THE ZARANKIEWICZ PROBLEM, CAGES, AND GEOMETRIES

GÁBOR DAMÁSDI, TAMÁS HÉGER, AND TAMÁS SZŐNYI

DEDICATED TO THE MEMORY OF ANDRÁS GÁCS AND ISTVÁN REIMAN

Abstract. In the paper we consider some constructions of $(k, 6)$-graphs that are isomorphic to an induced subgraph of the incidence graph of a finite projective plane, and present some unifying concepts. Also, we obtain new bounds on and exact values of Zarankiewicz numbers, mainly when the parameters are close to those of a design.

1. Introduction

This paper is dedicated to the memory of András Gács and István Reiman. We wish to present results on two well-known extremal graph theoretic problems, (k, g)-graphs (related to cages) and the Zarankiewicz problem, that András worked on in the last period of his life. These topics in some cases have close relations to finite geometry, and design theory. The first, pioneering results in exploring these connections are due to István Reiman [37, 38] in case of the Zarankiewicz problem. Although we formulate some results in more general settings, we mainly focus on issues that are related to finite projective planes. András had a major role in our work on (k, g)-graphs, and also took part in obtaining our first results on the Zarankiewicz problem. Those results have been improved later on, and we wish to publish them now.

In this section we give the preliminary definitions and notations, and introduce the two problems. In the paper we only consider finite structures, and all graphs are simple (without loops or multiple edges). The set of the neighbors of a vertex v will be denoted by $N(v)$, and $|N(v)|$ will be referred to as the degree of v or $\deg(v)$. A graph is k-regular if all of its vertices have degree k. The girth of a graph is the length of the shortest cycle in it. $K_{n,m}$ and C_n denote the complete bipartite graph on $n + m$ vertices and the cycle of length n, respectively. Note that $K_{2,2}$ is isomorphic to C_4. The number of edges of a graph G will be denoted by $e(G)$.

Definition 1.1. *A* (k, g)-graph *is a k-regular graph of girth g. A* (k, g)-cage *is a (k, g)-graph with as few vertices as possible. We denote the number of vertices of a (k, g)-cage by $c(k, g)$.*

Date: March 19, 2013.

Tamás Héger and the Tamás Szőnyi were supported by OTKA Grant K 81310. Gábor Damásdi was a participant of the ELTE Kutatódiák Program, and the TÁMOP 4.23-08/1/KMR project. Tamás Héger was also supported by ERC Grant No. 227701 DISCRETECONT. Tamás Szőnyi was partly supported by the Slovenian–Hungarian Intergovernmental Scientific and Technological Cooperation Project, Grant No. TÉT 10-1-2011-0606, and fruitful discussions with Boštjan Kuzman are also gratefully acknowledged.

A bipartite graph G with vertex classes A and B, and edge-set E will be denoted by $G = (A, B; E)$; we may omit the edge-set and write simply (A, B). We call $(|A|, |B|)$ the size of G; we may also say that G is a bipartite graph on $(|A|, |B|)$ vertices.

DEFINITION 1.2. *A bipartite graph $G = (A, B; E)$ is $K_{s,t}$-free if it does not contain s nodes in A and t nodes in B that span a subgraph isomorphic to $K_{s,t}$. The maximum number of edges a $K_{s,t}$-free bipartite graph of size (m, n) may have is denoted by $Z_{s,t}(m, n)$, and is called a* Zarankiewicz number.

Note that a $K_{s,t}$-free bipartite graph is not necessarily $K_{t,s}$-free if $s \neq t$.

We remark that Zarankiewicz's question in its original form was formulated via matrices in the following way: what is the minimum number of 1's in an $m \times n$ $0-1$ matrix that ensures the existence of an $s \times t$ submatrix all of whose entries are 1s? This quantity clearly equals $Z_{s,t}(m, n) + 1$, and it is also used as the definition of a Zarankiewicz number (e.g., in [23]).

Determining the exact values of $c(k, g)$ and $Z_{s,t}(m, n)$ is extremely hard in general. As a bipartite graph does not contain cycles of odd length, a $K_{2,2} = C_4$-free bipartite graph automatically has girth at least 6. In fact, the incidence graph of a finite projective plane of order n is known to be an extremal $K_{2,2}$-free graph of size $(n^2 + n + 1, n^2 + n + 1)$, and it is an $(n + 1, 6)$-cage as well. Projective planes can be considered as designs or as generalized polygons as well, which are incidence structures with special properties, and are also closely related to the Zarankiewicz problem and cage graphs, respectively.

An incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a triplet of the sets $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are referred to as points and lines (or blocks; then we write $\mathcal{B}$ instead of $\mathcal{L}$), respectively, and $\mathcal{I}$ is called the incidence relation. The incidence (or Levi) graph of an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the bipartite graph $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, that is, the two classes of vertices correspond to the point-set and the line-set of the structure, while edges are the flags (incident point-line pairs). As bipartite graphs and incidence structures are basically the same, we will mix the terminologies of the two notions without any further warning. In this manner, we may call the vertices of a graph a point or a line, or we may talk about a subgraph of an incidence structure. By the degree of a point or a line in an incidence structure we will mean the degree of the corresponding vertex in the incidence graph. The dual of the incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is $(\mathcal{L}, \mathcal{P}, \mathcal{I}^T)$, where $(l, P) \in \mathcal{I}^T \iff (P, l) \in \mathcal{I}$, that is, we only interchange the words point and line (block). We will usually omit the indication of the set $\mathcal{I}$ of incidences from the triplet, and we will use the notation $P \in l$ instead of $(P, l) \in \mathcal{I}$. Conventionally, a line $l \in \mathcal{L}$ (or block $B \in \mathcal{B}$) may be identified with the set of points it is incident with, and hence we may also write for example $|B|$ to indicate the size of a block B. Also, if the elements of $\mathcal{L}$ are considered as lines, then we say that the points $P_1, \dots, P_k$ are collinear if there exists a line $l \in \mathcal{L}$ incident with each P_i $(1 \leq i \leq k)$.

DEFINITION 1.3. *Let $x, y \in \mathcal{P} \cup \mathcal{L}$ be two objects of some incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$. Then the* distance $d(x, y)$ of x and y *is the distance of x and y in the incidence graph, that is, the length of the shortest path between x and y. Should there be no such path, let $d(x, y) = \infty$.*

DEFINITION 1.4. *Let $G = (V, E)$ be a graph with vertex-set V. For two (finite) vertex-sets X and Y let $d(X, Y) = \min\{d(x, y) \colon x \in X, y \in Y\}$. If X or Y has one element only,* *we write, for example,* $d(x, Y)$ *instead of* $d(\{x\}, Y)$*. A* ball *of center* v *and radius* r *is* $B(v, r) = \{u \in V : d(v, u) \leq r\}$*.*

Definition 1.5 (Generalized polygon, GP). *An incidence structure* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ *is a* generalized n-gon of order (s, t) *if and only if the following hold:*

GP1: *every point is incident with* $s + 1$ *lines;*
GP2: *every line is incident with* $t + 1$ *points;*
GP3: *the diameter and the girth of the incidence graph is* n *and* $2n$*, respectively.*

From GP3 it follows that if $d(x, y) \leq n - 1$, then there is a unique path of length $\leq n - 1$ connecting x to y. Note that the axioms of generalized polygons are symmetric in points and lines, that is, the dual of a GP of order (s, t) is a GP of order (t, s). By definition, the incidence graph of a generalized n-gon of order (q, q) is a $(q + 1, 2n)$-graph; moreover, it is a cage. Generalized n-gons of order (q, q) exist only if $n = 3$, 4 or 6, and are called a *generalized triangle* or *projective plane*, a *generalized quadrangle (GQ)*, and a *generalized hexagon (GH)* of order q, respectively. If q is a power of a prime, such generalized polygons of order q do exist, but none is known otherwise. We also mention that one can give alternative definitions of a GP. For example, a projective plane is commonly defined as an incidence structure satisfying the following three properties: (i) any two lines have a unique point in common; (ii) any two points have a unique line incident with both; (iii) there exist four points in general position (that is, no three of them are collinear). From these properties it follows that there exists a number q such that our incidence structure is a generalized triangle of order (q, q). In case of generalized quadrangles, GP3 is commonly rephrased as GQ3: for all $P \in \mathcal{P}$ and $l \in \mathcal{L}$ such that $P \notin l$, there exists a unique line $e \in \mathcal{L}$ such that $P \in e$ and e intersects l.

Definition 1.6. *Let* $\emptyset \neq K \subset \mathbb{Z}^+$*. An incidence structure* $(\mathcal{P}, \mathcal{B})$ *is called a* $t - (v, K, \lambda)$ design*, if* $|\mathcal{P}| = v$*,* $\forall B \in \mathcal{B}$: $|B| \in K$*, and every* t *distinct points are contained in precisely* λ *distinct blocks. If* $K = \{k\}$*, we write simply* $t - (v, k, \lambda)$*.*

The total number $|\mathcal{B}| = b$ of blocks, and the number r of blocks incident with an arbitrary fixed point in a $t-(v, k, \lambda)$ design are $b = \lambda \binom{v}{t} / \binom{k}{t}$, $r = bk/v = \lambda \binom{v-1}{t-1} / \binom{k-1}{t-1}$, respectively. We always assume that $k < v$ and $\lambda \geq 1$.

The incidence graph of a $t - (v, k, \lambda)$ design is $K_{t,\lambda+1}$-free of size (v, b) by definition, and they turn out to have the most possible number of edges among such graphs.

Definition 1.7. *We call the parameters* (t, v, k, λ) admissible*, if they are positive integers satisfying* $2 \leq t$*,* $t \leq k < v$*, furthermore,* $b := \lambda \binom{v}{t} / \binom{k}{t}$ *and* $r := bk/v = \lambda \binom{v-1}{t-1} / \binom{k-1}{t-1}$ *are also integers.*

A projective plane of order q can be considered as a generalized triangle of order (q, q), or as a $2 - (q^2 + q + 1, q + 1, 1)$ design. The main concept this paper considers is to look for small $(k, 6)$-graphs or C_4-free graphs with many edges as subgraphs of the incidence graph of a projective plane (or more generally, of a GP or a design), and we also propose the systematic study of this idea.

Section 2 is devoted to (k, g)-graphs ($g = 6, 8, 12$) as *induced* subgraphs of generalized polygons. *Induced* regular subgraphs of GPs are obtained by deleting *vertices only* from the incidence graph of the GP. In [19], t-good structures were introduced to examine this idea. We show that many former constructions that we are to list can be unified with this concept. We believe that t-good structures are useful to better understand the constructions obtained by several authors and different methods, and sometimes they even help to give new constructions.

One may look for *non-induced* regular subgraphs of a GP, that is, we are allowed to delete vertices and *edges as well* to obtain a regular graph from the incidence graph of the GP. Several recent papers use these kinds of ideas, see for example [3], [6]. This method might be examined through a natural generalization of t-good structures that is due to Araujo-Pardo and Balbuena [5]. In many cases the (k, g)-graphs obtained in this way are smaller than the induced ones. Also, one can extend the concept of t-good structures to obtain biregular graphs, which we will do only in order to give a better understanding of some 1-good structures in GQs. These ideas are rather unexplored yet, and will not be covered by this article. We wish only to detail the results in connection with t-good structures; for a general and recent survey on (k, g) graphs, we refer to [15]. We do not consider constructions that use different ideas, like [16] or [1].

Section 3 is devoted to the Zarankiewicz problem, particularly the case of $K_{2,2}$-free graphs. Among others, we prove the following (more detailed formulation is given in Section 3).

THEOREM 1.8. *Assume that a projective plane of order n exists, and let $n \geq 15$ in the first, and $n \geq 4$ in the fourth case. Then*

$$\begin{aligned} Z_{2,2}(n^2+n+1-c, n^2+n+1) &= (n^2+n+1-c)(n+1) && (0 \leq c \leq n/2), \\ Z_{2,2}(n^2+c, n^2+n) &= n^2(n+1)+cn && (0 \leq c \leq n+1), \\ Z_{2,2}(n^2-n+c, n^2+n-1) &= (n^2-n)(n+1)+cn && (0 \leq c \leq 2n), \\ Z_{2,2}(n^2-2n+1+c, n^2+n-2) &= (n^2-2n+1)(n+1)+cn && (0 \leq c \leq 3(n-1)). \end{aligned}$$

Other exact values of Zarankiewicz numbers are also obtained if the parameters are small, or they are close enough to those of a design.

2. (k, g)-GRAPHS

For details and results on cages, we refer to the online available dynamic survey of Exoo and Jajcay [15]. Connections with the degree/diameter problem and Moore graphs can be found in [35].

A general lower bound on the number of vertices of a (k, g)-cage, known as the *Moore bound*, is a simple consequence of the fact that the vertices at distance $0, 1, \ldots, \lfloor (g-1)/2 \rfloor$ from a vertex (if g is odd), or an edge (if g is even) must be distinct.

PROPOSITION 2.1 (Moore bound).

$$c(k,g) \geq M(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-1}{2}-1} & \textit{for } g \textit{ odd;} \\ 2\left(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\frac{g}{2}-1}\right) & \textit{for } g \textit{ even.} \end{cases}$$

As $(k, 2n+1)$-graphs with $M(k, 2n+1)$ vertices coincide with Moore graphs of valency k and diameter n, the term Moore graph is extended to any (k, g)-graph on $M(k, g)$ vertices. Such graphs may also be referred to as *Moore cages*. It is easy to see that $k+1$-regular Moore graphs with girth $2n$ are precisely the incidence graphs of generalized n-gons of order (k,k). Note that the cases $g=3$ and $g=4$ are trivial, the corresponding Moore cages are complete graphs and regular complete bipartite graphs, respectively.

2.1. Some constructions of (k,g)-graphs ($g=6,8,12$).

From now on we focus on constructions and results regarding generalized polygons, that is, the cases $g=6,8,12$.

Starting from a projective plane of order q, Brown ([11], 1967) constructed $(k,6)$-graphs for arbitrary $4 \le k \le q$ by deleting some properly chosen points and lines from the plane, that is, by removing vertices from the incidence graph of the plane. This is equivalent to finding a k-regular induced subgraph of the incidence graph. The $(k,6)$-graphs Brown obtained have $2kq$ number of vertices, hence from the distribution of primes it follows that $c(k,6) \sim 2k^2$. Although Brown himself only gave one specific construction, we refer to this construction method (deleting vertices from a projective plane of order q to obtain a $(k,6)$-graph, $k \le q$) as *Brown's method.* It may be generalized to the idea of finding (k',g)-graphs as induced subgraphs of (k,g)-cages, $k' < k$.

In 1997, Lazebnik, Ustimenko, and Woldar [33] proved the following.

Result 2.2. *Let $k \ge 2$ and $g \ge 5$ be integers, and let q denote the smallest odd prime power for which $k \le q$. Then*

$$c(k,g) \le 2kq^{\frac{3}{4}g-a},$$

where $a = 4$, $11/4$, $7/2$, $13/4$ for $g \equiv 0$, 1, 2, $3 \pmod 4$, respectively.

In particular, for $g=6$, 8, 12 this gives $c(k,6) \le 2kq$, $c(k,8) \le 2kq^2$, $c(k,12) \le 2kq^5$, where q is the smallest odd prime power not smaller than k. Combined with the Moore bound, this yields $c(k,8) \sim 2k^3$.

Using the addition and multiplication tables of $\mathrm{GF}(q)$, Abreu, Funk, Labbate and Napolitano ([2], 2006) constructed two infinite families of $(k,6)$, $k \le q$ graphs via their incidence matrices. The number of vertices of the graphs in the first and the second family are $2kq$ and $2(kq+(q-1-k))$, respectively. The second construction yields a graph smaller than the previously known ones for $k=q$, resulting $c(q,6) \le 2(q^2-1)$ for any prime power q. Moreover, Abreu et al. settled a conjecture on the incidence matrices of $\mathrm{PG}(2,q)$, q square, in connection with the partition of the point-set and line-set of $\mathrm{PG}(2,q)$ into Baer subplanes. They verified the conjecture for $q=4$, 9, and 16, which allowed them to construct $(k,6)$ graphs of size $2(kq-(q-k)(\sqrt{q}+1)-\sqrt{q}) \ge c(k,6)$ for $q=4$, 9, 16 and $k \le q$.

Deleting vertices from the incidence graph of a generalized quadrangle or hexagon, Araujo, González, Montellano-Ballesteros and Serra ([7], 2007) showed $c(k,8) \le 2kq^2$ and also $c(k,12) \le 2kq^4$, $k \le q$, q a prime power. Their construction uses only elementary combinatorial properties of generalized polygons. Their upper bound on $c(k,8)$ is the same as that of Lazebnik et al.'s [33], but the bound on $c(k,12)$ is better, and leads to $c(k,12) \sim 2k^5$.

Note that the above results yield $c(k,2n) \sim 2k^{n-1}$ for $n=2,3,4,6$.

2.2. Brown's method reformulated: t-good structures, a unifying concept.

Regarding the cases $g=6$, 8, and 12, Gács and Héger [19] (2008) present a point of view that unifies all the above constructions (except Lazebnik, Ustimenko, and Woldar's for $g = 12$) using the concept of a t-good structure, and also started to study them systematically.

DEFINITION 2.3. *A* t-good structure *in a generalized polygon is a pair* $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ *consisting of a proper subset of points* $\mathcal{P}_0$ *and a proper subset of lines* $\mathcal{L}_0$*, with the property that there are exactly* t *lines in* $\mathcal{L}_0$ *through any point not in* $\mathcal{P}_0$*, and exactly* t *points in* $\mathcal{P}_0$ *on any line not in* $\mathcal{L}_0$*.*

Removing the points and lines of a t-good structure $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ from the incidence graph of a generalized n-gon of order q results in a $(q+1-t)$-regular graph of girth at least $2n$, and hence provides an upper bound on $c(q+1-t, 2n)$. It is easy to see that $|\mathcal{P}_0| = |\mathcal{L}_0|$ for every t-good structure $\mathcal{T}$, hence the *size* of $\mathcal{T}$ is defined as $|\mathcal{P}_0|$, and may be denoted by $|\mathcal{T}|$. Trivially, the larger t-good structure we find for a fixed t, the smaller $(q+1-t)$ regular graph we obtain. Note that this concept works in any GP.

Most known t-good structures follow the same, general pattern we give here.

The neighboring balls construction. Recall that $d(x, y)$ denotes the distance of x and y. Let $\mathcal{L}^* = \{l_1, \ldots, l_t\}$ and $\mathcal{P}^* = \{P_1, \ldots, P_t\}$ be a collection of distinct lines and points such that $\forall 1 \le i < j \le t$ the following hold:

- (i) $d(l_i, l_j) = 2$ (the lines are pairwise intersecting);
- (ii) the unique point at distance one from l_i and l_j (their intersection point) is an element of $\mathcal{P}^*$;
- (i') $d(P_i, P_j) = 2$ (the points are pairwise collinear);
- (ii') the unique line at distance one from P_i and P_j (the line joining them) is an element of $\mathcal{L}^*$.

PROPOSITION 2.4. *Let* $(\mathcal{P}^*, \mathcal{L}^*)$ *satisfy the conditions above, and let* $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ *be the collection of points and lines that are at distance at most* $n-2$ *from some element of* $\mathcal{P}^*$ *or* $\mathcal{L}^*$*, that is,* $\mathcal{P}_0 \cup \mathcal{L}_0 = \bigcup_{i=1}^{t} \{B(P_i, n-2)\} \cup \bigcup_{i=1}^{t} \{B(l_i, n-2)\}$*. Then* $\mathcal{T}$ *is* t*-good.*

Proof. Let $Q \notin \mathcal{P}_0$. Then for every i $(1 \le i \le t)$, $d(Q, l_i) = n-1$ or n, and $d(Q, P_i) = n$ or $n-1$, depending on n being even or odd, respectively. We may assume that n is even (the odd case is analogous). Then for all i $(1 \le i \le t)$ there is a unique a line e_i such that $d(Q, e_i) = 1$ and $d(e_i, l_i) = n-2$, and these are precisely the lines of $\mathcal{L}_0$ that are incident with Q. Hence we must show that these are distinct. Suppose to the contrary that $e_i = e_j = e$ for some $i \ne j$. Let $P \in \mathcal{P}^*$ be the point incident with l_i and l_j. Since $d(Q, P) = n$, $d(P, e) = n-1$. But then there are two distinct paths of length $n-1$ from P to e, one through l_i and another one through l_j, a contradiction. The same (dual) arguments hold for lines. $\square$

Note that if we allow $\mathcal{P}^*$ and $\mathcal{L}^*$ to have different sizes, s and t respectively, and define $\mathcal{T}$ in the same way, then the same arguments show that after deleting $\mathcal{T}$, every point not in $\mathcal{T}$ has degree $q+1-s$ or $q+1-t$, and line not in $\mathcal{T}$ has degree $q+1-t$ or $q+1-s$, depending on n being odd or even, respectively. Hence in order to obtain biregular graphs, we could define (s, t)-good structures, as we will do in Subsection 2.2.2, but mainly restrict its use to construct 1-good structures.

We will use the next definition usually in the context of a t-good structure.

DEFINITION 2.5. *Let $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ be a pair of a point-set and a line-set in a GP $(\mathcal{P}, \mathcal{L})$. Then a point P is* $\mathcal{T}$-complete, *if $P \in \mathcal{P}_0$, and every line incident with P is in $\mathcal{L}_0$. We define a $\mathcal{T}$-complete line dually.*

2.2.1. *t-good structures in projective planes.* In the $n = 3$ case, that is, if we start from an arbitrary projective plane, the conditions (i) and (i') of the general construction hold automatically, while conditions (ii) and (ii') claim that $(\mathcal{P}^*, \mathcal{L}^*)$ should be a (possibly degenerate) subplane. We call a set of points and lines a *degenerate subplane*, if the intersection point of its lines and the lines joining two of its points belong to it, but it does not have four points in general position. Note that in a projective plane $d(x, y) \leq n-2 = 1$ means that $x = y$ or x is incident with y. Hence $(\mathcal{P}_0, \mathcal{L}_0)$ consists of points and lines that are incident with a subplane, that is, we put the points and the lines of $\mathcal{P}^*$ and $\mathcal{L}^*$ completely into $\mathcal{T}$ and delete them; thus this construction is called a *completely deleted subplane* by Gács, Héger and Weiner [20].

There are two types of degenerate subplanes:

- **type π_1:** there is an incident point-line pair (P, l) such that all points are incident with l and all lines are incident with P;
- **type π_2:** there is a non-incident point-line pair (P, l) such that every point except P is incident with l and every line except l is incident with P.

In a degenerate subplane of type π_1 and π_2 there are at most two or three points in general position, respectively. Brown's construction [11] and the first infinite family of Abreu et al. [2] can be obtained by completely deleting degenerate subplanes (CDDS) of type π_1 from a finite projective plane, while the second family of Abreu et al. can be constructed by CDDS of type π_2, see [19]. We remark that the constructions of Abreu et al. [2] correspond to t-good structures in $\mathrm{PG}(2, q)$, while Brown's construction works in an arbitrary finite projective plane. Also, note that a subplane has the same number of points and lines except if it is degenerate of type π_1; in that case, it may have a different number of points and lines, hence it can be used to obtain biregular graphs.

A different construction is also given in [19]. Let $\mathcal{T}$ consist of the points and the lines of t pairwise disjoint Baer subplanes. Then, using a result of Svéd [40], it can be shown that $\mathcal{T}$ is t-good. It is well known that $\mathrm{PG}(2, q)$, q square, can be partitioned into (pairwise) disjoint Baer subplanes, hence we may take t of them to obtain a t-good structure. Note that if we take the union of t disjoint subplanes from the partition, it is easily seen to be t-good without the result of Svéd. However, the disjoint Baer subplanes construction works for arbitrary disjoint Baer subplanes. This construction is independent from the conjecture of Abreu et al. [2], and extends their result to arbitrary square prime powers.

Regarding the sizes, the t-good structure resulting from a degenerate subplane of type π_1 or π_2, or a non-degenerate subplane of order t_1, where $t = t_1^2 + t_1 + 1$, is of size $tq + 1$, $tq - t + 3$ and $tq - (t_1 - 1)t$, respectively. The disjoint Baer subplanes construction gives a t-good structure of size $t(q + \sqrt{q} + 1)$.

Gács et al. in [19] and [20] show that if t is small enough, then the Baer subplane construction is optimal. Moreover, there are no other t-good structures in $\mathrm{PG}(2, q)$ than the ones listed above. The precise results are the following.

Result 2.6. *Let $\mathcal{T}$ be a t-good structure in a projective plane of order q, $t \leq 2\sqrt{q}$. Then $|\mathcal{T}| \leq t(q+\sqrt{q}+1)$. If the plane is* $\mathrm{PG}(2,q)$ *and $t < \sqrt[4]{q}/2$, then in case of equality $\mathcal{T}$ is the union of t disjoint Baer subplanes.*

Result 2.7. *Let p be a prime and let $\mathcal{T}$ be a t-good structure in* $\mathrm{PG}(2,q)$, *$q = p^h$; furthermore,*

- *for $h = 1$ and $h = 2$, let $t < p^{1/2}/2$;*
- *for $h \geq 3$, let $t < \min\{p+1, c_p q^{1/6} - 1, q^{1/4}/2\}$, where $c_2 = c_3 = 1/8$ and $c_p = 1$ for $p > 3$.*

Then $\mathcal{T}$ is either a completely deleted degenerate subplane, or the union of t disjoint Baer subplanes.

2.2.2. *t-good structures in GQs and GHs.* In the cases $n = 4, 6$, that is, generalized quadrangles and hexagons, two or more pairwise collinear points must all be incident with a fixed line l_1. Hence to use the neighboring balls construction for $t \geq 2$, the points of $\mathcal{P}^*$ are all incident with l_1, and $l_1 \in \mathcal{L}^*$. Dually, the lines of $\mathcal{L}^*$ must all be incident with a point $P_1 \in \mathcal{P}^*$, and hence $P_1 \in l_1$. This construction, due to Araujo et al. [7], is analogous to the CDDS of type π_1 in a projective plane. In other words, it might be regarded as an extension of Brown's original construction from projective planes to generalized polygons. This gives a t-good structure of size $tq^{n-2} + q^{n-3} + \ldots + q + 1$.

If $t = 1$, we may choose $\mathcal{P}^* = \{P_1\}$ and $\mathcal{L}^* = \{l_1\}$ arbitrarily, the conditions on $\mathcal{P}^*$ and $\mathcal{L}^*$ are trivially satisfied; hence $P_1 \notin l_1$ is also admissible [19]. In projective planes, this corresponds to a degenerate subplane of type π_2. This construction gives a 1-good structure of size $q^{n-2} + 2q^{n-3} + q^{n-4} + \ldots + 1$, which is greater than the former one by q^{n-3}.

We may also define (s,t)-*good structures*, that is, a pair of a point-set and a line-set $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ such that every line outside $\mathcal{L}_0$ intersects $\mathcal{P}_0$ in s points, and every point outside $\mathcal{P}_0$ is covered by t lines of $\mathcal{L}_0$. By definition, $\mathcal{T}$ is t-good if and only if it is (t,t)-good. It is also straightforward to check that the union $\mathcal{T}$ of an (s_1,t_1)-good structure $\mathcal{T}_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and an (s_2,t_2)-good structure $\mathcal{T}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ is (s_1+s_2, t_1+t_2)-good if and only if in $\mathcal{T} = (\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{L}_1 \cup \mathcal{L}_2)$ every point in $\mathcal{P}_1 \cap \mathcal{P}_2$ and every line in $\mathcal{L}_1 \cap \mathcal{L}_2$ is $\mathcal{T}$-complete. Note that the points of a $(0,t)$-good, and the lines of an $(s,0)$-good structure must be $\mathcal{T}$-complete, hence their union is (s,t)-good. With this (unexplored) concept it is comfortable to construct 1-good structures as the union of a $(0,1)$ and a $(1,0)$-good structure.

From now on we consider a generalized quadrangle $(\mathcal{P}, \mathcal{L})$ of order q. For $U \subset \mathcal{P}$, $U^\perp$ denotes the set of points collinear with all points of U, and $U^{\perp\perp}$ the set of points collinear with all points of $U^\perp$. (Every point is considered to be collinear with itself.) One can similarly define $W^\perp$ and $W^{\perp\perp}$ for a set W of lines.

It is easy to see that for a pair of points $\{u,v\}$, $\left|\{u,v\}^\perp\right| = q+1$. A non-collinear point-pair u, v is called *regular* if $\left|\{u,v\}^{\perp\perp}\right| = q+1$ holds. The definition of a regular line pair is analogous.

Let $\{u_0,u_1\}$ be a regular point pair, and put $\{u_0,u_1\}^\perp \cup \{u_0,u_1\}^{\perp\perp}$ into $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ completely. In other words, let $\mathcal{P}_0 = \{u_0,u_1\}^\perp \cup \{u_0,u_1\}^{\perp\perp}$, and let $\mathcal{L}_0$ consist of the lines that intersect $\mathcal{P}_0$. It is not hard to check that $(\mathcal{P}_0, \mathcal{L}_0)$ is $(0,1)$-good. Similarly, a regular line pair results in a $(1,0)$-good structure. It is also easy to see that the points and the lines at distance at most $n-2=2$ from a fixed point P or a fixed line l (that is, a ball of radius two) form a $(1,0)$ or a $(0,1)$-good structure, respectively. Regular point or line pairs do not always exist, but if they do, we can use them to construct a 1-good structure as follows. These constructions can be found in [19], though not using the concept of (s,t)-good structures.

Suppose that there exists a $(0,1)$-good structure $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ arising from a regular point pair. Uniting $\mathcal{T}$ with a ball of center $P \notin \mathcal{T}$, we obtain a 1-good structure will be of size q^2+3q+1. If we find a regular line pair such that the lines in the resulting $(1,0)$-good structure are not incident with any point from $\mathcal{P}_0$, their union will be of size q^2+4q+3. In the classical generalized quadrangle $Q(4,q)$, the first construction always works, while the second works if $q>2$ is even.

Beukemann and Metsch ([10], 2011) studied one-good structures in arbitrary generalized quadrangles of order q, and in particular, in the classical one $Q(4,q)$. They give several examples that work for arbitrary prime power q that can be phrased in terms of $(0,1)$ and $(1,0)$-good structures as above. Besides the two such structures above, they use an ovoid or a spread to construct 1-good structures. An *ovoid* in a GQ is a set of q^2+1 points that intersect every line in one point. A *spread* is the dual of an ovoid, that is, a set of q^2+1 lines that cover all point once. If $\mathcal{O}$ is an ovoid, then $(\mathcal{O}, \emptyset)$ is $(1,0)$-good, while for a spread $\mathcal{S}$, $(\emptyset, \mathcal{S})$ is $(0,1)$-good, hence can be used to obtain 1-good structures. However, they find no larger construction than the two in [19] that works for general q. For $q=3$, they find a sporadic example of size $22 = q^2+4q+2$. Moreover, Beukemann and Metsch prove the following upper bound on the size of a 1-good structure in a GQ.

Theorem 2.8 ([10]). *Let Q be a generalized quadrangle of order q, $q>1$, and let $\mathcal{T}$ be a* 1-*good structure in Q. Then*

(1) $|\mathcal{T}| \le 2q^2+2q-1$;
(2) *If Q is $Q(4,q)$ and q is even, then* $|\mathcal{T}| \le 2q^2+q+1$.

It seems that understanding t-good structures in GQs is much more difficult than in projective planes. In the latter case the characterization of 1-good structures is almost immediate (cf. [19]).

2.2.3. *The construction by Lazebnik et al. as t-good structures.* Consider the construction of Lazebnik et al. [33]. In the cases $g=6$ and 8, the graphs they construct are of the same size as Brown's [11] and Araujo et al.'s [7], respectively. We show that just as the latter two, Lazebnik et al.'s construction can also be interpreted as a special case of Brown's method, that is, it is isomorphic to a graph obtained by deleting a t-good structure from a projective plane or a GQ.

First they construct an incidence structure $D(q)$ as follows. Points and lines of $D(q)$ are written inside a parenthesis () or brackets [], respectively. Consider the vectors (P) and $[l]$ of infinite length over $\mathrm{GF}(q)$:

$$\begin{aligned}(P) &= (p_1, p_{11}, p_{12}, p_{21}, p'_{22}, p_{23}, \ldots, p_{ii}, p'_{ii}, p_{i,i+1}, p_{i+1,i}, \ldots),\\ [l] &= [l_1, l_{11}, l_{12}, l_{21}, l'_{22}, l_{23}, \ldots, l_{ii}, l'_{ii}, l_{i,i+1}, l_{i+1,i}, \ldots].\end{aligned}$$

A point (P) and a line $[l]$ are incident if and only if the following infinite list of equations hold simultaneously:

$$\begin{aligned} l_{11} - p_{11} &= l_1 p_1 \\ l_{12} - p_{12} &= l_{11} p_1 \\ l_{21} - p_{21} &= l_1 p_{11} \\ l_{ii} - p_{ii} &= l_1 p_{i-1,i} \\ l'_{ii} - p'_{ii} &= l_{i-1,i} p_1 \\ l_{i,i+1} - p_{i,i+1} &= l_{i,i} p_1 \\ l_{i+1,i} - p_{i+1,i} &= l_1 p'_{ii}, \end{aligned}$$

where the last four equations are defined for all $i \geq 2$. For an integer $n \geq 2$, let $D(n, q)$ be derived from $D(q)$ by projecting every vector onto its initial n coordinates. Then the point-set $\mathcal{P}_n$ and the line-set $\mathcal{L}_n$ of $D(n, q)$ both have q^n elements, and incidence is defined by the first $n-1$ equations above. Note that those involve only the first n coordinates of (P) and $[l]$, hence apply to the points and lines of $D(n, q)$ unambiguously. $D(n, q)$ as a bipartite graph can be proved to be q-regular and have girth at least $n+4$ (thus at least $n+5$ if n is odd).

Let $R, S \subset \mathrm{GF}(q)$, where $|R| = r \geq 1$ and $|S| = s \geq 1$, and let

$$\mathcal{P}_R = \{(P) \in \mathcal{P}_n \colon p_1 \in R\}, \mathcal{L}_S = \{[l] \in \mathcal{L}_n \colon l_1 \in S\}.$$

The graph $D(n, q, R, S)$ is defined as the subgraph of $D(n, q)$ induced by $\mathcal{P}_R \cup \mathcal{L}_S$. It can be shown that every vertex in $\mathcal{P}_R$ or $\mathcal{L}_S$ in $D(n, q, R, S)$ has degree s and r, respectively.

In the case $n = 2$, $\mathcal{P}_2 = \{(p_1, p_{11}) \in \mathrm{GF}(q)^2\}$ and $\mathcal{L}_2 = \{[l_1, l_{11}] \in \mathrm{GF}(q)^2\}$, and a point $(x, y) \in \mathcal{P}_2$ is incident with the line $[m, b] \in \mathcal{L}_2$ if and only if $b - y = mx$. Let

$$\begin{aligned} \varphi\colon D(2, q) &\to \mathrm{AG}(2, q) \\ (x, y) &\mapsto (x, y) \\ [m, b] &\mapsto \{(x, y) \colon y = -mx + b\}. \end{aligned}$$

The mapping φ is clearly injective and preserves incidence, hence it is an embedding of $D(2, q)$ into $\mathrm{AG}(2, q) \subset \mathrm{PG}(2, q)$. Note that vertical lines are not in the image, hence $\varphi(D(2, q))$ can be obtained by deleting the ideal line together with its points and the vertical lines from $\mathrm{PG}(2, q)$. If we consider the induced subgraph $D(2, q, R, S)$, geometrically it means that we take points only on the vertical lines $X = x \colon x \in R$ and lines with slopes $-m \in S$. In other words, we delete (besides the formerly deleted points and lines) all the points of the vertical lines $X = x \colon x \notin R$, and we delete all lines having slopes $-m \notin S$; that is, we delete the lines that intersect the ideal line in a direction (or point) (m) with $-m \notin S$. Hence this construction corresponds to a $(q + 1 - r, q + 1 - s)$-good CDDS of type π_1.

To see why the construction for $n = 3$ (that is, $g = 8$) is isomorphic to an (s, t)-good structure in a GQ, we give an explicit description of $\mathrm{PG}(3, q)$ and the classical generalized quadrangle $W(q)$ first.

The projective space $\mathrm{PG}(3, q)$ can be represented as the system of non-zero dimensional subspaces of $\mathrm{GF}(q)^4$, that is, the points, the lines and the planes of $\mathrm{PG}(3, q)$ correspond to the one, two and three dimensional subspaces of $\mathrm{GF}(q)^4$, respectively. Hence, a point of $\mathrm{PG}(3,q)$ can be represented by a nonzero vector of $\mathrm{GF}(q)^4$ that is defined up to a non-zero scalar multiplier. We write this representative as $(x:y:z:w)$, where the colons express that the coordinates are homogeneous. A line l of $\mathrm{PG}(3,q)$ corresponds to a plane of $GF(q)^4$, and hence can be defined as the span of two vectors, that is, $l=\{\alpha(x:y:z:w)+\beta(x':y':z':w') \mid (\alpha,\beta)\in\mathrm{GF}(q)^2\setminus\{(0,0)\}\}$ for some distinct points $(x:y:z:w)$ and $(x':y':z':w')$ of $\mathrm{PG}(3,q)$.

The generalized quadrangle $W(q)$ is defined by a non-degenerate symplectic form over $\mathrm{PG}(3,q)$. Let q be an odd prime power. Take a matrix $A\in\mathrm{GF}(q)^{4\times4}$ such that $A^T=-A$, and for $x,y\in\mathrm{GF}(q)^4$, let $x\sim y$ (x *perpendicular* to y) if and only if $xAy=0$. Note that the relation $\sim$ is well defined over $\mathrm{PG}(3,q)$, and for all $x\in\mathrm{GF}(q)^4$: $x\sim x$. The points of $W(q)$ are those of $\mathrm{PG}(3,q)$, and the lines of $W(q)$ are those of $\mathrm{PG}(3,q)$ that are totally isotropic, that is, any two points of which are perpendicular. Note that if $x\sim y$, then $(\alpha x+\beta y)\sim(\gamma x+\delta y)$ for all $\alpha,\beta,\gamma,\delta\in\mathrm{GF}(q)$, hence two points x and y are collinear in $W(q)$ if and only if $x\sim y$. Thus a point is incident with a line in $W(q)$ if and only if it is perpendicular to at least two of its points (and hence to all of them). It can be proved that $W(q)$ is a generalized quadrangle of order (q,q).

Now the graph $D(3,q)$ has point-set $\mathcal{P}_3=\{(x,y,z)\in\mathrm{GF}(q)^3\}$ and line-set $\mathcal{L}_3\{[a,b,c]\in\mathrm{GF}(q)^3\}$, where $(x,y,z)\in[a,b,c]$ if and only if $b-y=ax$ and $c-z=bx$. Now let

$$\begin{aligned}\varphi\colon D(3,q) &\to \mathrm{PG}(3,q)\\ (x,y,z) &\mapsto (x:y:z:1)\\ [a,b,c] &\mapsto \{\alpha(1:-a:-b:0)+\beta(0:b:c:1)\mid(\alpha,\beta)\in\mathrm{GF}(q)^2\setminus\{(0,0)\}\},\end{aligned}$$

furthermore, let

$$A=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&1&-1&0\end{pmatrix}.$$

We claim that φ is an embedding of $D(3,q)$ into $W(q)$ defined by the symplectic form coming from A. It is clear that φ is injective. Moreover, $(x,y,z)\in[a,b,c]\iff b-y=ax$ and $c-z=bx\iff(x:y:z:1)A(1:-a:-b:0)=0$ and $(x:y:z:1)A(0:b:c:1)=0\iff(x:y:z:1)$ is on the line spanned by $(1:-a:-b:0)$ and $(0:b:c:1)$, hence φ preserves incidence.

Note that the q^2+q+1 points collinear with $P_1=(0:0:1:0)$ in $W(q)$ (that is, points of form $(x:y:z:0)$, or in other words, the points of the plane at infinity) are not in the image of φ; moreover, lines intersecting the line $l_1=\{(0:\alpha:\beta:0)\}$ are also excluded (no lines in the image contain a point with first and fourth coordinates both 0). This means that $\varphi(D(3,q))\subset W(q)$ is obtained from $W(q)$ by deleting every point collinear with P_1 and every line intersecting l_1. As $P_1\in l_1$, this corresponds to a 1-good neighboring balls construction.

Now the points $(x:y:z:1)$, with $x\notin R$ fixed, are precisely the q^2 points collinear to $P_x=(0:1:x:0)\in l_1$ not on l_1. The lines $\{\alpha(1:-a:-b:0)+\beta(0:b:c:1)\}$, with $a\notin S$ fixed, are precisely the q^2 lines intersecting the line $l_a=\{\gamma(1:-a:0:0)+\delta(0:$

$0:1:0)\}$ not in P_1. Hence $\varphi(D(3,q,R,S))$ can be obtained by deleting the balls around $\mathcal{P}^* = \{P_x \colon x \notin R\} \cup \{P_1\}$ and $\mathcal{L}^* = \{l_a \colon a \notin S\} \cup \{l_1\}$.

3. The Zarankiewicz problem

In the Introduction (see Definiton 1.2) we stated Zarankiewicz's problem. Here we focus on results for $s = t = 2$, that is, determining the maximum number of edges in $K_{2,2}$-free bipartite graphs. The history of the problem and early results are collected in Guy [23], so we only discuss some of the results. Kővári, T. Sós and Turán [32] proved $Z_{2,2}(m,n) < [n^{3/2}] + 2n$ and $\lim_{n\to\infty} Z_{2,2}(m,n)/n^{3/2} = 1$. They also observed, using finite affine planes, that $Z_{2,2}(p^2, p^2+p) = p^2(p+1)$ for p prime. The case $m = n$ was studied in detail by Reiman.

Theorem 3.1 (Reiman [37]). *Let G be a $K_{2,2}$-free bipartite graph of size (n,n). Then the number of edges in G satisfies the inequality*

$$e(G) \leq \frac{n}{2}\left(1 + \sqrt{4n-3}\right).$$

Equality holds if and only if $n = k^2 + k + 1$ for some k and G is the incidence graph of a projective plane of order k.

In the same paper Reiman proved $Z_{2,2}(m,n) \leq \frac{1}{2}\left(n + \sqrt{n^2 + 4nm(m-1)}\right)$ and clarified the connection of $Z_{2,2}(p^2, p^2+p) = p^2(p+1)$ with affine planes. Later Reiman [38] went on to study Zarankiewicz's problem for $s = 2$ and larger t, and proved $Z_{2,\lambda+1}(m,n) \leq \frac{1}{2}\left(n + \sqrt{n^2 + 4\lambda nm(m-1)}\right)$ with equality if and only if there is a $2-(m,k,\lambda)$-design, and the bipartite graph is the incidence graph of the design. Here $n = m(m-1)\lambda/(k(k-1))$ is the number of blocks in this design. This upper bound was also proved by Hyltén-Cavallius [25]. The connection of Zarankiewicz's problem for general s, t and block designs was noted in a particular case by Kárteszi [29, 30], and done in detail by Roman [39] (see Theorem 3.5). We give two more early results that provide exact values for $Z_{s,t}(m,n)$ if n is much larger than m.

Theorem 3.2 (Čulík [14]). *If $1 \leq s \leq m$ and $n \geq (t-1)\binom{m}{s}$, then*

$$Z_{s,t}(m,n) = (s-1)n + (t-1)\binom{m}{s}.$$

Theorem 3.3 (Guy [23]). *If $\ell(n,s,t) \leq n \leq (t-1)\binom{m}{s} + 1$, then*

$$Z_{s,t}(m,n) = \left\lfloor \frac{(s^2-1)n + (t-1)\binom{m}{s}}{s} \right\rfloor,$$

where $\ell(n,s,t)$ is approximately $(t-1)\binom{m}{s}/(s+1)$.

Irving [27] gave a method which can be used to explicitly calculate an upper bound for $Z_{s,t}(m,n)$ in case of given parameters; his idea was also investigated in [21]. One may also realte s and t to n and m (e.g., $s = n/2$, $t = m/2$); for such studies see [9], [22] and their references. For general bounds, we refer to Füredi [17, 18], Kollár-Rónyai-Szabó [31], Alon-Rónyai-Szabó [4], Nikiforov [36], and the references therein.

3.1. **Roman's inequality.** Let $I \subset \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ a strictly increasing convex function, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I \cap \mathbb{Z}$, $A := \sum_{i=1}^n x_i = np + r$ for some $p \in \mathbb{Z}$, $0 \leq r < p$. Then Jensen's inequality for integers claims $\sum_{i=1}^n f(x_i) \geq rf(p+1) + (n-r)f(p) = (A-np)f(p+1) + (n(p+1)-A)f(p) = A(f(p+1)-f(p)) - n(pf(p+1)-(p+1)f(p))$, that is, $A \leq \left(\sum_{i=1}^n f(x_i) + n(pf(p+1)-(p+1)f(p))\right)/(f(p+1)-f(p))$. Roman's ideas [39] can be used to prove this inequality for general $p \in \mathbb{Z}$.

Theorem 3.4 (Roman's inequality). *Let $I \subset \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ a strictly increasing convex or a strictly decreasing concave function, $n \in \mathbb{N}$, $x_1, \ldots, x_n, p, p+1 \in I \cap \mathbb{Z}$. Then*

$$\sum_{i=1}^n x_i \leq \frac{\sum_{i=1}^n f(x_i)}{f(p+1)-f(p)} + n \cdot \frac{pf(p+1)-(p+1)f(p)}{f(p+1)-f(p)}.$$

Equality holds if and only if $x_i \in \{p, p+1\}$ for every $1 \leq i \leq n$ or $\{x_1, \ldots, x_n, p, p+1\} \subset I'$ for an interval I' on which f is linear.

It can be shown that the best choice of p is indeed $\lfloor A/n \rfloor$, hence Roman's inequality follows from Jensen's one. We note that Irving's method [27] for $s = t = 2$ is nothing else but Jensen's inequality for integers; however, for higher values of s and t it may give much better results. The advantage of Roman's bound is that we may choose the parameter p freely to obtain an upper bound on $A = \sum x_i$ in a comfortable way, while in Jensen's inequality one has to use $\lfloor A/n \rfloor$, where we are about to estimate A. We will use the following bound that was explicitly proved in [39].

Theorem 3.5 (Roman's bound [39]). *Let $G = (A, B; E)$ be a $K_{s,t}$-free bipartite graph of size (m, n), and let $p \geq s-1$. Then the number of edges in G satisfy*

$$e(G) \leq \frac{(t-1)}{\binom{p}{s-1}} \binom{m}{s} + n \cdot \frac{(p+1)(s-1)}{s}.$$

Equality holds if and only if every vertex in B has degree p or $p+1$ and every s-tuple in A has exactly $t-1$ common neighbors in B.

Definition 3.6. *For $s, t, m, n, p \in \mathbb{N}$, $p \geq s-1$, let*

$$R(s, t, m, n, p) := \frac{(t-1)}{\binom{p}{s-1}} \binom{m}{s} + n \cdot \frac{(p+1)(s-1)}{s}.$$

Remark 3.7. *If (t, v, k, λ) are admissible parameters in the sense of Definition 1.7, then $R(t, \lambda+1, v, b, k) = bk = rv$ is integer.*

The incidence graphs of $t-(v, \{k, k+1\}, \lambda)$ designs are $K_{t,\lambda+1}$-free, and these are precisely the graphs that satisfy the conditions of equality in Roman's bound. Bipartite graphs that are in some sense very close to $2-(v, \{k, k+1\}, 1)$ designs were also considered in [12].

Example 3.8. ***a)*** *If we delete one point arbitrarily from a $t-(v, k, \lambda)$ design $\mathcal{D}$, we obtain a $t-(v-1, \{k-1, k\}, \lambda)$ design $\mathcal{D}'$.*

b) *Take a $2-(v, k, 1)$ design $\mathcal{D}$ and delete a block from it with all, or all but one of its points. The obtained structure $\mathcal{D}'$ will be a $2-(v-k+a, \{k-1, k\}, 1)$ design, $a \in \{0, 1\}$.*

c) *Delete two intersecting lines from an affine plane of order n (a $2-(n^2, n, 1)$ design). In this way we get a $2-(n^2-2n+1, \{n-2, n-1\}, 1)$ design.*

3.2. **Results on the Zarankiewicz problem.** To prove our first result, we need a theorem of Metsch.

Result 3.9 (Metsch [34]). *Let $n \geq 15$, $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure with $|\mathcal{P}| = n^2+n+1$, $|\mathcal{L}| \geq n^2+2$ such that every line in $\mathcal{L}$ is incident with $n+1$ points of $\mathcal{P}$ and every two lines have at most one point in common. Then a projective plane Π of order n exists and $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ can be embedded into $\mathcal{P}$.* □

Lemma 3.10. *Let $n \geq 15$, $G = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence graph with $|\mathcal{P}| = n^2+n+1$, $|\mathcal{L}| \geq n^2+2$ such that every line in $\mathcal{L}$ is incident with* at least *$n+1$ points of $\mathcal{P}$, and every two lines have at most one point in common. Then a projective plane Π of order n exists, and $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ can be embedded into $\mathcal{P}$; specially, every line in $\mathcal{L}$ is incident with exactly $n+1$ points of $\mathcal{P}$.*

Proof. By deleting edges from G, we can obtain a graph $G' = (\mathcal{P}, \mathcal{L}, \mathcal{I}')$ in which the vertices of $\mathcal{L}$ have degree exactly $n+1$. Then, by Theorem 3.9, G' is a subgraph of a projective plane Π of order n. Now suppose that there is a line l in $\mathcal{L}$ that has degree at least $n+2$ in G. This means that there exists a point P such that l is incident with P in G, but not in Π. Then each of the $n+1$ lines passing through P in Π intersects l in a point different from P. As $|\mathcal{L}| \geq n^2+1$, at least one of these lines is a line of G as well, but it intersects l in at least two points in G, a contradiction. Hence every line has $n+1$ points in G. □

Theorem 3.11. *Let $n \geq 15$, and $c \leq n/2$. Then*

$$Z_{2,2}(n^2+n+1-c, n^2+n+1) \leq (n^2+n+1-c)(n+1).$$

Equality holds if and only if a projective plane of order n exists. Moreover, graphs giving equality are subgraphs of the incidence graph of a projective plane of order n.

Proof. If a projective plane of order n exists, deleting c of its lines yields a graph on $(n^2+n+1-c, n^2+n+1)$ vertices and $(n^2+n+1-c)(n+1)$ edges.

Suppose that $G = (A, B; E)$ is a $K_{2,2}$-free graph on $(n^2+n+1-c, n^2+n+1)$ vertices and $e(G) \geq |A|(n+1)$ edges. Let m be the number of vertices in A of degree at most n (low-degree vertices). Assume that $m \geq n-c$. Delete $(n-c)$ low-degree vertices to obtain a graph G' on (n^2+1, n^2+n+1) vertices with at least $(n^2+1)(n+1)+(n-c)$ edges. By Roman's bound with $p = n$, $Z_{2,2}(n^2+1, n^2+n+1) \leq (n^2+1)(n+1)+(n-1)/2$, hence $n-c \leq (n-1)/2$. This contradicts $c \leq n/2$, thus $m < n-c$ must hold.

Now delete all the low-degree vertices from A to obtain a graph G' on the vertex sets (A', B) with $|A'| \geq n^2+2$, $|B| = n^2+n+1$. Then every vertex in A' has degree at least $n+1$, hence we can apply Lemma 3.10 to derive that G' can be embedded into a projective plane Π of order n, therefore every vertex in A' has degree $n+1$, which combined with $e(G) \geq |A|(n+1)$ yields that every vertex in A has degree $n+1$ (in G), thus G itself can be embedded into Π. □

Remark 3.12. *If we knew $Z_{2,2}(n^2+1, n^2+n+1) \leq (n^2+1)(n+1)+\delta$, then the above argument would hold for $c < n-\delta$. Removing n points (or lines) from a projective plane* *of order n we get $Z_{2,2}(n^2+1, n^2+n+1) \geq (n^2+1)(n+1)$. Note that an affine plane plus an extra line containing a single point shows $Z_{2,2}(n^2, n^2+n+1) \geq n^2(n+1)+1$.*

QUESTION 3.13. *Is it true that $Z_{2,2}(n^2+1, n^2+n+1) \leq (n^2+1)(n+1)$ (if n is large enough)?*

REMARK 3.14. *The upper bound on the number of edges in Theorem 3.11 is a direct consequence of Roman's bound if $c(c-1) < 2n$ without assuming $n \geq 15$.*

The next result is based on a very simple observation, which was also pointed out by Guy [23], p138, point C. Let $\mathcal{F}$ be a subgraph-closed family of bipartite graphs, that is, if $G \in \mathcal{F}$ and H is a subgraph of G, then $H \in \mathcal{F}$. For example, $K_{s,t}$-free graphs clearly form a subgraph-closed family. Let $\mathcal{F}(m,n) = \{G = (A,B;E) \in \mathcal{F} \colon |A| = m, |B| = n\}$, and let $\mathrm{ex}_{\mathcal{F}}(m,n) = \max\{e(G) \colon G \in \mathcal{F}(m,n)\}$, and let $\mathrm{Ex}_{\mathcal{F}}(m,n) = \{G \in \mathcal{F}(m,n) \colon e(G) = \mathrm{ex}_{\mathcal{F}}(m,n)\}$. Graphs of $\mathrm{Ex}_{\mathcal{F}}(m,n)$ are called extremal.

THEOREM 3.15. *Let $\mathcal{F}$ be a subgraph-closed family of bipartite graphs, suppose that $\mathrm{ex}_{\mathcal{F}}(m,n) \leq e$, and let $c \in \mathbb{N}$. Then*

(1) $\mathrm{ex}_{\mathcal{F}}(m+c,n) \leq e + c\lfloor e/m \rfloor$;
(2) $\mathrm{ex}_{\mathcal{F}}(m,n+c) \leq e + c\lfloor e/n \rfloor$.

Moreover, if equality holds in, say, (1) for some $c \geq 1$, then equality holds for all $c' \in \mathbb{N}$, $0 \leq c' < c$ as well, and any $G \in \mathrm{Ex}_{\mathcal{F}}(m+c,n)$ has an induced subgraph that is in $\mathrm{Ex}_{\mathcal{F}}(m+c-1,n)$.

Proof. It is enough to prove (1), as (2) is completely analogous. We prove the assertion by induction on c. The statement is trivial if $c = 0$. Let $d = \lfloor e/m \rfloor$. Suppose $\mathrm{ex}_{\mathcal{F}}(m+c,n) \geq e+cd$, and let $G = (A,B;E) \in \mathrm{Ex}_{\mathcal{F}}(m+c,n)$. There is no vertex of degree strictly smaller than d in A, otherwise removing such a vertex we would obtain a graph in $\mathcal{F}(m+c-1,n)$ with more than $e+(c-1)d$ edges, which is not possible by the inductive hypothesis. Consider an arbitrary subgraph of G on (m,n) vertices. By the definition of d, we find a vertex in A of degree d. Removing this vertex we obtain a graph of $\mathcal{F}(m+c-1,n)$ with at least, hence (by the inductive hypothesis) exactly $e+(c-1)d$ edges. Thus $\mathrm{ex}_{\mathcal{F}}(m+c-1,n) = e+(c-1)d$, and $\mathrm{ex}_{\mathcal{F}}(m+c,n) = e(G) = e+cd$. □

For example, the above theorem can be used if we start from a design or a $2-(v,\{k,k+1\},1)$ design obtained by deleting a block from a $2-(v',k+1,1)$ (Example 3.8 **b)**).

COROLLARY 3.16. *(i) Let (t,v,k,λ) be admissible parameters (with $b = \lambda\binom{v}{t}/\binom{k}{t}$, $r = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$), and let $0 \leq c \in \mathbb{N}$. Then*

$$Z_{t,\lambda+1}(v+1+c,b) \leq rv + \lambda\frac{\binom{v}{t-1}}{\binom{k}{t-1}} + c(r-1). \tag{3.1}$$

(ii) Let $(2,v,k,1)$ be admissible parameters. Then

$$Z_{2,2}(v-k+c,b-1) \leq (v-k)r + c(r-1). \tag{3.2}$$

Moreover, if a $2-(v,k,1)$ design exists, then equality holds in (3.2) for all $0 \leq c \leq k$.

Proof. (i) We apply Theorem 3.15 with $m = v + 1$, $n = b$. By Roman's bound we see $rv = R(t, \lambda + 1, v, b, k) = \lambda\binom{v}{t}/\binom{k}{t-1} + b(k+1)(t-1)/t$, furthermore

$$Z_{2,\lambda+1}(v+1, b) \leq e := R(t, \lambda+1, v+1, b, k) = \lambda\frac{\binom{v+1}{t}}{\binom{k}{t-1}} + \frac{b(k+1)(t-1)}{t} = rv + \lambda\frac{\binom{v}{t-1}}{\binom{k}{t-1}}.$$

It is easy to see that $r < \lambda\frac{\binom{v}{t-1}}{\binom{k}{t-1}}$, thus $\lfloor e/(v+1)\rfloor = r - 1$.

(ii) Here $r = (v-1)/(k-1)$. Simple computations show that $Z_{2,2}(v-k, b-1) \leq R(2, 2, v-k, b-1, k-1) = r(v-k)$, thus the case $c = 0$ is verified. As $Z_{2,2}(v-k+1, b-1) \leq e := R(2, 2, v-k+1, b-1, k-1) = r(v-k) + (v-k)/(k-1) < r(v-k) + r$, Theorem 3.15 with $m = v-k+1$, $n = b-1$ proves the assertion. □

We remark that a $t{-}(v, k, 1)$ design is also called a *Steiner system*; in particular, $2{-}(v, 3, 1)$ and $3 - (v, 4, 1)$ designs are also known as *Steiner triple systems (STS)* and *Steiner quadruple systems (SQS)*, respectively (see e.g. [13]). For $k = 3, 4$ or 5, a $2 - (v, k, 1)$ design exists whenever $v \equiv 1$ or $3 \pmod 6$, $v \equiv 1$ or $4 \pmod{12}$, or $v \equiv 1$ or $5 \pmod{20}$, respectively. These can be used to obtain some exact values of $Z_{2,2}(m, n)$.

In case of affine planes, embeddibility theorems are available, thus we can formulate stronger results. Recall that an affine plane of order n is always embeddable into a projective plane of order n. Totten [41] also has a result on the complement of two lines in a projective plane (that is, we delete one line and all its points from an affine plane).

Result 3.17 (Totten [41]). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a finite linear space (that is, an incidence structure where any two distinct points are contained in a unique line) with $|\mathcal{P}| = n^2 - n$, $|\mathcal{L}| = n^2 + n - 1$, $2 \leq n \neq 4$, and every point having degree $n+1$. Then $\mathcal{S}$ can be embedded into a projective plane of order n.*

Corollary 3.18. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a finite partial linear space (that is, an incidence structure where any two distinct points are contained in at most one line) with $|\mathcal{P}| = n^2 - n$, $|\mathcal{L}| = n^2 + n - 1$, $n > 4$, in which the number of flags is at least $(n^2 - n)(n + 1)$. Then $\mathcal{S}$ is a linear space, and it can be embedded into a projective plane of order n.*

Proof. As $R(2, 2, n^2-n, n^2+n-1, n-1) = (n^2-n)(n+1)$, each line in $\mathcal{L}$ has degree $n-1$ or n, and any two distinct points must be contained in a unique line. The average degree of a point is $n+1$. Now suppose that there is a point P of degree at least $n+2$. Then the number of points on the lines incident with P is at least $1 + (n+2)(n-2) = n^2 - 3 > |\mathcal{P}| = n^2 - n$ (by $n > 4$). Hence every point has degree $n + 1$, so by Totten's Result 3.17, $\mathcal{S}$ is the complement of two lines in a projective plane of order n. □

Corollary 3.19. *Let $c \in \mathbb{N}$. Then*

$$Z_{2,2}(n^2 + c, n^2 + n) \leq n^2(n + 1) + cn, \tag{3.3}$$

$$Z_{2,2}(n^2 - n + c, n^2 + n - 1) \leq (n^2 - n)(n + 1) + cn, \tag{3.4}$$

$$Z_{2,2}(n^2 - 2n + 1 + c, n^2 + n - 2) \leq (n^2 - 2n + 1)(n + 1) + cn, \text{ if } n \geq 4. \tag{3.5}$$

Equality can be reached in all three inequalities if a projective plane of order n exists and $c \leq n + 1$, $c \leq 2n$, or $c \leq 3(n - 1)$, respectively.

Moreover, if $c \le n+1$, or $c \le 2n$ and $n > 4$, then graphs reaching the bound in (3.3) *or* (3.4)*, respectively, can be embedded into a projective plane of order n.*

Proof. The parameters of an affine plane, $(2, n^2, n, 1)$ (with $b = n^2 + n$, $r = n + 1$) are admissible. Hence (3.3) and (3.4) follow from Corollary 3.16. To apply Theorem 3.15 in (3.5), simply calculate that $R(2, 2, n^2 - 2n + 1, n^2 + n - 2, n - 2) = (n^2 - 2n + 1)(n + 1) = e$, and that $R(2, 2, n^2 - 2n + 2, n^2 + n - 2, n - 2) = e + n + 1/(n - 2) < (n^2 + 2n + 2)(n + 1)$ $(n \ge 4)$.

By taking a projective plane of order n, and deleting one, two, or three of its lines and all but c of their points each of which is contained in only one of the deleted lines, we can reach equality in (3.3), (3.4), and (3.5), respectively.

In (3.3), Theorem 3.15 also provides an affine plane of order n as an induced subgraph in graphs obtaining equality. Now the c extra points of degree n must be incident with pairwise non-intersecting lines to avoid C_4's in the graph; that is, they can be considered as the common points of c distinct parallel classes. Adding the missing $n + 1 - c$ ideal points and the line at infinity, we obtain a projective plane of order n.

In (3.4), Theorem 3.15 provides us an extremal C_4-free subgraph $G = (A, B)$ on $(n^2 - n, n^2 + n - 1)$ vertices and $(n^2 - n)(n + 1)$ edges in graphs reaching equality. By Corollary 3.18, G can be embedded into a projective plane of order n. As before, it is easy to see that the embedding extends to the c extra points as well. $\square$

Next we prove a straightforward recursive inequality. For a bipartite graph $G = (A, B; E)$ and vertex-sets $X \subset A$ and $Y \subset B$, let $G[X, Y]$ denote the subgraph of G induced by $X \cup Y$.

PROPOSITION 3.20. *Let $U_{s,t}(m, n, \alpha, \beta) = Z_{s-\alpha,t}(m-\alpha, \beta) + Z_{s,t}(m-\alpha, n-\beta) + (\alpha-1)n + \beta$. Then*

$$Z_{s,t}(m, n) \le \min_{\alpha} \max_{\beta} \min\{Z_{\alpha,\beta+1}(m, n),\ U_{s,t}(m, n, \alpha, \beta)\colon 1 \le \alpha < s,\ t - 1 \le \beta \le n\}.$$

Proof. Let $G = (A, B; E)$ be a maximal $K_{s,t}$-free bipartite graph on $m + n$ vertices. Let $1 \le \alpha < s$, and let β be the largest integer for which $K_{\alpha,\beta}$ is a subgraph of G (the ordering of the classes does matter). Then $|E| \le Z_{\alpha,\beta+1}(m, n)$ follows from G being $K_{\alpha,\beta+1}$-free. Now let $S \subset A$ and $T \subset B$ induce a $K_{\alpha,\beta}$, and let $U = A \backslash S$, $V = B \backslash T$. Then $G[U, T]$ must be $K_{s-\alpha,t}$-free, $G[U, V]$ is $K_{s,t}$-free; moreover, since no $K_{\alpha,\beta+1}$ can be found in G, every vertex in V may have at most $\alpha - 1$ neighbors in S. Summing up the maximum number of edges in each part, we get $|E| \le \alpha\beta + Z_{s-\alpha,t}(m-\alpha, \beta) + Z_{s,t}(m-\alpha, n-\beta) + (\alpha-1)(n-\beta) = U_{s,t}(m, n, \alpha, \beta)$. As G is maximal, it must contain a $K_{\alpha,t-1}$ for all $\alpha < s$, hence we have $\beta \ge t - 1$. $\square$

REMARK 3.21. *In particular, the case $\alpha = 1$ of this inequality investigates the vertex with largest degree. $Z_{s,t}(m, 0)$ is defined to be zero (which occurs above for $\beta = n$). Note that we may interchange the role of the classes, that is, write up the above inequality for $Z_{t,s}(n, m)$. We will call this the* transpose *of Proposition 3.20.*

REMARK 3.22. *In case of $\alpha = s - 1$, the function $U_{s,t}(m, n, s - 1, \beta)$ is non-increasing in β ($\beta \ge t - 1$), while $Z_{s-1,\beta+1}(m, n)$ is clearly non-decreasing. Thus the maximum of the minimum of these two values in β can be found easily.*

Proof.

$$U_{s,t}(m,n,s-1,\beta) = Z_{1,t}(m-s+1,\beta) + Z_{s,t}(m-s+1,n-\beta) + (s-2)n+\beta = (t-1)(m-s+1) + (s-2)n + \beta + Z_{s,t}(m-s+1,n-\beta).$$

By adding a vertex of degree $t-1$, we have $Z_{s,t}(m-s+1,n-\beta) \geq Z_{s,t}(m-s+1,n-(\beta+1)) + t-1$. □

This recursion is useful in some cases. For example, Roman's bound with $p = 4$ or 5 yields $Z_{3,3}(7,7) \leq 35$. We show $Z_{3,3}(7,7) \leq 33$. (Here, in fact, equality holds.) Let $\alpha = 2$. For $\beta \leq 4$ we have $Z_{2,\beta+1}(7,7) \leq R(2,5,7,7,5) = 33$, while $U_{3,3}(7,7,2,4) = Z_{1,3}(5,4) + Z_{3,3}(5,3) + 7 + 4 = 33$. By Remark 3.22, we are done. Other examples that prove this recursion useful are the balanced C_4-free graphs.

PROPOSITION 3.23. *Let $2 \leq q \in \mathbb{N}$, $3-q \leq c \leq 1+q$. Then*

$$Z_{2,2}(q^2+c,q^2+c) \leq (q^2+c)(q+\frac{1}{2}) + \left(\frac{c}{2}-1\right)q + \frac{c}{2} + \frac{(c-1)(c-2)}{2(q-1)}.$$

Proof. Consider the bounds in Corollary 3.22 with $s = t = 2$. If $\beta \leq q$, then $Z_{1,\beta+1}(q^2+c,q^2+c) \leq q(q^2+c)$, which is smaller than the bound stated provided that $c \geq 3-q$. Hence we may assume $\beta \geq q+1$. Then the second expression is $(q^2+c-1)+\beta+Z_{2,2}(q^2+c-1,q^2+c-\beta) \leq q^2+q+c+Z_{2,2}(q^2+c-1,q^2+c-q-1)$. Applying Roman's bound with $p = q-1$ to $Z_{2,2}(q^2+c-q-1,q^2+c-1)$, we get the desired result. □

REMARK 3.24. *It is easy to calculate that for $3-q \leq c \leq 1+q$, Roman's upper bound on $Z_{2,2}(q^2+c,q^2+c)$ gives the best result if we set $p = q$. The bound in Proposition 3.23 is smaller than Roman's one by*

$$\frac{q-c}{2} + \frac{(2q-c)(c-1)}{2q(q-1)}.$$

In the rest of this section we tackle Roman's bound and the recursive idea to establish some results that are tight if we are close to a design. Without a strong embedding theorem like Result 3.9, we obtain weaker results. The next proposition is a direct consequence of Roman's bound.

PROPOSITION 3.25. *Assume that the parameters (t,v,k,λ) are admissible, and let c_0 be the largest integer such that $\lambda\left(\binom{v-c_0}{t} + c_0\binom{v-1}{t-1} - \binom{v}{t}\right) < \binom{k-1}{t-1}$. Then for every $0 \leq c \leq c_0$,*

$$Z_{t,\lambda+1}(v-c,b) \leq r(v-c).$$

Equality can be reached if a $t-(v,k,\lambda)$-design exists. Moreover, if $c < c_0$, then in the graphs obtaining equality, the vertices in the class of size $v-c$ have degree r. In particular, the condition for $t = 2$ is $c_0(c_0-1) < 2(k-1)/\lambda$.

Proof. Removing c points from the incidence graph of a $t-(v,k,\lambda)$ design we obtain a $K_{t,\lambda+1}$-free graph on $(v-c,b)$ nodes and $r(v-c)$ edges.

On the other hand, using $rv = bk$ and $bk/t = \lambda\binom{v}{t}/\binom{k-1}{t-1}$, Roman's bound with $p = k-1$ yields

$$Z_{t,\lambda+1}(v-c,b) \leq \left\lfloor \frac{\lambda}{\binom{k-1}{t-1}}\binom{v-c}{t} + b\cdot\frac{k(t-1)}{t} \right\rfloor = r(v-c) + \left\lfloor \frac{\lambda\left(\binom{v-c}{t} + c\binom{v-1}{t-1} - \binom{v}{t}\right)}{\binom{k-1}{t-1}} \right\rfloor.$$

Suppose that $G = (A, B)$ is $K_{t,\lambda+1}$-free on $(v - c, b)$ vertices and $(v - c)r$ edges, $c < c_0$. Assume that there is a vertex $u \in A$ with degree smaller than r. Removing u from A, we obtain a graph on $(v - c - 1, b)$ vertices and more than $(v - c - 1)r$ edges, which contradicts our upper bound. $\square$

The recursive inequality of Proposition 3.20 can be used to achieve another bound in a more special case.

PROPOSITION 3.26. *Let $(2, v, k, 1)$ be admissible parameters. Then*

$$Z_{2,2}(v + 1, b) \leq bk + b - k(r - 1).$$

Proof. Let $G = (A, B; E)$ be an extremal $K_{2,2}$-free bipartite graph of size $(v+1, b)$. Then there must be a vertex in B with degree at least $k + 1$. Thus by Remark 3.22, we may use the transpose of Proposition 3.20 with $\alpha = 1$, $\beta = k + 1$ to obtain

$$e(G) \leq U_{2,2}(b, v + 1, 1, k + 1) = (b - 1) + k + 1 + Z_{2,2}(b - 1, v - k).$$

Now $Z_{2,2}(b-1, v-k) \leq (v-k)r$, as deleting a block and its points from a $2-(v, k, 1)$ design would result in a structure seen in Example 3.8 (so $R(2, 2, v - k, b - 1, k - 1) = (v - k)r$). Hence $e(G) \leq k + b + (v - k)r = bk + b - k(r - 1)$. $\square$

COROLLARY 3.27. *Let $n \geq 2$. Then $Z_{2,2}(n^2 + n + 2, n^2 + n + 1) \leq (n^2 + n + 1)(n + 1) + 1$, and equality holds if and only if a projective plane of order n exists. Moreover, any graph G reaching equality can be obtained in the following way: take a projective plane $(\mathcal{P}, \mathcal{L})$ of order n, let $A = \mathcal{P} \cup \{u_0\}$ ($u_0 \notin \mathcal{L} \cup \mathcal{P}$), $B = \mathcal{L}$. Take any point $v \in \mathcal{L}$, and let $\{u_1, \dots, u_{n+1}\}$ be its neighbors in $\mathcal{P}$. Let H be any subset of the neighbors of u_1, for which $v \notin H$. Delete the edges u_1v' for all $v' \in H$, and add the edges u_0v and u_0v' for all $v' \in H$. In particular, there must be a vertex in A with degree at most $n/2 + 1$.*

Proof. Proposition 3.26 applied to a projective plane of order n (with parameters $v = b = n^2+n+1$, $t = 2$, $\lambda = 1$, $k = n+1$) yields $Z_{2,2}(n^2+n+1, n^2+n+2) \leq (n^2+n+1)(n+1)+1$. Now let $G = (A, B)$ be a C_4-free graph on $(n^2 + n + 2, n^2 + n + 1)$ vertices and $(n^2 + n + 1)(n + 1) + 1$ edges. Then there must be a vertex $v \in B$ of degree at least $n + 2$. Consider the proof of Proposition 3.26. As $U_{2,2}(b, v+1, 1, k+2) = n^2+n+n+3+Z_{2,2}(n^2+n, n^2) \leq n^2 + 2n + 3 + (n^2 - 1)(n + 1) = (n^2 + n)(n + 1) + 2 < (n^2 + n + 1)(n + 1) + 1$, v must have degree $n + 2$. To reach equality, the decomposition in the proof of Proposition 3.20 (with $\alpha = 1$, $\beta = n+2$) assures that removing v and its neighbors $N(v) = \{u_0, \dots, u_{n+1}\}$ from G, we find an affine plane of order n, whose points and lines correspond to $A \setminus N(v)$ and $B \setminus \{v\}$, respectively; moreover, the degree of the vertices of $B \setminus \{v\}$ in G is $n + 1$. As these vertices have precisely n neighbors in $A \setminus N(v)$, each one has to be adjacent to one of the u_is. On the other hand, any u_i ($0 \leq i \leq n + 1$) may be adjacent only to the n lines of one parallel class (besides v), hence $\deg(u_i) \leq n + 1$. Let $\mathcal{L}_i \subset A \setminus \{v\}$ be the parallel classes of $\mathcal{L}$ ($1 \leq i \leq n + 1$). We may assume that $N(u_i) \setminus \{v\} \subset \mathcal{L}_i$ for all $1 \leq i \leq n + 1$. Let $H = N(u_0) \setminus \{v\}$; we may assume $H \subset \mathcal{L}_1$. Then $N(u_i) = \{v\} \cup \mathcal{L}_i$ for all $2 \leq i \leq n + 1$, and $N(u_1) = \{v\} \cup \mathcal{L}_1 \setminus H$. Then $\deg(u_0) + \deg(u_1) = n + 2$. $\square$

PROPOSITION 3.28. *Let $c \geq 1$ and $n \geq 2$. Then $Z_{2,2}(n^2+n+2+c, n^2+n+1) \leq (n^2+n+1)(n+1)+cn+1$. If $n \geq 3$, then $Z_{2,2}(n^2+n+2+c, n^2+n+1) \leq (n^2+n+1)(n+1)+cn$.*

Proof. Let $\mathcal{F}$ be the family of C_4-free graphs. The first statement follows from Proposition 3.27 and Theorem 3.15 (with $m = n^2 + n + 2$ and $d = n$). Now suppose $n \geq 3$ and that equality holds for some $c \geq 1$, thus for $c = 1$ as well. Then any $G \in \mathrm{Ex}_{\mathcal{F}}(n^2 + n + 3, n^2 + n + 1)$ induces a graph from $\mathrm{Ex}_{\mathcal{F}}(n^2 + n + 2, n^2 + n + 1)$, which has a vertex with degree at most $n/2 + 1$ by Proposition 3.27. Deleting this vertex from G we would have $\mathrm{ex}_{\mathcal{F}}(n^2+n+2, n^2+n+1) \geq (n^2+n+1)(n+1)+n+1-(n/2+1) > (n^2+n+1)(n+1)+1$, a contradiction. □

There are ad hoc ideas that may help when determining Zarankiewicz numbers for small parameters, see Guy [23], p138. The next proposition illustrates such a case.

Proposition 3.29. $Z_{2,2}(16, 17) \leq 70$.

Proof. Suppose to the contrary that there exist a C_4-free bipartite graph $G = (A, B; E)$, where $|A| = 16$, $|B| = 17$, $|E| = 71$. As $Z_{2,2}(16, 16) = Z_{2,2}(15, 17) = 67$, every vertex in G has degree at least four. Corollary 3.22 yields that there can be no vertex of degree six. Hence the degree sequence of A and B are $\{4^9, 5^7\}$, $\{4^{14}, 5^3\}$, where the superscripts denote the multiplicity of that degree. Let $v \in A$, $\deg(v) = 5$, and let $N(v) = \{u_1, \ldots, u_5\}$. Then $\deg(u_i) = 4$ for $1 \leq i \leq 5$, otherwise the pairwise disjoint sets $N(u_i) \setminus \{v\} \subset A \setminus \{v\}$, $1 \leq i \leq 5$, would have more than 15 elements. Let $v_i \in A$ a vertex with degree 5, $1 \leq i \leq 5$. Then $|N(v_1) \cup \ldots \cup N(v_5)| \geq 5 + 4 + 3 + 2 + 1 = 15$, but there are only 14 vertices of degree four in B. □

3.3. **Lower bounds for $s = t = 2$.** Now let us collect some constructions regarding the case $s = t = 2$. As a general principle, if we have an extremal graph $G = (A, B)$, we can always delete the lowest degree vertex from A (or B) to obtain a graph on $(|A| - 1, |B|)$ (or $(|A|, |B| - 1)$) vertices with many edges. This trivial method gives good results in many cases. Another simple idea is that if we find k points in A such that no two of them has a common neighbor, then we can add one vertex to B and connect it with those vertices. Note that $k = 1$ always works. Without the sake of completeness, we illustrate these methods in the upcoming propositions.

Proposition 3.30. $Z_{2,2}(14, 25) = 80$.

Proof. For basic facts about ovals we refer to [24]. Let O be an oval in $\mathrm{PG}(2, 5)$, and let $\mathcal{L}_0$ be the set of its six tangent lines. Let $\mathcal{P}_0$ be the set of $\binom{6}{2} = 15$ outer points of O together with two arbitrarily chosen points of O. Delete $\mathcal{P}_0$ and $\mathcal{L}_0$ from $\mathrm{PG}(2, 5)$. The resulting graph clearly has size $(14, 25)$. Any inner point of O is incident with zero tangent to O, whereas a point of O is incident with precisely one tangent to O. Thus the number of edges is $4 \cdot 5 + 10 \cdot 6 = 80$. On the other hand, $R(2, 2, 14, 25, 3) < 81$. □

Proposition 3.31. *Let $\mathcal{D}$ be a $2-(v, k, 1)$ design, and let $\ell^{\mathcal{D}}(i)$ be the least number of points that the union of i blocks may cover in $\mathcal{D}$. Let $f^{\mathcal{D}}(c)$ be the maximal value of i for which $\ell^{\mathcal{D}}(i) \leq c$. Then*

$$Z_{2,2}(v - c, b) \geq (v - c)r + f^{\mathcal{D}}(c).$$

Proof. By definition of $f^{\mathcal{D}}(c)$, we can delete c points from $\mathcal{D}$ so that $f^{\mathcal{D}}(c)$ blocks become empty. We can connect these blocks with any one of the points without creating a C_4, so we can add altogether $f^{\mathcal{D}}(c)$ edges to the $(v-c)r$ edges that remain after the deletion. □

n \ m	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
7	**21**																
8	**22**	**24**															
9	**24**	**26**	**29**														
10	**25**	**28**	**31**	**34**													
11	**27**	**30**	**33**	**36**	**39**												
12	**28**	**32**	**36**	**39**	**42**	**45**											
13	**30**	**33**	**37**	**40**	**44**	**48**	**52**										
14	**31**	**35**	**39**	**42**	**45**	**49**	**53**	**56**									
15	**33**	**36**	**40**	**44**	**47**	**51**	**55**	**58**	**60**								
16	**34**	**38**	**42**	**46**	**50**	**53**	**57**	**60**	***64***	***67***							
17	**36**	**39**	**43**	**47**	**51**	**55**	**59**	**63**	***67***	***70***	***74***						
18	**37**	**41**	**45**	**49**	**53**	**57**	**61**	**65**	**69**	***73***	***77***	***81***					
19	**39**	**42**	**46**	**51**	**55**	**60**	**64**	**68**	**72**	***76***	***80***	***84***	***88***				
20	**40**	**44**	**48**	**52**	**57**	**61**	**66**	**70**	**75**	**80**	***84***	***88***	***92***	***96***			
21	**42**	**45**	**49**	**54**	**59**	**63**	**67**	**72**	**77**	***81***	86	***90***	***95***	***100***	**105**		
22	**43**	**47**	**51**	**55**	**60**	**65**	**69**	**73**	***78***	83	88	93	97	***101***	***106***	110	
23	**44**	**48**	**52**	**57**	**62**	**66**	**71**	**75**	***80***	85	90	95	100	105	110	113	116
24	**45**	**50**	**54**	**58**	**63**	**68**	**73**	***78***	83	88	93	98	102	107	112	117	120
25	**46**	**51**	**55**	**60**	**65**	**70**	**75**	***80***	***85***	***90***	95	100	105	110	115	120	125
26	**47**	**53**	**57**	**61**	**66**	**72**	**78**	**81**	***86***	***91***	***96***	***101***	***106***	***111***	***116***	121	126
27	**48**	**54**	**58**	**63**	**68**	**73**	**79**	**83**	***88***	***93***	***98***	***103***	***108***	***113***	***118***	***123***	***128***
28	**49**	**56**	**60**	**64**	**69**	**75**	**81**	**85**	91	***96***	***101***	***106***	***111***	***116***	***121***	***126***	***131***
29	**50**	**57**	**61**	**66**	**71**	**76**	**82**	88	93	98	103	109	***114***	***120***	***125***	***130***	***135***
30	**51**	**58**	**63**	**67**	**72**	**78**	**84**	90	95	100	105	111	117	122	127	***132***	***138***
31	**52**	**59**	**64**	**69**	**74**	**79**	**85**	91	97	102	107	113	119	125	130	135	140

TABLE 1. The table contains the best upper bounds on $Z_{2,2}(m,n)$ up to our knowledge. Bold numbers indicate equality. An exact value is in italic shape if it was not reported by Guy in [23]. In some cases we did rely on the exact values reported by Guy. Possibly undiscovered inaccuracies there may result in inaccurate values here as well.

Note that we can dualize the above proposition: if we delete vertices that represent blocks, we may add an edge to each of the points all of whose neighbors have been removed. Next we give the exact value of $\ell^{\mathcal{D}}(i)$ in some cases.

REMARK 3.32. (1) *For any* $2-(v,k,1)$ *design* $\mathcal{D}$, $\ell^{\mathcal{D}}(i) = ik - \binom{i}{2}$ *for* $1 \le i \le 3$.

(2) *Let* $\mathcal{D} = \mathrm{PG}(2,q)$, $i \le q+1$. *Then* $\ell^{\mathcal{D}}(i) = i(q+1) - \binom{i}{2}$.

(3) *Let* $\mathcal{D} = \mathrm{AG}(2,q)$, $i \le q$. *Then* $\ell^{\mathcal{D}}(i) = iq - \binom{i}{2}$.

Proof. In general, as any two blocks of a $2-(v,k,1)$ design intersect in at most one point, $i \le k+1$ blocks cover at least $k + (k-1) + \ldots + (k-i+1) = ik - \binom{i}{2}$ points. This can be reached if and only if there exist i pairwise intersecting blocks in general position (no three of them have a common point). As $k \ge 2$, one can easily find three such blocks. In $\mathrm{PG}(2,q)$, a dual conic is well-known to be a set of $q+1$ lines in general position. One taken as the line at infinity, we obtain q lines in general position in $\mathrm{AG}(2,q)$. □

PROPOSITION 3.33. *Let* q *be a square prime power, and let* $v = q^2+q+1$, $w = q+\sqrt{q}+1$. *Suppose that* $1 \le c \le q-\sqrt{q}$, $0 \le d \le cw$, $0 \le h \le w-2$. *Then*

(1) $Z_{2,2}(v-c(w-1), v-d) \ge (v-c(w-1))(q+1) + c\sqrt{q} - d(q-\sqrt{q}+2-c)$;

(2) $Z_{2,2}(v-c(w-1)-h, v) \ge (v-c(w-1)-h)(q+1) + c\sqrt{q}$;

(3) $Z_{2,2}(v-cw, v-cw) \ge (v-cw)(q+1-c)$.

m	n	Lower b.	$Z_{2,2}$	Upper b.
8	8	24^{d}	24	$24^{\alpha=1,\,\beta=3}$
8	9	26^{d}	26	$26^{\alpha=1,\,\beta=4}$
8	10	28^{d}	28	28^{g}
9	9	29^{d}	29	$29^{\alpha=1,\,\beta=4}$
9	10	31^{d}	31	$31^{\alpha=1,\,\beta=4}$
9	11	33^{d}	33	33^{Aff}
10	10	34^{d}	34	$34^{\alpha=1,\,\beta=4}$
10	11	36^{d}	36	36^{Aff}
10	12	39^{d}	39	39^{Re}
10	13	40^{d}	40	$40^{\alpha=1,\,\beta=4}$
10	14	42^{d}	42	43^{Re}
10	15	44^{d}	44	44^{g}
10	16	46^{d}	46	46^{Re}
10	17	47^{d}	47	47^{g}
11	11	39^{d}	39	39^{Aff}
11	12	42^{d}	42	42^{Re}
11	13	44^{d}	44	44^{Re}
11	14	$45^{\mathrm{p,d}}$	45	46^{Re}
11	15	47^{d}	47	48^{Re}
11	16	50^{d}	50	50^{Re}
11	17	51^{d}	51	51^{g}
11	18	53^{Aff}	53	53^{Aff}
11	19	55^{d}	55	55^{g}
12	12	45^{d}	45	45^{Aff}
12	13	48^{d}	48	48^{Re}
12	14	$49^{\mathrm{p,d}}$	49	$49^{\alpha=1,\,\beta=5}$
12	15	51^{d}	51	52^{Re}
12	16	53^{d}	53	54^{Re}
12	17	55^{d}	55	55^{g}
12	18	57^{Aff}	57	57^{Aff}
12	19	60^{Aff}	60	60^{Re}
12	20	61^{d}	61	62^{Re}
12	21	63^{d}	63	64^{Re}
12	22	64^{d}	65	65^{g}
12	23	66^{d}	66	67^{Re}
12	24	68^{d}	68	68^{g}
13	13	52	52	52^{Re}
13	14	53^{p}	53	$53^{\alpha=1,\,\beta=5}$
13	15	54^{d}	55	55^{p}
13	16	57^{d}	57	58^{Re}
13	17	59^{d}	59	59^{g}
13	18	61^{Aff}	61	61^{Aff}
13	19	64^{Aff}	64	64^{Re}
13	20	$66^{\mathrm{B,d}}$	66	66^{Re}
13	21	67^{B}	67	68^{Re}
13	22	69^{d}	69	70^{Re}
13	23	71^{d}	71	72^{Re}
13	24	73^{d}	73	73^{g}
13	25	75^{d}	75	75^{g}
14	14	56^{B}	56	$56^{\alpha=1,\,\beta=4}$
14	15	58^{d}	58	$58^{\alpha=1,\,\beta=5}$
14	16	60^{d}	60	61^{g}
14	17	63^{d}	63	63^{g}
14	18	65^{Aff}	65	65^{Aff}
14	19	68^{Aff}	68	$68^{p=3}$
14	20	70^{d}	70	$70^{p=3}$
14	21	72^{B}	72	$72^{p=3}$
14	22	73^{d}	73	$74^{p=3}$
14	23	75^{d}	75	$76^{p=3}$
14	24	78^{d}	$78^{\star}$	$78^{p=3}$
14	25	80^{d}	$80^{\star}$	$80^{p=3}$
14	26	81^{d}	81	$82^{p=3}$
14	27	83^{d}	83	84^{Re}
14	28	84^{d}	85	86^{Re}
15	15	60^{d}	60	$62^{\alpha=1,\,\beta=5}$
15	16	64^{d}	$64^{\star}$	$64^{\alpha=-1,\,\beta=4}$
15	17	67^{d}	$67^{\star}$	67^{g}
15	18	69^{Aff}	69	69^{Aff}
15	19	72^{Aff}	72	72^{Aff}
15	20	75^{d}	75	75^{Re}
15	21	77^{B}	77	77^{Re}
16	20	80	80	80^{Re}

Table 2. The table contains the best lower and upper bounds on $Z_{2,2}(m,n)$ that can be obtained using the results presented in this paper. The parameters n and m range over the region where the general results 3.2 and 3.3 do not apply, but Guy published the exact values of $Z_{2,2}(m,n)$ in [23]. The marks are the following: $^{\mathrm{d}}$: deletion principle (e.g., 3.31); $^{\mathrm{B}}$: 3.33; $^{\mathrm{p}}$: 3.27 and 3.28; $^{\mathrm{Re}}$: [37], [25] and [32]; $^{p=k}$: Roman's bound 3.5 (with $p=k$); $^{\mathrm{g}}$: 3.15; $^{\mathrm{Aff}}$: 3.19; $^{\alpha=x,\beta=y}$: 3.20 (if $\alpha<0$, then the transposed version); *: the value is inaccurate in [23]. If more than one bounds give the stated result, we refer to the historically first one.

Proof. Let $\mathrm{PG}(2,q) = (\mathcal{P},\mathcal{L})$, and let $\mathcal{B}_1 = (\mathcal{P}_1,\mathcal{L}_1),\dots,\mathcal{B}_c = (\mathcal{P}_c,\mathcal{L}_c)$ be c pairwise disjoint Baer subplanes in it. Let $\mathcal{P}_0 = \cup_{i=1}^c \mathcal{P}_i$, $\mathcal{L}_0 = \cup_{i=1}^c \mathcal{L}_i$.

(1) Define $G = (A,B)$ in the following way. Let $A = \mathcal{P}\backslash\mathcal{P}_0 \cup \{\mathcal{B}_1,\dots,\mathcal{B}_c\}$ ($|A| = v-cw+c$), $B = \mathcal{L}$. The edges between $A\cap\mathcal{P}$ and B are those defined by $\mathrm{PG}(2,q)$; furthermore, connect the vertex $\mathcal{B}_i$ to all the vertices of $\mathcal{L}_i \subset B$, $1\le i\le c$. (That is, we contract the points of the Baer subplanes.) As any two lines of $\mathcal{L}_i$ had an intersection in $\mathcal{P}_i$, we do not create a C_4. Note that every $\mathcal{P}_i$ is a blocking set, so every line not in $\mathcal{L}_0$ looses precisely c neighbors. Thus the $v-cw$ vertices of $A\cap\mathcal{P}$ have degree $q+1$, the c new vertices have degree $w = q+\sqrt{q}+1$, thus there are $(v-cw+w)(q+1)+c\sqrt{q}$ edges in G. Let $\ell\in\mathcal{L}_i\subset\mathcal{L}_0$. Then $|\ell\cap\mathcal{P}_j|$ equals one for all $1\le j\le c$ except for $j=i$, in which case it equals $\sqrt{q}+1$. Hence $\deg(\ell) = q+1-\sqrt{q}-(c-1)$ in G. There are $c(q+\sqrt{q}+1)$ lines in $\mathcal{L}_0$, so we may delete any d of them to obtain a graph G' with the stated parameters.

(2) Every point of $A\cap\mathcal{P}$ has degree $q+1$ in G, so we may delete any h of them. It is not worth deleting more than $w-2$ points since we can contract another Baer subplane.

(3) Consider the graph induced by $\mathcal{P}\setminus\mathcal{P}_0$ and $\mathcal{L}\setminus\mathcal{L}_0$. Here every vertex has degree $q+1-c$. □

3.4. **Some remarks and open problems.** For small values of m and n, we have computed the best results one can obtain on C_4-free graphs using these ideas. These values can be found in Tables 1 and 2.

Illés and Krarup [26] use the formulation of Zarankiewicz's problem in terms of integer programming. They introduce Problem (R), that is, to find

$$r(n) = \max\left\{\sum_{j=1}^n x_j\colon \sum_{j=1}^n \binom{x_j}{2} \le \binom{n}{2},\ \text{where } x_j\ge 0, x_j\in\mathbb{Z} \text{ for all } 1\le j\le n\right\}.$$

The cost of a solution $\mathbf{x} = (x_1,\dots,x_n)$ is $\sum_j \binom{x_j}{2}$. They call a solution $\mathbf{x}$ *realizable* if there exists an $n\times n$ $J_2 = \begin{pmatrix}1&1\\1&1\end{pmatrix}$-free $0-1$ matrix in which the jth column contains x_j ones. In Remark 6, page 129 they claim: "It is conjectured that a necessary condition for realizability is that the corresponding optimal solution to (R) is a least cost solution." Note that the transpose of an optimal $n\times n$ J_2-free $0-1$ matrix is also an optimal matrix of that kind, hence the conjecture claims that the rows also correspond to a least cost optimal solution. As $\binom{x}{2}$ is convex, the cost of a solution is minimal if and only if $|x_i - x_j|\le 1$ for all $1\le i<j\le n$. In terms of C_4-free bipartite graphs of size (n,n), this is equivalent with saying that if such a graph has the maximum possible number of edges, then the degrees inside both classes must differ by at most one. This conjecture is false. Let $n=8$. Then $Z_{2,2}(8,8) = 24$. Let $G=(A,B)$ be the incidence graph of the Fano plane, and let $a\in A$ and $b\in B$ two non-adjacent vertices. Add two new vertices, u and v to A and B, respectively, and let $\{u,v\},\{a,v\},\{u,b\}$ be edges. The resulting graph is C_4-free, has $21+3=24$ edges, and the degrees in both classes take the values $2,3$ and 4. However, deleting a line l and a point P not on l, together with all the points and lines incident with l and P from $\mathrm{PG}(2,3)$, we obtain a three-regular bipartite graph on $(8,8)$ vertices.

We say that a vertex class of a bipartite graph is *nearly regular*, if the degrees in that class differ by at most one. We end this section by posing some questions that, to the best of our knowledge, are open. Let $2 \le t \le n \le m$ be arbitrary integers.

QUESTION 3.34. *Does there exist an extremal $K_{t,t}$-free graph on (n,n) vertices whose classes are both nearly regular?*

QUESTION 3.35. *Does there exist an extremal $K_{t,t}$-free graph on (n,m) vertices with at least one nearly regular class?*

Corollary 3.27 shows that extremal C_4-free bipartite graphs on (n^2+n+1, n^2+n+2) vertices, n a power of a prime, can not have two nearly regular classes.

QUESTION 3.36 (See [21]). *Is it true that $Z_{t,t}(n,m) \le Z_{t,t}\left(\lfloor (n+m)/2 \rfloor, \lceil (n+m)/2 \rceil\right)$?*

References

[1] , M. Abreu, G. Araujo-Pardo, C. Balbuena, D. Labbate, Families of small regular graphs of girth 5. *Discrete Math.* **312** (2012), no. 18, 2832–2842.

[2] M. Abreu, M. Funk, D. Labbate, V. Napolitano, On (minimal) regular graphs of girth 6. *Australas. J. Combin.* **35** (2006), 119–132.

[3] M. Abreu, M. Funk, D. Labbate, V. Napolitano, Deletions, extensions, and reductions of elliptic semiplanes. *Innov. Incidence Geom.* **11** (2010), 139–155.

[4] N. Alon, L. Rónyai, T. Szabó, Norm-graphs: Variations and Applications. *Journal of Combinatorial Theory, (Series B)*, **76** (1999), 280–290.

[5] G. Araujo-Pardo, C. Balbuena, Constructions of small regular bipartite graphs of girth 6. *Networks* **57** (2011), no. 2, 121–127.

[6] G. Araujo-Pardo, C. Balbuena, T. Héger, Finding small regular graphs of girths 6, 8 and 12 as subgraphs of cages. *Discrete Math.* **310** (2010), no. 8, 1301–1306.

[7] G. Araujo, D. González, J. J. Montellano-Ballesteros, O. Serra, On upper bounds and connectivity of cages. *Australas. J. Combin.* **38** (2007), 221–228.

[8] R. D. Baker, Elliptic semi-planes I. Existence and classification. *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing* (Louisiana State Univ., Baton Rouge, La., 1977), pp. 61–73. Congressus Numerantium, No. XIX, Utilitas Math., Winnipeg, Man., 1977.

[9] C. Balbuena, P. García-Vázquez, X. Marcote, J. C. Valenzuela, Extremal $K_{(s,t)}$-free bipartite graphs. *Discrete Math. Theor. Comput. Sci.* **10** (2008), no. 3, 35–48.

[10] L. Beukemann, K. Metsch, Regular graphs constructed from the classical generalized quadrangle $Q(4,q)$. *J. Combin. Des.* **19** (2011), Issue 1, 70–83.

[11] W. G. Brown, On Hamiltonian regular graphs of girth six. *J. London Math. Soc.* **42** (1967), 514–520.

[12] D. E. Bryant, Hung-Lin Fu, C_4-saturated bipartite graphs. *Discrete Math.* **259** (2002), 263–268.

[13] C. J. Colbourn, J. H. Dinitz, Handbook of Combinatorial Designs (2nd Edition ed.) *Boca Raton: Chapman & Hall / CRC* (2007)

[14] K. Čulík, Teilweise Lösung eines verallgemeinerten Problem von K. Zarankiewicz. *Ann. Soc. Polon. Math.* **3** (1956), 165–168.

[15] G. Exoo, R. Jajcay, Dynamic cage survey. *The electronic journal of combinatorics* **18** (2011), #DS16.

[16] M. Funk, D. Labbate, V. Napolitano, Tactical (de-)compositions of symmetric configurations. *Discrete Math.* **309** (2009), no. 4, 741–747.

[17] Z. Füredi, An upper bound on Zarankiewicz' problem. *Combinatorics, Probability and Computing* **5** (1996), 29-33.

[18] Z. Füredi, New asymptotics for bipartite Turán numbers. *Journal of Combinatorial Theory, Series A* **75** (1996), no. 1, 141–144.

[19] A. Gács, T. Héger, On geometric constructions of (k, g)-graphs. *Contributions to Discrete Mathematics* **3** (2008) 63–80.

[20] A. Gács, T. Héger, Zs. Weiner, On regular graphs of girth six arising from projective planes. *European Journal of Combinatorics* **34** (2013), No. 2, pp. 285–296.

[21] W. Goddard, M. Henning,O. Oellermann, Bipartite Ramsey numbers and Zarankiewicz numbers. *Discrete Math.* **219** (2000), no. 1-3, 85–95.

[22] J. Griggs, C. Ho, On the half-half case of the Zarankiewicz problem. *Discrete Math.* **249** (2002), no. 1-3, 95–104.

[23] R. K. Guy, The many faceted problem of Zarankiewicz in: "The Many Facets of Graphs Theory". Lecture Notes in Maths **110**, Springer, (1969), 129–148.

[24] J. W. P. Hirschfeld, Projective geometries over finite fields. *Clarendon Press, Oxford,* 1979, 2nd edition, 1998.

[25] C. Hyltén-Cavallius, On a combinatorial problem. *Colloq. Math.* **6** (1958), 59–65.

[26] T. Illés, J. Krarup, Maximum 4-block-free matrices and knapsack-type relaxations. *PU.M.A.* **10** (1999), No. 2, pp. 115–131.

[27] R. W. Irving, A bipartite Ramsey problem and the Zarankiewicz numbers. *Glasgow Math. J.* **19** (1978), 13–26.

[28] F. Kárteszi, Piani finiti ciclici come risoluzioni di un certo problema di minimo (Italian). *Boll. Un. Mat. Ital. (3)* **15** (1960) 522–528.

[29] F. Kárteszi, Finite Möbius planes as solutions of a combinatorial extremum problem (Hungarian). *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **21** (1973), 73–76.

[30] F. Kárteszi, Su certi grafi estremali (Italian). *Atti del Convegno di Geometria Combinatoria e sue Applicazioni* (Univ. Perugia, Perugia (1970)), pp. 301–305 Ist. Mat., Univ. Perugia, Perugia, (1971).

[31] J. Kollár, L. Rónyai, T. Szabó, Norm-graphs and Bipartite Turn Numbers. *Combinatorica*, **16** (1996), no. 3, 399–406.

[32] T. Kővári, V. T. Sós, P. Turán, On a problem of K. Zarankiewicz. *Colloq. Math.* **3** (1954) 50–57.

[33] F. Lazebnik, V. A. Ustimenko, A. J. Woldar, New upper bounds on the order of cages. *Electron. J. Combin.* **4** (1997), no. 2, Research Paper 13, approx. 11 pp. (electronic).

[34] K. Metsch, On the maximum size of a maximal partial plane. *Rend. Mat. Appl.* (7) **12** (1992), no. 2, 345–355.

[35] M. Miller, J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem. *The electronic journal of combinatorics* (2005), #DS14.

[36] V. Nikiforov, A contribution to the Zarankiewicz problem. *Linear Algebra Appl.* **432** (2010), no. 6, 1405–1411.

[37] I. Reiman, Über ein Problem von Zarankiewicz (in German), *Acta. Math. Acad. Sci. Hungar.* **9** (1958), 269–273.

[38] I. Reiman, Su una proprietàdei 2-disegni (in Italian). *Rend. Mat.* (6) **1** (1968) 75–81.

[39] S. Roman, A Problem of Zarankiewicz. *Journal of Combinatorial Theory* **Vol. 18** (No. 2) (1975).

[40] M. Svéd, Baer subspaces in the n-dimensional projective space, *Combinatorial mathematics, X* (Adelaide, 1982), 375–391, *Lecture Notes in Math.*, **1036**, Springer, Berlin, 1983.

[41] J. Totten, Embedding the complement of two lines in a finite projective plane, *J. Austral. Math. Soc. Ser. A* **22** (1976), no. 1, 27–34.

[42] K. Zarankiewicz, Problem of P101. *Colloq. Math.*, **2** (1951), p. 301.

Authors addresses:

Gábor Damásdi, Tamás Héger, Tamás Szőnyi:
Eötvös Loránd University, Faculty of Science, Institute of Mathematics
1117 Budapest, Pázmány Péter sétány 1/C, Budapest, HUNGARY.
`gabor.damasdi@gmail.com; hetamas@cs.elte.hu; szonyi@cs.elte.hu`