Linear orders on rings

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Abstract. The general ideas introduced in [8] are adopted in order to investigate the quasi cones and the cones of a ring. Using the finite extension property for cones, we answer the question, when a partial order in a partially ordered ring has a compatible linear extension (equivalently, when the positive cone is contained in a full cone). It turns out, that if there is no such extension, then it is caused by a finite system of polynomial like equations satisfied by some elements of a certain finite subset of the ring and some positive elements. As a consequence, we prove that a partial order can be linearly extended if and only if it can be linearly extended on every finitely generated subring.

1. INTRODUCTION

A central problem in the theory of ordered algebraic structures is to find necessary and sufficient conditions for the existence of a compatible linear extension of a given compatible partial order. The first step to answer the above question in arbitrary algebraic structures was taken in [8]. Surprisingly, the general approach presented in this paper provided new results even in such classical structures as semigroups and groups. Unfortunately, the results of [8] can not be directly applied to obtain linear extensions of partial orders in rings.

A partial order $\leq$ on the base set R of a ring $(R, +, \cdot, 0)$ is called compatible, if $x \leq y$, $x' \leq y'$ and $0 \leq z$ imply that

$$x + x' \leq y + y' \text{ and } xz \leq yz \ , \ zx \leq zy$$

for all $x, x', y, y', z \in R$. A partially ordered ring is a pair $(R, \leq)$ of a ring R and a compatible partial order $\leq$ on R. Since the multiplication of the ring does not entirely preserve the partial order relation, the use of the main theorems in [8] is impossible.

There is a one to one correspondence between the compatible partial orders and the cones of a ring. A similar correspondence can be established between the compatible quasi orders an the so called quasi cones. The main aim of the present paper is to adopt the general ideas introduced in [8], in order to investigate the quasi cones and the cones of a ring. Instead of the linear extensions of a partial order, we shall consider the full extensions of a cone. Our development is self contained and based on the use of the so called finite extension property of a (quasi) cone. We shall see, that if a cone can not be extended to a full cone, then it is caused by a finite

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subset of the ring and a certain system of polynomial like equations satisfied by some elements of this finite set and of the given cone. As a consequence, we derive a complete answer to the question of existence of a compatible linear extension in a partially ordered ring. The remarkable fact, that a partial order has a compatible linear extension if and only if it has a compatible linear extension on every finitely generated subring is also proved.

Any compatible linear order can be simply viewed as a compatible linear extension of the trivial partial order (identity). In view of this observation, we obtain a necessary and sufficient condition for the existence of a compatible linear order on an arbitrary ring. Note, that for domains (unitary rings without zero divisors), we have classical results about the existence of a compatible linear order: it is equivalent to the fact, that the sum of (non-zero) even elements can not be equal to zero (see [5]). A more refined result in [9] states, that the existence of a compatible linear order on a skew field is equivalent to the fact, that the sum of finite products of (non-zero) square elements can not be equal to -1. Unfortunately, our condition in Theorem 2.11. contains not only one prohibited equation (as the above cited conditions), but a prohibited system of equations. An obvious reason of this situation is the different approach, probably an other one is the possible presence of zero divisors.

2. CONES WITH THE FINITE EXTENSION PROPERTY

A subset $Q \subseteq R$ in a ring $(R, +, \cdot, 0)$ is called quasi cone, if $0 \in Q$ and Q is closed with respect to the addition and the multiplication of the ring, i.e. $x + y \in Q$ and $xy \in Q$ holds for any choice of the elements $x, y \in Q$. A quasi cone $Q \subseteq R$ is a cone, if $x \in Q$ and $-x \in Q$ imply $x = 0$ for all $x \in R$. If $Q \subseteq R$ is a quasi cone and $Q \subseteq P$ for some cone $P \subseteq R$, then Q is also a cone. The following properties of a quasi cone are equivalent:

(1) For $m \geq 1$ and $x_1, x_2, ..., x_m \in Q$ the equality $x_1 + x_2 + ... + x_m = 0$ implies that $x_1 = x_2 = ... = x_m = 0$.
(2) Q is a cone.

A well-known fact is, that any cone P induces a compatible partial order $\leq_P$ on R:

$$x \leq_P y \Longleftrightarrow y - x \in P.$$

Clearly, any compatible partial order $\leq$ on R is induced by a unique cone, which is the cone of its positive elements: $\leq$ coincides with $\leq_P$, where

$$P = \{x \in R \mid 0 \leq x\}.$$

Let P' be an other cone of R, then $\leq_{P'}$ is an extension of $\leq_P$ if and only if $P \subseteq P'$. If $S \subseteq R$ is a subring, then $P \cap S$ is a cone of S and the induced partial order $\leq_{P \cap S}$ on S is the same as $\leq_P$ considered between the elements of S. We say that a cone $P \subseteq R$ is full (or linear) if either $x \in P$ or $-x \in P$ holds for all $x \in R$. It is easy to see, that a compatible partial order $\leq$ on R is a linear order if and only if the cone of its positive elements is full.

We shall make use of the following operations on a subset $A \subseteq R$ of a ring R: A^{Σ} denotes the set of all sums $a_1 + a_2 + ... + a_m$ and A^{Π} denotes the set of all products $a_1 a_2 ... a_m$, where $m \geq 1$ and $a_i \in A$ for all $1 \leq i \leq m$. The quasi cone $[A] \subseteq R$

generated by A is the intersection of all quasi cones of R containing A, that is $[A] = (A^{\Pi})^{\Sigma}$ (if $a \in R$, then we write $[a]$ instead of $[\{a\}]$). Let

$$\mathrm{Qucone}(R) = \left\{Q \subseteq R \mid 0 \in Q \ , \ Q^{\Sigma} \subseteq Q \text{ and } Q^{\Pi} \subseteq Q\right\}$$

denote the set of all quasi cones of R, then the relation $\subseteq$ provides a natural lattice structure on Qucone(R). The infimum and supremum of the quasi cones $Q_i \in$Qucone(R), $i \in I$ are as follows

$$\underset{i\in I}{\wedge} Q_i = \underset{i\in I}{\cap} Q_i \text{ and } \underset{i\in I}{\vee} Q_i = [\underset{i\in I}{\cup} Q_i].$$

Let $H \subseteq R \setminus \{0\}$ be a subset, a function $\delta : H \longrightarrow R$ is called an orientation of H, if $\delta(a) \in \{a, -a\}$ for all $a \in H$. If $P \subseteq R$ is a cone, then we say that $\delta : H \longrightarrow R$ is a P-extending orientation, if for $z \in P$ and for any choice of functions

$$u_i : \{1, 2, ..., n_i, n_i + 1\} \longrightarrow P \cup \{1_*\} \text{ and } h_i : \{1, 2, ..., n_i\} \longrightarrow H, \ 1 \leq i \leq k$$

an equality of the form

$$z + \sum_{i=1}^{k} u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i + 1) = 0$$

implies that

$$z = 0 \text{ and } u_i(1)h_i(1)u_i(2)h_i(2)...u_i(n_i)h_i(n_i)u_i(n_i + 1) = 0$$

for all $1 \leq i \leq k$ (if $u_i(t) = 1_*$ then $u_i(t)$ is the empty symbol in the above products). Since

$$u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i + 1) = \\ = \pm u_i(1)h_i(1)u_i(2)h_i(2)...u_i(n_i)h_i(n_i)u_i(n_i + 1),$$

we obtain that each summand of the above sum is zero.

2.1.Lemma. *If R is a ring, $Q \subseteq R$ is a quasi cone, $H \subseteq R \setminus \{0\}$ a subset and $\delta : H \longrightarrow R$ is an orientation, then the following conditions are equivalent.*

(1) *The quasi cone $Q \subseteq R$ is a cone and δ is Q-extending.*
(2) *The supremum*

$$Q_\delta(H) = Q \vee \left(\underset{a\in H}{\vee}[\delta(a)]\right)$$

in Qucone(R) *is a cone.*
(3) *There exists a cone $P \subseteq R$ such that $Q \cup \delta(H) \subseteq P$.*

Proof. (1) $\Longrightarrow$ (2) : Since $Q_\delta(H)$ is a quasi cone, it is enough to prove that $x \in Q_\delta(H)$ and $-x \in Q_\delta(H)$ imply $x = 0$. It is easy to verify, that

$$Q_\delta(H) = \left[Q \cup \left(\underset{a\in H}{\cup}[\delta(a)]\right)\right] = [Q \cup \{\delta(a) \mid a \in H\}] = \left((Q \cup \{\delta(a) \mid a \in H\})^{\Pi}\right)^{\Sigma}.$$

The cone Q is multiplicatively closed, whence we obtain that an element of

$$(Q \cup \{\delta(a) \mid a \in H\})^{\Pi}$$

is either in Q or is of the form

$$v(1)g_1v(2)g_2...v(n)g_nv(n + 1),$$

where $n \geq 1$ is an integer,

$$v : \{1, 2, ..., n, n + 1\} \longrightarrow Q \cup \{1_*\}$$

is a function and $g_t = \delta(a_t)$ for some $a_t \in H$. Thus an element of $Q_\delta(H)$ can be written as

$$x = u + \sum_{i=1}^{k} u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i+1)$$

with $u \in Q$ and functions

$$u_i : \{1, 2, ..., n_i, n_i + 1\} \longrightarrow Q \cup \{1_*\}\ ,\ h_i : \{1, 2, ..., n_i\} \longrightarrow H,\ 1 \leq i \leq k.$$

We also have

$$-x = u' + \sum_{j=1}^{l} u'_j(1)\delta(h'_j(1))u'_j(2)\delta(h'_j(2))...u'_j(m_j)\delta(h'_j(m_j))u'_j(m_j+1)$$

with $u' \in Q$ and functions

$$u'_j : \{1, 2, ..., m_j, m_j + 1\} \longrightarrow Q \cup \{1_*\}\ ,\ h'_j : \{1, 2, ..., m_j\} \longrightarrow H,\ 1 \leq j \leq l.$$

Since $u + u' \in Q$, the Q-extending property of δ and

$$0 = x + (-x) = (u + u')+$$
$$+\left(\sum_{i=1}^{k} u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i+1)\right)+$$
$$+\left(\sum_{j=1}^{l} u'_j(1)\delta(h'_j(1))u'_j(2)\delta(h'_j(2))...u'_j(m_j)\delta(h'_j(m_j))u'_j(m_j+1)\right)$$

ensure, that $u + u' = 0$ and

$$u_i(1)h_i(1)u_i(2)h_i(2)...u_i(n_i)h_i(n_i)u_i(n_i+1) = 0,$$
$$u'_j(1)h'_j(1)u'_j(2)h'_j(2)...u'_j(m_j)h'_j(m_j)u'_j(m_j+1) = 0$$

for all $1 \leq i \leq k$ and $1 \leq j \leq l$. Now $u, u' \in Q$, whence we obtain first $u = u' = 0$ and then $x = 0$.

(2) $\Longrightarrow$ (3) : Since $Q \cup \delta(H) \subseteq Q_\delta(H)$, we can take $P = Q_\delta(H)$.

(3) $\Longrightarrow$ (1) : The fact, that Q is a cone, immediately follows from $Q \subseteq P$. Consider an element $z \in Q$ and the functions

$$u_i : \{1, 2, ..., n_i, n_i + 1\} \longrightarrow Q \cup \{1_*\}\ ,\ h_i : \{1, 2, ..., n_i\} \longrightarrow H,\ 1 \leq i \leq k$$

such that

$$z + \sum_{i=1}^{k} u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i+1) = 0.$$

Now $Q \cup \delta(H) \subseteq P$ implies that $z \in P$ and that each (non-empty) factor of the product

$$u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i+1)$$

is in P. Thus the product itself is also an element of P. The cone properties of P ensure, that all summands (including z) in the above sum are equal to 0. Hence

$$u_i(1)h_i(1)u_i(2)h_i(2)...u_i(n_i)h_i(n_i)u_i(n_i+1) = 0$$

for all $1 \leq i \leq k$ and so δ is Q-extending. □

The quasi cone $Q \subseteq R$ has the finite extension property (FEP), if for any finite subset $H \subseteq R \setminus \{0\}$, there exists a cone $P \subseteq R$ such that $Q \subseteq P$ and either $a \in P$

or $-a \in P$ holds for all $a \in H$. In view of Lemma 2.1., it is easy to see that the FEP of Q is equivalent to the following equivalent conditions:

- The quasi cone Q is a cone and for any finite subset $H \subseteq R \setminus \{0\}$, there exists a Q-extending orientation $\delta : H \longrightarrow R$.
- For any finite subset $H \subseteq R \setminus \{0\}$, there exists an orientation $\delta : H \longrightarrow R$ such that the supremum
$$Q_\delta(H) = Q \vee \left(\bigvee_{a \in H} [\delta(a)] \right)$$
in Qucone(R) is a cone.
- For any finite subset $H \subseteq R \setminus \{0\}$, there exists a cone $P \subseteq R$ such that $Q \cup \delta(H) \subseteq P$ for some orientation $\delta : H \longrightarrow R$.

2.2.Lemma. *Let $0 \neq a \in R$ and $Q \subseteq R$ be a quasi cone of the ring R with the FEP. Then one (or both) of the following quasi cones has the FEP:*
$$Q_a = Q \vee [a] \text{ and } Q_{-a} = Q \vee [-a].$$

Proof. Suppose that neither Q_a nor Q_{-a} has the FEP. Then we have finite subsets $H_1, H_2 \subseteq R \setminus \{0\}$, such that for any choice of the orientations $\delta' : H_1 \longrightarrow R$ and $\delta'' : H_2 \longrightarrow R$ the sets $Q_a \cup \delta'(H_1)$ and $Q_{-a} \cup \delta''(H_2)$ are not contained in the cones of R. Consider the finite subset
$$H_1 \cup H_2 \cup \{a\} \subseteq R \setminus \{0\},$$
the FEP of Q provides a cone $P \subseteq R$, such that
$$Q \cup \delta(H_1 \cup H_2 \cup \{a\}) \subseteq P$$
for some orientation
$$\delta : H_1 \cup H_2 \cup \{a\} \longrightarrow R.$$
If $\delta(a) = a$, then $a \in P$ implies that $[a] \subseteq P$, whence we obtain first $Q \cup [a] \subseteq P$ and next $Q_a = Q \vee [a] \subseteq P$. Now we have $Q_a \cup \delta'(H_1) \subseteq P$ with $\delta' = \delta \restriction H_1$, a contradiction. If $\delta(a) = -a$, then we get $Q_{-a} \cup \delta''(H_2) \subseteq P$ with $\delta'' = \delta \restriction H_2$, an other contradiction. $\square$

2.3.Lemma. *Let R be a ring and $Q_w \subseteq R$, $w \in W$ is a chain (with respect to $\subseteq$) of quasi cones of R, such that all Q_w have the FEP. Then $\bigcup_{w \in W} Q_w$ is a quasi cone of R with the FEP.*

Proof. It is easy to see that $\bigcup_{w \in W} Q_w$ is a cone of R, so we have to prove only the FEP. Suppose that $Q = \bigcup_{w \in W} Q_w$ has no FEP, then there exists a finite set $H \subseteq R \setminus \{0\}$ such that there is no Q-extending orientation $\delta : H \longrightarrow R$. In consequence, for each orientation $\delta : H \longrightarrow R$, we can find an element $z^\delta \in Q$ and functions
$$u_i^\delta : \{1, 2, ..., n_i(\delta), n_i(\delta)+1\} \longrightarrow Q \cup \{1_*\} \ , \ h_i^\delta : \{1, 2, ..., n_i(\delta)\} \longrightarrow H, \ 1 \leq i \leq k(\delta)$$
such that
$$z^\delta + \sum_{i=1}^{k(\delta)} u_i^\delta(1)\delta(h_i^\delta(1))u_i^\delta(2)\delta(h_i^\delta(2))...u_i^\delta(n_i(\delta))\delta(h_i^\delta(n_i(\delta)))u_i^\delta(n_i(\delta)+1) = 0$$

and

$$z^{\delta} \neq 0 \text{ or } u_t^{\delta}(1)h_t^{\delta}(1)u_t^{\delta}(2)h_t^{\delta}(2)...u_t^{\delta}(n_t(\delta))h_t^{\delta}(n_t(\delta))u_t^{\delta}(n_t(\delta)+1) \neq 0$$

for some $1 \leq t \leq k(\delta)$. Since the function u_i^{δ} takes only finitely many values in $Q \cup \{1_*\}$, there exists an index $w(\delta, i) \in W$ with the property, that

$$u_i^{\delta} : \{1, 2, ..., n_i(\delta), n_i(\delta)+1\} \longrightarrow Q_{w(\delta,i)} \cup \{1_*\}.$$

Now $z^{\delta} \in Q_{w(\delta,\Diamond)}$ for some $w(\delta, \Diamond) \in W$ and

$$\left\{Q_{w(\delta,i)} \mid \delta \text{ is an orientation of } H,\ i \in \{\Diamond, 1, 2, ..., k(\delta)\}\right\}$$

is a finite subset of the chain $\{Q_w \mid w \in W\}$, thus we can find an orientation $\sigma : H \longrightarrow R$ and an element $j \in \{\Diamond, 1, 2, ..., k(\sigma)\}$, such that $Q_{w(\delta,i)} \subseteq Q_{w(\sigma,j)}$ for any choice of δ and $i \in \{\Diamond, 1, 2, ..., k(\delta)\}$. Now

$$z^{\delta} \in Q_{w(\sigma,j)} \text{ and } u_i^{\delta} : \{1, 2, ..., n_i(\delta), n_i(\delta)+1\} \longrightarrow Q_{w(\sigma,j)} \cup \{1_*\}$$

for all δ and $1 \leq i \leq k(\delta)$. It follows, that there is no orientation $\delta : H \longrightarrow R$, which is $Q_{w(\sigma,j)}$-extending. Thus we obtained the contradiction, that $Q_{w(\sigma,j)}$ does not have the FEP. □

2.4.Theorem. *Let $P \subseteq R$ be a cone of the ring R, then the following conditions on P are equivalent:*

(1) *P has the FEP.*
(2) *P has a full extension, i.e. there exists a full cone $T \subseteq R$ of R such that $P \subseteq T$.*

Proof. $(1) \Longrightarrow (2)$: Take

$$\mathcal{Q} = \{Q \mid P \subseteq Q \subseteq R,\ Q \text{ is a quasi cone of } R \text{ with the FEP}\},$$

then Lemma 2.3. enables us to use Zorn's Lemma to obtain a maximal element T in the partially ordered set $(\mathcal{Q}, \subseteq)$. We claim that T is a full cone. Since the elements of $\mathcal{Q}$ are cones, it is enough to prove that either $a \in T$ or $-a \in T$ holds for all $a \in R$. If $0 \neq a \in R$, then Lemma 2.2. ensures that one of the joins

$$T_a = T \vee [a] \text{ and } T_{-a} = T \vee [-a]$$

is a quasi cone of R with the FEP. Since $P \subseteq T \subseteq T_a$ and $P \subseteq T \subseteq T_{-a}$, we obtain that either $T_a \in \mathcal{Q}$ or $T_{-a} \in \mathcal{Q}$. The maximality of T in $(\mathcal{Q}, \subseteq)$ implies that $T_a = T$ in the first case and $T_{-a} = T$ in the second case. In view of $a \in T_a$ and $-a \in T_{-a}$, we get that T is full.
$(2) \Longrightarrow (1)$: Now $P \subseteq T$ and for any subset $H \subseteq R \setminus \{0\}$, we have either $a \in T$ or $-a \in T$ for all $a \in H$, thus P has the FEP. □

2.5.Corollary. *Let R be a ring, then the following conditions are equivalent:*

(1) *The zero cone $\{0\}$ of R has the FEP.*
(2) *There exists a full cone of R.*

Using P-extending orientations, the reformulation of Theorem 2.4. gives an answer to the problem of the existence of a (compatible) linear extension of a compatible partial order $\leq$ of a ring R.

2.6.Theorem. *Let $(R,\leq)$ be a partially ordered ring with positive cone P, then the following conditions are equivalent:*

(1) *For any finite subset $H\subseteq R\backslash\{0\}$, there exists an orientation $\delta : H\longrightarrow R$ such that for any choice of an element $z\in P$ and functions*

$$u_i : \{1,2,...,n_i,n_i+1\}\longrightarrow P\cup\{1_*\}\ ,\ h_i : \{1,2,...,n_i\}\longrightarrow H,\ 1\leq i\leq k$$

an equality of the form

$$z+\sum_{i=1}^{k}u_i(1)\delta(h_i(1))u_i(2)\delta(h_i(2))...u_i(n_i)\delta(h_i(n_i))u_i(n_i+1)=0$$

implies that

$$z=0 \text{ and } u_i(1)h_i(1)u_i(2)h_i(2)...u_i(n_i)h_i(n_i)u_i(n_i+1)=0$$

for all $1\leq i\leq k$.

(2) *There exists a full cone $T\subseteq R$ of R such that $P\subseteq T$ (thus the compatible linear order $\leq_T$ is an extension of $\leq$).*

2.7.Corollary. *Let R be a ring, then the following conditions are equivalent:*

(1) *For any finite subset $H\subseteq R\backslash\{0\}$, there exists an orientation $\delta : H\longrightarrow R$ such that for any choice of functions $h_i : \{1,2,...,n_i\}\longrightarrow H$, $1\leq i\leq k$ an equality of the form*

$$\sum_{i=1}^{k}\delta(h_i(1))\delta(h_i(2))...\delta(h_i(n_i))=0$$

implies that

$$h_i(1)h_i(2)...h_i(n_i)=0$$

for all $1\leq i\leq k$.

(2) *There exists a compatible linear order $\leq$ of R.*

2.8.Theorem. *Let $P\subseteq R$ be a cone of the ring R, then the following conditions are equivalent:*

(1) *For any finitely generated subring $S\subseteq R$ of R, the cone $P\cap S$ of S has a full extension, i.e. there exists a full cone $T_S\subseteq S$ of S such that $P\cap S\subseteq T_S$.*

(2) *The cone P has a full extension, i.e. there exists a full cone $T\subseteq R$ of R such that $P\subseteq T$.*

Proof. $(1)\Longrightarrow(2)$: Suppose that there is no full cone T of R with $P\subseteq T$. The application of Theorem 2.6. provides a finite subset $H\subseteq R$, such that for each orientation $\delta : H\longrightarrow R$ we can find an element $z^\delta\in P$ and functions

$$u_i^\delta : \{1,2,...,n_i(\delta),n_i(\delta)+1\}\longrightarrow P\cup\{1_*\}\ ,\ h_i^\delta : \{1,2,...,n_i(\delta)\}\longrightarrow H,\ 1\leq i\leq k(\delta)$$

such that

$$z^\delta+\sum_{i=1}^{k(\delta)}u_i^\delta(1)\delta(h_i^\delta(1))u_i^\delta(2)\delta(h_i^\delta(2))...u_i^\delta(n_i(\delta))\delta(h_i^\delta(n_i(\delta)))u_i^\delta(n_i(\delta)+1)=0$$

and

$$z^\delta\neq 0 \text{ or } u_t^\delta(1)h_t^\delta(1)u_t^\delta(2)h_t^\delta(2)...u_t^\delta(n_t(\delta))h_t^\delta(n_t(\delta))u_t^\delta(n_t(\delta)+1)\neq 0$$

for some $1 \leq t \leq k(\delta)$. Now consider the subring $S \subseteq R$ of R, generated by the finite set

$$\{z^\delta \mid \delta : H \longrightarrow R \text{ is an orientation}\} \cup \{u_i^\delta(j) \mid 1 \leq j \leq n_i(\delta)+1, 1 \leq i \leq k(\delta)\} \cup H,$$

then $H \longrightarrow S$ and $H \longrightarrow R$ orientations coincide, $z^\delta \in P \cap S$ for all orientations $\delta : H \longrightarrow S$ and

$$u_i^\delta : \{1, 2, ..., n_i(\delta), n_i(\delta)+1\} \longrightarrow (P \cap S) \cup \{1_*\} \text{ for } 1 \leq i \leq k(\delta).$$

In view of Theorem 2.6., the finite set H and the above equalities ensure, that there is no full cone $T_S \subseteq S$ of S with $P \cap S \subseteq T_S$. Thus we obtained a contradiction.
$(2) \Longrightarrow (1)$: We can take $T_S = T \cap S$. □

2.9.Corollary. *Let $(R, \leq)$ be a partially ordered ring, then the following conditions are equivalent:*

(1) *For any finitely generated subring $S \subseteq R$ of R, the partial order $\leq$ considered on S has a compatible linear extension on S.*
(2) *There exists a compatible linear order on R extending $\leq$.*

2.10.Corollary. *Let R be a ring, then the following conditions are equivalent:*

(1) *For any finitely generated subring $S \subseteq R$ of R, there exists a compatible linear order on S.*
(2) *There exists a compatible linear order on R.*

Let

$$H = \{x_1, x_2, ..., x_m\}$$

be a subset of the ring R. If $x_1, x_2, ..., x_m \in R \setminus \{0\}$ are distinct elements and $h : \{1, 2, ..., n\} \longrightarrow H$ is a function, then the sign type of the product $h(1)h(2)...h(n)$ is $(j_1, j_2, ..., j_t)$, where $1 \leq j_1 < j_2 < ... < j_t \leq m$ (a subset of $\{1, 2, ..., m\}$) such that x_j has an odd number of occurrences among $h(1), h(2), ..., h(n)$ if and only if $j \in \{j_1, j_2, ..., j_t\}$ (note that the empty subset is also a sign type).
A sum of the form

$$\sum_{i=1}^{k} h_i(1)h_i(2)...h_i(n_i)$$

with the functions $h_i : \{1, 2, ..., n_i\} \longrightarrow H$, $1 \leq i \leq k$ is called sign homogenous (over H) of type $(j_1, j_2, ..., j_t)$, if the sign type of each product $h_i(1)h_i(2)...h_i(n_i)$ is $(j_1, j_2, ..., j_t)$. If

$$h_i(1)h_i(2)...h_i(n_i) \neq 0$$

for some $1 \leq i \leq k$, then the above sum is called non-trivial. An empty sum is zero and considered to be trivial and sign homogenous with respect to any sign type.

2.11.Theorem. *Let R be a ring, then the following conditions are equivalent:*

(1) *There are distinct elements $x_1, x_2, ..., x_m \in R \setminus \{0\}$ such that for each function*

$$\varepsilon : \{1, 2, ..., m\} \longrightarrow \{0, 1\}$$

and for each sign type $(j_1, j_2, ..., j_t)$ *we can find a sign homogenous sum* $s_\varepsilon(j_1, j_2, ..., j_t)$ *of type* $(j_1, j_2, ..., j_t)$ *over* $\{x_1, x_2, ..., x_m\}$ *with the property, that for any fixed* ε *at least one of* $s_\varepsilon(j_1, j_2, ..., j_t)$ *is non trivial and*

$$\sum_{(j_1, j_2, ..., j_t)} (-1)^{\varepsilon(j_1)+\varepsilon(j_2)+...+\varepsilon(j_t)} s_\varepsilon(j_1, j_2, ..., j_t) = 0$$

holds for all ε *(the sum runs over all sign types).*

(2) *There is no compatible linear order on* R.

Proof. (2) $\Longrightarrow$ (1) : According to Corollary 2.7., the fact that there is no compatible linear order on R is equivalent to the existence of a finite subset

$$H = \{x_1, x_2, ..., x_n\} \subseteq R \setminus \{0\}$$

such that for each orientation $\delta : H \longrightarrow R$ we can find functions

$$h_i^\delta : \{1, 2, ..., n_i(\delta)\} \longrightarrow H \ , \ 1 \leq i \leq k(\delta)$$

satisfying

$$\sum_{i=1}^{k(\delta)} \delta(h_i^\delta(1))\delta(h_i^\delta(2))...\delta(h_i^\delta(n_i(\delta))) = 0$$

and

$$\left\{i \mid 1 \leq i \leq k(\delta) \text{ and } h_i^\delta(1)h_i^\delta(2)...h_i^\delta(n_i(\delta)) \neq 0\right\} \neq \varnothing.$$

Clearly, for any $\varepsilon : \{1, 2, ..., m\} \longrightarrow \{0, 1\}$ and $1 \leq j \leq m$ we can take $\delta(x_j) = (-1)^{\varepsilon(j)} x_j$. Now

$$\delta(h_i^\delta(1))\delta(h_i^\delta(2))...\delta(h_i^\delta(n_i(\delta))) = (-1)^{\varepsilon(j_1)+\varepsilon(j_2)+...+\varepsilon(j_t)} h_i^\delta(1)h_i^\delta(2)...h_i^\delta(n_i(\delta)),$$

where $(j_1, j_2, ..., j_t)$ is the sign type of the product $h_i^\delta(1)h_i^\delta(2)...h_i^\delta(n_i(\delta))$. On collecting the products of similar sign types, we obtain that

$$\sum_{i=1}^{k(\delta)} \delta(h_i^\delta(1))\delta(h_i^\delta(2))...\delta(h_i^\delta(n_i(\delta))) = \sum_{(j_1, j_2, ..., j_t)} (-1)^{\varepsilon(j_1)+\varepsilon(j_2)+...+\varepsilon(j_t)} s_\varepsilon(j_1, j_2, ..., j_t),$$

where $s_\varepsilon(j_1, j_2, ..., j_t)$ is the sum of all products $h_i^\delta(1)h_i^\delta(2)...h_i^\delta(n_i(\delta))$ of sign type $(j_1, j_2, ..., j_t)$. Thus the proof of (2) $\Longrightarrow$ (1) is complete.

(1) $\Longrightarrow$ (2) : Let $\leq$ be a compatible linear order on R, a function

$$\overline{\varepsilon} : \{1, 2, ..., m\} \longrightarrow \{0, 1\}$$

can be defined in such a way, that we have $(-1)^{\overline{\varepsilon}(j)} x_j \geq 0$ for all $1 \leq j \leq m$. Then

$$(-1)^{\overline{\varepsilon}(j_1)+\overline{\varepsilon}(j_2)+...+\overline{\varepsilon}(j_t)} s(j_1, j_2, ..., j_t) \geq 0$$

holds for any sign homogenous sum $s(j_1, j_2, ..., j_t)$ of type $(j_1, j_2, ..., j_t)$. The sum of positive elements in a ring R can be equal to zero if each summand is zero. Thus an equality of the form

$$\sum_{(j_1, j_2, ..., j_t)} (-1)^{\overline{\varepsilon}(j_1)+\overline{\varepsilon}(j_2)+...+\overline{\varepsilon}(j_t)} s_{\overline{\varepsilon}}(j_1, j_2, ..., j_t) = 0,$$

with sign homogenous sums $s_{\overline{\varepsilon}}(j_1, j_2, ..., j_t)$ of type $(j_1, j_2, ..., j_t)$, implies that

$$s_{\overline{\varepsilon}}(j_1, j_2, ..., j_t) = 0$$

for all sign types $(j_1, j_2, ..., j_t)$. Since the summands in $s_{\overline{\varepsilon}}(j_1, j_2, \ldots, j_t)$ are of the same sign with respect to $\leq$, we obtain that all these summands are equal to zero. The above property of $\overline{\varepsilon}$ contradicts to (1). $\square$

2.12.Example. Let $E = \mathbb{Q}\langle v_1, v_2, ... \mid v_i v_j + v_j v_i = 0\rangle$ be the Grassmann (exterior) algebra generated by the pairwise anticommutative generators $v_1, v_2, ...$ For a function $\varepsilon : \{1, 2\} \longrightarrow \{0, 1\}$ and for the sign types $\varnothing, (1), (2), (1, 2)$ we define the sign homogenous sums over $H = \{v_1, v_2\}$ as follows: $s_\varepsilon(\varnothing), s_\varepsilon(1), s_\varepsilon(2)$ are empty sums and

$$s_\varepsilon(1, 2) = v_1 v_2 + v_2 v_1 = 0.$$

Since $v_1 v_2 \neq 0 \neq v_2 v_1$, the sum $s_\varepsilon(1, 2)$ is non trivial. Now

$$(-1)^0 s_\varepsilon(\varnothing) + (-1)^{\varepsilon(1)} s_\varepsilon(1) + (-1)^{\varepsilon(2)} s_\varepsilon(2) + (-1)^{\varepsilon(1)+\varepsilon(2)} s_\varepsilon(1, 2) = 0$$

holds for all ε, thus the application of the above Theorem 2.11. yields, that there is no compatible linear order on E.

2.13.Theorem. *Let R be a commutative ring, then the following conditions are equivalent:*

(1) *There are distinct elements $x_1, x_2, ..., x_m \in R \setminus \{0\}$ such that for each function*

$$\varepsilon : \{1, 2, ..., m\} \longrightarrow \{0, 1\}$$

and for each sign type $(j_1, j_2, ..., j_t)$ we can find a sum $d_\varepsilon(j_1, j_2, ..., j_t)$ of squares of the form $(x_1^{l_1} x_2^{l_2} ... x_m^{l_m})^2$ with the property, that for any fixed ε at least one of the products

$$x_{j_1} x_{j_2} ... x_{j_t} (x_1^{l_1} x_2^{l_2} ... x_m^{l_m})^2$$

is not zero and

$$\sum_{(j_1, j_2, ..., j_t)} (-1)^{\varepsilon(j_1)+\varepsilon(j_2)+...+\varepsilon(j_t)} x_{j_1} x_{j_2} ... x_{j_t} d_\varepsilon(j_1, j_2, ..., j_t) = 0$$

holds for all ε (the sum runs over all sign types).

(2) *There is no compatible linear order on R.*

Proof. In a commutative ring R any product of sign type $(j_1, j_2, ..., j_t)$ over $\{x_1, x_2, ..., x_m\}$ can be written as

$$h(1)h(2)...h(n) = x_{j_1} x_{j_2} ... x_{j_t} (x_1^{l_1} x_2^{l_2} ... x_m^{l_m})^2,$$

where $h : \{1, 2, ..., n\} \longrightarrow H$ is a function and $l_1 \geq 0, l_2 \geq 0, ..., l_m \geq 0$ are integers. Thus Theorem 2.13. is an immediate consequence of Theorem 2.11. $\square$

The following simple proposition together with Theorem 2.6. can be used to determine the intersection of the full extensions of a given cone.

2.14.Proposition. *Let $P \subseteq R$ be a cone of the ring R and $0 \neq a \in R$, then the following conditions are equivalent:*

(1) *The element a is contained in any full extension of P, i.e. $a \in T$ holds for any full cone T of R with $P \subseteq T \subseteq R$.*

(2) *The quasi cone $P \vee [-a]$ has no FEP.*

Proof. (1) $\Longrightarrow$ (2) : If $a \in T$ holds for some cone $T \subseteq R$, then $-a \notin T$. Thus we have $P \vee [-a] \nsubseteq T$ for all full cones T of R (either $P \nsubseteq T$ or $P \subseteq T$ implies $-a \notin T$). On applying Theorem 2.4., we obtain that $P \vee [-a]$ has no FEP.
(2) $\Longrightarrow$ (1) : If $P \vee [-a]$ has no FEP, then there is no full extension of $P \vee [-a]$ by Theorem 2.4. Thus $P \vee [-a] \nsubseteq T$ holds for all full cones T of R. In consequence, we get first $-a \notin T$ and then $a \in T$ for all full cones T with $P \subseteq T$ ($-a \in T$ and $P \subseteq T$ would imply $P \vee [-a] \subseteq T$). □

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