On additions of interactive fuzzy numbers

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Abstract: In this paper we will summarize some properties of the extended addition operator on fuzzy numbers, where the interactivity relation between fuzzy numbers is given by their joint possibility distribution.

1 Introduction

A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. Any fuzzy number can be described with the following membership function,

$$
A(t) = \begin{cases}
L\left(\dfrac{a-t}{\alpha}\right) & \text{if } t \in [a-\alpha, a] \\[2mm]
1 & \text{if } t \in [a,b], a \leq b, \\[2mm]
R\left(\dfrac{t-b}{\beta}\right) & \text{if } t \in [b, b+\beta] \\[2mm]
0 & \text{otherwise}
\end{cases}
$$

where $[a,b]$ is the peak of A; a and b are the lower and upper modal values; L and R are shape functions: $[0,1] \to [0,1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ which are non-increasing, continuous mappings. We shall call these fuzzy numbers of LR-type and use the notation $\tilde{a} = (a,b,\alpha,\beta)_{LR}$. If $R(x) = L(x) = 1 - x$, we denote $\tilde{a} = (a,b,\alpha,\beta)$. The family of fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy number A is defined by $[A]^\gamma = \{t \in \mathbb{R} | A(t) \geq \gamma\}$, if $\gamma > 0$ and $[A]^\gamma = \mathrm{cl}\{t \in \mathbb{R} | A(t) > 0\}$ (the closure of the support of A) if $\gamma = 0$.

An n-dimensional possibility distribution C is a fuzzy set in $\mathbb{R}^n$ with a normalized membership function of bounded support. The family of n-dimensional possibility distribution will be denoted by $\mathcal{F}_n$.

Let us recall the concept and some basic properties of joint possibility distribution introduced in [27]. If $A_1, \ldots, A_n \in \mathcal{F}$ are fuzzy numbers, then $C \in \mathcal{F}_n$ is said to be their joint possibility distribution if $A_i(x_i) = \max\{C(x_1, \ldots, x_n) \mid x_j \in \mathbb{R}, j \neq i\}$, holds for all $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. Furthermore, A_i is called the i-th marginal possibility distribution of C. For example, if C denotes the joint possibility distribution of $A_1, A_2 \in \mathcal{F}$, then C satisfies the relationships

$$\max_y C(x_1, y) = A_1(x_1), \quad \max_y C(y, x_2) = A_2(x_2),$$

for all $x_1, x_2 \in \mathbb{R}$. Fuzzy numbers $A_1, \ldots, A_n$ are said to be non-interactive if their joint possibility distribution C satisfies the relationship

$$C(x_1, \ldots, x_n) = \min\{A_1(x_1), \ldots, A_n(x_n)\},$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a triangular norm (t-norm for short) iff T is symmetric, associative, non-decreasing in each argument, and $T(x, 1) = x$ for all $x \in [0, 1]$. Recall that a t-norm T is Archimedean iff T is continuous and $T(x, x) < x$ for all $x \in]0, 1[$. Every Archimedean t-norm T is representable by a continuous and decreasing function $f : [0, 1] \to [0, \infty]$ with $f(1) = 0$ and

$$T(x, y) = f^{[-1]}(f(x) + f(y))$$

where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)] \\ 0 & \text{otherwise} \end{cases}$$

The function f is the additive generator of T. Let T_1, T_2 be t-norms. We say that T_1 is weaker than T_2 (and write $T_1 \leq T_2$) if $T_1(x, y) \leq T_2(x, y)$ for each $x, y \in [0, 1]$.

The basic t-norms are (i) the minimum: $\min(a, b) = \min\{a, b\}$; (ii) Łukasiewicz: $T_L(a, b) = \max\{a + b - 1, 0\}$; (iii) the product: $T_P(a, b) = ab$; (iv) the weak:

$$T_W(a, b) = \begin{cases} \min\{a, b\} & \text{if } \max\{a, b\} = 1 \\ 0 & \text{otherwise} \end{cases}$$

(v) Hamacher [7]:

$$T_\gamma^H(a, b) = \frac{ab}{\gamma + (1 - \gamma)(a + b - ab)}, \ \gamma \geq 0$$

and (vi) Yager

$$T_p^Y(a, b) = 1 - \min\{1, \sqrt[p]{[(1 - a)^p + (1 - b)^p]}\}, \ p > 0.$$

Using the concept of joint possibility distribution we introduced the following extension principle in [3].

Definition 1.1. *[3] Let C be the joint possibility distribution of (marginal possibility distributions) $A_1, \ldots, A_n \in \mathcal{F}$, and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then*

$$f_C(A_1, \ldots, A_n) \in \mathcal{F},$$

will be defined by

$$f_C(A_1, \ldots, A_n)(y) = \sup_{y=f(x_1,\ldots,x_n)} C(x_1, \ldots, x_n). \tag{1}$$

We have the following lemma, which can be interpreted as a generalization of Nguyen's theorem [25].

Lemma 1. *[3] Let $A_1, A_2 \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then,*

$$[f_C(A_1, \ldots, A_n)]^\gamma = f([C]^\gamma),$$

for all $\gamma \in [0, 1]$. Furthermore, $f_C(A_1, \ldots, A_n)$ is always a fuzzy number.

Let C be the joint possibility distribution of (marginal possibility distributions) $A_1, A_2 \in \mathcal{F}$, and let $f(x_1, x_2) = x_1 + x_2$ be the addition operator. Then $A_1 + A_2$ is defined by

$$(A_1 + A_2)(y) = \sup_{y=x_1+x_2} C(x_1, x_2). \tag{2}$$

If A_1 and A_2 are non-interactive, that is, their joint possibility distribution is defined by

$$C(x_1, x_2) = \min\{A_1(x_1), A_2(x_2)\},$$

then (2) turns into the extended addition operator introduced by Zadeh in 1965 [26],

$$(A_1 + A_2)(y) = \sup_{y=x_1+x_2} \min\{A_1(x_1), A_2(x_2)\}.$$

Furthermore, if $C(x_1, x_2) = T(A_1(x_1), A_2(x_2))$, where T is a t-norm then we get the t-norm-based extension principle,

$$(A_1 + A_2)(y) = \sup_{y=x_1+x_2} T(A_1(x_1), A_2(x_2)). \tag{3}$$

For example, if A_1 and A_2 are fuzzy numbers, T is the product t-norm then the sup-product extended sum of A_1 and A_2 is defined by

$$(A_1 + A_2)(y) = \sup_{x_1+x_2=y} A_1(x_1)A_2(x_2),$$

and the sup-T_γ^H extended addition of A_1 and A_2 is defined by

$$(A_1 + A_2)(y) = \sup_{x_1+x_2=y} \frac{A_1(x_1)A_2(x_2)}{\gamma + (1-\gamma)(A_1(x_1) + A_2(x_2) - A_1(x_1)A_2(x_2))}.$$

If T is an Archimedean t-norm and $\tilde{a}_1$, $\tilde{a}_2 \in \mathcal{F}$ then their T-sum

$$A_2 := \tilde{a}_1 + \tilde{a}_2$$

can be written in the form

$$A_2(z) = f^{[-1]}(f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2))), z \in \mathbb{R},$$

where f is the additive generator of T. By the associativity of T, the membership function of the T-sum $A_n := \tilde{a}_1 + \cdots + \tilde{a}_n$ can be written as

$$A_n(z) = \sup_{x_1 + \cdots + x_n = z} f^{[-1]}\left(\sum_{i=1}^{n} f(\tilde{a}_i(x_i))\right), z \in \mathbb{R}.$$

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non-increasing, we have

$$A_n(z) = f^{[-1]}\left(\inf_{x_1 + \cdots + x_n = z} \sum_{i=1}^{n} f(\tilde{a}_i(x_i))\right), z \in \mathbb{R}.$$

2 Additions of interactive fuzzy numbers

Dubois and Prade published their seminal paper on additions of interactive fuzzy numbers in 1981 [4]. Since then the properties of additions of interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm have been extensively studied in the literature [1-3, 5-24]. In 1992 Fullér and Keresztfalvi [6] generalized and extended the results presented in [4]. Namely, they determined the exact membership function of the t-norm-based sum of fuzzy intervals, in the case of Archimedean t-norm having strictly convex additive generator function and fuzzy intervals with concave shape functions. They proved the following theorem,

Theorem 2.1. *[6] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, b_i, \alpha, \beta)_{LR}$, $i = 1, \ldots, n$, be fuzzy numbers of LR-type. If L and R are twice differentiable, concave functions, and f is twice differentiable, strictly convex function then the membership function of the T-sum $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$ is*

$$\tilde{A}_n(z) = \begin{cases} 1 & \text{if } A_n \leq z \leq B_n \\ f^{[-1]}\left(n \times f\left(L\left(\dfrac{A_n - z}{n\alpha}\right)\right)\right) & \text{if } A_n - n\alpha \leq z \leq A_n \\ f^{[-1]}\left(n \times f\left(R\left(\dfrac{z - B_n}{n\beta}\right)\right)\right) & \text{if } B_n \leq z \leq B_n + n\beta \\ 0 & \text{otherwise} \end{cases}$$

where $A_n = a_1 + \cdots + a_n$ and $B_n = b_1 + \cdots + b_n$.

The results of Theorem 2.1 have been extended to wider classes of fuzzy numbers and shape functions by many authors.

In 1994 Hong and Hwang [8] provided an upper bound for the membership function of T-sum of LR-fuzzy numbers with different spreads. They proved the following theorem,

Theorem 2.2. *[8] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$, be fuzzy numbers of LR-type. If L and R are concave functions, and f is a convex function then the membership function of the T-sum $\tilde{A}_2 = \tilde{a}_1 + \tilde{a}_2$ is less than or equal to*

$$A_2^*(z) =$$

$$
\begin{cases}
f^{[-1]}\left(2f\left(L\left(1/2 + \dfrac{(A_2 - z) - \alpha^*}{(2\alpha_*)}\right)\right)\right) & \text{if } A_2 - \alpha_1 - \alpha_2 \leq z \leq A_2 - \alpha^* \\[3mm]
f^{[-1]}\left(2f\left(L\left(\dfrac{A_2 - z}{2\alpha^*}\right)\right)\right) & \text{if } A_2 - \alpha^* \leq z \leq A_2 \\[3mm]
f^{[-1]}\left(2f\left(R\left(\dfrac{z - A_2}{2\beta^*}\right)\right)\right) & \text{if } A_2 \leq z \leq A_2 + \beta^* \\[3mm]
f^{[-1]}\left(2f\left(R\left(1/2 + \dfrac{(z - A_2) - \beta^*}{2\beta_*}\right)\right)\right) & \text{if } A_2 + \beta^* \leq z \leq A_2 + \beta_1 + \beta_2 \\[3mm]
0 & \text{otherwise}
\end{cases}
$$

where $\beta^* = \max\{\beta_1, \beta_2\}$, $\beta_* = \min\{\beta_1, \beta_2\}$, $\alpha^* = \max\{\alpha_1, \alpha_2\}$, $\alpha_* = \min\{\alpha_1, \alpha_2\}$ and $A_2 = a_1 + a_2$.

The In 1995 Hong [9] proved that Theorem 2.1 remains valid for concave shape functions and convex additive t-norm generator. In 1996 Mesiar [22] showed that Theorem 2.1 remains valid if both $L \circ f$ and $R \circ f$ are convex functions. In 1997 Mesiar [23] generalized Theorem 2.1 to the case of nilpotent t-norms (nilpotent t-norms are non-strict continuous Archimedean t-norms). In 1997 Hong and Hwang [11] gave upper and lower bounds of T-sums of LR-fuzzy numbers $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, with different spreads where T is an Archimedean t-norm. They proved the following two theorems,

Theorem 2.3. *[11] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, be fuzzy numbers of LR-type. If $f \circ L$ and $f \circ R$ are concvex functions, then the membership function of their T-sum $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$*

is less than or equal to

$$A_n^*(z) = \begin{cases} f^{[-1]}\left(nf\left(L\left(\dfrac{1}{n}I_L\left(A_n - z\right)\right)\right)\right) & \text{if } A_n - \sum_{i=1}^n \alpha_i \leq z \leq A_n \\[4mm] f^{[-1]}\left(nf\left(R\left(\dfrac{1}{n}I_R\left(z - A_n\right)\right)\right)\right) & \text{if } A_n \leq z \leq A_n + \sum_{i=1}^n \beta_i \\[4mm] 0 & \text{otherwise,} \end{cases}$$

where

$$I_L(z) = \inf\left\{\frac{x_1}{\alpha_1} + \cdots + \frac{x_n}{\alpha_n} \,\middle|\, x_1 + \cdots + x_n = z,\ 0 \leq x_i \leq \alpha_i,\ i = 1, \ldots, n\right\},$$

and

$$I_R(z) = \inf\left\{\frac{x_1}{\beta_1} + \cdots + \frac{x_n}{\beta_n} \,\middle|\, x_1 + \cdots + x_n = z,\ 0 \leq x_i \leq \beta_i,\ i = 1, \ldots, n\right\}.$$

Theorem 2.4. *[11] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, be fuzzy numbers of LR-type. Then*

$$\tilde{A}_n(z) \geq A_n^{**}(z) =$$

$$\begin{cases} f^{[-1]}\left(nf\left(L\left(\dfrac{A_n - z}{\alpha_1 + \cdots + \alpha_n}\right)\right)\right) & \text{if } A_n - (\alpha_1 + \cdots + \alpha_n) \leq z \leq A_n \\[4mm] f^{[-1]}\left(nf\left(R\left(\dfrac{A_n - z}{\beta_1 + \cdots + \beta_n}\right)\right)\right) & \text{if } A_n \leq z \leq A_n + (\beta_1 + \cdots + \beta_n) \\[4mm] 0 & \text{otherwise,} \end{cases}$$

In 1997, generalizing Theorem 2.1, Hwang and Hong [15] studied the membership function of the t-norm-based sum of fuzzy numbers on Banach spaces and they presented the membership function of finite (or infinite) sum (defined by the sup-t-norm convolution) of fuzzy numbers on Banach spaces, in the case of Archimedean t-norm having convex additive generator function and fuzzy numbers with concave shape function. In 1998 Hwang, Hwang and An [16] approximated the strict triangular norm-based addition of fuzzy intervals of L-R type with any left and right spreadss. In 2001 Hong [12] showed a simple method of computing T-sum of fuzzy intervals having the same results as the sum of fuzzy intervals based on the weakest t-norm T_W.

2.1 Shape preserving arithmetic operations

Shape preserving arithmetic operations of LR-fuzzy intervals allow one to control the resulting spread. In practical computation, it is natural to require the preservation of

the shape of fuzzy intervals during addition and multiplication. Hong [13] showed that T_W , the weakest t-norm, is the only t-norm T that induces a shape-preserving multiplication of LR-fuzzy intervals. In 1995 Kolesarova [19, 20] Kolesarova proved the following theorem,

Theorem 2.5. *(a) Let T be an arbitrary t-norm weaker than or equal to the Łukasiewicz t-norm T_L; $T(x,y) \leq T_L(x,y) = \max(0, x+y-1)$, $x,y \in [0,1]$. Then the addition $\oplus$ based on T coincides on linear fuzzy intervals with the addition $\oplus$ based on the weakest t-norm T_W; i.e.,*

$$(a_1, b_1, \alpha_1, \beta_1) \oplus (a_2, b_2, \alpha_2, \beta_2) =$$
$$(a_1 + a_2, b_1 + b_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)).$$

(b) Let T be a continuous Archimedean t-norm with convex additive generator f . Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is a member of Yager's family of nilpotent t-norms with parameter $p \in [1, \infty)$, $T = T_p^Y$, and $f(x) = (1-x)^p$. Then $T_1^Y = T_L$ and for $p \in (0, \infty)$,

$$(a_1, b_1, \alpha_1, \beta_1) \oplus (a_2, b_2, \alpha_2, \beta_2) =$$
$$(a_1 + a_2, b_1 + b_2, (\alpha_1^q + \alpha_2^q)^{1/q}, (\beta_1^q + \beta_2^q)^{1/q}),$$

where $1/p + 1/q = 1$, i.e. $q = p/(p-1)$.

In 1997 Mesiar [24] studied the triangular norm-based additions preserving the LR-shape of LR-fuzzy intervals and conjectured that the only t-norm-based additions preserving the linearity of fuzzy intervals are those described in Theorem 2.5. He proved the following theorem,

Theorem 2.6. *[24] Let a continuous t-norm T be not weaker than or equal to T_L (i.e., there are some $x, y \in [0,1]$ so that $T(x,y) > x+y-1 > 0$). Let the addition based on T preserve the linearity of fuzzy intervals. Then either T is the strongest t-norm, $T = T_M$, or T is a nilpotent t-norm.*

In 2002 Hong [14] proved Mesiar's conjecture.

Theorem 2.7. *[14] Let a continuous t-norm T be not weaker than or equal to T_L. Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is either T_M or a member of Yagers family of nilpotent t-norms with parameter $p \in (1, \infty)$, $T = T_p^Y$, and $f(x) = (1-x)^p$.*

2.2   Additions of interactive fuzzy numbers

Until now we have summarized some properties of the addition operator on interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm. It is clear that in (3) the joint possibility distribution is defined *directly* and *pointwise*

from the membership values of its marginal possibility distributions by an aggregation operator. However, the interactivity relation between fuzzy numbers may be given by a more general joint possibility distribution, which can not be directly defined from the membership values of its marginal possibility distributions by any aggregation operator.

Drawing heavily on [3] we will now consider some properties of the addition operator on completely correlated fuzzy numbers, where the interactivity relation is given by their joint possibility distribution.

Let C be a joint possibility distribution with marginal possibility distributions A and B, and let

$$f(x_1, x_2) = x_1 + x_2,$$

the addition operator in $\mathbb{R}^2$. In [3] we introduced the notation,

$$A +_C B = f_C(A, B).$$

Definition 2.1. *[5] Fuzzy numbers A and B are said to be completely correlated, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution is defined by*

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2), \quad (4)$$

where $\chi_{\{qx_1 + r = x_2\}}$, stands for the characteristic function of the line

$$\{(x_1, x_2) \in \mathbb{R}^2 | qx_1 + r = x_2\}.$$

In this case we have,

$$[C]^\gamma = \left\{ (x, qx + r) \in \mathbb{R}^2 \big| x = (1 - t)a_1(\gamma) + ta_2(\gamma), t \in [0, 1] \right\}$$

where $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$; and $[B]^\gamma = q[A]^\gamma + r$, for any $\gamma \in [0, 1]$.

Definition 2.2. *[5] Fuzzy numbers A and B are said to be completely positively (negatively) correlated, if q is positive (negative) in (4).*

Now let us consider the extended addition of two completely correlated fuzzy numbers A and B,

$$(A +_C B)(y) = \sup_{y = x_1 + x_2} C(x_1, x_2).$$

That is,

$$(A +_C B)(y) = \sup_{y = x_1 + x_2} A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2).$$

Then from (2) and (4) we find,

$$[A +_C B]^\gamma = (q + 1)[A]^\gamma + r, \tag{5}$$

for all $\gamma \in [0, 1]$. If A and B are completely negatively correlated with $q = -1$, that is, $[B]^\gamma = -[A]^\gamma + r$, for all $\gamma \in [0, 1]$, then $A +_C B$ will be a crisp number. Really, from (5) we get $[A +_C B]^\gamma = 0 \times [A]^\gamma + r = r$, for all $\gamma \in [0, 1]$.
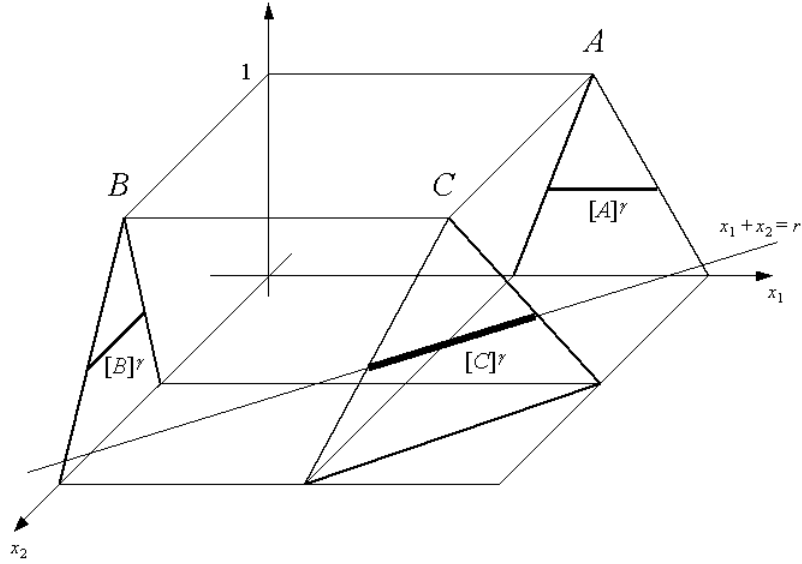
Figure 1: Completely negatively correlated fuzzy numbers with $q = -1$.

That is, the interactive sum, $A +_C B$, of two completely negatively correlated fuzzy numbers A and B with $q = -1$ and $r = 0$, i.e.

$$A(x) = B(-x), \forall x \in \mathbb{R},$$

will be (crisp) zero. On the other hand, a γ-level set of their non-interactive sum, $A + B$, can be computed as,

$$[A + B]^\gamma = [a_1(\gamma) - a_2(\gamma), a_2(\gamma) - a_1(\gamma)],$$

which is a fuzzy number.

In this case (i.e. when $q = -1$) any γ-level set of C are included by a certain level set of the addition operator, namely, the relationship,

$$[C]^\gamma \subset \{(x_1, x_2) \in \mathbb{R} | x_1 + x_2 = r\},$$

holds for any $\gamma \in [0, 1]$ (see Fig. 1). On the other hand, if $q \neq -1$ then the fuzziness of $A +_C B$ is preserved, since

$$[A +_C B]^\gamma = (q + 1)[A]^\gamma + r \neq \text{constant},$$

for all $\gamma \in [0, 1]$ and $y \in \mathbb{R}$. (see Fig. 2).

Really, in this case the set $\{(x_1, x_2) \in [C]^\gamma | x_1 + x_2 = y\}$ consists of a single point at most for any $\gamma \in [0, 1]$ and $y \in \mathbb{R}$.

Figure 2: Completely negatively correlated fuzzy numbers with $q \neq -1$.

Note 2.1. *The interactive sum of two completely negatively correlated fuzzy numbers A and B with $A(x) = B(-x)$ for all $x \in \mathbb{R}$ will be (crisp) zero.*

3 Summary

In this paper we have summarized some properties of the addition operator on interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm or by a more general type of joint possibility distribution.

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