ON THE ROOTS OF EDGE COVER POLYNOMIALS OF GRAPHS

PÉTER CSIKVÁRI AND MOHAMMAD REZA OBOUDI

ABSTRACT. Let G be a simple graph of order n and size m. An edge covering of the graph G is a set of edges such that every vertex of the graph is incident to at least one edge of the set. Let $e(G,k)$ be the number of edge covering sets of G of size k. The edge cover polynomial of G is the polynomial

$$E(G,x)=\sum_{k=1}^{m}e(G,k)x^k.$$

In this paper we obtain some results on the roots of the edge cover polynomials. We show that for every graph G with no isolated vertex, all the roots of $E(G,x)$ are in the ball

$$\{z\in\mathbb{C}\,:\,|z|<\frac{(2+\sqrt{3})^2}{1+\sqrt{3}}\simeq 5.099\}.$$

We prove that if every block of the graph G is K_2 or cycle, then all real roots of $E(G,x)$ are in the interval $(-4,0]$. We also show that for every tree T of order n we have

$$\xi_{\mathbb{R}}(K_{1,n-1})\leq\xi_{\mathbb{R}}(T)\leq\xi_{\mathbb{R}}(P_n),$$

where $-\xi_{\mathbb{R}}(T)$ is the smallest real root of $E(T,x)$, and $P_n,K_{1,n-1}$ are the path and the star of order n, respectively.

1. INTRODUCTION

The concept of the edge cover polynomial was introduced by Saieed Akbari and Mohammad Reza Oboudi [1]. The edge cover polynomial is defined as follows.

Let G be a graph on n vertices and m edges. Let $e(G,k)$ denote the number of ways one can choose k edges of G that cover all vertices of the graph G. We call any subset of edges of G that covers all vertices an *edge covering* of G; the cardinality of the smallest edge covering is the *edge covering number* of G, which is denoted by $\rho(G)$ [3]. We call the polynomial

$$E(G,x)=\sum_{k=1}^{m}e(G,k)x^k$$

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the *edge cover polynomial* of the graph G. Clearly, if the graph G has an isolated vertex then the edge cover polynomial is 0. We let $E(G,x) = 1$ when $n = m = 0$.

Our motivation is the following question posed by László Lovász [13].

Question: Is there any upper bound for the roots of edge cover polynomials?

We answer this question affirmatively. We will prove the following theorems.

Theorem 3.1 All roots of the edge cover polynomial lie in the ball $\{z \in \mathbb{C} : |z| < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}}\}$.

Recall that the block of a graph is a maximal induced subgraph without cut-vertex.

Theorem 4.5 Let G be a graph of order n. If every block of G is K_2 or cycle, then all real roots of $E(G,x)$ are in the interval $[-2-2\cos\frac{\pi}{n}, 0]$.

There are many papers on the locations of the roots of other graph polynomials such as chromatic polynomial, matching polynomial, independence polynomial, characteristic polynomial. In [16], Thomassen showed that the chromatic polynomials have no root in the intervals $(-\infty, 0), (0, 1)$ and $(1, \frac{32}{27}]$. Moreover, he proved that the roots of the chromatic polynomials are dense in the interval $[\frac{32}{27}, \infty)$. He also showed that if the chromatic polynomial of a graph has a non-integer root less than or equal to 1.29559..., then the graph has no Hamiltonian path [15]. In [4], Brown, Hickman, and Nowakowski proved that the real roots of the independence polynomials are dense in the interval $(-\infty, 0]$, while the complex roots are dense in the complex plane. There are many results on the roots of the matching polynomials as well. In [10], it was proved that all roots of the matching polynomials are real. Also it was shown that if a graph has a Hamiltonian path, then all roots of its matching polynomial are simple (see Theorem 4.5 of [8]). It is well known that all roots of the characteristic polynomials are real. For every tree, the matching polynomial and the characteristic polynomial are equal to each other [8]. Since for every n, $\sqrt{n}$ is a root of the characteristic polynomial of $K_{1,n+1}$, we can conclude that there is no constant bound for the roots of the characteristic polynomials and matching polynomials. For more details on characteristic polynomials see [5]. There are also many bounds for the roots of these polynomials in terms of other parameters of the graphs. For instance, in [14] Sokal proved that for every graph G, the absolute value of any root of the chromatic polynomial of G is at most $8\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of the graph G. On the other hand, $\chi(G)-1$ is clearly a root of the chromatic polynomial where $\chi(G)$ denotes the chromatic number. Therefore, there is no constant bound for the roots of these polynomials (chromatic polynomial , matching polynomial, independence polynomial, characteristic polynomial). Surprisingly, in this paper, we will show that there is a constant bound for the roots of the edge cover polynomials.

Recently, Averbouch, Godlin, and Makowsky [2] introduced a new graph polynomial, the *edge elimination polynomial* that is denoted by $\xi(G, x, y, z)$, for every graph G, which generalizes some well known graph polynomials such as (di)chromatic polynomials, matching polynomials, independence polynomials, and Tutte polynomials. Using Theorem 4 of the paper [2] one can easily see that this graph polynomial generalizes the edge cover polynomial as well. In fact, for every graph G, we have $E(G, x) = \xi(G, 0, x, x)$.

The structure of this paper is the following. In the next section we introduce the concept of the set-generating function which provides many identities for the edge cover polynomial. In Section 3, we will prove Theorem 3.1, while in Section 4, we prove Theorem 4.5.

Notation: Throughout this paper we will consider only graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The *order* and the *size* of G are the number of vertices and the number of edges of G, respectively. Let $S \subseteq V(G)$. By $G|_S$ we mean the induced subgraph of G on the vertex set S. For simplicity, we write $E(S, x)$ instead of $E(G|_S, x)$. We denote the *complete graph*, the *cycle*, and the *path* of order n by K_n, C_n and P_n, respectively. We denote the *complete bipartite graph* with part sizes m and n by $K_{m,n}$ and we call $K_{1,n}$ a *star* of order $n + 1$. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident to v and is denoted by $d_G(v)$. For the sake of simplicity, we write $d(v)$ instead of $d_G(v)$. A *pendant* vertex is a vertex with degree one. By a *pendant* edge we mean an edge, one of whose end points is a pendant vertex. We denote the minimum and the maximum degree of the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively.

For an arbitrary graph G with no isolated vertex, we define $\xi_{\mathbb{R}}(G)$ and $\xi_{\mathbb{C}}(G)$ as follows:

$$\xi_{\mathbb{R}}(G) = \max\{|z| \ : \ z \in \mathbb{R},\ E(G, z) = 0\},$$

and

$$\xi_{\mathbb{C}}(G) = \max\{|z| \ : \ z \in \mathbb{C},\ E(G, z) = 0\}.$$

Clearly, $\xi_{\mathbb{C}}(G) \geq \xi_{\mathbb{R}}(G)$. If G has an isolated vertex we let $\xi_{\mathbb{C}}(G) = \xi_{\mathbb{R}}(G) = \infty$. Note that $-\xi_{\mathbb{R}}(G)$ is the smallest real root of $E(G, x)$.

2. The set-generating function of the edge cover polynomial

This section is strongly motivated by the paper of Bodo Lass on matching polynomials [11] in which he derived many identities for matching polynomials by the aid of set-generating functions.

Let G be a graph on the vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G)$. Let m denote the size of $E(G)$. Let us consider the ring

$$\mathbb{D} = \mathbb{R}[a_1, \dots, a_n]/\langle a_i^2 - a_i \ (i = 1, \dots, n)\rangle.$$

This means that $\mathbb{D}$ is a commutative ring in which the elements a_i are idempotents, i.e., $a_i^2 = a_i$.

An element of this ring is of the form

$$F = \sum_{S \subseteq V(G)} f_S \prod_{i \in S} a_i,$$

where $f_S \in \mathbb{R}$. If we have the element

$$H = \sum_{S \subseteq V(G)} h_S \prod_{i \in S} a_i$$

then

$$F \cdot H = \sum_{S \subseteq V(G)} \Big(\sum_{S_1 \cup S_2 = S} f_{S_1} h_{S_2} \Big) \prod_{i \in S} a_i.$$

Let

$$\mathcal{E}(G, \lambda) = \prod_{\substack{(i,j) \in E(G), \\ i<j}} (1 + \lambda a_i a_j).$$

The main importance of this set-generating function lies in the fact that

$$\mathcal{E}(G, \lambda) = \sum_{S \subseteq V} E(S, \lambda) \prod_{i \in S} a_i.$$

If $e = (i,j) \in E(G)$, we obtain by $\mathcal{E}(G, \lambda) = (1 + \lambda a_i a_j) \mathcal{E}(G - e, \lambda)$ that

$$E(G, \lambda) = (1+\lambda) E(G \setminus e, \lambda) + \lambda(E(G \setminus i, \lambda) + E(G \setminus j, \lambda) + E(G \setminus \{i,j\}, \lambda)).$$

This recursive formula will play an important role in the proof of Theorem 3.1.

By multiplying the set-generating function $\mathcal{E}(G, \lambda)$ and $\mathcal{E}(G, \mu)$ we immediately get the identity

$$\sum_{S_1 \cup S_2 = S} E(S_1, \lambda) E(S_2, \mu) = E(S, \lambda + \mu + \lambda\mu),$$

for every subset S of the vertex set of G.

Note that we can write up an elegant formula for the derivative of $\mathcal{E}(G, \lambda)$:

$$\frac{d}{d\lambda} \mathcal{E}(G, \lambda) = \frac{1}{1+\lambda} \left(\sum_{\substack{(i,j) \in E(G), \\ i<j}} a_i a_j \right) \mathcal{E}(G, \lambda).$$

This means that

$$(1+\lambda) \frac{d}{d\lambda} E(G, \lambda) = m E(G, \lambda) + \sum_{i \in V} d(i) E(G \setminus i, \lambda) + \sum_{\substack{(i,j) \in E(G), \\ i<j}} E(G \setminus \{i,j\}, \lambda).$$

Recall that $d(i)$ denotes the degree of vertex i in the graph G. (Indeed, after multiplying by $1+\lambda$ the "coefficient" of the term $\prod_{i \in V(G)} a_i$ is $(1+\lambda) \frac{d}{d\lambda} E(G, \lambda)$ on the left hand side and

$$\sum_{\substack{(i,j) \in E(G), \\ i<j}} (E(G, \lambda) + E(G \setminus i, \lambda) + E(G \setminus j, \lambda) + E(G \setminus \{i,j\}, \lambda))$$

on the right hand side and this latter one is the same as the right hand side in the previous identity.)

Combining this with the previous recursive formula we obtain that

$$mE(G,\lambda) = \lambda \frac{d}{d\lambda} E(G,\lambda) + \sum_{e \in E(G)} E(G \setminus e, \lambda).$$

Finally, we collected the most important recursive formulas for the edge cover polynomial in order to be able to refer it. Note that many parts of the following theorem have been proved in [1].

Theorem 2.1. *Let G be a graph with m edges. Then the following hold:*

i) *Let $e = (u, v)$ be an edge of G. Then*

$$E(G,x) = (x+1)E(G \setminus e, x) + x\big(E(G \setminus u, x) + E(G \setminus v, x) + E(G \setminus \{u,v\}, x)\big).$$

ii) *Let u be a pendant vertex of G with the unique neighbor v. Then*

$$E(G,x) = x\big(E(G \setminus u, x) + E(G \setminus \{u,v\}, x)\big).$$

iii) *If H and K are disjoint graphs then*

$$E(H \cup K, x) = E(H,x)E(K,x).$$

iv)

$$mE(G,x) = x \frac{d}{dx} E(G,x) + \sum_{e \in E(G)} E(G \setminus e, x).$$

We will also use the following nice formula for the edge cover polynomial which was proved in [1].

Theorem 2.2. *For every graph G we have*

$$E(G,x) = \sum_{S \subseteq V(G)} (-1)^{|S|}(x+1)^{|E(G \setminus S)|}.$$

3. General bounds

In this section we prove Theorem 3.1.

Theorem 3.1. *All roots of the edge cover polynomial lie in the ball*

$$\left\{ z \in \mathbb{C} \; : \; |z| < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}} \simeq 5.099 \right\}.$$

In other words, for every graph G with no isolated vertex $\xi_{\mathbb{C}}(G) < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}}$.

Remark 3.2. For other graph polynomials such as chromatic polynomials, matching polynomials, independence polynomials, and characteristic polynomials, as we mentioned in the Introduction, there are no constant bounds for their roots, but surprisingly, previous theorem shows that all roots of edge cover polynomial are bounded.

Remark 3.3. To make it easier to understand the proof Theorem 3.1 and to avoid technical difficulties we prove a slightly weaker result, namely, $\xi_{\mathbb{C}}(G) < 6$.

Lemma 3.4. *Let G be a graph and e is an edge and v is a vertex of G. Let $|z| \geq 6$. Then*

(1) $|E(G,z)| \geq |E(G \setminus e, z)|$.

(2) If v is not an isolated vertex of G, then $|E(G,z)| \geq (|z|-2)|E(G \setminus v, z)|$.

Proof. We prove the assertion by induction on the pair (n,m) as follows, where n and m are the order and the size of G, respectively. We prove (1) for (n,m) assuming that (1) and (2) are already satisfied for (n',m'), where $n' \leq n$ and $m' \leq m$ and one of the inequality is strict. Also we prove (2) for (n,m) assuming that (1) and (2) are already true for (n',m'), where $n' \leq n$ and $m' \leq m$, and one of the inequality is strict and (1) holds for the pair (n,m).

Let us prove (2). Assume that the edges $e_1, \dots, e_k$ are incident to v. Using (1) we have

$$|E(G,z)| \geq |E(G \backslash e_1, z)| \geq |E(G \backslash \{e_1, e_2\}, z)| \geq \cdots \geq |E(G \backslash \{e_1, \dots, e_{k-1}\}, z)|.$$

Note that at the first step we have used (1) for the pair (n,m). Let $G' = G \setminus \{e_1, \dots, e_{k-1}\}$ and $e_k = (v,u)$. Then

$$E(G',z) = z(E(G' \setminus v, z) + E(G' \setminus \{u,v\}, z)),$$

and so

$$|E(G',z)| \geq |z||E(G' \setminus v, z)| - |z||E(G' \setminus \{u,v\}, z)|.$$

If u is an isolated vertex in the graph $G' \setminus v = G \setminus v$ then the claim is trivial: $|E(G,z)| \geq (|z|-2)|E(G \setminus v, z)| = 0$. If u is not an isolated vertex in the graph $G' \setminus v$ we can use the induction hypothesis:

$$|E(G' \setminus v, z)| \geq (|z|-2)|E(G' \setminus \{u,v\}, z)|.$$

Thus

$$|E(G',z)| \geq |z||E(G' \setminus v, z)| - |z||E(G' \setminus \{u,v\}, z)| \geq$$

$$(|z| - \frac{|z|}{|z|-2})|E(G' \setminus v)| \geq (|z|-2)|E(G' \setminus v, z)| = (|z|-2)|E(G \setminus v, z)|.$$

Hence $|E(G,z)| \geq |E(G',z)| \geq (|z|-2)|E(G \setminus v, z)|$.

Now to complete the proof, we prove (1). Let $e = v_1v_2$. If e is a pendant edge of G, then $E(G \setminus e, z) = 0$. So we are done. Now suppose that $d_G(v_1)$ and $d_G(v_2)$ both are at least two. By Theorem 2.1 we have

$$E(G,z) = (z+1)E(G \backslash e, z) + zE(G \backslash v_1, z) + zE(G \backslash v_2, z) + zE(G \backslash \{v_1, v_2\}, z).$$

Hence

$$|E(G,z)| \geq (|z|-1)|E(G \backslash e, z)| - |z|(|E(G \backslash v_1, z)| + |E(G \backslash v_2, z)| + |E(G \backslash \{v_1, v_2\}, z)|).$$

Now, we use the induction hypothesis (part (2)) for $G \setminus e$ and $G \setminus v_1$ to obtain that

$$|E(G \backslash e, z)| \geq \max((|z|-2)|E(G \backslash v_1, z)|, (|z|-2)|E(G \backslash v_2, z)|, (|z|-2)^2|E(G \backslash \{v_1, v_2\}, z)|).$$

Hence

$$|E(G,z)| \geq (|z| - 1 - \frac{|z|}{|z|-2} - \frac{|z|}{|z|-2} - \frac{|z|}{(|z|-2)^2})|E(G \setminus e, z)|.$$

For $|z| \geq 6$ we have

$$|z| - 1 - \frac{|z|}{|z|-2} - \frac{|z|}{|z|-2} - \frac{|z|}{(|z|-2)^2} \geq 1.$$

This proves our claim.

□

Proof of Theorem 3.1. Here we show that $\xi_{\mathbb{C}}(G) < 6$. Let S be an edge covering of G with smallest cardinality (i.e., with cardinality $\rho(G)$). Note that S is a disjoint union of stars. Set $G' = (V(G), S)$, clearly $E(G', z) = z^{|S|}$. If $|z| \geq 6$, then by Lemma 3.4,

$$|E(G,z)| \geq |E(G',z)| = |z|^{|S|} > 0.$$

This completes the proof. □

Remark 3.5. To prove the original inequality stated in Theorem 3.1, one need to use the following version of Lemma 3.4.

Lemma 3.4'. Let G be a graph and e is an edge and v is a vertex of G. Let $|z| \geq \frac{(2+\sqrt{3})^2}{1+\sqrt{3}}$. Then

(1) $|E(G,z)| \geq |E(G \setminus e, z)|$.

(2) If v is not an isolated vertex of G, then $|E(G,z)| \geq (2+\sqrt{3})|E(G \setminus v, z)|$.

The proof is almost identical to the original one.

We conjecture that for every graph G we have $\xi_{\mathbb{C}}(G) < 4$. The next theorem shows that this inequality is true if $\delta(G)$ is large.

Theorem 3.6. *Let G be a graph of order n with no isolated vertex. If $\delta(G) > \sqrt{2n \ln n}$ and n is large enough, then $\xi_{\mathbb{C}}(G) < 4$.*

Proof. We prove the assertion for $n \geq 381$. Let $\delta = \delta(G)$ and m be the size of G. To obtain the result, we use Theorem 2.2 which states that

$$E(G,x) = \sum_{S \subseteq V(G)} (-1)^{|S|}(x+1)^{|E(G \setminus S)|}.$$

We recall that $|E(G \setminus S)|$ is the number of edges of $G \setminus S$. Let $x + 1 = y$. We will show that if $|y| \geq 3$ then

$$|y|^m > \sum_{\substack{S \subseteq V(G) \\ S \neq \emptyset}} |y|^{|E(G \setminus S)|}.$$

This would prove that $x = y - 1$ cannot be a root of $E(G,x)$. Let

$$K = \left\lfloor \frac{\delta \ln 3 - \ln 2}{\ln n} \right\rfloor.$$

One can easily see that $K \geq 1$. We cut the sum

$$\sum_{\substack{S\subseteq V(G)\\ S\neq\emptyset}} |y|^{|E(G\setminus S)|}$$

into two parts L_1 and L_2 according to $|S| \leq K$ or $|S| > K$. For the first part we apply the bound

$$L_1 = \sum_{\substack{S\subseteq V(G)\\ 1\leq |S|\leq K}} |y|^{|E(G\setminus S)|} \leq \sum_{j=1}^{K} \binom{n}{j} |y|^{m-\delta} \leq n^K |y|^{m-\delta}.$$

For the second part we use that $|E(G \setminus S)| \leq m - \frac{K\delta}{2}$. Hence

$$L_2 = \sum_{\substack{S\subseteq V(G)\\ |S|>K}} |y|^{|E(G\setminus S)|} \leq 2^n |y|^{m-K\delta/2}.$$

We will show that if $\delta > \sqrt{2n\ln n}$, then $L_1 \leq \frac{1}{2}|y|^m$ and $L_2 < \frac{1}{2}|y|^m$. We have

$$n^K|y|^{m-\delta} \leq \exp\left(\ln n \cdot \frac{\delta\ln 3 - \ln 2}{\ln n}\right)|y|^{m-\delta} = \frac{1}{2}3^{\delta}|y|^{m-\delta} \leq \frac{1}{2}|y|^m.$$

On the other hand,

$$\ln(|y|^{K\delta/2}) \geq \left(\frac{\delta\ln 3 - \ln 2}{\ln n} - 1\right)\frac{\delta}{2}\ln(|y|) \geq \frac{\delta^2}{2\ln n}\ln(|y|) > n\ln 3 > (n+1)\ln 2.$$

(In the above inequalities we have used that $\delta > \sqrt{2n\ln n} > \frac{\ln 2n}{\ln 3 - 1}$ and $n\ln 3 > (n+1)\ln 2$ both hold for $n \geq 381$.) Hence

$$\frac{1}{2}|y|^m > 2^n|y|^{m-K\delta/2}.$$

This completes the proof.

□

4. The real roots of tree-like objects

In this section we prove Theorem 4.5. To do this we need some preparation.

Let G and H be two disjoint graphs. Let $u \in V(G)$ and $v \in V(H)$. By $G \cdot uv \cdot H$ we denote the graph which obtained by identifying the vertices u and v (see Figure 1). One can easily prove the following lemma on the edge cover polynomial of $G \cdot uv \cdot H$.

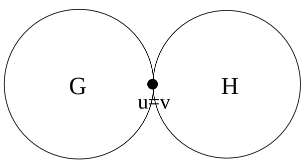

Figure 1. The graph $G \cdot uv \cdot H$.

Lemma 4.1. *Let G and H be two disjoint graphs. Let $u \in V(G)$ and $v \in V(H)$. Then*

$$E(G \cdot uv \cdot H, x) = E(G, x)E(H, x) + E(G \setminus u, x)E(H, x) + E(G, x)E(H \setminus v, x).$$

Remark 4.2. [1] The edge cover polynomial of the graphs P_n and C_n are the following

$$E(P_n, x) = \sum_{k=1}^{n-1} \binom{k-1}{n-k-1} x^k,$$

and

$$E(C_n, x) = x^n + \sum_{k=1}^{n-1} \frac{n}{n-k} \binom{k-1}{n-k-1} x^k.$$

Let $T_n(x), U_n(x)$ be the Chebyshev polynomials of the first and second kinds, respectively. It is well known that for every $n \geq 0$,

$$T_n(x) = x^n \sum_{k=0}^{n} \binom{n}{2k} (1 - x^{-2})^k,$$

and

$$U_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

We have the following relationship between the edge cover polynomial of paths and cycles and Chebyshev polynomials.

$$E(P_n, -4x^2) = (-1)^{n-1}(2x)^n U_{n-2}(x)$$

and

$$E(C_n, -4x^2) = (-1)^n 2^{n+1} x^n T_n(x).$$

Since the roots of $T_n(x)$ and $U_n(x)$ are well known, the following lemma is easy to check.

Lemma 4.3. *The roots of paths and cycles are the following:*

i) *For every natural number $n \geq 2$, zero is the root of $E(P_n, x)$ with multiplicity $\lceil \frac{n}{2} \rceil$. Also the set of all non-zero roots of $E(P_n, x)$ is*

$$\left\{ -2 - 2\cos\frac{2k\pi}{n-1},\ k = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1 \right\}.$$

ii) *Let $n \geq 3$. Then zero is the root of $E(C_n, x)$ with multiplicity $\lceil \frac{n}{2} \rceil$. The set of all non-zero roots of $E(C_n, x)$ is*

$$\left\{ -2 - 2\cos\frac{(2k+1)\pi}{n},\ k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1 \right\}.$$

As an immediate consequence, we obtain the following corollary. In the next results, by the notation C_2 we simply mean P_2. Some parts of the next result have been obtained in [1] by a different method.

Corollary 4.4. *The following statements hold for the roots of the paths and cycles.*

i) *For every* $n \geq 2$, *all the roots of* $E(P_n, x)$ *and* $E(C_n, x)$ *are in the interval* $(-4, 0]$.
ii) *The roots of families* $\{E(P_n, x)\}_{n=2}^{\infty}$ *and* $\{E(C_n, x)\}_{n=3}^{\infty}$ *are dense in the interval* $[-4, 0]$.
iii) *For every* $n \geq 3$, $\xi_{\mathbb{R}}(C_n) = 2 + 2\cos\frac{\pi}{n}$ *and* $\xi_{\mathbb{R}}(P_n) = 2 + 2\cos\frac{2\pi}{n-1}$.
iv) *For every* $n \geq 3$, $\xi_{\mathbb{R}}(C_n) > \xi_{\mathbb{R}}(P_n)$.
v) *For every* $n \geq 3$, $\xi_{\mathbb{R}}(C_n) > \xi_{\mathbb{R}}(C_{n-1})$ *and for every* $n \geq 4$, $\xi_{\mathbb{R}}(P_n) > \xi_{\mathbb{R}}(P_{n-1})$.

Now we are ready to prove the main theorem of this section.

Theorem 4.5. *Let* G *be a graph of order* n. *If every block of* G *is* K_2 *or a cycle, then*

$$\xi_{\mathbb{R}}(G) \leq \xi_{\mathbb{R}}(C_n).$$

In other words, all real roots of $E(G, x)$ *are in the interval* $[-2 - 2\cos\frac{\pi}{n}, 0]$. *Moreover, the equality holds if and only if* $G = C_n$.

Proof. Let $G \neq C_n$. We will show that $\xi_{\mathbb{R}}(G) < \xi_{\mathbb{R}}(C_n)$. Let m be the size of G. To obtain the result it is enough to show that the sign of $E(G, x)$ is $(-1)^m$ in the interval $(-\infty, -\xi_{\mathbb{R}}(C_n)]$. We proceed by induction on n. For $n = 2, 3$ there is nothing to prove. Suppose $n \geq 4$. If G is disconnected, then by the induction hypothesis and the fifth part of Corollary 4.4 the proof is complete.

Now let G be a connected graph. If $\Delta(G) = 2$, then $G = P_n$. Therefore, by the fourth part of Corollary 4.4 we obtain the result. Now suppose that $u \in V(G)$ and $d_G(u) \geq 3$. We can find two subgraphs H and K of G such that $u \in V(H) \cap V(K)$ and $d_H(u) = 2$ and consider G as $G = H \cdot uv \cdot K$. Let n_1, n_2 be the order of H, K, and m_1, m_2 be the size of H, K, respectively. Therefore $n = n_1 + n_2 - 1$ and $m = m_1 + m_2$. Note that $n_1 \geq 3$ and $n_2 \geq 2$. By Lemma 4.1, one has

$$E(G, x) = E(H, x)(E(K, x) + E(K \setminus v, x)) + E(H \setminus u, x)E(K, x).$$

Now consider P_2 with vertices a, b. So by Lemma 4.1 we can write

$$E(G, x) = \frac{1}{x}E(K \cdot va \cdot P_2, x)E(H, x) + E(H \setminus u, x)E(K, x).$$

Note that the order of $K \cdot va \cdot P_2$ is $n_2 + 1 \leq n - 1$. By the induction hypothesis and Part (v) of Corollary 4.4, the signs of the edge cover polynomials of the graphs $K \cdot va \cdot P_2$, K, H, $H \setminus u$ on the interval $(-\infty, -\xi_{\mathbb{R}}(C_n)]$ are $(-1)^{m_2+1}, (-1)^{m_2}, (-1)^{m_1}, (-1)^{m_1-2}$, respectively (if $H \setminus u$ has some isolated vertices, then for every real number x, $E(H \setminus u, x) = 0$). This shows that the sign of $E(G, x)$ is $(-1)^m$ on the interval $(-\infty, -\xi_{\mathbb{R}}(C_n)]$. So we are done. □

There are many theorems on eigenvalues and Laplacian eigenvalues of trees of the following kind:

Theorem 4.6. [12] *Let* $\Lambda(T)$ *be the largest eigenvalues of* T. *Then for every tree* T *of order* n *we have*

$$\Lambda(P_n) \leq \Lambda(T) \leq \Lambda(K_{1,n-1}).$$

Theorem 4.7. [9] *Let $\Upsilon(T)$ be the largest Laplacian eigenvalues of T. Then for every tree T of order n we have*

$$\Upsilon(P_n) \leq \Upsilon(T) \leq \Upsilon(K_{1,n-1}).$$

Here we state the next theorem that is similar to the previous theorems. Similar to the proof of Theorem 4.5 one can prove the following theorem. This theorem shows that all real roots of trees are in the interval $(-4, 0]$. Note that $-\xi_{\mathbb{R}}(T)$ is the smallest real root of $E(T, x)$.

Theorem 4.8. *Let T be a tree of order n. Then*

$$\xi_{\mathbb{R}}(K_{1,n-1}) \leq \xi_{\mathbb{R}}(T) \leq \xi_{\mathbb{R}}(P_n).$$

In other words, all real roots of $E(T, x)$ are in the interval $[-2 - 2\cos\frac{2\pi}{n-1}, 0]$ (for $n \geq 2$). Moreover, on the right hand side equality holds if and only if $T = P_n$.

Remark 4.9. One can prove this theorem also by the aid of the generalized tree shift [6, 7].

Note that $\xi_{\mathbb{R}}(K_{1,n-1}) = 0$. Surprisingly, there are infinitely many trees T with $\xi_{\mathbb{R}}(T) = 0$ (see Figure 2). Therefore, on the above theorem in the left hand side equality holds for infinitely many trees.

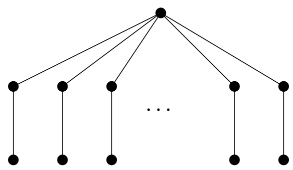

FIGURE 2. Let $SK_{1,n}$ be the subdivision of the star $K_{1,n}$, so it has $2n + 1$ vertices. If n is odd, then the only real root of $E(SK_{1,n}, x) = x^n((x + 1)^n - 1)$ is 0.

5. Open problems

We end this paper by some conjectures.

Conjecture 5.1. Let G be a graph with no isolated vertex. Then $\xi_{\mathbb{C}}(G) < 4$.

As we have already seen, this conjecture is true for graphs with large smallest degree. Similarly to the proof of Theorem 3.6 one can see that this conjecture is valid for complete graphs and complete bipartite graphs. We note that by Corollary 4.4, $\xi_{\mathbb{C}}(C_n) \longrightarrow 4$ as $n \longrightarrow \infty$. Therefore if the conjecture is true, then 4 is the best possible upper bound for $\xi_{\mathbb{C}}(G)$.

Conjecture 5.2. Let G be a graph with $\delta(G) = 2$. If $E(G, x)$ has only real roots, then all connected components of G are cycles.

Note that for $\delta(G) = 1$ in [1] it was shown that if the pendant edges of G forming a perfect matching, then all roots of $E(G, x)$ are real (in fact the roots are $0, -1$).

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