

## PARTIAL COVERING OF A CIRCLE BY EQUAL CIRCLES. PART I: THE MECHANICAL MODELS

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ABSTRACT. How must  $n$  equal circles of given radius be placed so that they cover as great a part of the area of the unit circle as possible? To analyse this mathematical problem, mechanical models are introduced. A generalized tensegrity structure is associated with a maximum area configuration of the  $n$  circles, whose equilibrium configuration is determined numerically with the method of dynamic relaxation, and the stability of equilibrium is investigated by means of the stiffness matrix of the tensegrity structure. In this Part I, the principles of the models are presented, while an application will be shown in the forthcoming Part II.

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### 1 Introduction

One of the well-known problems of discrete geometry concerning finite circle arrangements in the plane is the following [5]: How must  $n$  equal circles be packed in the unit circle without overlapping so that the radius of the circles will be as large as possible? This *packing* problem has a dual counterpart in *covering*: How must the unit circle be covered by  $n$  equal circles without interstices so that the radius of the circles will be as small as possible? For a given  $n$ , let  $r_{\max}^{(n)}$  and  $R_{\min}^{(n)}$  denote the maximum radius in the packing problem and the minimum radius in the covering problem, respectively. For circles with radius  $r$  such that  $r_{\max}^{(n)} \leq r \leq R_{\min}^{(n)}$ , Zahn [40] posed a problem *intermediate* between these two: How must  $n$  equal circles of given radius  $r$  be arranged so that the area of the part of the unit circle covered by the  $n$  circles will be as large as possible? This intermediate problem is the topic of the present paper. Here and throughout the paper, following Fejes Tóth [14], we use the term *circle* also for a circular disc.

Proven solutions to the packing problem have been established for values of  $n$  up to 10 by Pirl [32], for  $n = 11$  by Melissen [29], and for  $n = 12, 13, 14$  and 19 by Fodor [18, 19, 20, 17]. Conjectured solutions are known for  $15 \leq n \leq 18$  and for  $20 \leq n \leq 25$  [11]. Graham and co-workers [25] extended this range of  $n$  and provided putative solutions up to  $n = 65$ . The covering problem is somewhat more difficult than the packing problem. Certainly, fewer results are known for the covering problem. Although the solution is trivial up to  $n = 4$  and for  $n = 7$  further solutions are settled only up to  $n = 10$ : proven solutions have been provided for  $n = 5$  and 6 by Bezdek [4, 3], and for  $n = 8, 9, 10$  by Fejes Tóth [16]. Conjectural solutions have been given for  $n = 11$  and 12 by Melissen [30] (improved

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by Nurmela [31]) and for  $13 \leq n \leq 35$  by Nurmela [31]. The intermediate problem is the least tractable of the three. To our knowledge, the only paper dedicated to the intermediate problem in the plane is that by Zahn [40] who provided a proven solution for  $n = 2$  (for  $0.5 < r < 1$ ) and conjectured solutions for  $3 \leq n \leq 10$  (for between three and six selected values of  $r$ ). Apart from these results there are not even conjectures for other values of  $n$ . (It should be noted that a similar problem concerning partial covering of the spherical surface by  $n$  equal circles was raised and proven solutions for  $n = 2, 3, 4, 6, 12$  were established by Fejes Tóth [14, 15], with putative solutions for  $n = 5, 7$  to 11, and 13 produced later by Fowler and Tarnai [21, 22].)

Recently Connelly [9] studied the intermediate problem. He considered the case of  $n = 5$  as an example, and wanted to know how, for a continuous increase in  $r$ , the circle configuration changes in the transition from the maximum packing to the minimum covering. If  $r$  is close to the maximum packing radius, then the circles have only double overlaps. In this case, the maximum area can be determined with a formula of Csikós [12]. Connelly [9] worked out a stress interpretation of Csikós's formula, and showed how a tensegrity structure can be associated with the maximum area configuration. However, if  $r$  is close to the minimum covering radius then the circles can have some triple overlaps for which, although Csikós's formula remains valid, it is not known how to construct an adequate mechanical model to represent the maximum area configuration.

The primary aim of this paper is to set up a mechanical model to analyse the intermediate problem, which works also for triple overlaps, and by which the solution to the problem can be found, or which results in an at least locally optimum arrangement. In the case of triple overlaps, the equivalent mechanical model of the maximum area configuration of circles is a generalized tensegrity structure, a new feature of which is that, in addition to struts and cables, it contains also triangle elements (known as in-plane loaded plate elements in finite element techniques).

In the investigations we consider the intermediate problem in an analogous mechanical interpretation. The area function is very complicated and the determination of its critical point and its maximum is quite difficult. The advantage of the mechanical analogy is that it simplifies the process to find the solution to the problem, and provides some additional information about the solution. Instead of determining the extremum of the area function, equilibrium of forces is investigated. In structural mechanics, there are known iteration methods to find equilibrium configurations, which in the intermediate problem directly provide the critical point of the area function. In these methods the (tangent) stiffness matrices play a crucial role [28].

An additional aim is to introduce mechanical models also for solving the packing and covering problems. These models, however, are not entirely new in principle, because we have already used similar mechanical models for finding (locally) optimal circle packings and coverings in other domains (e.g. polygons, 2-sphere) [35, 36, 37, 38]. These models are different a little from those used in [2, 7].

An example of application of the mechanical models will be shown in Part II [24] where conjectured solutions to the intermediate problem for  $n = 5$  are given as  $r$  varies from the maximum packing radius to the minimum covering radius.

The outline of the rest of the paper is the following. In Section 2, we define the basic terms and introduce the basic relationships we use in the investigations of the generalized tensegrities. We present mechanical models used for packing (Section 3) and covering (Section 4) constructions. In Section 5, the generalized tensegrity model used for determination of the optimal partial coverings is presented, and formulae of internal forces and stiffness matrices of elements (cables, struts, triangle elements) of the generalized tensegrity are given. We show, for given value of circle radius  $r$ , how an equilibrium arrangement can be determined by the method of dynamic relaxation; how the individual equilibrium configurations can be characterized from the point of view of stability, and how the singular points of the equilibrium paths can be calculated.

## 2 Basic definitions and basic relationships

### 2.1 Tensegrity

The definitions below are taken from [8, 10]. Consider a finite *configuration* of points  $\mathbf{p}_i, i = 1, 2, \dots, v$  in the Euclidean  $d$ -dimensional space  $\mathbb{R}^d$ . This is denoted as  $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_v)$ . A *tensegrity graph*  $G$  is an abstract graph on the vertices  $1, 2, \dots, v$  (with no loops and multiple edges), where each edge is designated as either a *cable*, a *strut*, or a *bar*. A *realization* of  $G$ , where a point  $\mathbf{p}_i$  is assigned for the  $i$ th vertex of  $G$ , is called a *tensegrity framework* and it is denoted as  $G(\mathbf{p})$ . An edge of  $G$  (and consequently that of  $G(\mathbf{p})$  as well) is called a *member* or *element*. In  $G(\mathbf{p})$ , cables cannot increase in length, struts cannot decrease in length, and bars can neither increase nor decrease in length. In structural mechanics, it is said that cables are members only able to resist tension, struts are members only able to resist compression, and bars are members able to resist both tension and compression. If all edges of the graph  $G$  are only bars,  $G(\mathbf{p})$  is called a *bar framework* that, in structural mechanics, is known as a *pin-jointed framework*, a *truss* or a *bar-and-joint assembly* [34].

A *stress*  $\omega$  on a tensegrity framework  $G(\mathbf{p})$  is an assignment of a scalar  $\omega_{ij} = \omega_{ji}$  for each edge  $\{i, j\}$  of  $G$ , where  $\omega = (\dots, \omega_{ij}, \dots)$ . A stress  $\omega$  on a tensegrity framework is a *self-stress* if the following equilibrium equation holds at each vertex  $i$ :

$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0 \quad (1)$$

where the sum is taken over all vertices  $j$  adjacent to  $i$ . A *proper self-stress* is a stress  $\omega$  such that  $\omega_{ij} \geq 0$  if  $\{i, j\}$  is a cable, and  $\omega_{ij} \leq 0$  if  $\{i, j\}$  is a strut (there is no condition on a bar). In this way self-stress  $\omega$  (as a row vector) is a solution to the set of homogeneous linear equations  $\omega \mathbf{R}(\mathbf{p}) = 0$ , where the  $e$ -by- $dv$  matrix  $\mathbf{R}(\mathbf{p})$  is called the *rigidity matrix*. Here  $e$  is the number of edges of  $G$ .

Structural mechanics [34] provides a somewhat different interpretation of the above-mentioned vector equilibrium. Though the scalar  $\omega_{ij}$  is known as *tension coefficient* or *force density* of member  $\{i, j\}$  in structural mechanics, it is used only in special engineering problems. Since *force* is a central term, tensegrity frameworks (tensegrity structures) are

also investigated in terms of member forces which are closely related to the scalars  $\omega_{ij}$ . Let  $L_{ij}$  denote the length of the member  $\{\mathbf{p}_i, \mathbf{p}_j\}$  of the tensegrity framework:  $L_{ij} = L_{ji} = |\mathbf{p}_j - \mathbf{p}_i|$ , and let  $S_{ij} = S_{ji}$  denote the force in that member. The relationship between  $\omega_{ij}$  and  $S_{ij}$  is  $\omega_{ij} = S_{ij}/L_{ij}$ . Introducing this into the equation (1) we have

$$\sum_j S_{ij} \mathbf{e}_{ij} = 0 \quad (2)$$

where  $\mathbf{e}_{ij}$  is the unit vector pointing from point  $\mathbf{p}_i$  to point  $\mathbf{p}_j$ , and the sum is taken over all vertices  $j$  adjacent to  $i$ . Obviously  $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$ . By arranging the member forces into a vector  $\mathbf{s}^T = [\dots S_{ij} \dots]$ , where  $(\cdot)^T$  is the transpose operation, the set of equilibrium equations is obtained in the form  $\mathbf{A}\mathbf{s} = 0$  where  $\mathbf{A}$  is a  $dv$ -by- $e$  matrix called *equilibrium matrix*. We note here that the transpose of  $\mathbf{A}$  is called *compatibility matrix*, and it is related (but not identical) to the rigidity matrix  $\mathbf{R}(\mathbf{p})$ . In case  $\mathbf{A}\mathbf{s} = 0$  is fulfilled, it is said that the structure is in a *state of self-stress*.

Usually, it is not easy to find a realization of  $G$ , which has a proper self-stress. For problems of maximum packing and minimum covering with equal circles, however, even the tensegrity graph itself, associated with the circle arrangement, is unknown. Determination of the graph  $G$  and its realization with a proper self-stress is usually possible with an iteration process under certain constraints. Such a constraint is that the member lengths are all equal. All members are struts for packing, and all members are cables for covering. The iteration starts with a configuration of  $v$  points  $\tilde{\mathbf{p}} = (\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \dots, \tilde{\mathbf{p}}_v)$ , and with a properly selected graph  $\hat{G}$  where its representation  $\hat{G}(\tilde{\mathbf{p}})$  has no self-stress. In the case of packing (covering), by a uniform small increase (decrease) in the length of all members, a new configuration and a new representation of  $\hat{G}$ , without self-stress, is obtained. By repeating this step, a configuration is arrived at where the representation of  $\hat{G}$  has a self-stress. It also can occur that self-stress appears only after we modify the graph  $\hat{G}$  with adding or removing edges to or from it. It can happen that the detected self-stress is not proper. If it is so, then the strut  $\{i, j\}$  for which  $\omega_{ij} > 0$  (the cable  $\{i, j\}$  for which  $\omega_{ij} < 0$ ) should be removed from the graph, and the process should be repeated for this modified graph until the obtained tensegrity framework has a proper self-stress.

In structural mechanics, cables and struts represent unilateral constraints between joints which are connected by them, and the relationships for them can be written in the form of inequalities. Bars represent bilateral constraints, and the relationships for them can be written in the form of equations. In actual numerical calculations, it is more advantageous and simpler to use equations instead of inequalities. Therefore, bars are used for both cables and struts, and at the end of the calculation it is checked whether a tensional bar force is in the respective cable, and a compression bar force is in the respective strut. This is why all edges of the graphs associated with packings and coverings of equal circles are designated as bars.

If we imagine that the members of the respective tensegrity framework are equal bars made of steel, for instance, then each bar can expand at the same rate that the *temperature* of the bar increases. According to this thermal effect, a concerted equal rise in temperature (that is to say uniform “*heating*”) of all bars causes a uniform expansion of all bars, which

simply corresponds to an increase in the radius of the circles in packing. Similarly, a uniform drop in temperature (that is to say uniform “cooling”) corresponds to a uniform contraction of all bars, and so to a decrease in the radius of the circles in covering.

In the case of the circle packing problem, it is important that for the tensegrity framework with a proper self-stress, the compression forces in the bars form a *stable* state of self-stress. Let the tensegrity framework have  $v$  joints and  $e$  members. Since here  $d = 2$ ,  $e > 2v$ . The state of self-stress is stable if the rank of the  $2v$ -by- $e$  equilibrium matrix  $\mathbf{A}$  is equal to  $2v$ .

## 2.2 Generalized tensegrity

Tensegrity structures can be generalized in several different ways. We consider the generalization that is used in the problem of the maximum partial covering by equal circles.

Start with a tensegrity framework  $G(\mathbf{p})$  defined by the configuration  $\mathbf{p}$  of points  $\mathbf{p}_i \in \mathbb{R}^d, i = 1, 2, \dots, v$ , and by the graph  $G$  on the vertices  $i = 1, 2, \dots, v$ , where each edge of  $G$  is designated as a cable or a strut. If the vertices  $i, j, k$  of  $G$  are mutually adjacent, that is, edges  $\{i, j\}, \{j, k\}, \{k, i\}$  form a triangle, and if these edges are designated as struts, then a triangle  $T(i, j, k)$ , called a *triangle element*, can be added to the graph  $G$  where the vertices of  $T(i, j, k)$  coincide with the vertices  $i, j, k$ . Formally, the graph  $G$  is supplemented with the edges  $\{i, j\}^k, \{j, k\}^i, \{k, i\}^j$  of triangle  $T(i, j, k)$ , and the obtained graph has three double edges. If  $G$  has more triangles, then it can be supplemented with more triangle elements. The resulting graph, denoted by  $G_T$ , has some multiple edges. A realization of  $G_T$ , where a point  $\mathbf{p}_i$  is assigned for the  $i$ th vertex of  $G_T$ , is called a *generalized tensegrity structure* and it is denoted as  $G_T(\mathbf{p})$ .

A *state of stress* of a generalized tensegrity structure  $G_T(\mathbf{p})$  is an assignment of a scalar force  $S_{ij} = S_{ji}$  for each edge  $\{i, j\}$  of  $G_T$  (that is, for each member  $\{i, j\}$  of the underlying  $G$ ), and a scalar force  $Q_{ij}^k = Q_{ji}^k$  for the edge  $\{i, j\}^k$  of each triangle element  $T(i, j, k)$ . Here, the forces  $S_{ij}$  and  $Q_{ij}^k$  cannot be assigned arbitrarily, since they are dependent on lengths. (At circle packings and coverings, there are constraints for the member lengths, but the magnitudes of the member forces can be given relatively freely. At partial covering, the lengths are relatively free to be assigned, but the magnitudes of the member forces and edge forces depend on the lengths.) The member force  $S_{ij}$  is a function of the length  $L_{ij}$  of the member  $\{\mathbf{p}_i, \mathbf{p}_j\}$ , and the edge force  $Q_{ij}^k$  is a function of the edge lengths  $L_{ij}, L_{jk}, L_{ki}$  of the triangle  $T(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)$ . A *state of self-stress* on a generalized tensegrity structure is a state of stress, if equilibrium holds for each vertex  $i$  of  $G_T(\mathbf{p})$ :

$$\sum_j \left( S_{ij} + \sum_k Q_{ij}^k \right) \mathbf{e}_{ij} = 0 \quad (3)$$

where the sum for  $j$  is taken over all vertices adjacent to  $i$ , and the sum for  $k$  is taken over all triangle elements containing edge  $\{i, j\}$ . A *proper state of self-stress* is a state of self-stress where  $S_{ij} \geq 0$  if  $\{i, j\}$  is a cable,  $S_{ij} \leq 0$  if  $\{i, j\}$  is a strut, and  $Q_{ij}^k \geq 0$  if  $\{i, j\}^k$  is an edge of the triangle element  $T(i, j, k)$ . Denote the sum of the forces appearing between vertices  $i$  and  $j$  by  $s_{ij}$ :

$$s_{ij} = S_{ij} + \sum_k Q_{ij}^k$$

and arrange the forces  $s_{ij}$  into a vector  $\hat{\mathbf{s}}^T = [\dots s_{ij} \dots]$ , then the set of homogeneous linear equations  $\mathbf{A}\hat{\mathbf{s}} = 0$  is obtained where  $\mathbf{A}$  is the equilibrium matrix based on the original (non-generalized) graph  $G$ .

In the case of the problem of the maximum partial covering of the unit circle by  $n$  equal circles, the generalized tensegrity structure associated with the circle arrangement in this problem is constructed in the following way. Let  $d = 2$ , and  $v = n + 1$ , then  $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$  are the centres of the equal circles (discs)  $C_1, C_2, \dots, C_n$  of radius  $r$ , and  $\mathbf{q}_{n+1}$  is the centre of the unit circle (disc)  $C_{n+1}$ . The member  $\{\mathbf{q}_i, \mathbf{q}_j\}$  is a cable, if  $i \leq n, j = n + 1$ . The member  $\{\mathbf{q}_i, \mathbf{q}_j\}$  is a strut, if  $i \leq n, j \leq n$ . The triangle  $T(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)$  is a triangle element if  $i \leq n, j \leq n, k \leq n$ . A tensional force  $S_{ij} > 0$  appears in the cable  $\{\mathbf{q}_i, \mathbf{q}_{n+1}\}$  if  $C_i$  and the exterior of the unit circle  $C_{n+1}$  intersect (have an external overlap  $C_i \cap C_{n+1}^c \neq \emptyset$ ). A compression force  $S_{ij} < 0$  appears in the strut  $\{\mathbf{q}_i, \mathbf{q}_j\}$  if circles  $C_i$  and  $C_j$  intersect (have a double overlap  $C_i \cap C_j \neq \emptyset$ ). Tensional edge forces  $Q_{ij}^k > 0, Q_{jk}^i > 0, Q_{ki}^j > 0$  appear in the edges  $\{\mathbf{q}_i, \mathbf{q}_j\}^k, \{\mathbf{q}_j, \mathbf{q}_k\}^i, \{\mathbf{q}_k, \mathbf{q}_i\}^j$ , if circles  $C_i, C_j, C_k$  have a part in common (have a triple overlap  $C_i \cap C_j \cap C_k \neq \emptyset$ ).

Overlaps determine parts that could be removed from the equal circles without changing the coverage of the unit circle. The sum of their areas is called *surplus area*, and is denoted as  $A(\mathbf{q})$  for a configuration  $\mathbf{q}$  of the centres of the circles. For a given radius  $r$  of the equal circles, the mathematical task is to find the configuration  $\mathbf{p}$  where the surplus area  $A(\mathbf{p})$  is a minimum. In order that function  $A(\mathbf{q})$  have a local extremum at point  $\mathbf{p}$ , a necessary condition is that  $\mathbf{p}$  be a critical (or stationary) point, that is,  $\text{grad}A(\mathbf{p}) = \mathbf{0}$ . The function  $A(\mathbf{q})$  has a local minimum at  $\mathbf{p}$  if here, additionally, the Hessian matrix  $\mathbf{H}(A(\mathbf{p}))$  of the surplus area function is positive definite.

Since  $A(\mathbf{q})$  is a positive-valued function, the above mentioned mathematical problem has a mechanical formulation. Consider a generalized tensegrity structure  $G_T(\mathbf{q})$  defined on the configuration  $\mathbf{q}$ . Let  $A(\mathbf{q})$  be the internal potential energy for any configuration  $\mathbf{q}$ . The first derivatives of the overlap areas with respect to the respective member lengths (edge lengths) define member forces (edge forces). If  $\mathbf{p}$  is a critical point for the potential energy function  $A$ , then the member forces and edge forces provide a vector equilibrium at each vertex according to (3).

The generalized tensegrity is in a *stable state of self-stress*, if at point  $\mathbf{p}$ , the potential energy  $A$  has a local minimum. According to this, the equilibrium is *stable* if the Hessian matrix  $\mathbf{H}(A(\mathbf{p}))$ , called *tangent stiffness matrix* in structural mechanics [1, 28], is positive definite. We note that, in this case, in the rigidity theory of mathematics, it is said that the (generalized) tensegrity is *prestress stable*. This stability principle is different from that used in circle coverings [2]. The equilibrium is *unstable*, if at point  $\mathbf{p}$  the potential energy  $A$  has no local minimum, so if  $\mathbf{H}(A(\mathbf{p}))$  has a negative eigenvalue. The equilibrium is *critical* if  $\mathbf{H}(A(\mathbf{p}))$  is singular.

Let  $A_{ij}(\mathbf{q}_i, \mathbf{q}_j)$ , and  $A_{ijk}(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)$  denote the area of an external or a double overlap, and a triple overlap, respectively. Then the surplus area

$$A = \sum A_{ij}(\mathbf{q}_i, \mathbf{q}_j) - \sum A_{ijk}(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k).$$

Here the sum should be extended over all overlaps, but one overlap can be considered only once. Thus, say, it is supposed that  $i < j < k$ , and so the sum cannot contain both  $A_{ij}$  and  $A_{ji}$ . The relationships will be simpler if the areas are expressed with the member lengths and edge lengths:

$$A = \sum A_{ij}(L_{ij}(\mathbf{q}_i, \mathbf{q}_j)) - \sum A_{ijk}(L_{ij}(\mathbf{q}_i, \mathbf{q}_j), L_{jk}(\mathbf{q}_j, \mathbf{q}_k), L_{ki}(\mathbf{q}_k, \mathbf{q}_i)).$$

At generating the  $l$ th block of the gradient of function  $A$ , that is at  $\partial A / \partial \mathbf{q}_l$ , each overlap area, associated with a member or triangle element which is connected to vertex  $l$ , should be differentiated. For a member  $\{\mathbf{q}_i, \mathbf{q}_j\}$  (if  $i = l$ , or  $j = l$ )

$$\frac{\partial A_{ij}}{\partial \mathbf{q}_l} = \frac{dA_{ij}}{dL_{ij}} \frac{\partial L_{ij}}{\partial \mathbf{q}_l},$$

for a triangle element  $T(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)$  (if, for instance,  $i = l$ )

$$\frac{\partial A_{ijk}}{\partial \mathbf{q}_l} = \frac{\partial A_{ijk}}{\partial L_{ij}} \frac{\partial L_{ij}}{\partial \mathbf{q}_l} + \frac{\partial A_{ijk}}{\partial L_{ki}} \frac{\partial L_{ki}}{\partial \mathbf{q}_l}.$$

If we introduce member forces  $S_{ij}$  and edge forces  $Q_{ij}^k$

$$S_{ij} = \frac{dA_{ij}}{dL_{ij}}, \quad Q_{ij}^k = \frac{\partial A_{ijk}}{\partial L_{ij}}, \quad (4)$$

and we take into account that

$$\frac{\partial L_{ij}}{\partial \mathbf{q}_i} = -\mathbf{e}_{ij}^T, \quad \frac{\partial L_{ij}}{\partial \mathbf{q}_j} = \mathbf{e}_{ij}^T$$

then the blocks of the gradient of  $A$  result in the equilibrium equations of form (3).

The Hessian matrix of function  $A$ , that is, the tangent stiffness matrix of the generalized tensegrity is also produced from the Hessian matrices of the overlap areas associated with members and triangle elements. For instance, the block in the principal diagonal of the Hessian matrix for a member  $\{\mathbf{q}_i, \mathbf{q}_j\}$  (if  $i = l$ ) is

$$\frac{\partial \left( \frac{\partial A_{ij}}{\partial \mathbf{q}_l} \right)^T}{\partial \mathbf{q}_l} = -\frac{\partial \mathbf{e}_{ij}}{\partial \mathbf{q}_l} S_{ij} - \mathbf{e}_{ij} \frac{dS_{ij}}{dL_{ij}} \frac{\partial L_{ij}}{\partial \mathbf{q}_l},$$

where

$$\frac{\partial \mathbf{e}_{ij}}{\partial \mathbf{q}_l} = \frac{1}{L_{ij}} (\mathbf{E} - \mathbf{e}_{ij} \mathbf{e}_{ij}^T).$$



Here  $\mathbf{E}$  is the second-order unit matrix. If  $j = l$ , then the same expression is obtained. For the block off the principal diagonal, the same expression is obtained but with the negative sign. In this way, the stiffness matrix of a single cable or strut  $\{\mathbf{q}_i, \mathbf{q}_j\}$  has the simple form

$$\mathbf{K}_{ij} = \begin{bmatrix} \mathbf{e}_{ij} \\ -\mathbf{e}_{ij} \end{bmatrix} \left[ \frac{dS_{ij}}{dL_{ij}} \right] \begin{bmatrix} \mathbf{e}_{ij}^T & -\mathbf{e}_{ij}^T \end{bmatrix} + \frac{S_{ij}}{L_{ij}} \begin{bmatrix} \mathbf{E} - \mathbf{e}_{ij}\mathbf{e}_{ij}^T & -\mathbf{E} + \mathbf{e}_{ij}\mathbf{e}_{ij}^T \\ -\mathbf{E} + \mathbf{e}_{ij}\mathbf{e}_{ij}^T & \mathbf{E} - \mathbf{e}_{ij}\mathbf{e}_{ij}^T \end{bmatrix}.$$

This matrix can also be written with the help of member stress  $\omega_{ij}$  [26]:

$$\mathbf{K}_{ij} = \begin{bmatrix} \mathbf{q}_j - \mathbf{q}_i \\ \mathbf{q}_i - \mathbf{q}_j \end{bmatrix} \left[ \frac{1}{L_{ij}} \frac{d\omega_{ij}}{dL_{ij}} \right] \begin{bmatrix} (\mathbf{q}_j - \mathbf{q}_i)^T & (\mathbf{q}_i - \mathbf{q}_j)^T \end{bmatrix} + \omega_{ij} \begin{bmatrix} \mathbf{E} & -\mathbf{E} \\ -\mathbf{E} & \mathbf{E} \end{bmatrix}.$$

If the equal circles have no triple overlaps, then the tangent stiffness matrix of the entire structure, expressed with member forces, is

$$\mathbf{K}(\mathbf{q}) = \mathbf{A}(\mathbf{q})\mathbf{C}\mathbf{A}(\mathbf{q})^T + \mathbf{G}$$

where  $\mathbf{A}(\mathbf{q})$  is the  $2(n+1)$ -by- $e$  equilibrium matrix,  $\mathbf{C} = \langle \dots \frac{dS_{ij}}{dL_{ij}} \dots \rangle$  is an  $e$ -by- $e$  diagonal matrix, the symmetric  $2(n+1)$ -by- $2(n+1)$  matrix  $\mathbf{G}$  is the so called *geometric matrix* [33]. The same tangent stiffness matrix, expressed with member stresses, is

$$\mathbf{K}(\mathbf{q}) = \mathbf{R}(\mathbf{q})^T \mathbf{D} \mathbf{R}(\mathbf{q}) + \mathbf{\Omega} \otimes \mathbf{E}$$

where  $\mathbf{D} = \langle \dots \frac{1}{L_{ij}} \frac{d\omega_{ij}}{dL_{ij}} \dots \rangle$  is an  $e$ -by- $e$  diagonal matrix;  $\mathbf{\Omega}$  is an  $(n+1)$ -by- $(n+1)$  symmetric matrix, the so called *stress matrix* [6], whose  $(i, j)$ th entry is  $-\omega_{ij}$  if  $i \neq j$ , and  $\sum_k \omega_{ik}$  if  $i = j$  (in the entries,  $\omega_{ij} = 0$  if  $i$  and  $j$  are not connected by a member). The symbol  $\otimes$  denotes the tensor product of two matrices. If the equal circles have triple overlaps, the tangent stiffness matrix  $\mathbf{K}(\mathbf{q})$  becomes more complicated.

The tangent stiffness matrix has twofold importance in the partial covering problem.

(i) For an approximate configuration  $\mathbf{q}$ , the member forces and edge forces are determined by (4), and the force vectors at vertex  $i$ , in general are not in equilibrium, but their resultant is a nodal force (nodal load)  $\mathbf{f}_i$  which is not equilibrated

$$\sum_j \left( S_{ij} + \sum_k Q_{ij}^k \right) \mathbf{e}_{ij} = \mathbf{f}_i.$$

Let  $\mathbf{f}$  denote the vector of the nodal forces:  $\mathbf{f}^T = [\mathbf{f}_1^T \quad \mathbf{f}_2^T \quad \dots \quad \mathbf{f}_{n+1}^T]$ . We eliminate  $\mathbf{f}$  by changing the configuration  $\mathbf{q}$ . For this, the Newton-Raphson method is used where the tangent stiffness matrix  $\mathbf{K}(\mathbf{q})$  is the derivative. The  $m$ th step of the iteration takes the form  $\mathbf{q}^{m+1} = \mathbf{q}^m - \mathbf{K}^{-1}(\mathbf{q}^m)\mathbf{f}(\mathbf{q}^m)$ .

(ii) If  $\mathbf{f}$  is vanished, then  $\mathbf{q} = \mathbf{p}$ , and equations (3) are fulfilled. The generalized tensegrity associated with the partial circle covering is in a stable state of self-stress if  $\mathbf{K}(\mathbf{p})$  is positive definite. If it is so, it means that the surplus area has a local minimum at  $\mathbf{p}$ .



### 3 Model of the maximum packing

The optimal (maximum) packing can be found with uniform “heating” of a planar bar-and-joint assembly associated with the circle system. The joints of the assembly are the centres of the equal circles, and the bars connect the centres of two circles touching each other. Thus the length of each bar is  $2r$ . The joints at a distance of  $1-r$  from the centre of the unit circle are supported against displacement in the radial direction. One of these joints is also supported against displacement in the tangential direction in order to prevent rotation of the whole assembly about the centre of the unit circle. As an example, the bar-and-joint assembly associated with the optimal packing of seven circles is shown in Fig. 1.

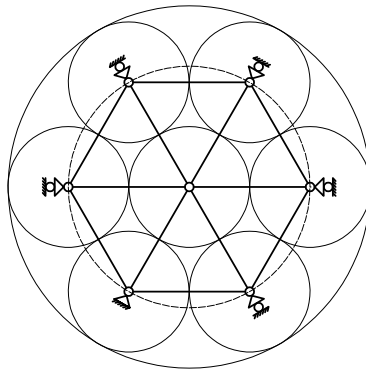


Figure 1: The bar-and-joint assembly associated with the maximum packing of  $n = 7$  circles

If such a bar-and-joint assembly subjected to uniform heating is able to change its state with the bars remaining free of stress, then the bars become longer and we find a packing of non-overlapping circles for a greater radius. Hence, the starting configuration cannot have been optimal. (We note here that, owing to heating, the radius of the circle passing through the supported joints decreases; we must therefore check whether the distance between two such joints, which was formerly greater than  $2r$ , has become equal to the new diameter of the circles. If this has happened, then the structure should be supplemented with a new bar.)

If a rise in temperature causes stresses in bars, then any bars in tension can be removed from the bar-and-joint assembly, and the heating procedure can be repeated for the modified structure. If the associated bar-and-joint assembly is in a stable state of self-stress such that there is no tensile force among the forces in bars, then the circle arrangement is locally optimal. For example, forces in all bars of the bar-and-joint assembly in Fig. 1 are equal compressive forces and form a stable state of self-stress. It should be noted that, in the case of a locally maximum packing, all bars can be replaced with struts, since it is more natural to consider a strut framework (as a tensegrity framework with a proper self-stress) as in [7].

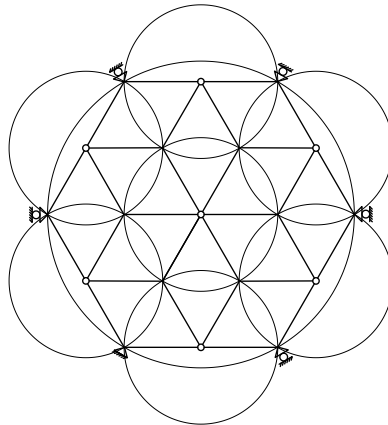


Figure 2: The bar-and-joint assembly associated with the minimum covering by  $n = 7$  circles

#### 4 Model of the minimum covering

The optimal (minimum) covering can be found with uniform “cooling” of a planar bar-and-joint assembly associated with the circle system. The network of the bar-and-joint assembly is a bipartite graph. This follows from the fact that the assembly has two kinds of joints, connected in a specific way. The joints of the first kind are the centres of the equal circles. The joints of the second kind are those points where the boundaries of three or more equal circles (or the boundary of the unit circle and the boundaries of two or more equal circles) intersect, and which are not interior points of any of the equal circles. The bars are always inside a circle and connect a joint of the first kind to one of the second. Thus the length of each bar is  $r$ . The joints on the boundary of the unit circle are supported against displacement in the radial direction. One of these joints is also supported against displacement in the tangential direction. As an example, the bar-and-joint assembly associated with the optimal covering of the unit circle by seven circles is shown in Fig. 2, where the joints of the first kind are marked with small circles.

If such a bar-and-joint assembly subjected to uniform cooling is able to change its state in such a way that the bars remain free of stress, then the bars become shorter and we have found circle covering without interstices for a smaller radius. Hence, in this case, the starting configuration cannot have been optimal.

We note, if  $k > 3$  bars are connected to a joint of the second kind, then this joint is considered as a multiple joint that should be split into  $k - 2$  joints such that, multiplying the connecting bars appropriately, 3 bars will be connected to each of the joints obtained by splitting. The idea of multiplying (splitting) was introduced independently by [2] and [37]. Multiplying the connecting bars is appropriate if

either

(i) there exists a new joint where, among the angles between the connecting bars, there is no angle greater than  $\pi$  (and of course, angles have been defined so that the third bar is not inside the angular domain); in this case the unit circle will remain covered,

or

(ii) there are two new joints, where at each joint, two of the connecting bars (and therefore all four bars) coincide with the same straight line, and the third bars are on different sides of that line.

These joints, which were in coincidence, can become separated during cooling. Different realizations of bar multiplication can lead to different configurations. From these finitely many configurations, the best should be selected.

Figure 3 presents a detail of a covering where 5 circles intersect at a point. Slightly distorted, the 3 bars connected to each of the new joints are also shown.

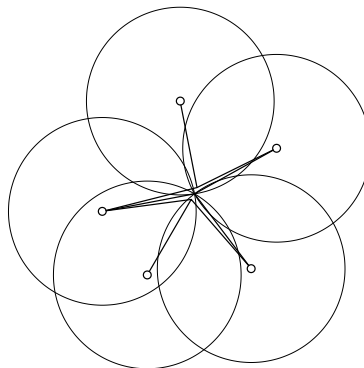


Figure 3: Detail of the bar-and-joint assembly

If a drop in temperature causes stresses in bars, then the bars in compression can be removed from the bar-and-joint assembly, and the cooling procedure can be repeated for the modified structure. If for a circle arrangement, the associated bar-and-joint assembly is in a state of self-stress such that there is no compressive force among the forces in bars, then the arrangement is locally optimal. The above-mentioned multiplication of joints and bars is effective only if the bars meeting at the joint carry non-zero stress in the state of self-stress. It should be noted that, in the case of a locally minimum covering, all bars can be replaced with cables, since it is more natural to consider a cable framework (as a tensegrity framework with a proper self-stress) as in [2].

## 5 Model of the maximum partial covering

### 5.1 Construction of the model

If radius  $r$  of the equal circles lies in the interval  $(r_{\max}^{(n)}, R_{\min}^{(n)})$  then, at the optimal arrangement, the equal circles can intersect the unit circle and also can intersect each other, but the unit circle will have a part uncovered by circles. The sum of the areas of the  $n$  equal circles is given, and hence minimizing the uncovered area is the same problem as minimizing the sum of two areas: the area of the unit circle that is multiply covered by the equal circles, the area of the parts of the equal circles that lie outside the unit circle. If a certain area is covered by each of  $k$  equal circles, then this area has a  $(k - 1)$ -fold surplus coverage. For the sake of simplicity it is supposed that, in the investigated arrangements, at most three circles cover the same area, that is, the equal circles can have at most triple overlaps.

The proposed mechanical model is composed of three different types of elements: (a) cable, (b) strut, (c) triangle element. A *cable* element connects the centre of the unit circle with the centre of one of the equal circles. It is active (that is, a tensile force arises in it) if the circle intersects the unit circle. A *strut* element connects the centres of two equal circles. It is active (that is, a compressive force arises in it) if the two equal circles intersect. A *triangle element* connects the centres of three equal circles. It is active (that is, tensile force arises at least in one of the edges of the triangle) if the three equal circles have a non-zero area in common.

The centre of the unit circle is supported against all displacements, but the centre of one of the equal circles is supported against tangential displacement. Figure 4 shows the active elements in a (non-optimal) arrangement. Cables and struts are represented by dashed lines and solid lines, respectively, the only triangle element is represented by a shaded triangle.

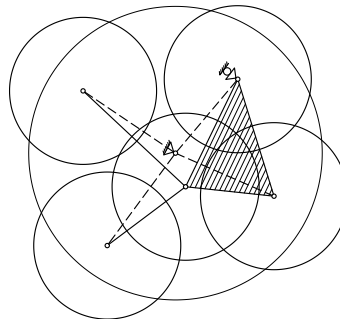


Figure 4: Active elements of the model of partial covering

If it is not known in advance which equal circles will intersect the unit circle and each other, then it is possible to consider all cables, struts and triangle elements simultaneously, but then many of the internal forces will be equal to zero. If for given values of  $n$  and  $r$  the whole model is in a stable state of self-stress, then, according to the explanation given in 2.2, the arrangement is locally optimal. This means that the surplus area has a local

minimum and the covered area of the unit circle has a local maximum.

## 5.2 Constitutive equations

### (a) Cable element

In the case where the unit circle and one of the equal circles intersect, the magnitude of the tensile force arising in the cable is defined as the derivative with respect to the cable length of the area of the part of the circle lying outside the unit circle. This is exactly equal to the length of the chord connecting the two points of intersection. Thus, the magnitude of the force  $S$  arising in the cable of length  $L$  is

$$S = \begin{cases} 0 & \text{if } L \leq 1 - r \\ \sqrt{4L^2 - (1 + L^2 - r^2)^2}/L & \text{if } L > 1 - r \end{cases}, \quad (5)$$

and the cable length  $L_s$  corresponding to cable force  $S$  is

$$L_s = \begin{cases} L_r & \text{if } S = 0 \\ \sqrt{1 + r^2 - S^2/2 - \sqrt{4r^2 - S^2 - r^2S^2 + S^4/4}} & \text{if } S > 0 \end{cases} \quad (6)$$

where  $L_r$  is the distance between the joints connected by the (passive) cable.

### (b) Strut element

In the case where two equal circles intersect, the magnitude of the compressive force arising in the strut is defined as the derivative of the common area with respect to the strut length. This is equal to the length of the chord connecting the two points of intersection. Thus, the magnitude of the force  $S$  arising in the strut of length  $L$  is

$$S = \begin{cases} -\sqrt{4r^2 - L^2} & \text{if } L < 2r \\ 0 & \text{if } L \geq 2r \end{cases}, \quad (7)$$

and the strut length  $L_s$  corresponding to strut force  $S$  is

$$L_s = \begin{cases} \sqrt{4r^2 - S^2} & \text{if } S < 0 \\ L_r & \text{if } S = 0 \end{cases} \quad (8)$$

where  $L_r$  is the distance between the joints connected by the (passive) strut.

### (c) Triangle element

In the case where three equal circles intersect, the magnitude of the tensile force  $Q_i$  arising in the  $i$ th edge of the triangle element is defined as the partial derivative of the common area  $A$  with respect to the  $i$ th edge length:

$$Q_i = \frac{\partial A(L_1, L_2, L_3)}{\partial L_i}.$$

(The force in the edge is a tensile force since, with compressive forces in the strut elements running along the three edges, a surplus coverage of the common area has already been subtracted three times, and so this common area has become uncovered.) For intersection of two equal circles, the strut force is calculated as the chord length. If the common area of these two equal circles is intersected by a third circle, then the force in the strut will be equal to the edge length of the Voronoi cell, corresponding to this strut. Therefore, the respective edge force of the triangle element is equal to the difference between the chord length and edge length of the cell as shown in Fig. 5.

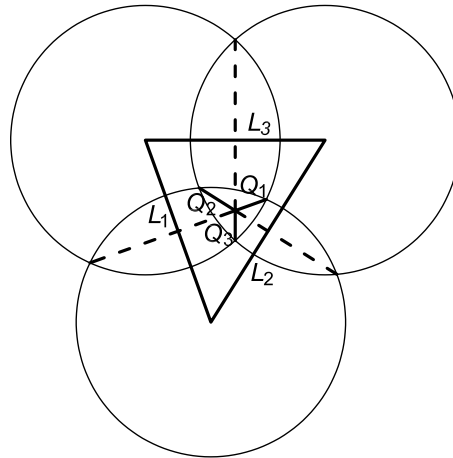


Figure 5: Edge forces of the triangle element

For calculating edge forces, the cyclic order  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$  of subscripts is used. The applied parameters are defined in Fig. 6.

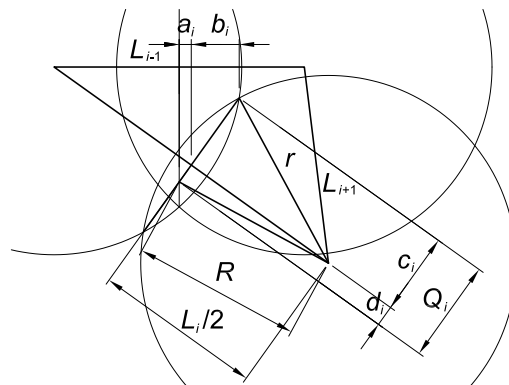


Figure 6: Definition of the applied parameters

Angles of the triangle element are

$$\beta_i = \arccos \frac{L_{i-1}^2 + L_{i+1}^2 - L_i^2}{2L_{i-1}L_{i+1}}.$$

Radius of the circumcircle of the triangle (for any  $i$ ) is

$$R = \frac{L_i}{2\sin\beta_i}.$$

If none of the edge lengths of the triangle element is greater than  $2r$ , then by introducing the parameters

$$a_i = \frac{L_i^2 - L_{i-1}^2 - L_{i+1}^2}{4L_{i-1}}, \quad b_i = c_i \sin\beta_{i+1}, \quad (9a)$$

$$c_i = \sqrt{r^2 - L_i^2/4}, \quad d_i = \sqrt{R^2 - L_i^2/4}, \quad (9b)$$

the edge forces are obtained as

$$Q_i = \begin{cases} 2c_i & \text{if } 0 < a_i - b_i \\ c_i + d_i & \text{if } a_i - b_i < 0 < a_i \\ c_i - d_i & \text{if } a_i < 0 < a_i + b_i \\ 0 & \text{if } a_i + b_i < 0 \end{cases}. \quad (10)$$

From this it follows that, if for all three edges ( $i = 1, 2, 3$ ) the inequality  $a_i + b_i < 0$  holds, then the triangle element is passive. We note that it may happen that only one of the edge forces is different from zero. This happens when the common part of two circles is contained by the third circle, that is, the common part of the three circles is the union of two equal circle segments. In this case, the edge force is just the opposite of the strut force arising in the strut element running along that edge. In such a case, the system of internal forces and the stiffness matrix of the structure remain unchanged, if both the triangle element and the strut element running along the aforementioned edge are considered passive.

### 5.3 Solution with dynamic relaxation

For given  $r$  and  $n$ , the equilibrium arrangement can be determined with the method of dynamic relaxation [13, 39] by using (5), (7) and (10). If  $n$  is fixed, and beginning with  $r_{\max}^{(n)}$ ,  $r$  is increased in small steps (and at each step, the previous position is used for starting the iteration), then points of an equilibrium path are obtained. If beginning with  $R_{\min}^{(n)}$ , we go backwards along the same steps of  $r$ , then we do not always obtain the same arrangement as before at each step. The reason for that is that more than one equilibrium arrangement can correspond to a single value of  $r$ . With the dynamic relaxation, the procedure mostly converges to a stable equilibrium configuration, but sometimes it tends to an unstable configuration close to the starting one. The dynamic relaxation does not investigate stability of the equilibrium configuration. For this investigation, the stiffness matrix of the model should be determined. With the stiffness matrix, also the critical points of the equilibrium path can be determined to an arbitrary desired degree of accuracy.



## 5.4 Element stiffnesses, stiffness matrices

The *stiffness* of an element depends strongly on the actual arrangement. It shows how the internal forces (cable force, strut force, edge forces) of an element change with a (differential) change in the element size. The (tangent) *stiffness matrix* provides the relationship between the differential change in the position of the nodes of the element and the differential change in the nodal force system keeping the element in equilibrium in this position. The stiffness matrix is composed of two parts:

(i) the *primary stiffness matrix* which shows the effect of the change in the magnitude of the internal forces caused by the change in the element size,

(ii) the *secondary stiffness matrix* which shows the effect of the change in the direction of already existing internal forces.

Let us investigate the three types of elements.

### (a) Cable element

Stiffness  $H$  of the element is

$$H = \frac{dS}{dL_s} = \begin{cases} 0 & \text{if } S = 0 \\ \frac{1}{L_s} \left( \frac{2(1 - L_s^2 + r^2)}{S} - S \right) & \text{if } S > 0 \end{cases} \quad (11)$$

The structure of stiffness matrix  $\mathbf{K}$  of the element is of the form

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0 & -\mathbf{K}_0 \\ -\mathbf{K}_0 & \mathbf{K}_0 \end{bmatrix} \quad (12)$$

where the block in the main diagonal is

$$\mathbf{K}_0 = \mathbf{e}\mathbf{e}^T \left( H - \frac{S}{L_r} \right) + \mathbf{E} \frac{S}{L_r}; \quad (13)$$

here  $\mathbf{e}$  is the unit vector pointing from the starting point to the end point of the element, and  $\mathbf{E}$  is the second-order unit matrix.

### (b) Strut element

The stiffness of the element is

$$H = \frac{dS}{dL_s} = \begin{cases} \sqrt{4(r/S)^2 - 1} & \text{if } S < 0 \\ 0 & \text{if } S = 0 \end{cases} \quad (14)$$

The stiffness matrix can be produced according to (8) and (9) also in this case.

### (c) Triangle element

Between the increment of the edge forces and the increment of the edge lengths the third-

order matrix  $\mathbf{H}$  makes relationship:

$$\begin{bmatrix} dQ_1 \\ dQ_2 \\ dQ_3 \end{bmatrix} = \mathbf{H} \begin{bmatrix} dL_1 \\ dL_2 \\ dL_3 \end{bmatrix}. \quad (15)$$

The entries of matrix  $\mathbf{H}$  take the form

$$H_{ij} = \frac{\partial Q_i}{\partial L_j} = \begin{cases} 2 \frac{\partial c_i}{\partial L_j} & \text{if } 0 < a_i - b_i \\ \frac{\partial c_i}{\partial L_j} + \frac{\partial d_i}{\partial L_j} & \text{if } a_i - b_i < 0 < a_i \\ \frac{\partial c_i}{\partial L_j} - \frac{\partial d_i}{\partial L_j} & \text{if } a_i < 0 < a_i + b_i \\ 0 & \text{if } a_i + b_i < 0 \end{cases} \quad (16)$$

where ( $\delta_{ij}$  is the Kronecker symbol)

$$\frac{\partial c_i}{\partial L_j} = \frac{-\delta_{ij} L_i}{4\sqrt{r^2 - L_i^2/4}}, \quad (17a)$$

$$\frac{\partial d_i}{\partial L_j} = \frac{R \frac{\partial R}{\partial L_j} - \delta_{ij} \frac{L_i}{4}}{\sqrt{R^2 - L_i^2/4}}, \quad (17b)$$

here

$$\frac{\partial R}{\partial L_j} = \frac{L_{j-1} L_{j+1} \left( L_j^4 - (L_{j-1}^2 - L_{j+1}^2)^2 \right)}{(2L_2^2 L_3^2 + 2L_1^2 L_3^2 + 2L_1^2 L_2^2 - L_1^4 - L_2^4 - L_3^4)^{3/2}}. \quad (17c)$$

The primary stiffness matrix ( $\mathbf{K}'$ ) of the triangle element provides the relationship between the displacement increments ( $d\mathbf{v}$ ) of the vertices and the nodal load increments ( $d\mathbf{f}$ ) equilibrating the edge force increments caused by the displacement increments:

$$\begin{bmatrix} df_{1x} \\ df_{1y} \\ df_{2x} \\ df_{2y} \\ df_{3x} \\ df_{3y} \end{bmatrix} = \begin{bmatrix} & -\mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{e}_1 & & -\mathbf{e}_3 \\ -\mathbf{e}_1 & \mathbf{e}_2 & \end{bmatrix} \mathbf{H} \begin{bmatrix} & \mathbf{e}_1^T & -\mathbf{e}_1^T \\ -\mathbf{e}_2^T & & \mathbf{e}_2^T \\ \mathbf{e}_3^T & -\mathbf{e}_3^T & \end{bmatrix} \begin{bmatrix} dv_{1x} \\ dv_{1y} \\ dv_{2x} \\ dv_{2y} \\ dv_{3x} \\ dv_{3y} \end{bmatrix} = \mathbf{K}' d\mathbf{v}, \quad (18)$$

and thus a block of  $\mathbf{K}'$  is

$$\mathbf{K}'_{ij} = \sum_{\alpha=1}^2 \sum_{\beta=1}^2 (-1)^{\alpha+\beta} \mathbf{e}_{i+\alpha} H_{i+\alpha, j+\beta} \mathbf{e}_{j+\beta}^T \quad (19)$$

where  $\mathbf{e}_i$  is the unit vector of the  $i$ th edge, and in the subscripts the previously defined cyclic order is valid.

The secondary (supplementary) stiffness matrix of the triangle element contains the effect of the change in direction of the already existing edge forces. The block in the main diagonal of the supplementary stiffness matrix of the  $i$ th edge is

$$\mathbf{K}_i^0 = \frac{Q_i}{L_i} (\mathbf{E} - \mathbf{e}_i \mathbf{e}_i^T), \quad (20)$$

and so the supplementary stiffness matrix  $\mathbf{K}''$  of the triangle element takes the form

$$\mathbf{K}'' = \begin{bmatrix} \mathbf{K}_2^0 + \mathbf{K}_3^0 & -\mathbf{K}_3^0 & -\mathbf{K}_2^0 \\ -\mathbf{K}_3^0 & \mathbf{K}_1^0 + \mathbf{K}_3^0 & -\mathbf{K}_1^0 \\ -\mathbf{K}_2^0 & -\mathbf{K}_1^0 & \mathbf{K}_1^0 + \mathbf{K}_2^0 \end{bmatrix}. \quad (21)$$

The complete (tangent) stiffness matrix  $\mathbf{K}$  of the triangle element is the sum of the two stiffness matrices:

$$\mathbf{K} = \mathbf{K}' + \mathbf{K}'' . \quad (22)$$

## 5.5 Equilibrium position, stability, singular points

Using the stiffness matrices of the elements and taking the supports into account we can compile the stiffness matrix of the entire structure in any state, according to the usual way in structural mechanics [27]. If, for a given radius  $r$ , the approximate position coordinates of the circle centres and the respective element forces are known, then their values can be made more exact with the following iteration steps.

(i) At every node, we calculate the sum of the force vectors appearing in the members and triangle element edges connected to the node. That provides the unequilibrated load at the node. From these nodal loads, the vector of the unequilibrated loads of the structure can be composed, that is the vector of the load errors. The differences between the lengths  $L_s$ , calculated from member forces, and distances  $L_r$  between the end points of the respective members provide the vector of the compatibility errors. If the norm of both error vectors is smaller than a prescribed error limit, then the iteration process can be stopped.

(ii) We generate the stiffness matrix of the structure and the vector of the reduced loads composed from the two error vectors. Considering them as the coefficient matrix and the right-hand side vector of a set of linear equations, the solution to this set of equations provides a linear approximation of the displacements of the nodes.

(iii) If the norm of the obtained displacement vector is greater than a prescribed value of a parameter defined to increase the radius of convergence of the iteration then, proportionally, we decrease the norm to the prescribed parameter value.

(iv) From the obtained displacement vector and the stiffness matrices of the elements, we calculate the linear increments of the element forces.

(v) By modifying the coordinates of the circle centres with the entries of the displacement vector, and the element forces with the calculated increments, we obtain a better approximation of them, and we can repeat the iteration from point (i).

If the stiffness matrix in the state obtained by iteration is positive definite, then the arrangement may be optimal, if it is not positive definite, then it is certain that there exists a better arrangement of the circles.

Plotting the equilibrium positions as a function of  $r$ , we obtain "equilibrium paths". Points of the equilibrium paths can be divided into four categories.

(a) The equilibrium path in a small enough neighbourhood of the point is a smooth curve. In this neighbourhood, the topology of the active elements is unchanged, the stiffness matrix is not singular at any point of the curve, and the number of its positive eigenvalues does not change.

(b) The equilibrium path in a small enough neighbourhood of the point is a curve, but the topology of the active elements on the two sides of the point is different. Since for the cable elements and strut elements the constitutive function (material law) is non-smooth (what is more the gradient at one side of the point is infinitely large), the potential energy function is non-smooth either. The stiffness matrix is not singular at any point of the curve in this neighbourhood and the number of its positive eigenvalues does not change.

(c) The stiffness matrix is singular at the point, and the equilibrium path bifurcates. In a small enough neighbourhood of the point, the topology of the active elements is the same on all branches of the path, and the potential energy of the structure is a smooth function in this neighbourhood. The critical points can be classified according to catastrophe theory [23].

(d) The equilibrium path bifurcates at the point, but the topology of the active elements is not the same on every branch. The function of the potential energy is non-smooth.

## 6 Conclusions

(a) With the tools of structural mechanics we have shown three mechanical models suitable for providing conjectural solutions to the circle arrangement problem.

(b) In the models of both packing and covering, bars that are equally able to bear both tensile and compressive forces are applied as structural elements. The reason for this is to avoid using non-smooth functions. These models are free of stress in the investigated domain, except at particular points. Where they can reach a state of self-stress, without using the constitutive equations, we define stresses. In this way it is possible to identify bars that can be removed from the assembly. We do not produce "equilibrium paths" here, but only "equilibrium points", that is, equilibrium positions for particular values of radius  $r$ .

In the generalized tensegrity model of the partial covering, we use all the geometrical, constitutive and equilibrium equations throughout. Here, stresses appear in members of the structure for all the possible values of  $r$ , and we can produce "equilibrium paths".

(c) In order to show how these mechanical models work and how to use them the case of  $n = 5$  will be presented as an example in Part II [24]. The generalized tensegrity model for  $n = 5$  provides a good occasion also to determine such “equilibrium paths” in cases where the stiffness matrix is not positive definite.

## 7 Acknowledgements

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