J. Aust. Math. Soc. **81** (2006), 245–252

MAXIMAL COMPATIBLE EXTENSIONS OF PARTIAL ORDERS

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(Received 4 February 2005; revised 14 June 2006)

Communicated by D. Easdown

Abstract

We give a complete description of maximal compatible partial orders on the mono-unary algebra (A, f), where $f : A \to A$ is an arbitrary unary operation.

2000 *Mathematics subject classification*: primary 06A10.
Keywords and phrases: compatible partial order, prohibited pair, quasilinear partial order.

1. Introduction

The well-known Szpilrajn theorem ([9]) asserts that any partial order $\leq_r$ (or r) on a set A can be extended to a linear order $\leq_R$. Recent work related to this early result includes ([2, 3, 4, 6, 7]). As a consequence of Szpilrajn's theorem we obtain that the maximal partial orders (with respect to the containment relation) on A are exactly the linear orders of A. A general scheme for extending Szpilrajn's theorem consists of restricting attention to orders with some prescribed property, and requiring that the linear extension also possess this property (see [1]). In particular, if $f : A \to A$ is a unary operation, then we can restrict our consideration to the so called *compatible* partial orders of (A, f), that is, to partial orders with the following property: $x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$. In the present paper we investigate the compatible extensions of a given r in a partially ordered mono-unary algebra $(A, f, \leq_r)$. Using f-prohibited pairs, for compatible partial orders we define the notion of f-quasilinearity. Our main result states, that a compatible partial order r on (A, f) can always be extended to a compatible f-quasilinear partial order R. As

The work of the first named author was partially supported by the European Community's Marie Curie Program (contract MTKD-CT-2004-003006). The second named author was supported by OTKA of Hungary No. T043034.

a consequence, we obtain that the maximal compatible partial orders on (A, f) are exactly the compatible f-quasilinear partial orders. It turns out, that a compatible f-quasilinear partial order is linear if and only if the function f has no proper cycle (*acyclic* according to the terminology of [8]). Thus the following main theorem of [8] will appear as a special case of our Theorem 4.2.

Let $f : A \to A$ be an acyclic function (there is no $c \in A$ such that $f(c) \neq c$ and $f^n(c) = c$ for some integer $n \geq 2$) and $r \subseteq A \times A$ a compatible partial order on (A, f). Then there exists a compatible linear order $R \subseteq A \times A$ on (A, f) with $r \subseteq R$.

On the other hand, we shall make extensive use of the above result in proving Theorem 4.2.

2. Components, cycles and distance

Let $f : A \to A$ be a function (unary operation on the set A). We define the relation $\sim_f$ as follows: for $x, y \in A$ let $x \sim_f y$ if $f^k(x) = f^l(y)$ for some integers $k \geq 0$ and $l \geq 0$. It is straightforward to see that $\sim_f$ is an equivalence on A. The equivalence class $[x]_f$ of an element $x \in A$ is called the *f-component* of x. Clearly, $[x]_f \subseteq A$ is a subalgebra in (A, f), that is, $f([x]_f) \subseteq [x]_f$. An element $c \in A$ is called *cyclic* with respect to f (or *cyclic* in (A, f)), if $f^m(c) = c$ for some integer $m \geq 1$. For a cyclic element c,

$$n = n(c) = \min\{m \mid m \geq 1 \text{ and } f^m(c) = c\}$$

is called the *period* of c or the *length* of the *cycle* $C = \{c, f(c), \ldots, f^{n-1}(c)\}$; it is easy to prove that C has exactly n elements, $f(C) = C$ and $f^k(c) = f^l(c)$ holds if and only if $k - l$ is divisible by n. A pair $(x, y) \in A \times A$ is called *f-prohibited*, if we can find integers $k \geq 0$, $l \geq 0$ and $m \geq 2$ such that m is not a divisor of $k - l$, the elements $f^k(x), f^{k+1}(x), \ldots, f^{k+m-1}(x)$ are distinct and $f^{k+m}(x) = f^k(x) = f^l(y)$. For an f-prohibited pair (x, y) and an integer $k \geq 0$ as above, we have $y \in [x]_f$, and $f^k(x)$ is a cyclic element in $[x]_f$ of period m. It is easy to verify, that a pair (x, y) is f-prohibited, if and only if $f^k(x) = f^l(y)$ is cyclic and $f^{k+l}(x) \neq f^{k+l}(y)$ for some integers $k \geq 0$ and $l \geq 0$ (the latter condition can be replaced by $f^t(x) \neq f^t(y)$ for all integers $t \geq 0$). The *distance* between an element $y \in [x]_f$ and a given cyclic element $c \in [x]_f$ is defined in part (1) of the following proposition, the proof of which is straightforward and hence omitted.

PROPOSITION 2.1. *Let $y \in [x]_f$ and $c \in [x]_f$ be a cyclic element of period $n \geq 1$. Then we have the following.*

(1) *There exists an integer* $t \geq 0$ *such that* $f^t(y) = c$. *Let*

$$d(y, c) = \min\{t \mid t \geq 0 \text{ and } f^t(y) = c\}$$

denote the distance of y *from* c.

(2) $d(f(c), c) = n - 1$ *and for* $y \neq c$, *we have* $d(f(y), c) = d(y, c) - 1$.

(3) *All cyclic elements of* $[x]_f$ *are in* $C = \{c, f(c), \ldots, f^{n-1}(c)\}$ *and each element in* C *is cyclic of period* n.

(4) *If* $l \geq 0$ *is an integer, then* $f^l(y) = c$ *holds if and only if* $l \geq d(y, c)$ *and* $l - d(y, c)$ *is divisible by* n.

(5) (x, y) *is* f*-prohibited if and only if* $d(x, c) - d(y, c)$ *is not divisible by* n.

PROPOSITION 2.2. *If* $(A, f, \leq_r)$ *is a partially ordered mono-unary algebra, then we have the following.*

(1) *If* $c \in A$ *is a cyclic element of period* $n \geq 1$, *then* $C = \{c, f(c), \ldots, f^{n-1}(c)\}$ *is an antichain with respect to* $\leq_r$.

(2) *If* $(x, y) \in A \times A$ *is an* f*-prohibited pair, then* x *and* y *are incomparable with respect to* $\leq_r$.

PROOF. (1) Take $c^* = f^i(c)$ and $t = j - i$. Then $f^t(c^*) = f^j(c)$. Now $c^* \leq_r f^t(c^*)$ implies $c^* \leq_r f^t(c^*) \leq_r f^{2t}(c^*) \leq_r \cdots \leq_r f^{nt}(c^*) = c^*$, in contradiction with $c^* \neq f^t(c^*)$. The reverse relation $f^t(c^*) \leq_r c^*$ leads to a similar contradiction.

(2) Let $f^k(x), \ldots, f^{k+m-1}(x)$ be distinct elements and $f^{k+m}(x) = f^k(x) = f^l(y)$ for some integers $k \geq 0$, $l \geq 0$ and $m \geq 2$ with $m \nmid k - l$. The assumption $x \leq_r y$ implies

$$f^{k+l}(x) \leq_r f^{k+l}(y)$$

for the elements $f^{k+l}(x)$ and $f^{k+l}(y) = f^k(f^l(y)) = f^k(f^k(x)) = f^{2k}(x)$ of the cycle $C = \{f^k(x), f^{k+1}(x), \ldots, f^{k+m-1}(x)\}$, which contradicts (1), since $m \nmid 2k - (k + l)$. The case $y \leq_r x$ can be treated similarly. □

3. The order components of $(A, f, \leq_r)$

Let $(A, f, \leq_r)$ be a partially ordered mono-unary algebra. Consider the factor set

$$B = A/\sim_f = \{[x]_f \mid x \in A\}.$$

We define the relation $\lhd_r$ on B as follows: $[x]_f \lhd_r [y]_f$ if $x_1 \leq_r y_1$ for some $x_1 \in [x]_f$ and $y_1 \in [y]_f$.

PROPOSITION 3.1. (1) $\lhd_r$ *is a quasiorder (reflexive and transitive) on* B.
(2) *If* $[x]_f \lhd_r [y]_f$ *and* $[y]_f \lhd_r [x]_f$ *for the* f*-components* $[x]_f \neq [y]_f$, *then there is no cyclic element* $c \in [x]_f \cup [y]_f$ *of period* $n \geq 1$.

PROOF. (1) In order to see transitivity, suppose $[x]_f \lhd_r [y]_f \lhd_r [z]_f$. Then $x_1 \leq_r y_1$ and $y_1' \leq_r z_1$ for some $x_1 \in [x]_f$, $y_1, y_1' \in [y]_f$ and $z_1 \in [z]_f$. Since $y_1 \sim_f y_1'$, we can find integers $k \geq 0$ and $l \geq 0$ such that $f^k(y_1) = f^l(y_1')$. However,

$$f^k(x_1) \leq_r f^k(y_1) = f^l(y_1') \leq_r f^l(z_1),$$

for $f^k(x_1) \in [x]_f$ and $f^l(z_1) \in [z]_f$, so $[x]_f \lhd_r [z]_f$.

(2) Suppose that $[x]_f \lhd_r [y]_f \lhd_r [x]_f$, $[x]_f \neq [y]_f$ and, without loss of generality, $c \in [x]_f$ is a cyclic element of period $n \geq 1$. There exist $x_1, x_2 \in [x]_f$ and $y_1, y_2 \in [y]_f$ with the properties $x_1 \leq_r y_1$ and $y_2 \leq_r x_2$. By part (1) of Proposition 2.1,

$$f^{t_1}(x_1) = c = f^{t_2}(x_2)$$

for some integers $t_1 \geq 0$ and $t_2 \geq 0$. Since $f^{t_1}(y_1) \sim_f f^{t_2}(y_2)$, we can find integers $k \geq 0$ and $l \geq 0$ such that

$$f^k(f^{t_1}(y_1)) = f^l(f^{t_2}(y_2)).$$

The compatibility of $\leq_r$ gives

$$f^k(c) = f^k(f^{t_1}(x_1)) \leq_r f^k(f^{t_1}(y_1)) = f^l(f^{t_2}(y_2)) \leq_r f^l(f^{t_2}(x_2)) = f^l(c),$$

where $f^k(c)$ and $f^l(c)$ are cyclic elements. Applying part (1) of Proposition 2.2, we obtain that $f^k(c) = f^k(f^{t_1}(y_1)) = f^l(c)$ in contradiction with $[x]_f \cap [y]_f = \varnothing$. □

The relation $\equiv_r$ is defined on $B = A/\sim_f$ as follows: for $x, y \in A$ let $[x]_f \equiv_r [y]_f$ if $[x]_f \lhd_r [y]_f$ and $[y]_f \lhd_r [x]_f$. It is well-known that starting from the quasiorder $\lhd_r$, the above definition provides an equivalence on B. We define the *order component* of x in $(A, f, \leq_r)$ by

$$\langle x \rangle = \bigcup_{y \in A \text{ and } [y]_f \equiv_r [x]_f} [y]_f.$$

Clearly, $[x]_f \subseteq \langle x \rangle \subseteq A$ and $\langle x \rangle$ is a subalgebra in (A, f), which corresponds to the $\equiv_r$ equivalence class $[[x]_f]_{\equiv_r}$ of $[x]_f$ in B. It is easy to see that $\{\langle x \rangle \mid x \in A\}$ is a partition of A.

If $c \in \langle x \rangle$ is a cyclic element, then part (2) of Proposition3.1 gives that $\langle x \rangle = [x]_f$. We make use of the partial order $\ll_r$ on $B/\equiv_r$, which can be derived from $\lhd_r$ in a natural way: $\langle x \rangle \ll_r \langle y \rangle$ if $[x]_f \lhd_r [y]_f$.

LEMMA 3.2. *Let $(A, f, \leq_r)$ be a partially ordered mono-unary algebra. If $x \in A$ and there is no cyclic element in $\langle x \rangle$, then there exists a linear order ρ on $\langle x \rangle$ with the following properties*:

(1) *ρ is compatible on $(\langle x \rangle, f)$,*
(2) *ρ is an extension of $\leq_r$ on the elements of $\langle x \rangle$.*

PROOF. The absence of cyclic elements ensures that $f : \langle x \rangle \longrightarrow \langle x \rangle$ is acyclic, preserving the partial order $r \cap (\langle x \rangle \times \langle x \rangle)$. A straightforward application of the Main Theorem in [8] gives the existence of the desired ρ. □

LEMMA 3.3. *Let $(A, f, \leq_r)$ be a partially ordered mono-unary algebra, $x \in A$ and $c \in \langle x \rangle$ a cyclic element of period $n \geq 1$. Then there exists a partial order ρ on $\langle x \rangle = [x]_f$ with the following properties*:

(1) *ρ is compatible on $([x]_f, f)$,*
(2) *ρ is an extension of $\leq_r$ on the elements of $[x]_f$,*
(3) *$[x]_f = E_0 \cup E_1 \cup \cdots \cup E_{n-1}$ is a pairwise disjoint union, where each set*

$$E_i = \{u \in [x]_f \mid d(u, c) - i \text{ is divisible by } n\}, \quad 0 \leq i \leq n-1,$$

is a chain with respect to ρ, and for $i \neq j$ the elements of $E_i \times E_j$ are f-prohibited pairs.

PROOF. Let $E = [x]_f$ and consider the equivalence relation $\varepsilon = \Delta_E \cup (C \times C)$ on E, where Δ_E is the diagonal of $E \times E$ and $C = \{c, f(c), \ldots, f^{n-1}(c)\}$ is the set of cyclic elements in E. Clearly, $[u]_\varepsilon = \{u\}$ if $u \in E \setminus C$ and $[u]_\varepsilon = C$ if $u \in C$. Using the factor set $E^* = E/\varepsilon$, define a function $f^* : E^* \to E^*$ and a relation $r^* \subseteq E^* \times E^*$ as follows: $f^*([u]_\varepsilon) = [f(u)]_\varepsilon$ and r^* is the transitive closure of the reflexive relation

$$s = \big\{([u]_\varepsilon, [v]_\varepsilon) \mid u, v \in E \text{ and } u' \leq_r v' \text{ for some } u' \in [u]_\varepsilon,\ v' \in [v]_\varepsilon\big\}.$$

Then f^* is well-defined since $f(C) \subseteq C$. It is immediate from the definitions that f^* preserves s, whence f^* preserves r^*. We claim, that r^* is a partial order on E^*. It is enough to show that there is no proper cycle in E^* with respect to s. If a proper cycle

$$[u_1]_\varepsilon s [u_2]_\varepsilon s \cdots s [u_k]_\varepsilon s [u_1]_\varepsilon$$

does not contain C, then we have

$$u_1 \leq_r u_2 \leq_r \cdots \leq_r u_k \leq_r u_1$$

implying that $u_1 = u_2 = \cdots = u_k$, a contradiction. If C appears in a proper cycle, then we can exhibit a segment of it as

$$C s [v_1]_\varepsilon s [v_2]_\varepsilon s \cdots s [v_l]_\varepsilon s C,$$

where $v_1, v_2, \dots, v_l \notin C$. Now we have

$$c' \leq_r v_1 \leq_r v_2 \leq_r \cdots \leq_r v_l \leq_r c''$$

for some $c', c'' \in C$. Applying part (1) of Proposition 2.2 gives that $c' = c''$. Thus the elements $v_1 = v_2 = \cdots = v_l = c' = c''$ are in C, a contradiction. The only cyclic element of (E^*, f^*) is C and $f^*(C) = C$, so we can apply the Main Theorem of [8] to the partially ordered algebra (E^*, f^*, r^*), in order to get a compatible linear order ρ^* on (E^*, f^*) with $r^* \subseteq \rho^*$. We claim that

$$\rho = \{(u, v) \mid u, v \in E\ ,\ ([u]_\varepsilon, [v]_\varepsilon) \in \rho^* \text{ and } n \mid d(u, c) - d(v, c)\}$$

is one of the desired relations on E.

The reflexive and transitive properties of ρ can be easily verified. Let $(u, v) \in \rho$ and $(v, u) \in \rho$. Then $([u]_\varepsilon, [v]_\varepsilon) \in \rho^*$ and $([v]_\varepsilon, [u]_\varepsilon) \in \rho^*$ imply $[u]_\varepsilon = [v]_\varepsilon$, whence $u = v$ or $u, v \in C$. If $u, v \in C$, then we also have $u = v$ since $n \mid d(u, c) - d(v, c)$, proving antisymmetry.

Suppose $(u, v) \in \rho$. Then $([u]_\varepsilon, [v]_\varepsilon) \in \rho^*$ and the compatibility of ρ^* provides that

$$([f(u)]_\varepsilon, [f(v)]_\varepsilon) = (f^*([u]_\varepsilon), f^*([v]_\varepsilon)) \in \rho^*.$$

Using part (2) of Proposition 2.1, we obtain $n \mid d(f(u), c) - d(f(v), c)$ as a consequence of the divisibility $n \mid d(u, c) - d(v, c)$, proving that $(f(u), f(v)) \in \rho$.

Suppose $u, v \in E$ and $u \leq_r v$. Then first we get $([u]_\varepsilon, [v]_\varepsilon) \in s$ and next $([u]_\varepsilon, [v]_\varepsilon) \in r^* \subseteq \rho^*$. If $n \nmid d(u, c) - d(v, c)$, then (u, v) is f-prohibited by part (5) of Proposition 2.1, contradicting part (2) of Proposition 2.2. Thus we have $n \mid d(u, c) - d(v, c)$ and $(u, v) \in \rho$, proving $r \subseteq \rho$.

For $u, v \in E_i$, the divisibility $n \mid d(u, c) - d(v, c)$ follows from $n \mid d(u, c) - i$ and $n \mid d(v, c) - i$. Since ρ^* is linear, either $([u]_\varepsilon, [v]_\varepsilon) \in \rho^*$ or $([v]_\varepsilon, [u]_\varepsilon) \in \rho^*$ holds. Thus we have either $(u, v) \in \rho$ or $(v, u) \in \rho$, proving that E_i is a chain with respect to ρ.

If $i \neq j$ and $(u, v) \in E_i \times E_j$, then $n \mid d(u, c) - i$ and $n \mid d(v, c) - j$ imply that $d(u, c) - d(v, c)$ is not divisible by n, so by part (5) of Proposition 2.1, (u, v) is f-prohibited. □

REMARK 3.4. According to [5, Proposition 3.6], the convexity of the antichain C implies that $\varepsilon = \Delta_E \cup (C \times C)$ is an order congruence of $(E, f, r \cap (E \times E))$.

4. The main results

A compatible partial order R on a mono-unary algebra (A, f) is called *f-quasilinear*, if $(x, y) \in R$ or $(y, x) \in R$ for all non f-prohibited pairs $(x, y) \in A \times A$. In view of part (2) of Proposition 2.2, we have the following simple observation.

PROPOSITION 4.1. *If a compatible partial order R on a mono-unary algebra (A, f) is f-quasilinear, then it is maximal (with respect to containment) among the compatible partial orders of (A, f).*

THEOREM 4.2. *If $(A, f, \leq_r)$ is a partially ordered mono-unary algebra, then there exists a compatible partial order R on (A, f) with the following properties:*

(1) *R is an extension of r,*
(2) *R is f-quasilinear.*

PROOF. Let $\ll_\lambda$ be an arbitrary linear extension of the partial order $\ll_r$ on the set $B/\equiv_r$ of order components in $(A, f, \leq_r)$, where $B = A/\sim_f$. Let $x \in A$. If there is no cyclic element in $\langle x \rangle$, then fix a compatible linear order $\rho_{\langle x \rangle}$ on $\langle x \rangle$ with the properties described in Lemma 3.2. If there is a cyclic element of period $n \geq 1$ in $\langle x \rangle$, then fix a compatible partial order $\rho_{\langle x \rangle}$ on $\langle x \rangle = [x]_f$ with the properties described in Lemma 3.3. We claim that

$$R = \{(x, y) \in A \times A \mid \langle x \rangle \ll_\lambda \langle y \rangle \text{ and } (x, y) \in \rho_{\langle x \rangle} \text{ in case of } \langle x \rangle = \langle y \rangle\}$$

satisfies (1) and (2).

The reflexive, antisymmetric and transitive properties of R can be easily verified. In order to prove the compatibility of R, it is enough to note that $\langle f(x) \rangle = \langle x \rangle$ and that $\rho_{\langle x \rangle}$ is a compatible partial order on $(\langle x \rangle, f)$.

Suppose $x \leq_r y$. Then $[x]_f \lhd_r [y]_f$, whence we obtain $\langle x \rangle \ll_r \langle y \rangle$ as well as $\langle x \rangle \ll_\lambda \langle y \rangle$. In the case of $\langle x \rangle = \langle y \rangle$, the relation $(x, y) \in \rho_{\langle x \rangle}$ follows from $r \cap (\langle x \rangle \times \langle x \rangle) \subseteq \rho_{\langle x \rangle}$. Thus we have $(x, y) \in R$, proving $r \subseteq R$. Therefore (1) holds.

Suppose now $x, y \in A$ are incomparable elements with respect to R. Then the linearity of $\ll_\lambda$ implies that $\langle x \rangle = \langle y \rangle$, $(x, y) \notin \rho_{\langle x \rangle}$ and $(y, x) \notin \rho_{\langle x \rangle}$. Since $\rho_{\langle x \rangle}$ is not linear, the order component $\langle x \rangle$ must contain a cyclic element c of period $n \geq 2$. In view of the properties of $\rho_{\langle x \rangle}$ described in Lemma 3.3, we obtain that $x \in E_i$ and $y \in E_j$ for some $i, j \in \{0, 1, \ldots, n-1\}$ with $i \neq j$. Now the last property of the E_i's guarantees that (x, y) is an f-prohibited pair. Thus (2) holds. □

COROLLARY 4.3. *A compatible partial order R on (A, f) is maximal (with respect to containment) if and only if R is f-quasilinear.*

Acknowledgement

The final version of this paper was prepared while the first named author was at the Alfred Renyi Institute of Mathematics, Hungarian Academy of Sciences.

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