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On the Sheffer stroke operation in fuzzy logic

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Abstract

From the beginnings of fuzzy logic, the Sheffer stroke operation has been overlooked and the efforts of the researchers have been devoted to other logical connectives. In this paper, the Sheffer stroke operation is introduced in fuzzy logic generalizing the classical operation when the truth values are restricted to $\{0, 1\}^2$. Similar to what happens in Boolean logic, the fuzzy Sheffer stroke is functionally complete and it can be used to generate any other fuzzy logical connective by combinations of itself. Two construction methods are presented and the close connection of this operation with a pair of fuzzy conjunction and negation is analysed.

Keywords: Sheffer stroke; Fuzzy conjunction; t-Norm; Fuzzy implication function

1. Introduction

In classical Boolean logic, the so-called Sheffer stroke [11,13] and Peirce arrow [8] (also known as NAND or NOR operations) highlight among other logical connectives due to the fact that they are functionally complete. Indeed, each one of these two operators can be considered by itself to constitute a logical formal system without the need to use any other logical connective. Therefore, all mathematically definable connectives in Boolean logic can be defined using either only Sheffer Stroke or only Peirce Arrow. No other unary or binary connective (or associated function) fulfils this property.

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Despite the metalogical importance of these two operations in Boolean logic, in the context of fuzzy logic, most of the theoretical efforts have been devoted to the study of fuzzy conjunctions, fuzzy disjunctions or fuzzy implication functions. In fact, up to our knowledge, neither Sheffer stroke nor Peirce arrow has been studied in the fuzzy logic framework. This fact is curious and one possible explanation could be that in the beginnings of fuzzy logic [14] the focus was set on fuzzy conjunctions and disjunctions to model the intersection and union of fuzzy sets and specially, on t-norms [5,10] due to their applications.

From the above discussion, the main goal of this paper is the proposal of the so-called fuzzy Sheffer stroke operation as a fuzzy logical connective which generalizes the Boolean Sheffer stroke when the truth values are restricted to $\{0, 1\}^2$. It will be proved that the defined operation is also functionally complete in the fuzzy logic framework being able to generate fuzzy conjunctions, disjunctions, negations and implication functions by itself. The generation of t-norms and t-conorms from this operation will be also studied. As a secondary goal of the paper, two construction methods of fuzzy Sheffer strokes are proposed. The first one relies on the use of a fuzzy conjunction and a fuzzy negation while the second one uses additive generators. From the first construction method, a connection between fuzzy Sheffer strokes and fuzzy conjunctions through a fuzzy negation is proved in an analogous way to the existing equivalence in Boolean logic.

The structure of the paper is as follows. After this introduction, those concepts and results related to fuzzy logical connectives which are necessary to understand the contents of the paper are recalled. Then in Section 3, the fuzzy Sheffer stroke operation is presented, some examples are given and the construction methods of the most important fuzzy logical connectives as combinations of fuzzy Sheffer stroke operations are introduced. In Section 4, two construction methods of families of fuzzy Sheffer strokes are proposed and some additional properties are studied. A potential application of this operation in real life is shown in Section 5. The paper ends with some conclusions and future work.

To end this introduction we want to stress that although this paper constitutes the first extensive study on the Sheffer stroke operation in fuzzy logic, some preliminary results were published in conference papers [4] and [7].

2. Preliminaries

This section encompasses the definitions and some immediate facts about the most important fuzzy logical connectives such as fuzzy negations, fuzzy conjunctions, fuzzy disjunctions and fuzzy implication functions to make the paper as self-contained as possible.

2.1. Fuzzy negations

Let us start recalling the definition of fuzzy negation and the additional properties they may have.

Definition 2.1 *([3]).* A decreasing function $N\colon [0,1] \to [0,1]$ is called a *fuzzy negation* if $N(0) = 1$ and $N(1) = 0$. Moreover, a fuzzy negation N is called

(i) *strict* if it is strictly decreasing and continuous;
(ii) *strong* if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Some important fuzzy negations which will be used throughout this paper are the classical negation (or standard negation) given by $N_C(x) = 1 - x$ for all $x \in [0,1]$, the least and the greatest fuzzy negations which are given respectively by

$$N_{D_1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases} \qquad N_{D_2}(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1, \end{cases}$$

and the Yager class and the Sugeno class of fuzzy negations defined respectively as

$$N^Y_\omega(x) = (1 - x^\omega)^{1/\omega}, \qquad N^S_\lambda(x) = \frac{1-x}{1+\lambda x},$$

for all $x \in [0,1]$, $\omega \in (0, +\infty)$ and $\lambda \in (-1, +\infty)$ (see [1]).

In order to recall the characterization theorem of strong fuzzy negations, we need previously the concept of φ-conjugation.

Definition 2.2 *([1]).* Let $\varphi\colon [0,1] \to [0,1]$ be an increasing bijection. We say that two functions $f, g\colon [0,1]^n \to [0,1]$ are φ-conjugated if $g = f_\varphi$, where

$$f_\varphi(x_1, \dots, x_n) = \varphi^{-1}(f(\varphi(x_1), \dots, \varphi(x_n))),$$

for all $x_1, \dots, x_n \in [0,1]$.

The next result provides the characterization theorem of strong fuzzy negations.

Theorem 2.3 *([12]). Let $N\colon [0,1] \to [0,1]$ be a function. The following statements are equivalent:*

(i) *N is a strong fuzzy negation.*
(ii) *There exists an increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that N and N_C are φ-conjugated, that is,*

$$N(x) = (N_C)_\varphi(x) = \varphi^{-1}(1 - \varphi(x)),$$

for all $x \in [0,1]$.

2.2. Fuzzy conjunctions and fuzzy disjunctions

Next, we recall the well-known definitions of fuzzy conjunctions and fuzzy disjunctions.

Definition 2.4 *([2]).* A function $C\colon [0,1]^2 \to [0,1]$ is called a *fuzzy conjunction* if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(C1) $C(x, y) \le C(z, y)$ for all $x \le z$,
(C2) $C(x, y) \le C(x, z)$ for all $y \le z$,
(C3) $C(0, 1) = C(1, 0) = 0$ and $C(1, 1) = 1$.

Definition 2.5 *([2]).* A function $D\colon [0,1]^2 \to [0,1]$ is called a *fuzzy disjunction* if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(D1) $D(x, y) \le D(z, y)$ for all $x \le z$,
(D2) $D(x, y) \le D(x, z)$ for all $y \le z$,
(D3) $D(0, 1) = D(1, 0) = 1$ and $D(0, 0) = 0$.

The most studied fuzzy conjunctions and fuzzy disjunctions are the families of t-norms and t-conorms, respectively.

Definition 2.6 *([5]).* An associative and commutative fuzzy conjunction $T\colon [0,1]^2 \to [0,1]$ with neutral element 1 is called a *t-norm*.

Definition 2.7 *([5]).* An associative and commutative fuzzy disjunction $S\colon [0,1]^2 \to [0,1]$ with neutral element 0 is called a *t-conorm*.

In Table 1, the t-norms and t-conorms which will be considered in this paper are collected (see [5]).

2.3. Fuzzy implication functions

The final part of this section is devoted to fuzzy implication functions. Nowadays, the most accepted definition of fuzzy implication functions is the following one.

Table 1
Some t-norms and t-conorms.

T-norm	Formula	T-conorm	Formula
Łukasiewicz	$T_{\mathbf{L}}(x, y) = \max\{x + y - 1, 0\}$	Łukasiewicz	$S_{\mathbf{L}}(x, y) = \min\{x + y, 1\}$
Minimum	$T_{\mathbf{M}}(x, y) = \min\{x, y\}$	Maximum	$S_{\mathbf{M}}(x, y) = \max\{x, y\}$
Product	$T_{\mathbf{P}}(x, y) = xy$	Prob. Sum	$S_{\mathbf{P}}(x, y) = x + y - xy$
Drastic	$T_{\mathbf{D}}(x, y) = \begin{cases} 0, & \text{if } (x, y) \in [0, 1)^2 \\ \min\{x, y\}, & \text{otherwise} \end{cases}$	Drastic	$S_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in (0, 1]^2 \\ \max\{x, y\}, & \text{otherwise} \end{cases}$

Table 2
Truth table of the classical Sheffer stroke.

p	q	$p \uparrow q$
0	0	1
0	1	1
1	0	1
1	1	0

Definition 2.8 *(see [1,3]).* A function $I : [0, 1]^2 \to [0, 1]$ is called a *fuzzy implication function* if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

(I1) $I(x, z) \geq I(y, z)$ for all $x \leq y$,
(I2) $I(x, y) \leq I(x, z)$ for all $y \leq z$,
(I3) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

It follows from the definition that $I(0, x) = 1$ and $I(x, 1) = 1$ for all $x \in [0, 1]$ whereas the symmetrical values $I(x, 0)$ and $I(1, x)$ are not derived from the axioms.

Many different fuzzy implication functions have been introduced in the literature (see [1]). In this paper, the Kleene-Dienes implication defined by $I_{\mathbf{KD}}(x, y) = \max\{1 - x, y\}$ for all $x, y \in [0, 1]$ will be considered.

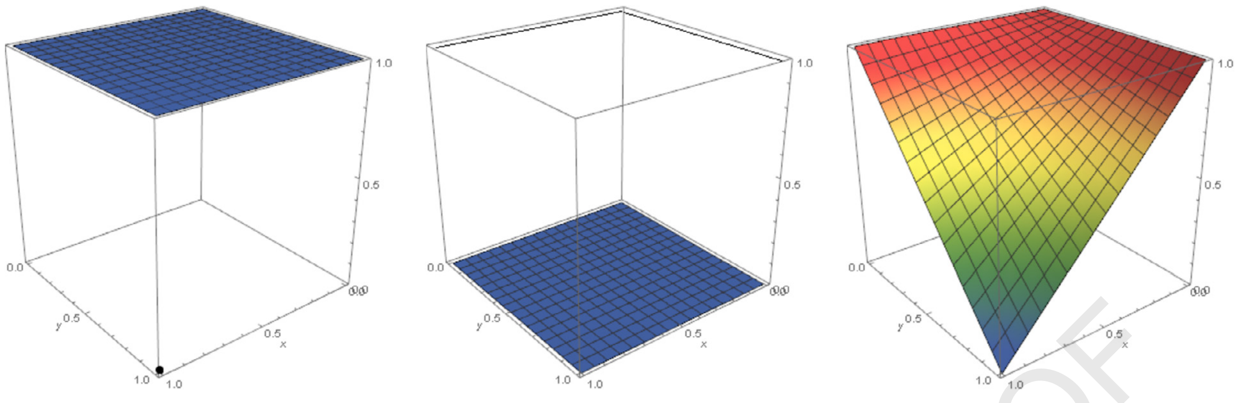
3. Fuzzy Sheffer stroke

In classical logic, Sheffer stroke, also called NAND or alternative denial, is one of the two operations that can be used by itself, without any other logical operations, to constitute a logical formal system. In spite of its importance in classical logic, this operator has not been investigated in the fuzzy logic framework where all the efforts have been devoted to other operators which are not functionally complete. The main goal of this paper is to introduce the Sheffer stroke operation in the context of fuzzy logic. Therefore, in this section the fuzzy Sheffer stroke operation is introduced as an operation which coincides with the classical Sheffer stroke when the truth values are restricted to $\{0, 1\}^2$. After that we also show how to construct all other main fuzzy logical connectives by using only combinations of the fuzzy Sheffer stroke operator.

3.1. Definition

In classical logic, Sheffer stroke operation is the operator which indicates whether at least one of its operands is false and it is logically equivalent to the negation of the conjunction. It is usually denoted by either $\uparrow$ or $|$ and its logical truth table is presented in Table 2.

Whenever a classical operator is generalized to the fuzzy logic framework, the first requirement is that the fuzzy logical operator restricted to the set $\{0, 1\}^2$ has to coincide with the corresponding classical operator. Therefore, any potential definition of the fuzzy Sheffer stroke must satisfy Table 2. Moreover, analogously to the classical operator, the fuzzy Sheffer stroke should provide a greater value as lesser is the truth value of any of its operands. This fact is equivalent to impose the antitonicity in each of the variables. Thus, all this discussion leads to the following definition of the fuzzy Sheffer stroke which contains axioms derived from the negation of the axioms of the fuzzy conjunction.



(a) $H_{\max}$ (b) $H_{\min}$ (c) H_3

Fig. 1. Plots of the fuzzy Sheffer strokes presented in Example 3.2.

Definition 3.1. A function $H: [0,1]^2 \to [0,1]$ is called a *fuzzy Sheffer stroke operation* (or fuzzy Sheffer stroke for short) if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(H1) $H(x,z) \geq H(y,z)$ for all $x \leq y$,
(H2) $H(x,y) \geq H(x,z)$ for all $y \leq z$,
(H3) $H(0,1) = H(1,0) = 1$ and $H(1,1) = 0$.

Note that, using the monotonicities, from the definition it is derived that $H(0,x) = 1$ and $H(x,0) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $H(x,0)$ and $H(1,x)$ are not derived from the definition. In particular, $H(0,0) = 1$ and it is satisfied that the fuzzy Sheffer stroke operator when restricted to $\{0,1\}^2$ coincides with the classical Sheffer stroke.

Example 3.2. Let us show some examples of operators fulfilling the conditions in Definition 3.1. Two important fuzzy Sheffer strokes are given by

$$H_{\max}(x,y) = \begin{cases} 0, & \text{if } (x,y) = (1,1), \\ 1, & \text{otherwise,} \end{cases} \qquad H_{\min}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in (0,1]^2, \\ 1, & \text{otherwise,} \end{cases}$$

which denote the maximum and the minimum fuzzy Sheffer strokes, respectively. On the other hand, $H_3(x,y) = 1 - xy$ for all $x, y \in [0,1]$ provides an example of a continuous fuzzy Sheffer stroke. The plots of these operators are shown in Fig. 1.

3.2. Construction of other fuzzy connectives from fuzzy Sheffer stroke

Once the fuzzy Sheffer stroke has been introduced, let us generate the main fuzzy logical connectives by using only combinations of this operator. Namely, in the rest of this section, we will prove that fuzzy negations, fuzzy conjunctions, fuzzy disjunctions and fuzzy implication functions can be obtained through adequate combinations of fuzzy Sheffer strokes. In order to generate these operators, we will apply some tautologies from classical logic.

3.2.1. Construction of fuzzy negations

From Definition 3.1 it can be straightforwardly seen that given a Sheffer stroke operation three fuzzy negations can be defined. These fuzzy negations are given by some sections of the Sheffer stroke as the following definition presents.

Definition 3.3. Let H be a Sheffer stroke operation.

(i) The function N_H^l defined by $N_H^l(x) = H(x,1)$ for all $x \in [0,1]$ is called the *left natural negation* of H.

(ii) The function N_H^r defined by $N_H^r(x) = H(1, x)$ for all $x \in [0, 1]$ is called the *right natural negation* of H.
(iii) The function N_H^d defined by $N_H^d(x) = H(x, x)$ for all $x \in [0, 1]$ is called the *diagonal natural negation* of H.

Next result proves that the three above defined operators are indeed fuzzy negations.

Proposition 3.4. *Let H be a Sheffer stroke operation. Then N_H^l, N_H^r and N_H^d are fuzzy negations.*

Proof. On the first hand, all of them are decreasing functions since H is decreasing in each variable. Moreover, the corner conditions are also satisfied since

$$\begin{aligned} N_H^l(0) = H(0,1) = 1, \quad & N_H^r(0) = H(1,0) = 1 \quad & N_H^d(0) = H(0,0) = 1, \\ N_H^l(1) = H(1,1) = 0, \quad & N_H^r(1) = H(1,1) = 0, \quad & N_H^d(1) = H(1,1) = 0. \quad \square \end{aligned}$$

Note that the three natural negations provided by each fuzzy Sheffer stroke introduced in Example 3.2 coincide. Indeed, $N_{H_{\max}}^l = N_{H_{\max}}^r = N_{H_{\max}}^d = N_{D_2}$, $N_{H_{\min}}^l = N_{H_{\min}}^r = N_{H_{\min}}^d = N_{D_1}$ and $N_{H_3}^l = N_{H_3}^r = N_{H_3}^d = N_C$. This is not always the case. Consider the fuzzy Sheffer stroke given by

$$H_4(x, y) = \begin{cases} 1, & \text{if } x + y \le 1, \\ 1 - y, & \text{otherwise.} \end{cases}$$

It can be checked that $N_{H_4}^l = N_{D_1}$, $N_{H_4}^r = N_C$ and

$$N_{H_4}^d(x) = \begin{cases} 1, & \text{if } x \le \frac{1}{2}, \\ 1 - x, & \text{otherwise.} \end{cases}$$

3.2.2. *Construction of fuzzy conjunctions and fuzzy disjunctions*

In classical logic, the following two tautologies:

$$\begin{aligned} & p \wedge q \equiv ((p \uparrow q) \uparrow (p \uparrow q)), \\ & p \vee q \equiv ((p \uparrow p) \uparrow (q \uparrow q)), \end{aligned}$$

provide the construction methods of conjunction and disjunction from classical Sheffer stroke. In the next results, we will prove that the corresponding formulas involving fuzzy logical connectives can be used analogously to generate fuzzy conjunctions and fuzzy disjunctions from fuzzy Sheffer strokes. First, let us generate fuzzy conjunctions from fuzzy Sheffer strokes.

Theorem 3.5. *Let H be a fuzzy Sheffer stroke. Then, the function $C_H\colon [0,1]^2 \to [0,1]$ given by*

$$C_H(x, y) = H(H(x, y), H(x, y)), \quad x, y \in [0, 1], \tag{1}$$

is a fuzzy conjunction.

Proof. First, we will prove that C_H satisfies the monotonicities, i.e., (**C1**) and (**C2**). Let $x, y, z \in [0, 1]$ be such that $x \le z$. By using the decreasingness of N_H^d and the one of H in the first variable, we get that

$$C_H(x, y) = H(H(x, y), H(x, y)) = N^d(H(x, y)) \le N^d(H(z, y)) = H(H(z, y), H(z, y)) = C_H(z, y).$$

Thus, C_H satisfies (**C1**). It can be shown analogously that C_H is increasing in the second variable and therefore, C_H also satisfies (**C2**). Finally, we show that C_H satisfies the boundary conditions, and consequently (**C3**), as follows:

$$\begin{aligned} & C_H(0, 1) = H(H(0, 1), H(0, 1)) = H(1, 1) = 0, \\ & C_H(1, 0) = H(H(1, 0), H(1, 0)) = H(1, 1) = 0, \\ & C_H(1, 1) = H(H(1, 1), H(1, 1)) = H(0, 0) = 1. \quad \square \end{aligned}$$

Example 3.6. Let us compute the fuzzy conjunctions generated from the fuzzy Sheffer strokes given in Example 3.2 by using Theorem 3.5.

(i) The fuzzy conjunction generated from $H_{\max}$ is given by

$$C_{H_{\max}}(x,y) = H_{\max}(H_{\max}(x,y), H_{\max}(x,y)) = \begin{cases} 1, & \text{if } (x,y) = (1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $C_{H_{\max}}$ is in fact the least fuzzy conjunction.

(ii) The fuzzy conjunction generated from $H_{\min}$ is given by

$$C_{H_{\min}}(x,y) = H_{\min}(H_{\min}(x,y), H_{\min}(x,y)) = \begin{cases} 1, & \text{if } (x,y) \in (0,1]^2, \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to check that $C_{H_{\min}}$ is just the greatest fuzzy conjunction.

(iii) Finally, the fuzzy conjunction generated from H_3 is given by

$$C_{H_3}(x,y) = H_3(H_3(x,y), H_3(x,y)) = 2xy - x^2y^2,$$

for all $x, y \in [0,1]$.

Next, let us provide a construction method of fuzzy disjunctions from fuzzy Sheffer strokes.

Theorem 3.7. *Let H be a fuzzy Sheffer stroke. Then, the following function*

$$D_H(x,y) = H(H(x,x), H(y,y)), \quad x, y \in [0,1], \tag{2}$$

is a fuzzy disjunction.

Proof. First, we will prove that D is increasing in each variable, i.e., (**D1**) and (**D2**). Let $x, y, z \in [0,1]$ be such that $x \le z$. Due to the decreasingness of N_H^d and the one of H in the first variable, we have that

$$\begin{aligned} D_H(x,y) &= H(H(x,x), H(y,y)) = H(N_H^d(x), H(y,y)) \\ &\le H(N_H^d(z), H(y,y)) = H(H(z,z), H(y,y)) = D_H(z,y) \end{aligned}$$

and D_H satisfies (**D1**). It can be shown analogously that D_H is increasing in the second variable satisfying also (**D2**). Finally, we show that D_H satisfies the boundary conditions, and therefore (**D3**), as follows:

$$\begin{aligned} D_H(0,0) &= H(H(0,0), H(0,0)) = H(1,1) = 0, \\ D_H(0,1) &= H(H(0,0), H(1,1)) = H(1,0) = 1, \\ D_H(1,0) &= H(H(1,1), H(0,0)) = H(0,1) = 1. \quad \square \end{aligned}$$

Example 3.8. Let us compute the fuzzy disjunctions generated from the fuzzy Sheffer strokes given in Example 3.2 by using Theorem 3.7.

(i) The fuzzy disjunction generated from $H_{\max}$ is given by

$$D_{H_{\max}}(x,y) = D_{\max}(D_{\max}(x,x), H_{\max}(y,y)) = \begin{cases} 0, & \text{if } (x,y) \in [0,1)^2, \\ 1, & \text{otherwise.} \end{cases}$$

Note that $D_{H_{\max}}$ is in fact the least fuzzy disjunction.

(ii) The fuzzy disjunction generated from $H_{\min}$ is given by

$$D_{H_{\min}}(x,y) = H_{\min}(H_{\min}(x,x), H_{\min}(y,y)) = \begin{cases} 0, & \text{if } (x,y) = (0,0), \\ 1, & \text{otherwise.} \end{cases}$$

It is straightforward to check that $D_{H_{\min}}$ is just the greatest fuzzy disjunction.

(iii) Finally, the fuzzy disjunction generated from H_3 is given by

$$D_{H_3}(x, y) = H_3(H_3(x, x), H_3(y, y)) = x^2 + y^2 - x^2y^2,$$

for all $x, y \in [0, 1]$.

Once we have already presented the generation methods of fuzzy conjunctions and fuzzy disjunctions from fuzzy Sheffer strokes, it is worthy to study when these methods provide in particular t-norms and t-conorms. In order to analyse this fact, we need to introduce several additional properties of fuzzy Sheffer strokes which will play an important role in the next results. These properties are the following ones:

(H4) $H(H(x, x), H(x, x)) = x$, for all $x \in [0, 1]$,
(H5) $H(1, x) = H(x, x)$, for all $x \in [0, 1]$,
(H6) $H(x, y) = H(y, x)$, for all $x, y \in [0, 1]$,
(H7) $H(H(H(x, y), H(x, y)), z) = H(x, H(H(y, z), H(y, z)))$, for all $x, y, z \in [0, 1]$.

Note that property (**H4**) can be rewritten as $N_H^d(N_H^d(x)) = x$ for all $x \in [0, 1]$, i.e., N_H^d is a strong negation. Next, (**H5**) is equivalent to $N_H^r = N_H^d$. (**H6**) means that H is symmetric and (**H7**) will prove to be related with the associativity of the generated fuzzy conjunction.

In this paper, we will restrict the study to the subcase of obtaining t-norms and t-conorms from fuzzy Sheffer strokes satisfying (**H4**). In Remark 3.11, it is proved that this condition is not necessary but it remains an open problem to analyse under which additional conditions a t-norm or a t-conorm can be obtained.

Let us start with a first lemma which will be useful to prove the next results.

Lemma 3.9. *Let $H\colon [0, 1]^2 \to [0, 1]$ be a fuzzy Sheffer stroke that satisfies (**H4**). Then the following statements are equivalent:*

(i) *H satisfies (**H5**).*
(ii) *$H(H(1, x), H(1, x)) = x$, for all $x \in [0, 1]$.*
(iii) *$H(1, H(x, x)) = x$, for all $x \in [0, 1]$.*

Proof. First assume that H satisfies (**H4**) and (**H5**) and let us prove (ii). Then, for all $x \in [0, 1]$, we have that

$$H(H(1, x), H(1, x)) = H(H(x, x), H(x, x)) = x,$$

and (ii) follows.

Let us assume now that H satisfies (**H4**) and (ii) and let us prove that (iii) is fulfilled. We have that

$$N_H^d(H(1, N_H^d(x))) = H(H(1, N_H^d(x)), H(1, N_H^d(x))) = N_H^d(x),$$

but since N_H^d is strong due to (**H4**) (and consequently one-to-one), this is equivalent to $H(1, N_H^d(x)) = x$ for all $x \in [0, 1]$ and therefore, $H(1, H(x, x)) = x$ for all $x \in [0, 1]$ holds.

Finally, assume now (**H4**) and $H(1, H(x, x)) = x$ for all $x \in [0, 1]$ and let us prove that (i) is fulfilled, i.e., (**H5**). It holds that

$$H(1, N_H^d(x)) = H(1, H(x, x)) = x = H(H(x, x), H(x, x)) = H(N_H^d(x), N_H^d(x)),$$

and since N_H^d is strong due to (**H4**) (and consequently surjective), we obtain $H(1, x) = H(x, x)$, for all $x \in [0, 1]$. □

Next theorem studies under which conditions a t-norm is obtained from fuzzy Sheffer strokes that satisfy (**H4**).

Theorem 3.10. *Let H be a fuzzy Sheffer stroke that satisfies (**H4**). Then, the following function*

$$T_H(x, y) = H(H(x, y), H(x, y)), \quad x, y \in [0, 1], \tag{3}$$

*is a t-norm if and only if H satisfies additionally (**H5**), (**H6**) and (**H7**).*

Proof. First of all, we will prove that if H is a fuzzy Sheffer stroke satisfying (**H4**), (**H5**), (**H6**) and (**H7**), then T_H is a t-norm. Indeed, Theorem 3.5 guarantees the increasingness of T_H in each variable. Moreover, Lemma 3.9 proves that T_H has neutral element 1 since H satisfies (**H5**). Next, it is clear that if property (**H6**) holds then T_H is commutative. Finally, the associativity of T_H follows from property (**H7**).

Reciprocally, if T_H is a t-norm, let us prove that H satisfies (**H5**), (**H6**) and (**H7**). Since T_H has neutral element 1, (**H5**) follows directly. From the commutativity of T_H, we get

$$N_H^d(H(x,y)) = H(H(x,y), H(x,y)) = T_H(x,y) = T_H(y,x) = H(H(y,x), H(y,x)) = N_H^d(H(y,x)),$$

for every $x, y \in [0,1]$, which proves that H satisfies (**H6**) since (**H4**) ensures that N_H^d is strong and therefore, one-to-one. Finally, let $x, y, z \in [0,1]$, then using the associativity of T_H, we have that

$$\begin{aligned} N_H^d(H(H(H(x,y), H(x,y)), z)) &= H(H(H(H(x,y), H(x,y)), z), H(H(H(x,y), H(x,y)), z)) \\ &= T_H(T_H(x,y), z) = T_H(x, T_H(y,z)) \\ &= H(H(x, H(H(y,z), H(y,z))), H(x, H(H(y,z), H(y,z)))) \\ &= N_H^d(H(x, H(H(y,z), H(y,z)))), \end{aligned}$$

and property (**H7**) holds due to the fact that again N_H^d is one-to-one. □

Remark 3.11. Although the study has been limited to the particular case of fuzzy Sheffer strokes satisfying (**H4**), there exist fuzzy Sheffer strokes not satisfying this additional property which provide also t-norms through Eq. (3). This is the case of the following fuzzy Sheffer stroke

$$H(x,y) = \begin{cases} 1-x, & \text{if } y = 1, \\ 1-y, & \text{if } x = 1, \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that H does not satisfy (**H4**) since $N_H^d = N_{D_2}$ but T_H is the well-known drastic t-norm $T_{\mathbf{D}}$.

Let us perform now a similar study but now for the construction of t-conorms.

Theorem 3.12. *Let H be a fuzzy Sheffer stroke that satisfies (**H4**). Then, the following function*

$$S_H(x,y) = H(H(x,x), H(y,y)), \quad x, y \in [0,1], \tag{4}$$

*is a t-conorm if and only if H satisfies additionally (**H5**), (**H6**) and (**H7**).*

Proof. First of all, Theorem 3.7 guarantees the increasingness of S_H in each variable. From Lemma 3.9 we obtain that S_H has neutral element 0 if and only if H satisfies (**H5**). It is straightforward to see that property (**H6**) implies the commutativity of S_H. Reciprocally, assume now that S_H is commutative, then

$$H(H(x,x), H(y,y)) = S_H(x,y) = S_H(y,x) = H(H(y,y), H(x,x)),$$

for all $x, y \in [0,1]$. This proves that H satisfies (**H5**) due to the exhaustivity of N_H^d. Let $x, y, z \in [0,1]$ and $a = H(x,x)$, $b = H(y,y)$, $c = H(z,z)$, then

$$\begin{aligned} S_H(x, S_H(y,z)) &= H(H(x,x), H(S(y,z), S(y,z))) \\ &= H(H(x,x), H(H(H(y,y), H(z,z)), H(H(y,y), H(z,z)))) \\ &= H(a, H(H(b,c), H(b,c))), S_H(S_H(x,y), z) \\ &= H(H(H(x,x), H(y,y)), H(z,z)) \\ &= H(H(H(H(x,x), H(y,y)), H(H(x,x), H(y,y))), H(z,z)) \\ &= H(H(H(a,b), H(a,b)), c). \end{aligned}$$

The above equations prove that S_H is associative if and only if H satisfies (**H7**) by using again the exhaustivity of N_H^d. □

Remark 3.13. Contrarily to the case of the construction of t-norms, it is still an open problem to determine whether property (**H4**) is necessary for a Sheffer stroke to generate a t-conorm by means of Eq. (4). Note that if we consider the same fuzzy Sheffer stroke H which provided a counterexample in Remark 3.11, in this case applying Eq. (4), we obtain the least fuzzy disjunction, which is not a t-conorm.

In the two previous theorems, it has been proved that those fuzzy Sheffer strokes satisfying properties (**H4**)-(**H7**) generate always t-norms and t-conorms through Eqs. (3) and (4), respectively. Due to their importance in these construction methods, the next theorem provides a characterization of these fuzzy Sheffer strokes.

Theorem 3.14. *Let $H\colon [0,1]^2 \to [0,1]$ be a binary function. Then the following statements are equivalent:*

(i) *H satisfies all the properties (**H1**)-(**H7**), i.e., H is a fuzzy Sheffer stroke satisfying properties (**H4**)-(**H7**).*
(ii) *There exists an increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that H is given by*

$$H(x, y) = \varphi^{-1}(1 - \varphi(\min\{x, y\})), \quad x, y \in [0,1].$$

Proof. We will prove first that (i) implies (ii). On the one hand, H satisfies properties (**H1**)-(**H3**) and consequently, it is a fuzzy Sheffer stroke. Now since (**H4**) is satisfied, then N_H^d is a strong fuzzy negation and by using Theorem 3.10, the function

$$T_H(x, y) = H(H(x, y), H(x, y)),$$

for all $x, y \in [0,1]$ is also a t-norm due to the fulfilment of properties (**H5**)-(**H7**). At this point, let us prove that $H(x, y) = N_H^d(T_H(x, y))$ for all $x, y \in [0,1]$. Indeed,

$$N_H^d(T_H(x, y)) = N_H^d(H(H(x, y), H(x, y))) = N_H^d(N_H^d(H(x, y))) = H(x, y)$$

and the equality holds. Now, we have that

$$N_H^d(x) = H(x, x) = N_H^d(T_H(x, x)),$$

for all $x \in [0,1]$ and since N_H^d is injective, we obtain that $x = T_H(x, x)$ for all $x \in [0,1]$. Thus, this leads to $T_H = T_{\mathbf{M}}$, the only idempotent t-norm, and therefore, using also Theorem 2.3, there exists an increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that H is given by

$$H(x, y) = N_H^d(T_H(x, y)) = \varphi^{-1}(1 - \varphi(\min\{x, y\})),$$

for all $x, y \in [0,1]$.

Reciprocally, let us prove that (ii) implies (i). First, it is clear that H is decreasing in each variable, i.e., it satisfies properties (**H1**) and (**H2**), and it satisfies also (**H3**) since $H(0,1) = \varphi^{-1}(1) = 1$, $H(1,0) = \varphi^{-1}(1) = 1$ and $H(1,1) = \varphi^{-1}(0) = 0$. Next, since both the diagonal and the right natural negation of H are equal to $(N_C)_\varphi$ which is a strong negation, property (**H4**) is satisfied. Note that $T_H = T_{\mathbf{M}}$, i.e., T_H is a t-norm. Indeed,

$$\begin{aligned} T_H(x, y) &= H(H(x, y), H(x, y)) = \varphi^{-1}(1 - \varphi(\min\{H(x, y), H(x, y)\})) \\ &= \varphi^{-1}(1 - \varphi(\varphi^{-1}(1 - \varphi(\min\{x, y\})))) = \min\{x, y\}, \end{aligned}$$

for all $x, y, z \in [0,1]$. From Theorem 3.10 the function H satisfies all the properties (**H1**)-(**H7**). □

Example 3.15. Some examples of fuzzy Sheffer strokes satisfying all the properties (**H1**)-(**H7**) are

$$H_1(x, y) = \max\{1 - x, 1 - y\}, \quad H_2(x, y) = \sqrt{\max\{1 - x^2, 1 - y^2\}}$$

for all $x, y \in [0,1]$ which are obtained by choosing $\varphi(x) = x$ and $\varphi(x) = x^2$, respectively, in Theorem 3.14.

3.2.3. Construction of fuzzy implication functions

In classical logic, there exist two tautologies to define the classical implication by using only the classical Sheffer stroke operation:

$$p \to q \equiv p \uparrow (q \uparrow q) \equiv \neg(p \wedge \neg(q \wedge q)), \tag{QQ}$$

$$p \to q \equiv p \uparrow (p \uparrow q) \equiv \neg(p \wedge \neg(p \wedge q)). \tag{PQ}$$

In this section, we will prove that when these tautologies are translated to the fuzzy logic framework, while the first construction method generates always a fuzzy implication function in the sense of Definition 2.8, the second method does not guarantee in general the decreasingness in the first variable.

Theorem 3.16. *Let H be a fuzzy Sheffer stroke. Then the function $I_H^{QQ}\colon [0,1]^2 \to [0,1]$ defined by*

$$I_H^{QQ}(x,y) = H(x, H(y,y)), \quad x, y \in [0,1], \tag{5}$$

is a fuzzy implication function.

Proof. Let $x, y, z \in [0,1]$ be such that $x \leq z$. Using the decreasingness of H in the first variable, we obtain

$$I_H^{QQ}(x,y) = H(x, H(y,y)) \geq H(z, H(y,y)) = I_H^{QQ}(z,y).$$

Analogously, let $x, y, z \in [0,1]$ be such that $y \leq z$. Using the decreasingness of N_H^d and the one of H in the second variable, we get

$$I_H^{QQ}(x,y) = H(x, H(y,y)) = H(x, N_H^d(y)) \leq H(x, N_H^d(z)) = H(x, H(z,z)) = I_H^{QQ}(x,z).$$

The border conditions are also satisfied:

$$I_H^{QQ}(0,0) = H(0, H(0,0)) = H(0,1) = 1,$$
$$I_H^{QQ}(1,0) = H(1, H(0,0)) = H(1,1) = 0,$$
$$I_H^{QQ}(1,1) = H(1, H(1,1)) = H(1,0) = 1. \quad \square$$

Example 3.17. Taking into account the fuzzy Sheffer strokes given in Example 3.2, let us compute which fuzzy implication functions are obtained by using Theorem 3.16.

(i) The fuzzy implication function generated from $H_{\max}$ is given by

$$I_{H_{\max}}^{QQ}(x,y) = H_{\max}(x, H_{\max}(y,y)) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y < 1, \\ 1, & \text{otherwise.} \end{cases}$$

(ii) The fuzzy implication function generated from $H_{\min}$ is given by

$$I_{H_{\min}}^{QQ}(x,y) = H_{\min}(x, H_{\min}(y,y)) = \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

(iii) Finally, the fuzzy implication function generated from H_3 is given by

$$I_{H_3}^{QQ}(x,y) = H_3(x, H_3(y,y)) = 1 - x + xy^2,$$

for all $x, y \in [0,1]$, a polynomial fuzzy implication of degree 3 (see [6] for further details).

On the other hand, if we consider the construction method derived from Eq. (PQ), properties (**I2**) and (**I3**) are the only properties which are always fulfilled.

Theorem 3.18. *Let H be a fuzzy Sheffer stroke. Then the function $I_H^{PQ}\colon [0,1]^2 \to [0,1]$ defined by*

$$I_H^{PQ}(x,y) = H(x, H(x,y)), \quad x, y \in [0,1],$$

*satisfies (**I2**) and (**I3**).*

Proof. Let $x, y, z \in [0,1]$ be such that $y \le z$. Using the decreasingness of H in the second variable, we obtain

$$I_H^{PQ}(x,y) = H(x, H(x,y)) \le H(x, H(x,z)) = I_H^{PQ}(x,z)$$

and (**I2**) follows. The border conditions are also satisfied:

$$I_H^{PQ}(0,0) = H(0, H(0,0)) = H(0,1) = 1,$$
$$I_H^{PQ}(1,0) = H(1, H(1,0)) = H(1,1) = 0,$$
$$I_H^{PQ}(1,1) = H(1, H(1,1)) = H(1,0) = 1. \quad \square$$

In order to check that property (**I1**) is not guaranteed in Theorem 3.18, consider the fuzzy Sheffer stroke H_3 given in Example 3.2. By using Eq. (PQ), the operator given by $I_{H_3}^{PQ}(x,y) = 1 - x + x^2 y$ for all $x, y \in [0,1]$ is obtained. This operator does not fulfil property (**I1**) since $I_{H_3}^{PQ}(0.5, 1) = 0.75 < 1 = I_{H_3}^{PQ}(1,1)$. On the other hand, if we consider the fuzzy Sheffer stroke $H_{\max}$ given in Example 3.2, the obtained operator:

$$I_{H_{\max}}^{PQ}(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y < 1, \\ 1 & \text{otherwise,} \end{cases}$$

satisfies (**I1**) and therefore, it is a fuzzy implication function. Another example of the fulfilment of (**I1**) is retrieved when we consider $H(x,y) = \min\{2 - x - y, 1\}$ for all $x, y \in [0,1]$. In that case, $I_H^{PQ} = I_{\mathbf{KD}}$, the well-known Kleene-Dienes implication.

Remark 3.19. Theorems 3.16 and 3.18 can be understood as generalizations of some results obtained in [7] for two new families of fuzzy implication functions denoted as SS_{pq} and SS_{qq}. Indeed, the families SS_{pq} and SS_{qq} are defined using the second equivalence in Eqs. (PQ) and (QQ) but taking t-norms and t-conorms as particular instances of fuzzy conjunctions and fuzzy disjunctions, respectively.

4. Construction methods of fuzzy Sheffer strokes

In this section, several different construction methods of fuzzy Sheffer strokes will be presented.

4.1. Construction method from a fuzzy conjunction and a fuzzy negation

In classical logic, Sheffer stroke is the negation of the conjunction (NAND), that is, $p \uparrow q = \neg(p \wedge q)$. This result is also valid in the fuzzy logic framework taking into account a fuzzy conjunction and a strict fuzzy negation.

Proposition 4.1. *Let C be a fuzzy conjunction and N a fuzzy negation. Then the function $H_{C,N}\colon [0,1]^2 \to [0,1]$ defined by*

$$H_{C,N}(x,y) = N(C(x,y)), \quad x, y \in [0,1], \tag{6}$$

is a fuzzy Sheffer stroke.

Proof. Let us prove that the operation $H_{C,N}$ is a fuzzy Sheffer stroke. Due to the monotonicity of C and N, we have that

$$H(x_1, y) = N(C(x_1, y)) \ge N(C(x_2, y)) = H(x_2, y),$$

for all $x_1, x_2, y \in [0,1]$ such that $x_1 \le x_2$, and therefore, H is non-increasing in the first variable. It can be shown analogously that H is non-increasing in the second variable. The border conditions are also satisfied:

$$H(0,1) = N(C(0,1)) = N(0) = 1,$$
$$H(1,0) = N(C(1,0)) = N(0) = 1,$$
$$H(1,1) = N(C(1,1)) = N(1) = 0.$$

Thus, $H(x,y) = N(C(x,y))$ is a fuzzy Sheffer stroke. $\square$

Table 3
Fuzzy Sheffer strokes $H_{C,N}$ obtained from the basic t-norms and $N = N_C$.

T-norm	Negation	$H_{C,N}$
$T_{\mathbf{M}}$	N_C	$H_{T_{\mathbf{M}},N_C}(x,y) = \max\{1-x, 1-y\}$
$T_{\mathbf{P}}$	N_C	$H_{T_{\mathbf{P}},N_C}(x,y) = 1-xy$
$T_{\mathbf{L}}$	N_C	$H_{T_{\mathbf{L}},N_C}(x,y) = \min\{2-x-y, 1\}$
$T_{\mathbf{D}}$	N_C	$H_{T_{\mathbf{D}},N_C}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [0,1)^2, \\ \max\{1-x, 1-y\}, & \text{otherwise.} \end{cases}$

Using Eq. (6), we can obtain fuzzy Sheffer strokes by considering some fuzzy conjunctions and fuzzy negations. Let us provide several examples of fuzzy Sheffer strokes generated through this construction method.

Example 4.2. The fuzzy Sheffer strokes that are obtained from the basic t-norms and the classical negation N_C are given in Table 3.

Example 4.3. If we consider the following family of fuzzy conjunctions $C_{\mathbf{P}}^{\mathbf{k}}(x,y) = (xy)^k$, for any $k > 0$, and the classical negation $N_{\mathbf{C}}$, we obtain

$$H_{C_{\mathbf{P}}^{\mathbf{k}},N_C}(x,y) = 1-(xy)^k.$$

Note that if $k = 1$, we recover $H_{T_{\mathbf{P}},N_C}$. Other two fuzzy Sheffer strokes, generated this time from the minimum t-norm $T_{\mathbf{M}}$ and strong negations from Sugeno and Yager classes are given by

$$H_{T_{\mathbf{M},\mathbf{N}_\omega^{\mathbf{Y}}}}(x,y) = (1-\min\{x,y\}^\omega)^{1/\omega}, \qquad H_{T_{\mathbf{M}},N_\lambda^S}(x,y) = \frac{1-\min\{x,y\}}{1+\lambda\min\{x,y\}},$$

respectively, with $\omega \in (0,+\infty)$ and $\lambda \in (-1,+\infty)$. It can be checked that $H_{T_{\mathbf{M},\mathbf{N}_\mathbf{1}^{\mathbf{Y}}}} = H_{T_{\mathbf{M},\mathbf{N}_\mathbf{0}^{\mathbf{S}}}} = H_{T_{\mathbf{M}},N_C}$.

Some of the fuzzy Sheffer strokes given in the previous examples are displayed in Fig. 2.

The construction method provided by Eq. (6) is of paramount importance. Not only provides a way to construct fuzzy Sheffer strokes, but in fact any fuzzy Sheffer stroke can be generated from a fuzzy conjunction and a fuzzy negation.

Theorem 4.4. *Let $H: [0,1]^2 \to [0,1]$ be a binary operation. Then the following statements are equivalent:*

(i) *H is a fuzzy Sheffer stroke.*
(ii) *There exist a fuzzy conjunction C and a strict fuzzy negation N such that $H(x,y) = N(C(x,y))$ for all $x,y \in [0,1]$.*

Moreover, in this case, $C(x,y) = N^{-1}(H(x,y))$ for all $x,y \in [0,1]$.

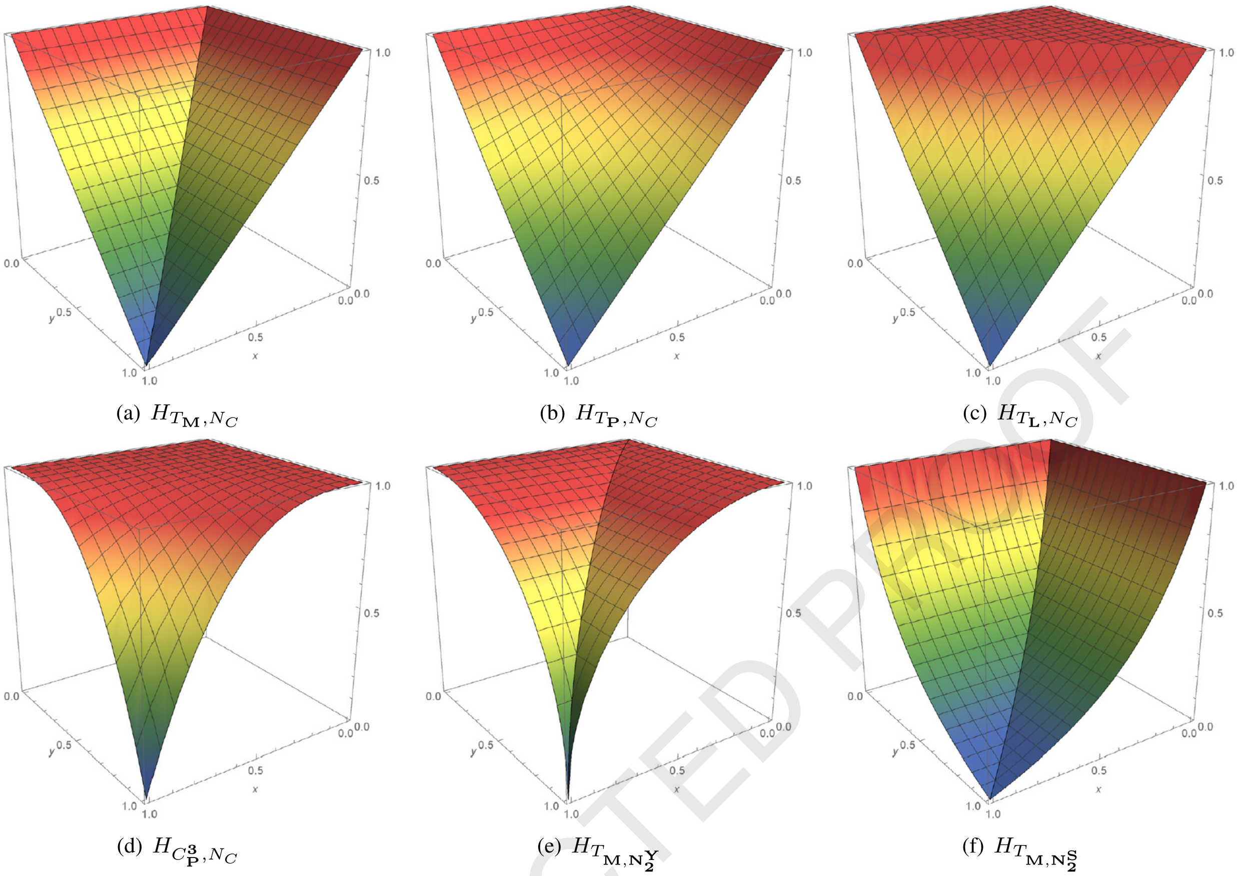
Proof. (ii) implies (i) is proved already by Proposition 4.1. Conversely, let us consider now a fuzzy Sheffer stroke operation H. Let us consider any strict fuzzy negation N and let us define C as the binary function given by

$$C(x,y) = N^{-1}(H(x,y)), \quad x,y \in [0,1].$$

We will prove that C is a fuzzy conjunction. Due to the decreasingness of H and N^{-1}, we have that for all $x_1, x_2, y \in [0,1]$, $x_1 \le x_2$,

$$C(x_1,y) = N^{-1}(H(x_1,y)) \le N^{-1}(H(x_2,y)) = C(x_2,y),$$

and therefore, C is increasing in the first variable. It can be shown analogously that C is increasing in the second variable. The border conditions are also satisfied:



(a) $H_{T_{\mathbf{M}},N_C}$ (b) $H_{T_{\mathbf{P}},N_C}$ (c) $H_{T_{\mathbf{L}},N_C}$

(d) $H_{C^{\mathbf{3}}_{\mathbf{P}},N_C}$ (e) $H_{T_{\mathbf{M}},\mathbf{N}^{\mathbf{Y}}_{\mathbf{2}}}$ (f) $H_{T_{\mathbf{M}},\mathbf{N}^{\mathbf{S}}_{\mathbf{2}}}$

Fig. 2. Plots of some of the fuzzy Sheffer strokes presented in Examples 4.2 and 4.3.

$$C(1,1) = N^{-1}(H(1,1)) = N^{-1}(0) = 1,$$
$$C(0,1) = N^{-1}(H(0,1)) = N^{-1}(1) = 0,$$
$$C(1,0) = N^{-1}(H(1,0)) = N^{-1}(1) = 0.$$

Finally, the result follows since

$$N(C(x,y)) = N(N^{-1}(H(x,y))) = H(x,y),$$

for all $x, y \in [0,1]$. □

For notation purposes, given a strict fuzzy negation N and a fuzzy Sheffer stroke H, we will denote by $C_{H,N}$ the fuzzy conjunction defined in Theorem 4.4.

Remark 4.5. Some remarks on the previous theorem are worthy to mention:

(i) The representation of a fuzzy Sheffer stroke in terms of a pair (C, N) is not unique. Indeed, any strict fuzzy negation N can be chosen. However, fixed a strict fuzzy negation N, the fuzzy conjunction C is unique.
(ii) Whenever one of the natural negations of the fuzzy Sheffer stroke is strict, it can be considered to represent the fuzzy Sheffer stroke. In this case, both the fuzzy negation and the fuzzy conjunction are defined from the expression of H.

Theorem 4.4 shows a strong connection between fuzzy Sheffer strokes and fuzzy conjunctions via strict negations. This connection could support the idea that any forthcoming study on fuzzy Sheffer strokes can be made through the corresponding study of the associated fuzzy conjunctions. However, from our point of view, this is not the case. Indeed, the connection between these two families of operators strongly depends on the fuzzy negation and the properties from one operator are not easily derived to the other operator due to the fuzzy negation. Moreover, in the literature there are analogous situations which provide strong evidence of our claim. It is well known that any fuzzy implication function I can be generated by means of a fuzzy disjunction D and a fuzzy negation N as the following result presented in [9] shows:

Theorem 4.6 *(see [9]). For a function* $I : [0,1]^2 \to [0,1]$*, the following statements are equivalent:*

1. *I is a fuzzy implication function.*
2. *There exists a fuzzy disjunction D such that $I(x, y) = D(1-x, y)$ for all $x, y \in [0,1]$.*

Moreover, in this case, $D(x, y) = I(1-x, y)$.

While this connection helps to study properties from fuzzy implication functions in terms of fuzzy disjunctions (e.g. exchange property vs associativity), other properties of fuzzy implication functions have no evident relation with known properties of the associated fuzzy disjunctions (e.g. invariance property with respect to powers of t-norms). This fact leads to different lines of research which evolve separately but colliding when the study needs it.

4.2. Construction method from univalued functions

In this section we present a method to generate fuzzy Sheffer strokes by means of two univalued functions with some specific properties.

Let us define this class of fuzzy Sheffer strokes and prove that they satisfy the axioms given in Definition 3.1.

Definition 4.7. Let $f : [0,1] \to [0,+\infty]$ be a decreasing function with $f(0) = +\infty$ and $f(1) = 0$, and let $g : [0,+\infty] \to [0,1]$ be an increasing function with $g(0) = 0$ and $g(+\infty) = 1$. The operator $H_{f,g} : [0,1]^2 \to [0,1]$ defined by

$$H_{f,g}(x, y) = g(f(x) + f(y)), \quad x, y, \in [0,1],$$

is called an (f, g)-Sheffer stroke. In this case, the pair of functions (f, g) is called the pair of additive generators of $H_{f,g}$.

Theorem 4.8. *Let* $f : [0,1] \to [0,+\infty]$ *be a decreasing function with* $f(0) = +\infty$*,* $f(1) = 0$*, and let* $g : [0,+\infty] \to [0,1]$ *be an increasing function with* $g(0) = 0$*,* $g(+\infty) = 1$*. Then* $H_{f,g}$ *is always a fuzzy Sheffer stroke.*

Proof. Let us begin proving that $H_{f,g}$ is decreasing in the first variable. Let $x_1, x_2, y \in [0,1]$ and $x_1 \le x_2$, then it holds that $f(x_1) \ge f(x_2)$ and $f(x_1) + f(y) \ge f(x_2) + f(y)$. Thus, we have that

$$g(f(x_1) + f(y)) \ge g(f(x_2) + f(y)).$$

It can be shown analogously that $H_{f,g}$ is decreasing in the second variable. Moreover, the border conditions are also satisfied as follows:

$$\begin{aligned} H_{f,g}(0,0) &= g(f(0) + f(0)) = g(+\infty) = 1, \\ H_{f,g}(0,1) = H_{f,g}(1,0) &= g(f(0) + f(1)) = g(+\infty) = 1, \\ H_{f,g}(1,1) &= g(f(1) + f(1)) = g(0) = 0. \end{aligned}$$

Thus, $H_{f,g}$ is a fuzzy Sheffer stroke. □

As it is usual when dealing with additive generators, this construction method can be also defined using multiplicative generators.

Remark 4.9. If (f, g) is a pair of additive generators of an (f, g)-Sheffer stroke, we can define the decreasing function $\varphi\colon [0, +\infty] \to [0, 1]$ given by $\varphi(x) = g(-\ln x)$ and the increasing function $\theta\colon [0, 1] \to [0, 1]$ given by $\theta(x) = e^{-f(x)}$, then it is obvious that for all $x, y \in [0, 1]$

$$H_{f,g}(x, y) = \varphi\left(\theta(x) \cdot \theta(y)\right).$$

Note that the additive generators of $H_{f,g}$ can be non-continuous functions as the following example shows.

Example 4.10. Let us consider the non-continuous functions $f(x) = \begin{cases} +\infty, & \text{if } x < 1, \\ 0, & \text{if } x = 1, \end{cases}$ and $g(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise,} \end{cases}$ which satisfy the requirements of Definition 4.7. Then we obtain the fuzzy Sheffer stroke given by

$$H_{\max}(x, y) = H_{f,g}(x, y) = \begin{cases} 0, & \text{if } (x, y) = (1, 1), \\ 1, & \text{otherwise,} \end{cases}$$

which is the maximum fuzzy Sheffer stroke given in Example 3.2.

From the representation theorem of fuzzy Sheffer strokes in terms of fuzzy conjunctions and negations, we know that given a fuzzy Sheffer stroke H and a strict fuzzy negation N, we can define the associated fuzzy conjunction $C_{H,N}$. Next result shows under which conditions the associated conjunction of an (f, g)-Sheffer stroke is a t-norm.

Proposition 4.11. *Let N be a strict fuzzy negation and (f, g) be a pair of additive generators of an (f, g)-Sheffer stroke with g continuous and strictly increasing. Then the following statements are equivalent:*

(i) $C_{H_{f,g},N} = N^{-1} \circ H_{f,g}$ *is a t-norm.*
(ii) $f = g^{-1} \circ N$.

In this case, the expression of the fuzzy Sheffer stroke is given by

$$H_{g,N}(x, y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)$$

for all $x, y \in [0, 1]$.

Proof. On the one hand, let $C_{H_{f,g},N} = N^{-1} \circ H_{f,g}$ be a t-norm and let us prove that $f = g^{-1} \circ N$. Since 1 is the neutral element of a t-norm, $N^{-1}(H_{f,g}(x, 1)) = x$ for all $x \in [0, 1]$. This implies that $H_{f,g}(x, 1) = N(x)$. Furthermore, as $f(1) = 0$ we deduce that

$$H_{f,g}(x, 1) = g\left(f(x) + f(1)\right) = g\left(f(x)\right).$$

Thus, $g\left(f(x)\right) = N(x)$ for all $x \in [0, 1]$ and since g is a continuous and strictly increasing function, we obtain that $f(x) = g^{-1}(N(x))$.

On the other hand, let us consider now $f = g^{-1} \circ N$ and let us prove that $C_{H_{f,g},N} = N^{-1} \circ H_{f,g}$ is a t-norm. Firstly, let us check that f is well defined. It is clear that $f\colon [0, 1] \to [0, +\infty]$ is decreasing and continuous. Furthermore,

$$f(0) = g^{-1}(N(0)) = g^{-1}(1) = +\infty, \quad f(1) = g^{-1}(N(1)) = g^{-1}(0) = 0.$$

Thus, (f, g) is indeed a pair of additive generators of a fuzzy Sheffer stroke. Now, we have that

$$\begin{aligned} C_{H_{f,g},N}(x, y) &= N^{-1}(H_{f,g}(x, y)) = N^{-1}\left(g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)\right) \\ &= (g^{-1} \circ N)^{-1}\left(g^{-1}(N(x)) + g^{-1}(N(y))\right), \end{aligned}$$

which is clearly a strict t-norm with $f = g^{-1} \circ N$ as additive generator. □

At this point, we will study whether the additional properties (**H4**)-(**H7**) introduced in Section 3.2.2 are fulfilled by the family of fuzzy Sheffer strokes given by $H_{g,N}$. It is evident that $H_{g,N}$ always satisfy (**H6**) for all strict negations N and continuous and strictly increasing functions $g\colon [0,+\infty] \to [0,1]$. However, they do not necessarily satisfy the other properties as the following results show.

Proposition 4.12. *Let $g\colon [0,+\infty] \to [0,1]$ be a continuous and strictly increasing function and N be a strict fuzzy negation. Then the following statements are equivalent:*

(i) *$H_{g,N}$ satisfies (**H4**).*

(ii) *There exists an automorphism $\varphi\colon [0,1] \to [0,1]$ such that $N(x) = g\left(\frac{g^{-1}((N_C)_\varphi(x))}{2}\right)$.*

Proof. On the one hand, if $H_{g,N}$ satisfies (**H4**), then we already know that $N^d_{H_{g,N}}$ is a strong fuzzy negation. Therefore, there exists an automorphism $\varphi\colon [0,1] \to [0,1]$ such that $N^d_{H_{g,N}}(x) = (N_C)_\varphi(x)$. Since $N^d_{H_{g,N}}(x) = g\left(2g^{-1}(N(x))\right)$ and g is a continuous and strictly increasing function we obtain that

$$N(x) = g\left(\frac{g^{-1}\left((N_C)_\varphi(x)\right)}{2}\right).$$

On the other hand, if (ii) holds, then:

$$H_{g,N}(x,y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right) = g\left(\frac{g^{-1}((N_C)_\varphi(x)) + g^{-1}((N_C)_\varphi(y))}{2}\right),$$

and consequently, $H_{g,N}(x,x) = (N_C)_\varphi(x)$. From this it follows that

$$H_{g,N}(H_{g,N}(x,x), H_{g,N}(x,x)) = H_{g,N}((N_C)_\varphi(x), (N_C)_\varphi(x)) = (N_C)_\varphi\left((N_C)_\varphi(x)\right) = x,$$

and $H_{g,N}$ satisfies (**H4**). □

Example 4.13. Consider $g(x) = \frac{x}{x+1}$, $\varphi(x) = x$ and $N(x) = \frac{1-x}{1+x}$ for all $x \in [0,1]$. Since in this case the conditions presented in Proposition 4.12-(ii) hold, the fuzzy Sheffer stroke $H_{g,N}(x,y) = \frac{x+y-2xy}{x+y}$ for all $x, y \in [0,1]$ satisfies Property (**H4**).

On the contrary, $H_{g,N}$ never satisfies (**H5**).

Proposition 4.14. *Let $g\colon [0,+\infty] \to [0,1]$ be a continuous and strictly increasing function and N be a strict fuzzy negation. Then $H_{g,N}$ never satisfies (**H5**).*

Proof. On the contrary, if $H_{g,N}$ satisfies (**H5**), then for all $x \in [0,1]$, we have that

$$N(x) = g\left(g^{-1}(N(1)) + g^{-1}(N(x))\right) = H_{g,N}(1,x) = H_{g,N}(x,x) = g(2g^{-1}(N(x))).$$

This implies that $2g^{-1}(N(x)) = g^{-1}(N(x))$ and $g^{-1}(N(x)) = 0$, which would lead to $N(x) = g(0) = 0$ for all $x \in [0,1]$, arising a contradiction. □

It remains still an open problem whether there exist some g and N such that $H_{g,N}$ satisfies (**H7**).

5. An example application of the fuzzy Sheffer stroke

In this section, a potential application of the fuzzy Sheffer stroke is presented in order to prove the importance of disposing of such operation in the fuzzy logic framework. Let us consider a refrigerator with two sensors connected to an alarm system which warns the user if the internal temperature of the refrigerator is abnormally high. One of the sensors controls the opening angle of the door of the refrigerator and the other one quantifies the internal temperature of the refrigerator.

Let $A\colon X \to [0,1]$ and $B\colon Y \to [0,1]$ be two fuzzy sets where X is the set of possible opening angles of the door and Y is the set of the possible internal temperatures. Thus, A and B are fuzzy sets which represent the closedness and coldness of the refrigerator, respectively. For instance, on the one hand, $A(0) = 1$, i.e., if the door is closed, closedness is maximum and $A(x_{\max}) = 0$ if $x_{\max}$ is the angle described by the door whenever it is completely open. On the other hand, $B(y_{\min}) = 1$ if $y_{\min}$ is the desired internal temperature that the user has set and $B(y_{\max}) = 0$ if $y_{\max}$ is the highest potential temperature.

Consider that the alarm system has a wide range of intensities. Let $F\colon A \times B \to [0,1]$ be the function that models the intensity of this alarm. If $F(a,b) = 0$ the alarm is off, but as $F(a,b)$ increases, so does the intensity of the alarm. The intensity can be represented in real life either with the volume of a sound or the period of a blinking light. Let us analyse the expected behaviour of this function. The following extremal cases are straightforward:

- If the door is completely open, $A(x) = 0$, then the intensity must be at its peak. Thus $F(0,b) = 1$ for all $b \in [0,1]$.
- If the internal temperature of the fridge is very hot, $B(y) = 0$, then the intensity must be also at its peak. Thus, $F(a,0) = 1$ for all $a \in [0,1]$.
- If the door is closed, $A(x) = 1$, and the internal temperature of the fridge is the desired one, $B(y) = 1$, then the alarm should be off. Thus, $F(1,1) = 0$.

Furthermore, on the one hand, when the opening angle of the door decreases, so should decrease the intensity of the alarm. That is, if $a_1 \le a_2$, then $F(a_1,b) \ge F(a_2,b)$ for all $b \in [0,1]$. On the other hand, when the internal temperature decreases, the intensity of the alarm should also decrease. That is, if $b_1 \le b_2$, then $F(a,b_1) \ge F(a,b_2)$ for all $a \in [0,1]$.

In this way, it is evident that we can use a fuzzy Sheffer stroke to model the alarm behaviour of this refrigerator. This operator can represent the fuzzy modelization that it is needed in this case saving energy and letting the user to approximately know the urgency degree of the problem.

6. Conclusions and future work

In this paper, the Sheffer stroke operation in the context of fuzzy logic has been introduced. This operation is of paramount importance in classical logic since it can be used to generate all the other logical connectives by itself without the need of any other one. The results presented in this paper prove that this is also the case in the fuzzy logic framework in which the fuzzy Sheffer stroke can be used to define fuzzy negations, conjunctions, disjunctions and implication functions by itself. Moreover, it can also generate t-norms and t-conorms under some conditions. In Section 4, two construction methods for fuzzy Sheffer strokes have been presented. Furthermore, the close connection of this operation with a pair of a fuzzy conjunction and a fuzzy negation has been analysed. Finally, a real-life application where a fuzzy Sheffer stroke can play an important role has been proposed.

Several open problems have appeared throughout the paper. Specifically,

- Under which additional conditions of a fuzzy Sheffer stroke a t-norm or a t-conorm can be obtained through Eqs. (1) and (2)?
- Characterize the subfamily of $H_{g,N}$ fuzzy Sheffer strokes satisfying (**H7**).

In addition to study these open problems, as a future work, we want to study in depth the families of fuzzy implication functions I_H^{QQ} and I_H^{PQ}, analyse their additional properties and to characterize under which conditions I_H^{PQ} is a fuzzy implication function in the sense of Definition 2.8.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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