“Whiteboxing” the content of a formal mathematical text in a dynamic geometry environment

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Abstract

In this article, we provide an empirical example of how digital technology, in this case GeoGebra, may assist students in uncovering—or whiteboxing—the content of a mathematical proof, in this case that of Proposition 41 from Euclid's Elements. In the discussion of the example, we look into the impact of GeoGebra's 'dragging' functionality on students' interactions and the possession and development of students' proof schemes. The study and accompanying analyses illustrate that despite the positive whiteboxing effects in relation to the mathematical content of the proposition, whiteboxing through dragging calls for caution in relation to students' work with proof and proving—in particular in relation to students seeing the necessity for formal proof. In particular, caution must be paid, e.g. by teachers, so that students do not jump to conclusions and in the process develop inexpedient mathematical proof schemes upon which they may stumble in their future mathematical work.

Introduction and inspiration from the History and Pedagogy of Mathematics (HPM) research

An important lesson to be learned when using digital technology (DT) is that it should always serve a purpose. The study in this article addresses DT as a so-called *whiteboxing* tool in students' work with a formal mathematical proof. Whiteboxing—understood as the opposite of blackboxing (see later)—is when DT serves the purpose of revealing mathematical aspects otherwise hidden or inaccessible to the student. The inspiration for the study came about from the topic of using history of mathematics in mathematics education (e.g. Jankvist, 2009), as we shall briefly account for below.

Studies related to the role and use of history of mathematics in the teaching and learning of mathematics usually reside in the so-called *History and Pedagogy of Mathematics* (HPM—an

ICMI affiliated international study group) literature of mathematics education research.[1] From the HPM literature it is known that although students' work with historical mathematics and in particular primary historical sources are considered among the most rewarding in terms of developing mathematical understanding, it is also one of the most demanding (e.g. Jahnke et al. 2000). This is due to the fact that historical sources may provide unfamiliarity to pieces of mathematics with which the students believed themselves to be otherwise in confidence—in a sense it puts them on foreign ground, often referred to as *dépaysement* (Barbin, 1997; Pengelley, 2011). Studies show that the educationally rewarding aspects of such *dépaysement*, are that students become more inclined to keep an open mind in relation to mathematical concepts (Furinghetti, Jahnke & van Maanen, 2006), and that students may even experience so-called meta-level learning (Kjeldsen & Blomhøj, 2012, with reference to Sfard, 2008). Only a few HPM-related studies also consider DT (e.g. Aguilar & Zavaleta, 2015; Baki & Guven, 2009; Chorlay, 2015; Isoda, 2004; Papadopoulos, 2014; Thomsen, 2021; Zengin, 2018). More recently, it has been suggested that DT can come to act as a 'tin opener' for the mathematical content of otherwise inaccessible sources in a whiteboxing manner (Balsløv, 2018; Jankvist & Geraniou, 2019; Jankvist, Misfeldt, & Aguilar, 2019a; Olsen & Thomsen, 2017; Thomsen & Jankvist, 2020).

Some mathematical texts are almost timeless in their presentation of mathematical knowledge. Euclid's *Elements* is one such, of course (e.g. Fitzpatrick, 2008). While being historical *per se*, its presentation of many of the propositions are identical to modern day presentations. Still, the classical, formal approach of Euclidean geometry is foreign to most lower school students. Hence, whether or not Euclid's *Elements* can be considered a primary historical source in the usual HPM sense of such (Janke et al., 2000), it on the one hand provides an element of *dépaysement*, while on the other hand also being somewhat inaccessible to many lower secondary school students. Thus, we were inspired by the discussions in the HPM literature of using DT as a 'tin opener' for accessing the mathematical content of a mathematical text. Furthermore, it appeared to us that for students unfamiliar with the formal mathematics present in traditional mathematics texts, the use of DT to access the content of these may potentially provide an element of *familiarity* and thus ease the access to the mathematical content. In particular, we were interested in this in relation to presentations of propositions and their proofs in Euclid's *Elements*, and the use of DT as a 'tin opener' in a whiteboxing manner.

From research on proofs and proving in mathematics education in relation to the use of Dynamic Geometry Environments (DGE), it is clear that in particular the *dragging potential* may serve as a whiteboxing element by providing “students with strong perceptual evidence that a certain property is true” (Mariotti, 2006, p.193). On the other hand, it is also clear that such a use may lead students to jump to conclusions, i.e. that exploration via dragging is sufficient to guarantee truth (Mason, 1991). Formulated differently, the DT can come to act as an authority for the students in matters of establishing truth, thus promoting a kind of *techno-authoritarian external conviction proof scheme* with the students (Misfeldt & Jankvist, 2018) (see later).

In this article, we address the question: *How and to what extent may the dragging functionality of a DGE (GeoGebra) assist secondary school students in a whiteboxing manner as part of their own work with a somewhat challenging mathematical text, while avoiding that the students jump to conclusions?*

In a classroom setting, Thomsen and Jankvist (2020) have emphasized the importance of the teacher being aware not to support students’ activation of techno-authoritarian proof schemes, when working with proof and proving in a setting of DT. In the present study, we operate in a different “peer-to-peer” setting of two students working together. The proposition and accompanying proof that we have chosen to work with is that of Proposition 41 from Euclid’s *Elements*. This proposition is sufficiently challenging to cause some kind of *dépaysement* for a secondary student, while it also holds a potential for *dragging* serving as a *whiteboxing* element in students’ exploration of the mathematical content of the proposition.

We address the notions of proof scheme and whiteboxing in the following section along with a few other theoretical constructs and underpinnings that we apply. Next, we present our research method along with the proposition and accompanying task presented to two British secondary school students; one familiar with the DT in use and one not. We then display the empirical case of the students’ work. Finally, based on the empirical case, we discuss the students’ benefit of having GeoGebra at their disposal when working with the proposition.

Theoretical constructs and underpinnings

In relation to symbolic manipulations with DT, more precisely Computer Algebra Systems (CAS), Buchberger (1990) argues that in a stage, where a given mathematical area, say X, is new “to the students, the use of a symbolic software system realizing the algorithms of area X as black boxes would be a disaster" (p. 13). Negative *blackbox effects* are well described in the literature (e.g. Buchberger, 2002; Lagrange, 2005), and so are the results of students who are able to perform CAS-based mathematical activities, but unable to account for the underlying processes (e.g. Jankvist & Misfeldt, 2015; Jankvist, Misfeldt & Aguilar, 2019b). Hence, blackboxing may leave students dependent on DT and with little experience of performing low-level mathematical processes (Nabb, 2010), which are deemed cognitively important in terms of mental mathematical concept formation (e.g. Dubinsky, Dautermann, Leron, & Zazkis ,1994). According to Buchberger (1990), before students should be allowed to use DT in a blackboxing manner, they need to have studied the "area thoroughly, i.e. they should study problems, basic concepts, theorems, proofs, algorithms based on the theorems, examples, hand calculations" (p. 13). This work with the mathematical content, is what Buchberger (1990; 2002) calls the "whiteboxing stage"—and the order of the whiteboxing stage prior to any blackboxing is what he refers to as the "whitebox/blackbox principle" for using CAS in mathematics education. Of course, Buchberger formulated this principle three decades ago, and in the context of mathematics symbolic calculators and software, and much has happened since not least in relation to the development of DGS. Still, he seems to more or less disregard the potential of DTs themselves serving a whiteboxing purpose in students' work with a to them unknown mathematical area. Cedillo and Kieran (2003), on the other hand, acknowledge this role of DTs by introducing "grayboxing". The notion of *grayboxing* combines blackboxing and whiteboxing by acknowledging that learning can take place in an environment combining the two. In the context of algebra, they argue that mathematical learning can indeed take place with a DT serving as "a mediator of algebra learning—a tool that helps create simultaneous meaning for the objects and the transformations of algebra" (p. 221).

As mentioned in the introduction, the use of DT in relation to mathematical proofs and proving is not completely unproblematic, neither in relation to CAS (e.g. Jankvist & Misfeldt, 2019) nor to DGE (e.g. Mariotti, 2006). In particular, Mason (1991) has pointed out that the contribution of DGE to finding a proof is unclear, which must be seen in the light of mathematical proof being indeed difficult for students. The Education Committee of the EMS (2011) states in their series of "Solid Findings" articles that "the concept of formal proof is completely outside mainstream thinking" (p. 51), while Dreyfus (1999) sums up research results to claim that "most high school and college students don't know what a proof is nor what it is supposed to achieve" (p. 94). Drawing on the work of Harel and Sowder (1998), Misfeldt and Jankvist (2018) coined the term *techno-authoritarian external conviction proof schemes*, "techno-authoritarian proof schemes being technical because the proof only makes sense in reference to a specific technology [...], and authoritarian because the scheme builds on blackboxing [...] in the sense that the students need to trust the technology in order to believe in the proof, i.e. as if the technology is an authority" (p. 379). A person's *proof scheme* consists "of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 2007, p. 809). Overall, a proof scheme belongs to one of three classes, each composed of different subclasses. The first class consists of so-called external conviction proof schemes, which may be manifested as: an authoritarian proof scheme, e.g., that something is held to be true because the textbook, the teacher or some other authority says so; a ritual proof scheme, e.g., that a geometry proof in the USA must have a two-column format; or a non-referential symbolic proof scheme, e.g., that a proof must display symbols and symbolic manipulations, which do not carry any meaning for the learners. The second class consists of the empirical proof schemes. These fall in two subclasses: either inductive proof schemes, e.g., you are being convinced by one or several specific empirical examples, or by what is perceived as a "crucial" generic example; or perceptual proof schemes, e.g., when a conjecture is validated on the basis of rudimentary mental "images that consist of perceptions and a coordination of perceptions but lack the ability to transform or to anticipate the results of a transformation" (Harel & Sowder, 1998, p. 255). The third class is made up of deductive proof schemes, such as direct proof, including axiomatic proofs, proof by contradiction, proofs by mathematical induction, combinatorial proofs, etc., all governed by logical deduction. The distinction between pragmatic and epistemic mediations is useful when it comes to DT in mathematics education. *Pragmatic mediations* concern a person's actions on objects, while *epistemic mediations* concern how the person gains knowledge of the objects' properties through the use of a given tool (Rabardel & Bourmaud, 2003). Both pragmatic and epistemic mediations serve meaningful purposes when using DT, yet any use that is only, or mainly, pragmatic is of little (or negative) educational value (Artigue, 2010). Epistemic mediations are connected to proofs that provide explanation and thus support deductive proof schemes, while pragmatic mediations "may be connected to one or more of the different proof schemes, including the empirical proof scheme, by providing necessary but laborious calculations and manipulations required for a certain argument" (Thomsen & Jankvist, 2020, p. 484).

As Lopez-Real and Leung (2006) argue, "dragging in DGE is a powerful dynamic tool to acquire mathematical knowledge" (p. 666). It supports students' justifications, but also provides them with strong perceptual evidence for the truthfulness of statements (Mariotti, 2006). When investigating students' interactions in Cabri with problem solving activities, Healy and Hoyles (2002) referred to the 'dragging test', i.e. dragging certain points of a figure to explore if the figure maintains its initial properties and potentially explore whether a statement regarding this figure is true. This dragging test supported students' transition from argumentation to logical deduction. "Interacting with Cabri can help learners to explore, conjecture, construct and explain geometrical relationships, and can even provide them with a basis from which to build deductive proofs" (Healy & Hoyles, 2002, p. 251). Lopez-Real and Leung (2006) argue that "dragging in DGE can open up some kind of semantic space (meaning potential) for mathematical concept formation in which dragging modalities (strategies) are temporal-dynamic semiotic mediation instruments that can create mathematical meanings, that is, a window to enter into a new semiotic environment of how geometry can be re-presented (re-shaped)" (p. 666). There are different modes of dragging. For example, Arzerello et al. (2002) and Mariotti (2006) refer to: *wandering*, which is moving basic points randomly on the screen without a plan; *bound dragging*, which is moving a semi-draggable point already linked to an object; *guided dragging*, which is dragging basic points of a figure to obtain a particular shape; *"dummy locus" dragging*, which covers moving a basic point so that a figure keeps a discovered property; *line dragging*, covering drawing new points on the ones that keep the regularity of a figure; *linked dragging*, which is linking a point to an object and moving it onto that object; and finally, the *dragging test*, which is to move draggable or semi-draggable points to see whether the figure keeps its initial properties (if so, it passes the test). Lopez-Real and Leung (2006) argue that "the process of making meaning, hence learning, is a semiotic process and dragging can be regarded as a prototypical form of it in DGE" (p. 666). Certainly, dragging can support students' empirical dynamic explorations, leading them to make certain observations, or in other words conjectures, before deducing a formal proof for those conjectures. Guven (2008) argues that DGEs "can provide an opportunity to link between empirical and deductive reasoning", and they "can be utilized to gain insight into a deductive argument" (p. 261). More precisely, in the context of using Cabri, he referred to a linear process of four steps, which can be adapted to DGE:

1. *Experimental results*, in which a problem is explored empirically;
2. *Towards a proof*, in which a DGE is used to make observations that eventually will lead to a deductive proof;
3. *Proof*, where results are proved deductively drawing on experimental DGE results;
4. *Some generalizations*, in which the proof is extended using additional DGE observations.

Even though Guven (2008) referred to these four steps in a students' reasoning process, Mariotti (2002) focused on two main processes/actions taking place when students work on construction tasks within the Cabri environment. She claimed that "the justifications provided by the students assume the form of a statement and a proof: the hypothesis drawn from the construction are correctly related to the thesis, while the justification explicitly refers to the system of principles, shared and stated within the class community" (ibid, p. 279). Dragging, as a DGE tool, thus plays an important role in different types of mediation. It influences a person's actions on objects (pragmatic mediation). It allows a person to explore an object's properties and therefore allows the person to gain knowledge of these properties (epistemic mediation). The combination of such mediations, while interacting with a DGE, are likely to affect a student's development of his/her proof schemes (Thomsen & Jankvist, 2020).

Research method

Laborde (2000) addresses the didactic complexity of students working with mathematical proof in a DGE setting, saying, "DGE itself without an adequately organized milieu would not prompt the need of a proof. It is a common feature [...] to have constructed a rich milieu with which the student is interacting during the solving process and the elaboration of a proof" (p. 154). The mentioning of milieu of course refers to Brousseau's (1997) notion of such. We wanted to orchestrate such a "rich milieu", by carefully selecting two students, that worked well together, could challenge each other's thinking and argumentation, while ensuring that at least one had experience with GeoGebra and proof, so as they could act as a 'guide', or in other words the 'more knowledgeable other' based on Vygotsky's (1978) Zone of Proximal Development theory. We expected such a pair of students would allow us to observe their deductive reasoning, their mediations and ultimately their proving strategies.

Two British students, Oscar (15 years old) and Alice (13 years old), were chosen to work as a pair. Oscar was familiar with GeoGebra, whereas Alice had never used GeoGebra before this study. The notion of ‘proof’ is not always introduced to school students in England, as it has not been part of the National mathematics curriculum in recent years. Therefore, it lies upon individual schools and mathematics teachers, who may (or may not) decide to mention or introduce mathematical proof. Oscar was familiar with the idea of ‘formal mathematical proof’ and had experience with geometrical reasoning as part of his extracurricular activities at his school, e.g. taking part in mathematics extra-curricular activities in afterschool clubs, which trained him in the use of mathematical reasoning. However, Alice had no experience with any activities involving mathematical proof and was not yet inclined to the use of more formal mathematical reasoning. Both students were deemed quite capable at mathematics by their teachers and concerning prior knowledge regarding the task in question. Furthermore, both students knew the formulas for the area of a parallelogram and a triangle from their primary school education and knew that the diagonals of a parallelogram ‘split’ the area of the parallelogram into two triangles of equal area.

Even though our overall research focus remained on investigating the power of GeoGebra in *whiteboxing* Euclid’s Proposition 41 for students, we decided that such a methodological approach would encourage Alice to ‘open up’ to Oscar’s prompts and suggestions and also provide us with a better window onto both students’ mathematical thinking and reasoning, as argued earlier. To ensure the students some familiarity with GeoGebra, we presented them with the GeoGebra tool and asked them to construct a house. During this familiarization task, our aim was to encourage them to use the tool that creates parallel lines and a polygon and ensure a prior knowledge recap: What are parallel lines? What do you know about triangles? What do you know about parallelograms? The students were also given a short introduction to who Euclid was and what he is known for, before being presented with the main task of this research study.

We used a task-based interview approach, video recording the study. Students worked together on a shared laptop and used pen and paper, whenever necessary.

Empirical case and analysis

In this section, we present Oscar and Alice’s interaction with Proposition 41 and GeoGebra, by following our adaptation of Guven’s (2008) four steps presented earlier. We include subsections: (Step 1) *Exploration in GeoGebra*, in which we discuss the pair’s exploration of the problem in

GeoGebra; (Step 2) *Towards a proof*, in which we analyze the pair's discussions about their constructions in GeoGebra and their observations which could have led them to a deductive proof; (Step 3) *Proof*, in which we show Oscar's and Alice's proving steps as inspired by their interactions in GeoGebra, but also as they are based on the experimental results acquired from GeoGebra; (Step 4) *Generalizations*, in which we argue about the pair's claims regarding the truthfulness of Proposition 41 for any parallelogram and any triangle that meets the criteria as stated in the proposition. Yet, before presenting this sequence of events, we include a section—step 0—on pre-GeoGebra preparatory work on Proposition 41.

Step 0: Initial pen-and-paper preparation

We presented the two students with Proposition 41, without any diagrams that could provide further elaboration of the proposition. The rationale was for the students to interpret the worded proposition (cf. figure 1), describe it in their own words and/or create their own diagram on paper and therefore consider what the proposition states to better prepare themselves for the subsequent steps in this activity.

μα΄.	**Proposition 41**
Ἐὰν παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστί τὸ παραλληλόγραμμον τοῦ τριγώνου.	**If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.**

Figure 1. The first sentence of Proposition 41 in its original language Greek (left) and its translation in English (right) (as presented by Fitzpatrick, 2008). The used English translation mentions "(the area)", which the original does not.

Alice immediately thought of an isosceles triangle and drew one (see the triangle on the top left in figure 2). Oscar then stopped her, saying that the proposition refers to a parallelogram and a triangle. He advised her to draw a parallelogram first, which she did. She also drew a triangle with one side being the same as one of the sides of the parallelogram (see the diagram on the top right in figure 2). Alice highlighted this triangle and argued:

(1) *Alice*: These two [pointing at the two (unhighlighted) smaller triangles] add up to this triangle [pointing at the highlighted triangle].

Oscar challenged her thinking and asked her a few times to repeat what she was claiming, which led her to articulate the statement better:

(2) *Alice*: The sum of the areas of the two [unhighlighted] smaller triangles is equal to the area of this triangle [the yellow highlighted triangle].

Oscar then added the symbolic representations for the areas seemingly to elaborate further Proposition 41 statement (see middle in figure 2). Then, the researcher asked the students to describe in their own words Proposition 41. Oscar started by saying:

(3) *Oscar*: So, if we can draw the shape again, we can extend this line [top parallel line of the parallelogram], so basically if these are the two parallels and we say this one is our base, if you draw a triangle with a top point anywhere along this top parallel line, the area of the triangle will always be half the area of the original parallelogram.

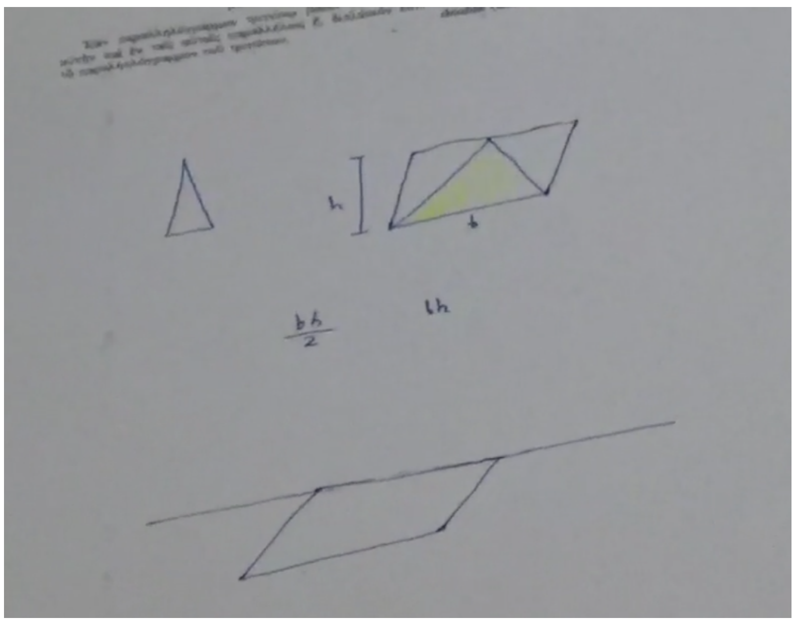
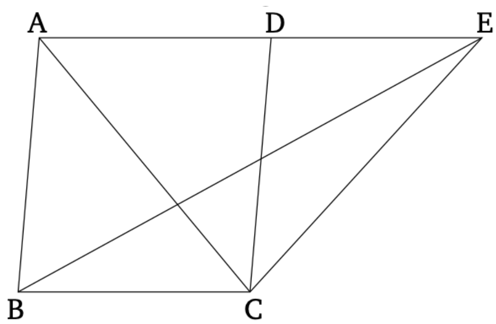


Figure 2. Oscar's and Alice's pen-and-paper diagram for Proposition 41.

Then, the researcher encouraged Alice to describe the proposition in her own words:

(4) *Alice*: So, basically the same as Oscar. If the base of the triangle equals the base of the parallelogram [pointing at the figure she drew on top], and the top vertex of the triangle is along the top parallel line of the parallelogram [pointing at the figure Oscar drew at the bottom], that must mean that the area of the triangle is half the area of the parallelogram.

The students got the correct diagram and seemed to interpret Proposition 41 correctly. Next, they were shown the full original description of Proposition 41 of the Euclid's *Elements Book 1* (the Fitzpatrick translation into English, i.e. Fitzpatrick, 2008), which is presented in figure 3 below.



For let parallelogram $ABCD$ have the same base BC as triangle EBC, and let it be between the same parallels, BC and AE. I say that parallelogram $ABCD$ is double (the area) of triangle BEC.

For let AC have been joined. So triangle ABC is equal to triangle EBC. For it is on the same base, BC, as (EBC), and between the same parallels, BC and AE [Prop. 1.37]. But, parallelogram $ABCD$ is double (the area) of triangle ABC. For the diagonal AC cuts the former in half [Prop. 1.34]. So parallelogram $ABCD$ is also double (the area) of triangle EBC.

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

Figure 3. The proof of Euclid's Proposition 41 as presented in the *Elements* (Fitzpatrick, 2008, p. 41).

The students commented that the full description was a more complicated version of what they had claimed earlier:

(5) *Oscar*: I think it's basically a more complicated version of what I said. And it also provides a proof, an example of which... It just gives an example that shows that it actually does work. For example, the one that cuts it in half. The *ABC* example. So, it's just got a bit added to it.

Both students seemed to recognize that their pen-and-paper drawings and the full description of Proposition 41 provide a specific 'static' case of a parallelogram and a triangle. Even though these cases are not specific in the sense of numerical cases, it is not clear if Alice recognized the generalizability and 'truthfulness' of the Proposition. Oscar, on the other hand, seemed to correctly identify why this diagram is included and the special case of the triangle *ABC*, which clearly (in his mind) had an area, which is half the area of the parallelogram *ABCD*. It seems that Oscar does possess a deductive proof scheme.

Step 1: Exploration in GeoGebra

Oscar and Alice were then asked to 'create' the described situation using GeoGebra. They got prompts such as: What is the first thing we construct? How can we make parallel lines? How can we make a parallelogram? How can we make a triangle with the same base as one of the parallel lines? Do you think that this proposition is true for any parallelogram? How do you know? The objective was for students to create something similar to figure 4 below,[2] and to be able to explain Proposition 41 using their own words and arguments in order to showcase their understanding of the content of the proposition.

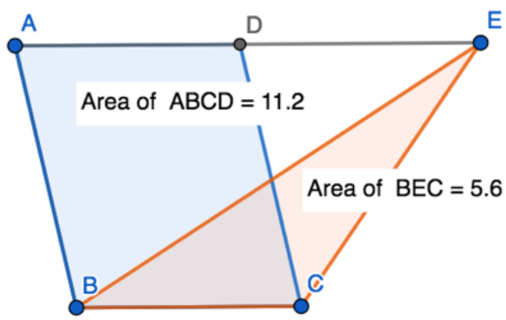


Figure 4. Proposition 41 created using GeoGebra.

Alice and Oscar started by figuring out how to make a parallelogram and then decided to make a square for simplicity, which, as they both claimed, is a special case of a parallelogram. They extended the top side of the square by creating a parallel line to the base, before constructing a triangle with the same base as the square but with its third vertex being on the top parallel line (point *E* in Figure 5a). Then, Alice said:

(6) *Alice*: We need to draw the line in half [pointing at the base of their constructed square].

(7) *Oscar*: No, you don't need to do that. [...] See if you can drag *E* along that line.

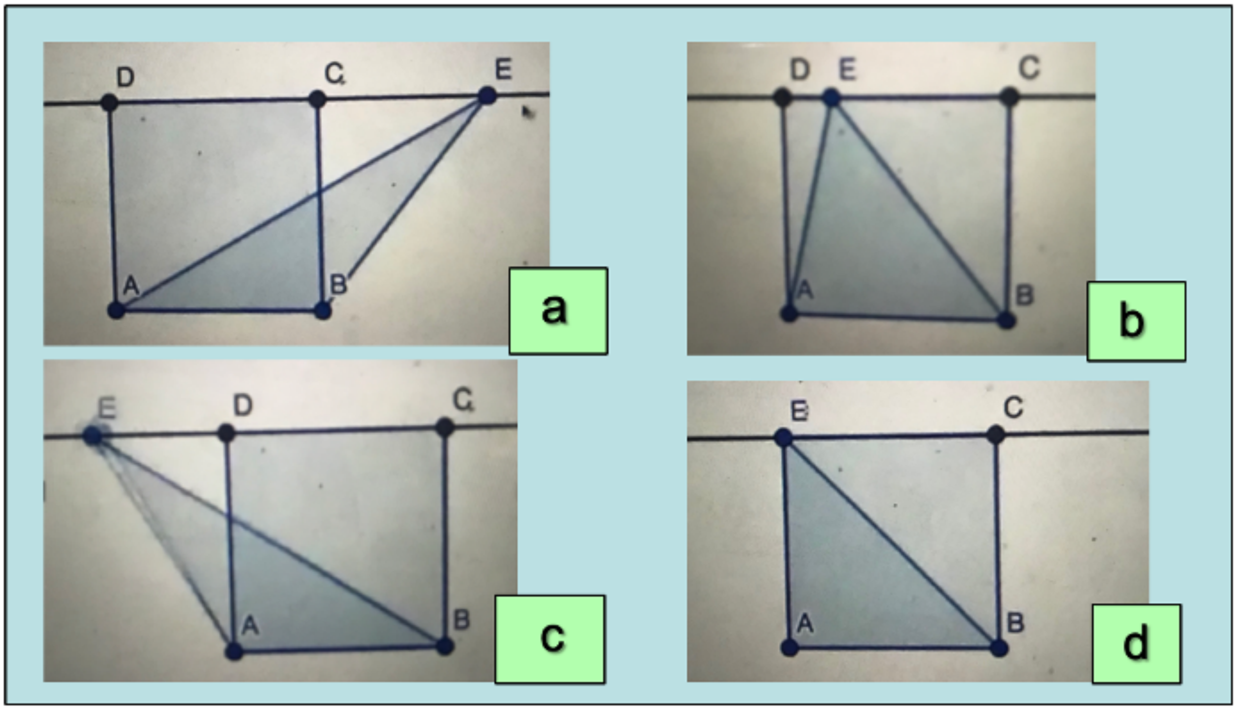


Figure 5: Students' constructions of Proposition 41 in GeoGebra.

Upon dragging point *E*, right to left and back a number of times (see figure 5a, 5b and 5c), Alice placed point *E* on top of point *D* (see figure 5d) and sighed in recognition (as it seemed to us) of the area of the triangle being half of the area of the square. The following conversation then took place:

(8) *Oscar:* So, the example shows [...] that the triangle *ABE* is exactly half and you do exactly that [pointing at their diagram as shown in Figure 5d].

(9) *Alice*: Oh... I... that's actually clear now.

(10) *Researcher:* So you created this for a square, which is a special case of a parallelogram, as you claimed earlier, do you think that it's true for any parallelogram?

(11) *Oscar:* I think so, because... the area of any parallelogram is base times height and the area of any triangle is base times height over 2... and that shows you that the triangle is half the parallelogram because of the formula.

(12) *Researcher:* So, if you drag that point *E* along this line, you think that it would…

(13) *Oscar:* Yes, you see here [pointing at the base of the square] the base will always be the same and the height of the triangle and the parallelogram [gesturing vertically while pointing at their GeoGebra construction to show that the distance between the

> base and the top parallel line] is always going to be the same. so, *b* and *h* are going to be the same and you just have to divide by 2... and yeah in any case the triangle will be half the parallelogram.

As seen above, once the students constructed a similar diagram in GeoGebra, following Oscar's prompt, Alice dragged the top vertex of the triangle and moved it along the parallel line for quite a while (figure 5), showcasing how GeoGebra enabled her to experiment and trial numerous cases, something which could not have been possible on paper. Alice's dragging can be characterized as *guided dragging*, since it was done with a particular goal in mind, following Oscar's prompt. Firstly, it offered her strong perceptual evidence (Mariotti, 2006) that one can create an infinite amount of triangles with the same base, while the third vertex of each of these triangles 'lies' on a parallel to the base line. Secondly, it was possible to 'see' that one of these triangles is the triangle *ABD*, and therefore conjecture that any of these triangles has indeed the same area as triangle *ABD*. This guided dragging promoted an epistemic mediation, as Alice gained knowledge about the relationship between these infinite triangles. When constructing the diagram for Proposition 41 in GeoGebra, Alice's initial statement was "we need to draw the line in half" referring to the base line *AB*, seemingly trying to 'cut' the square in half so as to find half the area of the square, which is relevant to what Proposition 41 states. Oscar, as the more knowledgeable other, intervened by advising Alice to explore what happened when dragging point *E*. This mediation allowed Alice to act upon the constructed objects and gain knowledge about the properties of the constructed objects, before reflecting on her initial statement and recognizing why the area of any triangle *ABE* is half the area of the square *ABCD*.

Following the researcher's prompt, Oscar used GeoGebra to display that the base of both the parallelogram (in their case the square) and the triangle are indeed the same by construction. Similarly, he claimed that since the top line going through points *D*, *C* and *E* (see figure 5) is parallel to the base, then the height will remain constant. It is interesting that Oscar used GeoGebra to display and justify the equal lengths, base and height, for the two shapes, but at the same time referred to the formulas for the area of the two shapes respectively. Surprisingly, Oscar did not use the area functionality of GeoGebra to illustrate that the area of the triangle *ABE* remains the same for any triangle created by dragging point *E*, while also using the area of the square *ABCD* to illustrate that this is always double the area of the triangle *ABE* (as shown in figure 4). It may be argued that it is his past experiences with formal mathematical proofs, as mentioned earlier, that may have resulted in this outcome, i.e. being confident about his reasoning being correct and therefore using GeoGebra only to further validate his thinking. Alternatively, Oscar may have not been familiar with the area feature or he might have forgotten about it.

Step 2: Towards a proof

Even though Alice's reasoning is not as clear as Oscar's and her articulated arguments are not mathematically valid compared to Oscar's, she seemed rather convinced from both their GeoGebra interactions and Oscar's argumentation.

(14) *Researcher:* What do you think, Alice? Do you agree?

(15) *Alice:* Um… yes... I think because it says for a parallelogram, so... any way it would work… and because with the triangle, if the parallelogram was tilted more this way [moving point *E* and recreating figure 5a], then the triangle can also do the same thing [moving point *E* further assuming that the parallelogram is also tilted more to the right].

(16) *Researcher:* Do you want to create the parallelogram and see if it's true?

The students then created a parallelogram and a triangle with the same base. Following a similar process for when they worked with the special case of a parallelogram, that of a square, Alice placed the top vertex of the triangle on each of the two top vertices of the parallelogram in turn and argued that the area of the triangle is half the area of the parallelogram.

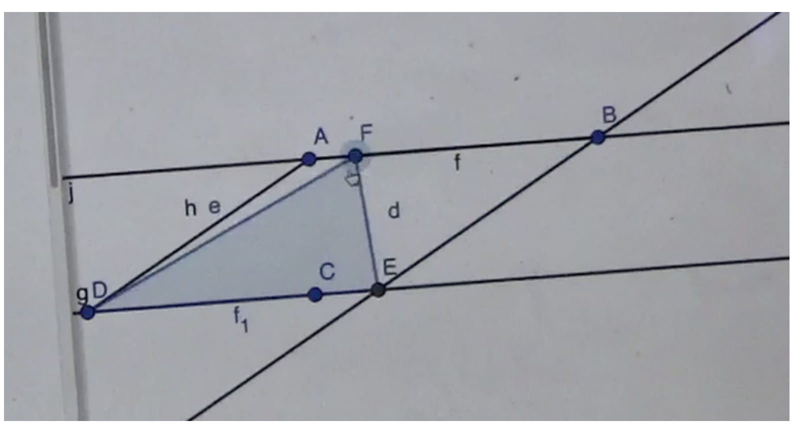


Figure 6. Students' second construction of Proposition 41, which includes a parallelogram (instead of a square).

In this process, Alice was guided by Oscar on how to construct a parallelogram *DABE* in GeoGebra by constructing two sets of parallel lines and a triangle *DFE* (see figure 6). In fact, Alice accidentally created two triangles with the same base as the parallelogram, i.e. *DFE* and *DGE* (see figure 7).

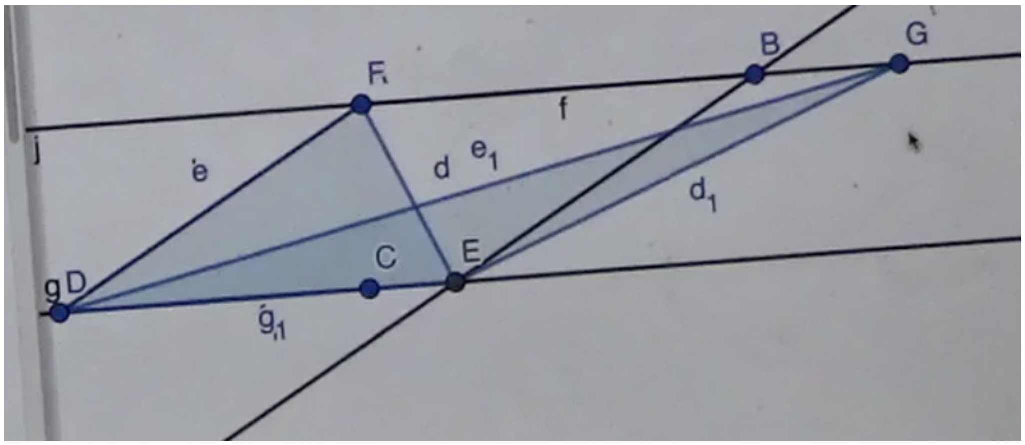


Figure 7. Students' second construction of Proposition 41, which includes a parallelogram (instead of a square) and two triangles with the same base as the parallelogram, i.e. *DFE* and *DGE*.

She decided to drag point *F* on top of the top left vertex of the parallelogram, *A*, and after being prompted by Oscar, she dragged the top vertex, *G* of the second triangle *DGE* along the top parallel line (see figure 8).

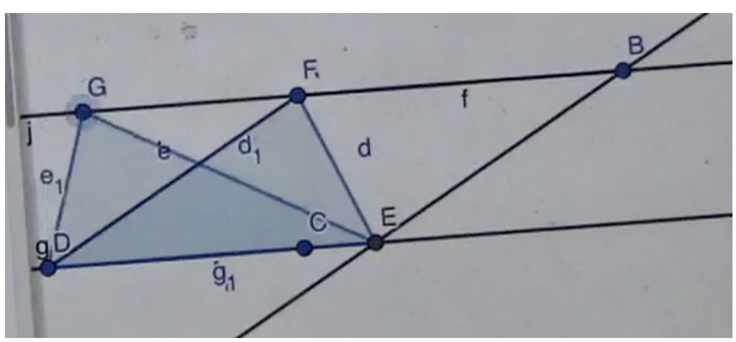


Figure 8. Alice dragged point *G* along the top parallel line to explore different cases of the triangle *DGE*.

It seemed as if Alice wanted to have one triangle placed within the parallelogram in such a way that one of its sides coincided with the 'invisible' diagonal *AE* of the parallelogram. At that point, the researcher asked again if they were convinced that the proposition was true for any parallelogram. After dragging point *G* for some time, Oscar and Alice seemed convinced that the proposition is true for any parallelogram. Both students replied 'yes'. At that point, the researcher decided to introduce the term 'proof'. When the researcher asked about what proof is, the following dialogue took place:

(17) *Oscar*: It is hard to describe it without using the word 'prove'. I guess it's true for every case.

(18) *Researcher*: So, GeoGebra showcased that it's true for the cases you chose. If I asked you to prove Proposition 41, what would you do?

(19) *Oscar:* I would go back and use the formulas.

(20) *Researcher*: Okay.

(21) *Oscar*: First, let's make sure we understand what's going on before we write the proof and make sure you [referring to Alice] understand before we move to the proof. So, explain to me why you think the triangle is half the parallelogram.

(22) *Alice*: Because it's between two parallel lines. And it's the same base and height.

(23) *Oscar*: Okay, so you just said the proposition again. You haven't really proved it.

(24) *Alice*: Because we've done two experiments and we proved the point?

Throughout their interactions with GeoGebra, the two students were encouraged to reflect on what GeoGebra offered and how they could 'prove' Proposition 41 by relying on their mathematical knowledge and abilities as well as GeoGebra's functionalities, such as the dragging functionality mentioned above. Not surprisingly, Oscar referred to the algebraic 'proof' by calculating verbally (but also on paper later on) the area for the parallelogram, bh, and the area for the triangle, $bh/2$. We could claim that Oscar possessed a *deductive proof scheme*, even prior to his interactions with GeoGebra, as he was able to articulate a *direct proof* for the proposition (lines 8 and 11). He was influenced by their GeoGebra interactions, relied on his prior knowledge to highlight the formulae for the two areas in question and showcased their mathematical relationship, i.e. one is half of the other (lines 8 and 11). On the other hand, we might also claim that Oscar's attention to formulas potentially may have prevented him from visualizing "twice the area" of the triangle on the dynamic figure.

Alice's claim "because we've done two experiments and we proved the point?" reveals her inexperience with proof, but also a potential lack of understanding of what 'proof' is (Dreyfus, 1999). In her mind, her experimentation of Proposition 41 using GeoGebra and the fact that she can 'see' the truthfulness of Proposition 41 is enough to convince her that the proposition holds for any parallelogram and any triangle that meet the stated conditions. In a sense, we witness a classical case of a student, who jumps to general conclusions based on exploration via dragging as described by Mariotti (2006) and Mason (1991), which could be viewed as a development of an *empirical proof scheme.*

Still, in the two special cases where point *E* coincided with either vertex of the parallelogram, both Oscar and Alice seemed comfortable with the proposition and their argumentation could be deemed as inductive, since they argued for the outcome of these two special cases being the same for any case (lines 13 and 15). Quite crucially, we argue that the above events indicate how a simple task given by a researcher (or a teacher), encouraged both Oscar and Alice to think about a given mathematical statement by experimenting using GeoGebra. Judging the earlier advice Oscar gave to Alice and also the quick transition to the algebraic proof, which Oscar further elaborated by combining the steps in the algebraic proof with the GeoGebra construction (line 13), it is obvious that Oscar possesses a more or less *deductive proof scheme*, with no need to be supported nor influenced by GeoGebra other than for demonstration purposes. We could argue though that Oscar guides Alice's interactions with GeoGebra, bearing in mind how GeoGebra's features can support Alice's perception of Proposition 41.

Another reflection regarding the potential reasons for Oscar's actions is that GeoGebra does not encourage students to compare numerically nor algebraically the two areas, unless of course the researcher intervened to show them how to calculate the area of the two shapes in GeoGebra. Through their own exploration in GeoGebra, they were able to translate Proposition 41 into a GeoGebra construction, visually compare the two areas of the triangle and the parallelogram, and focus on the two special cases, when the top vertex of the triangle coincided with either the top left or top right vertex of the parallelogram (cf. figure 5d and figure 7). This particular action was due to the guided dragging Alice performed advised by Oscar. The same action for Oscar seemed to be a *dragging test*, since he was very much aware of what to expect, compared to Alice who is at the early stages of interpreting Proposition 41 and becoming convinced that it is true. Earlier on, both Oscar and Alice argued about the base of the parallelogram and the triangle being the same. Oscar in particular gestured by pointing at their GeoGebra construction to show that the height for both the parallelogram and the triangle was also the same (line 13). Alice agreed and 'dragged' the top vertex of the triangle in their GeoGebra construction, seemingly to convince herself that the height of 'any' triangle remains the same (*dragging test*). In her eyes, placing the top vertex of the triangle so as to coincide with one of the two vertices of the parallelogram (cf. figure 5d and figure 7) was the special case that helped her deduce that it should be true for 'any' parallelogram and 'any' triangle and this action was feasible due to GeoGebra's dragging functionality. It also reveals that Alice appears to be on the brink of developing a *techno-authoritarian external conviction proof scheme*.

Step 3: Proof

The dialogue regarding finding a proof for Proposition 41 continued with Oscar making sure that Alice was up to date with calculating areas of triangles and parallelograms, respectively (see figure 9).

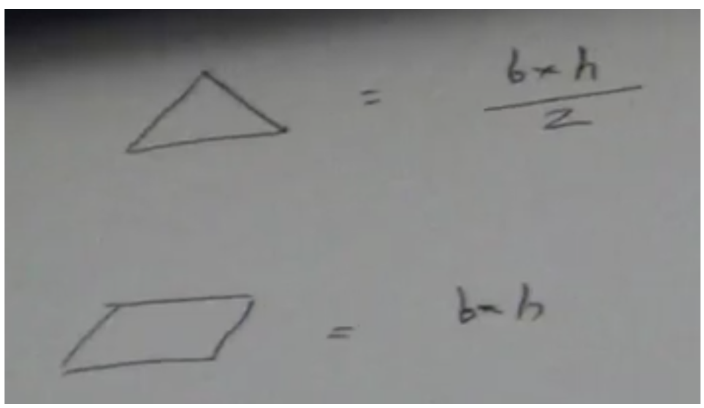


Figure 9. Oscar made diagrams regarding formulas on area for a triangle and a parallelogram.

(25) *Oscar*: Okay, so what we need to understand is that Proposition 41 basically makes us have a parallelogram and a triangle of the same base. So, both bases in each case will be equal and then because it's between the two...

(26) *Alice*: Oh, I know why it's half the area! It's because it's the same base and same height but if you half the parallelogram it would be a triangle... and it's because it's the same base and height that's why. [Throughout her explanation Alice points at their construction in GeoGebra and the objects she refers to, base of parallelogram, base of triangle and the height—cf. figure 7.]

(27) *Oscar*: Okay. So, the base will be the same because it does say that they will share the same base. And then the height will be the same because... well first of all we draw the parallelogram and it will be the same height. But then it says that the triangle must be drawn in between the two parallel lines... which means that wherever you draw the top vertex of the triangle, the height will always be the same.

Here Oscar performs the dragging test once again to show that the height will be the same for any triangle. GeoGebra allowed him and Alice to discern that the height of any ‘triangle between the two parallel lines’ remains the same. There is an interesting point to make here in that the height, the actual line representing the height of the triangle and the parallelogram, is not ‘visible’, as in it is not constructed, but yet again both of them are confident that the height remains the same for whichever triangle is ‘formed’ when they move the top vertex. Throughout their exploration, the parallelogram remains ‘static’ and they only interact with the constructed triangle. GeoGebra allowed both students to reason about the relationship between the two areas, that of the parallelogram and that of *any* triangle between the two parallel lines.

(28) *Oscar*: So now, we know that the bases will be equal and the two heights will be equal. So we can call the base... *b* and the height *h*. [moves to paper]... and they are the same for both the triangle and for the parallelogram… So what it’s saying is basically that the area of the parallelogram, which is base times height, right? Is half... no, it’s double the area of the triangle.

(29) *Alice*: Because it’s base times height over 2.

(30) *Oscar*: So, base times height is double the area of the triangle and we know that the area of the triangle is base times height divided by 2. [Writes on paper: $bh = 2 \text{ x } (bh/2)$. See figure 10.] ... and we can simplify this. Cancel these two out [referring to the 2’s].

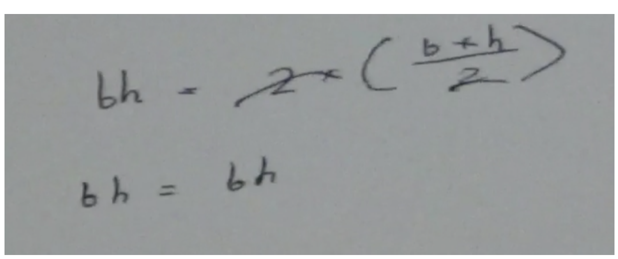


Figure 10. Oscar’s proof on paper.

Next follows a rather long dialogue between Oscar and Alice, in which Oscar explains to Alice his reasoning and proving of the proposition. In order to assess Alice's understanding of what is going on, we display this dialogue in its full length.

(31) *Oscar*: And this shows that the two areas are the same... because then they are equal.
(32) *Alice*: What?
(33) *Oscar*: Because of *bh* and *bh*.
(34) *Alice*: The measurements of *bh*? ... not *b* times *h*?
(35) *Oscar*: Well, when I write *bh* I mean *b* times *h*.
(36) *Alice*: but then that means that the triangle is no longer a triangle… well it is, but not in the...
(37) *Oscar*: I know. But I'm just... I'm proving it. So obviously, I need to change it a tiny bit.
(38) *Alice*: What are you proving though?
(39) *Oscar*: This... Proposition 41. I'm just trying to say... Okay. So, let's go back to what the proposition says. It says that the area of the parallelogram, which is *bh* [and points at the *bh* he wrote on paper] or *b* times *h* is equal to twice...
(40) *Alice*: I understand the *bh*.
(41) *Oscar*: Okay. So, you also understand that the base, the *b*, for both the triangle and the parallelogram are the same.
(42) *Alice*: Are the same, yes.
(43) *Oscar*: So, we can call them the same thing, *b* and *h*, for both the triangle [pointing at the *b* and *h* at the right side of the equation] and the parallelogram [pointing at the *b* and *h* at the left side of the equation]. And what I'm trying to show you here is that... What Proposition 41 says is that twice the area of the triangle equals the area of the parallelogram, which means that if we multiply the area of the triangle by 2 we should get the area of the parallelogram, which is exactly what I've done here [pointing at the written equations on paper]. And we get *b* times *h* and that equals *b* times *h*.
(44) *Alice*: Yeah.
(45) *Oscar*: And that proves that it's true for every case. For every value for *bh* put in. Give me two numbers.
(46) *Alice*: 1 and 3

(47) *Oscar*: Okay. 1 and 3; *b* is 1 and *h* is 3... *bh* is 3 on that side [pointing at left side of equation] and *bh* is 3 on that side [pointing at right side of equation].

(48) *Alice*: So are you saying that this [pointing at Oscar's written equation *bh* = *bh*] proves Proposition 41?

(49) *Oscar*: Yes.

(50) *Alice*: But shouldn't it be *bh* = ... uh... Okay.

(51) *Oscar*: Okay? Do you understand?

(52) *Alice*: Yes.

For Alice, GeoGebra allowed her to create an interactive resource, or a dynamic construction as referred to relevant mathematics education literature regarding DT (e.g. Monaghan, Trouche & Borwein, 2016), that enabled her to 'see' the truth in Proposition 41, but at the same time seemingly also adopt a *techno-authoritarian external conviction proof scheme*. Oscar, possessing a *deductive proof scheme*, is clear on the fact that a mathematical proof must be provided in another medium than that of GeoGebra, e.g. on pen and paper. From the lengthy dialogue above, it appears that Alice is able to follow the stepwise argumentation of Oscar as well as to accept the conclusion, i.e. the proof.

Step 4: Generalizations

As Guven (2008) stated, "in typical use of dynamic geometry environments, attention tends not to be focused on proving and proof but rather on the software's potential in aiding the transition from particular to general cases" (p. 261). This was evidenced in Steps 2 and 3 presented above, as the students were reminded to explore different cases for the triangle and the parallelogram in their GeoGebra constructions, and were asked whether their statements were 'true' for *any* parallelogram and *any* triangle. Alice and Oscar started with specific cases on paper, then the special case of a parallelogram, i.e. a square, before creating 'a' parallelogram and experimenting with different instances of a triangle that has the same base as the parallelogram. Based on their reflections, we can claim that they were able to 'see' Proposition 41's generalizability with the help of GeoGebra.

We believe that GeoGebra did generate a "powerful interplay between empirical explorations and formal proofs" (ibid., p. 261), as in the case presented by Guven (2008), and provided the students with an opportunity “to link between empirical and deductive reasoning, and how such software can be utilized to gain insight into a deductive argument” (ibid.).

Discussion

Even though Oscar and Alice were presented with Euclid’s proof for Proposition 41, as presented in the *Elements* (see figure 3), their exploration with GeoGebra led to a different argumentation process and ‘type of proof’ compared to that of Euclid. The first step in Euclid’s proof assumed the drawing of line segment *AC*, which was one of the diagonals of the parallelogram *ABCD* (see figure 3). While experimenting with GeoGebra, Oscar and Alice landed upon this special case of triangle *ABD* (or *ABC* in Euclid’s triangle), when *E* coincided with *D* (see figure 5d), which gave them the clue to ‘proving’ the proposition. Therefore, we may argue that GeoGebra served as an epistemic mediator by bridging the knowledge gap between the experienced (Euclid or the teacher or a knowledgeable other like Oscar) and the amateur mathematician (student, e.g. Alice).

As mentioned earlier, Oscar had to rely upon his algebraic knowledge to convince himself and ‘prove’ Proposition 41 to Alice and the researcher. Following their interactions on GeoGebra, Oscar produced an algebraic proof using a static diagram (copy of their GeoGebra construction) on paper, by writing an expression for the area of the parallelogram, bh, and the area of the triangle, $(bh)/2$, and then $bh=2\text{x}(bh)/2$. This incident somewhat puzzled us and made us wonder about what a great influence prior knowledge and experiences may have on students, when using DT. Oscar relied heavily on his algebraic knowledge and seemed to have used GeoGebra mainly as an additional tool for further convincing himself. In this sense, we might claim that his prior knowledge and attention to formulas blocked him from visualizing ‘twice the area’ of the triangle on the dynamic figure. Quite crucially, though, he also used GeoGebra as a tool for ‘teaching’ and supporting his peer, Alice, to understand the content of Proposition 41. He prompted her to experiment with the tool using the dragging feature. Alice responded by trialing different cases, but quite interestingly without any prompts from Oscar nor the researcher, she chose to focus on the special cases, which we described above. Such a process could not have been possible without GeoGebra.

Cedillo and Kieran (2003) state that Buchberger’s division between blackboxing and whiteboxing indicates that students use “the black box simply as a tool for problem-solving after they have learned to reason about the problem-solving situation" (p. 221). This is to say that the digital tool only serves purposes of pragmatic mediation. Although not necessarily acknowledged fully by Buchberger (1990) three decades ago, it is clear today that digital tools can indeed serve purposes of epistemic mediation as well (e.g. Artigue, 2002; Kieran & Drijvers, 2006; Trouche, 2005). In the empirical case presented, both students argued at the end of their interactions, the historical source was somewhat inaccessible and complex to understand. From this perspective, GeoGebra did help Alice and Oscar get a better sense of what Euclid claimed in Proposition 41. Hence, in line with what has been suggested previously in the HPM-related literature (e.g. Balsløv, 2018; Jankvist & Geraniou, 2019; Olsen & Thomsen, 2017), the DT acted as a kind of 'tin opener'—here for Euclid's Proposition 41 of the *Elements*. Now, the way that this takes place is in fact in a whiteboxing manner, since no blackboxing as such is involved in the students' activities—and for that reason the tool use cannot be classified as grayboxing either.

As mentioned earlier, in 'Step 0' of their interactions, Oscar and Alice were shown the proof of the proposition on paper, as it is presented in Euclid's *Elements* (Fitzpatrick, 2008). But they did not revisit that proof at the end of their GeoGebra interactions, for example in 'Step 4: Generalizations'. Instead, they concentrated on generating their own proof with the help of GeoGebra, which 'looks' very different to the one in the *Elements* (presented in Figure 3). In the process for Alice (step 2), GeoGebra still came to act as the tool that provided the 'truth' of Proposition 41. So, on the one hand, while GeoGebra served as a truly whiteboxing element (Buchberger, 1990; 2002) in providing Alice with a means for unpacking the content of Proposition 41, not least through the dragging functionality, the very same use of this functionality, on the other hand became the vehicle for developing an inexpedient mathematical proof scheme. There appear to be two possibilities as to the actual nature of this proof scheme. Either it is the so-called *techno-authoritarian external conviction proof scheme* (Jankvist & Misfeldt, 2019; Thomsen & Jankvist, 2020), if she considered GeoGebra to be the authority that established the truth of Proposition 41. Alternatively, if it was in fact the case that Alice was convinced by the empirical examples that GeoGebra provided due to its dragging functionality, she might also be considered to possess an *empirical proof scheme* (Harel & Sowder, 2007). Yet, due to the dragging functionality as a provider of empirical examples, and for the untrained formal deductivist, as a provider of what seems to be 'all' possible empirical examples, it may be difficult to distinguish the nature of dragging from that of proving. Oscar's final 'example' of asking Alice for random numbers for *b* and *h* does not necessarily help this situation—in fact, GeoGebra's area functionality might have done the job better, although it would of course only have provided a sequence of empirical examples. Nevertheless, Alice's comment "because we've done two experiments and we proved the point" may suggest a somewhat *empirical proof scheme*. Surely, Alice's remark may also refer to Euclid's proof, which they were shown in the beginning of the session ('Step 0'), yet at the time, she did not seem to fully grasp its content. One question, however, that we, upon having scrutinized the data and dialogue in step 3, still ask ourselves, is whether Alice truly grasps the need for a formal proof, i.e. the crucial difference between the convincing explorations and their strong perceptual evidence that Proposition 41 is true and the formal proof activity that Oscar finally walked her through. Or put differently; to what extent did this activity assist her on her path towards a more *deductive proof scheme*? Further activities would be needed to approach an answer to this question.

Conclusion

The research question we initially asked was how and to what extent the dragging functionality of a DGE, more precisely GeoGebra, can assist secondary school students in a whiteboxing manner as part of their own work with a somewhat challenging mathematical text, such as Proposition 41 and its proof from Euclid's *Elements*, while avoiding that they jump to conclusions.

Based on the empirical trail, it seems clear that DGEs can indeed assist by providing whiteboxing of the mathematical content of a formal piece of mathematics. In the case of Oscar's and Alice's interactions, GeoGebra gave them an opportunity to perceive that all "triangles between two parallel lines" are of equal area and to explore the "special cases", that is the square and the triangles that are created when 'drawing' the two diagonals of the square. This exploration subsequently allowed (especially Alice) to reason about the mathematical relationship between the two areas in question: that of a parallelogram and that of any triangle that has the same base and is between the same parallel lines. Hence, on the one hand, this study provides some further evidence to the claim that DT may act as 'tin openers' of the mathematical content in otherwise difficult to access mathematical texts (historical or not). Surely, more difficult to access mathematical texts than Euclid's *Elements* can be thought of, and further investigations of the potential reach and scope of the 'tin opener' effect are certainly in order. As for the argument of DT providing an element of familiarity in the work carried out on the "foreign ground" of the mathematical text, this appeared true for Oscar in our case, although not for Alice. Still, Oscar's use of the 'dragging test' to whitebox the content of the proposition illustrates the point. We thus find it fair to say that this study illustrates the fruitful potential of an interplay between mathematical texts (which may or may not be historical) and digital technologies, when students work with reasoning and proof, as described and analyzed here in terms of the construct of proof schemes and GeoGebra's dragging feature. The two students use GeoGebra to uncover mathematical content and to understand what Proposition 41 actually says. The digital tool serves as a 'vehicle' to eventually also establish the truth of the theorem—in Oscar's case for sure, possibly also in Alice's. This is whiteboxing in a different manner than that originally described by Buchberger (1990). As such, the study can be seen to question the original formulation of the "whitebox/blackbox principle" as not involving DT as such. Instead, we propose to define whiteboxing, still as the opposite of blackboxing, but to be when digital technology serves the purpose of revealing mathematical aspects otherwise hidden or inaccessible to the students. Despite the positive whiteboxing effects in relation to the mathematical content of the proposition, a fly in the ointment is that of whiteboxing through dragging. As illustrated through the case of Alice, whiteboxing through dragging calls for caution, when it comes to generalizability of mathematical results and the necessity of formal proofs. It thus also appears that Buchberger's (1990) comment that an area X must have been studied thoroughly, before involving DT, not only applies to specific mathematical areas, but also to epistemological aspects of mathematics—in our case, the need and function of mathematical proof. Hence, caution must be paid so that students do not jump to conclusions and in the process develop inexpedient mathematical proof schemes, such as those techno-authoritarian ones, upon which they may stumble in their future mathematical work.

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[1] https://www.mathunion.org/icmi/organizationaffiliated-organizations/hpm or http://www.clab.edc.uoc.gr/hpm/about%20HPM.htm

[2] https://www.geogebra.org/geometry/sfycrm76