Multiplicative loops of 2-dimensional topological quasifields

Abstract

We determine the algebraic structure of the multiplicative loops for locally compact 2-dimensional topological connected quasifields. In particular, our attention turns to multiplicative loops which have either a normal subloop of positive dimension or which contain a 1-dimensional compact subgroup. In the last section we determine explicitly the quasifields which coordinatize locally compact translation planes of dimension 4 admitting an at least 7-dimensional Lie group as collineation group.

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1. Introduction

Locally compact connected topological non-desarguesian translation planes have been a popular subject of geometrical research since the seventies of the last century ([18], [2]-[9], [13], [15]). These planes are coordinatized by locally compact quasifields Q such that the kernel of Q is either the field $\mathbb{R}$

of real numbers or the field $\mathbb{C}$ of complex numbers (cf. [11], IX.5.5 Theorem, p. 323). If the quasifield Q is 2-dimensional, then its kernel is $\mathbb{R}$.

The classification of topological translation planes $\mathcal{A}$ was accomplished by reconstructing the spreads corresponding to $\mathcal{A}$ from the translation complement which is the stabilizer of a point in the collineation group of $\mathcal{A}$. In this way all planes $\mathcal{A}$ having an at least 7-dimensional collineation group have been determined ([3]-[8], [15]).

Although any spread gives the lines through the origin and hence the multiplication in a 2-dimensional quasifield Q coordinatizing the plane $\mathcal{A}$, to the algebraic structure of the multiplicative loop Q^* of a proper quasifield Q is not given special attention apart from the facts that the group topologically generated by the left translations of Q^* is the connected component of $\mathrm{GL}_2(\mathbb{R})$, the group topologically generated by the right translations of Q^* is an infinite-dimensional Lie group (cf. [14], Section 29, p. 345) and any locally compact 2-dimensional semifield is the field of complex numbers ([17]).

Since in the meantime some progress in the classification of compact differentiable loops on the 1-sphere has been achieved (cf. [10]), we believe that loops could have more space in the research concerning 4-dimensional translation planes. Using the images of differentiable sections $\sigma : G/H \to G$, where $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$, we classify the C^1-differentiable multiplicative loops Q^* of 2-dimensional locally compact quasifields Q by functions, the Fourier series of which are described in [10].

The multiplicative loops Q^* of 2-dimensional locally compact left quasifields Q for which the set of the left translations of Q^* is the product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where $\mathcal{T}$ is the set of the left translations of a 1-dimensional compact loop and $\mathcal{K}$ is the set of the left translations of Q^* corresponding to the the kernel K_r of Q, form an important subclass of loops, that we call decomposable loops. Namely, if Q^* has a normal subloop of positive dimension or if it contains the group $\mathrm{SO}_2(\mathbb{R})$, then Q^* is decomposable. Moreover, we show that any 1-dimensional C^1-differentiable compact loop is a factor of a decomposable multiplicative loop of a locally compact connected quasifield coordinatizing a 4-dimensional translation plane. A 2-dimensional locally compact quasifield Q is the field of complex numbers if and only if the multiplicative loop Q^* contains a 1-dimensional normal compact subloop.

Till now mainly those simple loops have been studied for which the group generated by their left translations is a simple group. If the group generated by the left translations of a loop L is simple, then L is also simple (cf. Lemma 1.7 in [14]). The multiplicative loops Q^* of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.

In the last section we use Betten's classification to determine in our framework the multiplicative loops Q^* of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes $\mathcal{A}$ admitting an at least seven-dimensional collineation group and to study their properties. The results obtained there yield the following

Theorem *Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group Γ. If the quasifield Q coordinatizing $\mathcal{A}$ is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of Γ or contain the set of the fixed points of Γ, then the multiplicative loop Q^* of Q is decomposable if and only if one of the following cases occurs:*

(a) Γ is 8-dimensional, the translation complement C is the group $\mathrm{GL}_2(\mathbb{R})$ and acts reducibly on the translation group $\mathbb{R}^4$;

(b) Γ is 7-dimensional, the translation complement C fixes two distinct lines of $\mathcal{A}$ and leaves on one of them, one or two 1-dimensional subspaces invariant;

(c) Γ is 7-dimensional, the translation complement C fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W but does not act transitively on the product space $P_S \times P_W$, where P_S and P_W are the sets of all 1-dimensional subspaces of S, respectively of W.

2. Preliminaries

A binary system $(L, \cdot)$ is called a quasigroup if for any given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions which we denote by $y = a \backslash b$ and $x = b/a$. If a quasigroup L has an element 1 such that $x = 1 \cdot x = x \cdot 1$ holds for all $x \in L$, then it is called a loop and 1 is the identity element of L. The left translations $\lambda_a : L \to L, x \mapsto a \cdot x$ and the right translations $\rho_a : L \to L, x \mapsto x \cdot a$, $a \in L$, are bijections of L. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isotopic if there exist three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for all $x, y \in L_1$. A binary system $(K, \cdot)$ is called a subloop of $(L, \cdot)$ if $K \subset L$, for any given $a, b \in K$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions in K and $1 \in K$. The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L, i.e. a subloop of L such that

$$x \cdot N = N \cdot x, \ (x \cdot N) \cdot y = x \cdot (N \cdot y), \ (N \cdot x) \cdot y = N \cdot (x \cdot y) \qquad (1)$$

hold for all $x, y \in L$. A loop L is called simple if $\{1\}$ and L are its only normal subloops.

A loop L is called topological, if it is a topological space and the binary operations $(a, b) \mapsto a \cdot b$, $(a, b) \mapsto a \backslash b$, $(a, b) \mapsto b/a : L \times L \to L$ are continuous. Then the left and right translations of L are homeomorphisms of L. If L is a connected differentiable manifold such that the loop multiplication and the left division are continuously differentiable mappings, then we call L an almost $\mathcal{C}^1$-differentiable loop. If also the right division of L is continuously differentiable, then L is a $\mathcal{C}^1$-differentiable loop. A connected topological loop is quasi-simple if it contains no normal subloop of positive dimension. Every topological, respectively almost $\mathcal{C}^1$-differentiable, connected loop L having a Lie group G as the group topologically generated by the left translations of L corresponds to a sharply transitive continuous, respectively $\mathcal{C}^1$-differentiable section $\sigma : G/H \to G$, where $G/H = \{xH | x \in G\}$ consists of the left cosets of the stabilizer H of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates G. The section σ is sharply transitive if the image $\sigma(G/H)$ acts sharply transitively on the factor space G/H, i.e. for given left cosets xH, yH there exists precisely one $z \in \sigma(G/H)$ which satisfies the equation $zxH = yH$.

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ such that $(Q, +)$ is an abelian group with neutral element 0, $(Q \setminus \{0\}, \cdot)$ is a loop with identity element 1 and between these operations the (left) distributive law $x \cdot (y+z) = x \cdot y + x \cdot z$ holds. A locally compact connected topological quasifield is a locally compact connected topological space Q such that $(Q, +)$ is a topological group, $(Q \setminus \{0\}, \cdot)$ is a topological loop, the multiplication $\cdot : Q \times Q \to Q$ is continuous and the mappings $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ with $0 \neq a \in Q$ are homeomorphisms of Q. If for any given $a, b, c \in Q$ the equation $x \cdot a + x \cdot b = c$ with $a + b \neq 0$ has precisely one solution, then Q is called planar. A translation plane is an affine plane with transitive group of translations; this is coordinatized by a planar quasifield (cf. [16], Kap. 8).

The kernel K_r of a (left) quasifield Q is a skewfield defined by

$$K_r = \{k \in Q;\ (x+y)\cdot k = x\cdot k + y\cdot k \text{ and } (x\cdot y)\cdot k = x\cdot(y\cdot k) \text{ for all } x, y \in Q\}.$$

In this paper we consider left quasifields Q. Then Q is a right vector space over K_r. Moreover, for all $a \in Q$ the map $\lambda_a : Q \to Q, x \mapsto a \cdot x$ is K_r-linear. According to [12], Theorem 7.3, p. 160, every quasifield that has finite dimension over its kernel is planar.

Let F be a skewfield and let V be a vector space over F. A collection $\mathcal{B}$ of subspaces of V with $|\mathcal{B}| \geq 3$ is called a spread of V if for any two different elements $U_1, U_2 \in \mathcal{B}$ we have $V = U_1 \oplus U_2$ and every vector of V is contained in an element of $\mathcal{B}$.

If S and W are different subspaces of the spread $\mathcal{B}$, then V can be coordinatized in such a way that $S = \{0\} \times X$ and $W = X \times \{0\}$. Any spread of $V = X \times X$ can be described by a collection $\mathcal{M}$ of linear mappings $X \to X$ satisfying the following conditions:

(M_1) For any $\omega_1 \neq \omega_2 \in \mathcal{M}$ the mapping $\omega_1 - \omega_2$ is bijective.

(M_2) For all $x \in X \setminus \{0\}$ the mapping $\phi_x : \mathcal{M} \to X : \omega \mapsto \omega(x)$ is surjective.

Namely, if $\mathcal{M}$ is a collection of linear mappings satisfying (M_1) and (M_2), then the sets $U_\omega = \{(x, \omega(x)), x \in X\}$ and $\{0\} \times X$ yield a spread of $V = X \times X$. Conversely, every component $U \in \mathcal{B} \setminus \{S\}$ of V is the graph of a linear mapping $\omega_U : W \to S$ and the set of ω_U gives a collection $\mathcal{M}$ of linear mappings of X satisfying (M_1) and (M_2) (cf. [13], Proposition 1.11.). The mapping ω_W is the zero mapping. For this reason any collection $\mathcal{M}$ of linear mappings of X a spread set of X.

Every translation plane can be obtained from a spread set of a suitable vector space $V = X \times X$ (cf. [13], Theorem 1.5, p. 7, and [1]). As every translation plane can be coordinatized by a quasifield and a quasifield contains 0 and 1, the associated spread set contains the zero endomorphism and the identity map. This is not true for arbitrary spread sets $\mathcal{M}$, but if $\omega_0, \omega_1 \in \mathcal{M}$ are distinct, then $\mathcal{M}' = \{(\omega - \omega_0)(\omega_1 - \omega_0)^{-1},\ \omega \in \mathcal{M}\}$ is a normalized spread of X which contains the zero and the identity map and the translation planes obtained from $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic (cf. [13], Lemma 1.15, p. 13). Let $\mathcal{M}$ be a normalized spread of X, $e \in X \setminus \{0\}$ and let $\phi_e : \mathcal{M} \to X$ be defined by $\phi_e(\omega) = \omega(e)$. Then the multiplication $\circ : X \times X \to X$ defined by $m \circ x = (\phi_e^{-1}(m))(x)$ yields a multiplicative loop of a left quasifield Q coordinatizing the translation plane $\mathcal{A}$ belonging to the spread $\mathcal{M}$ of X.

If we fix a basis of Q over its kernel K_r and identify X with the vector space of pairs $\{(x, y)^t,\ x, y \in K_r\}$, then the set $\mathcal{M}$ consists of matrices $C(\alpha, \beta, \gamma, \delta) =$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha, \beta, \gamma, \delta \in K_r$. If $e = (1,0)^t$, then we get $\phi_e(C(\alpha,\beta,\gamma,\delta)) = C(\alpha,\beta,\gamma,\delta)(e) = (\alpha,\gamma)^t$. Since $\mathcal{M}$ is a spread of X the set of vectors $(\alpha,\gamma)^t$ consists of all vectors of X. Hence if $(\alpha,\gamma)^t$ is an element of X, then there exists a unique matrix of $\mathcal{M}$ having $(\alpha,\gamma)^t$ as the first column.

We consider multiplicative loops of locally compact connected topological quasifields Q of dimension 2 coordinatizing 4-dimensional non-desarguesian topological translation planes. Then the kernel K_r of Q is isomorphic to the field of the real numbers, $(Q,+)$ is the vector group $\mathbb{R}^2$ and the multiplicative loop $(Q\backslash\{0\},\cdot)$ is homeomorphic to $\mathbb{R} \times S^1$, where S^1 is the circle.

4. Multiplicative loops of 2-dimensional quasifields

Let $(Q,+,*)$ be a real topological (left) quasifield of dimension 2. Let e_1 be the identity element of the multiplicative loop $Q^* = (Q \setminus \{0\}, *)$ of Q, which generates the kernel $K_r = \mathbb{R}$ of Q as a vector space and let $B = \{e_1, e_2\}$ be a basis of the right vector space Q over K_r. Once we fix B, we identify Q with the vector space of pairs $(x,y)^t \in \mathbb{R}^2$ and K_r with the subspace of pairs $(x,0)^t$. The element $(1,0)^t$ is the identity element of Q^*. According to [14], Theorem 29.1, p. 345, the group G topologically generated by the left translations of Q^* is the connected component of the group $\mathrm{GL}_2(\mathbb{R})$. As $\dim Q^* = 2$ and the stabilizer H of the identity element of Q^* in G does not contain any non-trivial normal subgroup of G we assume that H is the subgroup $\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$. The elements g of G have a unique decomposition as the product

$$g = \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}$$

with suitable elements $u \in \mathbb{R}\backslash\{0\}$, $k > 0$, $l \in \mathbb{R}$, $t \in [0, 2\pi)$. Hence the loop Q^* corresponds to a continuous section $\sigma : G/H \to G$;

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix} \tag{2}$$

where the pair of continuous functions $a(u,t), b(u,t) : \mathbb{R}\backslash\{0\} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$a(u,t) > 0, \quad a(1, 2\pi k) = 1, \quad b(1, 2\pi k) = 0 \ \text{ for all } \ k \in \mathbb{Z}.$$

As Q is a left quasifield, any $(x, y)^t \in Q^*$ induces a linear transformation $M_{(x,y)} \in \sigma(G/H)$. More precisely one has

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = M_{(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} a(r,\varphi) & b(r,\varphi) \\ 0 & a^{-1}(r,\varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3}$$

where $x = r\cos(\varphi)a(r,\varphi)$, $y = -r\sin(\varphi)a(r,\varphi)$. The kernel K_r of Q consists of $(0,0)^t$ and $(ra(r,0), 0)^t$, $r \in \mathbb{R}\backslash\{0\}$, such that the matrices corresponding to the elements $(ra(r,0), 0)^t$ have the form

$$M(ra(r,0), 0) = \begin{pmatrix} ra(r,0) & rb(r,0) \\ 0 & ra^{-1}(r,0) \end{pmatrix}.$$

The identity matrix I corresponds to the identity $(1,0)^t$ of Q^*. Since to each real number $ra(r,0)$ corresponds precisely one matrix $M(ra(r,0),0)$, the function $f(r) = ra(r,0)$ is strictly monotone. If the function $a(r,0)$ is differentiable, then for every $r \in \mathbb{R} \setminus \{0\}$ the derivative $a(r,0) + ra'(r,0)$ is either always positive or negative. This is equivalent to the fact that the derivative $[\ln(a(r,0))]'$ is always greater or smaller than $-r^{-1}$.

Remark 1. *The set $\mathcal{K} = \{M(ra(r,0),0); r \in \mathbb{R} \setminus \{0\}\}$ of the left translations of Q^* corresponding to the kernel K_r of Q is $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, r \in \mathbb{R} \setminus \{0\} \right\}$ if and only if one has $a(r,0) = 1$, $b(r,0) = 0$ for all $r \in \mathbb{R} \setminus \{0\}$.*

The section σ given by (2) is sharply transitive precisely if for all pairs (u_1,t_1), (u_2,t_2) in $\mathbb{R}\setminus\{0\} \times [0,2\pi)$ there exists precisely one $(u,t) \in \mathbb{R}\setminus\{0\} \times [0,2\pi)$ and $k > 0$, $l \in \mathbb{R}$ such that

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix} \begin{pmatrix} u_1\cos t_1 & u_1\sin t_1 \\ -u_1\sin t_1 & u_1\cos t_1 \end{pmatrix} =$$

$$\begin{pmatrix} u_2\cos t_2 & u_2\sin t_2 \\ -u_2\sin t_2 & u_2\cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}. \tag{4}$$

As the determinant of the matrices on both sides of (4) are equal we get that $u = u_1^{-1}u_2$. Therefore the system (4) of equations is uniquely solvable if and only if for any fixed $u \in \mathbb{R}\backslash\{0\}$ the mapping

$$\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix}$$

determines a quasigroup F_u homeomorphic to S^1. One may take as the points of F_u the vectors $(ua(u,t)a^{-1}(u,0)\cos t, -ua(u,t)a^{-1}(u,0)\sin t)^t$ and as the section the mapping

$$\sigma_u : \begin{pmatrix} ua(u,t)a^{-1}(u,0)\cos t \\ -ua(u,t)a^{-1}(u,0)\sin t \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u,t)a^{-1}(u,0) & b(u,t) \\ 0 & a^{-1}(u,t)a(u,0) \end{pmatrix} =$$

$$\begin{pmatrix} a(u,t)a^{-1}(u,0)\cos t & b(u,t)\cos t + a^{-1}(u,t)a(u,0)\sin t \\ -a(u,t)a^{-1}(u,0)\sin t & -b(u,t)\sin t + a^{-1}(u,t)a(u,0)\cos t \end{pmatrix}. \quad (5)$$

In this way we see that the quasigroup F_u has the right identity $(u,0)^t$ since

$$\sigma_u \begin{pmatrix} ua(u,t)a^{-1}(u,0)\cos t \\ -ua(u,t)a^{-1}(u,0)\sin t \end{pmatrix} \cdot \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} ua(u,t)a^{-1}(u,0)\cos t \\ -ua(u,t)a^{-1}(u,0)\sin t \end{pmatrix}.$$

The quasigroup F_u is a loop, i.e. $(u,0)^t$ is the left identity of F_u, if and only if

$$\sigma_u \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} a(u,0)a^{-1}(u,0)\cos 0 & b(u,0)\cos 0 \\ 0 & a^{-1}(u,0)a(u,0)\cos 0 \end{pmatrix} = \begin{pmatrix} 1 & b(u,0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means $b(u,0) = 0$ for all $u \in \mathbb{R} \setminus \{0\}$. The almost $\mathcal{C}^1$-differentiable loop Q^* belonging to the sharply transitive $\mathcal{C}^1$-differentiable section σ given by (2) is $\mathcal{C}^1$-differentiable precisely if the mapping $(xH, yH) \mapsto z : G/H \times G/H \to \sigma(G/H)$ determined by $zxH = yH$ is $\mathcal{C}^1$-differentiable (cf. [14], p. 32), i.e. the solutions $u \in \mathbb{R} \setminus \{0\}$, $t \in [0, 2\pi)$ of the matrix equation (4) are continuously differentiable functions of $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, $t_1, t_2 \in$

$[0, 2\pi)$. The function $u = u_1^{-1} u_2$ is continuously differentiable. If for each fixed $u \in \mathbb{R} \setminus \{0\}$ the section σ_u given by (5) yields a 1-dimensional $\mathcal{C}^1$-differentiable compact loop, then the function $t(u_1, u_2, t_1, t_2) = t_{(u_1,u_2)}(t_1, t_2)$ is continuously differentiable (cf. [14], Examples 20.3, p. 258). Indeed, the function $t_{(u_1,u_2)}(t_1, t_2)$ is determined implicitly by equations which depend continuously differentiably also on the parameters u_1 and u_2. Applying the above discussion we can prove the following:

Theorem 2. *Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q. Then Q^* is diffeomorphic to $S^1 \times \mathbb{R}$ and belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form*

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix},$$

with $b(u,0) = 0$ for all $u \in \mathbb{R} \setminus \{0\}$ if and only if for each fixed $u \in \mathbb{R} \setminus \{0\}$ the function $a_u^{-1}(t) := a(u,0)a^{-1}(u,t)$ has the shape

$$a_u^{-1}(t) = e^t\Big(1 - \int_0^t R(s)e^{-s}\, ds\Big)$$

where $R(s)$ is a continuous function, the Fourier series of which is contained in the set $\mathcal{F}$ of Definition 1 in [10] and converges uniformly to R. Moreover, $b_u(t) := b(u,t)$ is a periodic $\mathcal{C}^1$-differentiable function with $b_u(0) = b_u(2\pi) = 0$ such that

$$b_u(t) > -a_u(t) \int_0^t \frac{(a_u^2(s) - a_u'^2(s))}{a_u^4(s)}\, ds \quad for \quad all \quad t \in (0, 2\pi).$$

Proof. The section σ_u given by (5) yields a 1-dimensional $\mathcal{C}^1$-differentiable compact loop having the group $\mathrm{SL}_2(\mathbb{R})$ as the group topologically generated by its left translations if and only if for each fixed $u \in \mathbb{R}\setminus\{0\}$ the continuously differentiable functions $a(u,0)a^{-1}(u,t) := \bar{a}_u(t)$, $-b(u,t) := \bar{b}_u(t)$ satisfy the conditions

$$\bar{a}_u'^2(t) + \bar{b}_u(t)\bar{a}_u'(t) + \bar{b}_u'(t)\bar{a}_u(t) - \bar{a}_u^2(t) < 0, \;\; \bar{b}_u'(0) < 1 - \bar{a}_u'^2(0) \qquad (6)$$

(cf. [14], Section 18, (C), p. 238, [10], pp. 132-139). The solution of the differential inequalities (6) is given by Theorem 6 in [10], pp. 138-139. This proves the assertion. □

Proposition 3. *Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q. Assume that for each fixed $u \in \mathbb{R} \setminus \{0\}$ the function $a_u(t) := a^{-1}(u,0)a(u,t)$ is the constant function 1 and that $b(u,0) = 0$ is satisfied for all $u \in \mathbb{R} \setminus \{0\}$. Then Q^* belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form (2) if and only if for each fixed $u \in \mathbb{R} \setminus \{0\}$ one has $b_u(t) := b(u,t) > -t$ for all $0 < t < 2\pi$.*

Proof. If for each fixed $u \in \mathbb{R}\setminus\{0\}$ the function $a(u,0)a^{-1}(u,t) = a_u^{-1}(t) = \bar{a}_u(t)$ is constant with value 1, then the section σ_u given by (5) yields a $\mathcal{C}^1$-

differentiable compact loop L if and only if for each fixed $u \in \mathbb{R}\backslash\{0\}$ the continuously differentiable function $\bar{b}_u(t) := -b_u(t)$ satisfies the differential inequality $\bar{b}'_u(t) < 1$ with the initial condition $\bar{b}'_u(0) < 1$ (cf. (6)). This is the case precisely if one has $b_u(t) > -t$ for all $0 < t < 2\pi$. □

Proposition 4. *Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q. Assume that for each fixed $u \in \mathbb{R}\backslash\{0\}$ the function $b(u,t)$ is the constant function 0. Then Q^* belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form (2) precisely if for each fixed $u \in \mathbb{R}\backslash\{0\}$ one has $e^{-t} < a(u,t)a^{-1}(u,0) < e^t$ for all $0 < t < 2\pi$.*

Proof. If for each fixed $u \in \mathbb{R}\backslash\{0\}$ the function $b(u,t) = -\bar{b}_u(t)$ is constant with value 0, then the section σ_u given by (5) determines a $\mathcal{C}^1$-differentiable compact loop L if and only if for each fixed $u \in \mathbb{R}\backslash\{0\}$ the following inequalities are satisfied:

$$(\bar{a}'_u(t) - \bar{a}_u(t))(\bar{a}'_u(t) + \bar{a}_u(t)) < 0, \quad 0 < 1 - \bar{a}'^2_u(0),$$

where $\bar{a}_u(t) = a(u,0)a^{-1}(u,t)$. This is the case precisely if either one has $\bar{a}'_u(t) - \bar{a}_u(t) < 0$ and $\bar{a}'_u(t) + \bar{a}_u(t) > 0$ or one has $\bar{a}'_u(t) - \bar{a}_u(t) > 0$ and $\bar{a}'_u(t) + \bar{a}_u(t) < 0$. Now we consider the first case. Then the function $\bar{a}_u(t)$ determines a loop if and only if for each fixed $u \in \mathbb{R}\backslash\{0\}$ it is a subfunction of a differentiable function $h_u(t) := h(u,t)$ with $h_u(0) = 1$, $h'^2_u(0) = 1$,

$h'_u(t) = h_u(t)$ and an upper function of a differentiable function $l_u(t) := l(u,t)$ with $l_u(0) = 1$, $l'^2_u(0) = 1$, $l'_u(t) = -l_u(t)$ (cf. [19], p. 66). Hence for each fixed $u \in \mathbb{R}\backslash\{0\}$ the function $\bar{a}_u(t)$ is a subfunction of the function e^t and an upper function of the function e^{-t} for all $t \in (0, 2\pi)$. Therefore, any continuously differentiable function $\bar{a}_u(t)$ such that for each fixed $u \in \mathbb{R}\backslash\{0\}$ and for all $t \in (0, 2\pi)$ one has $e^{-t} < \bar{a}_u(t)^{-1} < e^t$ determines a $\mathcal{C}^1$-differentiable compact loop L.

In the second case an analogous consideration as in the first case gives that for all fixed $u \in \mathbb{R}\backslash\{0\}$ the function $a(u,t)a^{-1}(u,0)$ must be a subfunction of the function e^{-t} and an upper function of the function e^t for all $t \in (0, 2\pi)$. Hence in this case the function $a(u,t)a^{-1}(u,0)$ does not exist. □

Proposition 5. *Let*

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, u \in \mathbb{R}\setminus\{0\}, t \in \mathbb{R} \quad (7)$$

with $b(u,0) = 0$ for all $u \in \mathbb{R}\setminus\{0\}$ be a section belonging to a multiplicative loop Q^* of a locally compact 2-dimensional connected topological quasifield Q. Then Q^* contains for any $u \in \mathbb{R}\setminus\{0\}$ a 1-dimensional compact subloop.

Proof. The image of the section (7) acts sharply transitively on the point set $\mathbb{R}^2\setminus\{(0,0)^t\}$. Since the subgroup $\left\{\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u \in \mathbb{R}\setminus\{0\}\right\}$ leaves any line through $(0,0)^t$ fixed, the subset

$$\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in \mathbb{R} \right\} \tag{8}$$

acts sharply transitively on the oriented lines through $(0,0)^t$ for any $u \in \mathbb{R} \setminus \{0\}$. Therefore $\mathcal{T}$ corresponds to a 1-dimensional compact loop since $b(u,0) = 0$ for all $u \in \mathbb{R} \setminus \{0\}$. □

As $\mathcal{T}$ given by (8) is the image of a section corresponding to a 1-dimensional compact subloop of Q^*, every element of $\mathcal{T}$ is elliptic.

Proposition 6. *Every element of the set $\mathcal{T}$ given by (8) is elliptic if and only if the following holds:*

1) if for all $t \in \mathbb{R}$ and $u \in \mathbb{R} \setminus \{0\}$ one has $b(u,t) = 0$, then the function $a(1,t)$ satisfies the inequalities:

$$\frac{1 - |\sin(t)|}{|\cos(t)|} \leq a(1,t) \leq \frac{1 + |\sin(t)|}{|\cos(t)|}, \tag{9}$$

2) if the function $b(u,t)$ is different from the constant function 0, then for $\sin(t) > 0$ one has

$$\frac{(a(1,t) + a(1,t)^{-1})\cos(t) - 2}{\sin(t)} < b(u,t) < \frac{(a(1,t) + a(1,t)^{-1})\cos(t) + 2}{\sin(t)}, \tag{10}$$

for $\sin(t) < 0$ we have

$$\frac{(a(1,t) + a(1,t)^{-1})\cos(t) + 2}{\sin(t)} < b(u,t) < \frac{(a(1,t) + a(1,t)^{-1})\cos(t) - 2}{\sin(t)}. \tag{11}$$

Proof. Any element of (8) is elliptic if and only if the inequality

$$|\cos(t)(a(1,t)+a(1,t)^{-1})-\sin(t)b(u,t)|\leq 2 \tag{12}$$

holds, where the equality sign occurs only for $t=k\pi$, $k\in\mathbb{Z}$. If $b(u,t)=0$, then inequality (12) reduces to $a^2(1,t)|\cos(t)|-2a(1,t)+|\cos(t)|\leq 0$ which is equivalent to inequalities (9). If $b(u,t)\neq 0$, then inequality (12) is equivalent for all $t\neq k\pi$, $k\in\mathbb{Z}$, to

$$(a(1,t)+a(1,t)^{-1})^2\cos^2(t)-2(a(1,t)+a(1,t)^{-1})\sin(t)\cos(t)b(u,t)+\sin^2(t)b^2(u,t)<4. \tag{13}$$

Solving the quadratic equation

$$(a(1,t)+a(1,t)^{-1})^2\cos^2(t)-2(a(1,t)+a(1,t)^{-1})\sin(t)\cos(t)x+\sin^2(t)x^2=4 \tag{14}$$

we get

$$x=\frac{2(a(1,t)+a(1,t)^{-1})\cos(t)\sin(t)\pm 4\sin(t)}{2\sin^2(t)}=\frac{(a(1,t)+a(1,t)^{-1})\cos(t)\pm 2}{\sin(t)}.$$

Comparing (13) and (14) one obtains

$$\left(b(u,t)-\frac{(a(1,t)+a(1,t)^{-1})\cos(t)-2}{\sin(t)}\right)\left(b(u,t)-\frac{(a(1,t)+a(1,t)^{-1})\cos(t)+2}{\sin(t)}\right)<0$$

which yields inequalities (10) and (11). □

Proposition 7. *The multiplicative loop Q^* of a locally compact connected topological quasifield Q of dimension 2 is the field $\mathbb{C}$ of complex numbers if and only if it contains a 1-dimensional compact normal subloop.*

Proof. If Q is the field of complex numbers, then Q^* is the group $\mathrm{SO}_2(\mathbb{R})\times\mathbb{R}$ and the assertion is true. Assume that the loop Q^* contains a 1-dimensional compact normal subloop. If Q^* is a proper loop, then the group topologically generated by its left translations is the connected component $\mathrm{GL}_2^+(\mathbb{R})$ of $\mathrm{GL}_2(\mathbb{R})$ (cf. [14], Theorem 29.1, p. 345). By Lemma 1.7, p. 19, in [14], the left translations of a normal subloop of Q^* generate a normal subgroup N of

$\mathrm{GL}_2^+(\mathbb{R})$ which can be only the group $SL_2(\mathbb{R})$. This contradiction proves the assertion. $\square$

Lemma 8. *If the multiplicative loop Q^* of a 2-dimensional locally compact connected topological quasifield Q is not quasi-simple, then the set $\mathcal{K} = \{M(ra(r,0),0); r \in \mathbb{R} \setminus \{0\}\}$ of the left translations of Q^* corresponding to the kernel K_r of Q has the form $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$, which is a normal subgroup of the set Λ_{Q^*} of all left translations of Q^*.*

Proof. If Q is the field of complex numbers, then the assertion is true. If the loop Q^* is proper and not quasi-simple, then the set Λ_{Q^*} of the left translations of Q^* must contain the group $\mathcal{K} < GL_2^+(\mathbb{R})$ as a normal subgroup. $\square$

Assume that the set $\mathcal{K}$ of the left translations of the loop Q^* having $(1,0)^t$ as identity corresponding to the elements of the kernel K_r of Q has the form given in Lemma 8. According to (3) the element

$$\begin{pmatrix} ra(r,\varphi)\cos\varphi & rb(r,\varphi)\cos\varphi + ra^{-1}(r,\varphi)\sin\varphi \\ -ra(r,\varphi)\sin\varphi & -rb(r,\varphi)\sin\varphi + ra^{-1}(r,\varphi)\cos\varphi \end{pmatrix}$$

corresponds to the left translation of $(ra(r,\varphi)\cos\varphi, -ra(r,\varphi)\sin\varphi)^t$. Let N^* be the subgroup of Q^* corresponding to the normal subgroup $\mathcal{K}$ of Λ_{Q^*}. We show that $N^* := \{(s,0)^t, s \in \mathbb{R} \setminus \{0\}\}$ is normal in Q^*. For all elements $x := (\cos\varphi, -\sin\varphi)^t$, $y := (u,v)^t$ of Q^* the condition $(N^* * x) * y = N^* * (x * y)$

of (1) is satisfied if and only if we have

$$\left[\begin{pmatrix} s \\ 0 \end{pmatrix} * \begin{pmatrix} cos\varphi \\ -sin\varphi \end{pmatrix}\right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} s' \\ 0 \end{pmatrix} * \left[\begin{pmatrix} cos\varphi \\ -sin\varphi \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix}\right]$$

for all $\varphi \in \mathbb{R}$, $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with suitable $s, s' \in \mathbb{R} \setminus \{0\}$. This is the case precisely if one has

$$\begin{pmatrix} usa(s,\varphi)\cos\varphi + vsb(s,\varphi)\cos\varphi + vsa^{-1}(s,\varphi)\sin\varphi \\ -usa(s,\varphi)\sin\varphi - vsb(s,\varphi)\sin\varphi + vsa^{-1}(s,\varphi)\cos\varphi \end{pmatrix} =$$

$$\begin{pmatrix} s'a(1,\varphi)\cos\varphi u + vs'b(1,\varphi)\cos\varphi + vs'a^{-1}(1,\varphi)\sin\varphi \\ -s'a(1,\varphi)\sin\varphi u - vs'b(1,\varphi)\sin\varphi + vs'a^{-1}(1,\varphi)\cos\varphi \end{pmatrix}$$

or equivalently for all $\varphi \in \mathbb{R}$, $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ we have

$$[ua(s,\varphi)\cos\varphi + vb(s,\varphi)\cos\varphi + va^{-1}(s,\varphi)\sin\varphi] \cdot [-ua(1,\varphi)\sin\varphi - vb(1,\varphi)\sin\varphi + va^{-1}(1,\varphi)\cos\varphi] =$$

$$[-ua(s,\varphi)\sin\varphi - vb(s,\varphi)\sin\varphi + va^{-1}(s,\varphi)\cos\varphi] \cdot [ua(1,\varphi)\cos\varphi + vb(1,\varphi)\cos\varphi + va^{-1}(1,\varphi)\sin\varphi].$$

The last equation holds if and only if

$$a(s,\varphi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi)a(1,\varphi))uv + (b(s,\varphi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi)b(1,\varphi))v^2 = 0$$

and hence

$$(a(s,\varphi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi)a(1,\varphi) = 0, b(s,\varphi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi)b(1,\varphi) = 0.$$

As $a(s, \varphi)$ is positive we have $a(s, \varphi) = a(1, \varphi)$ and $b(s, \varphi) = b(1, \varphi)$ for all $s \in \mathbb{R} \setminus \{0\}$, $\varphi \in \mathbb{R}$. By (1) the group N^* is a normal subgroup of Q^* if and only if for all φ and all $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ one has

$$\left[\begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} * \begin{pmatrix} s \\ 0 \end{pmatrix}\right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} * \left[\begin{pmatrix} s' \\ 0 \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix}\right] \quad \text{or}$$

$$\begin{pmatrix} sa(1,\varphi)a(1,\varphi)\cos\varphi & sa(1,\varphi)b(1,\varphi)\cos\varphi + ssin\varphi \\ -sa(1,\varphi)a(1,\varphi)\sin\varphi & -sa(1,\varphi)b(1,\varphi)\sin\varphi + scos\varphi \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} =$$

$$\begin{pmatrix} a(1,\varphi)\cos\varphi & b(1,\varphi)\cos\varphi + a^{-1}(1,\varphi)sin\varphi \\ -a(1,\varphi)\sin\varphi & -b(1,\varphi)\sin\varphi + a^{-1}(1,\varphi)cos\varphi \end{pmatrix} \begin{pmatrix} s'u \\ s'v \end{pmatrix}$$

for suitable $s, s' \in \mathbb{R} \setminus \{0\}$. This is equivalent to

$$\begin{pmatrix} sua(1,\varphi)^2\cos\varphi + sv[a(1,\varphi)b(1,\varphi)\cos\varphi + \sin\varphi] \\ -sua(1,\varphi)^2\sin\varphi + sv[-a(1,\varphi)b(1,\varphi)\sin\varphi + \cos\varphi] \end{pmatrix} =$$

$$\begin{pmatrix} us'a(1,\varphi)\cos\varphi + s'v[b(1,\varphi)\cos\varphi + a^{-1}(1,\varphi)\sin\varphi] \\ -us'a(1,\varphi)\sin\varphi + s'v[-b(1,\varphi)\sin\varphi + a^{-1}(1,\varphi)\cos\varphi] \end{pmatrix}.$$

A direct computation yields that

$$[ua(1,\varphi)^2\cos\varphi + va(1,\varphi)b(1,\varphi)\cos\varphi + v\sin\varphi]\cdot[-ua(1,\varphi)\sin\varphi - vb(1,\varphi)\sin\varphi + va^{-1}(1,\varphi)\cos\varphi] =$$

$$[-ua(1,\varphi)^2\sin\varphi - va(1,\varphi)b(1,\varphi)\sin\varphi + vcos\varphi]\cdot[ua(1,\varphi)\cos\varphi + vb(1,\varphi)\cos\varphi + va^{-1}(1,\varphi)\sin\varphi].$$

Using Proposition 7, Lemma 8 and the discussion above we have the following

Theorem 9. *The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is not quasi-simple if and only if for all $r \in \mathbb{R}\backslash\{0\}$, $\varphi \in \mathbb{R}$ one has $a(r,0) = 1$, $b(r,0) = 0$, $a(r,\varphi) = a(1,\varphi)$ and $b(r,\varphi) = b(1,\varphi)$. Then Q^* is a split extension of a 1-dimensional normal subgroup N^* by a subloop homeomorphic to the 1-sphere. Moreover, one has*

a) N^* is isomorphic to $\mathbb{R}$ or to $\mathbb{R} \times Z_2$, where Z_2 is the group of order 2.

b) This extension is the direct product precisely if Q is the field $\mathbb{C}$.

Proof. We have only to prove a) and b). According to Lemma 8 and the above discussion the only possibility for a normal subloop of positive dimension is the group N^*. The intersection of a compact subloop of Q^* with N^* has cardinality at most 2 (cf. Proposition 5 and Lemma 8). Hence the claim a) is proved. The claim of b) follows from Proposition 7. $\square$

The set Λ_{Q^*} of the left translations of Q^* with a normal subloop of positive dimension has the form

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1,t) & ub(1,t) \\ 0 & ua^{-1}(1,t) \end{pmatrix}, u \in \mathbb{R}\backslash\{0\}, t \in [0, 2\pi) \right\}. \tag{15}$$

5. Decomposable multiplicative loops of 2-dimensional quasifields

Definition 1. *We call the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q* decomposable*, if the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \le 2$, where $\mathcal{T}$ is the set of all left translations of a 1-dimensional compact loop of the form (8) and $\mathcal{K}$ is the set of all left translations of Q^* corresponding to the kernel K_r of Q.*

If the loop Q^* is decomposable, then it contains compact subloops for any $u \in \mathbb{R} \setminus \{0\}$ corresponding to the section (7). From now on we choose $u = 1$.

Then one has

$$\begin{pmatrix} \cos ta(1,t) & \cos tb(1,t)+\sin ta^{-1}(1,t) \\ -\sin ta(1,t) & -\sin tb(1,t)+\cos ta^{-1}(1,t) \end{pmatrix} \left[\begin{pmatrix} ra(r,0) & rb(r,0) \\ 0 & ra^{-1}(r,0) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] =$$

$$\begin{pmatrix} r\cos ta(r,t) & r\cos tb(r,t)+r\sin ta^{-1}(r,t) \\ -r\sin ta(r,t) & -r\sin tb(r,t)+r\cos ta^{-1}(r,t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{16}$$

Equation (16) yields that $a(r,t) = a(1,t)a(r,0)$.

Now we give sufficient and necessary conditions for the loop Q^* to be decomposable.

Proposition 10. *The multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is decomposable if and only if for all $r \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ one has*

$$a(r,t) = a(1,t)a(r,0) \text{ and } b(r,t) = a(1,t)b(r,0) + a^{-1}(r,0)b(1,t).$$

Proof. The point $(x,y)^t$ is the image of the point $(1,0)^t$ under the linear mapping $M_{(x,y)}$ and the set $\{M_{(x,y)}; (x,y)^t \in Q^*\}$ acts sharply transitively on Q^*. The matrix equation

$$\begin{pmatrix} \cos ta(1,t) & \cos tb(1,t)+\sin ta^{-1}(1,t) \\ -\sin ta(1,t) & -\sin tb(1,t)+\cos ta^{-1}(1,t) \end{pmatrix} \left[\begin{pmatrix} ra(r,0) & rb(r,0) \\ 0 & ra^{-1}(r,0) \end{pmatrix} \begin{pmatrix} u\cos\varphi a(u,\varphi) \\ -u\sin\varphi a(u,\varphi) \end{pmatrix} \right] =$$

$$\begin{pmatrix} r\cos ta(r,t) & r\cos tb(r,t)+r\sin ta^{-1}(r,t) \\ -r\sin ta(r,t) & -r\sin tb(r,t)+r\cos ta^{-1}(r,t) \end{pmatrix} \begin{pmatrix} u\cos\varphi a(u,\varphi) \\ -u\sin\varphi a(u,\varphi) \end{pmatrix} \tag{17}$$

holds precisely if the identities of the assertion are satisfied. □

Theorem 11. *If the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q is not quasi-simple, then Q^* is decomposable.*

Proof. By Theorem 9 the loop Q^* is not quasi-simple if and only if for all $r \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ one has $a(r,0) = 1$, $b(r,0) = 0$, $a(r,t) = a(1,t)$ and $b(r,t) = b(1,t)$. Therefore the identities given in the assertion of Proposition 10 are satisfied. □

If Q^* is decomposable, then $|\mathcal{T} \cap \mathcal{K}| = 1$ if and only if one has $a(1,0) = a(-1,0) = a(1,\pi) = 1$ and $b(1,0) = b(-1,0) = b(1,\pi) = 0$, since $a(-1,\pi) = a(-1,0)a(1,\pi) = 1$ as well as $b(-1,\pi) = b(-1,0)a(1,\pi) = 0$. In this case the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{W}$ with $\mathcal{T} \cap \mathcal{W} = I$, where $\mathcal{W}$ is the set of all left translations corresponding to the connected component of the kernel K_r of Q. We say in this case that Q^* is *positively decomposable.*

Proposition 12. *The $\mathcal{C}^1$-differentiable multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q is decomposable precisely if for the inverse function $\bar{a}(1,t) = a^{-1}(1,t)$ and for $\bar{b}(1,t) = -b(1,t)$ the differential inequalities*

$$\bar{a}'^2(1,t) + \bar{b}(1,t)\bar{a}'(1,t) + \bar{b}'(1,t)\bar{a}(1,t) - \bar{a}^2(1,t) < 0, \ \textit{and}$$

$$\bar{b}'(1,0) < 1 - \bar{a}'^2(1,0) \tag{18}$$

are satisfied.

Proof. If Q^* is a $\mathcal{C}^1$-differentiable multiplicative loop of a quasifield Q, then the continuously differentiable functions $a(u,t) = \bar{a}^{-1}(u,t)$, $b(u,t) = -\bar{b}(u,t)$

satisfy the conditions in (6). The set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$ if and only if $a(u,t) = a(u,0)a(1,t)$ and $b(u,t) = a(1,t)b(u,0) + a^{-1}(u,0)b(1,t)$ (cf. Proposition 10). Putting this into (6) we get

$$a'^2(1,t) + b(1,t)a'(1,t)a^2(1,t) - b'(1,t)a^3(1,t) - a^2(1,t) < 0 \text{ and}$$

$$b'(1,0) > a'^2(1,0) - 1. \tag{19}$$

Inequalities (19) are equivalent to the inequalities (18) with $\bar{a}(1,t) = a^{-1}(1,t)$ and $\bar{b}(1,t) = -b(1,t)$. □

Corollary 13. *Let T be any 1-dimensional $\mathcal{C}^1$-differentiable connected compact loop such that the set $\mathcal{T}$ of its left translations has the form (8) and let $\mathcal{K}$ be any set of matrices of the form*

$$\mathcal{K} = \left\{ \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix}, 0 \neq u \in \mathbb{R} \right\},$$

where $a(u,0) > 0$ and $b(u,0)$ are continuously differentiable functions such that $ua(u,0)$ is strictly monotone. Then the product $\mathcal{T}\mathcal{K}$ is the set of all left translations of a $\mathcal{C}^1$-differentiable decomposable multiplicative loop Q^* of a 2-dimensional locally compact connected quasifield Q.

Proof. As

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(1,t) \\ 0 & a(1,t)^{-1} \end{pmatrix} \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix} =$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u,0)a(1,t) & ub(u,0)a(1,t) + ub(1,t)a^{-1}(u,0) \\ 0 & ua^{-1}(u,0)a(1,t)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u,t) & ub(u,t) \\ 0 & ua^{-1}(u,t) \end{pmatrix}$$

and for the continuously differentiable functions $a(1,t)$, $b(1,t)$ the inequalities (19) hold, for each fixed $u \in \mathbb{R}\backslash\{0\}$ the continuously differentiable functions $\bar{a}^{-1}(u,t) = a(u,t) = a(u,0)a(1,t)$, $-\bar{b}(u,t) = b(u,t) = b(u,0)a(1,t) + b(1,t)a^{-1}(u,0)$ satisfy inequalities (6). Hence the product $\mathcal{TK}$ given in the assertion is the image of a $\mathcal{C}^1$-differentiable section of a multiplicative loop Q^* of a quasifield Q. □

Proposition 14. *The set Λ_{Q^*} of all left translations of the multiplicative loop Q^* for a locally compact connected topological 2-dimensional quasifield Q contains the group $\mathrm{SO}_2(\mathbb{R})$ if and only if Λ_{Q^*} has the form*

$$\Lambda_{Q^*} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\} \tag{20}$$

where $a(u,0)$, $b(u,0)$ are arbitrary continuous functions with $a(u,0) > 0$ such that $ua(u,0)$ is strictly monotone. In this case Q^* is positively decomposable.

Proof. If the set Λ_{Q^*} contains the group $\mathrm{SO}_2(\mathbb{R})$, then for each fixed $u \in \mathbb{R} \setminus \{0\}$ the function $a_u(t)$ is constant with value 1 and the function $b_u(t)$ is constant with value 0. So the functions $a(u,t) = a(u,0)$, $b(u,t) = b(u,0)$

do not depend on the variable t. Hence the identities in Proposition 10 are satisfied and the set Λ_{Q^*} has the form as in the assertion.

Conversely, if $ua(u,0)$ is a strictly monotone continuous function, then for arbitrary continuous functions $a(u,0), b(u,0)$ with $a(u,0) > 0$ the set given by (20) is the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* of a locally compact quasifield such that Λ_{Q^*} contains the group $\mathrm{SO}_2(\mathbb{R})$.

Furthermore, Q^* is positively decomposable because $a(1,\pi) = 1$, $b(1,\pi) = 0$, $a(-1,\pi) = a(-1,0)a(1,\pi) = 1$ and $b(-1,\pi) = b(-1,0)a(1,\pi) = 0$. $\square$

6. Betten's classification of 4-dimensional translation planes

Using 2-dimensional spreads, Betten in [3], [4], [5], [6], [7], [8], see also [13] and [15], has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group. His normalized 2-dimensional spreads are images of sharply transitive sections $\sigma' : G/H' \to G$, where G is the connected component of the group $\mathrm{GL}_2(\mathbb{R})$, H' is the subgroup $\left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, d > 0, c \in \mathbb{R} \right\}$ (cf. [2], [3]) and $\sigma'(G/H')$ consists of the elements

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r,t) & 0 \\ 0 & r^{-1}a^{-1}(r,t) \end{pmatrix} \begin{pmatrix} 1 & b(r,t)a^{-1}(r,t) \\ 0 & r^2 \end{pmatrix}.$$

With respect to the stabilizer $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ the sharply transitive section σ' transforms into a sharply transitive section $\sigma : G/H \to G$ given by (2), because the elements of $\sigma'(G/H')$ coincide with

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a(r,t) & b(r,t) \\ 0 & a^{-1}(r,t) \end{pmatrix}.$$

Proposition 15. *Let $\mathcal{A}$ be a 4-dimensional non-desarguesian translation plane admitting an 8-dimensional collineation group such that $\mathcal{A}$ is coordinatized by the locally compact topological quasifield Q. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*

a) Λ_{Q^*} has the form (15) with $a(1,t) = 1$ and $b(1,t) = 0$ for $0 \le t \le \pi$, $a(1,t) = 1/\sqrt{\cos^2 t + \frac{\sin^2 t}{w}}$ and $b(1,t) = a(1,t)\frac{1-w}{w}\sin t \cos t$ for $\pi < t < 2\pi$. The quasifields Q_w, $w > 1$, correspond to a one-parameter family of planes $\mathcal{A}_w$.

b) Λ_{Q^*} is the range of the section given by (2) such that for $\alpha \ge \frac{-3\beta^2}{4}$ one has $a(r,t) = \sqrt{\frac{\alpha^2+\beta^2}{(\alpha+\beta^2)^2}}$ and $b(r,t) = \varepsilon\frac{\beta(-\alpha+1)}{\sqrt{\alpha^2+\beta^2}}$, where $\varepsilon = 1$ for $\alpha + \beta^2 > 0$ and $\varepsilon = -1$ for $\alpha + \beta^2 < 0$ with $r\cos(t) = \alpha\sqrt{\frac{(\alpha+\beta^2)^2}{\alpha^2+\beta^2}}$, $r\sin(t) = -\beta\sqrt{\frac{(\alpha+\beta^2)^2}{\alpha^2+\beta^2}}$. For $\alpha < \frac{-3\beta^2}{4}$ we have $a(r,t) = 3\sqrt{\frac{\alpha^2+\beta^2}{\alpha^2}}$ and $b(r,t) = \beta\sqrt{\frac{\alpha^2+\beta^2}{\alpha^2}} + \frac{\beta\alpha}{3\sqrt{\alpha^2(\alpha^2+\beta^2)}}$ with $r\cos(t) = \frac{\alpha}{3}\sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}}$, $r\sin(t) = -\frac{\beta}{3}\sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}}$. The quasifield Q coordinatizes a single plane.

c) Λ_{Q^*} *is the range of the section given by (2) such that* $a(r,t) = \sqrt{\frac{v^2+s^2}{\frac{s^4}{3}+s^2v+v^2}}$*,* $b(r,t) = \frac{-\frac{s^3v}{3}+s^3+sv}{\sqrt{(\frac{s^4}{3}+s^2v+v^2)(s^2+v^2)}}$ *with*

$$r\cos(t) = v\sqrt{\frac{\frac{s^4}{3}+s^2v+v^2}{s^2+v^2}}, \quad r\sin(t) = -s\sqrt{\frac{\frac{s^4}{3}+s^2v+v^2}{s^2+v^2}}.$$

The quasifield Q *coordinatizes a single plane.*

In case a) the multiplicative loop Q_w^* *is positively decomposable and a split extension of the normal subgroup* $N^* \cong \mathbb{R}$ *corresponding to the connected component of* $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ *with a subloop homeomorphic to the* 1*-sphere. In cases b) and c) the set of the left translations of* Q^* *corresponding to the kernel* K_r *of the quasifield* Q *has the form* $\mathcal{K}$*. The multiplicative loops* Q^* *are not decomposable and quasi-simple.*

Proof. If the translation complement of $\mathcal{A}$ is the group $\mathrm{GL}_2(\mathbb{R})$ and acts reducibly on $\mathbb{R}^4$, then one obtains the one-parameter family $\mathcal{A}_w$, $w > 1$, of the non-desarguesian translation planes corresponding to the following spreads:

$$\{S\} \cup \left\{ \begin{pmatrix} s & -v \\ v & s \end{pmatrix}, s,v \in \mathbb{R}, v \geq 0 \right\} \cup \left\{ \begin{pmatrix} s & \frac{-v}{w} \\ v & s \end{pmatrix}, s,v \in \mathbb{R}, v < 0 \right\},$$

$w > 1$ (cf. [3], Satz 5, p. 144). Any such spread coincides with the set Λ in (15) with $a(1,t)$ and $b(1,t)$ as in assertion a). By Theorem 9 the multiplicative loop Q_w^* is a split extension of a normal subgroup N^* with a

1-dimensional compact loop. By Theorem 11 the loop Q_w^* is decomposable. As $a(\pm 1, 0) = a(1, \pi) = 1$, $b(\pm 1, 0) = b(1, \pi) = 0$ the loop Q_w^* is positively decomposable. Hence N^* has the form as in the assertion.
If the translation complement $\mathrm{GL}_2(\mathbb{R})$ acts irreducibly on $\mathbb{R}^4$, then one obtains a single plane $\mathcal{A}$ generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & -\alpha\beta - \beta^3 \\ \beta & \alpha + \beta^2 \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha \geq \frac{-3\beta^2}{4} \right\} \cup \left\{ \begin{pmatrix} \alpha & \frac{1}{3}\alpha\beta \\ \beta & \frac{\alpha}{9} + \frac{\beta^2}{3} \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha < \frac{-3\beta^2}{4} \right\}, \tag{21}$$

(cf. [5], Satz, p. 553).

If the translation complement is solvable, then one gets a single plane $\mathcal{A}$ generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\}, \tag{22}$$

(cf. [4], Satz 2 (b), p. 331).

The spread (21), respectively (22) coincides with the image of the section σ in (2) with the well defined functions $a(r, t)$ and $b(r, t)$ given in assertion b), respectively c). Since in both cases one has $a(r, 0) = 1$, $b(r, 0) = 0$, Remark 1 gives the form $\mathcal{K}$ of the assertion.

For decomposable Q^*, the identity $a(r, t) = a(1, t)$ holds for all $r \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ (cf. Proposition 10). In case b) for $-3 \leq \alpha \leq 1$ one has $a(1, t) = \sqrt{\alpha^2 - \alpha + 1}$ which yields a contradiction. In case c) we have $a(r, \frac{\pi}{4}) = \sqrt{\frac{2}{1 - s + \frac{s^2}{3}}}$, $s \in \mathbb{R} \setminus \{0\}$ and the condition $a(r, \frac{\pi}{4}) = a(1, \frac{\pi}{4})$ gives a contradiction. Hence in both cases Q^* is not decomposable and therefore quasi-simple (cf. Theorem 11). $\square$

If the translation complement of a 4-dimensional topological plane $\mathcal{A}$ has dimension 3, then the point ∞ of the line $S = \{(0,0,u,v), u,v \in \mathbb{R}\}$ is fixed under the seven-dimensional collineation group Γ of $\mathcal{A}$.

Proposition 16. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the 7-dimensional collineation group Γ of $\mathcal{A}$ acts transitively on the points of $W \setminus \{\infty\}$, where W is the translation axis of $\mathcal{A}$ and the kernel of the action of the translation complement on the line S has dimension 1. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*

a) Λ_{Q^*} is the range of the section (2) such that

$$a(r,t) = \sqrt{\frac{s^2+v^2}{s^2v+v^2+\frac{s^4}{3}+s^2}} \text{ and } b(r,t) = \frac{s^3-\frac{s^3v}{3}}{\sqrt{(s^2v+v^2+\frac{s^4}{3}+s^2)(s^2+v^2)}}$$

with $r\cos(t) = v\sqrt{\frac{s^2v+v^2+\frac{s^4}{3}+s^2}{s^2+v^2}}$, $r\sin(t) = -s\sqrt{\frac{s^2v+v^2+\frac{s^4}{3}+s^2}{s^2+v^2}}$. The quasifield Q corresponds to a single plane.

b) Λ_{Q^*} is the range of the section given by (2) such that

$$a(r,t) = \sqrt{\frac{v^2+u^2+2\gamma^2(1-\cos(u))-2v\gamma\sin(u)-2\gamma u\cos(u)+2\gamma u}{v^2+u^2-2\gamma^2+2\gamma^2\cos(u)}} \text{ and}$$

$$b(r,t) = \frac{-2u\gamma\sin u+2v\gamma\cos u-2v\gamma}{\sqrt{v^2+u^2+2\gamma^2(1-\cos u)-2v\gamma\sin u-2\gamma u\cos u+2\gamma u}\sqrt{v^2+u^2-2\gamma^2(1-\cos u)}}$$

with

$$r\cos(t) = (v-\gamma\sin(u))\sqrt{\frac{v^2+u^2-2\gamma^2+2\gamma^2\cos u}{v^2+u^2+2\gamma^2(1-\cos(u))-2v\gamma\sin(u)-2\gamma u\cos(u)+2\gamma u}},$$

$$r\sin(t) = (u-\gamma(\cos(u)-1))\sqrt{\frac{v^2+u^2-2\gamma^2+2\gamma^2\cos u}{v^2+u^2+2\gamma^2(1-\cos(u))-2v\gamma\sin(u)-2\gamma u\cos(u)+2\gamma u}}.$$

The quasifields Q_γ coordinatize a one-parameter family of planes $\mathcal{A}_\gamma, 0 < |\gamma| \le 1$.

In all cases the multiplicative loop Q^* is not decomposable and quasi-simple. The set $\mathcal{K}$ of the left translations of Q^* corresponding to the kernel of the quasifield Q has the form $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$.

Proof. If the translation complement C leaves a 1-dimensional subspace of S invariant, then one obtains a single plane $\mathcal{A}$ which corresponds to the following spread:

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} - s \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\} \tag{23}$$

(cf. [18], 73.10., [4], pp. 330-331).

If the translation complement acts transitively on the 1-dimensional subspaces of S, then one gets a one-parameter family $E_\gamma, 0 < |\gamma| \le 1$, of planes which are generated by the normalized spread

$$\{S\} \cup \left\{ \begin{pmatrix} v - \gamma \sin u & u + \gamma(\cos u - 1) \\ \gamma(\cos u - 1) - u & v + \gamma \sin u \end{pmatrix}, u, v \in \mathbb{R} \right\}, \tag{24}$$

([8], Satz, p. 128, [13], Proposition 5.8). The spread (23), respectively (24) coincides with the image of the section σ in (2) such that the well defined functions $a(r,t)$ and $b(r,t)$ are given in assertion a), respectively b). Since in both cases one has $a(r,0) = 1, b(r,0) = 0$, Remark 1 gives the form of $\mathcal{K}$. Moreover, in case a) one has $a(r, \frac{\pi}{4}) = \sqrt{\frac{2}{2+v+\frac{v^2}{3}}}$, $v \in \mathbb{R} \setminus \{0\}$. In case b) for $v = 1$ we get

$$a(r_j, t_j) = \sqrt{\frac{1 + u^2 + 2\gamma^2(1 - \cos u) - 2\gamma \sin u - 2\gamma u \cos u + 2\gamma u}{1 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}}, \quad a(1, t_j) = 1.$$

For decomposable Q^* one has $a(r,t) = a(1,t)$ for all $r \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ (cf.

Proposition 10) which yields a contradiction. Thus in both cases Q^* is not decomposable and hence quasi-simple (cf. Theorem 11). □

Proposition 17. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group Γ of $\mathcal{A}$ has an orbit of dimension 1 on $W \setminus \{0\}$, C leaves in the set of lines through the origin only S fixed and the kernel of its action on S has positive dimension. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*
a) Λ_{Q^*} is the range of the section (2) such that for $\beta \geq 0$ one has

$$a(r,t) = \sqrt{\frac{\alpha^2+\beta^2}{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}} \text{ and } b(r,t) = \frac{w\alpha\beta^{\frac{1-s}{1+s}} + \alpha\beta + z\beta^{\frac{2+s}{1+s}}}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}}$$

$$\text{with } r\cos(t) = \alpha\sqrt{\frac{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}{\alpha^2+\beta^2}},\ r\sin(t) = -\beta\sqrt{\frac{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}{\alpha^2+\beta^2}}.$$

For $\beta < 0$ one gets

$$a(r',t) = \sqrt{\frac{\alpha^2+\beta^2}{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}} \text{ and } b(r',t) = \frac{p\alpha(-\beta)^{\frac{1-s}{1+s}} + \alpha\beta - q(-\beta)^{\frac{2+s}{1+s}}}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}}$$

with

$$r'\cos(t) = \alpha\sqrt{\frac{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}{\alpha^2+\beta^2}} \text{ and } r'\sin(t) = -\beta\sqrt{\frac{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}{\alpha^2+\beta^2}}.$$

The quasifields $Q_{s,w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{s,w,z,p,q}$ such that the parameters s, w, z, p, q satisfy the conditions $0 < s < 1$, $z^2 + 4w(1-s^2) \leq 0$, $q^2 - 4p(1-s^2) \leq 0$.

b) Λ_{Q^*} is the range of the section (2) such that for $\beta \geq 0$ we have

$$a(r,t) = \sqrt{\frac{\alpha^2+\beta^2}{\alpha^2+z\alpha\beta-w\beta^2+2\alpha\beta\ln\beta+z\beta^2\ln\beta+\beta^2(\ln\beta)^2}} \quad and$$

$$b(r,t) = \frac{(w+1)\alpha\beta+z\beta^2-z\alpha\beta\ln\beta-\alpha\beta(ln\beta)^2+2\beta^2\ln\beta}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2+z\alpha\beta+2\alpha\beta\ln\beta-w\beta^2+z\beta^2\ln\beta+\beta^2(\ln\beta)^2}}$$

with

$$r\cos(t) = \alpha\sqrt{\frac{\alpha^2+z\alpha\beta-w\beta^2+2\alpha\beta\ln\beta+z\beta^2\ln\beta+\beta^2(\ln\beta)^2}{\alpha^2+\beta^2}},$$

$$r\sin(t) = -\beta\sqrt{\frac{\alpha^2+z\alpha\beta-w\beta^2+2\alpha\beta\ln\beta+z\beta^2\ln\beta+\beta^2(\ln\beta)^2}{\alpha^2+\beta^2}}.$$

For $\beta < 0$ we obtain

$$a(r',t) = \sqrt{\frac{\alpha^2+\beta^2}{\alpha^2-q\alpha\beta+p\beta^2+(2\alpha\beta-q\beta^2)\ln(-\beta)+\beta^2(\ln(-\beta))^2}} \quad and$$

$$b(r',t) = \frac{(1-p)\alpha\beta-q\beta^2+(2\beta^2+q\alpha\beta)\ln(-\beta)-\alpha\beta(\ln(-\beta))^2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2-q\alpha\beta+p\beta^2+(2\alpha\beta-q\beta^2)\ln(-\beta)+\beta^2(\ln(-\beta))^2}}$$

with

$$r'\cos(t) = \alpha\sqrt{\frac{\alpha^2-q\alpha\beta+p\beta^2+(2\alpha\beta-q\beta^2)\ln(-\beta)+\beta^2(\ln(-\beta))^2}{\alpha^2+\beta^2}},$$

$$r'\sin(t) = -\beta\sqrt{\frac{\alpha^2-q\alpha\beta+p\beta^2+(2\alpha\beta-q\beta^2)\ln(-\beta)+\beta^2(\ln(-\beta))^2}{\alpha^2+\beta^2}}.$$

The quasifields $Q_{w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{w,z,p,q}$ such that for the parameters w,z,p,q the relations $\left(\frac{z}{2}\right)^2 \leq -w-1$, $\left(\frac{q}{2}\right)^2 \leq p-1$ hold.
c) Λ_{Q^*} is the range of the section given by (2) such that $a(r,0)=1=a(r,\pi)$ and $b(r,0)=0=b(r,\pi)$ with $r=\beta$ for $t=0$ and $r=-\beta$ for $t=\pi$.
For $\beta > 0$, we get

$$a(r,t) = \sqrt{\frac{u^2+\sin^2(l)(w^2+2zu+z^2)+\cos^2(l)-(2uw+2u+2z)\sin(l)\cos(l)}{u^2+uz-w}},$$

$$b(r,t) = \frac{\cos^2(l)(2uw+2u+2z)+\sin(l)\cos(l)(1-w^2-z^2-2uz)-(u+z+uw)}{\sqrt{(u^2+\sin^2(l)(w^2+2zu+z^2)+\cos^2(l)-(2uw+2u+2z)\sin(l)\cos(l))(u^2+uz-w)}}$$

with

$$r\cos(t) = \beta\left(u-(w+1)\sin(l)\cos(l)+z\sin^2(l)\right)\sqrt{\frac{u^2+uz-w}{u^2+\sin^2(l)(w^2+2zu+z^2)+\cos^2(l)-(2uw+2u+2z)\sin(l)\cos(l)}},$$

$$r\sin(t) = \beta\left(w\sin^2(l)+z\sin(l)\cos(l)-\cos^2(l)\right)\sqrt{\frac{u^2+uz-w}{u^2+\sin^2(l)(w^2+2zu+z^2)+\cos^2(l)-(2uw+2u+2z)\sin(l)\cos(l)}},$$

where $l = \frac{1}{k}\ln\beta$. For $\beta < 0$ one gets

$$a(r',t') = \sqrt{\frac{u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2u+2q-2up)\sin(l_1)\cos(l_1)}{u^2+uq+p}}$$

$$b(r',t') = \frac{\sin(l_1)\cos(l_1)(1-2uq-p^2-q^2)+\sin^2(l_1)(2q+2u-2up)+(up-q-u)}{\sqrt{(u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2q+2u-2up)\sin(l_1)\cos(l_1))(u^2+uq+p)}}$$

with

$$r'\cos(t') = \beta\left((p-1)\sin(l_1)\cos(l_1) - q\sin^2(l_1) - u\right)\cdot$$
$$\sqrt{\frac{u^2+uq+p}{u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2u+2q-2up)\sin(l_1)\cos(l_1)}},$$
$$r'\sin(t') = -\beta\left(\cos^2(l_1) + q\sin(l_1)\cos(l_1) + p\sin^2(l_1)\right)\cdot$$
$$\sqrt{\frac{u^2+uq+p}{u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2u+2q-2up)\sin(l_1)\cos(l_1)}},$$

where $l_1 = \frac{1}{k}\ln(-\beta)$.

The quasifields $Q_{k,w,z,p,q}$ *coordinatize a family of planes* $\mathcal{A}_{k,w,z,p,q}$ *such that for the parameters* k, w, z, p, q *one has* $k \neq 0$, $(4+k^2)(z^2+(w+1)^2) \leq k^2(1-w)^2$, $(4+k^2)(q^2+(p-1)^2) \leq k^2(p+1)^2$, $(w, z, p, q) \neq (-1, 0, 1, 0)$.

In all cases Q^* *is not decomposable and quasi-simple. The set of the left translations of* Q^* *belonging to the kernel of* Q *is* $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$.

Proof. If the translation complement C fixes two 1-dimensional subspaces of S, then we have a family of translation planes $\mathcal{A}_{s,w,z,p,q}$ such that the normalized spreads belonging to these planes are given as follows:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta^{\frac{1-s}{1+s}} \\ \beta & z\beta^{\frac{1}{1+s}} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & p(-\beta)^{\frac{1-s}{1+s}} \\ \beta & q(-\beta)^{\frac{1}{1+s}} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\}, \qquad (25)$$

(cf. [6], Satz 1, pp. 411-412).
If the translation complement C fixes only one 1-dimensional subspace of S, then there is a family of translation planes $\mathcal{A}_{w,z,p,q}$ such that the corresponding normalized spreads have the form:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta - z\beta\ln\beta - \beta(\ln\beta)^2 \\ \beta & \alpha + z\beta + 2\beta\ln\beta \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & -p\beta - \beta(\ln(-\beta))^2 + q\beta\ln(-\beta) \\ \beta & q(-\beta) + \alpha + 2\beta\ln(-\beta) \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\} \qquad (26)$$

(cf. Satz 2, [6], pp. 418-419).
If the translation complement C acts transitively on the 1-dimensional subspaces of S, then we have a family of translation planes $\mathcal{A}_{k,w,z,p,q}$ such that the normalized spreads belonging to these planes have the form

$$\{S\} \cup \left\{ \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{R} \right\} \cup$$

$$\left\{\beta\begin{pmatrix} u-(w+1)\sin(l)\cos(l)+z\sin^2(l) & w\cos^2(l)-z\sin(l)\cos(l)-\sin^2(l) \\ \cos^2(l)-z\sin(l)\cos(l)-w\sin^2(l) & z\cos^2(l)+(w+1)\sin(l)\cos(l)+u \end{pmatrix}, u\in\mathbb{R}, \beta>0\right\}\cup$$

$$\left\{\beta\begin{pmatrix} (p-1)\sin(l_1)\cos(l_1)-q\sin^2(l_1)-u & q\sin(l_1)\cos(l_1)-p\cos^2(l_1)-\sin^2(l_1) \\ \cos^2(l_1)+q\sin(l_1)\cos(l_1)+p\sin^2(l_1) & (1-p)\sin(l_1)\cos(l_1)-q\cos^2(l_1)-u \end{pmatrix}, u\in\mathbb{R}, \beta<0\right\}, \qquad (27)$$

where $l = \frac{1}{k}\ln\beta$, $l_1 = \frac{1}{k}\ln(-\beta)$ (cf. [15], Proposition 4.1, p. 6, and [6], Satz 3, pp. 422-423). The spreads (25), respectively (26), respectively (27) coincide with the image of the section σ in (2) such that the well defined functions $a(r,t)$ and $b(r,t)$ are given in assertion a), respectively b), respectively c). Since in all three cases we have $a(r,0) = 1$, $b(r,0) = 0$, Remark 1 shows that $\mathcal{K}$ has the form as in the assertion. In case a), respectively b) for $\beta > 0$ one gets $a(r,\frac{\pi}{4}) = \sqrt{\frac{2\beta^2}{\beta^2 - z\beta^{\frac{2+s}{1+s}} - w\beta^{\frac{2}{1+s}}}}$, respectively $a(r,\frac{\pi}{4}) = \sqrt{\frac{2}{1-z-w-2\ln\beta+z\ln\beta+(\ln\beta)^2}}$. In case c) for $u = 0$, $\beta > 0$ we get that $a(1,t_j)$ is constant. These relations give a contradiction to the condition $a(r,t) = a(1,t)$ of Proposition 10. Hence in all cases Q^* is not decomposable and quasi-simple (cf. Theorem 11). □

Proposition 18. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group Γ of $\mathcal{A}$ has an orbit of dimension 1 on $W \setminus \{0\}$, C leaves only S in the set of lines through the origin fixed and the kernel of its action on S is zero-dimensional. Then the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* is given by the range of the section (2) defined as follows: For $\alpha \geq -\frac{\beta^2}{2}$ one has*

$$a(r,t) = \sqrt{\frac{\alpha^2+\beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r,t) = \frac{\frac{p}{q}\alpha\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} - \frac{p}{q}\left(\alpha^2+\beta^2\right) + \frac{1-q}{q}\beta\alpha^2 + \frac{\alpha\beta^3}{6q} - \frac{\beta^3\alpha}{2} + \frac{\beta^3}{2q} + \frac{\beta^3}{2} + \alpha\beta}{\sqrt{\alpha^2+\beta^2}\sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{(q-1)}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r\cos(t) = \alpha\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}},$$

$$r\sin(t) = -\beta\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}}.$$

For $\alpha < -\frac{\beta^2}{2}$ *we get*

$$a(r,t) = \sqrt{\frac{\alpha^2 + \beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r,t) = \frac{\frac{w}{q}\alpha\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{p}{q}\left(-\alpha^2 - \beta^2\right) + \left(\frac{z+1}{q}\alpha\beta - \frac{z\beta}{q}\right)\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\alpha\beta^3}{3q} + \frac{\beta^3}{2q}}{\sqrt{\alpha^2+\beta^2}\sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r\cos(t) = \alpha\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}},$$

$$r\sin(t) = -\beta\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}}.$$

The quasifields $Q_{w,z,p,q}$ *coordinatize a family of planes* $\mathcal{A}_{w,z,p,q}$ *such that the parameters* w, z, p, q *satisfy* $(3w)^2 \leq -16z(z+1)$, $(3p)^2 \leq 16q(q-1)$, $q > 0$, $z < 0$ *and* $(w,z,p,q) \neq (0, -\frac{1}{3}, 0, 3)$.

The multiplicative loops $Q^*_{w,z,p,q}$ *of the quasifields* $Q_{w,z,p,q}$ *are not decomposable and quasi-simple. The left translations of* $Q^*_{w,z,p,q}$ *corresponding to the kernel of* $Q_{w,z,p,q}$ *have the form* $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, $0 \neq r \in \mathbb{R}$, *if and only if* $w = p = 0$.

Proof. By Satz 5 in [6], the planes $\mathcal{A}_{w,z,p,q}$ are determined by the normalized spreads which have the form

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & \frac{-p}{q}\alpha + \frac{p}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(1-q)}{q}\beta\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & \frac{-p}{q}\beta + \frac{\beta^2}{2q} + \left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha \geq -\frac{\beta^2}{2} \right\} \cup$$

$$\left\{ \begin{pmatrix} \alpha & \frac{-p}{q}\alpha + \frac{w}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(z+1)}{q}\beta\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & \frac{-p}{q}\beta + \frac{\beta^2}{2q} - \frac{z}{q}\left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha < -\frac{\beta^2}{2} \right\}.$$

These spreads coincide with the image of the section σ in (2) such that the well defined functions $a(r,t)$ and $b(r,t)$ are given in the assertion.

For $\beta > 2$ we obtain

$$a\left(r, \frac{\pi}{4}\right) = \frac{\sqrt{2\beta^2}}{\sqrt{\frac{\beta^4}{3q} - \frac{\beta^3}{2q} + \left(\frac{\beta^2}{2} - \beta\right)\left(\frac{q-1}{q}\beta^2 - \beta\right) - \frac{p\beta}{q}\left(\frac{\beta^2}{2} - \beta\right)^{\frac{3}{2}}}}.$$

The loop $Q^*_{w,z,p,q}$ is not decomposable since we have a contradiction to the condition $a(r, \frac{\pi}{4}) = a(1, \frac{\pi}{4})$ for $r < 0$ (cf. Proposition 10). Hence $Q^*_{w,z,p,q}$ is quasi-simple (cf. Theorem 11). As $a(r,0) = 1$ and $b(r,0) = 0$ holds precisely if $w = p = 0$ the last assertion follows. □

Proposition 19. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group $\mathcal{A}$ fixes two distinct lines $\{S, W\}$ through the origin and leaves on S one or two 1-dimensional subspaces invariant. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* having the form (20):*

a)

$$a(r,0) = r^{\frac{1-w}{1+w}}, \quad b(r,0) = c\left(r^{\frac{w-1}{w+1}} - r^{\frac{1-w}{1+w}}\right),$$

with $r = s^{\frac{w+1}{2}}$, $s > 0$, $t = -\varphi$, where s and φ are variables of the spreads (28). The quasifields $Q_{w,c}$ coordinatize a family of planes $\mathcal{A}_{w,c}$ such that for the parameters $w \neq 1, c$ one has $0 < w$ and $(w-1)^2c^2 \leq 4w$.

b)

$$a(r,0)=1, \quad b(r,0)=\frac{\ln r}{d},$$

with $r=e^s$, $t=-\varphi$, *where* s *and* φ *are variables of the spreads (29). The quasifields* Q_d *coordinatize a one-parameter family of planes* $\mathcal{A}_d$ *such that* $4d^2 \geq 1$.

In both cases Q^* *is positively decomposable and contains the group* $\mathrm{SO}_2(\mathbb{R})$.

Proof. If the group C fixes two 1-dimensional subspaces of S, respectively only one 1-dimensional subspace of S, then one obtains a family of translation planes corresponding to the normalized spreads

$$\{S,W\} \cup \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} s & c(s^w - s) \\ 0 & s^w \end{pmatrix}, \ s,\varphi \in \mathbb{R}, s>0 \right\} \quad (28)$$

(cf. [7], Satz 1 and [9], p. 15), respectively

$$\{S,W\} \cup \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} e^s & e^s \frac{s}{d} \\ 0 & e^s \end{pmatrix}, s,\varphi \in \mathbb{R} \right\}, \quad (29)$$

(cf. [7], Satz 2 and [9], p. 15). In both cases these spreads coincide with the set $\Lambda = \mathrm{SO}_2(\mathbb{R})\mathcal{K}$ given in (20) such that the set $\mathcal{K}$ corresponding to the kernel K_r of Q is determined by the functions $a(r,0)$, $b(r,0)$ as in assertion a), respectively b). □

Remark 20. *In [2] D. Betten constructed* 4*-dimensional locally compact non-desarguesian planes* $\mathcal{A}_f$ *corresponding to continuous, non-linear, strictly* *monotone functions f defined for $0 \leq u \in \mathbb{R}$ with $f(0) = 0$ and $\lim_{u \to \infty} f(u) = \infty$. The planes $\mathcal{A}_f$ are determined by the normalized spreads*

$$\left\{ \begin{pmatrix} u \cos\varphi & -\frac{f(u)\sin\varphi}{f(1)} \\ u \sin\varphi & \frac{f(u)\cos\varphi}{f(1)} \end{pmatrix}, u > 0,\ \varphi \in [0, 2\pi) \right\}.$$

These spreads coincide with the set $\Lambda = \mathrm{SO}_2(\mathbb{R})\mathcal{K}$ given in (20) such that the set $\mathcal{K}$ corresponding to the kernel K_r of the quasifield Q_f coordinatizing $\mathcal{A}_f$ is determined by the functions $a(r,0) = \sqrt{\frac{uf(1)}{f(u)}}$, $b(r,0) = 0$ with $r = \sqrt{\frac{uf(u)}{f(1)}}$, $t = -\varphi$, $u \neq 0$. For $f(u) = f(1)u^w$ these planes are planes in Proposition 19 a) with $c = 0$ and $a(r,0) = r^{\frac{1-w}{1+w}}$. Otherwise the full collineation group of the planes $\mathcal{A}_f$ has dimension 6.

Proposition 21. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group of $\mathcal{A}$ fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W of all 1-dimensional subspaces of S, respectively W. Then the multiplicative loop Q^* of Q can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*
a) Λ_{Q^*} is the range of the section (2) with

$$a(r,u) = \sqrt{\frac{dD}{de^{2(qt-ps)} + de^{2q\pi} + e^{qt-ps+q\pi}\left(2d\cos s\cos t + (c^2+1+d^2)\sin s\sin t\right)}},$$

$$b(r,u) = \frac{e^{2(qt-ps)}\left[(-c^2-1+d^2)\cos t\sin t - c(c^2+1+d^2)\sin^2 t\right]}{\sqrt{dD\left[d(e^{2(qt-ps)} + e^{2q\pi}) + e^{qt-ps+q\pi}(2d\cos s\cos t + (d^2+c^2+1)\sin s\sin t)\right]}} +$$

$$+ \frac{e^{qt-ps+q\pi}\left(\cos s\cos t + d\sin s\sin t + c\cos s\sin t\right)}{\sqrt{dD\left[d(e^{2(qt-ps)} + e^{2q\pi}) + e^{qt-ps+q\pi}(2d\cos s\cos t + (d^2+c^2+1)\sin s\sin t)\right]}},$$

such that

$$r\cos u = \frac{e^{qt-ps}(\cos s\cos t + c\sin t\cos s + d\sin t\sin s) + e^{q\pi}}{1+e^{q\pi}} a^{-1}(r,u),$$

$$r\sin u = -\frac{e^{qt-ps}(d\cos s\sin t - \sin s\cos t - c\sin s\sin t)}{1+e^{q\pi}} a^{-1}(r,u),$$

$$D = e^{2(qt-ps)}\left((\cos t + c\sin t)^2 + d^2\sin^2 t\right) + e^{2q\pi} + 2e^{qt-ps+q\pi}(\cos s\cos t + c\cos s\sin t + d\sin s\sin t).$$

The quasifields $Q_{p,q,c,d}$ *coordinatize a family of planes* $\mathcal{A}_{p,q,c,d}$ *such that the parameters* p, q, c, d *satisfy the conditions*

$$\begin{array}{lll} p = q > 0 & and & -1 \leq d < 0, \\ q > 0, p = \frac{k-1}{k+1} q, k = 1, 2, 3, \cdots & and & d > 0, \end{array}$$

$$-(q+p)^2 A + (q-p)^2 B - 4AB \geq 0, \; where \; A = \frac{(d-1)^2 + c^2}{4d} \; and \; B = \frac{(d+1)^2 + c^2}{4d}.$$

The multiplicative loops Q^* *of the quasifields* $Q_{p,q,c,d}$ *are not decomposable and quasi-simple.*

b) Λ_{Q^*} *has the form (15) with*

$$a(1,u) = \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}, \; b(1,u) = \frac{\sin nt \cos nt (d^2 - 1 - c^2) - c \sin^2 nt (d^2 + 1 + c^2)}{d\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

such that

$$r \cos u = \frac{s(\cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}, \; r \sin u = \frac{s(d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

and $s \geq 0$.

The quasifields $Q_{m,n,c,d}$ *coordinatize a family of planes* $\mathcal{A}_{m,n,c,d}$ *such that the parameters* $m, n \in \mathbb{Z}$, $(m,n) = 1$, $c, d \in \mathbb{R}$ *satisfy the conditions*

$$\begin{array}{llll} m = n = 1 & & and & -1 \leq d < 0 \\ m = 1, 2, 3, \cdots & n = m + 1 & and & d > 0 \\ m = 1, 3, 5, \cdots & n = m + 2 & and & d > 0 \end{array}$$

$$(n-m)^2 B \geq (n+m)^2 A, \; where \; A = \frac{(d-1)^2 + c^2}{4d} \; and \; B = \frac{(d+1)^2 + c^2}{4d}.$$

The loops $Q^*_{m,n,c,d}$ *are split extensions of the normal subgroup* $N^* \cong \mathbb{R}$ *corresponding to the connected component of* $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 \neq u \in \mathbb{R} \right\}$ *with a subloop homeomorphic to the* 1*-sphere.*

Proof. If the translation complement C acts transitively on the product space $P_S \times P_w$, then there is a family of translation planes corresponding to the normalized spreads

$$\{S,W\}\cup\left\{\begin{pmatrix}\frac{\alpha(s,t)+e^{q\pi}}{1+e^{q\pi}} & \frac{\gamma(s,t)-c\alpha(s,t)}{d(1+e^{q\pi})}\\ \frac{\beta(s,t)}{1+e^{q\pi}} & \frac{\delta(s,t)-c\beta(s,t)+de^{q\pi}}{d(1+e^{q\pi})}\end{pmatrix}, s,t\in\mathbb{R}\right\}$$

such that $\alpha(s,t)=e^{qt-ps}(\cos s\cos t+c\sin t\cos s+d\sin t\sin s)$,

$\beta(s,t)=e^{qt-ps}(d\cos s\sin t-\sin s\cos t-c\sin s\sin t)$,

$\gamma(s,t)=e^{qt-ps}(d\cos t\sin s-\sin t\cos s+c\cos t\cos s)$,

$\delta(s,t)=e^{qt-ps}(d\cos t\cos s+\sin t\sin s-c\cos t\sin s)$ (cf. [7], Satz 3, pp. 135-136). These spreads coincide with the image of the section σ in (2) with the well defined functions $a(r,u)$ and $b(r,u)$ as in assertion a). For $s=0$ we get a contradiction to the condition $a(r_j,u_j)=a(r_j,0)a(1,u_j)$ which must hold for decomposable Q^*. It follows that Q^* is not decomposable and hence quasi-simple (cf. Theorem 11).

If the translation complement C does not act transitively on the product space $P_S\times P_W$, then there is a family of translation planes which correspond to the normalized spreads

$$\{S,W\}\cup\left\{\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}\begin{pmatrix}a_{11}(t) & -\frac{c}{d}a_{11}(t)+\frac{1}{d}a_{21}(t)\\ a_{12}(t) & -\frac{c}{d}a_{12}(t)+\frac{1}{d}a_{22}(t)\end{pmatrix}, s\ge 0, t\in\mathbb{R}\right\}$$

with $a_{11}(t)=\cos nt\cos mt+c\sin nt\cos mt+d\sin nt\sin mt$,

$a_{12}(t)=d\sin nt\cos mt-\cos nt\sin mt-c\sin nt\sin mt$,

$a_{21}(t)=d\cos nt\sin mt-\sin nt\cos mt+c\cos nt\sin mt$,

$a_{22}(t)=d\cos nt\cos mt+\sin nt\sin mt-c\cos nt\sin mt$ (cf. [7], Satz 4, pp.

142-144). These spreads coincide with the set Λ in (15) such that the periodic functions $a(1,t)$ and $b(1,t)$ are given in assertion b). As in the proof of Proposition 15 a) it follows that the loop $Q^*_{m,n,c,d}$ is a split extension as in the assertion. □

Corollary 22. *Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian topological plane which admits an at least 7-dimensional collineation group Γ. If the quasifield Q coordinatizing $\mathcal{A}$ is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of Γ or contain the set of the fixed points of Γ, then for the multiplicative loop Q^* of Q one of the following holds:*

a) Q^* is quasi-simple and not decomposable. Such quasifields Q are described by Propositions 15 b), 15 c), 16), 17), 18) and in Proposition 21 a).

b) Q^* is quasi-simple but decomposable and it is a product $SO_2(\mathbb{R})B$, where B is a 1-dimensional loop homeomorphic to $\mathbb{R}$. The quasifields Q of this type are described in Proposition 19.

c) Q^* is a split extension of the group $N^* \cong \mathbb{R}$ with a loop homeomorphic to the 1-sphere. The quasifields of this type are described in Propositions 15 a) and 21 b).

Proof. A locally compact topological quasifield coordinatizing the translation plane $\mathcal{A}$ and constructed with respect to two lines satifying the assumptions is isotopic to a quasifield given in Betten's classification (cf. [11], p. 321, [3] Satz 5). For isotopic loops Q_1^* and Q_2^* the following holds: The group generated by their left translations, every subgroup and all nuclei of them are isomorphic (cf. [14], Lemmata 1.9, 1.10, p. 20). From these facts we get: If Q_1 is quasisimple and not decomposable, then also Q_2 is quasisimple and not decomposable. If Q_1 contains the subgroup $SO_2(\mathbb{R})$, then also Q_2 contains the group $SO_2(\mathbb{R})$. If Q_1 is a split extension of N^* with a 1-dimensional compact loop, then the same holds for Q_2. □

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References

[1] André, J. Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. *Math. Z.* 60:156-186.

[2] Betten, D. Nicht-desarguessche 4-dimensionale Ebenen. *Arch. Math.* 21:100-102.

[3] Betten, D. 4-dimensionale Translationsebenen. *Math. Z.* 128:129-151.

[4] Betten, D. 4-dimensionale Translationsebenen mit 8-dimensionaler Kollineationsgruppe. *Geom. Ded.* 2:327-339.

[5] Betten, D. 4-dimensionale Translationsebenen mit irreduzibler Kollineationsgruppe. *Arch. Math.* 24:552-560. *Math. Z.* 132:249-259.

[6] Betten, D. 4-dimensionale Translationsebenen mit genau einer Fixrichtung. *Deom. Ded.* 3:405-440.

[7] Betten, D. 4-dimensionale Translationsebenen mit 7-dimensionaler Kollineationsgruppe. *J. Reine Angew. Math.* 285:126-148.

[8] Betten, D. 4-dimensionale Translationsebenen mit kommutativer Standgruppe. *Math. Z.* 154:125-141.

[9] Betten, D. (1997) On the classification of 4-Dimensional Flexible Projective Planes. In: Johnson, N. L., ed. *M*ostly finite geometries. Decker, pp. 9-33.

[10] Figula, Á., Strambach, K. Loops on spheres having a compact-free inner mapping group. *Monatsh. Math.* 156:123-140.

[11] Grundhöfer, T., Salzmann, H. H. (1990). Locally compact double loops and ternary fields. In: Chein, O., et al., ed. *Quasigroups and Loops: Theory and Applications.* Heldermann, pp. 313-355.

[12] Hughes, D. R., Piper, F. C. (1973). Projective Planes. Springer.

[13] Knarr, N. (1995). Translation Planes. Springer.

[14] Nagy, P. T., Strambach, K. (2002). Loops in group theory and Lie theory. Walter de Gruyter.

[15] Ortleb, S. A new family of locally compact 4-dimensional translation planes admitting a 7-dimensional collineation group. *Adv. Geom.* 9:1-12.

[16] Pickert, G. (1955) Projektive Ebenen. Springer.

[17] Plaumann, P., Strambach, K. Zweidimensionale Quasialgebren mit Nullteilern. *Aequationes Math.* 15:249-264.

[18] Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R., Stroppel, M. (1995). Compact projective planes. Walter de Gruyter.

[19] Walter, W. (1970). Differential and Integral Inequalities. Springer.