SALEM NUMBERS DEFINED BY COXETER TRANSFORMATION

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Abstract. A real algebraic integer $\alpha > 1$ is called a *Salem number* if all its remaining conjugates have modulus at most 1 with at least one having modulus exactly 1. It is known ([12], [10], [5]) that the spectral radii of Coxeter transformation defined by stars, which are neither of Dynkin nor of extended Dynkin type, are Salem numbers. We prove that the spectral radii of the Coxeter transformation of *generalized stars* are also Salem numbers. A generalized star is a connected graph without multiple edges and loops that has exactly one vertex of degree at least 3.

1. Introduction

A real algebraic integer $\alpha > 1$ is called a *Salem number* if all its remaining conjugates have modulus at most 1 with at least one having modulus exactly 1. The corresponding minimal polynomial $P(z)$ of these numbers, called a *Salem polynomial*, is reciprocal, that is $z^{\deg P}P(1/z) = P(z)$. Salem numbers have appeared in quite different branches of mathematics (harmonic analysis, knot theory, number theory etc).

The smallest known Salem number ($\approx 1.176281...$) is a zero of the reciprocal polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

This polynomial is the reciprocal polynomial of a star (in the sense of [11]) and meanwhile it is the Coxeter polynomial of the same graph (at any orientation) ([12]).

In the light of this, using a restricted class of graphs (star-like trees, and bipartite generalized stars) in [10] and [6] Salem numbers were constructed. Recently McKee and Smyth in [11] have found all trees that define Salem numbers. The reciprocal polynomial of any bipartite graph (in the sense of [11]) is the same as the Coxeter polynomial of the graph (at bipartite orientation)([12]).

In this paper using Coxeter polynomials of generalized stars we give a class of Salem numbers. Our result shows that even if the reciprocal polynomial of a non-bipartite graph is not a Salem polynomial, then the Coxeter polynomial of the graph (by any orientation without oriented cycles) might be a Salem polynomial.

The zeros of the Coxeter polynomial of bipartite graph (at bipartite orientation) are either on the unit circle or they are positive real numbers ([12]).

There exist Coxeter polynomials whose non-real zeros are not on the unit circle ([8]). There, the underlying graph has multiple edges. It is not known whether the zeros of Coxeter polynomials of non-bipartite graphs with simple edges are on $\mathbb{S} \cup \mathbb{R}^+$ ($\mathbb{S}$ is the unit circle). Our results settle this question for generalized stars.

Date: July 26, 2009.
1991 *Mathematics Subject Classification.* Primary 11R06, Secondary 11C08, 05C50, 20F55, 30C15 .
Key words and phrases. Coxeter polynomial, semi-reciprocal polynomials, Chebyshev transform, Salem number .
Research supported by Hungarian OTKA grants No. NK68040 and NK72523.

1. Notions and preliminary results

The concept of Coxeter transformation came from the representation theory of finite dimensional algebras and it is defined on finite oriented graphs without oriented cycles.

Let Δ be a connected (finite) simple graph (graph without multiple edges and loops) with set of vertices $I = \{1, 2, \ldots, n\}$ and set of edges $J = \{(i, j)|i, j \in I;\ i \neq j\}$, which are unordered pairs of vertices. An orientation $<$ of Δ assigns to each edge of Δ a unique direction, i.e. for $(i, j) \in J$, there is an arrow either from i to j or from j to i if $i \neq j$. Denote by $Q = (\Delta, <)$ the oriented graph derived from Δ.
In the sequel *we always consider orientations without oriented cycles*.

Let Q be an oriented graph and for all vertices i, j of Q let $b_{i,j}$ be the number of edges (0 or 1) from the vertex i to j. Then the matrix $B = \left(b_{i,j}\right)$ is called the *adjacency matrix* of Q.

There are different equivalent definitions of Coxeter transformation for an oriented graph (the first one in [3]), here we use the definition of [12].

The Coxeter transformation $\mathcal{C}_Q$ of Q (with respect to its order of vertices!) is defined by the matrix $\Phi_Q = -(E - B)^{-1}(E - B)^{\top}$, where E is the identity matrix, B is the adjacency matrix of Q. This integral matrix plays an important role in the study of representations of certain algebras. The characteristic polynomial of the Coxeter transformation $\mathcal{C}_Q$ (or Φ_Q) is called the *Coxeter polynomial* of Q and it is denoted by $\chi_Q(z)$. It is known ([12]) that *Coxeter polynomials are reciprocal polynomials.* The maximum of absolute values of the eigenvalues of the Coxeter transformation is called *the spectral radius of the Coxeter transformation.* If the underlying graph is not of Dynkin type, then the eigenvalue of maximal modulus is a real, simple eigenvalue of the Coxeter transformation (see [13]).

A vertex $x \in I$ is called a *sink* of $(\Delta, <)$ if there is no arrow in $(\Delta, <)$ leaving x and the vertex $y \in I$ is called a *source* if there is no arrow entering y. Let x be a source. Reverse the orientation of all edges starting at x. This way we have an oriented graph $(\Delta, \tilde{<})$ where the vertex x is a sink and we call such a change of orientation of Δ an *admissible change of the orientation*. The Coxeter transformation of $(\Delta, <)$ is similar to the Coxeter transformation of $(\Delta, \tilde{<})$ (see [13]), i.e. admissible changes of orientation preserve the Coxeter polynomial.

To determine the location of zeros of reciprocal polynomials we apply a *Chebyshev transformation* described in [7]. This transformation for the same purpose is used by Lenzing and Pena in [9] for a special class of polynomial. They used a terminology different from ours.

A polynomial $f(z) = \sum\limits_{k=0}^{2n} a_k z^k \in \mathbb{R}[z]$ is called a *semi-reciprocal polynomial* if

$$a_{2n-k} = a_k \quad (k = 0, \ldots, 2n).$$

We do not require $a_{2n} = 0$ as in case of reciprocal polynomials.

For a semi-reciprocal polynomial $f(z)$ we have

$$f(z) = z^n \left(a_n + \sum_{j=1}^{n} a_{n+j} \left(z^j + \frac{1}{z^j} \right) \right).$$

It is known that if $z+\dfrac{1}{z}=x$ then $z^j+\dfrac{1}{z^j}=C_j(x)$ $(j=1,2,\dots)$, (see e.g. [14], p. 224) where

$$C_j(x) := 2T_j\left(\frac{x}{2}\right) \quad (x\in\mathbb{C},\ j=1,2,\dots)$$

are the normalized Chebyshev polynomials of the first kind, and T_j are the jth Chebyshev polynomials of the first kind, defined by $T_j(\cos x)=\cos jx$ $(j=0,1,\dots)$. Defining $C_0(x)=T_0(x)=1$, $(x\in\mathbb{C})$ we have

$$f(z)=z^n\sum_{j=0}^{n}a_{n+j}C_j(x) \quad \left(z+\frac{1}{z}=x\right).$$

The polynomial $\sum\limits_{j=0}^{k}a_{n+j}C_j(x)$ here is called the *Chebyshev transform of* $f(z)$ and it is denoted by $\mathcal{T}f(x)$.

The Chebyshev transform is a linear operator which is an isomorphism of the (real) vector space $\mathcal{R}_{2n}$ (the set of all semi-reciprocal polynomials of degree at most $m=2n$ over $\mathbb{R}$) onto the set of all polynomials of degree at most n over $\mathbb{R}$ (see [4]).

The following Lemma is well known.

Lemma 1 ([4])**.** *Let $f(z)$ be a monic integral reciprocal polynomial of degree n and let $\hat{f}(z)$ be defined by*

$$\hat{f}(z)=\begin{cases} f(z) & \text{if } n \text{ is even,}\\ f(z)(z+1) & \text{if } n \text{ is odd.}\end{cases}$$

Then $f(z)$ is the product of a Salem polynomial and certain cyclotomic polynomials if and only if the Chebyshev transform $\mathcal{T}\hat{f}(x)$ has $\left[\frac{n-1}{2}\right]$ (the integer part of $\frac{n-1}{2}$) zeros in the interval $[-2,2]$.

A *generalized star* is a connected graph without multiple edges and loops that has exactly one vertex i_0 of degree at least 3. The vertex i_0 is called the *branching vertex.*

We remark that a generalized star (which does not consist of one or two cycles) can be created from a star graph by replacing each edge by a path or a cycle.

We may assume, without loss of generality, that the considered generalized star is not a tree and not a single cycle.

Let Δ be a generalized star. In the sequel let $\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_s$ $(s\ge 1)$ be the cycles in Δ with the number of edges $c_1,c_2,\dots c_s$, $(c_i>2, 1\le i\le s)$, respectively. Suppose that Δ consists of $r\ge 0$ number of arms of lengths $1\le p_1\le p_2\le\cdots\le p_r$, where $r\ge 1$ if $s=1$. Thus, the number of vertices of Δ is equal to

$$n=\sum_{i=1}^{r}p_i+1+\sum_{i=1}^{s}c_i-s.$$

For a generalized star with arbitrary orientation without cycles there is an orientation with a unique source having the same Coxeter polynomial. Denote by $Q=(\Delta,<)$ the orientation of the generalized star Δ and by i_0 the unique source and by a_i and b_i the number of arrows in $\mathcal{S}_i$ pointing to counterclockwise direction and to clockwise direction respectively. Consequently $c_i=a_i+b_i$, where $a_i,b_i\ge 1$.

For this Q also the detailed notation $Q=\Delta_{[(a_1,b_1),(a_2,b_2),\dots,(a_s,b_s),p_1,p_2,\dots,p_r]}$ will be used.

Denote by $\chi_Q(z) = \chi_{[(a_1,b_1),(a_2,b_2),\dots,(a_s,b_s),p_1,p_2,\dots,p_r]}(z)$ the Coxeter polynomial of Q. Throughout this paper we use the notation

$$v_m(z) = \frac{z^{m+1}-1}{z-1} = z^m + z^{m-1} + \dots + z^2 + z + 1,\ (m \in \mathbb{N}) \text{ and } v_0(z) = 1,\ v_{-1}(z) = 0.$$

Likewise, let

$$u_{m,h}(z) = 2v_m(z) + \left(z^{m-h} + z^h\right) \quad \left(m \in \mathbb{N} \ \ h = 1, \dots \left[\frac{m}{2}\right]\right).$$

For odd $m = 2n+1$, let

$$\hat{v}_{2n}(z) = \frac{v_{2n+1}(z)}{z+1} \text{ and } \hat{u}_{2n,h}(z) = \frac{u_{2n+1,h}(z)}{z+1} \quad (n \in \mathbb{N} \ \ h = 1, 2, \dots, n).$$

Using the reduction formulas for Coxeter polynomials of [1], we get that:

$$\begin{aligned} f(z) := \chi_Q(z) = \chi_{[(a_1,b_1),(a_2,b_2),\dots,(a_s,b_s),p_1,p_2,\dots,p_r]} = (z+1)\prod_{j=1}^{s} v_{a_j+b_j-1}(z)\prod_{i=1}^{r} v_{p_i}(z) \\ -z\sum_{k=1}^{s}\left((2v_{a_k+b_k-2}(z) + z^{a_k-1} + z^{b_k-1})\prod_{j=1,\, j\neq k}^{s} v_{a_j+b_j-1}(z)\prod_{i=1}^{r} v_{p_i}(z)\right) \\ -z\sum_{k=1}^{r}\left(v_{p_k-1}(z)\prod_{j=1}^{s} v_{a_j+b_j-1}(z)\prod_{i=s+1,\, i\neq k}^{s+r} v_{p_i}(z)\right). \end{aligned} \tag{1}$$

If Δ is not a tree then the Coxeter polynomial depends on the orientation. If Δ is a bipartite graph then the spectrum of the Coxeter transformation (at a particular orientation, provided that Δ is not a tree) can be derived from the spectrum of the adjacency matrix of the underlying graph and vice versa (see [3]).

In the bipartite case Theorem 3.3 of [11] gives sufficient conditions for a Coxeter polynomial to be Salem. In [6] we described a class of graphs with this property and have shown that our class is larger than the class given in [5]:

Theorem ([6]). *Let $(\Delta, <)$ be a bipartite orientation of a generalized star Δ with even cycles. The spectral radius of the Coxeter transformation of $(\Delta, <)$ is a Salem number.*

Our main result generalizes this Theorem for arbitrary (not necessarily bipartite) graphs.

2. Theorem and proof

Theorem . *The Coxeter polynomial of all oriented generalized stars without oriented cycles is a product of a Salem polynomial and certain cyclotomic polynomials.*

To count the changes of sign of the Chebyshev transform of (1) in the interval $[-2, +2]$ we have to describe how the terms $\mathcal{T}\hat{u}_{2n-2,h}(x)$ and $\mathcal{T}u_{2n,h}(x)$ (which appear in a modified form of this transform) behave at the zeros of $\mathcal{T}v_{2n}(x)$ and $\mathcal{T}\hat{v}_{2n}(x)$. First we have to recall some results of [7]. The next Lemma is a special case of Theorem 1 in [7].

Lemma 2 ([7]). *All zeros of the reciprocal polynomial $u_{m,h}(z)$ ($m \geq 2$, $h = 1, \dots, \left[\frac{m}{2}\right]$) of degree m are on the unit circle. Moreover in case $m = 2n$ all zeros of $u_{2n,h}$ can be given as $e^{\pm i x_j}$ ($j = 1, \dots, n$) where*

$$\frac{2j-1}{m+1}\pi < x_j < \frac{2j+1}{m+1}\pi \quad (j = 1, 2, \dots, n-1), \qquad \frac{2n-1}{2n+1}\pi < x_n \leq \pi. \tag{2}$$

In the last inequality $x_n \leq \pi$ equality holds if and only if h is odd, and then $-1 = e^{i\pi} = e^{-i\pi}$ is a double zero, all other zeros are simple.

In case of $m = 2n+1$ all zeros of $u_{2n+1,h}$ are simple, have the form $e^{\pm i x_j}$ ($j = 1, \dots, n$), $e^{i\pi}$ where these x_j's satisfy the first inequality of (2) for all $j = 1, 2, \dots, n$.

In [7] we found that

$$\begin{aligned} \mathcal{T} v_{2n}(x) &= U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right), & (3)\\ \mathcal{T} \hat{v}_{2n}(x) &= 2U_n\left(\frac{x}{2}\right), & (4)\\ \mathcal{T} u_{2n,h}(x) &= 2\mathcal{T} v_{2n}(x) + 2T_{n-h}\left(\frac{x}{2}\right) \quad (h = 1, \dots, n), & (5)\\ \mathcal{T} \hat{u}_{2n-2,h}(x) &= 2\mathcal{T} \hat{v}_{2n-2}(x) + \left(U_{n-h-1}\left(\frac{x}{2}\right) - U_{n-h-2}\left(\frac{x}{2}\right)\right) \quad (h = 1, \dots, n-1), & (6) \end{aligned}$$

where U_n and T_n are the nth Chebyshev polynomials of the second and first kind, respectively, and $U_{-1}(z) := 0$. Further the zeros of $\mathcal{T}\hat{v}_{2n}$ are $\beta_j = 2\cos y_j$ with $y_j = \frac{j\pi}{n+1}$ ($j = 1, \dots, n$) and the zeros of $\mathcal{T} v_{2n}$ are $\gamma_j = 2\cos z_j$ with $z_j = \frac{2j\pi}{2n+1}$ ($j = 1, \dots, n$).

Lemma 3. *Let α_j, β_j ($j = 1, 2, \dots n$) be the zeros of the polynomials $\mathcal{T} u_{2n,h}(x)$, $\mathcal{T}\hat{v}_{2n}(x)$ respectively, where $h = 1, 2, \dots n$ is fixed.*

(a) *The zeros α_j, β_j interlace, that is*

$$-2 \leq \alpha_n \leq \beta_n < \alpha_{n-1} \leq \beta_{n-1} < \cdots < \alpha_j \leq \beta_j < \alpha_{j-1} \leq \beta_{j-1} < \cdots \leq \alpha_1 < \beta_1 < 2.$$

(b) *If $\mathcal{T} u_{2n,h}(\beta_j) \neq 0$ for some $j = 1, 2, \dots, n$ then*

$$\operatorname{sgn} \mathcal{T} u_{2n,h}(\beta_j) = \operatorname{sgn} \mathcal{T} v_{2n}(\beta_j).$$

(c) *If $\mathcal{T} u_{2n,h}(\beta_j) = 0$ for some $j = 1, 2, \dots, n$ then for a suitable small $\varepsilon > 0$*

$$\operatorname{sgn} \mathcal{T} u_{2n,h}(\beta_j + \varepsilon) = \operatorname{sgn} \mathcal{T} v_{2n}(\beta_j).$$

(d) *If $\mathcal{T} u_{2n,h}(-2) = 0$ then for $\varepsilon > 0$ small enough*

$$\operatorname{sgn} \mathcal{T} u_{2n,h}(-2 - \varepsilon) = \operatorname{sgn} \mathcal{T} v_{2n}(-2).$$

Proof. Introducing the notation $\hat{z}_j = \frac{(2j+1)\pi}{2n+1}$, $\hat{\gamma}_j = 2\cos\hat{z}_j$ ($j = 1, \dots, n$) (the $\hat{z}_j$'s are near to z_j) we have by Lemma 3, that

$$-2 = \hat{\gamma}_n \leq \alpha_n < \hat{\gamma}_{n-1}, \quad \hat{\gamma}_j < \alpha_j < \hat{\gamma}_{j-1} \ (j = 1, \dots, n-1) \tag{7}$$

and for even h also the first inequality is strict: $-2 = \hat{\gamma}_n < \alpha_n < \hat{\gamma}_{n-1}$. Since $\hat{z}_j > y_j > \hat{z}_{j-1}$ ($j = 1, \dots, n$) and the cos function is strictly decreasing in $[0, \pi]$ we also have

$$\hat{\gamma}_j < \beta_j < \hat{\gamma}_{j-1} \ (j = 1, \dots, n). \tag{8}$$

By (7), (8) $\mathcal{T}u_{2n,h}(x)$, $\mathcal{T}\hat{v}_{2n}(x)$ both have one zero in each interval $[\hat{\gamma}_j, \hat{\gamma}_{j-1})$ $(j = 1, \dots, n)$ thus they are necessarily simple.

Next, we find the signs of $\mathcal{T}v_{2n}$ and $\mathcal{T}u_{2n,h}$ at β_j $(j = 1, \dots, n)$ and at -2.
Using (3) and some trigonometrical identities we get

$$2\mathcal{T}v_{2n}(\beta_j) = 2\left(U_n\left(\tfrac{\beta_j}{2}\right) + U_{n-1}\left(\tfrac{\beta_j}{2}\right)\right) = 2\frac{\sin\frac{2n+1}{2}y_j}{\sin\frac{1}{2}y_j} = 2\frac{\sin(1-\frac{1}{2n+2})j\pi}{\sin\frac{1}{2}\frac{j\pi}{n+1}} = 2(-1)^{j+1},$$

$$2\mathcal{T}v_{2n}(-2) = 2\left(U_n(-1) + U_{n-1}(-1)\right) = 2\left((-1)^n(n+1) + (-1)^{n-1}n\right) = 2(-1)^n.$$

Similarly, by (5) and some elementary calculation we obtain

$$\begin{aligned}\mathcal{T}u_{2n,h}(\beta_j) &= 2\mathcal{T}v_{2n}(\beta_j) + 2T_{n-h}\left(\tfrac{\beta_j}{2}\right) = 2\left((-1)^{j+1} + \cos\tfrac{(n-h)j\pi}{n+1}\right),\\ \mathcal{T}u_{2n,h}(-2) &= 2\mathcal{T}v_{2n}(-2) + 2T_{n-h}(-1) = 2(-1)^n + 2(-1)^{n-h}.\end{aligned}$$

Using these two formulae one can easily check that

$$\operatorname{sgn}\mathcal{T}u_{2n,h}(\beta_j) = \begin{cases} (-1)^{j+1} = \operatorname{sgn}\mathcal{T}v_{2n}(\beta_j) & \text{if } \frac{(h+1)j}{n+1} \notin 2\mathbb{Z},, \\ \\ 0 \neq (-1)^{j+1} = \operatorname{sgn}\mathcal{T}v_{2n}(\beta_j) & \text{if } \frac{(h+1)j}{n+1} \in 2\mathbb{Z}, \end{cases} \tag{9}$$

$$\operatorname{sgn}\mathcal{T}u_{2n,h}(-2) = \begin{cases} 0 \neq (-1)^n = \operatorname{sgn}\mathcal{T}v_{2n}(-2) & \text{if } h \notin 2\mathbb{Z}, \\ \\ (-1)^n = \operatorname{sgn}\mathcal{T}v_{2n}(-2) & \text{if } h \in 2\mathbb{Z}. \end{cases} \tag{10}$$

We also need the signs of $\mathcal{T}u_{2n,h}$ at $\hat{\gamma}_j$ $(j = 1, \dots, n)$. By (5) and some trigonometric identities we have for $j = 1, \dots, n-1$

$$\mathcal{T}u_{2n,h}(\hat{\gamma}_j) = 2\mathcal{T}v_{2n}(\hat{\gamma}_j) + 2\cos\frac{(n-h)(2j+1)\pi}{2n+1} = 2\frac{(-1)^j}{\sin\frac{(2j+1)\pi}{2(2n+1)}} + 2\cos\frac{(n-h)(2j+1)\pi}{2n+1},$$

and $\mathcal{T}u_{2n,h}(\hat{\gamma}_n) = \mathcal{T}u_{2n,h}(-2)$; therefore

$$\begin{aligned}\operatorname{sgn}\mathcal{T}u_{2n,h}(\hat{\gamma}_j) &= (-1)^j \quad \text{if } j = 1, \dots, n-1,\\ \operatorname{sgn}\mathcal{T}u_{2n,h}(\hat{\gamma}_n) &= \operatorname{sgn}\mathcal{T}u_{2n,h}(-2).\end{aligned} \tag{11}$$

By (11) $\operatorname{sgn}\mathcal{T}u_{2n,h}(\hat{\gamma}_{j-1}) = (-1)^{j-1}$, and $\mathcal{T}u_{2n,h}(\alpha_j) = 0$, thus the sign of $\mathcal{T}u_{2n,h}$ is $(-1)^{j-1} = (-1)^{j+1}$ on the interval $(\alpha_j, \hat{\gamma}_{j-1})$ and $(-1)^j$ on the interval $(\hat{\gamma}_j, \alpha_j)$.

If $\frac{(h+1)j}{n+1} \notin 2\mathbb{Z}$ then by (9) β_j must be in the interval $(\alpha_j, \hat{\gamma}_{j-1})$, implying $\alpha_j < \beta_j < \hat{\gamma}_{j-1} \leq \alpha_{j-1}$.

If $\frac{(h+1)j}{n+1} \in 2\mathbb{Z}$ then by (9) $\beta_j = \alpha_j$. This cannot happen for $j = 1$ as then $\frac{(h+1)j}{n+1} = \frac{h+1}{n+1} \notin 2\mathbb{Z}$. For $j = n$ this arises only if $\frac{(h+1)j}{n+1} = \frac{(h+1)n}{n+1} \in 2\mathbb{Z}$ i.e. if $h = n$ is an even number. In this case we have $-2 < \alpha_n = \beta_n$. This proves (a) and (b).

If $\operatorname{sgn}\mathcal{T}u_{2n,h}(\beta_j) = 0$ then taking an $\varepsilon > 0$ such that $\beta_j + \varepsilon \in (\beta_j, \hat{\gamma}_{j-1})$ the statement (c) clearly holds.

By (10) $\mathcal{T} u_{2n,h}(-2) = 0$ if and only if h is odd, and then $\alpha_n = \hat{\gamma}_n = -2$ is the only simple zero of $\mathcal{T} u_{2n,h}$ in the interval $[-2, \hat{\gamma}_{n-1})$. By (11) $\operatorname{sgn} \mathcal{T} u_{2n,h}(\hat{\gamma}_{n-1}) = (-1)^{n-1}$ hence the sign of $\mathcal{T} u_{2n,h}$ is $(-1)^n$ on the left hand side of -2, the same as $\operatorname{sgn} \mathcal{T} v_{2n}(-2) = (-1)^n$, proving (d). □

Lemma 4. *Let γ_j $(j = 1, \dots, n)$ be the zeros of $\mathcal{T} v_{2n}(x)$ and let λ_j $(j = 1, \dots, n-1)$ be the zeros of $\mathcal{T} \hat{u}_{2n-2,h}(x)$, where $h = 1, 2, \dots n-1$ is fixed.*

(a) *The polynomial $\mathcal{T} \hat{u}_{2n-2,h}(x)$ alternates sign at the zeros of $\mathcal{T} v_{2n}(x)$, i.e.*

$$-2 < \gamma_n < \lambda_{n-1} < \gamma_{n-1} < \dots \gamma_{j+1} < \lambda_j < \gamma_j < \dots < \gamma_2 < \lambda_1 < \gamma_1 < 2.$$

(b) *For $j = 1, \dots, n$ we have*

$$\operatorname{sgn} \mathcal{T} \hat{u}_{2n-2,h}(\gamma_j) = \operatorname{sgn} \mathcal{T} \hat{v}_{2n-2}(\beta_j).$$

Proof. We have seen earlier that $\gamma_j = 2\cos y_j$ where $y_j = \frac{2j\pi}{2n+1}$ $(j = 1, \dots, n)$. Using (4), (6) and the addition formulae for sin, we get

$$\mathcal{T} \hat{v}_{2n-2}(\gamma_j) = 2\, U_{n-1}\left(\frac{\gamma_j}{2}\right) = 2\frac{\sin n\frac{2j\pi}{2n+1}}{\sin\frac{2j\pi}{2n+1}} = 2\frac{\sin(j\pi - \frac{j\pi}{2n+1})}{\sin\frac{j\pi}{2n+1}\cos\frac{j\pi}{2n+1}} = 2\frac{(-1)^{j+1}}{2\cos\frac{j\pi}{2n+1}},$$

and similarly

$$\mathcal{T} \hat{u}_{2n-2,h}(\gamma_j) = 2\mathcal{T} \hat{v}_{2n-2}(\gamma_j) + \frac{\cos\frac{2n-2h-1}{2}\frac{2j\pi}{2n+1}}{2\cos\frac{2j\pi}{2(2n+1)}} == 2\left(\frac{(-1)^{j+1} + \cos\frac{2n-2h-1}{2}\frac{2j\pi}{2n+1}}{2\cos\frac{j\pi}{2n+1}}\right).$$

As $\cos\frac{j\pi}{2n+1} > 0$ we have for fixed $h = 1, 2, \dots n-1$

$$\operatorname{sgn} \mathcal{T} \hat{u}_{2n-2,h}(\gamma_j) = \mathcal{T} \hat{v}_{2n-2}(\gamma_j) = (-1)^{j+1} \quad (j = 1, \dots, n). \tag{12}$$

This shows that the polynomial $\mathcal{T} \hat{u}_{2n-2,h}(x)$ (of degree $n-1$) alternates sign at the points $\gamma_1, \dots, \gamma_n$ which implies (a) except the two inequalities at the far ends: $-1 < \gamma_n,\ \gamma_1 < 2$, which are obvious.

The validity of (b) follows from (12). □

Proof of the Theorem. For the sake of definiteness assume that $a_i \le b_i$ $(1 \le i \le s)$. Let $h_1 = a_i - 1$,

$$u_{2n_i,h_i} = \ 2v_{2n_i} + \left(z^{h_i} + z^{2n_i - h_i}\right) \qquad \text{if } a_i + b_i - 2 = 2n_i,$$

$$\hat{u}_{2n_i,h_i} = \ \frac{2v_{2n_i+1} + \left(z^{h_i} + z^{2n_i - h_i - 1}\right)}{z+1} \qquad \text{if } a_i + b_i - 2 = 2n_i + 1,$$

$$\text{and } m_i = \begin{cases} a_i + b_i - 1 & \text{if } 1 \le i \le s \\ p_{i-s} & \text{if } s < i \le s + r \end{cases}.$$

(i) *Suppose that m_i's are pairwise distinct* (positive integers) arranged such that odd m_i's are listed first, followed by the even m_i's, i.e. $m_i = 2n_i + 1$, for $1 \le i \le s_1$ and $m_i = 2n_i$ if $s_1 < i \le s_1 + s_2$ for some $0 \le s_1, s_2 \le s + r$ and $s_1 + s_2 = s + r$. Let l_1 be the number of u's with even and l_2 the number of u's with odd degree.

Taking out the maximal power of the factor $z+1$ from all terms of $f(z)$, splitting the second and third terms of $f(z)$ according the parity of the degrees of the u's and v's and rearranging the sums we get

$$f(z) = (z+1)^{s_1+1} \prod_{i=1}^{s_1} \hat{v}_{2n_i} \prod_{i=s_1+1}^{s_1+s_2} v_{2n_i} - z(z+1)^{s_1-1} \sum_{j=1}^{l_1} \left(u_{2n_j,h_j} \prod_{i=1,\, i\neq j}^{s_1} \hat{v}_{2n_i} \prod_{i=s_1+1}^{s_1+s_2} v_{2n_i} \right)$$

$$-z(z+1)^{s_1-1} \sum_{j=l_1+1}^{s_1} \left(v_{2n_j} \prod_{i=1,\, i\neq j}^{s_1} \hat{v}_{2n_i} \prod_{i=s_1+1}^{s_1+s_2} v_{2n_i} \right)$$

$$-z(z+1)^{s_1+1} \left(\sum_{j=s_1+1}^{s_1+l_2} \left(\hat{u}_{2n_j-2,h_j} \prod_{i=1}^{s_1} \hat{v}_{2n_i} \prod_{i=s_1+1,\, i\neq j}^{s_1+s_2} v_{2n_i} \right) + \sum_{j=s_1+l_2+1}^{s_1+s_2} \left(\hat{v}_{2n_j-2} \prod_{i=1}^{s_1} \hat{v}_{2n_i} \prod_{i=s_1+1,\, i\neq j}^{s_1+s_2} v_{2n_i} \right) \right)$$

$$= (z+1)^{s_1-1} \left((z+1)^2 f_1(z) - z f_2(z) - z(z+1)^2 f_3(z) \right)$$

for some $l_1 \leq s_1,\ l_2 \leq s_2$ with suitable polynomials f_1, f_2, f_3. The degree of f is (the degree of its first term)

$$N := (s_1+1) + \sum_{i=1}^{s_1+s_2} 2n_i = (s_1+1) + \sum_{i=1}^{s_1} (m_i - 1) + \sum_{i=s_1+1}^{s_1+s_2} m_i = \sum_{i=1}^{s_1+s_2} m_i + 1 = (s_1+1) + 2d,$$

where $d := \sum_{i=1}^{s_1+s_2} n_i$. Let

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } N \text{ or } s_1+1 \text{ is even,} \\ f(z)(z+1) & \text{if } N \text{ or } s_1+1 \text{ is odd.} \end{cases}$$

Instead of the Chebyshev transform of $\hat{f}$ we shall count the zeros of the function

$$F(x) := \frac{\mathcal{T}\hat{f}(x)}{(x+2)^{\left[\frac{s_1}{2}\right]}} = (x+2)\mathcal{T}f_1(x) - \mathcal{T}f_2(x) - (x+2)\mathcal{T}f_3(x).$$

By Lemma 1 our theorem is proved, if we show that $F(x)$ has $\left[\frac{N-1}{2}\right] - \left[\frac{s_1}{2}\right] = d$ zeros on the interval $[-2,2]$ (we deducted the multiplicity $\left[\frac{s_1}{2}\right]$ of the zero -2 of the denominator).

To find the number of zeros of $F(x)$ on the interval $[-2,2]$, we arrange the zeros $\delta_i,\ 1 \leq i \leq d$ of the function

$$\mathcal{T}f_1(x) = \prod_{i=1}^{s_1} \mathcal{T}\hat{v}_{2n_i}(x) \prod_{i=s_1+1}^{s_1+s_2} \mathcal{T}v_{2n_i}(x)$$

in decreasing order

$$-2 < \delta_d < \delta_{d-1} < \cdots < \delta_2 < \delta_1 \ (< 2),$$

and we check the sign of $F(x)$ at all the listed points.

Clearly, in agreement with our earlier notations, each δ_k equals one of the numbers $\beta_{j,i} = 2\cos\frac{2j\pi}{2n_i+2},\ (1 \leq i \leq s_1)$ or $\gamma_{j,i} = 2\cos\frac{2j\pi}{2n_i+1},\ (s_1 < i \leq s_1+s_2)$ where $j = 1, \ldots, n_i$.

If $\mathcal{T}\hat{v}_{2n_k}(\delta_j) = 0$ for some $1 \le k \le s_1$ and $1 \le j \le d$ then

$$F(\delta_j) = -\mathcal{T}f_2(\delta_j) = -\mathcal{T}u_{2n_k,h_k}(\delta_j) \prod_{i=1,\, i\neq k}^{s_1} \mathcal{T}\hat{v}_{2n_i}(\delta_j) \prod_{i=s_1+1}^{s_1+s_2} \mathcal{T}v_{2n_i}(\delta_j) \quad \text{if } k \le l_1 \ ,$$

$$F(\delta_j) = -\mathcal{T}f_2(\delta_j) = -\mathcal{T}v_{2n_k}(\delta_j) \prod_{i=1,\, i\neq k}^{s_1} \mathcal{T}\hat{v}_{2n_i}(\delta_j) \prod_{i=s_1+1}^{s_1+s_2} \mathcal{T}v_{2n_i}(\delta_j) \qquad \text{if } l_1 < k \le s_1.$$

If $\mathcal{T}v_{2n_k}(\delta_j) = 0$ for $s_1 < k \le s_1 + s_2$, then only non-zero terms in $F(\delta_j)$ are

$$F(\delta_j) = -(x+2)\mathcal{T}f_3(\delta_j) = -(x+2)\mathcal{T}\hat{u}_{2n_k-2,h_k}(\delta_j) \prod_{i=1}^{s_1} \mathcal{T}\hat{v}_{2n_i}(\delta_j) \prod_{i=s_1+1,\, i\neq k}^{s_1+s_2} \mathcal{T}v_{2n_i}(\delta_j)$$

if $s_1 < k \le s_1 + l_2$,

$$F(\delta_j) = -(x+2)\mathcal{T}f_3(\delta_j) = -(x+2)\mathcal{T}\hat{v}_{2n_k-2}(\delta_j) \prod_{i=1}^{s_1} \mathcal{T}\hat{v}_{2n_i}(\delta_j) \prod_{i=s_1+1,\, i\neq k}^{s_1+s_2} \mathcal{T}v_{2n_i}(\delta_j).$$

if $s_1 + l_2 < k \le s_1 + s_2$.

By an easy calculation we notice that $\mathcal{T}\hat{v}_{2n_i-2}$ and $\mathcal{T}v_{2n_i}$ alternate sign at the zeros of $\mathcal{T}v_{2n_i}$ and $\mathcal{T}\hat{v}_{2n_i}$ respectively.
Unfortunately $F(\delta_j)$ $(j = 1, \dots, d)$ or $F(-2)$ can be zero and then counting the sign change does not give the correct number of zeros. To avoid this problem, we evaluate $\mathcal{T}u_{2n_i,h_i}$ and $\mathcal{T}\hat{u}_{2n_i-2}$ at the points $\delta_j + \varepsilon$ (with suitable small $\varepsilon > 0$) instead of δ_j (based on Lemmas 4 and 5). We define the modified sign function in following way:

$$S_i(\delta_j) := \begin{cases} \operatorname{sgn} \mathcal{T}\hat{v}_{2n_i}(\delta_j) & \text{if } \mathcal{T}\hat{v}_{2n_i}(\delta_j) \neq 0 \quad 1 \le i \le s_1, \\ \operatorname{sgn} \mathcal{T}v_{2n_i}(\delta_j) & \text{if } \mathcal{T}\hat{v}_{2n_i}(\delta_j) = 0 \quad 1 \le i \le s_1, \\ \operatorname{sgn} \mathcal{T}v_{2n_i}(\delta_j) & \text{if } \mathcal{T}v_{2n_i}(\delta_j) \neq 0 \quad s_1 < i \le s_1 + s_2, \\ \operatorname{sgn} \mathcal{T}\hat{v}_{2n_i-2}(\delta_j) & \text{if } \mathcal{T}v_{2n_i}(\delta_j) = 0 \quad s_1 < i \le s_1 + s_2. \end{cases}$$

For $j = 1, 2, \dots, d-1$ this implies for $i = 1, 2, \dots, s_1 + s_2$ that

$$S_i(\delta_{j+1}) = \begin{cases} -S_i(\delta_j) & \text{if } \mathcal{T}v_{2n_i}(\delta_j) = 0 \text{ or } \mathcal{T}\hat{v}_{2n_i}(\delta_j) = 0 \\ S_i(\delta_j) & \text{otherwise .} \end{cases} \tag{13}$$

Thus, we can substitute $\operatorname{sgn} F(\delta_j)$ by $-\prod_{i=1}^{s_1+s_2} S_i(\delta_j)$ for $j = 2, 3 \dots, d-1$.
By the proof of Lemma 4 and 5, $\operatorname{sgn} F(\delta_1) = -1$. Lemma 4 and 5 imply that the function F has exactly one change of sign on $[\delta_i, \delta_{i+1})$ for $i = 1, 2, \dots, d-1$. Therefore there are $d-1 = \sum n_i - 1$ zeros of F in $[\delta_d, \delta_1)$. We show that F has one more zero on $[-2, \delta_d)$.
We have to show that either $F(-2) = 0$ or $\operatorname{sgn} F(-2) \neq \operatorname{sgn} F(\delta_1)$. Since for $k > s_1$ we have $F(-2) = 0$, we consider only the case $k \le s_1$.
We have $\operatorname{sgn} \mathcal{T}\hat{v}_{2n_i}(-2) = (-1)^{n_i}$ and $\operatorname{sgn} \mathcal{T}v_{2n_i}(-2) = (-1)^{n_i}$ and by Lemma 4 (d)

$$F(-2) = -\mathcal{T}f_2(-2) = -\mathcal{T}v_{2n_k}(-2) \prod_{i=1,\, i\neq k}^{s_1} \mathcal{T}\hat{v}_{2n_i}(-2) \prod_{i=s_1+1}^{s_1+s_2} \mathcal{T}v_{2n_i}(-2) = (-1)^{\sum_{i=1}^{s_1+s_2} n_i} = (-1)^d.$$

Since all v's and $\hat{v}$'s are positive at δ_d we have $\operatorname{sgn} F(\delta_d) = -1$, and $F(\delta_{d-i}) = (-1)^{i-1}$ and by (13) $\operatorname{sgn} F(\delta_1) = (-1)^d$ proving that $F(x)$ has a zero on $[-2, \delta_d)$.

Consequently, F has $d = \sum_{i=1}^{s} n_i$ zeros in the interval $[-2, 2]$.

(ii) *Next, assume that* $m_1, m_2, \dots m_s$ *are not pairwise distinct*, e.g. $m_1 = m_2 = \dots = m_{l_1}$, $m_{l_1+1} = m_{l_1+2} = \dots = m_{l_2}, \dots$ etc. Then we perform a small analytic perturbation in the terms of F corresponding to $m_2, \dots m_{l_1}$ etc. and apply Rouché's theorem. Thus, we may count the zeros of F the same way as in case of (i) (with multiplicities).
or all orientation of generalized stars without oriented cycle, and the Theorem follows. □

In our proof it was essential that the number of cycles s in our graph is > 1. If $s = 0$ then $F(x)$ may have $d + 1$ zeros on $[-2, 2]$ (only if $F(2) \leq 0$). In these cases we have trees, whose Coxeter polynomials do not depend on orientation and they are well described in [11].

Example. Take the oriented $\Delta_{[(9,3),4]}$, i.e. the graph with 16 vertices. In this case $s_1 = 1$ and $S_1(\delta_4) = -1$, since $\mathcal{T}u_{10,3}(-1+\varepsilon) < 0$ and a $S_1(\delta_5) > 0$. The Coxeter polynomial is

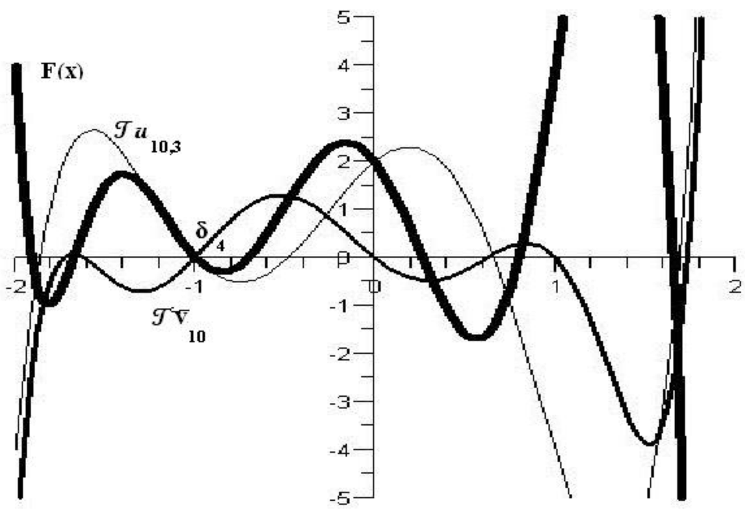
$$\hat{f}(z) = f(z) = (z+1)^2 \hat{v}_{10} v_4 - z u_{10,3} v_4 - z(z+1)^2 \hat{v}_{10} \hat{v}_3.$$

As $\left[\frac{s_1}{2}\right] = 0$, $F(x)$ is exactly the Chebyshev transform of $\hat{f}(x)$. This transform is

$$F(x) = (x+2)\mathcal{T}\hat{v}_{10}\mathcal{T}v_4 - \mathcal{T}u_{10,3}\mathcal{T}v_4 - (x+2)\mathcal{T}\hat{v}_{10}\mathcal{T}\hat{v}_3 =$$

$$(x+2)(x^5-4x^3+3x)(x^2+x-1)-(2x^5+2x^4-7x^3-6x^2+3x+2)(x^2+x-1)-(x^5-4x^3+3x)x.$$

The next figure shows the graphs of the functions $F(x), \mathcal{T}u_{10,3}$ and $\mathcal{T}v_{10}(= x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1)$:



The zeros of $\mathcal{T}\bar{v}_{10} = U_5(\frac{x}{2}) = x^5 - 4x^3 + 3x$ are $\delta_7, \delta_5, \delta_4, \delta_2, \delta_1$ and the zeros of $\mathcal{T}v_4 = U_2(\frac{x}{2}) + U_1(\frac{x}{2}) = x^2 + x - 1$ are δ_6, δ_3. We have $S_1(\delta_4) = -1$, since $\operatorname{sgn} \mathcal{T}u_{10,3}(-2) = \operatorname{sgn} \mathcal{T}v_{10}(-2) < 0$ and $S_1(\delta_5) > 0$.

	-2	$\delta_7 \simeq -1.73$	$\delta_6 \simeq -1.62$	$\delta_5 = -1$	$\delta_4 = 0$	$\delta_3 \simeq .62$	$\delta_2 = 1$	$\delta_1 \simeq 1.73$
$S_1(\delta_j)$	-1	1	1	-1	1	1	-1	1
$S_2(\delta_j)$	1	1	-1	-1	-1	1	1	1
sgn F	1	-1	1	-1	1	-1	1	-1

Since $\mathcal{T}u_{10,3}(x)$ has a zero on $[-2,\delta_7)$, the polynomial $\mathcal{T}f(z) = F(x)$ has 7 zeros on $[-2,2]$.

We show that the Salem numbers given by our Theorem (defined by the Coxeter polynomials of generalized stars) extends the set defined in [5].

Denote by T the set of all Salem numbers and by T' the set of spectral radii of Coxeter transformation defined by wild stars, which are Salem numbers ([5]). Then $T' \neq T$ holds, since the Salem number ρ_1 listed in [2] and defined by the reciprocal polynomial $z^{10} - z^6 - z^5 - z^4 + 1$ is not in the set T'.
It is known that the spectral radius of the Coxeter transformation of a subgraph of a tree is not greater than the spectral radius of the same tree. The list of stars with spectral radius $1.7 > \rho > 1.6$ of its Coxeter transformation is not large (see [6]). There is a gap between the spectral radii of Coxeter transformations of stars with length of arms $2,3,4$ and $2,3,5$, i.e. between spectral radii of $\mathcal{C}_{\Delta_{([2,3,4])}} \approx 1.6574$ and $\mathcal{C}_{\Delta_{([2,3,5])}} \approx 1.6935$. The only zero non-cyclotomic factors of the Coxeter polynomial of the bipartite graph $\Delta_{[(6,6),1]}$ is $z^8 - z^7 - z^6 - z^2 - z + 1$, which has a zero (being also a Salem number) on the above interval and it is not in the set T'. The oriented graph $\Delta_{([(6,7),1])}$ is non-bipartite, however for the spectral radius of its Coxeter transformation $\rho_2 = \mathcal{C}_{\Delta_{([(6,7),1])}} \approx 1.6733$ we also have $\rho_2 \notin T'$.

Acknowledgement. The author thanks the referee for several useful remarks on the presentation of the paper.

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