turn at point A (a); development of the block along the string: any finite displacement of the chord AA along the arc with a central angle $\pi/2$ preserves compatibility (b). Small grey markers help identification of faces.

the entire (two-loop) configuration (symmetry is used here to ensure that the other loop also runs along a closed geodesic). It is left only now to check whether or not a finite displacement exists for such a pair of loops: it is relatively easy since the displacement of a (straight) development of the loop is sufficient to look at. Keeping in mind that two identical faces marked at their corner must be transformable into each other by a rotation by $\pi/2$, the centre C of rotation is located $a + b/2$ to the right and $a + b + c/2$ down from the leftmost face centre A. If (and only if) line AA is moved into the position $A'A'$ (with smaller thickness in the figure) along the circle centered at C, the closed shape of the loop with rectangular turn is preserved. Thus, a compatible path of point A of one loop is convex from the side of the loop itself, but the same holds for its neighbour; therefore no finite motion exists for such a pair of loops. It allows us to formulate

Theorem 4. *For some dimensions a, b, c there exists a pair of closed, simple and convex loops aligned to some self-intersecting geodesics such that they provide a stable binding for the rectangular block.*

Five comments should be made to this result:
(i) The total length obtained by double loops is indeed smaller than that found in Theorem 3, since

$$(2\overline{AA})^2 = (4a + 3b + c)^2 + (b + c)^2$$

$$\leq (4a + 3b + c + (b + c))^2 < (4a + 4b + 4c)^2. \quad (2)$$

(ii) This optimum does not belong to an unique shape: as closed geodesics can be translated (see the proof of Theorem 1), the binding point can be moved from the symmetric position (the symmetry itself can be lost) but finite motions still cannot exist for the two-loop assembly for reasons quite similar to those given at Theorem 4.
(iii) The net and geodesic segment AA shown in Fig. 4b do not always exist. Fig. 5 shows 6 possibilities for individual (simple) loops with three internal vertices, from which case (a) corresponds to Fig. 4. One can check by these schematic 'geodesics' that such loops on a cube can only be of type (c), since no other lines are straight on a cubic net.
(iv) Just for the sake of completeness, let us remark that two-loop configurations for general a, b, c may appear aligned with generically asymmetric complex geodesics (i.e., for which no translation of binding point to either a face centre or edge midpoint exists) but this is left out of consideration here.
(v) A detailed parametric study proves that all six cases presented above are possible for values c not excessively large compared to a and b: long and slender packages can only be bound in this way by pairs of complex loops; making thus possible infinitely many developments of those loops in addition to two cases shown in Fig. 5a,d. This problem is not discussed here either.

4.2 *Some numeric optima*

This section compares some newly found optimal configurations in terms of total length of loops. The basis of comparison will be the minimum length $4(a + b + c)$ for four-loop solutions discussed in Section 3. Consider first the case (c) of Fig. 5 for testing the optimum of the cube. In general, the ratio r_c of optimal and rectangular binding can be written as

$$r_c = \frac{\sqrt{(4a + 3b + 2c)^2 + (b + 2c)^2}}{4(a + b + c)}. \quad (3)$$

Although it is always smaller than 1 because of the triangle inequality (even smaller for data where c is predominant, but c is limited by the net of block), a particular number can be given for the cube, i.e., when $a = b = c$:

$$r_{c,cube} = \frac{\sqrt{10}}{4}. \quad (4)$$

Another example we look at is of case (b). The reason for doing so is that there is a loop over less faces, hopefully, with a 'better' optimum. Check first the minimum value of c in function of a and b (the straight geodesic should pass above the left angle $3\pi/2$ of the net):

$$\frac{c}{2a + 2b} > \frac{2b}{2a + c} \rightarrow c > 2b. \quad (5)$$

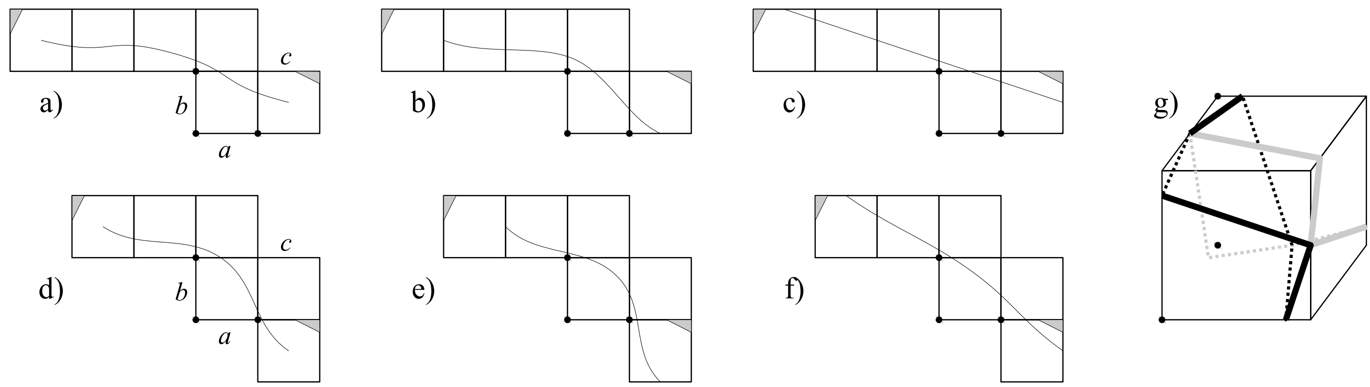

Figure 5: Possible arrangements of loops pertaining to a symmetric pair on rectangular blocks: three dots indicate vertices inside a loop. Cases (a)-(c) and (d)-(f) are different in the relative position of the binding point to the three vertices; binding points in cases(a) and (d) are face centres, otherwise edge midpoints. case (g) shows 3D realisation of (c) with cubic geometry; binding points can be moved off the edge by a small translation of the entire geodesic according to the initial assumptions.

Let us choose now for simplicity this lower limit of c and $a = b = c/2$; now the ratio r_b is

$$r_b = \frac{\sqrt{(4a+2b+c)^2 + (2b+c)^2}}{4(a+b+c)} = \frac{\sqrt{5}}{4}. \quad (6)$$

It is clearly visible therefore that a prism of aspect ratio 1:1:2 can be bound even more efficiently by two oblique strings than a cube.

4.3 *Tetrahedra*

Regular tetrahedra have no complex (non-simple) geodesics (Fuchs & Fuchs 2007), therefore no statement like of Theorem 4 holds for them. By intuition, a four-fold binding similar to that shown in Fig. 3c seems to be stable and indeed proves to be rigid to the second order, see Fig. 6a. We note that such configuration does not obey to the initial assumption that all bindings are lie off the edges. Interestingly, the lack of self-crossing geodesics excludes only two-loop configurations with a single binding (more precisely, if the angular defect is more than π at two or more vertices, two loops with a single binding could occur even with different angles γ at turns), it is still possible, however, to look for two-loop solutions with two or more bindings. Intuitively again, let us use symmetry and look at the configuration shown in Fig. 6b: both loops are triangular, convex and complex (due to triple connection). Numeric tests showed its second-order rigidity against any mobilisation, which is a proof for the following

Theorem 5. *There exists a stable binding for a regular tetrahedron by two convex and complex loops.*

Remarks: (i) If the perimeter of the central (grey) loop approaches zero, the total length of loops approaches four times the height of triangular faces. In the course of this transition, no angles are modified, therefore stability can be extended to all this family of two-loop configurations. Compared to the length of loops in the case (a), the following ratio is obtained:

$$r_{tetra} = \frac{4(a\sqrt{3}/2)}{12(a/2)} = \frac{\sqrt{3}}{3}. \quad (7)$$

(ii) The configuration (b) seems to be realizable as a stable binding for irregular tetrahedra or even other n-gonal pyramids but it is not proved here.

5 DISCUSSION

Following the common style of package binding with rectangular crossings on rectangular boxes, four- and three-loop stable configurations were found by binding strings upon each other; theoretical minimum of their total length was found at $4(a + b + c)$. In another type of binding, a minimum of two simple convex loops (aligned with self-crossing geodesics) were found for such blocks in the neighbourhood of six symmetric configurations. The possible development of loops and the net of block is sensibly dependent on block edge lengths, and the categorization into six configuration types is not proved to be complete. Without the aim of completeness again, numeric optima (with respect to the length of four rectangular loops) were calculated for some geometries, which showed considerable reduction in total length compared to the traditional binding patterns. In parallel, some regular tetrahedral bindings have also been investigated: an optimal solution both in number (two) and total length of loops is found. The results seem to be extendible to irregular solids as well but still no rigorous proof is found for that. It must be emphasized that, even conjectured, no rigorous proofs are given either for the statement that solutions described as optimal cannot be improved in some way; it is left for future investigations. Another open question is the optimum analysis of irregular solids topologically identical to the cube: there may not exist close geodesics at all, but pairs (or triplets) of loops in a stable configuration might still be found.

ACKNOWLEDGEMENTS

This research was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by the OTKA under grant no. K 81146.

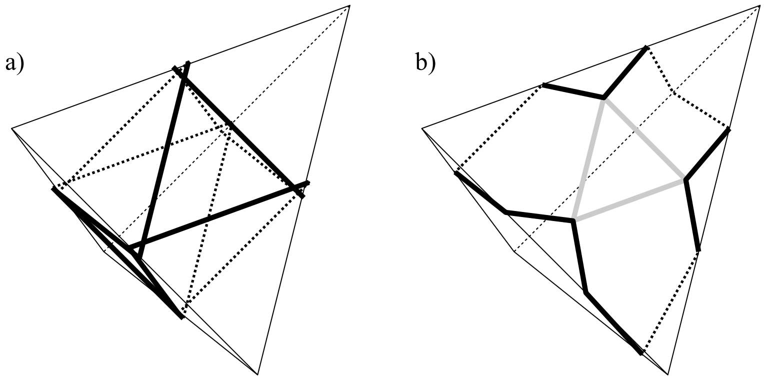

Figure 6: Binding regular tetrahedra: with four simple convex loops (a); with two convex loops and 3 binding points (b).

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