Order
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A Half-Space Approach to Order Dimension

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Abstract The aim of the present paper is to investigate the half-spaces in the convexity structure of all quasiorders on a given set and to use them in an alternative approach to classical order dimension. The main result states that linear orders can almost always be replaced by half-space quasiorders in the definition of the dimension of a partially ordered set.

Keywords Convexity · Quasiorder · Preorder · Half-space · Dimension

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1 Introduction

Within the framework of the general theory of abstract convexity (van de Vel [9]), strict quasiorders (irreflexive and transitive relations) on a set A can be thought of as convex subsets of $\{(x, y) \in A \times A \mid x \neq y\}$:

(1) $\{(x, y) \in A \times A \mid x \neq y\}$ is a strict quasiorder,
(2) Any intersection of strict quasiorders is a strict quasiorder,
(3) Any nested union of strict quasiorders is a strict quasiorder.

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In general, a half-space is defined as a convex subset of the base set with a convex set complement. Abstract convexity theory addresses questions such as the representation of convex sets as intersections of half-spaces. For technical reasons, instead of the strict quasiorders in $\{(x, y) \in A \times A \mid x \neq y\}$, we shall consider the ordinary (reflexive) quasiorders in $A \times A$ (there is a natural one to one correspondence between them). We can use half-space quasiorders to define the half-space dimension of a quasiordered set, in a similar way as linear orders are used to define the order dimension of a partially ordered set. The aim of the present paper is to investigate the half-space quasiorders and to study the above dimension concept for quasiorders, along the lines of the classical theory of order dimension (see e.g. [1, 2, 7, 8]). Our main result (Theorem 2.16) states that linear orders can almost always be replaced by half-space quasiorders in the definition of the order dimension. Since there are considerably more half-spaces than linear orders, establishing upper bounds on order dimension can be easier using representations of partial orders as intersections of half-spaces. In order to demonstrate this, we give a simple proof for the "difficult" part of the classical Dushnik–Miller theorem (in [2]) about the dimension of the direct product of chains.

In Section 2 we provide some simple characterizations of half-spaces and examine the relationship between half-spaces and linear orders. A standard construction together with a complete description of half-spaces is also given. In the rest of Section 2, we show the tight connection between half-space dimension and classical order dimension. It turns out, that the half-space dimension and the order dimension of a partially ordered set can be different only for half-space partial orders.

In Section 3 we deal with direct products. First we prove that the direct product of quasiorders can be a half-space only in one exceptional situation. Then we use half-spaces to obtain the exact upper bound for the dimension in the above mentioned theorem of Dushnik and Miller.

2 Half-Spaces and the Dimension of Quasiordered Sets

A *quasiorder* γ on the set A is a reflexive and transitive relation:

$$\Delta_A = \{(a, a) \mid a \in A\} \subseteq \gamma \subseteq A \times A$$

and $(x, y) \in \gamma$, $(y, z) \in \gamma$ imply $(x, z) \in \gamma$ for all $x, y, z \in A$. The containment relation $\subseteq$ provides a natural complete lattice structure on the set $\mathrm{Quord}(A)$ of all quasiorders on A: $(\mathrm{Quord}(A), \vee, \cap)$. If γ is a partial order, then we frequently use the standard notations $x \leq_\gamma y$ and $x <_\gamma y$ for $(x, y) \in \gamma$ and for $(x, y) \in \gamma, x \neq y$. For a relation γ, the inverse of γ is $\gamma^{-1} = \{(y, x) \mid (x, y) \in \gamma\}$ and for a quasiorder the intersection $\gamma \cap \gamma^{-1}$ is an equivalence on A. The equivalence class of an element $a \in A$ is denoted by $[a]_{\gamma\cap\gamma^{-1}}$, thus

$$A/(\gamma \cap \gamma^{-1}) = \{[a]_{\gamma\cap\gamma^{-1}} \mid a \in A\}.$$

It is well known that γ induces a natural partial order r_γ (in order to avoid repeated indices, we write $\leq^\gamma$ instead of $\leq_{r_\gamma}$) on the above quotient set: for $a, b \in A$

$$[a]_{\gamma\cap\gamma^{-1}} \leq^\gamma [b]_{\gamma\cap\gamma^{-1}} \text{ if and only if } (x, y) \in \gamma \text{ for some } x \in [a]_{\gamma\cap\gamma^{-1}} \text{ and } y \in [b]_{\gamma\cap\gamma^{-1}}.$$

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Also $[a]_{\gamma\cap\gamma^{-1}} \leq^{\gamma} [b]_{\gamma\cap\gamma^{-1}}$ holds if and only if $(x, y) \in \gamma$ for all $x \in [a]_{\gamma\cap\gamma^{-1}}$ and for all $y \in [b]_{\gamma\cap\gamma^{-1}}$.

A quasiorder $\alpha \subseteq A \times A$ is said to be a *half-space on* A if it has a "strong" complement in the lattice $(\mathrm{Quord}(A), \subseteq)$, i.e. if $\alpha \cup \beta = A \times A$ and $\alpha \cap \beta = \Delta_A$ hold for some quasiorder $\beta \subseteq A \times A$. Clearly, this complement β is also a half-space and it is uniquely determined by α: $\beta = \Delta_A \cup ((A \times A) \setminus \alpha)$. It follows, that α is a half-space if and only if $\Delta_A \cup ((A \times A) \setminus \alpha)$ is transitive. The simplest examples of half-spaces are linear orders, the identity Δ_A and the full relation $A \times A$ on any set A. Complementary half-spaces are put into a pair of the form $\alpha \updownarrow \beta$ and can be characterized in the lattice $(\mathrm{Quord}(A), \vee, \cap)$ as follows.

Proposition 2.1 *For any quasiorders $\alpha, \beta \in \mathrm{Quord}(A)$ the following are equivalent:*

(1) *$\alpha \updownarrow \beta$ is a pair of complementary half-spaces, i.e. $\alpha \cap \beta = \Delta_A$ and $\alpha \cup \beta = A \times A$.*
(2) *$\alpha \cap \beta = \Delta_A$ and $(\alpha \cap \gamma) \vee (\beta \cap \gamma) = \gamma$ for all $\gamma \in \mathrm{Quord}(A)$.*
(3) *$\alpha \cap \beta = \Delta_A$ and $(\alpha \cap \gamma) \cup (\beta \cap \gamma) = \gamma$ for all $\gamma \in \mathrm{Quord}(A)$.*

Proof (1) $\Longrightarrow$ (2):

$$\gamma = (A \times A) \cap \gamma = (\alpha \cup \beta) \cap \gamma = (\alpha \cap \gamma) \cup (\beta \cap \gamma) \subseteq (\alpha \cap \gamma) \vee (\beta \cap \gamma) \subseteq \gamma.$$

(2) $\Longrightarrow$ (1): Suppose that $\alpha \cup \beta \neq A \times A$, then $(a, b) \notin \alpha \cup \beta$ for some $a, b \in A$. Since $\gamma(a, b) = \Delta_A \cup \{(a, b)\}$ is a quasiorder on A, we have

$$(\alpha \cap \gamma(a, b)) \vee (\beta \cap \gamma(a, b)) = \gamma(a, b)$$

in contradiction with $\alpha \cap \gamma(a, b)) = \beta \cap \gamma(a, b) = \Delta_A$.

(1) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (2) trivially. □

For a half-space α the inverse relation α^{-1} is also a half-space, if $\alpha \updownarrow \beta$ for $\alpha, \beta \in \mathrm{Quord}(A)$, then $\alpha^{-1} \updownarrow \beta^{-1}$. If $B \subseteq A$ is a subset, then the restriction of a quasiorder to B yields a quasiorder on B and a similar statement holds for half-spaces, $\alpha \updownarrow \beta$ implies that $\alpha \cap (B \times B) \updownarrow \beta \cap (B \times B)$. This observation leads to another characterization of half-spaces, which will be repeatedly used in the sequel.

Proposition 2.2 *For a quasiorder $\alpha \in \mathrm{Quord}(A)$ the following are equivalent:*

(1) *α is a half-space.*
(2) *$\alpha \cap (B \times B)$ is a half-space (on B) for any three element subset $B \subseteq A$.*
(3) *For any $x, y, z \in A$ the relations $(x, y) \notin \alpha$, $(y, x) \notin \alpha$ and $(x, z) \in \alpha$, $z \neq x$ imply that $(y, z) \in \alpha$.*
(4) *For any $x, y, z \in A$ the relations $(z, y) \notin \alpha$, $(y, z) \notin \alpha$ and $(x, z) \in \alpha$, $x \neq z$ imply that $(x, y) \in \alpha$.*

Proof (1) $\Longrightarrow$ (2): This is a special case of our claim preceding Proposition 2.2.

(2) $\Longrightarrow$ (3): Let $(x, y) \notin \alpha$, $(y, x) \notin \alpha$ and $(x, z) \in \alpha$, $z \neq x$ for the elements $x, y, z \in A$ and take the three element subset $B = \{x, y, z\}$ of A. Suppose that $(y, z) \notin \alpha$ and consider the complementary half-space $\delta \subseteq B \times B$ of $\alpha \cap (B \times B)$.

Now

$$(\alpha \cap (B \times B)) \cup \delta = B \times B$$

implies that $(x, y) \in \delta$ and $(y, z) \in \delta$, whence $(x, z) \in (\alpha \cap (B \times B)) \cap \delta = \Delta_B$ can be derived in contradiction with $z \neq x$.

(3) $\Longrightarrow$ (4): Let $(z, y) \notin \alpha$, $(y, z) \notin \alpha$ and $(x, z) \in \alpha$, $x \neq z$ for the elements $x, y, z \in A$ and suppose that $(x, y) \notin \alpha$. Clearly, $(y, x) \in \alpha$ would imply $(y, z) \in \alpha$, a contradiction. Thus $(x, y) \notin \alpha$, $(y, x) \notin \alpha$ and $(x, z) \in \alpha$, $x \neq z$, whence we obtain that $(y, z) \in \alpha$, a contradiction. It follows that $(x, y) \in \alpha$.

(4) $\Longrightarrow$ (1): In order to see the transitivity of $\beta = \Delta_A \cup ((A \times A) \setminus \alpha)$ let $(x, y) \in \beta$, $(y, z) \in \beta$, $x \neq y$ and suppose that $(x, z) \notin \beta$. We have either $(z, y) \notin \alpha$ or $(z, y) \in \alpha$. In the first case $(z, y) \notin \alpha$, $(y, z) \notin \alpha$ and $(x, z) \in \alpha$, $x \neq z$ would imply that $(x, y) \in \alpha \cap \beta = \Delta_A$, a contradiction. In the second case $(x, z) \in \alpha$ and $(z, y) \in \alpha$ would imply that $(x, y) \in \alpha \cap \beta = \Delta_A$, a contradiction again. Thus we have $(x, z) \in \beta$. □

Proposition 2.3 *If α is a half-space quasiorder on A, then the induced partial order r_α is a half-space on $A/(\alpha \cap \alpha^{-1})$.*

Proof We can use part (3) in Proposition 2.2. If $([x]_{\alpha\cap\alpha^{-1}}, [y]_{\alpha\cap\alpha^{-1}}) \notin r_\alpha$, $([y]_{\alpha\cap\alpha^{-1}}, [x]_{\alpha\cap\alpha^{-1}}) \notin r_\alpha$ and $([x]_{\alpha\cap\alpha^{-1}}, [z]_{\alpha\cap\alpha^{-1}}) \in r_\alpha$, $[z]_{\alpha\cap\alpha^{-1}} \neq [x]_{\alpha\cap\alpha^{-1}}$, then we have $(x, y) \notin \alpha$, $(y, x) \notin \alpha$ and $(x, z) \in \alpha$, $z \neq x$. Since α is a half-space, we obtain first $(y, z) \in \alpha$ and then $([y]_{\alpha\cap\alpha^{-1}}, [z]_{\alpha\cap\alpha^{-1}}) \in r_\alpha$. □

Proposition 2.4 *If $\gamma \subseteq A \times A$ is a quasiorder and $\gamma \subseteq \alpha$ for some half-space α on A, then there exists a half-space τ on A, such that $\gamma \subseteq \tau \subseteq \alpha$ and $\tau \cap \tau^{-1} = \gamma \cap \gamma^{-1}$.*

Proof Let R be a linear extension of the induced partial order r_γ and define the relation $\tau \subseteq A \times A$ as follows:

$$\tau = \alpha \setminus \{(a, b) \in \alpha \cap \alpha^{-1} \mid [b]_{\gamma\cap\gamma^{-1}} <_R [a]_{\gamma\cap\gamma^{-1}}\}.$$

Since $(x, y) \in \gamma$ implies that $(x, y) \in \alpha$ and $[x]_{\gamma\cap\gamma^{-1}} \leq_R [y]_{\gamma\cap\gamma^{-1}}$, we obtain that $(x, y) \in \tau$. Thus $\gamma \subseteq \tau \subseteq \alpha$ and $\gamma \cap \gamma^{-1} \subseteq \tau \cap \tau^{-1}$. If $(x, y) \in \tau \cap \tau^{-1}$, then the relations $[y]_{\gamma\cap\gamma^{-1}} <_R [x]_{\gamma\cap\gamma^{-1}}$ and $[x]_{\gamma\cap\gamma^{-1}} <_R [y]_{\gamma\cap\gamma^{-1}}$ are not satisfied, whence $[x]_{\gamma\cap\gamma^{-1}} = [y]_{\gamma\cap\gamma^{-1}}$ and $(x, y) \in \gamma \cap \gamma^{-1}$ can be derived. It follows, that $\tau \cap \tau^{-1} \subseteq \gamma \cap \gamma^{-1}$ and hence $\tau \cap \tau^{-1} = \gamma \cap \gamma^{-1}$.

In order to see the transitivity of τ take $(x, y) \in \tau$ and $(y, z) \in \tau$. Now $(x, y) \in \alpha$ and $(y, z) \in \alpha$ imply that $(x, z) \in \alpha$. Suppose that $(x, z) \notin \tau$, whence $(x, z) \in \alpha \cap \alpha^{-1}$ and $[z]_{\gamma\cap\gamma^{-1}} <_R [x]_{\gamma\cap\gamma^{-1}}$ follow. The relations $(y, z) \in \alpha$ and $(z, x) \in \alpha$ imply that $(y, x) \in \alpha$ and hence $(x, y) \in \alpha \cap \alpha^{-1}$. Similarly, $(z, x) \in \alpha$ and $(x, y) \in \alpha$ imply that $(y, z) \in \alpha \cap \alpha^{-1}$. In view of $(x, y) \in \tau$ and $(y, z) \in \tau$ we have $[x]_{\gamma\cap\gamma^{-1}} \leq_R [y]_{\gamma\cap\gamma^{-1}}$ and $[y]_{\gamma\cap\gamma^{-1}} \leq_R [z]_{\gamma\cap\gamma^{-1}}$, whence we obtain that $[x]_{\gamma\cap\gamma^{-1}} \leq_R [z]_{\gamma\cap\gamma^{-1}}$, a contradiction.

In order to prove that τ is a half-space we can use part (3) of Proposition 2.2. Take $x, y, z \in A$ such that $(x, y) \notin \tau$, $(y, x) \notin \tau$ and $(x, z) \in \tau$, $z \neq x$. Now $(x, y) \notin \tau$ implies that either $(x, y) \notin \alpha$ or $(x, y) \in \alpha \cap \alpha^{-1}$ with $[y]_{\gamma\cap\gamma^{-1}} <_R [x]_{\gamma\cap\gamma^{-1}}$. Similarly, $(y, x) \notin \tau$ implies that either $(y, x) \notin \alpha$ or $(y, x) \in \alpha \cap \alpha^{-1}$ with $[x]_{\gamma\cap\gamma^{-1}} <_R [y]_{\gamma\cap\gamma^{-1}}$. It is easy to check that the only possibility to have $(x, y) \notin \tau$ and $(y, x) \notin \tau$ at the same time is the case when $(x, y) \notin \alpha$ and $(y, x) \notin \alpha$. Since α is a half-space, $(x, y) \notin \alpha$, $(y, x) \notin \alpha$ and $(x, z) \in \alpha$, $z \neq x$ imply that $(y, z) \in \alpha$. Suppose that $(y, z) \in \alpha \cap \alpha^{-1}$, then $(x, z) \in \alpha$ and the transitivity of α imply that $(x, y) \in \alpha$, a contradiction. Thus we have $(y, z) \notin \alpha \cap \alpha^{-1}$, whence $(y, z) \in \tau$ follows. □

Proposition 2.5 *Let the partial order α on A be a half-space. If λ is a linear order on A, then*

$$\alpha[\lambda] = \alpha \cup (\lambda \setminus (\alpha \cup \alpha^{-1}))$$

is a linear extension of α on A and $\alpha = \alpha[\lambda] \cap \alpha[\lambda^{-1}]$.

Proof In order to see the transitivity of $\alpha[\lambda]$ take $(x, y) \in \alpha[\lambda]$ and $(y, z) \in \alpha[\lambda]$ with $x \neq y \neq z$. Clearly, $(x, y) \in \alpha$ and $(y, z) \in \alpha$ imply $(x, z) \in \alpha$. If $(x, y) \in \alpha$ and $(y, z) \in \lambda \setminus (\alpha \cup \alpha^{-1})$, then $(y, z) \notin \alpha$, $(z, y) \notin \alpha$ and $(x, y) \in \alpha$, $x \neq y$, whence $(x, z) \in \alpha$ can be derived by part (4) of Proposition 2.2. Similarly, $(x, y) \in \lambda \setminus (\alpha \cup \alpha^{-1})$ and $(y, z) \in \alpha$ imply $(x, z) \in \alpha$ by part (3) of Proposition 2.2. If we have $(x, y) \in \lambda \setminus (\alpha \cup \alpha^{-1})$ and $(y, z) \in \lambda \setminus (\alpha \cup \alpha^{-1})$, then $(x, y) \in \lambda$ and $(y, z) \in \lambda$ imply $(x, z) \in \lambda$. Since $(x, y) \notin \alpha \cup \alpha^{-1}$ and $(y, z) \notin \alpha \cup \alpha^{-1}$ imply that $(x, y) \in \beta \cap \beta^{-1}$ and $(y, z) \in \beta \cap \beta^{-1}$ (here β is the complementary half-space of α), the transitivity of $\beta \cap \beta^{-1}$ gives that $(x, z) \in \beta \cap \beta^{-1}$, i.e. that $(x, z) \notin \alpha \cup \alpha^{-1}$. It follows that $(x, z) \in \lambda \setminus (\alpha \cup \alpha^{-1})$.

Suppose that $(x, y) \in \alpha[\lambda]$ and $(y, x) \in \alpha[\lambda]$, then $(x, y) \in \alpha$ and $(y, x) \in \lambda \setminus (\alpha \cup \alpha^{-1})$ is impossible. Similarly, $(x, y) \in \lambda \setminus (\alpha \cup \alpha^{-1})$ and $(y, x) \in \alpha$ is also impossible. Thus we have either $(x, y) \in \alpha$, $(y, x) \in \alpha$ or $(x, y) \in \lambda \setminus (\alpha \cup \alpha^{-1})$, $(y, x) \in \lambda \setminus (\alpha \cup \alpha^{-1})$, in both cases $x = y$ follows by the antisymmetric properties of α and λ, respectively.

Suppose that $(x, y) \notin \alpha$ and $(y, x) \notin \alpha$, then $(x, y) \notin \alpha \cup \alpha^{-1}$. Now $(x, y) \in \lambda$ implies $(x, y) \in \lambda \setminus (\alpha \cup \alpha^{-1})$ and $(y, x) \in \lambda$ implies $(y, x) \in \lambda \setminus (\alpha \cup \alpha^{-1})$. We proved that $\alpha[\lambda]$ is a linear order.

Using $\alpha \cap (\lambda \setminus (\alpha \cup \alpha^{-1})) = \alpha \cap (\lambda^{-1} \setminus (\alpha \cup \alpha^{-1})) = \varnothing$ and $\lambda \cap \lambda^{-1} = \Delta_A$, it is straightforward to see that $\alpha = \alpha[\lambda] \cap \alpha[\lambda^{-1}]$. □

Corollary 2.6 *If α is a half-space quasiorder on A, then the induced partial order is of the form $r_\alpha = R_1 \cap R_2$ for some linear orders R_1 and R_2 on $A/(\alpha \cap \alpha^{-1})$, i.e. r_α has order dimension at most 2.*

Proof The partial order r_α is a half-space on $A/(\alpha \cap \alpha^{-1})$ by Proposition 2.3. If R is an arbitrary linear order on $A/(\alpha \cap \alpha^{-1})$, then $R_1 = r_\alpha[R]$ and $R_2 = r_\alpha[R^{-1}]$ are linear orders on $A/(\alpha \cap \alpha^{-1})$ with $r_\alpha = r_\alpha[R] \cap r_\alpha[R^{-1}]$ by Proposition 2.5. □

We remark that Corollary 2.6 does not characterize half-spaces entirely.

As already noted, any linear order λ on A is an example of a half-space: $\lambda \updownarrow \lambda^{-1}$. Let $f : A \longrightarrow X$ be a function, $Y \subseteq X$ a subset, and R a linear order on X. Define the following relations on A:

$$\ker_Y(f) = \Delta_A \cup \{(a, b) \in A \times A \mid f(a) = f(b) \in Y\},$$

$$f^{-1}(R) = \Delta_A \cup \{(a, b) \in A \times A \mid f(a) <_R f(b)\}.$$

Note that $\ker_X(f)$ is the ordinary kernel

$$\ker(f) = \{(a, b) \in A \times A \mid f(a) = f(b)\}.$$

The following is a standard construction of a half-space using a linear order.

Proposition 2.7 *Let (A, γ) be a quasiordered set, (X, ρ) a partially ordered set and $f : A \longrightarrow X$ a (γ, ρ) quasiorder preserving function: $(x, y) \in \gamma \Longrightarrow (f(x), f(y)) \in \rho$ for all $x, y \in A$. If $Y \subseteq X$ is a subset, R is a linear extension of ρ on X and $\gamma \cap \ker(f) \subseteq \ker_Y(f)$, then*

$$\alpha = \ker_Y(f) \cup f^{-1}(R)$$

is a half-space extension of γ and $\alpha \cap \alpha^{-1} = \ker_Y(f)$.

If $Y = \varnothing$, then $\ker_Y(f) = \Delta_A$ and $\ker_Y(f) \cup f^{-1}(R) = f^{-1}(R)$ is a partial order.

If $Y = X$, then $\ker_Y(f) = \ker(f)$ (now $\gamma \cap \ker(f) \subseteq \ker_Y(f)$ is automatically satisfied) and $\ker_Y(f) \cup f^{-1}(R) = \ker(f) \cup f^{-1}(R)$ is a half-space extension of γ. In particular, if $\kappa : A \longrightarrow A/(\gamma \cap \gamma^{-1})$ is the canonical surjection and R is a linear extension of the induced partial order r_γ on $A/(\gamma \cap \gamma^{-1})$, then $\ker(\kappa) \cup \kappa^{-1}(R) = (\gamma \cap \gamma^{-1}) \cup \kappa^{-1}(R)$ is a half-space extension of γ.

Proof The containment $\gamma \subseteq \ker_Y(f) \cup f^{-1}(R)$ is a consequence of $\rho \subseteq R$, $\gamma \cap \ker(f) \subseteq \ker_Y(f)$ and of the quasiorder preserving property of f. It is easy to see that $\ker_Y(f) \cup f^{-1}(R)$ and $\ker_{X \setminus Y}(f) \cup f^{-1}(R^{-1})$ are quasiorders on A. We have

$$(\ker_Y(f) \cup f^{-1}(R)) \cup (\ker_{X \setminus Y}(f) \cup f^{-1}(R^{-1})) = A \times A$$

and

$$(\ker_Y(f) \cup f^{-1}(R)) \cap (\ker_{X \setminus Y}(f) \cup f^{-1}(R^{-1})) = \Delta_A,$$

thus $\ker_Y(f) \cup f^{-1}(R) \updownarrow \ker_{X \setminus Y}(f) \cup f^{-1}(R^{-1})$. Also $\alpha \cap \alpha^{-1} = \ker_Y(f)$ is obvious. To conclude the proof, it is enough to note that κ is a (γ, r_γ) quasiorder preserving function. □

Proposition 2.8 *Let (A, γ) be a quasiordered set, (X, ρ) a partially ordered set and $f : A \longrightarrow X$ a completely (γ, ρ) quasiorder preserving function: $(x, y) \in \gamma \Longleftrightarrow (f(x), f(y)) \in \rho$ for all $x, y \in A$. If $Y_i \subseteq X$, $i \in I$ is a collection of subsets, $\gamma \cap \ker(f) \subseteq \ker_{Y_i}(f)$ for all $i \in I$ and $\{R_i \mid i \in I\}$ is a set of linear extensions of ρ with $\bigcap_{i \in I} R_i = \rho$, then*

$$\bigcap_{i \in I} (\ker_{Y_i}(f) \cup f^{-1}(R_i)) = \gamma,$$

where the half-spaces $\ker_{Y_i}(f) \cup f^{-1}(R_i)$, $i \in I$ are described in Proposition 2.7. In particular, if $\kappa : A \longrightarrow A/(\gamma \cap \gamma^{-1})$ is the canonical surjection and $\{R_i \mid i \in I\}$ is a set of linear extensions of the induced partial order r_γ on $A/(\gamma \cap \gamma^{-1})$ with $\bigcap_{i \in I} R_i = r_\gamma$, then

$$\bigcap_{i \in I} (\ker(\kappa) \cup \kappa^{-1}(R_i)) = \bigcap_{i \in I} ((\gamma \cap \gamma^{-1}) \cup \kappa^{-1}(R_i)) = \gamma.$$

Proof We only have to show that

$$\bigcap_{i \in I} (\ker_{Y_i}(f) \cup f^{-1}(R_i)) \subseteq \gamma.$$

In view of the definition of $\ker_{Y_i}(f) \cup f^{-1}(R_i)$, the relation

$$(a, b) \in \bigcap_{i \in I} (\ker_{Y_i}(f) \cup f^{-1}(R_i))$$

ensures that $f(a) \leq_{R_i} f(b)$ for all $i \in I$. Now $\bigcap_{i \in I} R_i = \rho$ implies $f(a) \leq_\rho f(b)$, whence we obtain $(a, b) \in \gamma$. To conclude the proof, it is enough to note that κ is completely (γ, r_γ) quasiorder preserving. □

The following is now a straightforward consequence.

Theorem 2.9 *Any quasiorder on A can be obtained as an intersection of half-space quasiorders on A.*

In terms of the classification of convexities by separation axioms (van de Vel [9]) the above theorem means that the convexity on the base set $\{(x, y) \in A \times A \mid x \neq y\}$ whose convex sets are the strict quasiorders on A is an S_3 convexity, i.e. a convex set K can be always separated from any element of the base set not in K by complementary half-spaces. This is the case in the standard convexity of a Euclidean space and, as Szpilrajn's theorem [7] shows, in the convexity on the base set $\{(x, y) \in A \times A \mid x \neq y\}$ whose convex sets are the strict partial orders on A plus $\{(x, y) \in A \times A \mid x \neq y\}$ itself. However, it is not difficult to see that, unlike in Euclidean space, in quasiorder convexity or in the coarser partial order convexity, disjoint convex sets cannot always be separated by complementary half-spaces. A counterexample with respect to both the quasiorder and partial order convexities is provided, for $A = \{1, 2, 3, 4\}$, by the partial orders $\{(1, 2), (3, 4)\}$ and $\{(1, 4), (3, 2)\}$.

Theorem 2.9 enables us to define a *half-space realizer* of a quasiorder $\gamma \subseteq A \times A$ as a set $\{\alpha_i \mid i \in I\}$ of half-spaces on A with $\bigcap_{i \in I} \alpha_i = \gamma$. The *half-space dimension* $\operatorname{hsdim}(A, \gamma)$ of a quasiordered set (A, γ) is the minimum of the cardinalities of the half-space realizers of γ. The close analogy between the half-space dimension and the usual order dimension of a partially ordered set can be seen immediately. The observation preceding Proposition 2.2 guarantees that

$$\operatorname{hs\,dim}(B, \gamma \cap (B \times B)) \leq \operatorname{hs\,dim}(A, \gamma)$$

for any subset $B \subseteq A$. Since any linear order is a half-space, for a partially ordered set (A, γ) we have $\operatorname{hsdim}(A, \gamma) \leq \dim(A, \gamma)$, where dim denotes the order dimension. In general, here we can not expect equality. The partial order of the four element Boolean lattice M_2 is a half-space, thus $\operatorname{hsdim}(M_2, \leq) = 1$, while $\dim(M_2, \leq) = 2$. The next inequality is also a straightforward consequence of Proposition 2.8.

Corollary 2.10 *For a quasiordered set (A, γ) we have*

$$\operatorname{hs\,dim}(A, \gamma) \leq \dim(A/(\gamma \cap \gamma^{-1}), r_\gamma).$$

The following theorem gives a complete description of half-space quasiorders.

Theorem 2.11 *If $\alpha \subseteq A \times A$ is a relation, then the following are equivalent.*

(1) *α is a half-space quasiorder on A.*

(2) *There exists an equivalence relation* ε *on* A*, a linear order* R *on the factor set* A/ε *and a function* $t : A/\varepsilon \longrightarrow \{0, 1\}$ *with* $t([a]_\varepsilon) = 0$ *where* $[a]_\varepsilon = \{a\}$ *such that*

$$\alpha = \Delta_A \cup \{(a, b) \in A \times A \mid [a]_\varepsilon = [b]_\varepsilon \text{ and } t([a]_\varepsilon) = 1\} \cup \{(a, b) \in A \times A \mid [a]_\varepsilon <_R [b]_\varepsilon\}.$$

(3) *There exist a set* X*, a subset* $Y \subseteq X$*, a linear order* R *on* X *and a function* $f : A \longrightarrow X$ *such that* $\alpha = \ker_Y(f) \cup f^{-1}(R)$.

(4) *There exists an equivalence relation* ε *on* A *such that* α *is either the full or the identity relation on each* ε*-equivalence class, and any irredundant set of representatives of the* ε*-equivalence classes is linearly ordered by* α.

Proof (1) $\Longrightarrow$ (2): Let $\alpha \updownarrow \beta$ be complementary half-spaces and take

$$\varepsilon = (\alpha \cap \alpha^{-1}) \cup (\beta \cap \beta^{-1}).$$

Clearly, ε is reflexive and symmetric. Assume that $(x, y) \in \alpha \cap \alpha^{-1}$ and $(y, z) \in \beta \cap \beta^{-1}$. Since $\alpha \cup \beta = A \times A$, we have either $(x, z) \in \alpha$ or $(x, z) \in \beta$. In the first case $(y, x) \in \alpha$ implies that $(y, z) \in \alpha \cap \beta = \Delta_A$. In the second case $(z, y) \in \beta$ implies that $(x, y) \in \alpha \cap \beta = \Delta_A$. Thus $(x, y) \in \alpha \cap \alpha^{-1}$ and $(y, z) \in \beta \cap \beta^{-1}$ imply $x = y$ or $y = z$. Similarly, $(x, y) \in \beta \cap \beta^{-1}$ and $(y, z) \in \alpha \cap \alpha^{-1}$ also imply $x = y$ or $y = z$. In view of the above observations, it is easy to see that ε is transitive. We also have $[a]_\varepsilon = [a]_{\alpha\cap\alpha^{-1}} \cup [a]_{\beta\cap\beta^{-1}}$ and $[a]_{\alpha\cap\alpha^{-1}} = \{a\}$ or $[a]_{\beta\cap\beta^{-1}} = \{a\}$ for all $a \in A$.

We claim that $(a, b) \in \alpha$ and $[a]_\varepsilon \neq [b]_\varepsilon$ imply that $(x, y) \in \alpha$ for all $x \in [a]_\varepsilon$ and for all $y \in [b]_\varepsilon$. Suppose that $(x, y) \notin \alpha$, then $(x, y) \in \beta$. In view of $(x, a), (y, b) \in \varepsilon$ we have the following cases. (1) $(x, a), (b, y) \in \alpha$, whence $(x, y) \in \alpha$ can be obtained, a contradiction. (2) $(x, a) \in \alpha$ and $(y, b) \in \beta$, whence $(x, b) \in \alpha \cap \beta = \Delta_A$ can be obtained in contradiction with $[x]_\varepsilon = [a]_\varepsilon \neq [b]_\varepsilon$. (3) $(a, x) \in \beta$ and $(b, y) \in \alpha$, whence $(a, y) \in \alpha \cap \beta = \Delta_A$ can be obtained in contradiction with $[a]_\varepsilon \neq [b]_\varepsilon = [y]_\varepsilon$. (iv) $(a, x), (y, b) \in \beta$, whence $(a, b) \in \alpha \cap \beta = \Delta_A$ can be obtained in contradiction with $[a]_\varepsilon \neq [b]_\varepsilon$. Thus the claim is proved.

Using our claim it is straightforward to check that

$$R = \{([a]_\varepsilon, [b]_\varepsilon) \mid (a, b) \in \alpha\}$$

is a linear order on A/ε. For $a \in A$ let

$$t([a]_\varepsilon) = \begin{cases} 1 \text{ if } [a]_\varepsilon = [a]_{\alpha\cap\alpha^{-1}} \neq \{a\} \\ 0 \text{ otherwise} \end{cases}.$$

Clearly, t is well defined, moreover $[a]_\varepsilon = \{a\}$ implies $[a]_\varepsilon = [a]_{\alpha\cap\alpha^{-1}} = \{a\}$ and $t([a]_\varepsilon) = 0$. If $t([a]_\varepsilon) = 1$, then $[a]_\varepsilon = [a]_{\alpha\cap\alpha^{-1}}$ and $[a]_\varepsilon = [b]_\varepsilon$ implies that $(a, b) \in \alpha$. It follows that

$$\Delta_A \cup \{(a, b) \in A \times A \mid [a]_\varepsilon = [b]_\varepsilon \text{ and } t([a]_\varepsilon) = 1\} \cup \{(a, b) \in A \times A \mid [a]_\varepsilon <_R [b]_\varepsilon\} \subseteq \alpha.$$

If $[a]_\varepsilon = [b]_\varepsilon$, then $(a, b) \in \alpha$ and $a \neq b$ implies that $[a]_\varepsilon = [a]_{\alpha\cap\alpha^{-1}} \neq \{a\}$, whence

$$\alpha \subseteq \Delta_A \cup \{(a, b) \in A \times A \mid [a]_\varepsilon = [b]_\varepsilon \text{ and } t([a]_\varepsilon) = 1\} \cup \{(a, b) \in A \times A \mid [a]_\varepsilon <_R [b]_\varepsilon\}$$

can be obtained.

(2) $\Longrightarrow$ (3): It is straightforward to see that $\alpha = \ker_Y(f) \cup f^{-1}(R)$, where $X = A/\varepsilon$, $Y = \{[a]_\varepsilon \mid a \in A \text{ and } t([a]_\varepsilon) = 1\}$ and $f : A \longrightarrow X$ is the canonical surjection. Thus any half-space quasiorder can be obtained by the standard construction of Proposition 2.7.

(3) $\Longrightarrow$ (1): This implication is a part of Proposition 2.7.
(2) $\Longleftrightarrow$ (4): Condition (4) is simply a reformulation of (2). □

Remark 2.12 The triple $(f, Y \subseteq X, R)$ given in the (2) $\Longrightarrow$ (3) part of the above proof has the following universal property. If $g : A \longrightarrow U$ is a function, $V \subseteq U$ is a subset and S is a linear order on U such that

$$\alpha = \ker_V(g) \cup f^{-1}(S),$$

then there exists a unique function $h : X \longrightarrow U$ with $h \circ f = g$, moreover $h(Y) \subseteq V$, $g^{-1}(\{y\})$ is a one element set for all $y \in h(X \setminus Y) \cap V$ and h is $(<_R, <_S)$ strict order preserving

In view of the above characterization of the half-space α, an equivalence class $[a]_\varepsilon$ is called a *box* of α, such a box is called *full* if $t([a]_\varepsilon) = 1$ and *empty* if $t([a]_\varepsilon) = 0$ (note that a one element box is always empty). A subset $B \subseteq A$ is a box of the half-space α, if and only if there are no elements $b_1, b_2 \in B$ such that $(b_1, b_2) \in \alpha$, $(b_2, b_1) \notin \alpha$ and B is maximal with respect to this property. A box is empty if $\alpha \cap (B \times B) = \Delta_B$ and full if $|B| > 1$ and $B \times B \subseteq \alpha$.

In certain situations it is also convenient to give a half-space as

$$\alpha = (B_w, w \in W, \leq_W, t),$$

where the subsets $B_w \subseteq A$, $w \in W$ are the boxes of α, the linear order $\leq_W$ is given on the index set W and $t(B_w) = 1$ or $t(B_w) = 0$ shows that B_w is full or empty. If W is finite, then we can write $W = \{1, 2, ..., n\}$ and $\alpha = (B_1 < B_2 < ... < B_n, t)$. If $\alpha \updownarrow \beta$ is a complementary pair of half-spaces, then α and β have the same boxes, a full α-box is an empty β-box and a full β-box is an empty α-box, moreover the linear order of the boxes in α and β are opposite to each other. It is also clear that $[a]_{\alpha \cap \alpha^{-1}} = \{a\}$ if $[a]_\varepsilon$ is empty and $[a]_{\alpha \cap \alpha^{-1}} = [a]_\varepsilon$ if $[a]_\varepsilon$ is full.

With reference to the terminology of interval decompositions and lexicographic sums of partial orders and more general relations (see e.g. [3–6]), it is clear from Condition (4) of Theorem 2.11 that half-space quasiorders are precisely the lexicographic relational sums of trivial and full binary relations over a linear order, i.e. they are the binary relations decomposable into intervals such that the restriction to each interval is a trivial or full relation and the quotient is a linear order.

Theorem 2.13 *If (A, γ) is a quasiordered set and $\{\alpha_i \mid i \in I\}$ is a half-space realizer of γ with $|I| \geq 2$, then there exists an I-indexed family R_i, $i \in I$ of linear extensions of the induced partial order r_γ on $A/(\gamma \cap \gamma^{-1})$ such that*

$$\bigcap_{i \in I} R_i = r_\gamma.$$

Proof By Proposition 2.4, for each $i \in I$ there exists a half-space τ_i on A such that $\gamma \subseteq \tau_i \subseteq \alpha_i$ and $\tau_i \cap \tau_i^{-1} = \gamma \cap \gamma^{-1}$. Clearly, $\bigcap_{i \in I} \alpha_i = \gamma$ implies that $\bigcap_{i \in I} \tau_i = \gamma$, whence

$$\bigcap_{i \in I} r_{\tau_i} = r_\gamma$$

can be derived for the induced partial orders $r_{\tau_i}, i \in I$ on $A/(\tau_i \cap \tau_i^{-1}) = A/(\gamma \cap \gamma^{-1})$. Using the notation $\pi_i = r_{\tau_i}$, Proposition 2.3 ensures that each partial order π_i is a half-space on $P = A/(\gamma \cap \gamma^{-1})$.

We claim, that

$$\rho = \Delta_P \cup \left(\left(\underset{i \in I}{\cup} \pi_i \right)^{-1} \setminus \left(\underset{i \in I}{\cup} \pi_i \right) \right)$$

is partial order on P. The reflexive and antisymmetric properties of ρ can be immediately seen. In order to prove the transitivity of ρ consider the pairs $(x, y) \in \rho$ and $(y, z) \in \rho$ with $x, y, z \in P$ being different. We have $(y, x) \in \pi_j$, $(z, y) \in \pi_k$ for some $j, k \in I$ and $(x, y) \notin \underset{i \in I}{\cup} \pi_i$, $(y, z) \notin \underset{i \in I}{\cup} \pi_i$. If $(z, y) \in \pi_j$, then the transitivity of π_j implies $(z, x) \in \pi_j$. If $(z, y) \notin \pi_j$, then $(y, z) \notin \pi_j$ and the half-space property of π_j imply that $(z, x) \in \pi_j$ (see part (3) of Proposition 2.2). It follows that $(x, z) \in \left(\underset{i \in I}{\cup} \pi_i \right)^{-1}$. Suppose that $(x, z) \in \underset{i \in I}{\cup} \pi_i$, then $(x, z) \in \pi_t$ for some $t \in I$. If $(z, y) \in \pi_t$, then the transitivity of π_t gives that $(x, y) \in \pi_t$, a contradiction. If $(z, y) \notin \pi_t$, then $(y, z) \notin \pi_t$ and the half-space property of π_t gives that $(x, y) \in \pi_t$ (see part (4) of Proposition 2.2), another contradiction. Thus $(x, z) \notin \underset{i \in I}{\cup} \pi_i$, whence $(x, z) \in \rho$ follows.

Let $\sigma_i \subseteq P \times P$ denote the complementary half-space of π_i and consider the following equivalence relation:

$$\Theta = \underset{i \in I}{\cap} (\sigma_i \cap \sigma_i^{-1})$$

on P. Since $\pi_i^{-1} \cap \sigma_i^{-1} = \Delta_P$ for all $i \in I$, we have $\rho \cap \Theta = \Delta_P$ and hence $\rho^{-1} \cap \Theta = \Delta_P$. Now we prove the containments $\Theta \circ \rho \subseteq \rho$ and $\rho \circ \Theta \subseteq \rho$. If $(x, y) \in \Theta$ and $(y, z) \in \rho$ for the elements $x, y, z \in P$ with x, y, z being different, then $(z, y) \in \pi_j$ for some $j \in I$ and $(y, z) \notin \underset{i \in I}{\cup} \pi_i$. In view of $(x, y) \in \sigma_j \cap \sigma_j^{-1}$, we have $(x, y) \notin \pi_j$ and $(y, x) \notin \pi_j$. Using part (4) in Proposition 2.2, we obtain that $(z, x) \in \pi_j$ and $(x, z) \in \left(\underset{i \in I}{\cup} \pi_i \right)^{-1}$. Suppose that $(x, z) \in \underset{i \in I}{\cup} \pi_i$, then $(x, z) \in \pi_k$ follows for some $k \in I$. Since $(x, y) \in \sigma_k \cap \sigma_k^{-1}$ implies that $(x, y) \notin \pi_k$ and $(y, x) \notin \pi_k$, the application of part (3) in Proposition 2.2 yields $(y, z) \in \pi_k$, a contradiction. Thus we have $(x, z) \notin \underset{i \in I}{\cup} \pi_i$, whence $(x, z) \in \rho$ follows. A similar argument shows that $\rho \circ \Theta \subseteq \rho$.

Fix a linear order μ on P, then $\mu \cap \Theta$ and $\mu^{-1} \cap \Theta$ are partial orders. Using the above properties of ρ and Θ, it is straightforward to see that $\rho \cup (\mu \cap \Theta)$ and $\rho \cup (\mu^{-1} \cap \Theta)$ are also partial orders on P.

Let $\lambda \supseteq \rho \cup (\mu \cap \Theta)$ and $\lambda^* \supseteq \rho \cup (\mu^{-1} \cap \Theta)$ be linear extensions on P and fix an index $i^* \in I$. In view of Proposition 2.5, we can consider the linear orders $R_i = \pi_i[\lambda]$, $i \in I \setminus \{i^*\}$ and $R_{i^*} = \pi_{i^*}[\lambda^*]$ on P (note that $I \setminus \{i^*\}$ is not empty). Since $\pi_i \subseteq R_i$ for all $i \in I$, the inclusion

$$r_\gamma = \bigcap_{i \in I} \pi_i \subseteq \bigcap_{i \in I} R_i$$

is obvious. In order to prove the reverse containment let $(x, y) \notin \bigcap_{i \in I} \pi_i$ for some $x, y \in P$. We have $(x, y) \notin \pi_j$ for some $j \in I$. If $(y, x) \in \bigcup_{i \in I} \pi_i$, then $(y, x) \in \pi_k \subseteq R_k$ and hence $(x, y) \notin R_k$ for some $k \in I$. If $(y, x) \notin \bigcup_{i \in I} \pi_i$, then we distinguish two cases.

First suppose that $(x, y) \in \bigcup_{i \in I} \pi_i$. Then $(y, x) \in \rho \subseteq \lambda \cap \lambda^*$ and the relations $(x, y) \notin \pi_j$, $(y, x) \notin \pi_j$ imply that $(y, x) \in \pi_j[\lambda]$ (or $(y, x) \in \pi_{i^*}[\lambda^*]$ if $j = i^*$), whence $(x, y) \notin R_j$ follows.

Next suppose that $(x, y) \notin \bigcup_{i \in I} \pi_i$. Then $(x, y) \in \Theta$ and the linearity of μ gives that we have either $(y, x) \in \mu \cap \Theta$ or $(y, x) \in \mu^{-1} \cap \Theta$. If $(y, x) \in \mu \cap \Theta \subseteq \lambda$, then $(y, x) \in \pi_i[\lambda]$ and hence $(x, y) \notin \pi_i[\lambda] = R_i$ for all $i \in I \setminus \{i^*\}$. If $(y, x) \in \mu^{-1} \cap \Theta \subseteq \lambda^*$, then $(y, x) \in \pi_{i^*}[\lambda^*]$ and hence $(x, y) \notin \pi_{i^*}[\lambda^*] = R_{i^*}$. □

Remark 2.14 Another possibility to construct the linear orders R_i in the above proof is the following. Fix a well ordering $<$ on I and for $i \in I \setminus \{i^*\}$ let

$$R_i = \pi_i \cup ((\sigma_i \cap \sigma_i^{-1}) \cap \Lambda) \cup (\Theta \cap \mu),$$

$$R_{i^*} = \pi_{i^*} \cup ((\sigma_{i^*} \cap \sigma_{i^*}^{-1}) \cap \Lambda) \cup (\Theta \cap \mu^{-1}),$$

where $\Lambda = \{(x, y) \mid (y, x) \in \pi_k \text{ and } (x, y) \in \bigcap_{i \in I, i < k} (\sigma_i \cap \sigma_i^{-1}) \text{ for some } k \in I\}$.

In view of Corollaries 2.6 and 2.10, the above Theorem 2.13 yields the following.

Theorem 2.15 *If* (A, γ) *is a quasiordered set and* $\operatorname{hsdim}(A, \gamma) = 1$, *then* γ *is a half-space and*

$$\dim(A/(\gamma \cap \gamma^{-1}), r_\gamma) = 1 \text{ if } \gamma \text{ has no empty box with more than one element,}$$

$$\dim(A/(\gamma \cap \gamma^{-1}), r_\gamma) = 2 \text{ if } \gamma \text{ has an empty box with more than one element.}$$

If $\operatorname{hsdim}(A, \gamma) \geq 2$, *then we have*

$$\dim(A/(\gamma \cap \gamma^{-1}), r_\gamma) = \operatorname{hs\,dim}(A, \gamma).$$

Theorem 2.16 *If* (A, γ) *is a partially ordered set and* $\operatorname{hsdim}(A, \gamma) = 1$, *then* γ *is a half-space and*

$$\dim(A, \gamma) = 1 \text{ if } \gamma \text{ is a linear order,}$$

$$\dim(A, \gamma) = 2 \text{ if } \gamma \text{ is not a linear order.}$$

If $\operatorname{hsdim}(A, \gamma) \geq 2$, *then we have*

$$\dim(A, \gamma) = \operatorname{hs\,dim}(A, \gamma).$$

3 Direct Product Irreducibility of Half-Space Quasiorders

If (A_i, γ_i), $i \in I$ is a family of quasiordered sets, then

$$\prod_{i\in I}\gamma_i = \{(\mathbf{a},\mathbf{b}) \mid \mathbf{a},\mathbf{b} \in \prod_{i\in I} A_i \text{ and } (\mathbf{a}(i),\mathbf{b}(i)) \in \gamma_i \text{ for all } i \in I\}$$

is a quasiorder on the product set $\prod_{i\in I} A_i$ (here $\mathbf{a}$ and $\mathbf{b}$ are functions $I \longrightarrow \bigcup_{i\in I} A_i$ such that $\mathbf{a}(i), \mathbf{b}(i) \in A_i$ for all $i \in I$). We call $(\prod_{i\in I} A_i, \prod_{i\in I}\gamma_i)$ the direct product of the above family. The kernel of the natural surjection

$$\varphi : \prod_{i\in I} A_i \longrightarrow \prod_{i\in I} A_i/(\gamma_i \cap \gamma_i^{-1})$$

is $\prod_{i\in I}(\gamma_i \cap \gamma_i^{-1})$, whence we obtain a natural bijection

$$\left(\prod_{i\in I} A_i\right) / \left(\prod_{i\in I}(\gamma_i \cap \gamma_i^{-1})\right) \longrightarrow \prod_{i\in I} A_i/(\gamma_i \cap \gamma_i^{-1}).$$

It is easy to see that

$$\left(\prod_{i\in I}\gamma_i\right) \cap \left(\prod_{i\in I}\gamma_i\right)^{-1} = \prod_{i\in I}(\gamma_i \cap \gamma_i^{-1}) \text{ and } r = \prod_{i\in I} r_{\gamma_i},$$

where r is the partial order on $\prod_{i\in I} A_i/(\gamma_i \cap \gamma_i^{-1})$ induced by the quasiorder $\prod_{i\in I}\gamma_i$.

The product of non-trivial partial orders is never a linear order. In contrast, the product of two half-spaces can be a half-space again: the four element Boolean lattice M_2 is a product of two-element chains. We show that this is the only possibility to get a non-trivial half-space as a product of quasiorders.

Lemma 3.1 *Let (A_i, γ_i), $i \in I$ be a family of quasiordered sets and let $j, k \in I$, $j \neq k$ be indices such that $a_j \neq c_j$, $(a_j, c_j) \in \gamma_j$, $(a_j, b_j) \notin \gamma_j$ for some $a_j, b_j, c_j \in A_j$ and $\gamma_k \neq A_k \times A_k$ with $|A_k| > 1$. Then $\prod_{i\in I}\gamma_i$ is not a half-space on $\prod_{i\in I} A_i$.*

Proof Let $\mathbf{u} \in \prod_{i\in I} A_i$ be an arbitrary element and $x_k, y_k \in A_k$ such that $(x_k, y_k) \notin \gamma_k$. Define $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \prod_{i\in I} A_i$ as follows: for an index $i \in I$ let

$$\mathbf{a}(i) = \begin{cases} a_j \text{ if } i = j \\ y_k \text{ if } i = k \\ \mathbf{u}(i) \text{ if } i \in I \setminus \{j,k\} \end{cases}, \quad \mathbf{b}(i) = \begin{cases} b_j \text{ if } i = j \\ x_k \text{ if } i = k \\ \mathbf{u}(i) \text{ if } i \in I \setminus \{j,k\} \end{cases},$$

$$\mathbf{c}(i) = \begin{cases} c_j \text{ if } i = j \\ y_k \text{ if } i = k \\ \mathbf{u}(i) \text{ if } i \in I \setminus \{j,k\} \end{cases}.$$

Clearly, $(a_j, b_j) \notin \gamma_j$ implies $(\mathbf{a}, \mathbf{b}) \notin \prod_{i\in I}\gamma_i$ and $(x_k, y_k) \notin \gamma_k$ implies $(\mathbf{b}, \mathbf{a}) \notin \prod_{i\in I}\gamma_i$. Since $(\mathbf{a}, \mathbf{c}) \in \prod_{i\in I}\gamma_i$ and $(x_k, y_k) \notin \gamma_k$ implies $(\mathbf{b}, \mathbf{c}) \notin \prod_{i\in I}\gamma_i$, we can use part (3) in Proposition 2.2

to see that $\prod_{i\in I}\gamma_i$ is not a half-space (we note that $\mathbf{c} \neq \mathbf{a}$ is an immediate consequence of $a_j \neq c_j$). □

Lemma 3.2 *If (A, γ) is a quasiordered set such that there are no elements $a, b, c \in A$ with $a \neq c$, $(a, c) \in \gamma$ and $(a, b) \notin \gamma$, then $\gamma \in \{\Delta_A, A \times A\}$ or $\gamma = (B_1 < B_2)$ is a half-space with a full lower box B_1 (or $|B_1| = 1$) and an empty upper box B_2.*

Proof If $\gamma \notin \{\Delta_A, A \times A\}$ satisfies the above conditions, then for each $a \in A$ we have either $(a, x) \in \gamma$ for all $x \in A$ or $(a, y) \notin \gamma$ for all $y \in A$. Take

$$B_1 = \{a \in A \mid (a, x) \in \gamma \text{ for all } x \in A\} \text{ and } B_2 = \{a \in A \mid (a, y) \notin \gamma \text{ for all } y \in A\},$$

then $B_1 \cup B_2 = A$, $B_1 \cap B_2 = \varnothing$ and $\gamma = B_1 \times A = (B_1 \times B_1) \cup (B_1 \times B_2)$ is a half-space, with a full lower box B_1 (or $|B_1| = 1$) and an empty upper box B_2. Thus we can write $\gamma = (B_1 < B_2)$. □

Lemma 3.3 *Let $\gamma_i = (B_{i1} < B_{i2})$, $1 \leq i \leq 2$ be half-spaces on A_i with full lower boxes B_{i1} (or $|B_{i1}| = 1$) and empty upper boxes B_{i2}. Then we have the following.*

(1) $\Delta_{A_1\times A_2} \neq \gamma_1 \times \gamma_2 \neq (A_1 \times A_2) \times (A_1 \times A_2)$ *and take* $\mathbf{a} = (a_{12}, a_{21})$, $\mathbf{b} = (a_{11}, a_{21})$, $\mathbf{c} = (a_{12}, a_{22})$, *where $a_{ij} \in B_{ij}$, $i, j \in \{1, 2\}$ are arbitrary elements. Then* $\mathbf{a} \neq \mathbf{c}$, $(\mathbf{a}, \mathbf{c}) \in \gamma_1 \times \gamma_2$ *and* $(\mathbf{a}, \mathbf{b}) \notin \gamma_1 \times \gamma_2$.

(2) *$\gamma_1 \times \gamma_2$ is a half-space if and only if $\left|B_{ij}\right| = 1$ for all $i, j \in \{1, 2\}$.*

Proof

(1): Obvious.

(2): If $\left|B_{ij}\right| = 1$ for all $i, j \in \{1, 2\}$, then it is clear that $A_1 \times A_2$ is a four element set and $\gamma_1 \times \gamma_2$ is a partial order relation on $A_1 \times A_2$ providing a lattice isomorphic to M_2, which is a half-space as we have already noted.

Suppose now, that $|B_{11}| > 1$ and take $a', a'' \in B_{11}$ such that $a' \neq a''$. Let $\mathbf{z} = (a', b)$, $\mathbf{x} = (a'', b)$ and $\mathbf{y} = (a, c)$, where $a \in B_{12}$, $b \in B_{22}$, $c \in B_{21}$ are arbitrary elements. Since $(\mathbf{x}, \mathbf{y}) \notin \gamma_1 \times \gamma_2$, $(\mathbf{y}, \mathbf{x}) \notin \gamma_1 \times \gamma_2$ and $(\mathbf{x}, \mathbf{z}) \in \gamma_1 \times \gamma_2$, $(\mathbf{y}, \mathbf{z}) \notin \gamma_1 \times \gamma_2$, we can apply part (3) in Proposition 2.2 to derive that $\gamma_1 \times \gamma_2$ is not a half-space.

If $|B_{12}| > 1$ then take $a', a'' \in B_{12}$ such that $a' \neq a''$. Let $\mathbf{z} = (a', b)$, $\mathbf{x} = (a', c)$ and $\mathbf{y} = (a'', c)$, where $b \in B_{22}$, $c \in B_{21}$ are arbitrary elements. Since $(\mathbf{x}, \mathbf{y}) \notin \gamma_1 \times \gamma_2$, $(\mathbf{y}, \mathbf{x}) \notin \gamma_1 \times \gamma_2$ and $(\mathbf{x}, \mathbf{z}) \in \gamma_1 \times \gamma_2$, $(\mathbf{y}, \mathbf{z}) \notin \gamma_1 \times \gamma_2$, we can apply part (3) in Proposition 2.2 to derive that $\gamma_1 \times \gamma_2$ is not a half-space.

The cases $|B_{21}| > 1$ and $|B_{22}| > 1$ can be treated analogously. □

Theorem 3.4 *If (A_i, γ_i), $i \in I$ is a family of non-trivial quasiordered sets (i.e. $\Delta_{A_i} \neq \gamma_i \neq A_i \times A_i$ for all $i \in I$), then the following are equivalent.*

(1) *$\prod_{i\in I}\gamma_i$ is a half-space on $\prod_{i\in I} A_i$.*

(2) *Either $I = \{1\}$ and γ_1 is a half-space or $I = \{1, 2\}$ and (A_1, γ_1), (A_2, γ_2) are two-element chains.*

Proof

(2) $\Longrightarrow$ (1): It is an immediate consequence of part (2) in Lemma 3.3.

(1) $\Longrightarrow$ (2): It is enough to deal with the case $|I| \geq 2$. Using Lemma 3.1, we obtain that there is no $j \in I$ such that $a_j \neq c_j$, $(a_j, c_j) \in \gamma_j$, $(a_j, b_j) \notin \gamma_j$ for some $a_j, b_j, c_j \in A_j$. In view of Lemma 3.2, each γ_j is a half-space on A_j of the form $\gamma_j = (B_{j1} < B_{j2})$ with a full lower box B_{j1} (or $|B_{j1}| = 1$) and an empty upper box B_{j2}. If $|I| \geq 3$, then we have different indices $i_1, i_2, i_3 \in I$ and

$$\prod_{i \in I} \gamma_i = (\gamma_{i_1} \times \gamma_{i_2}) \times \gamma_{i_3} \times \left(\prod_{i \in I \setminus \{i_1, i_2, i_3\}} \gamma_i \right),$$

where $\gamma_{i_1} \times \gamma_{i_2}$ has the property described in part (1) of Lemma 3.3. Since $\gamma_{i_3} \neq A_{i_3} \times A_{i_3}$ with $|A_{i_3}| > 1$, Lemma 3.1 ensures that our product is not a half-space, a contradiction. Thus $|I| = 2$ and part (2) in Lemma 3.3 gives that (A_1, γ_1) and (A_2, γ_2) are two-element chains (here we assumed $I = \{1, 2\}$). □

Remark 3.5 If $\gamma_j = \Delta_{A_j}$ and $|A_j| > 1$ for some $j \in I$, then $\prod_{i \in I} \gamma_i$ is disconnected, hence not a non-trivial half-space (because $\prod_{i \in I} \gamma_i = \Delta$ would be the only possibility to get a half-space). If $\gamma_j = A_j \times A_j$ for some $j \in I$, then γ_j has no effect on wether the product $\prod_{i \in I} \gamma_i$ is a half-space (in other words $\prod_{i \in I} \gamma_i$ is a half-space if and only if $\prod_{i \in I \setminus \{j\}} \gamma_i$ is a half-space).

Now, as promised in the introduction, we illustrate the use of half-spaces in a short proof of the following statement.

Theorem 3.5 (Dushnik–Miller) *If (A_i, R_i), $i \in I$ is a family of non-trivial linearly ordered sets (chains) with $|I| \geq 2$, then*

$$\dim \left(\prod_{i \in I} A_i, \prod_{i \in I} R_i \right) \leq |I|.$$

Proof For an index $j \in I$ let π_j denote the natural $\prod_{i \in I} A_i \longrightarrow A_j$ projection. We have

$$\bigcap_{j \in I} \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \prod_{i \in I} A_i \text{ and } \mathbf{a}(j) \leq_{R_j} \mathbf{b}(j)\} = \prod_{i \in I} R_i,$$

for the half-spaces

$$\{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \prod_{i \in I} A_i \text{ and } \mathbf{a}(j) \leq_{R_j} \mathbf{b}(j)\} = \ker(\pi_j) \cup \pi_j^{-1}(R_j), \quad j \in I$$

(see Proposition 2.7). If $\prod_{i \in I} R_i$ is not a half-space, then

$$\dim \left(\prod_{i \in I} A_i, \prod_{i \in I} R_i \right) = \operatorname{hs\,dim} \left(\prod_{i \in I} A_i, \prod_{i \in I} R_i \right) \leq |I|$$

by Theorem 2.16. If $\prod_{i \in I} R_i$ is a half-space, then $|I| = 2$ and $(\prod_{i \in I} A_i, \prod_{i \in I} R_i)$ is the four element Boolean lattice by Theorem 3.4. □

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