A characterization of multiple $(n-k)$-blocking sets in projective spaces of square order

S. Ferret L. Storme P. Sziklai* Zs. Weiner

Abstract

In [10], it was shown that small t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$, $q = p^h$, p prime, $h \geq 1$, intersect every k-dimensional space in $t \pmod{p}$ points. We characterize in this article all t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$, q square, $q \geq 661$, $t < c_p q^{1/6}/2$, $|B| < tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.

1 Introduction

Throughout this paper, $\mathrm{PG}(n,q)$ will denote the n-dimensional projective space over the Galois field $\mathrm{GF}(q)$, where $q = p^h$, p prime.

A *t-fold $(n-k)$-blocking set* B of $\mathrm{PG}(n,q)$, with $0 < k < n$, is a set of points of $\mathrm{PG}(n,q)$ intersecting every k-dimensional subspace of $\mathrm{PG}(n,q)$ in at least t points.

A 1-fold $(n-k)$-blocking set B of $\mathrm{PG}(n,q)$ containing an $\mathrm{PG}(n-k,q)$ is called *trivial*.

A point r of B is called *essential* if there is a k-dimensional subspace through r intersecting B in precisely t points. The t-fold blocking set B is called *minimal* if all of its points are essential. A 1-fold $(n-k)$-blocking set is also called an *$(n-k)$-blocking set*. A t-fold 1-blocking set in $\mathrm{PG}(2,q)$ is also called a *t-fold blocking set*, or a *t-fold planar blocking set*.

These latter t-fold planar blocking sets have been studied in great detail.

Theorem 1.1 (Blokhuis *et al.* [8]) *Let B be a t-fold blocking set in* $\mathrm{PG}(2,q)$*, $q = p^h$, p prime, of size $t(q+1)+c$. Let $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for*

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$p > 3$.
(1) *If* $q = p^{2d+1}$ *and* $t < q/2 - c_p q^{2/3}/2$, *then* $c \geq c_p q^{2/3}$, *unless* $t = 1$ *in which case* B, *with* $|B| < q + 1 + c_p q^{2/3}$, *contains a line.*
(2) *If* $4 < q$ *is a square,* $t < c_p q^{1/6}$ *and* $c < c_p q^{2/3}$, *then* $c \geq t\sqrt{q}$ *and* B *contains the union of* t *pairwise disjoint Baer subplanes, except for* $t = 1$ *in which case* B *contains a line or a Baer subplane.*
(3) *If* $q = p^2$, p *prime, and* $t < q^{1/4}/2$ *and* $c < p\lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \rceil$, *then* $c \geq t\sqrt{q}$ *and* B *contains the union of* t *pairwise disjoint Baer subplanes, except for* $t = 1$ *in which case* B *contains a line or a Baer subplane.*

Theorem 1.2 (Ball [1]) *A* t*-fold blocking set in* $\mathrm{PG}(2, q)$ *which does not contain a line has at least* $tq + \sqrt{tq} + 1$ *points.*

If B *is a* t*-fold blocking set in* $\mathrm{PG}(2, p)$, *where* $p > 3$ *is prime, and if* $1 < t < p/2$, *then* $|B| \geq (t+1/2)(p+1)$, *while if* $t > p/2$, *then* $|B| \geq (t+1)p$.

In the theory of 1-fold planar blocking sets, $1 \pmod{p}$ results for *small* 1-fold planar blocking sets play an important role.

Definition 1.3 *A blocking set of* $\mathrm{PG}(2, q)$ *is called* small *when it has less than* $3(q + 1)/2$ *points.*

If $q = p^h$, p *prime,* $h \geq 1$, *the* exponent e *of the minimal blocking set* B *of* $\mathrm{PG}(2, q)$ *is the maximal integer* e *such that every line intersects* B *in* $1 \pmod{p^e}$ *points.*

Theorem 1.4 *Let* B *be a small minimal* 1*-fold blocking set in* $\mathrm{PG}(2, q)$, $q = p^h$, p *prime,* $h \geq 1$. *Then* B *intersects every line in* $1 \pmod{p}$ *points, so for the exponent* e *of* B, *we have* $1 \leq e \leq h$. (Szőnyi [18])

In fact, this exponent e *is a divisor of* h. (Sziklai [17])

This result was extended by Szőnyi and Weiner [19] to 1-fold $(n - k)$-blocking sets in $\mathrm{PG}(n, q)$.

Definition 1.5 *A* 1*-fold* $(n-k)$*-blocking set of* $\mathrm{PG}(n, q)$ *is called* small *when it has less than* $3(q^{n-k} + 1)/2$ *points.*

If $q = p^h$, p *prime,* $h \geq 1$, *the* exponent e *of the minimal* 1*-fold* $(n - k)$*-blocking set* B *is the maximal integer* e *such that every hyperplane intersects* B *in* $1 \pmod{p^e}$ *points.*

A most interesting question of the theory of blocking sets is to classify the small blocking sets. A natural construction (blocking the k-subspaces of $\mathrm{PG}(n, q)$) is a subgeometry $\mathrm{PG}(h(n - k)/e, p^e)$, if it exists (recall $q = p^h$, so $1 \leq e \leq h$ and $e|h$).

It is easy to see that the projection of a blocking set, w.r.t. k-dimensional subspaces, from a vertex V onto an r-dimensional subspace of $\mathrm{PG}(n,q)$, is again a blocking set, w.r.t. the $(k+r-n)$-dimensional subspaces of $\mathrm{PG}(r,q)$ (where $\dim(V)=n-r-1$ and V is disjoint from the blocking set).

A blocking set of $\mathrm{PG}(r,q)$, which is a projection of a subgeometry of $\mathrm{PG}(n,q)$, is called *linear*. (Note that the trivial blocking sets are linear as well.) Linear blocking sets were defined by Lunardon, and they were first studied by Lunardon, Polito and Polverino [12], [13].

Conjecture 1.6 (Linearity Conjecture [17]) *In* $\mathrm{PG}(n,q)$*, every small minimal blocking set, with respect to k-dimensional subspaces, is linear.*

There are some cases of the Conjecture that are proved already.

Theorem 1.7 *For $q=p^h$, p prime, $h\geq 1$, every small minimal non-trivial blocking set w.r.t. k-dimensional subspaces is linear, if*

(a) $n=2$, $k=1$ *(so we are in the plane) and:*

- *(i)* (Blokhuis [5]) $h=1$ *(i.e. there is no small non-trivial blocking set at all);*
- *(ii)* (Szőnyi [18]) $h=2$ *(the only non-trivial example is a Baer subplane with p^2+p+1 points);*
- *(iii)* (Polverino [14]) $h=3$ *(there are two examples, one with p^3+p^2+1 and another with p^3+p^2+p+1 points);*
- *(iv)* (Blokhuis, Ball, Brouwer, Storme, Szőnyi [6], Ball [2]) *if $p>2$ and there exists a line ℓ intersecting B in $|B\cap\ell|=|B|-q$ points (so a blocking set of Rédei type);*

(b) for general k:

- *(i)* (Szőnyi and Weiner [19]) *if $h(n-k)\leq n$, $p>2$, and B is not contained in an $(h(n-k)-1)$-dimensional subspace;*
- *(ii)* (Storme and Weiner [16] (*for* $k=n-1$), Bokler [9] and Weiner [20]) $h=2$*,* $q\geq 16$*;*
- *(iii)* (Storme and Sziklai [15]) *if $p>2$ and there exists a hyperplane H intersecting B in $|B\cap H|=|B|-q^{n-k}$ points (so a blocking set of* Rédei type*).*

The following $\pmod p$ result is known.

Theorem 1.8 (Szőnyi and Weiner [19]) *A minimal* 1*-fold* $(n-k)$*-blocking set in* $\mathrm{PG}(n,q)$*,* $q=p^h$*,* $p>2$ *prime, of size less than* $\frac{3}{2}(q^{n-k}+1)$ *intersects every subspace in zero or in* $1 \pmod p$ *points.*

There is an even more general version of the Conjecture. A t-fold blocking set w.r.t. k-dimensional subspaces is a point set which intersects each k-dimensional subspace in at least t points. Multiple points may be allowed as well.

Conjecture 1.9 (Linearity Conjecture for multiple blocking sets [17]) *In* $\mathrm{PG}(n,q)$*, any* t*-fold minimal blocking set* B*, with respect to* k*-dimensional subspaces, is the union of some (not necessarily disjoint) linear point sets* $B_1,...,B_s$*, where* B_i *is a* t_i*-fold blocking set w.r.t.* k*-dimensional subspaces and* $t_1+\cdots+t_s=t$*; provided that* t *and* $|B|$ *are small enough (*$t \le T(n,q,k)$ *and* $|B| \le S(n,q,k)$ *for two suitable functions* T *and* S*).*

Again, some cases of this conjecture have been proved already; in this paper, we cover many new cases which provide "evidence" to the Linearity Conjecture for multiple blocking sets.

Note that there exists a $(\sqrt[4]{q}+1)$-fold blocking set in $\mathrm{PG}(2,q)$, constructed by Ball, Blokhuis and Lavrauw [3], which is *not* the union of smaller blocking sets. (This multiple blocking set is a linear point set.)

The $1 \pmod p$ result in $\mathrm{PG}(2,q)$, $q=p^h$, p prime, was extended by Blokhuis *et al.* to a $t \pmod p$ result on *small* minimal t-fold blocking sets in $\mathrm{PG}(2,q)$ [7].

Definition 1.10 *A* t*-fold blocking set of* $\mathrm{PG}(2,q)$ *is called* small *when it has less than* $(t+1/2)(q+1)$ *points.*

If $q=p^h$*,* p *prime, the* exponent e *of the minimal* t*-fold blocking set* B *in* $\mathrm{PG}(2,q)$ *is the maximal integer* e *such that every line intersects* B *in* $t \pmod{p^e}$ *points.*

Theorem 1.11 (Blokhuis *et al.* [7]) *Let* B *be a small minimal* t*-fold blocking set in* $\mathrm{PG}(2,q)$*,* $q=p^h$*,* p *prime,* $h \ge 1$*. Then* B *intersects every line in* $t \pmod p$ *points.*

Regarding characterization results on small minimal 1-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$, we mention the following results.

In the next theorem, θ_m denotes the size of an m-dimensional space $\mathrm{PG}(m,q)$.

Theorem 1.12 (Bokler [9]) *The minimal* $(n-k)$*-blocking sets of cardinality at most* $\theta_{n-k} + \theta_{n-k-1}\sqrt{q}$ *in projective spaces* $\mathrm{PG}(n,q)$ *of square order* q*,* $q \geq 16$*, are Baer cones with an* m*-dimensional vertex* $\mathrm{PG}(m,q)$ *and base a Baer subgeometry* $\mathrm{PG}(2(n-k-m-1),\sqrt{q})$*, for some* m *with* $\max\{-1, n-2k-1\} \leq m \leq n-k-1$*.*

In the following theorem, $s(q)$ denotes the size of the smallest blocking set in $\mathrm{PG}(2,q)$, q square, not containing a line or Baer subplane.

Theorem 1.13 (Storme and Weiner [16]) *Let* K *be a minimal* 1*-blocking set in* $\mathrm{PG}(n,q)$*,* q *square,* $q = p^h$*,* $h \geq 1$*,* $p > 3$ *prime,* $n \geq 3$*, with* $|K| \leq s(q)$*. Then* K *is a line or a minimal planar blocking set of* $\mathrm{PG}(n,q)$*.*

Theorem 1.14 (Storme and Weiner [16]) *In* $\mathrm{PG}(n,q^3)$*,* $q = p^h$*,* $h \geq 1$*,* p *prime,* $p \geq 7$*,* $n \geq 3$*, a minimal* 1*-blocking set* K *of cardinality at most* $q^3 + q^2 + q + 1$ *is either:*

(1) *a line;*

(2) *a Baer subplane when* q *is a square;*

(3) *a minimal blocking set of cardinality* q^3+q^2+1 *in a plane of* $\mathrm{PG}(n,q^3)$*;*

(4) *a minimal blocking set of cardinality* $q^3 + q^2 + q + 1$ *in a plane of* $\mathrm{PG}(n,q^3)$*;*

(5) *a subgeometry* $\mathrm{PG}(3,q)$ *in a* 3*-dimensional subspace of* $\mathrm{PG}(n,q^3)$*.*

The following result was the first characterization result to use the 1 (mod p) result of Theorem 1.8.

Theorem 1.15 (Weiner [20]) *Let* B *be a* 1*-fold* $(n-k)$*-blocking set in* $\mathrm{PG}(n, q = p^{2h})$*,* $p > 2$ *prime,* $q \geq 81$*, of size* $|B| < 3(q^{n-k}+1)/2$ *and intersecting every* k*-space in* 1 (mod $\sqrt{q}$) *points. Then* B *is a Baer cone with an* m*-dimensional vertex* $\mathrm{PG}(m,q)$ *and base a Baer subgeometry* $\mathrm{PG}(2(n-k-m-1),\sqrt{q})$*, for some* m *with* $\max\{-1, n-2k-1\} \leq m \leq n-k-1$*.*

Regarding characterizations of small minimal t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$, we mention the following result.

Theorem 1.16 (Barát and Storme [4]) *Let* B *be a* t*-fold* 1*-blocking set in* $\mathrm{PG}(n,q)$*,* $q = p^h$*,* p *prime,* $q \geq 661$*,* $n \geq 3$*, of size* $|B| < tq + c_p q^{2/3} - (t-1)(t-2)/2$*, with* $c_2 = c_3 = 2^{-1/3}$*,* $c_p = 1$ *when* $p > 3$*, and with* $t < \min(c_p q^{1/6}, q^{1/4}/2)$*. Then* B *contains a union of* t *pairwise disjoint lines and/or Baer subplanes.*

Recently, in [10], the following t (mod p) result on weighted t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$ has been obtained.

Theorem 1.17 (Ferret *et al.* [10]) *Let B be a minimal weighted t-fold $(n-k)$-blocking set of* $\mathrm{PG}(n,q)$*, $q=p^h$, p prime, $h \geq 1$, of size $|B| = tq^{n-k}+t+k'$, with $t+k' \leq (q^{n-k}-1)/2$.*

Then B intersects every k-dimensional space in $t \pmod p$ points.

We now use this $t \pmod p$ result to characterize multiple blocking sets. We present in this article characterization results on small t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$, q square, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.

2 Intervals for minimal t-fold $(n-k)$-blocking sets

The following interval theorems on the size of minimal t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$ will play a crucial role in our arguments.

Theorem 2.1 (Ferret *et al.* [10]) *Let B be a minimal t-fold $(n-k)$-blocking set in* $\mathrm{PG}(n,q)$*, $n \geq 2$, $|B| = tq^{n-k}+t+k'$, with $t+k' \leq (q^{n-k}-1)/2$. Assume that $q=p^h$, p prime, $h \geq 1$, and that B intersects every k-dimensional space in $t \pmod E$ points, with $E=p^e$, and with e the largest integer for which this is true.*

If $2t < E$, then

$$tq^{n-k} + \frac{q^{n-k}}{p^e+1} - 1 \leq |B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E}.$$

Theorem 2.2 (Ferret *et al.* [10]) *Let B be a t-fold $(n-k)$-blocking set in* $\mathrm{PG}(n,q)$*. Assume that $q=p^h$, p prime, $h \geq 1$, and that B intersects every k-dimensional space in $t \pmod E$ points, with $E=p^e$, and with e the largest integer for which this is true.*

If $\max\{2t,4\} < E$, then

$$|B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E} \quad \text{or} \quad |B| \geq Eq^{n-k}+t.$$

3 t-Fold 1-blocking sets

In Theorem 1.16, see also [4], Barát and Storme presented characterization results on t-fold 1-blocking sets in $\mathrm{PG}(n,q)$. These results were obtained before the $t \pmod p$ results (Theorems 1.11 and 1.17) were known.

Repeating their arguments, but now including the $t \pmod p$ results, leads to the following theorem.

Theorem 3.1 *Let B be a t-fold 1-blocking set in $\mathrm{PG}(n,q)$, $q=p^h$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq + c_p q^{2/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$. Then B contains a union of t pairwise disjoint lines and/or Baer subplanes.*

The following result, which relies on the preceding classification of t-fold 1-blocking sets, plays an important role in the proofs of the characterization results which will follow.

From now on, let B be a minimal t-fold $(n-k)$-blocking set in $\mathrm{PG}(n,q)$, $q=p^h$, p prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq^{n-k} + c_p q^{n-k-1/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.

Lemma 3.2 *Let B be a minimal t-fold $(n-k)$-blocking set in $PG(n,q)$, $k \geq 2$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.*

If Π is a $(k+1)$-dimensional space intersecting B in a non-minimal t-fold 1-blocking set, then

$$|\Pi \cap B| \geq q\sqrt{q} + t.$$

Proof: Since $\Pi \cap B$ intersects every k-dimensional space in Π in $t \pmod{\sqrt{q}}$ points, either $|\Pi \cap B| \leq tq + 2t\sqrt{q}$ or $|\Pi \cap B| \geq q\sqrt{q} + t$ (Theorem 2.2). Assume that $|\Pi \cap B| \leq tq + 2t\sqrt{q}$, then by Theorem 3.1, $\Pi \cap B$ contains a union of t pairwise disjoint lines and/or Baer subplanes. Let S_1 be the minimal part of $\Pi \cap B$, consisting of those t pairwise disjoint lines and/or Baer subplanes, and let S_2 be the remaining part of $\Pi \cap B$.

Let $r \in S_2$. Consider a line L of Π through r only intersecting B in r. We now prove that it is possible to find a $(k-1)$-dimensional space Π_{k-1} of Π through L only intersecting B in r. This is immediately true for $k=2$. Let $k \geq 3$, then there are $q^{n-2} + q^{n-3} + \cdots + q + 1$ planes through L. Since there are at most $tq^{n-k} + q^{n-k-1/3} < q^{n-2} + \cdots + q + 1$ points in B, it is possible to find a plane Π_2 through L only intersecting B in r. Repeating this argument, a 3-dimensional space Π_3 through Π_2 only intersecting B in r can be found, a 4-dimensional space Π_4 through Π_3 only intersecting B in r can be found, $\ldots$, a $(k-1)$-dimensional space Π_{k-1} through Π_{k-2} only intersecting B in r can be found since there are $q^{n-k+1} + \cdots + q + 1$ $(k-1)$-dimensional spaces through Π_{k-2} and $|B| < tq^{n-k} + q^{n-k-1/3}$.

There are $q+1$ k-dimensional spaces in Π through Π_{k-1}, all intersecting S_1 in $t \pmod{\sqrt{q}}$ points. Since these k-dimensional spaces intersect B in $t \pmod{\sqrt{q}}$ points, every such hyperplane intersects S_2 in r and in at least $\sqrt{q}-1$ other points. So $|\Pi \cap B| \geq 1 + (q+1)(\sqrt{q}-1) + t(q+1)$. This contradicts $|\Pi \cap B| \leq tq + 2t\sqrt{q}$. $\square$

4 t-Fold 2-blocking sets

Let B be a minimal t-fold 2-blocking set in $\mathrm{PG}(n, q)$ intersecting every $(n-2)$-dimensional space in $t \pmod{\sqrt{q}}$ points. Assume that

$$|B| \leq tq^2 + 2tq\sqrt{q} < tq^2 + c_p q^{5/3},$$

with $q \geq 661$ and with $t < c_p q^{1/6}/2$.

The $t \pmod{\sqrt{q}}$ assumption implies that every $(n-1)$-dimensional space intersects B in at most $tq + 2t\sqrt{q}$ points or in at least $q\sqrt{q} + t$ points (Lemma 3.2).

We will show that B is the union of t pairwise disjoint planes, Baer cones with a point as vertex and a Baer subplane $\mathrm{PG}(2, \sqrt{q})$ as base, or subgeometries $\mathrm{PG}(4, \sqrt{q})$.

Remark 4.1 *(1) In this article, when we state that* a Baer subline L is contained in B, *then we mean that this Baer subline is effectively contained in B, but that the line $\widehat{L}$, defined over* $\mathrm{GF}(q)$, *defined by L is not completely contained in B.*

(2) In the next lemma, we state that a subset S of points on a line L can be written in a unique way as a union of at most t pairwise disjoint points and Baer sublines. *This has the following meaning. If S contains a Baer subline, then, first of all, the $\sqrt{q}+1$ points of this Baer subline must be considered in this description as a Baer subline and not as $\sqrt{q}+1$ distinct points, secondly, these Baer sublines and points contained in S are all pairwise disjoint, and thirdly, if you consider the different Baer sublines contained in S and then the remaining points of S, the total number of these different Baer sublines and remaining points is at most t.*

(3) Consider a Baer subline L, then $\widehat{L}$ will always denote the line, over GF(q), containing the Baer subline L.

Lemma 4.2 *A line L not contained in B shares at most $t(\sqrt{q}+1)$ points with B. This intersection $L \cap B$ can be written in a unique way as a union of at most t pairwise disjoint points and Baer sublines.*

Proof: By using the same arguments as in the proof of Lemma 3.2, it is possible to find an $(n-3)$-dimensional space through L containing no other points of B. It is then possible to select an $(n-2)$-dimensional space through this $(n-3)$-dimensional space containing at most t extra points of B since there are q^2+q+1 $(n-2)$-dimensional spaces through a given $(n-3)$-dimensional space, and $|B| < tq^2 + q^{5/3}$. Similarly, it is then possible to select a hyperplane π through this $(n-2)$-dimensional space containing at most $tq + 2t\sqrt{q}$ other points of B.

Then $|\pi \cap B| \leq q+t+tq+2t\sqrt{q} < q\sqrt{q}+t$, so by Theorem 2.2, $|\pi \cap B| \leq tq + 2t\sqrt{q}$, and then Theorem 3.1 and Lemma 3.2 imply that π intersects B in a union of t pairwise disjoint lines and Baer subplanes.

This implies that L intersects B in a number of points and/or Baer sublines.

Assume that L shares at least one Baer subline with B. Since $t < q^{1/6}/2$, and since two distinct Baer sublines share at most two points, it is only possible to partition the points of a Baer subline in $L \cap B$ over other Baer sublines in $L \cap B$ if $t \geq (\sqrt{q}+1)/2$.

This is not the case, so $L \cap B$ can be written in a unique way as a union of at most t pairwise disjoint points and Baer sublines. □

Lemma 4.3 *Every hyperplane Π intersects B in a union of t pairwise disjoint lines and/or Baer subplanes, or intersects B in at least $q\sqrt{q}+t$ points.*

Proof: By Theorem 2.2, since every $(n-2)$-dimensional space intersects B in $t \pmod{\sqrt{q}}$ points, B intersects every hyperplane in at most $tq+2t\sqrt{q}$ points or in at least $q\sqrt{q}+t$ points. Assume that a hyperplane Π intersects B in at most $tq+2t\sqrt{q}$ points, then this intersection $\Pi \cap B$ must be a minimal t-fold 1-blocking set in Π, since for a non-minimal intersection, $|\Pi \cap B| \geq q\sqrt{q}+t$ (Lemma 3.2).

Since for the case $|\Pi \cap B| \leq tq+2t\sqrt{q}$, the intersection must be a minimal t-fold 1-blocking set, Theorem 3.1 implies that $B \cap \Pi$ is a union of t pairwise disjoint lines and/or Baer subplanes. □

We know from Lemma 4.3 that every hyperplane Π intersects B in a union of t lines and/or Baer subplanes, or intersects B in at least $q\sqrt{q}+t$ points. Consequently, for every hyperplane Π, $|\Pi \cap B| \geq t(q+1)$.

Consider an $(n-2)$-dimensional space Δ sharing t distinct points with B. The $q+1$ hyperplanes through Δ all contain at least $tq+t$ points of B, so if we subtract $(q+1)tq$ from the size of B, at most $2tq\sqrt{q}-tq$ points in B remain. Dividing this number by $q\sqrt{q}-tq$ then implies that at most $2t$ hyperplanes through Δ contain at least $q\sqrt{q}+t$ points of B. The other, at least $q+1-2t$, hyperplanes through Δ share at most $tq+2t\sqrt{q}$ points with B, and therefore intersect B in a union of t pairwise disjoint lines and/or Baer subplanes (Lemma 4.3).

This shows that every point of $\Delta \cap B$ lies on at least $q+1-2t$ lines and/or Baer subplanes, contained in B.

Lemma 4.4 *Let $r \in \Delta \cap B$ and suppose that r lies in two Baer subplanes B_1 and B_2, contained in B, in distinct hyperplanes through Δ.*

Then B_1 and B_2 define a 4-dimensional Baer subgeometry completely contained in B.

Proof: Consider a Baer subline L_2 of B_2 through r. Then the line $\widehat{L_2}$, defined over GF(q), through L_2 shares at most $t(\sqrt{q}+1)$ points with B (Lemma 4.2). By using the same arguments as in the proof of Lemma 3.2, it is possible to find an $(n-3)$-dimensional space Π_{n-3} through L_2 containing no other points of B, and intersecting the plane of B_1 only in r.

There are q^2+q+1 $(n-2)$-dimensional spaces through Π_{n-3}. Precisely $q+1$ of those $(n-2)$-dimensional spaces through Π_{n-3} intersect the plane PG$(2,q)$ containing the Baer subplane B_1 in a line through r, so q^2 of these $(n-2)$-dimensional spaces through Π_{n-3} only intersect the plane of B_1 in r. It is therefore possible to select an $(n-2)$-dimensional space Δ' through Π_{n-3} containing at most t extra points of B, and only intersecting the plane of B_1 in r. Then $|\Delta' \cap B| \leq t(\sqrt{q}+1)+t$ since there are at most $t(\sqrt{q}+1)$ points of B belonging to $\widehat{L_2}$ (Lemma 4.2).

Since $|\Delta' \cap B| \equiv t \pmod{\sqrt{q}}$, necessarily $|\Delta' \cap B| \leq t(\sqrt{q}+1)$.

Every hyperplane through Δ' contains at least $tq - t\sqrt{q}$ other points of B since every hyperplane shares at least $t(q+1)$ points with B (Lemma 4.3). If we subtract $(q+1)(tq-t\sqrt{q})$ from the size of B, at most $3tq\sqrt{q}-tq+t\sqrt{q}$ points in B remain. A hyperplane through Δ' containing at least $q\sqrt{q}+t$ points of B still contains at least $q\sqrt{q}-tq$ other points of B, so at most $3t$ hyperplanes through Δ' contain at least $q\sqrt{q}+t$ points of B.

This implies that at least $\sqrt{q}+1-3t$ hyperplanes through Δ' intersect B_1 in a Baer subline, and intersect B in a union of $t < q^{1/6}/2$ pairwise disjoint lines and/or Baer subplanes. Since such a hyperplane shares a Baer subline with B_1 and with B_2, both passing through the same point r, these two latter Baer sublines must be contained in a Baer subplane contained in B.

The preceding arguments show that at least $\sqrt{q}+1-3t$ Baer subplanes of the 3-dimensional Baer subgeometry $\langle B_1, L_2\rangle$, passing through L_2, are contained in B.

Assume that the Baer subgeometry $\langle B_1, L_2\rangle$ is not contained in B. Select a Baer subline N of $\langle B_1, L_2\rangle$ skew to L_2 which is not contained in B. Then this Baer subline N shares at least $\sqrt{q}+1-3t$ and at most $\sqrt{q}$ points with B.

By Lemma 4.2, it is possible to describe $N \cap B$ in a unique way as a union of at most $t < q^{1/6}/2$ pairwise disjoint points and Baer sublines.

Since $\sqrt{q}+1-3t > t$, some of the points of $N \cap B$ lying in $\langle B_1, L_2\rangle$ must lie in Baer sublines contained in $N \cap B$. Two distinct Baer sublines share at most two points. Since $\sqrt{q}+1-3t > 2t$, this is impossible, so the Baer subline $N \cap \langle L_2, B_1\rangle$ is completely contained in B.

This shows that the 3-dimensional Baer subgeometry $\langle L_2, B_1\rangle$ is completely contained in B. By letting vary L_2 over all Baer sublines of B_2 through r, the 4-dimensional Baer subgeometry $\langle B_1, B_2\rangle$ is completely contained in B. $\square$

This latter 4-dimensional Baer subgeometry $\langle B_1, B_2\rangle$ is either a Baer cone with a point as vertex and a Baer subplane as base, or a Baer subgeometry $\mathrm{PG}(4, \sqrt{q})$.

In both cases, they are 1-fold 2-blocking sets, and the $t \pmod{\sqrt{q}}$ result implies that $B \setminus \langle B_1, B_2\rangle$ is a $(t-1)$-fold 2-blocking set intersecting every $(n-2)$-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

Since we know from the calculations preceding Lemma 4.4 that every point of $\Delta \cap B$ lies on at least $q+1-2t$ lines or Baer subplanes contained in B, the preceding lemma and observations now imply that we can assume that every point of $\Delta \cap B$ lies on at least $q-2t$ lines contained in B. Since B is minimal, it is possible to assume that every point of B lies on at least $q-2t$ lines of B. We now show that there is a plane contained in B.

Lemma 4.5 *If every point of B lies on at least $q-2t$ lines contained in B, then there is a plane contained in B.*

Proof: Consider an $(n-2)$-dimensional space Δ intersecting B in exactly t points. The calculations preceding Lemma 4.4 indicate that at least $q+1-2t$ hyperplanes through Δ intersect B in a union of t lines and/or Baer subplanes. But none of the t points of $\Delta \cap B$ lies on two Baer subplanes of B in those hyperplanes. So, at least $q+1-2t-t$ hyperplanes Π through Δ intersect B in t pairwise disjoint lines $L_1, \dots, L_t$.

Let r be a point of $B \setminus \Pi$. This point r lies on at least $q-2t$ lines completely contained in B. These lines intersect Π in a point of $B \cap \Pi = L_1 \cup \cdots \cup L_t$. So at least one of the lines L_i is intersected by at least $(q-2t)/t$ lines of B passing through r.

Then the plane $\langle r, L_i\rangle$ intersects B in at least $(q-2t)/t$ lines passing through r. Then every line of this plane, not passing through r, shares already $(q-2t)/t$ points with B. If such a line is not contained in B, it shares at most $t(\sqrt{q}+1)$ points with B (Lemma 4.2).

Since $(q-2t)/t > t(\sqrt{q}+1)$, every line of $\langle L_i, r\rangle$, not passing through r, is contained in B, and so this plane $\langle L_i, r\rangle$ is contained in B. $\square$

The $t \pmod{\sqrt{q}}$ result again implies that $B \setminus \Pi$, Π a plane contained in B, is a $(t-1)$-fold blocking set intersecting every $(n-2)$-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

Repeating the preceding lemmas for this $(t-1)$-fold blocking set, the following characterization theorem is obtained.

Theorem 4.6 *Let B be a minimal t-fold 2-blocking set, of size at most $tq^2 + 2tq\sqrt{q} < tq^2 + c_p q^{5/3}$, in $\mathrm{PG}(n, q)$, $q \geq 661$, $t < c_p q^{1/6}/2$, intersecting every $(n-2)$-dimensional space in $t \pmod{\sqrt{q}}$ points.*

Then B is the union of t pairwise disjoint planes, Baer cones with a point as vertex and a Baer subplane as base, and 4-dimensional Baer subgeometries $\mathrm{PG}(4, \sqrt{q})$.

5 t-Fold $(n-k)$-blocking sets in $\mathrm{PG}(n, q)$

We now will present the characterization result on minimal t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n, q)$, with $1 \leq k < n-2$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points. The results of the preceding two sections will be the induction bases for the general characterization results.

The general induction hypothesis (IH) we rely on for classifying the minimal t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n, q)$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points, is as follows.

Induction hypothesis (IH): *For $1 \leq j \leq n-k-1$, let B_j be a minimal t-fold $(n-k-j)$-blocking set in $\mathrm{PG}(n, q)$, q square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B_j| \leq tq^{n-k-j} + 2tq^{n-k-j-1}\sqrt{q} < tq^{n-k-j} + c_p q^{n-k-j-1/3}$, intersecting every $(k+j)$-dimensional space in $t \pmod{\sqrt{q}}$ points.*

Then B_j is a union of t pairwise disjoint cones $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-j-1), \sqrt{q})\rangle$, $-1 \leq m_i \leq n-k-j-1$, $i = 1, \ldots, t$.

In the above description, if $m_i = n-k-j-1$, then $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-j-1), \sqrt{q})\rangle$ is a subspace $\mathrm{PG}(n-k-j, q)$, and if $m_i = -1$, then $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-j-1), \sqrt{q})\rangle$ is a Baer subgeometry $\mathrm{PG}(2(n-k-j), \sqrt{q})$.

The goal is to prove the following similar characterization result for t-fold $(n-k)$-blocking sets.

Let B be a minimal t-fold $(n-k)$-blocking set in $\mathrm{PG}(n, q)$, q square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + c_p q^{n-k-1/3}$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.

Then B is a union of t pairwise disjoint cones $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-1), \sqrt{q})\rangle$, $-1 \leq m_i \leq n-k-1$, $i = 1, \ldots, t$.

So, from now on, we assume that B is a minimal t-fold $(n-k)$-blocking set in $\mathrm{PG}(n,q)$, q square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + c_p q^{n-k-1/3}$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.

We first present some analogous lemmas to lemmas of Section 4.

Lemma 5.1 *Every $(k+1)$-dimensional space Π intersects B in a union of t pairwise disjoint lines and/or Baer subplanes, or intersects B in at least $q\sqrt{q}+t$ points.*

Proof: By Theorem 2.2, since every k-dimensional space intersects B in $t \pmod{\sqrt{q}}$ points, B intersects every $(k+1)$-dimensional space in at most $tq + 2t\sqrt{q}$ points or in at least $q\sqrt{q}+t$ points. Assume that a $(k+1)$-dimensional space Π intersects B in at most $tq+2t\sqrt{q}$ points, then this intersection $\Pi \cap B$ must be a minimal t-fold 1-blocking set in Π, since for a non-minimal intersection, $|\Pi \cap B| \geq q\sqrt{q}+t$ (Lemma 3.2).

Since for the case $|\Pi \cap B| \leq tq + 2t\sqrt{q}$, the intersection must be a minimal t-fold 1-blocking set, Theorem 3.1 implies that $B \cap \Pi$ is a union of t pairwise disjoint lines and/or Baer subplanes. $\Box$

For the description of the next lemma, we again rely on Remark 4.1 (2).

Lemma 5.2 *A line L, not contained in B, intersects B in at most $t(\sqrt{q}+1)$ points. This intersection can be described in a unique way as a union of at most t pairwise disjoint points and Baer sublines.*

Proof: We know that $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$.

By using the same arguments as in the proof of Lemma 3.2, it is possible to construct a $(k-1)$-dimensional space Π_{k-1} through L containing no other points of B. It is then possible to construct a k-dimensional space Π_k through Π_{k-1} containing at most t other points of B. So $|\Pi_k \cap B| \leq q+t$.

There are at most $tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$ points in B left. By the induction hypothesis (IH), the smallest t-fold 1-blocking sets which are the intersection of a $(k+1)$-dimensional space with B are the union of t pairwise disjoint lines, see also Lemma 4.3. Hence, every $(k+1)$-dimensional space through Π_k contains at least $(t-1)q$ extra points of B. So we observe that at most $tq^{n-k} + 2tq^{n-k-1}\sqrt{q} - (t-1)q(q^{n-k-1} + q^{n-k-2} + \cdots + q + 1)$ other points of B can remain.

If a $(k+1)$-dimensional space Π_{k+1} through Π_k contains at least $q\sqrt{q}+t$ points of B (Theorem 2.2 and Lemma 3.2), then it still contains at least $q\sqrt{q}-tq$ other points of $B\backslash\Pi_k$. Since $(q^{n-k-1}+q^{n-k-2}+\cdots+q+1)(q\sqrt{q}-tq) > tq^{n-k}+2tq^{n-k-1}\sqrt{q}-(t-1)q(q^{n-k-1}+q^{n-k-2}+\cdots+q+1)$, there is at least one $(k+1)$-dimensional space Π_{k+1} through Π_k with at most $tq+2t\sqrt{q}+t$ points of B. Then $|\Pi_{k+1}\cap B| \leq tq+2t\sqrt{q}$ (Theorem 2.2 and Lemma 3.2). This intersection $\Pi_{k+1}\cap B$ is a minimal t-fold 1-blocking set in Π_{k+1} (Lemma 3.2), so it is a union of t pairwise disjoint lines and/or Baer subplanes (Theorem 3.1). The line L shares zero or one points with the lines of $\Pi_{k+1}\cap B$, and zero, one, or $\sqrt{q}+1$ points with the Baer subplanes of $\Pi_{k+1}\cap B$. This proves the lemma. $\square$

Lemma 5.3 *Let r be a point of B lying on two lines L_0 and L_1 contained in B.*

Then the plane $\langle L_0, L_1\rangle$ is either contained in B, or $\langle L_0, L_1\rangle\cap B$ contains a cone with r as vertex and a Baer subline as base, containing L_0 and L_1.

Proof: Consider the plane $\langle L_0, L_1\rangle$. Whatever its intersection with B is, the intersection size is at most q^2+q+1.

By using the same arguments as in the proof of Lemma 3.2, construct a $(k-1)$-dimensional space Π_{k-1} through $\langle L_0, L_1\rangle$ containing no other points of B. Since $|B| < tq^{n-k}+q^{n-k-1/3}$, and since there are $q^{n-k}+\cdots+q+1$ different k-dimensional spaces through Π_{k-1}, it is possible to construct a k-dimensional space Π_k through Π_{k-1} containing at most t extra points of B. Similarly, since $|B| \leq tq^{n-k}+2tq^{n-k-1}\sqrt{q}$, it is possible to find a $(k+1)$-dimensional space Π_{k+1} through Π_k containing at most $tq+2t\sqrt{q}$ other points of B.

So, $|B\cap\Pi_{k+1}| \leq q^2+q+1+tq+2t\sqrt{q}+t$.

Consider all $q^{n-k-2}+\cdots+q+1$ $(k+2)$-dimensional spaces through Π_{k+1}. Since $|B| < tq^{n-k}+q^{n-k-1/3}$, it is possible to find a $(k+2)$-dimensional space Π_{k+2} through Π_{k+1} containing at most $tq^2+2tq\sqrt{q}$ other points of B. This certainly implies that $|\Pi_{k+2}\cap B| \leq (t+2)q^2$.

Since $|\Pi_{k+2}\cap B| \leq tq^2+2tq\sqrt{q}$ or $|\Pi_{k+2}\cap B| \geq q^2\sqrt{q}+t$ (Theorem 2.2 and Lemma 3.2), necessarily $|\Pi_{k+2}\cap B| \leq tq^2+2tq\sqrt{q}$.

Theorem 4.6 implies that $\Pi_{k+2}\cap B$ is a union of t pairwise disjoint planes, cones with a point as vertex and a Baer subplane as base, and Baer subgeometries $\mathrm{PG}(4,\sqrt{q})$.

Since L_0 and L_1 are intersecting lines of this intersection, the plane $\langle L_0, L_1\rangle$ either is contained in B, or its intersection with B contains a cone with $L_0\cap L_1$ as vertex and a Baer subline as base, which contains the lines

L_0 and L_1. $\square$

The following two lemmas are proven in exactly the same way as the preceding lemma. In the following lemma, a Baer cone with vertex s and base the Baer subline L_2, $s \notin \widehat{L_2}$, is the set of $\sqrt{q}+1$ lines through the point s and the points of the Baer subline L_2. We also recall Remark 4.1 (1); with a Baer subline L contained in B, we mean a Baer subline contained in B whose corresponding line $\widehat{L}$ over $\mathrm{GF}(q)$ is not contained in B.

Lemma 5.4 *Suppose that the point r of B lies on a line L_0 contained in B and on a Baer subline L_2 contained in B.*

Then there is a Baer cone completely contained in B, with a point of $L_0 \setminus \{r\}$ as vertex and with L_2 as base.

Lemma 5.5 *Suppose that the point $r \in B$ lies on two Baer sublines L_0 and L_1 contained in B, then the Baer subplane $\langle L_0, L_1\rangle$ is completely contained in B.*

Lemma 5.6 *Let L_2 be a Baer subline contained in B. Let v be a point not lying on the line $\widehat{L_2}$, defined over $\mathrm{GF}(q)$, by L_2. Suppose that the cone with vertex v and with base the Baer subline L_2 is contained in B.*

Let r be a point of L_2 and suppose that L_1 is an other Baer subline of B through r, not lying in the plane $\langle v, L_2\rangle$.

Then the Baer cone Ω with vertex v and with base the Baer subplane $\langle L_1, L_2\rangle$ is contained in B.

Proof: Let L_2' be a second Baer subline of the Baer cone $\langle v, L_2\rangle$ passing through r. Then the Baer subplane $\langle L_1, L_2'\rangle$ is contained in B (Lemma 5.5). This Baer subplane $\langle L_1, L_2'\rangle$ is projected from v onto the Baer subplane $\langle L_1, L_2\rangle$.

Letting vary L_2' over all Baer sublines of the Baer cone $\langle v, L_2\rangle$ through r, the preceding arguments prove that the Baer cone Ω with vertex v and with base $\langle L_1, L_2\rangle$ is completely contained in B, up to maybe some points on the line rv.

But let r' be an arbitrary point of the line $rv \setminus \{r, v\}$, and let L_3 be an arbitrary Baer subline of the Baer cone Ω through r'. This Baer subline is completely contained in B, up to maybe the point r'. So, L_3 contains $\sqrt{q}$ or $\sqrt{q}+1$ points of B. We prove that the Baer subline L_3 is completely contained in B. Let $\widehat{L_3}$ be the line over $\mathrm{GF}(q)$ defined by L_3, then the intersection of $\widehat{L_3}$ with B can be described in a unique way as the union of at most t pairwise disjoint points and Baer sublines (Lemma 5.2). If the Baer subline L_3 contains exactly $\sqrt{q}$ points of B, then these $\sqrt{q}$ points need to be partitioned over at most $t < q^{1/6}/2$ pairwise disjoint points and Baer sublines (Lemma 5.2). Since two distinct Baer sublines share at most two points, this is impossible. So the Baer subline L_3 is completely contained in B.

This proves that the Baer cone Ω with vertex v and with base the Baer subplane $\langle L_1, L_2 \rangle$ is completely contained in B. $\square$

Consider a point r from B and select a subspace $\Delta_k \simeq \mathrm{PG}(k, q)$ through r sharing t points with B.

There is at least one $(k+1)$-dimensional subspace through Δ_k sharing at most $tq + 2t\sqrt{q}$ points, not in Δ_k, with B, since these $(k+1)$-dimensional spaces through Δ_k cannot all contain $q\sqrt{q}$ other points of B (Theorem 2.2).

Then such a $(k+1)$-dimensional subspace Δ_{k+1} through Δ_k shares at most $tq + 2t\sqrt{q} + t$ points with B. By Theorem 2.2 and Lemma 3.2, $|\Delta_{k+1} \cap B| \leq tq + 2t\sqrt{q}$. Hence, Δ_{k+1} intersects B in t pairwise disjoint lines and/or Baer subplanes (Lemma 5.1). So Δ_{k+1} shares at most $tq + t\sqrt{q}$ other points with B. Select $\Delta_{k+1} \simeq \mathrm{PG}(k+1, q)$ through Δ_k sharing at most $tq + t\sqrt{q} + t$ points with B.

We now prove that we can find an $(n-2)$-dimensional space Δ_{n-2} through Δ_{k+1} sharing at most $t(q^{n-k-2} + q^{n-k-3}\sqrt{q} + q^{n-k-3} + \cdots + \sqrt{q} + 1)$ points with B. We heavily rely on the bounds of Theorem 2.2. Since B intersects every k-dimensional space in $t \pmod{\sqrt{q}}$ points, this theorem states that B intersects every $(k+i)$-dimensional space in either at most $tq^i + 2\sqrt{q}q^{i-1}$ points or in at least $\sqrt{q}q^i + t$ points. Consider all the $q^{n-k-2} + \cdots + q + 1$ different $(k+2)$-dimensional spaces through Δ_{k+1}. As $|\Delta_{k+1} \cap B| \leq tq + t\sqrt{q} + t$, it is impossible that all these $(k+2)$-dimensional spaces share at least $\sqrt{q}q^2 + t$ points with B since $|B| < tq^{n-k} + q^{n-k-1/3}$, so there is at least one $(k+2)$-dimensional space Δ_{k+2} through Δ_{k+1} sharing at most $tq^2 + 2\sqrt{q}q$ points with B. Repeating this argument by induction on i, it is possible to find a $(k+i+1)$-dimensional space Δ_{k+i+1}, sharing at most $tq^{i+1} + 2\sqrt{q}q^i$ points with B, through a given $(k+i)$-dimensional space Δ_{k+i}, sharing at most $tq^i + 2\sqrt{q}q^{i-1}$ points with B. This leads us eventually to the $(n-2)$-dimensional space Δ_{n-2} through Δ_{k+1} sharing at most $t(q^{n-k-2} + q^{n-k-3}\sqrt{q} + q^{n-k-3} + \cdots + \sqrt{q} + 1)$ points with B. The upper bound on $|\Delta_{n-2} \cap B|$ follows from the induction hypothesis (IH), which states that $\Delta_{n-2} \cap B$ is a union of t pairwise disjoint cones $\langle \Pi_{m_i}, \mathrm{PG}(2(n-k-2-m_i-1), \sqrt{q}) \rangle$, with $-1 \leq m_i \leq n-k-3$, $i = 1, \ldots, t$.

Now it is possible to find at least two hyperplanes H_1, H_2 through Δ_{n-2} containing at most $t(q^{n-k-1} + q^{n-k-2}\sqrt{q} + q^{n-k-2} + \cdots + \sqrt{q} + 1)$ points of B, since it is not possible that all hyperplanes through Δ_{n-2} share at least $q^{n-k-1}\sqrt{q} + t$ points with B (Theorem 2.2). By the induction hypothesis (IH), these two hyperplanes meet B in a union of t pairwise disjoint cones

$\langle \Pi_{m_i'}, \mathrm{PG}(2(n-k-1-m_i'-1), \sqrt{q})\rangle$, with $-1 \leq m_i' \leq n-k-2$, $i = 1, \ldots, t$.

We now prove a first major part of the characterization result for the t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$. Our goal is to prove that small minimal t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$, q square, are a union of t pairwise disjoint Baer cones $\langle \pi_m, \mathrm{PG}(2(n-k-m-1), \sqrt{q})\rangle$, $-1 \leq m \leq n-k-1$. For $m = n-k-1$, such a Baer cone is in fact an $(n-k)$-dimensional subspace $\mathrm{PG}(n-k,q)$, and for $m < n-k-1$, such a Baer cone is a cone with an m-dimensional subspace $\mathrm{PG}(m,q)$ as vertex and a base $\mathrm{PG}(2(n-k-m-1) \geq 2, \sqrt{q})$ which is a non-projected Baer subgeometry. For $m < n-k-1$, such a Baer cone contains Baer sublines. The following lemma shows that if there is a line not contained in B, sharing at least one Baer subline with B, then this implies that B contains a Baer cone $\langle \pi_m, \mathrm{PG}(2(n-k-m-1), \sqrt{q})\rangle$, $-1 \leq m < n-k-1$.

Lemma 5.7 *Let Δ be an $(n-2)$-dimensional space intersecting B in a union of t pairwise disjoint Baer cones $\langle \Pi_{m_i}, \mathrm{PG}(2(n-k-2-m_i-1), \sqrt{q})\rangle$, $-1 \leq m_i \leq n-k-3$, $i = 1, \ldots, t$.*

Assume that $m_i < n-k-3$ for at least one value $i \in \{1, \ldots, t\}$.

Then B contains a Baer cone $\langle \pi_{m''}, \mathrm{PG}(2(n-k-m''-1), \sqrt{q})\rangle$, $-1 \leq m'' < n-k-1$.

Proof: It is possible to find at least two hyperplanes H_1, H_2 through Δ containing at most $t(q^{n-k-1} + q^{n-k-2}\sqrt{q} + q^{n-k-2} + \cdots + \sqrt{q} + 1)$ points of B, since it is not possible that q hyperplanes through Δ share at least $q^{n-k-1}\sqrt{q}+t$ points with B (Theorem 2.2). By the induction hypothesis (IH), these two hyperplanes H_1, H_2 through Δ respectively intersect B in unions of t pairwise disjoint cones $\langle \Pi'_{m_i'}, \mathrm{PG}(2(n-k-1-m_i'-1), \sqrt{q})\rangle$, with $-1 \leq m_i' \leq n-k-2$, and t pairwise disjoint cones $\langle \Pi''_{m_i''}, \mathrm{PG}(2(n-k-1-m_i''-1), \sqrt{q})\rangle$, with $-1 \leq m_i'' \leq n-k-2$.

Since we assume that one of the t Baer cones of $\Delta \cap B$ is a cone $\langle \Pi_m, \mathrm{PG}(2(n-k-2-m-1), \sqrt{q})\rangle$, with $m < n-k-3$, so with base a Baer subspace $\mathrm{PG}(s = 2(n-k-2-m-1) \geq 2, \sqrt{q})$, at least one of these Baer cones in $B \cap \Delta$ contains a Baer subline L_2.

Then H_1, and similarly H_2, shares with B a cone of type either $\langle \Pi_m, \mathrm{PG}(s+2, \sqrt{q})\rangle$ or $\langle \Pi_{m+1}, \mathrm{PG}(s, \sqrt{q})\rangle$, intersecting Δ in this Baer cone $\langle \Pi_m, \mathrm{PG}(s, \sqrt{q})\rangle$. We denote this particular Baer cone in H_1 by $\langle \Pi_{m_1}, \mathrm{PG}(s_1, \sqrt{q})\rangle$, and this particular Baer cone in H_2 by $\langle \Pi_{m_2}, \mathrm{PG}(s_2, \sqrt{q})\rangle$.

Up to equivalence, there are three possibilities. The first possibility is $m = m_1 = m_2$, $s_1 = s_2 = s+2$. Then $\langle \Pi_{m_1}, \mathrm{PG}(s_1, \sqrt{q})\rangle$ and $\langle \Pi_{m_2}, \mathrm{PG}(s_2, \sqrt{q})\rangle$ define a Baer cone with vertex Π_m and base $\mathrm{PG}(s+4, \sqrt{q})$, or with an

$(m+1)$-dimensional vertex and base $\mathrm{PG}(s+2,\sqrt{q})$. Up to equivalence, the second possibility is $m_1 = m+1$ and $m_2 = m$, which then means that $s = s_1$ and $s_2 = s+2$. The smallest Baer cone containing $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$ and $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$ is the Baer cone with vertex Π_{m_1} and base $\mathrm{PG}(s_2,\sqrt{q})$. The last possibility is that $m_1 = m_2 = m+1$ and that $s = s_1 = s_2$. In this case, the smallest Baer cone containing $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$ and $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$ is the Baer cone with vertex the $(m+2)$-dimensional space $\langle \Pi_{m_1}, \Pi_{m_2}\rangle$ and with base $\mathrm{PG}(s,\sqrt{q})$.

The following arguments will show for all three cases that this smallest Baer cone B_0 containing $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$ and $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$ lies completely in B. In the first part of this proof, we prove our crucial result for proving that B_0 is contained in B.

Part 1. Consider a non-singular point x of $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$, not lying in Δ. Let $L_1 \subset B$ be a Baer subline of $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$, passing through x and containing a point r of the base $\mathrm{PG}(s,\sqrt{q})$ of the Baer cone $\langle \Pi_m, \mathrm{PG}(s,\sqrt{q})\rangle$ in Δ. We show that the Baer subgeometry defined by L_1 and $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$ lies completely in B.

Lemma 5.6 proves that if you have a Baer cone $\langle v, L_2\rangle$ of B, where $L_2 \simeq \mathrm{PG}(1,\sqrt{q})$, L_1 a Baer subline of B not in the plane of v and L_2, and $L_1 \cap L_2 \neq \emptyset$, then the cone with vertex v and base $\langle L_1, L_2\rangle \simeq \mathrm{PG}(2,\sqrt{q})$ lies completely in B.

By letting vary v over Π_{m_2} and by letting vary L_2 over all Baer sublines through r in $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$, we reach all points of $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$; the cone with vertex v and base $\langle L_1, L_2\rangle \simeq \mathrm{PG}(2,\sqrt{q})$ lies in B, hence the Baer subgeometry defined by Π_{m_2}, L_1, and $\mathrm{PG}(s_2,\sqrt{q})$ lies completely in B.

Part 2. The Baer cones $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$ and $\langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$ define a (projected) Baer subgeometry B_0 over $\mathrm{GF}(\sqrt{q})$.

Consider in B_0 an arbitrary Baer subgeometry Ω of dimension one larger than the Baer subgeometry $B_0 \cap H_2 = \langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$, passing through the Baer subgeometry $B_0 \cap H_2 = \langle \Pi_{m_2}, \mathrm{PG}(s_2,\sqrt{q})\rangle$. Then Ω intersects H_1 in a Baer cone of dimension at least one larger than $\langle \Pi_m, \mathrm{PG}(s,\sqrt{q})\rangle$, so Ω contains points of B_0 in $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$, not lying in Δ. This intersection $\Omega \cap H_1$ contains non-singular points of the Baer cone $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$, not lying in Δ. For, if there were only such singular points in $H_1 \cap \Omega$, then let r be a point of $\Pi_{m_1} \setminus \Delta$ lying in Ω. Consider the line rr' through r and a point r' of the base $\mathrm{PG}(s,\sqrt{q})$ of the Baer cone $B_0 \cap \Delta = \langle \Pi_m, \mathrm{PG}(s,\sqrt{q})\rangle$. This line already contains two points r and r' of Ω, so contains at least one Baer subline of Ω. Hence, Ω contains at least one non-singular point x of $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$, not lying in Δ. So $\Omega \cap H_1$ intersects $\langle \Pi_{m_1}, \mathrm{PG}(s_1,\sqrt{q})\rangle$ in a Baer subgeometry of dimension one larger than $\dim\langle \Pi_m, \mathrm{PG}(s,\sqrt{q})\rangle$. This intersection can be defined uniquely by $\langle \Pi_m, \mathrm{PG}(s,\sqrt{q})\rangle$ and a Baer subline

L_1 joining x to a non-vertex point in $\langle \Pi_m, \mathrm{PG}(s, \sqrt{q})\rangle$. We have proven in Part 1 that L_1 together with $\langle \Pi_{m_2}, \mathrm{PG}(s_2, \sqrt{q})\rangle$ defines a unique Baer cone, completely lying in B. This Baer cone is in fact Ω. Hence, Ω lies in B.

So we conclude that an arbitrary Baer cone in B_0, of dimension one larger than $\dim\langle \Pi_{m_2}, \mathrm{PG}(s_2, \sqrt{q})\rangle$, passing through the Baer subgeometry $B_0 \cap H_2 = \langle \Pi_{m_2}, \mathrm{PG}(s_2, \sqrt{q})\rangle$, lies completely in B. This shows that $B_0 \subseteq B$.

Hence, B contains a Baer cone $B_0 = \langle \pi_{m''}, \mathrm{PG}(2(n-k-m''-1), \sqrt{q})\rangle$, $-1 \le m'' < n-k-1$. □

Assume that the conditions of the preceding lemma are valid, then using the $t \pmod{\sqrt{q}}$ assumption, $B \setminus B_0$ is a $(t-1)$-fold $(n-k)$-blocking set, intersecting every k-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

Assume that there is a line L defined over $\mathrm{GF}(q)$ intersecting B in a set of at most $t(\sqrt{q}+1)$ points, containing a Baer subline L_1. Then, by using the same arguments as in the proof of Lemma 3.2, it is first of all possible to find a $(k-1)$-dimensional space Π_{k-1} through L containing no other points of B. Since $|B| < tq^{n-k} + q^{n-k-1/3}$, there is a k-dimensional space Δ_k through Δ_{k-1} containing at most t other points of B. Similarly, there is a $(k+1)$-dimensional space through Δ_k sharing at most $tq+2tq$ points with B since it is impossible that all these $(k+1)$-dimensional spaces through Δ_k contain at least $\sqrt{q}q+t$ points of B (Theorem 2.2). The same arguments as in the proof of Lemma 5.6 then prove that it is possible to find an $(n-2)$-dimensional space Δ through L intersecting B in a union of t pairwise disjoint Baer cones $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-2-m_i-1), \sqrt{q})\rangle$, $-1 \le m_i \le n-k-3$, $i = 1, \ldots, t$, where for at least one such Baer cone in $\Delta \cap B$, $m_i < n-k-3$.

Then the conditions of the preceding lemma are met, and it is possible to find a 1-fold $(n-k)$-blocking set B_0 in B, such that $B \setminus B_0$ is a $(t-1)$-fold $(n-k)$-blocking set.

To obtain the complete characterization of t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$ of size at most $tq^{n-k} + 2tq^{n-k-1}\sqrt{q}$, it suffices to consider the case that lines are either completely contained in B, or intersect B in at most t distinct points, since it is no longer necessary to assume that Baer sublines are contained in B.

We now show that this implies that B contains an $(n-k)$-dimensional space over $\mathrm{GF}(q)$.

Let Δ be an $(n-2)$-dimensional space intersecting B in at most $tq^{n-k-2} + 2tq^{n-k-3}\sqrt{q}$ points, so by the induction hypothesis (IH) and also using the fact that there are no Baer sublines contained in B, Δ shares t pairwise disjoint spaces $\mathrm{PG}(n-k-2, q)$ with B. Consider again two hyperplanes H_1 and H_2 through Δ intersecting B in at most $tq^{n-k-1} + 2tq^{n-k-2}\sqrt{q}$ points. By the induction hypothesis, and again using that no Baer sublines are contained in B, these two hyperplanes H_1 and H_2 intersect B in t pairwise disjoint subspaces $\mathrm{PG}(n-k-1,q)$.

Let Π_1 and Π_2 be two $(n-k-1)$-dimensional spaces in respectively H_1 and in H_2, both contained in B, and intersecting Δ in the same $(n-k-2)$-dimensional space Π. We now show that Π_1 and Π_2 define an $(n-k)$-dimensional space Π_{n-k} completely contained in B.

Let r be a point of Π and consider two lines L_1 and L_2, through r, lying in respectively Π_1 and in Π_2, but not lying in Δ. Then the plane $\langle L_1, L_2\rangle$ lies completely in B (Lemma 5.3).

Letting vary the point r in Π and letting vary the lines L_1 and L_2 in Π_1 and in Π_2, the $(n-k)$-dimensional space $\Pi_{n-k} = \langle \Pi_1, \Pi_2\rangle$ lies completely in B.

By using the $t \pmod{\sqrt{q}}$ assumption, $B \setminus \Pi_{n-k}$ is a $(t-1)$-fold $(n-k)$-blocking set of $\mathrm{PG}(n,q)$, intersecting every k-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

The preceding arguments now lead to the desired characterization result.

Theorem 5.8 *Let B be a minimal t-fold $(n-k)$-blocking set in $\mathrm{PG}(n,q)$, q square, $q \geq 661$, $t < c_p q^{1/6}/2$, of size at most $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + c_p q^{n-k-1/3}$, intersecting every k-dimensional space in $t \pmod{\sqrt{q}}$ points.*

Then B is a union of t pairwise disjoint cones $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-1),\sqrt{q})\rangle$, $-1 \leq m_i \leq n-k-1$, $i = 1,\ldots,t$.

Proof: Let Δ be an $(n-2)$-dimensional space intersecting B in at most $tq^{n-k-2} + 2tq^{n-k-3}\sqrt{q}$ points.

The preceding lemma and arguments show that it is possible to find a 1-fold $(n-k)$-blocking set B_0 in B such that $B \setminus B_0$ is a $(t-1)$-fold $(n-k)$-blocking set, intersecting every k-dimensional space in $(t-1) \pmod{\sqrt{q}}$ points.

By induction on t, this proves the theorem. □

The preceding result is not the end of the classification since such unions of $t \geq 2$ pairwise disjoint cones $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-1),\sqrt{q})\rangle$ only exist if $k \geq n/2$.

Theorem 5.9 *Let B be a minimal t-fold $(n-k)$-blocking set in $\mathrm{PG}(n,q)$, q square, $t \geq 2$, which is a union of t pairwise disjoint cones $\langle \pi_{m_i}, \mathrm{PG}(2(n-k-m_i-1),\sqrt{q})\rangle$, $\max\{-1, n-2k-1\} \leq m_i \leq n-k-1$. Then $k > n/2$ if*

B contains at least one $(n-k)$-dimensional space $\mathrm{PG}(n-k,q)$ *and* $k \geq n/2$ *in the other cases.*

Proof: If B contains at least two $(n-k)$-dimensional spaces $\mathrm{PG}(n-k,q)$ which are disjoint, then $k > n/2$. If B contains an $(n-k)$-dimensional space and a cone $\langle \pi_m, \mathrm{PG}(2(n-k-m-1), \sqrt{q})\rangle$, $\max\{-1, n-2k-1\} \leq m < n-k-1$, then since the Baer cone intersects every k-dimensional space, necessarily $n-k<k$, and again $k > n/2$.

We now assume that B does not contain $(n-k)$-dimensional spaces $\mathrm{PG}(n-k,q)$. A Baer cone $\langle \pi_m, \mathrm{PG}(2(n-k-m-1), \sqrt{q})\rangle$, $\max\{-1, n-2k-1\} \leq m < n-k-1$, is in fact a projected Baer subgeometry $\mathrm{PG}(2n-2k, \sqrt{q})$. This defines a vector space $V(2n-2k+1, \sqrt{q})$.

The projective space $\mathrm{PG}(n,q)$ defines a vector space $V(2n+2, \sqrt{q})$ over $\mathrm{GF}(\sqrt{q})$. If this $(2n+2)$-dimensional vector space over $\mathrm{GF}(\sqrt{q})$ contains two disjoint $(2n-2k+1)$-dimensional subspaces, necessarily $2(2n-2k+1) \leq 2n+2$, leading to $k \geq n/2$. $\square$

Remark 5.10 The lower bound $k \geq n/2$ is sharp as the following examples of t-fold $(n-k)$-blocking sets in $\mathrm{PG}(n,q)$ show.

Let $n = 2n'$. Consider t pairwise disjoint subgeometries $\mathrm{PG}(n, \sqrt{q})_i$, $i = 1, \ldots, t$, of $\mathrm{PG}(n = 2n', q)$. They are t pairwise disjoint 1-fold n'-blocking sets, so they form together a t-fold n'-blocking set.

If $n = 2n'+1$, then the lower bound on k is $k \geq n'+1$. Consider the example of the preceding paragraph, lying in $\mathrm{PG}(2n', q)$, and embed this $2n'$-dimensional space into a $(2n'+1)$-dimensional space. Then the example of the preceding paragraph forms a t-fold n'-blocking set in $\mathrm{PG}(n = 2n'+1, q)$, so a t-fold $(n-k)$-blocking set with $k = n'+1$.

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Address of the authors:
(S. Ferret: saferret@cage.ugent.be, http://cage.ugent.be/∼saferret)
Ghent University, Dept. of Mathematics, Krijgslaan 281-S22, 9000 Gent, Belgium

(L. Storme: ls@cage.ugent.be, http://cage.ugent.be/∼ls)
Ghent University, Dept. of Mathematics, Krijgslaan 281-S22, 9000 Gent, Belgium

(P. Sziklai: sziklai@cs.elte.hu, http://www.cs.elte.hu/∼sziklai)
Eötvös Loránd University Budapest, Dept. of Computer Science, Pázmány P. s. 1/c, Budapest, Hungary H-1117

(Zs. Weiner: weiner@cs.elte.hu)
Eötvös Loránd University Budapest, Dept. of Computer Science, Pázmány P. s. 1/c, Budapest, Hungary H-1117