Domination in transitive colorings of tournaments

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Abstract

An edge coloring of a tournament T with colors $1, 2, \dots, k$ is called k-*transitive* if the digraph $T(i)$ defined by the edges of color i is transitively oriented for each $1 \leq i \leq k$. We explore a conjecture of the second author:
For each positive integer k there exists a (least) $p(k)$ such that every k-transitive tournament has a dominating set of at most $p(k)$ vertices.

We show how this conjecture relates to other conjectures and results. For example, it is a special case of a well-known conjecture of Erdős, Sands, Sauer and Woodrow [15] (so the conjecture is interesting even if false). We show that the conjecture implies a stronger conjecture, a possible extension of a result of Bárány and Lehel on covering point sets by boxes. The principle used leads also to an upper bound $O(2^{2^{d-1}} d \log d)$ on the d-dimensional box-cover number that is better than all previous bounds, in a sense close to best possible. We also improve the best bound known in 3-dimensions from 3^{14} to 64 and propose possible further improvements through finding the maximum domination number over parity tournaments.

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1 Introduction.

1.1 Two conjectures.

A digraph is called *transitive* if its edges are transitively oriented, i.e. whenever ab, bc are edges ac is also an edge. An equivalent definition is obtained if vertices represent elements of a partially ordered set P and ab is an edge if and only if $a <_P b$. If orientations of the edges are disregarded, transitive digraphs define *comparability graphs.*

We call an edge coloring of a tournament T with colors $1, 2, \dots, k$ *transitive* if the digraph $T(i)$ defined by the edges of color i is transitive for each $i, 1 \le i \le k$. If a tournament has a transitive coloring with k colors, we say that it is k*-transitive.* Notice that a k-transitive tournament $T = \cup_{i \in [k]} T(i)$ remains k-transitive if for some index set $I \subseteq [k]$ we replace $T(i)$ by its reverse for all $i \in I$. This is called a *scrambling* of T and is denoted by T_I. Clearly, 1-transitive tournaments are equivalent to transitive tournaments, i.e. to acyclic tournaments. In fact, this is true for 2-transitive tournaments as well.

Proposition 1. *Suppose that a tournament T is transitively 2-colored. Then T is a transitive tournament.*

Proof. Observe that T can not contain a cyclic triangle because two of its edges would be colored by the same color and that contradicts transitivity of the digraph of that color. Then T can not have any oriented cycle either, thus T is transitive. $\square$

A vertex set S of a tournament is dominating if for every vertex $v \notin S$ there exists some $w \in S$ such that wv is an edge of the tournament. The size of a smallest dominating set in a tournament T is denoted by $dom(T)$. A well-known fact (an illustration of the probability method) is that there are tournaments T with arbitrary large $dom(T)$. Tournaments that cannot be dominated by k vertices are sometimes called k-paradoxical. The smallest k-paradoxical tournaments for $k = 1, 2, 3$ have $3, 7, 19$ vertices, respectively. As in [2], we associate a hypergraph $H(T)$ to any digraph T, with vertex set $V(T)$ and with hyperedges associated to $v \in V(T)$ as follows: the hyperedge $e(v)$ contains v and all vertices w for which $wv \in E(T)$. In terms of $H(T)$, a dominating set S is a transversal of $H(T)$ and $dom(T) = \tau(H(T))$, the transversal number of $H(T)$.

In this note we explore two conjectures of the second author and show that they are equivalent.

Conjecture 1. *For each positive integer k there is a (least) $p(k)$ such that for every k-transitive tournament T, $dom(T) \le p(k)$.*

Conjecture 1 is open for $k \geq 3$. In Section 3 we give some examples of 3-transitive tournaments and prove that Paley tournaments cannot provide counterexamples to Conjecture 1: for every k, Paley tournaments of order at least 2^{4k+4} are not k-transitive tournaments (Theorem 4).

Conjecture 1 relates to other conjectures and results. A set $S \subset V(T)$ is an *enclosure set* in a k-colored tournament $T = \cup_{i \in [k]} T(i)$ if for any $b \in V(T) \setminus S$ there exist an $i \in [k]$ and $a, c \in S$ such that $ab, bc \in E(T(i))$. We say in this case that b is *between* $\{a, c\}$. The smallest enclosure set of a tournament T is denoted by $encl(T)$.

Conjecture 2. *For each positive integer k there is a (least) $r(k)$ such that every for k-transitive tournament T, $encl(T) \leq r(k)$.*

1.2 Conjecture 1 implies Conjecture 2.

Notice that by definition $p(k) \leq r(k)$. Our main observation is that $r(k)$ is bounded in terms of $p(k)$, i.e. Conjecture 2 follows from Conjecture 1. Denote by T_I the 2^k possible scramblings of a k-transitive tournament T.

Theorem 1. $r(k) \leq \sum_{I \subseteq [k]} dom(T_I)$.

If we know that $p(k)$ is finite, Theorem 1 implies

Corollary 1. $r(k) \leq 2^k p(k)$.

Proof of Theorem 1. Select a dominating set S_I from each scrambling T_I. Clearly, it is enough to prove the following.

Claim 1. $R = \cup_{I \subseteq [k]} S_I$ *is an enclosure set of* T.

Indeed, suppose $v \in V(T) \setminus R$ and v is not between any two points of R. This means that for every color $i \in [k]$ either nobody dominates v from R, or nobody is dominated by v from R (or possibly both). Denote the latter set of colors by $I \subset [k]$. In this case nobody dominates v in T_I, which contradicts the choice of S_I. $\square$

1.3 d-coordinate tournaments, improved bound on the box-cover number.

Our main result comes by applying Corollary 1 to special transitively colored tournaments T, where $dom(T)$ is bounded. These tournaments are defined by finite sets $S \subset R^d$. The tournament T has vertex set S, the edge between $p = (p_1, \ldots, p_d)$

and $q = (q_1, \ldots, q_d)$ is oriented from p to q if $p_1 < q_1$ (we shall assume that coordinates of points of S are all different). The color of an edge is assigned according to the 2^{d-1} possible relation of the other $d-1$ coordinates of p and q. Thus we have here 2^{d-1} colors and clearly each determines a transitive digraph, thus T is a 2^{d-1}-transitive tournament. We call tournaments obtained from T by scrambling (i.e. reversing direction on any subset of colors) *d-coordinate tournaments*. Note that $2^{2^{d-1}}$ d-coordinate tournaments are defined by any $S \subset R^d$.

Set $g(d) = \max_T encl(T)$ over all d-coordinate tournaments T. According to Conjecture 2, $encl(T) \le r(2^{d-1})$. We shall describe this problem in a more geometric way as follows.

Given two points $p, q \in R^d$, define $box(p,q)$ as the smallest closed box whose edges are parallel to the coordinate axes and contains p and q. So $box(p,q) = \{x \in R^d \mid \forall i \ \min(p_i, q_i) \le x_i \le \max(p_i, q_i)\}$.

Denote by $g(d)$ the smallest number such that we can select a set $P, |P| \le g(d)$ from any finite collection $S \subset R^d$, such that for any $s \in S$ we have $p, q \in P$ for which $s \in box(p,q)$. Note that it is possible to give $2^{2^{d-1}}$ points in R^d so that none of them is in the box generated by two others, implying $2^{2^{d-1}} \le g(d)$. This lower bound comes from many sources, in fact related to Ramsey problems as well, see [4]. In fact, it is sharp: in any set of $2^{2^{d-1}} + 1$ points of R^d there is a point in the box generated by two of the other points. This nice proposition comes easily from iterating the Erdős - Szekeres monotone sequence lemma. (A proof is in [12] while [2] refers to this as an unpublished result of N. G. de Bruijn.)

On the other hand, Bárány and Lehel [4] have shown that $g(d)$ always exists and gave the upper bound $g(d) \le (2d^{2^d} + 1)^{d2^d}$, which was later improved to $g(d) \le 2^{2^{d+2}}$ by Pach [12], and finally to $g(d) \le 2^{2^d + d + \log d + \log\log d + O(1)}$ by Alon et al. [2]. Note that this last bound is about the square of the lower bound. We use Corollary 1 to improve their method from [2] and this gives our main result, an upper bound that is very close to the lower bound.

Theorem 2. $g(d) = O(2^{2^{d-1}} d \log d)$.

We give the proof of this theorem in Section 2.

1.4 3-coordinate tournaments and Parity tournaments.

Although Theorem 2 is “close” to the lower bound $2^{2^{d-1}}$, it is not useful for small d. Note that $g(1) = 2$ is obvious, $g(2) = 4$ is easy but the best known bound in 3 dimension was $g(3) \le 3^{14}$ by Bárány and Lehel. Here we look at the 3-dimensional case more carefully and obtain a more reasonable bound of $g(3)$.

In the 3-dimensional case any point set defines sixteen 3-coordinate tournaments. The obtained tournaments are of three kind.

Case 1. Six tournaments are "dictatorships", where one coordinate defines the edges and the other two are irrelevant. These are transitive and thus each is dominated by a single vertex.

Case 2. Eight tournaments are isomorphic to a 2-majority tournament, where uv is an edge if two coordinates of u are bigger than the respective coordinates of v. It was proved in [2] that such tournaments have a dominating set of three vertices (and this is sharp).

Case 3. The last two tournaments are inverse to each other, and we call them *parity tournaments* as uv is an edge if an even (resp. odd) number of the three coordinates of u are bigger than the respective coordinates of v. Set $m = \max_T dom(T)$ over all parity tournaments T. We strongly believe that m can be determined by a (relatively) simple, combinatorial argument (like in [2] for 2-majority tournaments) thus we pose

Problem 1. *Determine* $m = \max_T dom(T)$ *over all parity tournaments* T.

The observations above with Theorem 1 yield the following upper bound on $g(3)$.

Corollary 2. $g(3) \leq 6 \times 1 + 8 \times 3 + 2m = 30 + 2m$.

It is possible that $m = 3$ but we could prove only $m \leq 17$ by a careful modification of the probabilistic proof of Theorem 3 (for details, see the Appendix). Using this and Corollary 2 we get $g(3) \leq 64$, a first step to a "down to earth" bound.

1.5 A path-domination conjecture and majority tournaments.

Conjecture 1 also relates to a well-known conjecture of Erdős, Sands, Sauer and Woodrow [15].

Conjecture 3. *Is there for each positive integer* k *a (least) integer* $f(k)$*, such that every tournament whose edges are colored with* k *colors, contains a set* S *of* $f(k)$ *vertices with the following property: every vertex not in* S *can be reached by a monochromatic directed path with starting point in* S*. In particular, (a.) does* $f(3)$ *exist? (b.)* $f(3) = 3$*?*

For further developments on Conjecture 3 see [6, 7, 9, 11, 14].

Proposition 2. $p(k) \leq f(k)$.

Proof. Suppose T has a transitive k-coloring. From the definition of f, there exists $S \subset V(T)$ such that $|S| = f(k)$ and every vertex $w \notin S$ can be reached by a directed path of color i for some $1 \le i \le k$ with a starting point $v \in S$. From transitivity of color i, $vw \in E(T)$, thus S is a dominating set in T. □

Examples 2, 3 in Subsection 3.1 show $p(3) \ge 3$, thus Proposition 2 gives

Corollary 3. *If (b) is true in Conjecture 3 then* $p(3) = 3$.

A last remark relates transitive tournaments to *k-majority tournaments* T, defined by $2k-1$ linear orders on $V(T)$ by orienting each pair according to the majority. Note that every k-majority tournament is also a $(2k-1)$-coordinate tournament. It was conjectured that $dom(T)$ is bounded by a function $maj(k)$ for k-majority tournaments and Alon et al. [2] proved the conjecture with a very good bound $maj(k) = O(k \log k)$. The existence of $maj(k)$ would also follow from the existence of $p(k)$ (with a very poor bound).

Proposition 3.

$$maj(k) \le p\Big(\sum_{i=k}^{2k-1} \binom{2k-1}{i}\Big).$$

Proof. If T is defined by the majority rule with linear orders $L_1, \dots, L_{2k-1}$ then one can color each edge $xy \in T$ by the set of indices i for which $x <_{L_i} y$. It is easily seen that this coloring is transitive (using at most $\sum_{i=k}^{2k-1} \binom{2k-1}{i}$ colors). □

2 Proof of Theorem 2.

We may suppose that $S \subset R^d$ is in general position in the sense that no two points share a coordinate. Let T denote the tournament associated to S. To bound $dom(T)$, we follow the same strategy as in Alon et al. [2].

As mentioned in the introduction, a dominating set S is a transversal of $H(T)$, thus $dom(T) = \tau(H(T))$. We need a bound on $\tau(H(T))$ in terms of d. Let $\tau^*(H(T))$ be fractional transversal number of $H(T)$.

The following claim is from [2], we give a different proof for it.

Claim 2. *[Alon et al.] For any tournament T, $\tau^*(H(T)) < 2$.*

Proof. Let T be a tournament on n vertices, set $H = H(T)$. We will prove that the fractional matching number, $\nu^*(H)$, which equals $\tau^*(H)$, is less than 2. Denote by A the $n \times n$ incidence matrix of H, with rows indexed by vertices, columns with edges. Vectors are column vectors with n coordinates. Denote by $\mathbf{j}$ the all 1 vector, by J the all 1 $n \times n$ matrix. Since $\nu^* = \max\{\mathbf{j^T x} \mid \mathbf{Ax} \leq \mathbf{j}, \mathbf{0} \leq \mathbf{x} \leq \mathbf{j}\}$, let us take an $\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{j}$. This also implies $\mathbf{x^T A^T} \leq \mathbf{j^T}$. From this, using that $A + A^T$ contains 1-s except along the main diagonal, where all elements are 2-s, we have

$$(\mathbf{j}^T\mathbf{x})^2 = \mathbf{x}^T\mathbf{j}\mathbf{j}^T\mathbf{x} = \mathbf{x}^T J\mathbf{x} < \mathbf{x}^T(A + A^T)\mathbf{x} \leq \mathbf{j}^T\mathbf{x} + \mathbf{x}^T\mathbf{j},$$

implying $\mathbf{j}^T\mathbf{x} < 2$. $\square$

As in [2], we bound the VC-dimension of H, $VC(H)$, then we can use the following consequence of the Haussler-Welz theorem, formulated first by Komlós, Pach and Woeginger, [8].

Theorem 3. *$\tau(H) = O(VC(H)\tau^*(H)\log\tau^*(H))$.*

Note that this implies $dom(T) = O(VC(H(T)))$ in our case. Our next statement is a slight modification of a respective claim from [2].

Claim 3. *If T is a d-coordinate tournament and $h = VC(H(T))$, then $(h+1)^d \geq 2^h$ and so $h \leq (1 + o(1))d\log d$.*

Proof. Fix h vertices that are shattered. These vertices divide each of the d coordinates into $h+1$ parts. For two points that belong to the same $h+1$ parts in each of the d coordinates, the restriction of their dominating hyperedge to the h vertices is also the same. (Note that the h vertices themselves do not belong to any of the $(h+1)^d$ possibilities but they are equivalent to a vertex obtained by moving them slightly up or down in all coordinates.) Since we have 2^h different restrictions, the bound follows. $\square$

Putting all together we have

$$dom(T) = \tau(H(T)) = O(VC(H(T))\tau^*(H(T))\log\tau^*(H(T))) = O(d\log d).$$

Applying Corollary 1 and the above inequality to the 2^{d-1}-transitive tournament T, we get

$$encl(T) \leq 2^{2^{d-1}} dom(T) = O(2^{2^{d-1}} d\log d),$$

which establishes Theorem 2.

3 2- and 3-transitive tournaments, Paley tournaments.

3.1 2- and 3-transitive tournaments.

One can say more than stated in Proposition 1 about 2-transitive tournaments. For a permutation $\pi = x_1 \dots x_n$ of $\{1, 2, \dots, n\}$ define the 2-colored tournament $T(\pi)$ by orienting each edge $x_i x_j$ from smaller index to larger index and coloring it with red (resp. blue) if $x_i < x_j$ (resp. $x_i > x_j$).

Proposition 4. *Suppose that a tournament T is transitively 2-colored. Then $T = T(\pi)$ with a suitable permutation π of the vertices of T.*

Proof. Follows from a result of [13]. $\square$

Transitive 3-colorings can be quite complicated. Here we give some examples.

Example 1. *Suppose that the vertex set of T is partitioned into two parts, A, B. $T(1)$ is an arbitrary bipartite graph oriented from A to B. $T(2)$ is the bipartite complement, oriented from B to A. $T(3)$ is the union of two disjoint transitive tournaments, one is on A, the other is on B. Note that T can be dominated by two vertices. However, the hypergraph $H(T)$ can have arbitrarily large VC-dimension.*

Example 2. *([16]) The Paley tournament PT_q is defined for every prime power $q \equiv -1 \pmod 4$. Its vertex set is $GF(q)$ and xy is an edge if $y - x$ is a (non-zero) square in $GF(q)$. Let PT_7 be the Paley tournament on $[7]$ with edges $(i, i+1), (i, i+2), (i, i+4)$. A transitive 3-coloring of T is the following:*

$$T(1) = \{(1,2), (1,5), (3,4), (3,5), (3,7), (4,5), (6,7)\},$$

$$T(2) = \{(1,3), (2,3), (2,4), (2,6), (4,6), (5,6), (5,7)\},$$

$$T(3) = \{(4,1), (5,2), (6,3), (6,1), (7,1), (7,2), (7,4)\}.$$

Substitution of a colored tournament H into a vertex v of a colored tournament T is replacing v by a copy H^* of H and for any $w \in V(H^*), u \in V(T) \setminus V(H^*)$ the color and the orientation of wu is the same as of vu.

Example 3. *Let T be the tournament on nine vertices obtained from a 3-colored cyclic triangle by substituting it into each of its vertices.*

Observe that examples 2 and 3 both show $p(3) \geq 3$.

3.2 Paley tournaments.

By a theorem of Graham and Spencer, Paley tournaments PT_q are k-paradoxical if $q \geq k^2 4^k$. Thus Paley tournaments are potential counterexamples to Conjecture 1. This is eliminated by the next result.

Theorem 4. *For $q > 2^{4k+4}$, PT_q has no transitive k-coloring.*

Proof. We shall use the following lemma of Alon (Lemma 1.2 in Chapter 9 of [3], where it is stated only for primes but it comes from [1] where it is stated for prime powers).

Lemma 1. *Suppose A, B are subsets of vertices in the Paley tournament PT_q, for some prime power q. Then*

$$|e(A,B) - e(B,A)| \leq (|A||B|q)^{1/2}.$$

Suppose indirectly that $T = PT_q$ has a transitive k-coloring with colors $1, 2, \ldots, k$. Set $\nu = \frac{1}{2^{2k+2}}$ and define the type of a vertex $v \in V(T)$ as a pair of subsets $I(v), O(v)$ where $I(v), O(v) \subseteq [k]$ and $i \in I(v)$ if and only if $d_i^-(v) \geq \nu q$ and $i \in O(v)$ if and only if $d_i^+(v) \geq \nu q$. Thus the type of $v \in V(T)$ determines those colors in which the indegree or the outdegree of v is large.

Claim 4. *There is a vertex $w \in V(T)$ whose type (I, O) satisfies $I \cap O \neq \emptyset$.*

Proof of claim. There are at most 2^{2k} types so there is an $A \subset V(T)$ such that all vertices of A have the same type and

$$|A| \geq \frac{q}{2^{2k}}. \tag{1}$$

If for each $a \in A$, $I(a) \cap O(a) = \emptyset$, then $|E(A)|$, the number of edges in A can be counted as follows:

$$\binom{|A|}{2} = |E(A)| = \sum_{a \in A, i \in I(a)} d_i^+(a) + \sum_{a \in A, i \notin I(a)} d_i^-(a) < \nu q|A|$$

where the inequality comes from bounding each term by νq (in the first sum we used that $i \in I(a)$ implies $i \notin O(a)$). Therefore $|A| < 2q\nu + 1$, thus, using (1) we obtain that

$$\frac{q}{2^{2k}} \leq |A| < 2q\nu + 1$$

implying

$$2^{-(2k+1)} - (2q)^{-1} < \nu.$$

However, by $q > 2^{2k+1}$ (being generous here),

$$\nu = 2^{-(2k+2)} = 2^{-(2k+1)} - 2^{-(2k+2)} < 2^{-(2k+1)} - (2q)^{-1} < \nu$$

contradiction. This finishes the proof of the claim. □

Let $w \in V(T)$ be a vertex given by the claim, and set $A = N_i^-(w), B = N_i^+(w)$ for some $i \in I(w) \cap O(w)$. From transitivity of color i, all edges of $[A, B]$ are oriented from A to B (in color i). Therefore, using Lemma 1,

$$|A||B| = |E(A,B)| = |E(A,B)| - |E(B,A)| \leq (|A||B|q)^{1/2}$$

leading to $|A||B| \leq q$. However, from the definition of A, B, $\nu^2 q^2 \leq |A||B|$. Therefore we get $q \leq \frac{1}{\nu^2}$ contradicting the assumption $q > 2^{4k+4}$. □

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Appendix - Domination in parity tournaments

Here we prove that $m = \max_T dom(T) \leq 17$, where the maximum is over all parity tournaments T. The proof closely follows the proof of Theorem 3, for details, see e.g. [10].

Denote by H the hypergraph associated to some parity tournament. We have that $\tau^*(H) < 2$ and $\pi(n) \leq (n+1)^3$ where $\pi(n)$ is the *shatter function* of H, defined as

$$\max_{|S|=n} |\{S \cap F \mid F \in H\}|.$$

First we prove $m \leq 19$.

Pick a multiset of 38 points at random according to the distribution given by $\tau^*(H)$ and divide it randomly into two sets, A and B, with $a = 19$ and $b = 19$ points. Denote by E the event that there is an $F \in H$ which is disjoint from A and contains B. The probability that A is not a $\frac{1}{2}$-net is at most 2^b times the probability of E. For any given F, the probability of $F \cap A = \emptyset$ and $B \subset F$ is at most $\frac{1}{\binom{a+b}{b}}$. As $|F \cap (A \cup B)|$ can take at most $\pi(a+b)$ values, the probability that A is not a $\frac{1}{2}$-net is at most $\frac{2^b \cdot (a+b+1)^3}{\binom{a+b}{b}} = \frac{2^{19} \cdot 39^3}{\binom{38}{19}} < 1$.

Now this bound can be further improved using the fact that instead of working with π, for us it is enough to look at the subsets $F \cap (A \cup B)$ whose size is exactly b. We define $\pi_k(n)$ as

$$\max_{|S|=n} |\{S \cap F \mid |S \cap F| = k, F \in H\}|.$$

The above argument works if $\frac{2^b \cdot \pi_b(n)}{\binom{a+b}{b}} < 1$, which is equivalent to $\pi_b(n) < \binom{a+b}{b}/2^b$. Now we need some good bounds for $\pi_k(n)$.

A trivial observation is that $\pi_k(n)$ is upper bounded by

$$\sum_{i \equiv k \ mod \ 2} \pi_i(n) \leq \lceil (n+1)^3/2 \rceil.$$

(This is already sufficient to show $m \leq 18$.) We can further improve this in the following way. Consider the three times $n+1$ intervals to which the n points divide the three coordinates. We say that the *rank* of a point p is (x, y, z), if there are x, y and z points under it in the respective coordinates. So we have $0 \leq x, y, z \leq n$ and our first observation was that $x + y + z \equiv k \ mod \ 2$ is necessary. It is easy to see that we also need $x + y + z \geq k$ if at least k points are dominated by p and $x + y + z \leq 3n - (n - k)$ if at most k points are dominated by p. The cardinality of the points not satisfying the first, resp. second inequality, is $\binom{k+2}{3}$ and, resp. $\binom{n-k+2}{3}$.

Adding these constraint would make our parity argument a bit more complicated, so instead of the second inequality we only require $x+y+z \leq 3n-(n-k)+1$. Using these we get $\pi_k \leq ((n+1)^3 - \binom{k+2}{3} - \binom{n-k+1}{3})/2$. If we choose $n = 31$ and $k = 14$, we get that this is less than $\binom{n}{k}/2^k$, proving that $m \leq n - k = 17$.