THE DENSITY TURÁN PROBLEM

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ABSTRACT. Let H be a graph on n vertices and let the blow-up graph $G[H]$ be defined as follows. We replace each vertex v_i of H by a cluster A_i and connect some pairs of vertices of A_i and A_j if (v_i, v_j) was an edge of the graph H. As usual, we define the edge density between A_i and A_j as

$$d(A_i, A_j) = \frac{e(A_i, A_j)}{|A_i||A_j|}.$$

We study the following problem. Given densities γ_{ij} for each edge $(i, j) \in E(H)$. Then one has to decide whether there exists a blow-up graph $G[H]$ with edge densities at least γ_{ij} such that one cannot choose a vertex from each cluster so that the obtained graph is isomorphic to H, i.e, no H appears as a transversal in $G[H]$. We call $d_{crit}(H)$ the maximal value for which there exists a blow-up graph $G[H]$ with edge densities $d(A_i, A_j) = d_{crit}(H)$ $((v_i, v_j) \in E(H))$ not containing H in the above sense. Our main goal is to determine the critical edge density and to characterize the extremal graphs.

First in the case of tree T we give an efficient algorithm to decide whether a given set of edge densities ensures the existence of a transversal T in the blow-up graph. Then we give general bounds on $d_{crit}(H)$ in terms of the maximal degree. In connection with the extremal structure, the so-called star decomposition is proved to give the best construction for H-transversal-free blow-up graphs for several graph classes.

Our approach applies algebraic graph-theoretical, combinatorial and probabilistic tools.

1. INTRODUCTION

Given a simple, connected graph H, we define a blow-up graph $G[H]$ of H as follows. Replace each vertex $v_i \in V(H)$ by a cluster A_i and connect vertices between the clusters A_i and A_j (not necessarily all) if v_i and v_j were adjacent in H. As usual, we define the density between A_i and A_j as

$$d(A_i, A_j) = \frac{e(A_i, A_j)}{|A_i||A_j|},$$

where $e(A_i, A_j)$ denotes the number of edges between the clusters A_i and A_j. We say that the graph H is a transversal of $G[H]$ if H is a subgraph of $G[H]$ such

that we have a homomorphism $\varphi : V(H) \to V(G[H])$ for which $\varphi(v_i) \in A_i$ for all $v_i \in V(H)$. We will also use the terminology that H is a factor of $G[H]$.

The density Turán problem seeks to determine the critical edge density d_{crit} which ensures the existence of the subgraph H of $G[H]$ as a transversal. What does this mean? Assume that for all $e = (v_i, v_j) \in E(H)$ we have $d(A_i, A_j) > d_{crit}$. Then, no matter how the graph $G[H]$ looks like, it induces the graph H as a transversal. On the other hand, for any $d < d_{crit}(H)$ there exists a blow-up graph $G[H]$ such that $d(A_i, A_j) > d$ for all $(v_i, v_j) \in E(H)$ and it does not contain H as a transversal. Clearly, the critical edge density of the graph H is the largest one of the critical edge densities of its components. Thus we will assume throughout the paper that H is a connected graph.

The problem considered was studied in [12]. A very closely related variant of this problem was mentioned in the book Extremal Graph Theory by Bollobás [2] on page 324. There are many papers where density condition is replaced by the minimal degree constraint [3, 4, 10, 16].

It will turn out that it is useful to consider the following more general problem. Assume that a density γ_e is given for every edge $e \in E(H)$. Now the problem is to decide whether the densities $\{\gamma_e\}$ ensure the existence of the subgraph H as a transversal or one can construct a blow-up graph $G[H]$ such that $d(A_i, A_j) \geq \gamma_{ij}$, yet the graph H does not appear in $G[H]$ as a transversal. This more general approach allows us to use inductive proofs. We refer to this general setting as inhomogeneous condition on the edge densities while the above condition of having a common lower bound $d_{crit}(H)$ for the densities is called the homogeneous case.

Moreover, it will turn out that an even more general setting is worth considering, namely, weighted blow-up graphs (see Section 2).

The paper is organized as follows. We end this Introduction by setting out the notation. In Section 2 we introduce the most important concepts via an example and we sketch the main results of [12] and some useful lemmas. Section 3 is devoted to the case when H is a tree. This case is covered in [12] in the homogeneous case, showing that

$$d_{crit}(T) = 1 - \frac{1}{\lambda_{max}^2(T)},$$

where $\lambda_{max}(T)$ denotes the maximal eigenvalue of the adjacency matrix of the tree. In the inhomogeneous case a set of edge densities is given for a blow-up graph T_n, and we have to decide whether the edge densities ensure the existence of a factor T_n. We give an efficient algorithm to do this. The proof is based on the strong connection with the multivariate matching polynomial. In Section 4, by the application of the Lovász local lemma and its extension, we show that

$$d_{crit}(H) < 1 - \frac{1}{4(\Delta(H) - 1)},$$

where $\Delta(H)$ is the maximal degree of H. The extremal structures are investigated in Section 5. Here we give a recursive construction for blow-up graphs not containing the corresponding transversal, and examine for which classes of graphs it gives the extremal structure. These constructions also give lower bounds for the critical edge density in the homogeneous case.

$$\star \quad \star \quad \star$$

Throughout the paper, we use the following notation.

Notation 1.1. $H = (V(H), E(H))$ will be a connected graph on the labelled vertices $\{1, \ldots, n\}$.

$G[H]$ denotes a blow-up graph of H on n clusters, where the cluster A_i corresponds to the vertex $i \in V(H)$. If all densities equal 1 in $G[H]$ then we call it a complete blow-up graph of H.

Graph S_n, P_n, C_n denote the star, the path and the cycle on n vertices, respectively. As usual, K_n and $K_{m,n}$ denote the complete and complete bipartite graphs, respectively. T_n denotes an arbitrary tree on n vertices.

$d_{crit}(H)$ is the critical edge density assigned to H, while d_e is the edge density between A_i and A_j if $e = ij \in E(H)$.

$\Delta(H)$ will denote the maximum degree in H, while $N(z)$ will denote the neighborhood of vertex z.

Now we define the weighted version of the well-known independence and matching polynomials.

Notation 1.2. Let G be a graph and assume that a positive weight function $w : V(G) \to \mathbb{R}^+$ is given. Then let

$$I((G, \underline{w}); t) = \sum_{S \in \mathcal{I}} \left(\prod_{u \in S} w_u \right) (-t)^{|S|},$$

where the summation goes over the set $\mathcal{I}$ of all independent sets S of the graph G including the empty set. When $\underline{w} = \underline{1}$ we simply write $I(G, t)$ instead of $I((G, \underline{1}); t)$ and we call $I(G, t)$ the independence polynomial of G. Clearly,

$$I(G, t) = \sum_{k=1}^{n} i_k(G)(-1)^k t^k,$$

where $i_k(G)$ denotes the number of independent sets of size k in the graph G.

Let G be a graph and assume that a positive weight function $w : E(G) \to \mathbb{R}^+$ is given. Then let

$$M((G, \underline{w}); t) = \sum_{S \in \mathcal{M}} \left(\prod_{e \in S} w_e \right) (-1)^{|S|} t^{n-2|S|},$$

where the summation goes over the set $\mathcal{M}$ of all independent edge sets S of the graph G including the empty set. In the case when $\underline{w} = \underline{1}$ we call the polynomial

$$M(G, t) = M((G, \underline{1}); t) = \sum_{k=0}^{n/2} (-1)^k m_k(G) t^{n-2k}$$

the matching polynomial of G, where $m_k(G)$ denotes the number of k independent edges (i.e., the k-matchings) in the graph G.

A closely related variant of the weighted matching polynomial is the multivariate matching polynomial defined as follows. Let x_e's be variables assigned to each edge of a graph. The *multivariate matching polynomial F* is defined as follows:

$$F(\underline{x_e}, t) = \sum_{M \in \mathcal{M}} (\prod_{e \in M} x_e)(-t)^{|M|},$$

where the summation goes over the matchings of the graph including the empty matching.

Clearly, if L_G denotes the line graph of the graph G, we have

$$F(\underline{x_e}, t) = I((L_G, \underline{x_e}); t)$$

or in other words

$$t^n F(\underline{x_e}, \frac{1}{t^2}) = M((G, \underline{x_e}); t).$$

2. Preliminaries

In this section we motivate some key definitions via an example graph. The diamond is the unique simple graph on 4 vertices and 5 edges, generally denoted by K_4^-.
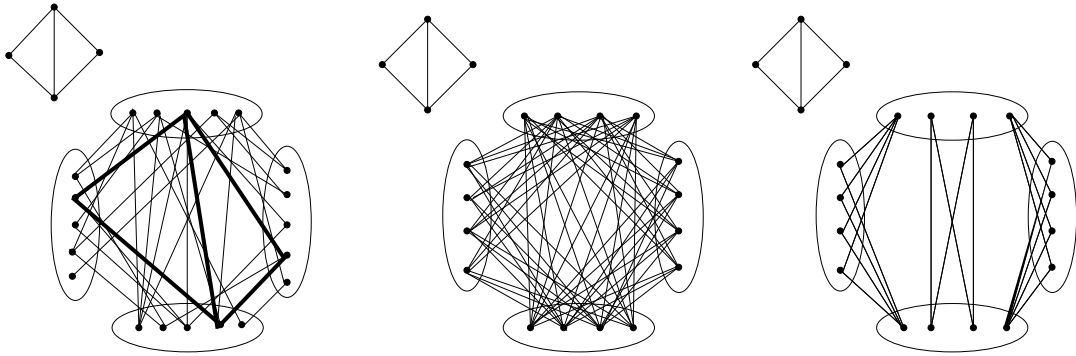

Figure 1. Blow-up graphs of diamonds.

In the figure above, the first blow-up graph of the diamond contains the diamond as a transversal. The second blow-up graph does not contain the diamond as a transversal, although the edge density is $3/4$ between any two clusters. To see it, we have given the *complement of the blow-up graph with respect to the complete blow-up graph*; in what follows we will simply call this graph the *complement graph* and we will denote it by $\overline{G[H]|H}$. In the "complement language" the claim is as follows: if one chooses one vertex from each cluster then we cannot avoid choosing both ends of a complementary edge. This is indeed true: whichever vertex we choose from the "right" and "left" clusters we cannot choose the rightmost and leftmost vertices of the upmost and downmost clusters; so we have to choose a vertex from the middle of these clusters, but they are all connected by complementary edges.

We also see that this construction was somewhat redundant in the sense that each vertex from the right and left clusters had the same role. This motivates the following definition.

Definition 2.1. A weighted blow-up graph is a blow-up graph where a non-negative weight $w(u)$ is assigned to each vertex u such that the total weight of each cluster is 1. The density between clusters A_i and A_j is

$$d_{ij} = \sum_{\substack{(u,v) \in E \\ u \in A_i, v \in A_j}} w(u)w(v).$$

This definition also has the advantage that we can now allow irrational weights too. (But this does not change the problem since we can approximate any irrational weight by rational weights and then we blow up the construction with the
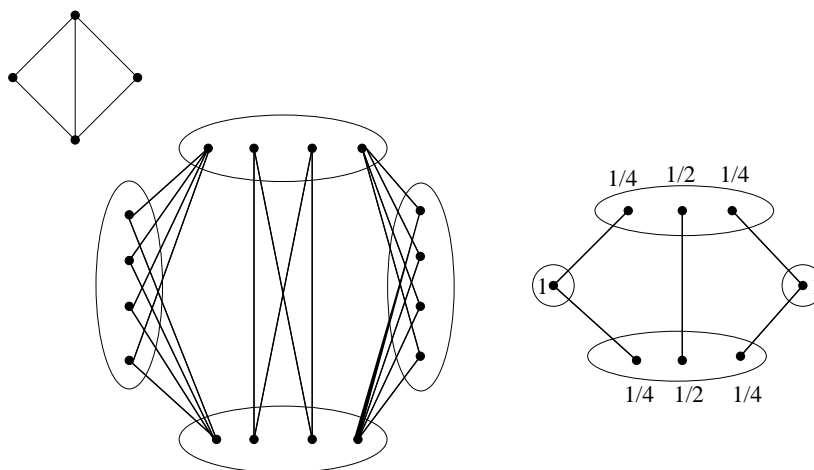
FIGURE 2. Weighted blow-up graph.

common denominator of the weights.) The following result of the second author [12] also shows that the problem in this framework is much more convenient. Note that this result is a simple generalisation of a statement of Bondy, Shen, Thomassé and Thomassen [5].

Theorem 2.2. [12] *If there is a construction of a blow-up graph $G[H]$ not containing H, then there is a construction of a weighted blow-up graph $G'[H]$ not containing H, where*

- *each edge density is at least as large as in $G[H]$,*
- *the cluster V_i contains at most as many vertices as the degree of the vertex v_i in the graph H.*

The importance of this theorem lies in the fact that if we are looking for the critical edge density we only have to check those constructions where each cluster contains a bounded number of vertices. So in fact, we have to check a finite number of configurations and we only have to decide that which configuration has a weighting providing the greatest density. In general, the number of possible configurations is very large, yet it has some notable consequences. For instance, there is a "best" construction in the sense that if we have a construction for $\gamma_e - \varepsilon$ for every ε, then we have a construction with edge densities γ_e. Indeed, we have a compact space (finite number of configurations) and the edge densities are continuous functions of the weights.

With a small extra idea one can prove the following important corollary of this theorem.

Theorem 2.3. [12] *There is a weighted blow-up graph $G[H]$ not containing H, where each edge density is exactly the critical edge density.*

From this theorem one can deduce the following results.

Proposition 2.4. [12] *If H_1 is a subgraph of H_2, then for the critical edge densities we have*

$$d_{crit}(H_1) \leq d_{crit}(H_2).$$

If H_2 is connected and H_1 is a proper subgraph of H_2, then the inequality is strict.

A general lower and upper bound was also proved in [12]. The lower bound is the consequence of Proposition 2.4 and the fact that $d_{crit}(S_n) = 1 - \frac{1}{n-1}$.

Proposition 2.5. $(1 - \frac{1}{\Delta(H)}) \le d_{crit}(H) \le (1 - \frac{1}{\Delta^2(H)})$.

The upper bound will be strengthened in Section 4. It was known that

$$d_{crit}(H) < 1 - \frac{1}{4(\Delta(H) - 1)}$$

also holds for trees. It turned out that it is a general upper bound.

Finally, let us mention a theorem by Bondy, Shen, Thomassé and Thomassen. On the one hand, it solves the inhomogeneous problem for $H = K_3$. On the other hand, it provides a base in some forthcoming proofs.

Lemma 2.6. [5] *Let α, β, γ be the edge densities between the clusters of a blow-up graph of the triangle. If*

$$\alpha\beta + \gamma > 1, \ \beta\gamma + \alpha > 1, \ \gamma\alpha + \beta > 1,$$

then the blow-up graph contains a triangle as a transversal. Otherwise there exists a weighted blow-up graph with the prescribed edge densities without containing a triangle.

3. INHOMOGENEOUS CASE: TREES

In this section we study the case when the graph H is a tree.

Theorem 3.1. *Let T be a tree, and let v_n be a leaf of T. Assume that for each edge of T a density $\gamma_e = 1 - r_e$ is given. Let T' be a tree obtained from T by deleting the leaf v_n (together with the edge $e_{n-1,n} = v_{n-1}v_n$). Let the densities γ'_e be defined as follows:*

$$\gamma'_e = \begin{cases} \gamma_e = 1 - r_e & \text{if } e \text{ is not incident to } v_{n-1}, \\ 1 - \frac{r_e}{1 - r_{e_{n-1,n}}} & \text{if } e \text{ is incident to } v_{n-1}. \end{cases}$$

Then the set of densities γ_e ensures the existence of the factor T if and only if all γ'_e are between 0 and 1 and the set of densities γ'_e ensures the existence of the factor T'.

Remark 3.2. Clearly, this theorem provides us with an efficient algorithm to decide whether a given set of densities ensures the existence of a factor (see Algorithm 3.3).

Proof. First we prove that if all the γ'_e are indeed densities and they ensure the existence of the factor T', then the original γ_e ensure the existence of a factor T.

Assume that $G[T]$ is a blow-up of T such that the density between A_i and A_j is at least γ_{ij}, where A_i is the blow-up of the vertex v_i of T. We need to show that it contains a factor T.

Let us define

$$R = \{v \in A_{n-1} \mid v \text{ is incident to some edge going between } A_{n-1} \text{ and } A_n\}.$$

First of all we show that the cardinality of R is large:

$$|R||A_n| \ge e(R, A_n) = \gamma_{n-1,n}|A_{n-1}||A_n|.$$

Thus $|R| \ge \gamma_{n-1,n}|A_{n-1}|$.

Next we show that many edges are incident to R. Let v_k be adjacent to v_{n-1}. Then we can bound the number of edges between R and A_k as follows:

$$e(R, A_k) \ge e(A_{n-1}, A_k) - (|A_{n-1}| - |R|)|A_k| = |R||A_k| + (\gamma_{k,n-1} - 1)|A_k||A_{n-1}| \ge$$

$$\geq |R||A_k| + (\gamma_{k,n-1} - 1)\frac{1}{\gamma_{n-1,n}}|R||A_k| =$$

$$= (1 - \frac{r_{k-1,n}}{1 - r_{n-1,n}})|R||A_k| = \gamma'_{k,n-1}|R||A_k|.$$

Now delete the vertex set A_n and $A_{n-1}\backslash R$ from $G[T]$. Then the obtained graph is a blow-up of T' with edge densities ensuring the factor T'. But this factor can be extended to a factor of T because of the definition of R.

Now we prove that if some $\gamma'_{k,n-1} < 0$, then there exists a construction for a blow-up of T having no factor of T. In fact $\gamma'_{k,n-1} < 0$ means that $\gamma_{k,n} + \gamma_{n-1,n} < 1$ and so we can conclude that some construction does not induce the path $u_k u_{n-1} u_n$ where $u_i \in A_i$ $(i \in \{k, n-1, n\})$.

Now assume that all γ'_e are proper densities, but there is a construction $G'[T']$ with edge-densities at least γ'_e, but which does not induce a factor T'. In this case we can easily construct a blow-up $G[T]$ of the tree not inducing T by setting $A_{n-1} = R^* \cup A'_{n-1}$ with an appropriate weight of $R^* = \{v^*_{n-1}\}$, and taking an $A_n = \{v_n\}$ which we connect to all elements of A'_{n-1} but do not connect to v^*_{n-1}. $\qquad\square$

Algorithm 3.3. *Step 0.* *Let there be given a tree T_0 and edge densities γ^0_e. Set $T := T_0$ and $r_e = 1 - \gamma^0_e$.*
Step 1. *Consider $(T, \underline{r_e})$.*

- *If $|V(T)| = 2$ and $0 \leq r_e < 1$ then **STOP**: the densities γ^0_e ensure the existence of the transversal T_0.*
- *If $|V(T)| \geq 2$ and there exists an edge for which $r_e \geq 1$ then **STOP**: the densities γ^0_e do not ensure the existence of the transversal T_0.*

Step 2. *If $|V(T)| \geq 3$ and $0 \leq r_e < 1$ for all edges $e \in E(T)$ then **do** pick a vertex v of degree 1, let u be its unique neighbor. Let $T' := T - v$ and*

$$r'_e = \begin{cases} r_e \text{ if } e \text{ is not incident to } u, \\ \frac{r_e}{1 - r_{(u,v)}} \text{ if } e \text{ is incident to } u. \end{cases}$$

Jump to Step 1 with $(T, \underline{r_e}) := (T', \underline{r'_e})$.

In what follows we analyse Algorithm 3.3. The following concept will be the key tool.

Let x_e's be variables assigned to each edge of a graph. Recall that we define the *multivariate matching polynomial F* as follows:

$$F(\underline{x_e}, t) = \sum_{M \in \mathcal{M}} (\prod_{e \in M} x_e)(-t)^{|M|},$$

where the summation goes over the matchings of the graph including the empty matching.

The following lemma is a straightforward generalization of the well-known fact that for trees the matching polynomial and the characteristic polynomial of the adjacency matrix coincide.

Lemma 3.4. *Let T be a tree on n vertices. Let us define the following matrix of size $n \times n$. The entry $a_{i,j} = 0$ if the vertices v_i and v_j are not adjacent and*

$a_{i,j} = \sqrt{x_e}$ if $e = v_i v_j \in E(T)$. *Let $\phi(\underline{x_e}, t)$ be the characteristic polynomial of this matrix. Then*

$$\phi(\underline{x_e}, t) = t^n F(\underline{x_e}, \frac{1}{t^2})$$

where $F(\underline{x_e}, t)$ is the multivariate matching polynomial.

Proof. Indeed when we expand the $\det(tI - A)$ we only get non-zero terms when the cycle decomposition of the permutation consist of cycles of length at most 2; but these terms correspond to the terms of the matching polynomial. $\square$

Proposition 3.5. *Let G be a tree and let $t_w(G)$ denote the largest real root of the polynomial $M((G, \underline{w}); t)$. Let G_1 be a subgraph of G then we have*

$$t_w(G_1) \leq t_w(G).$$

Proof. This is straightforward after applying Lemma 3.4 $\square$

Note that Proposition 3.5 holds for arbitrary graph G, but we do not use this stronger version.

Corollary 3.6. *Let T be a tree and, assume that for each edge $e \in E(T)$ a weight $w_e > 0$ is assigned. Furthermore, let T' be a subtree of T with the induced edge weights. Then the polynomial $F_T(\underline{w_e}, t)$ has a smaller positive root than the polynomial $F_{T'}(\underline{w_e}, t)$.*

Lemma 3.7. *Let T be a weighted tree with $\gamma_e = 1 - tr_e$ weights. Assume that after running Algorithm 3.3 we get the two node tree with edge weight 0. Then t is the root of the multivariate matching polynomial $F(\underline{r_e}, s)$ of the tree T.*

Proof. We prove the statement by induction on the number of vertices of the tree. If the tree consists of two vertices, then $0 = 1 - tr_e$ means exactly that t is the root of the multivariate matching polynomial of the tree.

Now assume that the statement is true for trees on at most $n-1$ vertices. Let T be a tree on n vertices and assume that we execute the algorithm for the pendant edge $e_{n-1,n} = (v_{n-1}, v_n)$ in the first step, where the degree of the vertex v_n is 1. Let $T' = T - v_n$. Now we continue executing the algorithm, obtaining the two node tree with edge weight 0. By induction we get that $F_{T'}(\underline{r'_e}, t) = 0$.

We can expand $F_{T'}$ according to whether a monomial contains $x_{k,n-1}$ ($e_{k,n-1} \in E(T')$) or not. Each monomial can contain at most one of the variables $x_{k,n-1}$ ($v_k \in N(v_{n-1})$). Thus

$$F_{T'}(\underline{x_e}, s) = Q_0(\underline{x_e}, s) - \sum_{v_k \in N(v_{n-1})} s x_{k,n-1} Q_k(\underline{x_e}, s),$$

where Q_0 consists of those terms which contain no $x_{k,n-1}$ and $-s x_{k,n-1} Q_k$ consists of those terms which contain $x_{k,n-1}$, i.e., these terms correspond to the matchings containing the edge (v_k, v_{n-1}). Observe that

$$F_T(\underline{x_e}, s) = (1 - s x_{n-1,n}) Q_0(\underline{x_e}, s) - \sum_{v_k \in N(v_{n-1})} s x_{k,n-1} Q_k(\underline{x_e}, s)$$

by the same argument.

Since

$$0 = F_{T'}(\underline{r'_e}, t) = Q_0(\underline{r_e}, t) - \sum_{v_k \in N(v_{n-1})} \frac{r_{k,n-1}}{1 - tr_{n-1,n}} Q_k(\underline{r_e}, t)$$

we have

$$0 = (1-tr_{n-1,n})F_{T'}(\underline{r}'_e,t) = (1-tr_{n-1,n})Q_0(\underline{r}_e,t)- \sum_{v_k\in N(v_{n-1})} r_{k,n-1}Q_k(\underline{r}_e,t) = F_T(\underline{r}_e,t).$$

Hence t is the root of $F_T(\underline{r}_e,s)$. $\qquad\square$

Theorem 3.8. *Let T be a tree and let $\gamma_e = 1 - r_e$ be edge densities. Then the edge densities ensure the existence of the tree T as a transversal if and only if for the multivariate matching polynomial we have*

$$F(\underline{r}_e,t) > 0$$

for all $t \in [0,1]$.

Remark 3.9. We mention that the really hard part of this theorem is that if

$$F(\underline{r}_e,t) > 0$$

for all $t \in [0,1]$ then the edge densities $\gamma_e = 1-r_e$ ensure the existence of the tree T as a transversal. Later we will prove that this is true for every graph H: see Theorem 4.4.

Proof. We prove the theorem by induction on the number of vertices. We will use Theorem 3.1. First we show that if the edge densities ensure the existence of the factor T then

$$F(\underline{r}_e,t) > 0$$

for all $t \in [0,1]$.

Clearly,

$$F(\underline{r}_e,t) = F(\underline{r}_e t,1).$$

It is also trivial that the densities $\gamma_e = 1 - r_e$ ensure the existence of a factor T, then the densities $\gamma_e = 1 - tr_e$ ($t \in [0,1]$) ensure the existence of factor T. Hence we only need to prove that if the densities $\gamma_e = 1 - r_e$ ensure the existence of factor T, then $F(\underline{r}_e,1) > 0$.

We will use the notation of Theorem 3.1. By induction and Theorem 3.1 we have $F_{T'}(\underline{r}'_e,1) > 0$. Now we repeat the argument of Lemma 3.7.

As before, we can expand $F_{T'}$ according to whether a monomial contains $x_{k,n-1}$ ($e_{k,n-1} \in E(T')$) or not. Each monomial can contain at most one of the variables $x_{k,n-1}$ ($v_k \in N(v_{n-1})$). Thus

$$F_{T'}(\underline{x}_e,t) = Q_0(\underline{x}_e,t) - \sum_{v_k\in N(v_{n-1})} tx_{k,n-1}Q_k(\underline{x}_e,t),$$

where Q_0 consists of those terms which contain no $x_{k,n-1}$ and $-tx_{k,n-1}Q_k$ consists of those terms which contain $x_{k,n-1}$, i.e., these terms correspond to the matchings containing the edge (v_k,v_{n-1}). We have

$$F_T(\underline{x}_e,t) = (1 - tx_{n-1,n})Q_0(\underline{x}_e,t) - \sum_{v_k\in N(v_{n-1})} tx_{k,n-1}Q_k(\underline{x}_e,t)$$

by the same argument.

Hence

$$0 < F_{T'}(\underline{r}'_e,1) = Q_0(\underline{r}_e,1) - \sum_{v_k\in N(v_{n-1})} \frac{r_{k,n-1}}{1 - r_{n-1,n}}Q_k(\underline{r}_e,1).$$

So we get that

$$0 < (1-r_{n-1,n})F_{T'}(r'_e, 1) = (1-r_{n-1,n})Q_0(\underline{r_e}, 1) - \sum_{v_k \in N(v_{n-1})} r_{k,n-1}Q_k(\underline{r_e}, 1) = F_T(\underline{r_e}, 1).$$

This completes one direction of the proof.

Now we assume that $F(\underline{r_e}, t) > 0$ for all $t \in [0,1]$. We prove by contradiction that the edge densities γ_e ensure the existence of factor T. Assume that the Algorithm 3.3 stops with some $r_{e^\circ} \geq 1$. We will call e° the violating edge. In the next step we show that for some $t \in [0,1]$ we can ensure that the algorithm stops with $r_{e^\circ}(t) = 1$ when we start with the edge densities $\gamma_e = 1 - tr_e$.

First of all, let us examine what happens if we decrease the r_e. If $0 < r_e \leq r_e^*$ and $0 < r_f \leq r_f^*$, then

$$\frac{r_e}{1-r_f} \leq \frac{r_e^*}{1-r_f^*}.$$

Hence all r_i decrease under the algorithm if we decrease t.

If we set $t = 0$, then for the edge densities $\gamma_e = 1 - tr_e$ the algorithm gives 1 for all densities which show up. Since we are changing t continuously, all densities will change continuously, and we can choose an appropriate $t \in [0,1]$ for which, by running our algorithm with tr_e instead of r_e, we can assume that the algorithm stops with $r_{e^\circ}(t) = 1$.

Now consider those vertices and edges, together with the violating edge which were deleted when executing the algorithm. These edges form a forest. Consider the component of this forest which contains the violating edge. Let us call this subtree T_1. According to Lemma 3.7 our chosen t is the root of the matching polynomial of T_1 (clearly, only the deleted edges modified the weight of the violating edge). On the other hand, we know from Corollary 3.6 that the matching polynomial of T has a smaller root than the matching polynomial of T_1. This means that the matching polynomial of T has a root in the interval $[0,1]$, contradicting the condition of the theorem.

$\square$

Corollary 3.10. *Let T be a tree and assume that all edge densities γ_e satisfy $\gamma_e > 1 - \frac{1}{\lambda(T)^2}$ where $\lambda(T)$ is the largest eigenvalue of the adjacency matrix of T. Then the densities γ_e ensure the existence of factor T. If all $\gamma = 1 - \frac{1}{\lambda(T)^2}$, then there exists a weighted blow-up of T not containing T as a transversal. In other words,*

$$d_{crit}(T) = 1 - \frac{1}{\lambda(T)^2}.$$

Proof. We can assume that all edge densities are equal to $1 - d > 1 - \frac{1}{\lambda^2}$. In this case $dt < \frac{1}{\lambda(T)^2}$ for all $t \in [0,1]$ and so

$$0 < \phi_T(\frac{1}{\sqrt{dt}}) = (dt)^{-n/2}F_T(\underline{dt}, 1) = (dt)^{-n/2}F_T(\underline{d}, t)$$

by Lemma 3.4. By Theorem 3.8 this implies that the set of edge densities $\{\gamma_e\}$ ensures the existence of factor T. Theorem 3.8 also implies that there exists a weighted blow-up with weights $\gamma = 1 - \frac{1}{\lambda(T)^2}$ of T not containing T as a transversal.

$\square$

Finally we recall a structure theorem concerning the critical edge density of trees.

Proposition 3.11. [12] *Let T be a tree. Let us consider the following blow-up graph $G[T]$ of T. Let the cluster A_i consist of the vertices v_{ij} where $j \in N(i)$. If $(i, j) \in E(T)$ then we connect all vertices of A_i and A_j except v_{ij} and v_{ji}. Then $G[T]$ does not contain T as a transversal.*
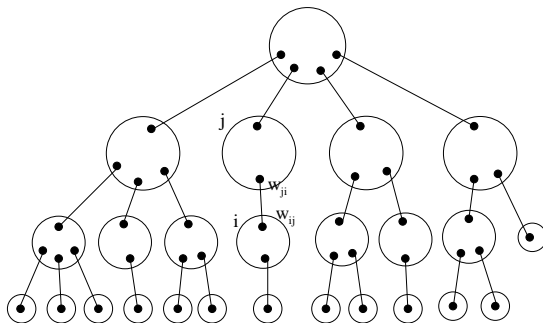

FIGURE 3. The complement of a special blow-up graph of a tree.

Proof. We have to prove that one cannot avoid choosing both end vertices of a complementary edge (v_{ij}, v_{ji}) if one chooses one vertex from each cluster. This is indeed true since the set of all vertices of $G[T]$ can be decomposed to $(n - 1)$ such pairs. Since we have to choose n vertices we have to choose both vertex from such a pair. □

We show that we can give weights to the vertices of $G[T]$ constructed above such that the density will be $1 - \frac{1}{\lambda^2}$ where $\lambda = \lambda(T)$. The following weighting was the idea of András Gács [8].

Recall that there exists a non-negative eigenvector $\underline{x}$ belonging to the largest eigenvalue λ of T. So, if v_i's are the vertices of T, then we have

$$\lambda x_i = \sum_{j \in N(i)} x_j$$

for all i. Now let us define the weight w_{ij} of the vertex v_{ij} of $G[T]$ as follows: $w_{ij} = \frac{x_j}{\lambda x_i} \geq 0$. Then we have

$$w(A_i) = \sum_{j \in N(i)} w_{ij} = \sum_{j \in N(i)} \frac{x_j}{\lambda x_i} = 1.$$

Furthermore,

$$d(A_i, A_j) = 1 - w_{ij} w_{ji} = 1 - \frac{x_j}{\lambda x_i} \frac{x_i}{\lambda x_j} = 1 - \frac{1}{\lambda^2}.$$

4. GENERAL BOUNDS

Our next aim is to prove good bounds on the critical edge density. Recall that $(1 - \frac{1}{\Delta(H)}) \leq d_{crit}(H) \leq (1 - \frac{1}{\Delta^2(H)})$ was known before, see Proposition 2.5. Our approach is probabilistic. First we give a bound applying the Lovász local lemma. In fact, we can copy the argument of [1].

Theorem 4.1. (Lovász local lemma, symmetric case, [1].) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of all other events, but at most Δ of them. Furthermore, assume that for each i,*

$$\mathbb{P}(A_i) \leq \frac{1}{e(\Delta + 1)},$$

where e is the base of the natural logarithm. Then

$$\mathbb{P}(\cap_{i=1}^n \overline{A_i}) > 0.$$

Theorem 4.2. *Let Δ be the largest degree of the graph H and let $d_{crit}(H)$ be the critical edge density. Then*

$$d_{crit}(H) \leq 1 - \frac{1}{e(2\Delta - 1)},$$

where e is the base of the natural logarithm.

Proof. We use proof by contradiction. Assume that there exists a blow-up graph $G[H]$ of the graph H with edge densities greater than $1 - \frac{1}{e(2\Delta-1)}$ which does not induce H.

We can assume that all classes of the blow-up graph $G[H]$ contain exactly N vertices. Indeed, we can approximate each weight by a rational number so that every edge density is still larger than $1 - \frac{1}{e(2\Delta-1)}$. Then we "blow up" the construction by the common denominator of all weights.

Let us choose a vertex from each class with equal probability $1/N$, independently of each other. Let f be an edge of the complement of the graph $G[H]$ with respect to H. Let A_f be the event that we have chosen both end nodes of the edge f (clearly a bad event we would like to avoid). Then $\mathbb{P}(A_f) = 1/N^2$ and A_f is independent from all events $A_{f'}$ where the edge f' has endvertices in different classes. Thus A_f is independent from all, but at most $(2\Delta - 1)rN^2$ bad events where $r = 1 - d_{crit}(H)$. Since $r < \frac{1}{e(2\Delta-1)}$, the condition of Lovász local lemma is satisfied, and gives that

$$\mathbb{P}(\cap_{f \in E(\overline{G[H]|H})} A_f) > 0.$$

which means that $G[H]$ induces the graph H (with positive probability), contradicting the assumption. $\qquad\square$

Next, we use a generalization of the Lovász local lemma to improve on the bound of Theorem 4.2.

Theorem 4.3. [Scott-Sokal] [13] *Assume that, given a graph G, there is an event A_i assigned to each node i. Assume that A_i is mutually independent of the events $\{A_k \mid (i,k) \in E(\overline{G})\}$. Set $\mathbb{P}(A_i) = p_i$.*

(a) Assume that $I((G, \underline{p}), t) > 0$ for all $t \in [0, 1]$. Then we have

$$\mathbb{P}(\cap_{i \in V(G)} \overline{A_i}) \geq I((G, \underline{p}), 1) > 0.$$

(b) Assume that $I((G, \underline{p}), t) = 0$ for some $t \in [0, 1]$. Then there exists a probability space and a family of events B_i with $\mathbb{P}(B_i) \leq p_i$ and with dependency graph G such that

$$\mathbb{P}(\cap_{i \in V(G)} \overline{B_i}) = 0.$$

Theorem 4.4. *Assume that for the graph H we have $F_H(\underline{r_e}, t) > 0$ for all $t \in [0, 1]$ and some weights $r_e \in [0, 1]$ assigned to each edge. Then the densities $\gamma_e = 1 - r_e$ ensure the existence of H as a transversal.*

Proof. As before, we choose a vertex from each cluster independently of each other. We choose the vertex u from the cluster V_i of the graph $G[H]$ with probability $w(u)$. We would like to show that we do not choose both end vertices of an edge of the complement $\overline{G[H]|H}$ with positive probability. Let $f = (u_1, u_2)$ be an edge of the $\overline{G[H]|H}$. Let A_f be the event that we have chosen both end nodes of the edge f (clearly, a bad event we would like to avoid). Then $\mathbb{P}(A_f) = w(u_1)w(u_2)$ and A_f is independent from all events $A_{f'}$, where the edge f' has end vertices in different classes. Now let us consider the weighted independence polynomial of the graph determined by the vertices A_f in which we connect A_f and $A_{f'}$ if there exists a cluster containing end vertices of both f and f'. In this graph, the events A_f, where f goes between the fixed clusters V_i, V_j, not only form a clique but it is also true that they are connected to the same set of events. Hence we can replace them by one vertex of weight

$$\sum_{(u_1, u_2) \in E(\overline{G[H]}(V_i \cup V_j))} w(u_1)w(u_2) = r_{ij}$$

without changing the weighted independence polynomial. But then the obtained weighted independence polynomial is

$$I((L_H, \underline{r_e}), t) = F_H(\underline{r_e}, t) > 0$$

for $t \in [0, 1]$. Then, by the Scott-Sokal theorem we have

$$\mathbb{P}(\cap_{f \in E(\overline{G[H]|H})} \overline{A_f}) \geq F((H, \underline{r_e}), 1) > 0.$$

$\square$

Corollary 4.5. *Let Δ be the largest degree of the graph H and $t(H)$ be the largest root of the matching polynomial. Then, for the critical edge density $d_{crit}(H)$ we have*

$$d_{crit}(H) \leq 1 - \frac{1}{t(H)^2}.$$

In particular,

$$d_{crit}(H) < 1 - \frac{1}{4(\Delta - 1)}.$$

Proof. Let $\gamma_e = 1 - r$ for every edge $e \in E(H)$, where $r < \frac{1}{t(H)^2}$ then

$$F_H(\underline{r}, t) = \sum_{k=0}^{n} (-1)^k m_k(H) r^k t^k = (rt)^{n/2} M(H, \frac{1}{\sqrt{rt}}) > (rt)^{n/2} M(H, t(H)) = 0$$

for $t \in [0, 1]$. Hence the set of densities $\{\gamma_e\}$ ensures the existence of the graph H. Thus $d_{crit}(H) \leq 1 - r$ for every $r < \frac{1}{t(H)^2}$. Hence

$$d_{crit}(H) \leq 1 - \frac{1}{t(H)^2}.$$

The second claim follows from the fact that $t(H) < 2\sqrt{\Delta - 1}$: see [11]. $\square$

5. Star decomposition

In this section we examine a large class of blow-up graphs which do not induce a given graph as a transversal. Assume that $H = H_1 \cup \{v_n\}$ and we have a blow-up graph of H_1 which does not induce H_1 as a transversal. We can construct a blow-up graph of H not inducing H as follows. Let $A_n = \{w_n\}$ be the blow-up of v_n. Furthermore, assume that $N_H(v_n) = \{v_1, v_2, \ldots, v_k\}$ with the corresponding clusters $A'_1, \ldots, A'_k$ in the blow-up of H_1. Then let $A_i = A'_i \cup \{w_i\}$ if $1 \leq i \leq k$, and we leave unchanged all other clusters. Let us connect w_n to each elements of A'_i $(1 \leq k \leq n)$ and connect w_i with every possible neighbor except w_n. All other pairs of vertices remain adjacent or non-adjacent as in the blow-up of H_1.

Now it is clear why we call this construction a star decomposition: the complement of the construction with respect to $G[H]$ consists of stars, see Figure 4.
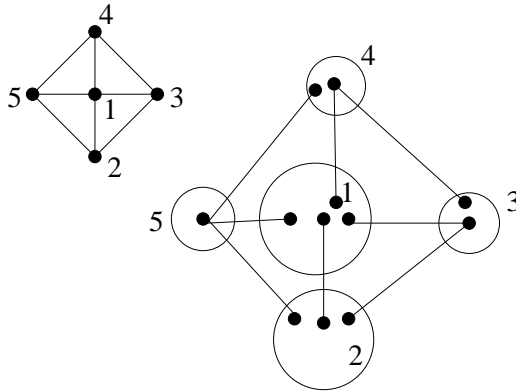

FIGURE 4. Star decomposition of the wheel, the complement of the construction.

This new blow-up graph will clearly not induce H as a transversal.

Although we gave a construction of a blow-up of the graph H not inducing H, this is only one half of a full construction, since we can vary the weights of the vertices of the blow-up graph. Of course, we would like to choose the weights optimally. But what does this mean? Assume that we are given densities for all edges of H and we wish to make a construction iteratively as we described in the previous paragraph, and now we would like to choose the weights so that the edge-densities are at least as large as the required edge-densities. To quantify this argument we need some definitions.

Definition 5.1. A proper labelling of the vertices of the graph H is a bijective function f from $\{1, 2, \ldots, n\}$ to the set of vertices such that the vertex set $\{f(1), \ldots, f(k)\}$ induces a connected subgraph of H for all $1 \leq k \leq n$.

Definition 5.2. Let there be weighted graph H with a proper labelling f, where the weights on the edges are between 0 and 1. The *weighted monotone-path tree* of H is defined as follows. The vertices of this graph are the paths of the form $f(i_1)f(i_2)\ldots f(i_k)$, where $1 = i_1 < i_2 < \cdots < i_k$, and two such paths are connected if one is the extension of the other with exactly one new vertex. The weight of the edge connecting $f(i_1)f(i_2)\ldots f(i_{k-1})$ and $f(i_1)f(i_2)\ldots f(i_k)$ is the weight of the edge $f(i_{k-1})f(i_k)$ in the graph H.
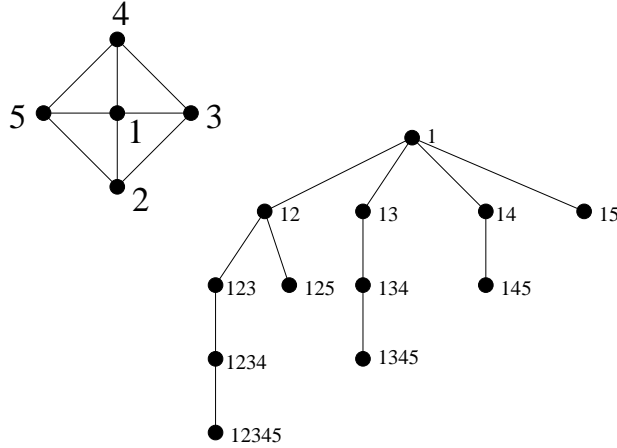
The monotone-path tree is the same without weights.

FIGURE 5. A monotone-path tree of the wheel on 5 vertices.

Theorem 5.3. *Let H be a properly labelled graph with edge densities γ_e, and let $T_f(H)$ be its weighted monotone-path tree with weights γ_e. Assume that these densities do not ensure the existence of the factor $T_f(H)$. Then there is a construction of a blow-up graph of H not inducing H as a transversal and all densities between the clusters are at least as large as the given densities.*

Remark 5.4. So this theorem provides a necessary condition for the densities ensuring the existence of factor H. In fact, this gives as many necessary conditions as there are proper labellings the graph H. The advantage of this theorem is that we already know the case of trees substantially.

Proof. We prove the statement by induction on the number of vertices of H. For $n = 1, 2$ the claim is trivial since $H = T_f(H)$. Now assume that we already know the statement for $n - 1$, and we need to prove it for $|V(H)| = n$.

We know from Theorem 3.1 that γ_e ensure the existence of factor $T = T_f(H)$ if the corresponding γ'_e ensure the existence of factor T'. Let us apply this theorem as follows. We delete all vertices (monotone-paths) of $T_f(H)$ which contains the vertex $f(n)$. The remaining tree will be a weighted path tree of $H_1 = H - \{f(n)\}$, where the new labelling is simply the restriction of f to the set $\{1, 2, \ldots, n-1\}$. (We will denote this restriction by f too.) By induction there exists a blow-up graph of H_1 not inducing H_1 as a transversal, and all densities between the clusters are at least $\gamma_e(T_f(H_1))$, where we can also assume that the total weight of each cluster is 1.

Now we can do the the construction described in the beginning of this section. Let $f(n) = u$ and $N_H(u) = \{u_1, \ldots, u_k\}$. Let the weight of the new vertex $w_i \in A_i$ be $(1 - \gamma_{uu_i})$ and the weights of the other vertices of the cluster be γ_{uu_i} times the original one. Clearly, between the clusters A_n and A_i $(1 \le i \le k)$, the weight is just γ_{uu_i} as required. What about the other densities? First of all let us examine the γ'_e. Let us consider the adjacent vertices $f(1) \ldots f(i)$ and $f(1) \ldots f(i)f(j)$ of $T_f(H_1)$. If both $f(i), f(j) \in N_H(u)$, then we deleted the vertices $f(1) \ldots f(i)f(n)$ and $f(1) \ldots f(i)f(j)f(n)$ from $T_f(H)$, changing $\gamma_e = 1 - r_e$ to $1 - \frac{r_e}{\gamma_{f(n)f(i)}\gamma_{f(n)f(j)}}$. If only one of the vertices $f(i)$ or $f(j)$ was connected to $f(n)$, then we can still easily follow the change: $\gamma'_e = 1 - \frac{r_e}{\gamma_{f(n)f(i)}}$ if $f(i)$ was connected to $f(n)$. If none of them was connected to $f(n)$, then there is no change. But in all cases we do

exactly the inverse of this operation at the blow-up graphs, ensuring that the new densities are at least γ_e. □

Corollary 5.5. *Let $S(H)$ be the set of proper labellings of the graph H. The critical density of the graph H is at least*

$$\max_{f \in S(H)} \left\{ 1 - \frac{1}{\lambda(T_f(H))^2} \right\}.$$

Remark 5.6. If each edge density is equal to $1 - \frac{1}{\lambda(T_f(H))^2}$ then there is a straight-forward connection between the weights of the constructed blow-up graph and the eigenvector of the tree $T_f(H)$ belonging to the eigenvalue $\lambda(T_f(H))$. This connection is very similar to the one given by András Gács.

5.1. **The Main conjecture and a counterexample.**

The following conjecture seems a natural one following the case for trees.

Conjecture 5.7 (General Star Decomposition Conjecture). *Let H be a graph with edge densities γ_e. Assume that for each proper labelling f, the weights as densities of the weighted monotone-path tree ensure the existence of the graph $T_f(H)$. Then the given densities ensure the existence of the graph H.*

The following conjecture states that the bound on the critical edge density coming from Corollary 5.5 is sharp.

Conjecture 5.8 (Uniform Star Decomposition Conjecture). *Let $S(H)$ be the set of proper labellings of the graph H. The critical density of the graph H satisfies*

$$d_{crit} = \max_{f \in S(H)} \left\{ 1 - \frac{1}{\lambda(T_f(H))^2} \right\}.$$

Remark 5.9. So the General Star Decomposition Conjecture asserts that for every graph and every weighting (or edge densities), the best we can do is to choose a good order of the vertices and construct the "stars". The Uniform Star Decomposition Conjecture is clearly a special case of this conjecture when all edge densities are the same for every edge.

The General Star Decomposition Conjecture is true for the triangle in the sense that, for every weighting the star decomposition of a suitable labelling gives the best construction or shows that there is no suitable blow-up graph. This is a theorem of Adrian Bondy, Jian Shen, Stéphan Thomassé and Carsten Thomassen: see Lemma 2.6 or [5]. As we have seen, this conjecture is also true for trees. We show that it also holds for cycles.

Theorem 5.10. *General Star Decomposition Conjecture holds for C_n.*

We only sketch the proof here, since the inhomogeneous condition of edge densities of C_n is needed here, which may build up in a similar way to the homogeneous case, described in the 4th section of [12]. The details will be left to the Reader.

Sketch of the proof. Notice that a key statement in the proof of $d(C_n) = d(P_{n+1})$ [12] was to make a correspondence between the constructions for C_n and P_{n+1}. In our terminology, this correspondence is exactly the one between the cycle and its monotone-path tree, which is in fact a path on $n + 1$ vertices.

Hence the proof can be constructed as follows. First, by applying Theorem 2.2 we can assume that each cluster has size at most 2. Then it turns out that, just like in the homogeneous case [12], there is only one candidate for the edge-construction to give the best construction with appropriate weighting. In fact, this edge construction is exactly Construction 4.1 in [12].

Then, slightly modifying Lemma 4.5 in [12], we can obtain that one may assume that one of the clusters has cardinality 1, which provides the correspondence of the construction of cycles and paths. In this case we have n different paths depending on the starting vertex, and these are exactly the monotone-path trees of the cycle.

$\square$

However, in the following we will show that the General Star Decomposition Conjecture is in general false. Thus it seems very unlikely that the Uniform Star Decomposition Conjecture is true. Still, it is a meaningful question to ask for which graphs one or both conjectures hold. The authors strongly believe that the Uniform Star Decomposition Conjecture is true for complete graphs and complete bipartite graphs.

Our counterexample for the General Star Decomposition Conjecture is a weighted bow-tie given by Figure 6. It is not a star decomposition in the sense we constructed it, while it is indeed a good construction: whatever we choose from the middle cluster, we cannot choose its neighbors (since it is the complement), but then we have to choose the other vertices from the corresponding clusters, but they are connected in the complement.
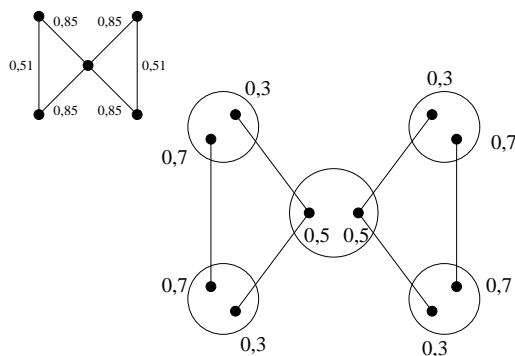

FIGURE 6. Weighted bow-tie and its weighted blow-up graph of the complement.

We will show that the given construction of the blow-up graph is the best possible in the following sense. If for some blow-up graph the edge densities are at least as large as the required densities and one of them is strictly greater, then it induces the bow-tie as a transversal. We will also show that no star decomposition can attain the same densities.

Before we prove it we need some preparation. We prove a lemma which can be considered as a generalization of Theorem 3.1.

Lemma 5.11. *Let H_1, H_2 be two graphs and let $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$. Let us denote by $H_1 : H_2$ the graph obtained by identifying the vertices u_1, u_2 in $H_1 \cup H_2$. Let $0 < m_1, m_2 < 1$ such that $m_1 + m_2 \leq 1$. Furthermore, assume that*

an edge density $\gamma_e = 1 - r_e$ is assigned to every edge. If the edge densities

$$\gamma_e' = \begin{cases} \gamma_e = 1 - r_e \text{ if } e \in E(H_1) \text{ is not incident to } u_1, \\ 1 - \frac{r_e}{m_1} \text{ if } e \in E(H_1) \text{ is incident to } u_1, \end{cases}$$

ensure the existence of a transversal H_1, and the edge densities

$$\gamma_e' = \begin{cases} \gamma_e = 1 - r_e \text{ if } e \in E(H_2) \text{ is not incident to } u_2, \\ 1 - \frac{r_e}{m_2} \text{ if } e \in E(H_2) \text{ is incident to } u_2. \end{cases}$$

ensure the existence of a transversal H_2, then the edge densities $\{\gamma_e\}$ ensure the existence of a transversal $H_1 : H_2$.

Proof. Let $G[H_1 : H_2]$ be a weighted blow-up graph of $H_1 : H_2$ with edge density $\{\gamma_e\}$. Let

$$R_1 = \{v \in A_{u_1=u_2} \mid v \text{ can be extended to a transversal } H_1 \subset G[H_1]\}$$

and

$$R_2 = \{v \in A_{u_1=u_2} \mid v \text{ can be extended to a transversal } H_2 \subset G[H_2]\}.$$

We show that

$$\sum_{v \in R_1} w(v) > 1 - m_1 \quad \text{and} \quad \sum_{v \in R_2} w(v) > 1 - m_2.$$

But then, since $m_1 + m_2 \leq 1$ there would be some $v \in R_1 \cap R_2$, which we could extend to a transversal of H_1 and H_2 as well, and thus we could find a transversal $H_1 : H_2$. Naturally, it is enough to prove that $\sum_{v \in R_1} w(v) > 1 - m_1$, because of the symmetry. We prove it by contradiction. Assume that $\sum_{v \in R_1} w(v) = 1 - t \leq 1 - m_1$. Let us erase all vertices belonging to R_1 from $A_{u_1=u_2}$, and let us give the weight $\frac{w(u)}{t}$ to the remaining vertices $u \in A_{u_1=u_2} - R_1$. Then we obtained a weighted blow-up graph $G'[H_1]$ in which every edge density is at least γ_e' ($e \in E(H_1)$). But then the assumption of the lemma ensures the existence of a transversal H_1, which contradicts the construction of $G'[H_1]$. $\square$

Now we are ready to prove that the construction given above is best possible.

Counterexample 5.12. *For graph H, let*

$$V(H) = \{v_1, v_2, v_3, v_4, v_5\}, \text{ and}$$
$$E(H) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_4v_5\}.$$

Furthermore, assume that the edge densities of the blow-up graph $G[H]$ satisfy the inequalities $\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15} \geq 0,85$, $\gamma_{23}, \gamma_{45} \geq 0,51$, and at least one of the inequalities is strict. Then $G[H]$ contains H as a transversal.

Proof. We can assume by symmetry that at least one of the strict inequalities $\gamma_{12} > 0,85$ or $\gamma_{23} > 0,51$ holds. Let us apply Lemma 5.11 with $H_1 = H(v_1, v_2, v_3)$ and $H_2 = H(v_1, v_4, v_5)$, $u_1 = u_2 = v_1$, densities γ_{ij} and $m_1 = 1/2 - \varepsilon$, $m_2 = 1/2 + \varepsilon$, where ε is a very small positive number chosen later. Then

$$\gamma_{ij}'\gamma_{jk}' + \gamma_{ik} - 1 = 1 - r_{12}' - r_{13}' - r_{23}' + r_{ij}'r_{jk}' > 0$$

for any permutation i, j, k of $\{1, 2, 3\}$. Indeed, since $0,3 = \frac{0,15}{0,5}$ we have

$$1 - 0,3 - 0,3 - 0,49 + 0,3 \cdot 0,49 > 1 - 0,3 - 0,3 - 0,49 + 0,3 \cdot 0,3 = 0,$$

and one of the r_{ij}'s is strictly smaller than $0,3$ or $0,49$ and so for small enough ε, the expression $1 - r'_{12} - r'_{13} - r'_{23} + r'_{ij}r'_{jk}$ is positive. Hence by Lemma 2.6 it ensures the existence of a triangle transversal. For the other triangle, $r'_{14} = \frac{r_{14}}{1/2+\varepsilon} < 0,3$ and similarly, $r'_{15} < 0,3$ and $r_{45} \leq 0,49$. Again by Lemma 2.6 it ensures the existence of a triangle transversal. By Lemma 5.11 we obtain that there exists a transversal H in $G[H]$. □

Proposition 5.13. *There is no weighted blow-up graph of the bow-tie arising from star decomposition which is at least as good as the weighted blow-up graph in Figure 7.*

Proof. Because of the symmetry, and since we only need to consider the star decompositions where the labelling is proper, we only have to consider two star decompositions. Because of Statement 5.12, all edge densities must be exactly the required one. This makes the whole computation routine.
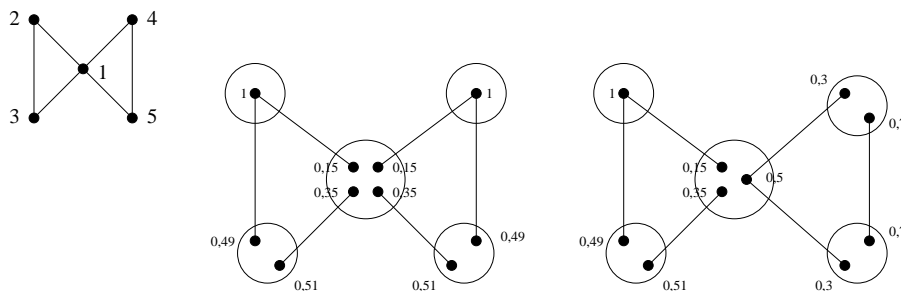

FIGURE 7. Star decompositions of bow-ties.

□

5.2. **The complete bipartite graph case.**

Let $d_{crit}(K_{n,m}) = d(n,m)$ be the critical edge density of the complete bipartite graph $K_{n,m}$. Let $d_s(n,m)$ be the best edge density coming from the star decomposition (s stands for star in d_s).

If one starts to do the star decomposition to $K_{n,m}$, then we have the recursion

$$d_s(n,m) = \frac{1}{2 - d_s(n,m-1)} \quad \text{or} \quad \frac{1}{2 - d_s(n-1,m)}$$

according to which class contains the vertex $f(n+m)$. Although we have two possibilities, the recursion has only one solution, namely

$$d_s(n,m) = 1 - \frac{1}{n+m-1}$$

since $d(1,1) = d_s(1,1) = 0$. From this we gain an interesting fact.

Theorem 5.14. *For any proper labelling f of the graph $K_{n,m}$, the tree $T_f(K_{n,m})$ has spectral radius $\sqrt{n+m-1}$.*

Remark 5.15. In this case a proper labelling simply means that $f(1)$ and $f(2)$ are elements of different classes in the bipartite graph.

For different proper labellings these trees can look very different, but as the theorem shows, their spectral radii are the same. In fact, it turns out that not

only their spectral radius, but all their eigenvalues too are of the form $\pm\sqrt{n}$, where n is a non-negative integer. These are the same trees defined in the paper [6].

Conjecture 5.16. $d_{crit}(K_{n,m}) = d_s(n,m) = 1 - \frac{1}{n+m-1}$.

Remark 5.17. Conjecture 5.8 clearly implies Conjecture 5.16, but the authors have the feeling that Conjecture 5.16 is true while Conjecture 5.8 may not hold.

If Conjecture 5.16 holds it would have an interesting consequence. In the case of trees and cycles the extremal construction is unique, and so it is conjectured for the complete graphs. However, this would not stand in the case of complete bipartite graphs: there would be several different types of constructions depending on the proper labelling (see Figure 8).
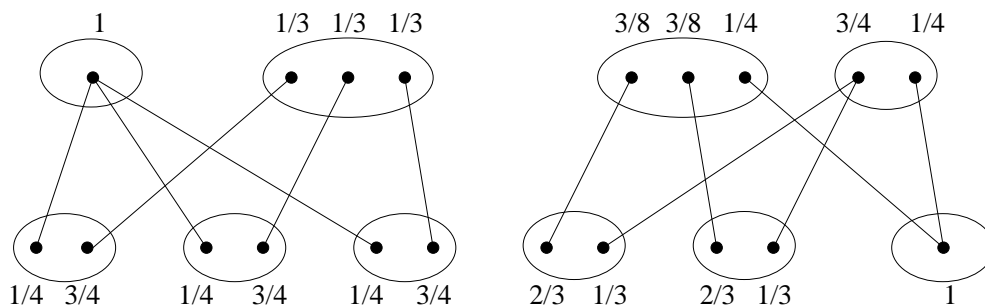

FIGURE 8. Two constructions for $G = K_{2,3}$ attaining $d_s(2,3)$

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