SUPERSYMMETRIC F-THEORY GUT MODELS

A Dissertation

by

YU-CHIEH CHUNG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2011

Major Subject: Physics

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Approved by:

Chair of Committee,	Katrin Becker
Committee Members,	Melanie Becker
	Christopher Pope
	Joseph Landsberg
Head of Department,	Edward Fry

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ABSTRACT

Supersymmetric F-theory GUT Models. (May 2011)

Yu-Chieh Chung, B.S., National Tsing Hua University;

M.S., National Tsing Hua University

Chair of Advisory Committee: Dr. Katrin Becker

F-theory is a twelve-dimensional geometric version of string theory and is believed to be a natural framework for GUT model building. The aim of this dissertation is to study how gauge theories realized by F-theory can accommodate GUT models.

In this dissertation, we focus on local and semi-local GUT model building in F-theory. For local GUT models, we build $SU(5)$ GUTs by using abelian $U(1)$ fluxes via the $SU(6)$ gauge group. Doing so, we obtain non-minimal spectra of the MSSM with doublet-triplet splitting by switching on abelian $U(1)^2$ fluxes. We also classify all supersymmetric $U(1)^2$ fluxes by requiring an exotic-free bulk spectrum. For semi-local GUT models, we start with an E_8 singularity and obtain lower rank gauge groups by unfolding the singularity governed by spectral covers. In this framework, the spectra can be calculated by the intersection numbers of spectral covers and matter curves. In particular, we use $SU(4)$ spectral covers and abelian $U(1)_X$ fluxes to build flipped $SU(5)$ models. We show that three-generation spectra of flipped $SU(5)$ models can be achieved by turning on suitable fluxes. To construct E_6 GUTs, we consider $SU(3)$ spectral covers breaking E_8 down to E_6. Also three-generation extended MSSM can be obtained by using non-abelian $SU(2) \times U(1)^2$ fluxes.

To my parents

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CHAPTER I

INTRODUCTION[1]

String theory is so far the most promising candidate for a unified theory. Building realistic models of particle physics to answer fundamental questions is one of the challenges in string theory. One of the main issues to be addressed from particle physics is the unification of gauge couplings. The natural solution to this question is the framework of grand unified theory (GUT). One task for string theory is whether it can accommodate GUT models. String theory makes contact with four-dimensional physics through various compactifications. There are two procedures to realize GUTs in string theory compactifications. The first is the top-down procedure in which the full compactification is consistent with the global geometry of extra dimensions; the spectrum is close to a GUT after breaking some symmetries [1]. In the bottom-up procedure, the gauge breaking can be understood in the decoupling limit of gravity [2–4], particularly in the framework that D-branes are introduced on the local regions within the extra dimensions in type IIB compactifications [2–5]. In this case we can neglect the effects from the global geometry for the time being, which makes the procedure more flexible and efficient. In addition, the construction of the local models can reveal the requirements for the global geometry. Eventually, the local models need to be embedded into some compact geometry for UV completion.

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In $SU(5)$ GUTs, there are two important Yukawa couplings, $\mathbf{10 10 5_H}$ and $\mathbf{10 \bar{5}_M \bar{5}_H}$. It is well-known that $\mathbf{10 10 5_H}$ is forbidden in perturbative type IIB theory. However, it was shown in [6, 7] that the Yukawa coupling $\mathbf{10 10 5_H}$ can be achieved by introducing non-perturbative corrections. From this perspective, the non-perturbative property is intrinsic for GUT model building in type IIB theory. F-theory is a non-perturbative twelve-dimensional theory built on the type IIB framework with an auxiliary two-torus [8–10]. For a nice review of F-theory, see [11]. The ordinary string extra dimensions are regarded as a base manifold and the two-torus is as a fiber over this base manifold. The modulus of the elliptic curve is identified as axion-dilaton in type IIB theory. Due to the $SL(2,\mathbb{Z})$ monodromy of the modulus, F-theory is essentially a non-perturbative completion of type IIB theory. There is an elegant correspondence between physical objects in type IIB theory and geometry in F-theory. The modular parameter of the elliptic fiber, identified with the axion-dilaton in type IIB theory, varies over the base. Singularities develop when the fibers degenerate. The loci of the singular fiberation indicate the locations of the seven-branes in type IIB theory and the type of the singularity determines the gauge group of the world-volume theory on seven-branes [12]. According to the classification of the singular fibration, there are singularities of types A, D, and E. The first two types have perturbative descriptions in Type IIB. More precisely, A-type and D-type singularities correspond to configurations of the $D7$-branes and $D7$-branes along O-planes, respectively [13]. For the singularity of type E, there is no perturbative description in type IIB theory, which means that F-theory captures a non-perturbative part of the type IIB theory. Under certain geometric assumptions, the full F-theory can decouple from gravity [14–17]. In such a way, one can focus on the gauge theory on seven-branes supported by the discriminant loci in the base manifold of an elliptically fibered Calabi-Yau fourfold. Extensive studies of GUT local models and their corresponding phenomenology in

F-theory have been undertaken in [14–39] [2]. In addition, supersymmetry breaking has been discussed in [41–45], and the application to cosmology has been studied in [46]. It is becoming clear that F-theory provides a promising framework for model building of supersymmetric GUTs. To build local $SU(5)$ GUTs in F-theory, one can start by engineering a Calabi-Yau fourfold with an A_4 singularity. To decouple from gravity, it is required that the volume of S, a component of the discriminant locus, is contractible to zero size.[3] We assume that S can contract to a point and thus possesses an ample canonical bundle K_S^{-1} [14–17]. In particular, we focus on the case that S is a del Pezzo surface [49,50] wrapped by seven-branes where one can engineer an eight-dimensional supersymmetric gauge theory with gauge group $G_S = SU(5)$ in $\mathbb{R}^{3,1} \times S$. Other components S_i' of the discriminant locus intersect S along the curves Σ_i. Due to the collision of the singularities, the gauge group G_S is enhanced to G_{Σ_i} on Σ_i and the matter in the bi-fundamental representations may be localized on the curves [51]. It was shown in [14–17] that the spectrum is given by the bundle-valued cohomology groups. The minimal $SU(5)$ GUT has been studied in [14–17]. In that case, the GUT group is broken into $G_{\mathrm{std}} \equiv SU(3) \times SU(2) \times U(1)_Y$ by a non-trivial $U(1)_Y$ gauge flux. Furthermore, one can obtain an exotic-free spectrum of the minimal supersymmetric Standard Model (MSSM) from those curves with doublet-triplet splitting but no rapid proton decay. The success of the minimal $SU(5)$ GUT model motivates us to pursue other local GUT models from higher rank gauge groups. The next simplest case is a gauge group of rank five: like $SO(10)$ and $SU(6)$. These two non-minimal $SU(5)$ GUTs have been studied in [33]. For the latter, one can get an

[2] For a review, see [40]

[3] There are two ways in which we could take $V_S \to 0$. The first way is by requiring S to contract to a point, and the second is by requiring S to contract to a curve of singularities. See [47,48] for the details.

exotic-free spectrum, but due to the lack of an extra $U(1)$ flux, the GUT group cannot be broken into $G_{\rm std}$. To avoid this difficulty, it is natural to study local F-theory models of $G_S = SU(6)$ and $G_S = SO(10)$ with supersymmetric $U(1)^2$ gauge fluxes, which consist of two supersymmetric $U(1)$ gauge fluxes and are associated with a rank two polystable bundles over S.

In chapter II we shall explicitly construct supersymmetric $U(1)^2$ gauge fluxes in local F-theory models of $G_S = SU(6)$ and $SO(10)$ and calculate the matter spectrum of the MSSM. For the case of $G_S = SO(10)$, there is a no-go theorem [15] which states that for an exotic-free spectrum, there are no solutions for $U(1)^2$ gauge fluxes. For the case of $G_S = SU(6)$, we can explicitly construct supersymmetric $U(1)^2$ gauge fluxes. It turns out that each flux configuration contains two fractional lines bundles. One of the gauge fluxes is universal and is the same as $U(1)_Y$ hypercharge flux in the minimal $SU(5)$ GUT [14–17]. The second one varies along with the configurations of the bulk zero modes. With suitable supersymmetric $U(1)^2$ gauge fluxes, the bulk spectrum can be exotic-free and the chiral matter comes from curves. The restriction of these $U(1)^2$ fluxes to the curves induce $U(1)$ fluxes over the curves, which breaks the enhanced gauge group G_Σ down to $G_{\rm std} \times U(1)$. In this case, the Higgs fields can be localized on curves $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$. On $\Sigma_{SU(7)}$, non-trivial induced fluxes break $SU(7)$ into $G_{\rm std} \times U(1)$. With suitable fluxes, doublet-triplet splitting can be achieved. However, the situations become more complicated on the curves with $G_\Sigma = SO(12)$. Since the dimension of the adjoint representation of $SO(12)$ is higher than $SU(7)$, one gets more constraints to solve for given field configurations, which results in difficulties for doublet-triplet splitting. By explicitly solving the allowed field configurations, one can find that there are still a few solutions for doublet-triplet splitting. To obtain a complete matter spectrum of the MSSM, we analyze the case of Σ_{E_6} in addition to $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$. It is extremely difficult to obtain the minimal spectrum of the MSSM without exotic fields. However, we found that in some cases, the exotic fields can form trilinear couplings with the doublets or triplets on the curves with $G_\Sigma = SU(7)$. When these fields get vacuum expectation values (vevs), the exotic fields will be decoupled from the low-energy spectrum. A way to do this is that we introduce extra curves supporting the doublets or triplets, which intersect the curves hosting the exotic fields to form the couplings. With the help of these doublets or triplets, it turns out that the non-minimal spectrum of the MSSM without doublet-triplet splitting problem can be achieved by local F-theory model of $G_S = SU(6)$ with supersymmetric $U(1)^2$ gauge fluxes. Constructing local GUT models is the first step toward global F-theory GUTs.[4] The middle step of F-theory GUT model building is to construct semi-local models by using spectral covers. In chapters III and IV, we shall focus on local and semi-local model building in F-theory.

Spectral cover construction [17, 47] originally was introduced in the heterotic string compactifications [59]. This construction has been used to build an $SU(5)$ GUT with an $SU(5)$ cover [16, 17, 39, 47, 52–54, 57, 58, 60–79], an $SO(10)$ with $SU(4)$ covers [80, 81], and an MSSM with an $SU(5) \times U(1)$ cover [82, 83]. For a systematic review of recent progress of F-theory compactifications and model buildings, see [84]. Systematic studies of how models of higher rank GUT groups, such as $SO(10)$, are embedded into the compact geometry in F-theory have not been fully investigated. To this end, we are interested in the $SO(10)$ subgroup $SU(5) \times U(1)_X$ which is realized as the flipped $SU(5)$ GUT [85–87]. Although local flipped $SU(5)$ models have been discussed in F-theory, we study the model as a semi-local construction. In chapter III we shall build flipped $SU(5)$ models by unfolding an E_8 singularity via the $SO(10)$ gauge group. To construct flipped $SU(5)$ models in four-dimensional spacetime, we

[4] Recent development for global GUT models can be found in [52–58].

compactify F-theory on an elliptically fibered Calabi-Yau fourfold X_4 with a base threefold B_3. We consider a del Pezzo surface S [49, 50] inside B_3 such that we can reduce full F-theory on X_4 to an effective eight-dimensional supersymmetric gauge theory on $\mathbb{R}^{3,1} \times S$. To construct flipped $SU(5)$ models from an $SO(10)$ gauge group, the singularities of types D_5, D_6, E_6, and E_7 have to be engineered in the Calabi-Yau fourfold. Because these singularities can be embedded into a single singularity E_8, it motivate us to build models by starting with an E_8 singularity and unfold it into a D_5 singularity.

Generally, one may turn on certain fluxes to obtain the chiral spectrum. In F-theory, there is a four-form G-flux, which consists of three-form fluxes and gauge fluxes. In type IIB theory, these three-form fluxes produce a back-reaction in the background geometry. It was shown in [37, 88] that the three-form fluxes induce non-commutative geometric structures and also modify the texture of the Yukawa couplings. F-theory in Fuzzy space also was studied in [89]. In this dissertation, we shall turn off these three-form fluxes and focus only on the gauge fluxes. The $U(1)_X$ gauge flux is able to break the gauge group $SO(10)$ down to $SU(5) \times U(1)_X$. It was shown in [17, 47] that the spectral cover construction naturally encodes the unfolding information of an E_8 singularity as well as the gauge fluxes. In chapter III we shall focus on the $SU(4)$ spectral cover encoding the D_5 singularity from unfolding E_8. The four-dimensional low-energy spectrum of the flipped $SU(5)$ model is then determined by the cover fluxes and the $U(1)_X$ flux.

The $SU(4)$ spectral cover has interesting properties. From the subgroup decomposition of E_8, one can find that there is no explicit presentation of $\overline{\mathbf{10}}$. In addition, the cover associated to the $\mathbf{10}$ representation forms a double-curve and along this curve there are co-dimension two singularities. After resolving the singularities along the curve, one finds that the net chirality of the $\mathbf{10}$ curve vanishes [52]. Since the background geometry generically determines the G flux, there are not many degrees of freedom left to adjust the chirality on the **16** curve to create three-generation models. These ideas motivate us to consider factorizing the spectral cover [53, 54, 57, 65, 66] to introduce additional parameters for model building. We consider two possibilities of splitting the $SU(4)$ spectral cover: (3,1) and (2,2) factorizations. The curve of the fundamental representation is then divided into two **16** curves, while generically the **10** curve is detached into three. However, due to the monodromy structure there are only two **10** curves in the (3,1) case.

In semi-local $SO(10)$ GUTs, there exists only the **16 16 10** Yukawa coupling from the enhancement to an E_7 singularity. The GUT Higgs fields coming from the adjoints or other representations such as **45**, **54**, or **120** are absent in the F-theory construction. Therefore, the most convincing way to break the $SO(10)$ gauge group is turning on the $U(1)_X$ flux on the GUT surface S. This $U(1)_X$ gauge field can be massless [3, 14, 16], so we can interpret the gauge group as the flipped $SU(5)$ model after turning on such a flux. With non-trivial restrictions to the curves, this $U(1)_X$ flux generically modifies the net chirality of matter localized on these curves. We may identify the flipped $SU(5)$ superheavy Higgs fields with one of the $\mathbf{10}+\overline{\mathbf{10}}$ vector-like pairs in the spectrum for further gauge breaking to MSSM.

In local models, an abelian or a non-abelian flux of the rank greater than two may be turned on the bulk to break the gauge group [15]. Following this idea, an MSSM model from breaking an $SU(6)$ model by an $U(1) \times U(1)$ gauge flux has been studied [38]. There are two kinds of rank three fluxes, $U(1)^3$ and $SU(2) \times U(1)^2$, both embedded in the E_6 gauge group with commutants including the Standard Model (SM) gauge structure. We are particularly interested in the second case containing a non-abelian $SU(2)$ gauge flux. In chapter IV we shall study the physics of the E_6 GUT model [90] broken by the $SU(2) \times U(1)^2$ fluxes in F-theory. There are many breaking routes from E_6 to a subgroup containing the SM gauge group, such as via $SO(10)$ and then $SU(5)$, via $SU(6)$, Pati-Salam, or trinification. These breaking routes end up with two resultant gauge groups, $G_1 : SU(3) \times SU(2)_L \times U(1)^3$ and $G_2 : SU(3) \times SU(2)_L \times SU(2) \times U(1)^2$. These two subgroups are referred to as extended MSSM models of rank 6. By suitable rotation of the $U(1)$ gauge groups and the third component of the $SU(2)$ gauge group, one can show that these two subgroups are equivalent. It was found [91] that the extended MSSM models can be obtained from an E_6 unification by an $SU(2) \times U(1)^2$ or $U(1)^3$ flux[5] in the heterotic string models. In the literature, the gauge group obtained by breaking E_6 can be rank 5 or rank 6 depending on the flux turned on [91, 93–100]. When a non-abelian flux $SU(2) \times U(1)^2$ is turned on, E_6 is broken directly to a rank 5 model with a gauge group $SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_\eta$ after rearranging the $U(1)$s. Normally rank 6 models have more degrees of freedom with which to solve the problems in phenomenology. However, the $U(1)$ gauge groups induce additional gauge bosons and increase exotic fields. By giving a large VEV to one of the $U(1)$ gauge groups, the rank 6 models can be further reduced to the so-called effective rank 5 models. By arranging the matter assignments, one can build many interesting low energy models, such as $SU(3) \times SU(2) \times U(1)_Y \times U(1)_N$. In the rank 6 model, $U(1)_N$ is inherited from the third $U(1)$ gaining a VEV, whereas in the rank 5 model, $U(1)_\eta$ is fixed and does not possess additional symmetries.

One of the motivations to consider models with an additional gauge group $U(1)'$ as a gauge extension of the Standard Model (NMSSM) is for solving the μ-problem. The minimum matter content for such a model with gauge group $SU(3) \times SU(2) \times U(1)_Y \times U(1)'$ includes the MSSM fermions, two Higgs doublets H and $\bar{H}$, an SM

[5]For breaking scenarios via discrete Wilson lines in the context of orbifold constructions, please see [92] and references therein.

singlet S with a non-zero $U(1)'$ charge, and exotic color triplets. The effective scale of μ-term can arise from the coupling $SH\bar{H}$ when the singlet S acquires a VEV. The radiative breaking of the $U(1)'$ gauge symmetry is usually achieved by the large Yukawa couplings between the singlet S and the exotic fields. This model can be naturally embedded in a model with the E_6 gauge group while the fields mentioned above are included in the three families of **27**-plets. For the desire of gauge unification without introducing anomalies, a pair of Higgs-like doublets from one or more additional $(\mathbf{27}+\overline{\mathbf{27}})$ is also needed. Recently, the minimum MSSM from the E_6 GUT has been studied, for example, in [101–105], and phenomenology such as the neutrino physics [106], leptogenesis [107], and baryogenesis [108] were also discussed.

In chapter IV we construct E_6 GUT models in F-theory by using the spectral cover construction and study their breaking down to the rank 5 extended MSSM by turning on non-abelian fluxes. We only consider the case that the Higgs multiplets are located on a different **27** due to the reasons of desiring for more degrees of freedom as well as the singularity structure of Yukawa coupling in F-theory. We represent a few examples corresponding to two spectral cover factorizations. In the example of $(2,1)$ factorization in dP_7, all the fermions are located on one **27** curve and the introduction of fluxes for gauge breaking results in extra copies of quarks and leptons which are exotic to the conventional three-generation E_6 models. We find a better model in the $(1,1,1)$ factorization where the fermions are from two different **27** curves and there is only a pair of vector-like triplet exotic field. Both examples in dP_7 contain exotic fields on the Higgs **27** curve, and we assume they obtain zero vacuum expectation values.

The organization of the rest of this dissertation is as follows: In chapter II we give a brief review of F-theory and build local MSSM models by using abelian $U(1)^2$ fluxes. In chapter III we build semi-local flipped $SU(5)$ models by using $SU(4)$

spectral covers and $U(1)_X$ fluxes. In chapter IV we construct semi-local E_6 GUTs by using $SU(3)$ spectral covers and non-abelian $SU(2) \times U(1)^2$ fluxes. We present matter spectra for these models and discuss their phenomenology. A summary can be found in chapter V.

CHAPTER II

LOCAL F-THEORY GUT MODELS[6]

In this chapter we briefly review F-theory and local GUT model building. In particular, we analyze abelian gauge fluxes in local F-theory models with $G_S = SU(6)$ and $SO(10)$. For the case of $G_S = SO(10)$, there is a no-go theorem which states that for an exotic-free spectrum, there are no solutions for $U(1)^2$ gauge fluxes. We explicitly construct the $U(1)^2$ gauge fluxes with an exotic-free bulk spectrum for the case of $G_S = SU(6)$. We also analyze the conditions for the curves supporting the given field content and discuss non-minimal spectra of the MSSM with doublet-triplet splitting.

A. F-theory and ADE Singularities

F-theory is a twelve-dimensional geometric version of type IIB theory. The construction of F-theory is motivated by $SL(2,\mathbb{Z})$ symmetry in type IIB action. The low energy field content of type IIB theory contains a metric g_{MN} of then-dimensional space M^{10}, an anti-symmetric two-tensor B_{MN}, a scalar dilaton ϕ, and form fields of even degrees C_p. The low energy type IIB action is as follows:

$$\begin{aligned} S_{IIB} &= \int_{M^{10}} d^{10}x\sqrt{-g}R - \frac{1}{2}\int_{M^{10}} \Big[\frac{d\tau_{IIB}\wedge\star d\bar{\tau}_{IIB}}{(\mathrm{Im}\tau_{IIB})^2} + \frac{dG_3\wedge\star d\overline{G_3}}{(\mathrm{Im}\tau_{IIB})} \\ &+ \frac{1}{2}\widetilde{F}_5\wedge\star\widetilde{F}_5 + C_3\wedge H_3\wedge F_3\Big], \end{aligned} \tag{2.1}$$

where R is the scalar curvature of M^{10}, $\star$ is the Hodge dual in M^{10}, $\tau_{IIB} = C_0 + ie^{-\phi}$, $H_3 = dB_2$, $F_{p+1} = dC_p$, $G_3 = F_3 - \tau_{IIB}H_3$, and $\widetilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$. This

[6] Portions of this chapter are reprinted from *Journal of High Energy Physics*, Volume 2010, Number 3, 6, Yu-Chieh Chung, Abelian Gauge Fluxes and Local Models in F-Theory, Copyright 2010, with permission from SISSA.

action has $SL(2,\mathbb{Z})$ symmetry under the following transformations:

$$\tau_{IIB} \to \frac{a\tau_{IIB}+b}{c\tau_{IIB}+d}, \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}, \ \widetilde{F}_5 \to \widetilde{F}_5,\ g_{MN} \ \to g_{MN}, \quad (2.2)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. The transformation acting on the axio-dilation τ_{IIB} is exactly the modular transformation on the complex structure τ_{T^2} of a torus, provided that one identifies τ_{IIB} with τ_{T^2}. This identification not only provides a geometric interpretation of $SL(2,\mathbb{Z})$ symmetry in type IIB theory, but also gave birth to F-theory. Motivated by the identity $\tau_{IIB} = \tau_{T^2}$, F-theory is defined on a twelve-dimensional manifold which admits an elliptic fibration and is dual to type IIB theory on the base manifold. To preserve supersymmetry, it is required that the twelve-dimensional manifold has to be a Calabi-Yau manifold. With the identity $\tau_{IIB} = \tau_{T^2}$, one can deduce the relation between the singularities of an elliptic fibration and the locations of seven-branes in type IIB theory. Various configurations of seven-brane locations determine eight-dimensional world volume theories with different gauge groups. It was shown [12] that the singularity of types A_n, D_n, and E_n correspond to $SU(n+1)$, $SO(2n)$, and E_n gauge groups, respectively. Consider F-theory compactified on an elliptic $K3$ surface with a base manifold $\mathbb{P}^1$. This surface can be described by the Weierstrass form

$$y^2 = x^3 + fx + g, \quad (2.3)$$

where f and g are respectively sections of $K_{\mathbb{P}^1}^{-4}$, and $K_{\mathbb{P}^1}^{-6}$, and $K_{\mathbb{P}^1}$ stands for the canonical bundle of $\mathbb{P}^1$. The fiber degeneration happens at the locus of $\Delta \equiv 4f^3 + 27g^2 = 0$ which generically determines twenty four locations of seven-branes with A_0 singularities. Generally the singularities of an elliptic fibration are classified by the vanishing orders of f, g, and Δ denoted by $\mathrm{ord}(f)$, $\mathrm{ord}(g)$, $\mathrm{ord}(\Delta)$, respectively.

Singularity	ord(f)	ord(g)	ord(Δ)	Gauge Group
A_n	0	0	$n+1$	$SU(n+1)$
D_{n+4}	$\geqslant 2$	3	$n+6$	$SO(2n+8)$
D_{n+4}	2	$\geqslant 3$	$n+6$	$SO(2n+8)$
E_6	$\geqslant 3$	4	8	E_6
E_7	3	$\geqslant 5$	9	E_7
E_8	$\geqslant 4$	5	10	E_8

Table I. *ADE* Singularities and Gauge Groups Correspondences.

A summary of the correspondence between *ADE* singularities and gauge groups of eight-dimensional gauge theories on seven-branes is in Table I.

B. Local Geometry for Model Building

Consider F-theory compactified on an elliptically fibred Calabi-Yau fourfold, $\pi_{X_4}: X_4 \to B_3$ with a section, which can be realized in the Weierstrass form:

$$y^2 = x^3 + fx + g, \tag{2.4}$$

where x and y are complex coordinates on the fiber, f and g are sections of the suitable line bundles over the base manifold B_3. The degrees of f and g are determined by the Calabi-Yau condition, $c_1(X_4) = 0$. The degenerate locus of fibers is given by the discriminant $\Delta = 4f^3 + 27g^2 = 0$, which is in general a codimension one reducible subvariety in the base B_3. In local models, we focus on one component S of the discriminant locus $\Delta = 0$, which will be wrapped by a stack of the seven-branes and supports the GUT model. In order to decouple from the gravitational sector, the anti-canonical bundle K_S^{-1} of the surface S is assumed to be ample. According to the classification theorem of algebraic surfaces, the surface S is a del Pezzo surface and birational to the complex projective plane $\mathbb{P}^2$. There are ten del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, and $dP_k,\ k = 1, 2, ..., 8$, which are blow-ups of k generic points on $\mathbb{P}^2$. In this dissertation we shall focus on the case of $S = dP_k,\ 2 \leqslant k \leqslant 8$ with (-2) 2-cycles[7]. It was shown that there are ten families of del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and the blow-ups of $\mathbb{P}^2$ at k generical points, where $1 \leqslant k \leqslant 8$ [49, 50]. In what follows, we shall briefly review the geometry of the del Pezzo surfaces.

The del Pezzo surface S is an algebraic surface with ample anti-canonical bundle, namely $K_S^{-1} > 0$. It follows that $h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0$[8] and that $\chi(S, \mathcal{O}_S) = \sum_{i=0}^{2}(-1)^i h^i(S, \mathcal{O}_S) = 1$. According to the classification theorem of algebraic surfaces, these surfaces are birational to the complex projective plane $\mathbb{P}^2$. It was shown in [14–16] that to obtain an exotic-free bulk spectrum, the gauge fluxes have to correspond to the dual of (-2) 2-cycles in S. Notice that the Picard group of $\mathbb{P}^2$ is generated by the hyperplane divisor h with intersection number $h \cdot h = 1$. Thus, there is no (-2) 2-cycle in $\mathbb{P}^2$. The Picard group of dP_k is generated by the hyperplane divisor h, which is inherited from $\mathbb{P}^2$ and the exceptional divisors $e_i,\ i = 1, 2, .., k$ from blow-ups with intersection numbers $h \cdot h = 1$, $h \cdot e_i = 0$, and $e_i \cdot e_j = -\delta_{ij},\ \forall\, i, j$. It is easy to see that dP_1 contains no (-2) 2-cycles. It follows that the candidates of the del Pezzo surfaces containing (-2) 2-cycles are dP_k with $2 \leqslant k \leqslant 8$. In what follows, I shall focus on the del Pezzo surfaces dP_k with $2 \leqslant k \leqslant 8$. The canonical divisor of dP_k is $K_S = -3h + e_1 +, ..., +e_k$. The first term comes from $K_{\mathbb{P}^2} = -3h$ and

[7] A (-2) 2-cycle is a 2-cycle with self-intersection number -2.

[8] It can be easily seen by the Kodaira vanishing theorem which states that for any ample line bundle $\mathcal{L}$, $h^i(S, K_S \otimes \mathcal{L}) = 0,\ \forall i > 0$.

Mori Cone	Generators	Number
$\overline{\mathrm{NE}}(dP_2)$	$e_i,\ h-e_1-e_2$	3
$\overline{\mathrm{NE}}(dP_3)$	$e_i,\ h-\sum_{m=1}^{2} e_{i_m}$	6
$\overline{\mathrm{NE}}(dP_4)$	$e_i,\ h-\sum_{m=1}^{2} e_{i_m}$	10
$\overline{\mathrm{NE}}(dP_5)$	$e_i,\ h-\sum_{m=1}^{2} e_{i_m},\ 2h-\sum_{n=1}^{5} e_{i_n}$	16
$\overline{\mathrm{NE}}(dP_6)$	$e_i,\ h-\sum_{m=1}^{2} e_{i_m},\ 2h-\sum_{n=1}^{5} e_{i_n}$	27
$\overline{\mathrm{NE}}(dP_7)$	$e_i,\ h-\sum_{m=1}^{2} e_{i_m},\ 2h-\sum_{n=1}^{5} e_{i_n},\ 3h-2e_i-\sum_{p=1}^{6} e_{i_p}$	56
$\overline{\mathrm{NE}}(dP_8)$	$e_i,\ h-\sum_{m=1}^{2} e_{i_m},\ 2h-\sum_{n=1}^{5} e_{i_n},\ 3h-2e_i-\sum_{p=1}^{6} e_{i_p},$ $4h-2\sum_{q=1}^{3} e_{i_q}-\sum_{r=1}^{5} e_{i_r},\ 5h-2\sum_{l=1}^{6} e_{i_l}-e_r-e_s,$ $6h-3e_i-2\sum_{m=1}^{7} e_{i_m}$	240

Table II. The generators of the Mori cone $\overline{\mathrm{NE}}(dP_k)$ for $k = 2, ...8$, where all indices are distinct.

the rest comes from the blow-ups, which lead to the exceptional divisors $e_1,\ e_2, ...,\ e_k$. For local models in F-theory, the curves supporting matter fields are required to be effective. Next we shall define effective curves and the Mori cone. Consider a complex surface Y and its homology group $H_2(Y, \mathbb{Z})$. Let C be a holomorphic curve in Y. Then $[C] \in H_2(Y, \mathbb{Z})$ is called an effective class if $[C]$ is equivalent to C. The Mori cone $\overline{\mathrm{NE}}(Y)$ is spanned by a countable number of generators of the effective classes [109, 110]. The Mori cones $\overline{\mathrm{NE}}(dP_k)$ of the del Pezzo surfaces dP_k are all finitely generated [49]. To be concrete, we list the generators of the Mori cones of dP_k for $2 \leqslant k \leqslant 8$ in Table II.

With the Mori cone, one can easily check that the anti-canonical divisor $-K_S$ is ample[9]. The dual of the Mori cone is the ample cone, denoted by $\mathrm{Amp}(dP_k)$, which is defined by $\mathrm{Amp}(dP_k) = \{\omega \in H_2(dP_k, \mathbb{R})|\ \omega \cdot \zeta > 0,\ \forall \zeta \in \overline{\mathrm{NE}}(dP_k)\}$. Each ample divisor ω in the ample cone is associated with a Kähler class ω_S. In this chapter we choose the "large volume polarization", namely $\omega = Ah - \sum_{i=1}^{k} a_k e_k$ with $A \gg a_k > 0$ [14, 15]. It is easy to check that this ω is ample. For the del Pezzo surfaces S and a line bundle $\mathcal{L}$ over S, there are two useful theorems. One is the Riemann-Roch theorem [109, 110], which says that

$$\chi(S, \mathcal{L}) = 1 + \frac{1}{2} c_1(\mathcal{L})^2 - \frac{1}{2} c_1(\mathcal{L}) \cdot K_S. \tag{2.5}$$

Another one is the vanishing theorem ([14], also see [111]), which states that for a non-trivial holomorphic vector bundle $\mathcal{V}$ over S satisfying the Hermitian Yang-Mills equations (2.11),

$$H^0_{\bar{\partial}}(S, \mathcal{V}) = H^2_{\bar{\partial}}(S, \mathcal{V}) = 0. \tag{2.6}$$

These two theorems simplify the calculation of the spectrum. Note that the vanishing theorem (2.6) holds when $\mathcal{V}$ is a line bundle. It follows from Eq. (2.5) and Eq. (2.6) that $h^1(S, \mathcal{L}) = -\chi(S, \mathcal{L}) = -(1 - \frac{1}{2} c_1(\mathcal{L}) \cdot K_S + \frac{1}{2} c_1(\mathcal{L})^2)$. In this case $h^1(S, \mathcal{L})$ is determined by intersection numbers $c_1(\mathcal{L}) \cdot K_S$ and $c_1(\mathcal{L})^2$.

In local models, we require that all curves be effective. That is, the homological classes of the curves in $H_2(S, \mathbb{Z})$ can be written as non-negative integral combinations of the generators of the Mori cone, namely $\Sigma = \sum_\beta n_\beta \mathcal{C}_\beta$ with $n_\beta \in \mathbb{Z}_{\geqslant 0}$[10]. To calculate the genus of the curve, we can apply the adjunction formula, which says

[9] One can apply the Nakai-Moishezon criterion which states that for any divisor D, D is ample if and only if $D \cdot D > 0$ and $D \cdot C_\alpha > 0$, where C_α are generators of the Mori cone.

[10] By abuse of notation, we use Σ to denote the homological class of the curve Σ.

that for a smooth, irreducible curve of genus g, the following equation holds

$$\Sigma \cdot (\Sigma + K_S) = 2g - 2. \tag{2.7}$$

In this chapter we shall choose genus zero curves to support the matter in GUTs or MSSM, which means that all matter curves satisfy the equation $\Sigma \cdot (\Sigma + K_S) = -2$. To calculate the spectrum from curves, we also need the Rieman-Roch theorem [109, 110] for algebraic curves. For an algebraic curve Σ, the Rieman-Roch theorem states that for a line bundle $\mathcal{L}$ over Σ,

$$h^0(\Sigma, \mathcal{L}) - h^1(\Sigma, \mathcal{L}) = 1 - g + c_1(\mathcal{L}). \tag{2.8}$$

In particular, for the case of $g = 0$, we have

$$h^0(\Sigma, K_\Sigma^{1/2} \otimes \mathcal{L}) = \begin{cases} c_1(\mathcal{L}), & \text{if } c_1(\mathcal{L}) \geqslant 0 \\ 0, & \text{if } c_1(\mathcal{L}) < 0, \end{cases} \tag{2.9}$$

where $K_\Sigma^{1/2}$ is the spin bundle of Σ and the Serre duality [109, 110] has been used. Eq. (2.9) will be useful to calculate the spectrum from curves.

C. Matter Spectrum

In the vicinity of S, the geometry of X may be regarded as an ALE fibration over S [112–117]. The singularity of the ALE fiberation determines the gauge group G_S of the eight-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory. Let us consider this eight-dimensional $\mathcal{N} = 1$ gauge theory compactified on S. To obtain unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions, it was shown [14–16] that a gauge connection A

and an adjoint Higgs field Φ have to satisfy the following BPS equations:

$$\begin{cases} F_A \wedge \omega_S + \frac{i}{2}[\Phi^\dagger, \Phi] = 0 \\ F_A^{2,0} = F_A^{0,2} = 0 \\ \bar{\partial}_A \Phi = 0, \end{cases} \tag{2.10}$$

where F_A is the curvature two-form of A and ω_S is a Kähler form of S. To solve BPS equations, one may take V as a holomorphic vector bundle over S with the connection A and Φ being holomorphic. The simplest solution for (A, Φ) is that Φ is diagonal and V is a polystable bundle. In this case $[\Phi^\dagger, \Phi] = 0$ and Eq. (2.10) is then reduced to the Hermitian Yang-Mills (HYM) equations

$$F_A^{2,0} = F_A^{0,2} = 0, \qquad\qquad F_A \wedge \omega_S = 0. \tag{2.11}$$

It was shown in [118,119] that a bundle admitting a hermitian connection solving Eq. (2.11) is equivalent to a polystable bundle, which is guaranteed by the Donaldson-Uhlebeck-Yau theorem. We shall in the next section define the stability of vector bundles and briefly review some facts about the equivalence. In this case, the low energy spectrum is therefore decoupled to Φ and only depends on the Hermitian Yang-Mills connection A. The eigenvalues of Φ characterize the locations of intersecting seven-branes. The unbroken gauge group in four dimensions is the commutant Γ_S of H_S in G_S, where H_S is the structure group of the bundle V. The spectrum from the bulk is given by the bundle-valued cohomology groups $H^i_{\bar{\partial}}(S, R_k)$ and their duals, where $R_k = V,\ \wedge^k V$, or EndV. The spectrum of the bulk transforms in the adjoint representation of G_S. The decomposition of adG_S into representations of $\Gamma_S \times H_S$ is

$$\mathrm{ad}G_S = \bigoplus_k \rho_k \otimes \mathcal{R}_k, \tag{2.12}$$

where ρ_k and $\mathcal{R}_k$ are representations of Γ_S and H_S, respectively. The matter fields are determined by the zero modes of the Dirac operator on S. It was shown in [15,16] that the chiral and anti-chiral spectrum is determined by the bundle-valued cohomology groups

$$H^0_{\bar{\partial}}(S, R_k^\vee)^\vee \oplus H^1_{\bar{\partial}}(S, R_k) \oplus H^2_{\bar{\partial}}(S, R_k^\vee)^\vee \tag{2.13}$$

and

$$H^0_{\bar{\partial}}(S, R_k) \oplus H^1_{\bar{\partial}}(S, R_k^\vee)^\vee \oplus H^2_{\bar{\partial}}(S, R_k) \tag{2.14}$$

respectively, where $\vee$ stands for the dual bundle and R_k is the vector bundle on S whose sections transform in the representation $\mathcal{R}_k$ of the structure group H_S. By the vanishing theorem of del Pezzo surfaces [15], the number of chiral fields ρ_k and anti-chiral fields ρ_k^* can be calculated by

$$n_{\rho_k} = -\chi(S, R_k) \tag{2.15}$$

and

$$n_{\rho_k^*} = -\chi(S, R_k^\vee), \tag{2.16}$$

respectively. In particular, when $V = L_1 \oplus L_2$ with structure group $U(1) \times U(1)$, according to Eq. (2.15), the chiral spectrum of $\rho_{r,s}$ is determined by

$$n_{\rho_{r,s}} = -\chi(S, {L_1}^r \otimes L_2^s), \tag{2.17}$$

where r and s correspond respectively to the $U(1)_1$ and $U(1)_2$ charges of the representations in the group theory decomposition. In order to preserve supersymmetry, the gauge bundle V has to obey the HYM equations (2.11), which is equivalent to the polystability conditions, namely

$$\omega_S \wedge c_1(L_1) = \omega_S \wedge c_1(L_2) = 0, \tag{2.18}$$

where ω_S is the Kähler form on S. We will discuss the polystability conditions in more detail in section E.

Another way to obtain chiral matter is from intersecting seven-branes along a curve, which is a Riemann surface. Let S and S' be two components of the discriminant locus Δ with gauge groups G_S and $G_{S'}$, respectively. The gauge group on the curve Σ will be enhanced to G_Σ, where $G_\Sigma \supset G_S \times G_{S'}$. Therefore, chiral matter appears as the bi-fundamental representations in the decomposition of adG_Σ

$$\mathrm{ad}G_\Sigma = \mathrm{ad}G_S \oplus \mathrm{ad}G_{S'} \oplus_k (\mathcal{U}_k \otimes \mathcal{U}'_k). \tag{2.19}$$

As mentioned above, the presence of H_S and $H_{S'}$ will break $G_S \times G_{S'}$ to the commutant subgroup when non-trivial gauge bundles on S and S' with structure groups H_S and $H_{S'}$ are turned on. Let $\Gamma = \Gamma_S \times \Gamma_{S'}$ and $H = H_S \times H_{S'}$, the decomposition of $\mathcal{U} \otimes \mathcal{U}'$ into irreducible representation is

$$\mathcal{U} \otimes \mathcal{U}' = \bigoplus_k (v_k, \mathcal{V}_k), \tag{2.20}$$

where v_k and $\mathcal{V}_k$ are representations of Γ and H, respectively. The light chiral fermions in the representation v_k are determined by the zero modes of the Dirac operator on Σ. It is shown in [15,16] that the net number of chiral fields v_k and anti-chiral fields v_k^* is given by

$$N_{v_k} \equiv n_{v_k} - n_{v_k^*} = \chi(\Sigma, K_\Sigma^{1/2} \otimes V_k), \tag{2.21}$$

where V_k is the vector bundle whose sections transform in the representation $\mathcal{V}_k$ of the structure group H. In particular, if H_S and $H_{S'}$ are $U(1) \times U(1)$ and $U(1)$, respectively, G_Σ can be broken into $G_M \times U(1) \times U(1) \times U(1) \subset G_S \times U(1)$. In this case, the bi-fundamental representations in Eq. (2.19) will be decomposed into

$$\bigoplus_j (\sigma_j)_{r_j,s_j,r'_j}, \tag{2.22}$$

where r_j, s_j and r'_j correspond to the $U(1)$ charges of the representations in the group theory decomposition and σ_j are representations in G_M. The representations $(\sigma_j)_{r_j,s_j,r'_j}$ are localized on Σ [15,16,51] and as shown in [15,16], the generation number of the representations $(\sigma_j)_{r_j,s_j,r'_j}$ and $(\bar{\sigma}_j)_{-r_j,-s_j,-r'_j}$ can be calculated by

$$n_{(\sigma_j)_{r_j,s_j,r'_j}} = h^0(\Sigma, K_\Sigma^{1/2} \otimes L_{1\Sigma}^{r_j} \otimes L_{2\Sigma}^{s_j} \otimes L'^{r'_j}_{\Sigma}) \tag{2.23}$$

and

$$n_{(\bar{\sigma}_j)_{-r_j,-s_j,-r'_j}} = h^0(\Sigma, K_\Sigma^{1/2} \otimes L_{1\Sigma}^{-r_j} \otimes L_{2\Sigma}^{-s_j} \otimes L'^{-r'_j}_{\Sigma}), \tag{2.24}$$

where $L_{1\Sigma} \equiv L_1|_\Sigma$, $L_{2\Sigma} \equiv L_2|_\Sigma$, and $L'_\Sigma \equiv L'|_\Sigma$ are the restrictions of the line bundles L_1, L_2 and L' to the curve Σ, respectively. Note that from Eq. (2.9) below, if $c_1(L_{1\Sigma}^{r_j} \otimes L_{2\Sigma}^{s_j} \otimes L'^{r'_j}_{\Sigma}) = 0$, then $N_{(\sigma_j)_{r_j,s_j,r'_j}} = N_{(\bar{\sigma}_j)_{-r_j,-s_j,-r'_j}} = 0$. If $c_1(L_{1\Sigma}^{r_j} \otimes L_{2\Sigma}^{s_j} \otimes L'^{r'_j}_{\Sigma}) \neq 0$, then only one of them is non-vanishing. Using these properties, we can solve the doublet-triplet splitting problem with suitable line bundles. In addition to the analysis of the spectrum, the pattern of Yukawa couplings also has been studied [14–16, 62]. By the vanishing theorem of del Pezzo surfaces [15,16], Yukawa couplings can form in two different ways. In the first way, the coupling comes from the interaction between two fields on the curves and one field on the bulk S. In the second way, all three fields are localized on the curves which intersect at a point where the gauge group G_p is further enhanced by two ranks. Recently, flavor physics in F-theory models has been studied in [?, 20, 22, 23, 28, 29, 36, 37, 62, 70]. When one turns on bulk three-form fluxes, the structure of the Yukawa couplings will be distorted and non-commutative geometry will emerge [37]. The case of rk(V) = 1 and minimal $SU(5)$ GUT model has been studied in [14–16]. In this article, we shall focus on the case that V is a polystable bundle of rank two. We will study non-minimal cases, namely $G_S = SU(6)$ and $SO(10)$, with these rank two polystable bundles and the spectrum of the MSSM.

D. $U(1)$ Gauge Fluxes

In this section we briefly review some ingredients of $SU(5)$ GUT Models with $G_S = SU(5)$, $SU(10)$ and $SU(6)$. In these models, we introduce a non-trivial $U(1)$ gauge flux to break gauge group G_S. We are primarily interested in doublet-triple splitting and an exotic-free spectrum of the MSSM. From now on, unless otherwise stated, the del Pezzo surface S is assumed to be dP_8.

1. $G_S = SU(5)$

Before discussing the case of $G_S = SO(10)$, $SU(6)$, let us review the case of $G_S = SU(5)$ [14–16]. On the bulk, we consider the following breaking pattern [120]:

$$\begin{aligned} SU(5) &\rightarrow SU(3) \times SU(2) \times U(1)_S \\ \mathbf{24} &\rightarrow (\mathbf{8},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0 + (\mathbf{3},\mathbf{2})_{-5} + (\bar{\mathbf{3}},\mathbf{2})_5 + (\mathbf{1},\mathbf{1})_0. \end{aligned} \tag{2.25}$$

The bulk zero modes are given by

$$(\mathbf{3},\mathbf{2})_{-5} \in H^0_{\bar\partial}(S,L^5)^\vee \oplus H^1_{\bar\partial}(S,L^{-5}) \oplus H^2_{\bar\partial}(S,L^5)^\vee \tag{2.26}$$

$$(\bar{\mathbf{3}},\mathbf{2})_5 \in H^0_{\bar\partial}(S,L^{-5})^\vee \oplus H^1_{\bar\partial}(S,L^5) \oplus H^2_{\bar\partial}(S,L^{-5})^\vee, \tag{2.27}$$

where $\vee$ stands for the dual and L is the supersymmetric line bundle associated with $U(1)_S$. Let $n_{(\mathbf{A},\mathbf{B})_c}$ be the number of the fields in the representation $(\mathbf{A},\mathbf{B})_c$ under $SU(3)\times SU(2)\times U(1)_S$, where c is the charge of $U(1)_S$. Note that $(\mathbf{3},\mathbf{2})_{-5}$ and $(\bar{\mathbf{3}},\mathbf{2})_5$ are exotic fields in the MSSM. In order to eliminate the exotic fields $(\mathbf{3},\mathbf{2})_{-5}$ and $(\bar{\mathbf{3}},\mathbf{2})_5$, it is required that $\chi(S,L^{\pm 5}) = 0$. It follows from the Riemann-Roch theorem

(2.5) that $c_1(L^{\pm 5})^2 = -2$ and $c_1(L^{\pm 5})$ correspond to a root of E_8, $e_i - e_j,\ i \neq j$, which leads to a fractional line bundle[11] $L = \mathcal{O}_S(e_i - e_j)^{\pm 1/5}$ [14–16]. In this case, all matter fields must come from the curves. Now we turn to the spectrum from the curves. In general, the gauge groups on the curves will be enhanced at least by one rank. With $G_S = SU(5)$, the gauge groups on the curves G_Σ can be enhanced to $SU(6)$ or $SO(10)$ [51]. We first focus on the curves supporting the matter fields in an $SU(5)$ GUT. To obtain complete matter multiples of the $SU(5)$ GUT, it is required that $L_\Sigma = \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$, where L' is a line bundle associated with $U(1)'$. Consider the following breaking patterns:

$$\begin{array}{rcl} SU(6) & \rightarrow & SU(5) \times U(1)' \\ \mathbf{35} & \rightarrow & \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{5}_6 + \bar{\mathbf{5}}_{-6} \end{array} \tag{2.28}$$

$$\begin{array}{rcl} SO(10) & \rightarrow & SU(5) \times U(1)' \\ \mathbf{45} & \rightarrow & \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_4 + \overline{\mathbf{10}}_{-4}. \end{array} \tag{2.29}$$

From the patterns (2.28) and (2.29), it can be seen by counting the dimension of the adjoint representations that matter fields $\mathbf{5}_6$ and $\bar{\mathbf{5}}_{-6}$ are localized on the curves with $G_\Sigma = SU(6)$ while $\mathbf{10}_4$ and $\overline{\mathbf{10}}_{-4}$ are localized on the curve with $G_\Sigma = SO(10)$. The Higgs fields localize on the curves with $G_\Sigma = SU(6)$ as well. Since on the matter curves L_Σ is required to be trivial, the only line bundle used to determine the spectrum is L'_Σ. With non-trivial L'_Σ, it is not difficult to engineer three copies of the matter fields, $3 \times \mathbf{5}_6$, $3 \times \bar{\mathbf{5}}_{-6}$, and $3 \times \mathbf{10}_4$. In order to get doublet-triplet splitting, it is required that $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. With non-trivial L_Σ and L'_Σ, G_Σ will be

[11] In this chapter, all indices appearing in the divisors will be assumed to be distinct unless otherwise stated.

Q	u^c	d^c	e^c	L	$\bar{h}$	h
$(\mathbf{3},\mathbf{2})_{1,4}$	$(\bar{\mathbf{3}},\mathbf{1})_{-4,4}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-6}$	$(\mathbf{1},\mathbf{1})_{6,4}$	$(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}$	$(\mathbf{1},\mathbf{2})_{3,6}$	$(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}$

Table III. Field content of the MSSM from $G_S = SU(5)$.

broken into $G_{\text{std}} \times U(1)'$. Consider the following breaking patterns,

$$\begin{aligned} SU(6) &\rightarrow SU(3) \times SU(2) \times U(1)_S \times U(1)' \\ \mathbf{35} &\rightarrow (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\bar{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{2})_{3,6} + (\mathbf{3},\mathbf{1})_{-2,6} + (\mathbf{1},\bar{\mathbf{2}})_{-3,-6} + (\bar{\mathbf{3}},\mathbf{1})_{2,-6} \end{aligned} \tag{2.30}$$

$$\begin{aligned} SO(10) &\rightarrow SU(3) \times SU(2) \times U(1)_S \times U(1)' \\ \mathbf{45} &\rightarrow (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\bar{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0} + [(\mathbf{3},\mathbf{2})_{1,4} + (\bar{\mathbf{3}},\mathbf{1})_{-4,4} + (\mathbf{1},\mathbf{1})_{6,4} + c.c]. \end{aligned} \tag{2.31}$$

From the patterns (2.30) and (2.31), the field content of the MSSM is identified as shown in Table III.

The superpotential is as follows:

$$\mathcal{W}_{\text{MSSM}} \supset Qu^c\bar{h} + Qd^ch + Le^ch + \cdots . \tag{2.32}$$

Note that the $U(1)_S$ in the patterns is consistent with $U(1)_Y$ in the MSSM and that this is the only way to consistently identify the fields in the patterns (2.30) and (2.31) with the MSSM. Now we are going to analyze the conditions for the curves to support the field content in Table III. We choose the curve $\Sigma_{SU(6)}$ to be a genus zero curve and let $(m_1, m_2) = (n_{(\bar{\mathbf{3}},\mathbf{1})_{2,-6}}, n_{(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}})$, where $n_{(\mathbf{A},\mathbf{B})_{a,b}}$ is the number of the fields in the representation $(\mathbf{A},\mathbf{B})_{a,b}$ under $SU(3) \times SU(2) \times U(1)_S \times U(1)'$, and a, b are the charges of $U(1)_S$ and $U(1)'$, respectively. Note that $(\mathbf{3},\mathbf{1})_{-2,6}$ is exotic in the MSSM.

Multiplet	(m_1, m_2)	Conditions	Σ
$3 \times d^c$	(3, 0)	$(e_i - e_j) \cdot \Sigma = -3$	$5h - 4e_j - e_i$
$3 \times L$	(0, 3)	$(e_i - e_j) \cdot \Sigma = 3$	$4h + 2e_j - e_i$
$1 \times h$	(0, 1)	$(e_i - e_j) \cdot \Sigma = 1$	$h - e_i - e_l$
$1 \times \bar{h}$	(0,-1)	$(e_i - e_j) \cdot \Sigma = -1$	$h - e_j - e_s$

Table IV. Field content of the $SU(6)$ Curve from $G_S = SU(5)$.

To avoid the exotic, we require that $m_1 \in \mathbb{Z}_{\geqslant 0}$. Given (m_1, m_2), the homological class of the curve $\Sigma_{SU(6)}$ has to satisfy the following equation:[12]

$$(e_i - e_j) \cdot \Sigma_{SU(6)} = m_2 - m_1, \tag{2.33}$$

where $L = \mathcal{O}_S(e_j - e_i)^{1/5}$ has been used. By Eq. (2.33), we can engineer three copies of d_R, three copies of L_L, one copy of H_d, and one copy of H_u on the individual curves as shown in Table IV.

Note that all field configurations in Table IV obey the conditions, $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. In local models, the curves are required to be effective. With Table II, it is not difficult to check that all curves in Table IV are effective. The results in Table IV show that the triplet and double states in $\mathbf{5}_6$ or $\bar{\mathbf{5}}_{-6}$ of $SU(5)$ can be separated by the restrictions of the supersymmetric line bundles to the curves. Next let us turn to the curve with $G_\Sigma = SO(10)$. Set $(l_1, l_2, l_3) = (n_{(\mathbf{3},\mathbf{2})_{1,4}}, n_{(\bar{\mathbf{3}},\mathbf{1})_{-4,4}}, n_{(\mathbf{1},\mathbf{1})_{6,4}})$. To avoid exotics in the MSSM, it is required that $l_k \in \mathbb{Z}_{\geqslant 0},\ k = 1, 2, 3$. Given (l_1, l_2, l_3), the

[12] $L_{\Sigma_{SU(6)}} = \mathcal{O}_{\Sigma_{SU(6)}}(\frac{(m_1 - m_2)}{5})$ and $L'_{\Sigma_{SU(6)}} = \mathcal{O}_{\Sigma_{SU(6)}}(-\frac{(3m_1 + 2m_2)}{30})$

curve $\Sigma_{SO(10)}$ has to satisfy the following equations:[13]

$$\begin{cases} (e_i - e_j)\cdot \Sigma_{SO(10)} = l_2 - l_1 \\ l_3 = 2l_1 - l_2. \end{cases} \tag{2.34}$$

To obtain the minimal spectrum of the MSSM, we require that $l_1, l_2 \leqslant 3$. Taking the conditions, $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$ into account, we have the following configurations:

$$(l_1, l_2, l_3) = \left\{ \ (1,2,0), (1,0,2), (2,1,3), (2,3,1) \ \right\}. \tag{2.35}$$

From the configurations in (2.35), it is clear that unlike with $G_\Sigma = SU(6)$, it is impossible to engineer the matter fields $3\times Q$, $3\times u^c$, and $3\times e^c$ on the individual curves with $G_\Sigma = SO(10)$, which correspond to $(l_1, l_2, l_3) = (3,0,0)$, $(0,3,0)$, and $(0,0,3)$, respectively, without extra matter fields. Fortunately, in this case all Higgs fields come from $\Sigma_{SU(6)}$ instead of $\Sigma_{SO(10)}$. Although the field content on $\Sigma_{SO(10)}$ is more complicated than that on $\Sigma_{SU(6)}$, we can engineer the spectrum of the MSSM as shown in Table V.

From Table V, we find that for the case of $G_S = SU(5)$, we can get an exotic-free, minimal spectrum of the MSSM with doublet-triplet splitting. In addition, by arranging $\bar{h}$ and h on different curves, rapid proton decay can be avoided [14–16].

2. $G_S = SO(10)$

For the case of $G_S = SO(10)$ [33], we first look at the spectrum from the bulk. Consider the following breaking pattern,

$$\begin{aligned} SO(10) &\ \rightarrow\ SU(5)\times U(1)_S \\ \mathbf{45} &\ \rightarrow\ \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_4 + \overline{\mathbf{10}}_{-4}. \end{aligned} \tag{2.36}$$

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[13] $L_{\Sigma_{SO(10)}} = \mathcal{O}_{\Sigma_{SO(10)}}(\frac{(l_1-l_2)}{5})$ and $L'_{\Sigma_{SO(10)}} = \mathcal{O}_{\Sigma_{SO(10)}}(\frac{(4l_1+l_2)}{20})$

Multiplet	Curve	Σ	g_Σ	L_Σ	L'_Σ
$1\times Q+2\times u^c$	$\Sigma^1_{SO(10)}$	$2h-e_2-e_3$	0	$\mathcal{O}_{\Sigma^1_{SO(10)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^1_{SO(10)}}(1)^{3/10}$
$2\times Q+1\times u^c$ $+3\times e^c$	$\Sigma^2_{SO(10)}$	$2h-e_1-e_4$	0	$\mathcal{O}_{\Sigma^2_{SO(10)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{SO(10)}}(1)^{9/20}$
$3\times d^c$	$\Sigma^1_{SU(6)}$	$5h-4e_1-e_2$	0	$\mathcal{O}_{\Sigma^1_{SU(6)}}(1)^{3/5}$	$\mathcal{O}_{\Sigma^1_{SU(6)}}(-1)^{3/10}$
$3\times L$	$\Sigma^2_{SU(6)}$	$4h+2e_1-e_2$	0	$\mathcal{O}_{\Sigma^2_{SU(6)}}(-1)^{3/5}$	$\mathcal{O}_{\Sigma^2_{SU(6)}}(-1)^{1/5}$
$1\times h$	$\Sigma^d_{SU(6)}$	$2h-e_2-e_4$	0	$\mathcal{O}_{\Sigma^d_{SU(6)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^d_{SU(6)}}(-1)^{1/15}$
$1\times \bar{h}$	$\Sigma^u_{SU(6)}$	$h-e_1-e_3$	0	$\mathcal{O}_{\Sigma^u_{SU(6)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^u_{SU(6)}}(1)^{1/15}$

Table V. A minimal spectrum of the MSSM from $G_S = SU(5)$, where $L=\mathcal{O}_S(e_1-e_2)^{1/5}$.

The bulk zero modes are determined by

$$\mathbf{10}_4 \in H^0_{\bar\partial}(S,L^{-4})^\vee \oplus H^1_{\bar\partial}(S,L^4)\oplus H^2_{\bar\partial}(S,L^{-4})^\vee \tag{2.37}$$

$$\overline{\mathbf{10}}_{-4} \in H^0_{\bar\partial}(S,L^{4})^\vee \oplus H^1_{\bar\partial}(S,L^{-4})\oplus H^2_{\bar\partial}(S,L^{4})^\vee . \tag{2.38}$$

To eliminate $\mathbf{10}_4$ and $\overline{\mathbf{10}}_{-4}$, it is required that $\chi(S,L^{\pm4})=0$, which give rise to the fractional line bundles $L=\mathcal{O}_S(e_i-e_j)^{\pm1/4}$. In this case, all chiral fields must come from the curves. Let us turn to the spectrum from the curves. With $G_S=SO(10)$, the gauge groups on the curve can be enhanced to $G_\Sigma=SO(12)$ or $G_\Sigma=E_6$. The

Multiplet	Curve	Σ	g_Σ	L_Σ	L'_Σ
$3\times\mathbf{10}_{-3,-1}$[14]	$\Sigma^1_{E_6}$	$4h+2e_1-e_2$	0	$\mathcal{O}_{\Sigma^1_{E_6}}(-1)^{3/4}$	$\mathcal{O}_{\Sigma^1_{E_6}}(-1)^{3/4}$
$3\times\bar{\mathbf{5}}_{-3,3}$[15]	$\Sigma^2_{E_6}$	$5h+3e_2-e_5$	0	$\mathcal{O}_{\Sigma^2_{E_6}}(1)^{3/4}$	$\mathcal{O}_{\Sigma^2_{E_6}}(-1)^{1/4}$
$1\times\mathbf{5}_{-2,2}$	$\Sigma^1_{SO(12)}$	$3h+e_3-e_1$	0	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{1/4}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(-1)^{1/4}$
$1\times\bar{\mathbf{5}}_{2,-2}$	$\Sigma^2_{SO(12)}$	$h-e_2-e_3$	0	$\mathcal{O}_{\Sigma^2_{SO(12)}}(-1)^{1/4}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/4}$

Table VI. An $SU(5)$ GUT model from $G_S = SO(10)$, where $L = \mathcal{O}_S(e_1-e_2)^{1/4}$.

breaking chains and matter content from the enhanced adjoints of the curves are

$$\begin{aligned} SO(12) &\rightarrow SO(10)\times U(1)' \rightarrow SU(5)\times U(1)'\times U(1)_S \\ \mathbf{66} &\rightarrow \mathbf{45}_0+\mathbf{1}_0 \rightarrow \mathbf{24}_{0,0}+\mathbf{1}_{0,0}+\mathbf{10}_{0,4}+\overline{\mathbf{10}}_{0,-4}+\mathbf{1}_{0,0} \\ &\quad +\mathbf{10}_2+\overline{\mathbf{10}}_{-2} \qquad +\mathbf{5}_{2,2}+\bar{\mathbf{5}}_{2,-2}+\bar{\mathbf{5}}_{-2,-2}+\mathbf{5}_{-2,2} \end{aligned} \tag{2.39}$$

$$\begin{aligned} E_6 &\rightarrow SO(10)\times U(1)' \rightarrow SU(5)\times U(1)'\times U(1)_S \\ \mathbf{78} &\rightarrow \mathbf{45}_0+\mathbf{1}_0 \rightarrow \mathbf{24}_{0,0}+\mathbf{1}_{0,0}+\mathbf{10}_{0,4}+\overline{\mathbf{10}}_{0,-4}+\mathbf{1}_{0,0} \\ &\quad +\mathbf{16}_{-3}+\overline{\mathbf{16}}_3 \qquad +(\mathbf{10}_{-3,-1}+\bar{\mathbf{5}}_{-3,3}+\mathbf{1}_{-3,-5}+c.c.). \end{aligned} \tag{2.40}$$

Note that the $U(1)_S$ charges of the fields localized on the curves should be conserved in each Yukawa coupling. The superpotential is as follows:

$$\mathcal{W} \supset \mathbf{10}_{-3,-1}\mathbf{10}_{-3,-1}\mathbf{5}_{-2,2}+\mathbf{10}_{-3,-1}\bar{\mathbf{5}}_{-3,3}\bar{\mathbf{5}}_{2,-2}+\cdots . \tag{2.41}$$

In order to get complete matter multiplets in the $SU(5)$ GUT, we require that L_Σ and L'_Σ are both non-trivial. With non-trivial L_Σ and L'_Σ, we can engineer field

[14]With six additional singlets

[15]With three additional singlets

content with minimal singlets as shown in Table VI [33].

However, because of the lack of extra $U(1)$ gauge fluxes or Wilson lines, the doublet-triplet splitting is not achievable in the present case. This motivates us to consider supersymmetric $U(1)^2$ fluxes.

3. $G_S = SU(6)$

To look at the spectrum from the bulk , we consider the following breaking pattern,

$$\begin{aligned} SU(6) &\rightarrow SU(5) \times U(1)_S \\ \mathbf{45} &\rightarrow \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{5}_6 + \bar{\mathbf{5}}_{-6}. \end{aligned} \tag{2.42}$$

The bulk zero modes are given by

$$\mathbf{5}_6 \in H^0_{\bar{\partial}}(S, L^{-6})^\vee \oplus H^1_{\bar{\partial}}(S, L^6) \oplus H^2_{\bar{\partial}}(S, L^{-6})^\vee \tag{2.43}$$

$$\bar{\mathbf{5}}_{-6} \in H^0_{\bar{\partial}}(S, L^6)^\vee \oplus H^1_{\bar{\partial}}(S, L^{-6}) \oplus H^2_{\bar{\partial}}(S, L^6)^\vee. \tag{2.44}$$

To eliminate $\mathbf{5}_6$ and $\bar{\mathbf{5}}_{-6}$, it is required that $\chi(S, L^{\pm 6}) = 0$, which gives rise to the fractional line bundles $L = \mathcal{O}_S(e_i - e_j)^{\pm 1/6}$ [33]. In this case, all chiral fields must come from the curves. Let us turn to the spectrum from the curves. With $G_S = SU(6)$, the gauge groups on the curve can be enhanced to $G_\Sigma = SU(7)$, $G_\Sigma = SO(12)$ or $G_\Sigma = E_6$.

The breaking chains and matter content from the enhanced adjoints of the curves are

$$\begin{aligned} SU(7) &\rightarrow SU(6) \times U(1)' &&\rightarrow SU(5) \times U(1)' \times U(1)_S \\ \mathbf{48} &\rightarrow \mathbf{35}_0 + \mathbf{1}_0 + \mathbf{6}_{-7} + \bar{\mathbf{6}}_7 &&\rightarrow \mathbf{24}_{0,0} + \mathbf{1}_{0,0} + \mathbf{5}_{0,6} + \bar{\mathbf{5}}_{0,-6} + \mathbf{1}_{0,0} \\ & && \quad + \mathbf{5}_{-7,1} + \mathbf{1}_{-7,-5} + \bar{\mathbf{5}}_{7,-1} + \mathbf{1}_{7,5} \end{aligned} \tag{2.45}$$

Multiplet	Curve	Σ	g_Σ	L_Σ	L'_Σ
$3\times \mathbf{10}_{2,2}$	$\Sigma^1_{SO(12)}$	$4h+2e_2-e_1$	0	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{1/2}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)$
$3\times \bar{\mathbf{5}}_{7,-1}$	$\Sigma^1_{SU(7)}$	$5h+3e_1-e_6$	0	$\mathcal{O}_{\Sigma^1_{SU(7)}}(-1)^{1/2}$	$\mathcal{O}_{\Sigma^1_{SU(7)}}(1)^{5/14}$
$1\times \mathbf{5}_{2,-4}$	$\Sigma^2_{SO(12)}$	$3h+e_1-e_3$	0	$\mathcal{O}_{\Sigma^2_{SO(12)}}(-1)^{1/6}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/6}$
$1\times \bar{\mathbf{5}}_{7,-1}$	$\Sigma^2_{SU(7)}$	$h-e_2-e_3$	0	$\mathcal{O}_{\Sigma^2_{SU(7)}}(-1)^{1/6}$	$\mathcal{O}_{\Sigma^2_{SU(7)}}(1)^{5/42}$

Table VII. An $SU(5)$ GUT model from $G_S = SU(6)$, where $L = \mathcal{O}_S(e_1-e_2)^{1/6}$.

$$\begin{aligned} SO(12) &\to SU(6)\times U(1)' &&\to SU(5)\times U(1)'\times U(1)_S \\ \mathbf{66} &\to \mathbf{35}_0+\mathbf{1}_0+\mathbf{15}_2+\overline{\mathbf{15}}_{-2} &&\to \mathbf{24}_{0,0}+\mathbf{1}_{0,0}+\mathbf{5}_{0,6}+\bar{\mathbf{5}}_{0,-6}+\mathbf{1}_{0,0} \\ & && \quad +\mathbf{10}_{2,2}+\mathbf{5}_{2,-4}+\overline{\mathbf{10}}_{-2,-2}+\bar{\mathbf{5}}_{-2,4} \end{aligned} \tag{2.46}$$

$$\begin{aligned} E_6 &\to SU(6)\times U(1)' &&\to SU(5)\times U(1)'\times U(1)_S \\ \mathbf{78} &\to \mathbf{35}_0+\mathbf{1}_0+\mathbf{1}_{\pm 2} &&\to \mathbf{24}_{0,0}+2\times\mathbf{1}_{0,0}+\mathbf{5}_{0,6}+\bar{\mathbf{5}}_{0,-6}+\mathbf{1}_{\pm 2,0} \\ & \quad +\mathbf{20}_1+\mathbf{20}_{-1} && \quad +\mathbf{10}_{1,-3}+\overline{\mathbf{10}}_{1,3}+\mathbf{10}_{-1,-3}+\overline{\mathbf{10}}_{-1,3}. \end{aligned} \tag{2.47}$$

In this case, the $U(1)_S$ charges of the fields localized on the curves should be conserved in each Yukawa coupling. The superpotential is:

$$\mathcal{W} \supset \mathbf{10}_{2,2}\mathbf{10}_{2,2}\mathbf{5}_{2,-4}+\mathbf{10}_{2,2}\bar{\mathbf{5}}_{7,-1}\bar{\mathbf{5}}_{7,-1}+\cdots. \tag{2.48}$$

With non-trivial L_Σ and L'_Σ, we can engineer configurations of the curves with desired field content but without any exotic fields as shown in Table VII [33].

Although in this case one can obtain an exotic-free spectrum in an $SU(5)$ GUT, the doublet-triplet splitting can not be achieved, similar to the case of $G_S = SO(10)$. Again this motivates us to consider supersymmetric $U(1)^2$ gauge fluxes. On the other hand, to get the spectrum of the MSSM, we also need some mechanisms to break $SU(5) \subset G_\Sigma$ into $SU(3) \times SU(2) \times U(1)_Y$. One possible way is to consider supersymmetric $U(1)^2$ gauge fluxes instead of U(1) fluxes. These supersymmetric $U(1)^2$ gauge fluxes correspond to polystable bundles of rank two with structure group $U(1)^2$. In the next section we shall discuss polystable bundles of rank two.

E. Gauge Bundles

In this section we shall briefly review the notion of stability of the vector bundle and the relation between polystable bundles and the HYM equations. In addition, we also discuss the semi-stable bundles of rank two, in particular, polystable bundles over S.

1. Stability

Let E be a holomorphic vector bundle over a projective surface S and $[\omega_S]$ be the dual ample divisor of Kähler form ω_S in the Kähler cone. The slope $\mu_{[\omega_S]}(E)$ is defined by

$$\mu_{[\omega_S]}(E) = \frac{\int_S c_1(E) \wedge \omega_S}{\text{rk}(E)}. \tag{2.49}$$

The vector bundle E is (semi)stable if for every subbundle or subsheaf $\mathcal{E}$ with $\text{rk}(\mathcal{E}) < \text{rk}(E)$, the following inequality holds

$$\mu[\omega_S](\mathcal{E}) < (\leqslant) \mu_{[\omega_S]}(E). \tag{2.50}$$

Assume that $E = \oplus_i^k \mathcal{E}_i$, then E is polystable if each $\mathcal{E}_i$ is a stable bundle with $\mu_{[\omega_S]}(\mathcal{E}_1) = \mu_{[\omega_S]}(\mathcal{E}_2) = ... = \mu_{[\omega_S]}(\mathcal{E}_k)$ [118, 119]. It is clear that every line bundle is stable and polystable bundle is a type of semistable bundle. The Donaldson-Uhlenbeck-Yau theorem [118,119] states that a (split) irreducible holomorphic bundle E admits a hermitian connection satisfying Eq. (2.11) if and only if E is (poly)stable. As mentioned in section 2.1, to preserve supersymmetry, the connection of the bundle has to obey the HYM equations (2.11), which is equivalent to the polystable bundle. In particular, when the bundle is split, supersymmetry requires that the bundle is polystable. In the next section we primarily focus on polystable bundles of rank two over S.

2. Rank Two Polystable Bundle

Here we are interested in the case $S = dP_k$. Consider the case of $V = L_1 \oplus L_2$, where L_1 and L_2 are line bundles over S and set $L_i = \mathcal{O}_S(D_i),\ i = 1, 2$, where D_i are divisors in S. Before writing down a more explicit expression for the bundle V, we first consider the stability condition of the polystable bundle. Recall that the bundle V is polystable if $\mu_{[\omega_S]}(L_1) = \mu_{[\omega_S]}(L_2)$ where μ is the slope defined by Eq. (2.49). To solve the HYM equation Eq. (2.18), it is required that $\mu_{[\omega_S]}(L_1) = \mu_{[\omega_S]}(L_2) = 0$. It follows that $c_1(L_1) \wedge \omega_S = c_1(L_2) \wedge \omega_S = 0$ or equivalently,

$$D_1 \cdot [\omega_S] = D_2 \cdot [\omega_S] = 0. \tag{2.51}$$

In particular, we choose "large volume polarization", namely $[\omega_S] = Ah - \sum_{i=1}^{k} a_i e_i$, $A \gg a_i > 0$ [14,15]. Note that Eq. (2.51) is exactly the BPS equations, $c_1(L_i) \wedge \omega_S = 0,\ i = 1, 2$ for supersymmetric line bundles. So the polystable bundle V is a direct sum of the supersymmetric line bundles L_1 and L_2. In section 5.2 we shall apply physical constraints to the polystable bundle that satisfies the Eq. (2.51) and derive the explicit expression of the $U(1)^2$ gauge fluxes L_1 and L_2.

3. Supersymmetric $U(1)^2$ Gauge Fluxes

Each supersymmetric $U(1)^2$ gauge flux configuration contains two fractional line bundles, which may not be well-defined themselves. It is natural to ask whether it makes sense for these configurations to be polystable vector bundles of rank two. In what follows, we shall show that supersymmetric $U(1)^2$ gauge fluxes can be associated with polystable vector bundles of rank two. Let us consider the case of $G_S = SU(6)$ and the breaking pattern through $SU(6) \to SU(5) \times U(1) \to SU(3) \times SU(2) \times U(1)_1 \times U(1)_2$. Let L_1 and L_2 be two supersymmetric line bundles, which associate to $U(1)_1$ and $U(1)_2$, respectively. Write $L_i = \mathcal{O}_S(D_i)$, $i = 1, 2$, where D_i are in general "$\mathbb{Q}$-divisors" which means that D_i are the linear combinations of the divisors in S with rational coefficients. Now we consider the rotation of the $U(1)$ charges, $U(1)_1$ and $U(1)_2$, given by

$$\widetilde{\mathbb{U}} = \mathrm{M}\mathbb{U} \tag{2.52}$$

with $\mathbb{U} = (U(1)_1, U(1)_2)^t$, $\widetilde{\mathbb{U}} = (\widetilde{U(1)}_1, \widetilde{U(1)}_2)^t$, and $\mathrm{M} \in GL(2, \mathbb{Q})$, where t represents the transpose. We define $\widetilde{L}_1$ and $\widetilde{L}_2$ to be two line bundles which associate to $\widetilde{U(1)}_1$ and $\widetilde{U(1)}_2$, respectively and write $\widetilde{L}_i = \mathcal{O}_S(\widetilde{D}_i)$, $i = 1, 2$. Let $(\mathbf{A}, \mathbf{B})_{c,d}$ and $(\mathbf{A}, \mathbf{B})_{\widetilde{c},\widetilde{d}}$ be representations in the breaking patten $SU(6) \to SU(3) \times SU(2) \times U(1)_1 \times U(1)_2$ and $SU(6) \to SU(3) \times SU(2) \times \widetilde{U(1)}_1 \times \widetilde{U(1)}_2$, respectively. Up to a linear combination of $U(1)$ charges, we have $n_{(\mathbf{A},\mathbf{B})_{c,d}} = n_{(\mathbf{A},\mathbf{B})_{\widetilde{c},\widetilde{d}}}$, which requires that the corresponding divisors be transferred as follows:

$$\widetilde{\mathbb{D}} = (\mathrm{M}^{-1})^t \mathbb{D}, \tag{2.53}$$

where $\mathbb{D} = (D_1, D_2)^t$, $\widetilde{\mathbb{D}} = (\widetilde{D}_1, \widetilde{D}_2)^t$. In general, $\widetilde{D}_i$ are $\mathbb{Q}$-divisors via the rotation (2.53). However, it is possible to get integral divisors $\widetilde{D}_i$ by a suitable choice of the matrix $\mathrm{M} = \mathrm{M}_*$. Once this is done, we obtain two corresponding line bundles, $\widetilde{L}_1$ and $\widetilde{L}_2$ since $\widetilde{D}_i \in H_2(S, \mathbb{Z})$, $i = 1, 2$. Moreover, if $\mu_{[\omega_S]}(\widetilde{L}_1) = \mu_{[\omega_S]}(\widetilde{L}_2) = 0$, we can construct the polystable bundle $V = \widetilde{L}_1 \oplus \widetilde{L}_2$. Note that when L_i are supersymmetric, which means that they satisfy the BPS condition (2.51), by the transformation (2.53) we have $\mu_{[\omega_S]}(\widetilde{L}_1) = \mu_{[\omega_S]}(\widetilde{L}_2) = 0$. As a result, each supersymmetric $U(1)^2$ gauge fluxes is associated with a polystable vector bundle of rank two if the suitable matrix M_* exists. To be concrete, let us consider the case of $G_S = SU(6)$. The breaking pattern via $G_{\rm std} \times U(1)$ is as follows:

$$\begin{aligned} SU(6) &\rightarrow SU(3)\times SU(2)\times U(1)_1\times U(1)_2 \\ \mathbf{35} &\rightarrow (\mathbf{8},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{3})_{0,0}+(\mathbf{3},\mathbf{2})_{-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{5,0}+(\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{2})_{3,6}+(\mathbf{3},\mathbf{1})_{-2,6}+(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{2,-6}. \end{aligned} \tag{2.54}$$

Let L_1 and L_2 be the supersymmetric line bundles associated to $U(1)_1$ and $U(1)_2$, respectively. Note that $U(1)_1$ can be identified as $U(1)_Y$ in the MSSM. The exotic-free spectrum from the bulk requires that L_1 and L_2 are fractional line bundles. The details could be found in section 5.2. Now consider the rotation

$$\mathrm{M} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{10} \\ 0 & \frac{1}{6} \end{pmatrix}. \tag{2.55}$$

Then we obtain

$$\begin{aligned} SU(6) &\rightarrow SU(3)\times SU(2)\times U(1)_1\times U(1)_2 \\ \mathbf{35} &\rightarrow (\mathbf{8},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{3})_{0,0}+(\mathbf{3},\mathbf{2})_{1,0}+(\bar{\mathbf{3}},\mathbf{2})_{-1,0}+(\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{2})_{0,1}+(\mathbf{3},\mathbf{1})_{1,1}+(\mathbf{1},\bar{\mathbf{2}})_{0,-1}+(\bar{\mathbf{3}},\mathbf{1})_{-1,-1} \end{aligned} \tag{2.56}$$

with $\widetilde{L}_1 = L_1^{-5}$ and $\widetilde{L}_2 = L_1^3 \otimes L_2^6$. It is clear that $n_{(\mathbf{A},\mathbf{B})_{c,d}} = n_{(\mathbf{A},\mathbf{B})_{\tilde{c},\tilde{d}}}$ with respect to (2.54) and (2.56). It turns out that $\widetilde{L}_1$ and $\widetilde{L}_2$ are truly line bundles. Furthermore, one can show that BPS condition (2.51) for (L_1, L_2) is equivalent to the stability conditions of the polystable bundle $V = \widetilde{L}_1 \oplus \widetilde{L}_2$ by the transformation (2.53). In this case, we know that supersymmetric $U(1)^2$ gauge fluxes are associated with polystable bundles of rank two with the same number of zero modes charged under $U(1)^2$. With this correspondence, we can avoid talking about the gauge bundle defined by the direct sum of two fractional line bundles. In other words, a supersymmetric $U(1)^2$ gauge flux (L_1, L_2) is well-defined in the sense that it can be associated with a well-defined polystable bundle of rank two. Form now on, we shall simply use the phrase $U(1)^2$ gauge fluxes in stead of polystable bundle in the following sections.

F. $U(1)^2$ Gauge Fluxes

In this section we consider $U(1)^2$ gauge fluxes in local F-theory models, in particular we focus on the case of $G_S = SO(10)$ and $SU(6)$. With the gauge fluxes, G_S can be broken into $G_{\text{std}} \times U(1)$. For the case of $G_S = SO(10)$, there is a no-go theorem which states that there do not exist $U(1)^2$ gauge fluxes such that the spectrum is exotic-free. This result was first shown in [15]. We review the case of $G_S = SO(10)$ in subsection 1 for completeness. For the case of $G_S = SU(6)$, with appropriate physical conditions, we shall show that there are finitely many supersymmetric $U(1)^2$ gauge fluxes with an exotic-free bulk spectrum and we obtain the explicit expression of these gauge fluxes as well. With these explicit flux configurations, we study doublet-triplet splitting and the spectrum of the MSSM. The details can be found in subsection 2 and 3.

1. $G_S = SO(10)$

a. $U(1)^2$ Gauge Flux Configurations

The maximal subgroups of $SO(10)$ which contain G_{std} and the consistent MSSM spectrum are as follows [15]:

$$SO(10) \supset SU(5) \times U(1) \supset G_{\text{std}} \times U(1) \tag{2.57}$$

$$SO(10) \supset SU(2) \times SU(2) \times SU(4) \supset G_{\text{std}} \times U(1) \tag{2.58}$$

For the latter, one of $SU(2)$ groups needs to be broken into $U(1)\times U(1)$ to get the consistent $U(1)_Y$ charge in the MSSM. It follows from the patterns (2.57) and (2.58) that up to linear combinations of the $U(1)$ charges in the breaking patterns, it is enough to analyze the case of $U(1)^2$ gauge fluxes which breaks $SO(10)$ via the sequence $SO(10)\to SU(5)\times U(1)\to G_{\rm std}\times U(1)$. The breaking pattern is as follows:

$$\begin{array}{lcl} SO(10) & \to & SU(3)\times SU(2)\times U(1)_1\times U(1)_2 \\ \mathbf{45} & \to & (\mathbf{8},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{3})_{0,0}+(\mathbf{3},\mathbf{2})_{-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{5,0}+(\mathbf{1},\mathbf{1})_{0,0} \\ & & +(\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{1})_{6,4}+(\bar{\mathbf{3}},\mathbf{1})_{-4,4}+(\mathbf{3},\mathbf{2})_{1,4}+(\mathbf{1},\mathbf{1})_{-6,-4} \\ & & +(\mathbf{3},\mathbf{1})_{4,-4}+(\bar{\mathbf{3}},\mathbf{2})_{-1,-4}. \end{array} \tag{2.59}$$

Note that $U(1)_1$ can be identified with $U(1)_Y$ in the MSSM. Let $\widetilde{L}_3$ and $\widetilde{L}_4$ be non-trivial supersymmetric line bundles associated with $U(1)_1$ and $U(1)_2$, respectively, in the breaking pattern (2.59). The bulk zero modes are given by

$$(\mathbf{3},\mathbf{2})_{-5,0}\in H^0_{\bar\partial}(S,\widetilde{L}_3^5)^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{-5})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^5)^\vee \tag{2.60}$$

$$(\bar{\mathbf{3}},\mathbf{2})_{5,0}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{-5})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{5})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{-5})^\vee \tag{2.61}$$

$$(\mathbf{3},\mathbf{2})_{1,4}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{-1}\otimes\widetilde{L}_4^{-4})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{1}\otimes\widetilde{L}_4^{4})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{-1}\otimes\widetilde{L}_4^{-4})^\vee \tag{2.62}$$

$$(\bar{\mathbf{3}},\mathbf{2})_{-1,-4}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{1}\otimes\widetilde{L}_4^{4})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{-1}\otimes\widetilde{L}_4^{-4})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{1}\otimes\widetilde{L}_4^{4})^\vee \tag{2.63}$$

$$(\mathbf{3},\mathbf{1})_{4,-4}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{-4}\otimes\widetilde{L}_4^{4})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{4}\otimes\widetilde{L}_4^{-4})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{-4}\otimes\widetilde{L}_4^{4})^\vee \tag{2.64}$$

$$(\bar{\mathbf{3}},\mathbf{1})_{-4,4}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{4}\otimes\widetilde{L}_4^{-4})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{-4}\otimes\widetilde{L}_4^{4})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{4}\otimes\widetilde{L}_4^{-4})^\vee, \tag{2.65}$$

$$(\mathbf{1},\mathbf{1})_{6,4}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{-6}\otimes\widetilde{L}_4^{-4})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{6}\otimes\widetilde{L}_4^{4})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{-6}\otimes\widetilde{L}_4^{-4})^\vee \tag{2.66}$$

$$(\mathbf{1},\mathbf{1})_{-6,-4}\in H^0_{\bar\partial}(S,\widetilde{L}_3^{6}\otimes\widetilde{L}_4^{4})^\vee\oplus H^1_{\bar\partial}(S,\widetilde{L}_3^{-6}\otimes\widetilde{L}_4^{-4})\oplus H^2_{\bar\partial}(S,\widetilde{L}_3^{6}\otimes\widetilde{L}_4^{4})^\vee. \tag{2.67}$$

To avoid exotics, it is clear that the line bundles $\widetilde{L}_3^5$, $\widetilde{L}_3^1\otimes\widetilde{L}_4^4$, $\widetilde{L}_3^4\otimes\widetilde{L}_4^{-4}$, and $\widetilde{L}_3^6\otimes\widetilde{L}_4^4$ cannot be trivial. Let $n_{(\mathbf{A},\mathbf{B})_{a,b}}$ be the number of the fields in the representation

$(\mathbf{A},\mathbf{B})_{a,b}$ under $SU(3)\times SU(2)\times U(1)_1\times U(1)_2$, where a and b are the charges of $U(1)_1$ and $U(1)_2$, respectively. By the vanishing theorem (2.6), the exotic-free spectrum requires that

$$n_{(\mathbf{3},\mathbf{2})_{-5,0}} = -\chi(S,E) = 0 \tag{2.68}$$

$$n_{(\bar{\mathbf{3}},\mathbf{2})_{5,0}} = -\chi(S,E^{-1}) = 0 \tag{2.69}$$

$$n_{(\bar{\mathbf{3}},\mathbf{2})_{-1,-4}} = -\chi(S,F^{-1}) = 0 \tag{2.70}$$

$$n_{(\mathbf{3},\mathbf{1})_{4,-4}} = -\chi(S,E^{-1}\otimes F^{-1}) = 0 \tag{2.71}$$

$$n_{(\mathbf{1},\mathbf{1})_{-6,-4}} = -\chi(S,E\otimes F^{-1}) = 0. \tag{2.72}$$

We define

$$n_{(\mathbf{3},\mathbf{2})_{1,4}} = -\chi(S,F) \equiv \beta_1, \tag{2.73}$$

$$n_{(\bar{\mathbf{3}},\mathbf{1})_{-4,4}} = -\chi(S,E\otimes F) \equiv \beta_2 \tag{2.74}$$

$$n_{(\mathbf{1},\mathbf{1})_{6,4}} = -\chi(S,E^{-1}\otimes F) \equiv \beta_3, \tag{2.75}$$

where $E=\widetilde{L}_3^{-5}$, $F=\widetilde{L}_3^{1}\otimes\widetilde{L}_4^{4}$ and $\beta_i\in\mathbb{Z}_{\geqslant 0}$, $i=1,2,3$. By Eqs. (2.68)-(2.70), and Eq. (2.73), we obtain the following equations

$$\begin{cases} c_1(E)^2 = -2 \\ c_1(F)^2 = -\beta_1 - 2 \\ c_1(E)\cdot K_S = 0 \\ c_1(F)\cdot K_S = \beta_1. \end{cases} \tag{2.76}$$

Then by Eq. (4.105) and Eq. (2.71), we obtain

$$c_1(E)\cdot c_1(F) = 1. \tag{2.77}$$

On the other hand, using Eq. (4.105) and Eq. (2.72), we have

$$c_1(E) \cdot c_1(F) = -1, \tag{2.78}$$

which leads to a contradiction. Therefore, there do not exist solutions for given $\beta_i \in \mathbb{Z}_{\geqslant 0},\ i = 1, 2, 3$ such that Eqs. (2.68)-(2.75) hold. This is a no-go theorem shown in [15]. Due to this no-go theorem, we are not going to study this case further. In the next section we turn to the case of $G_S = SU(6)$.

2. $G_S = SU(6)$

a. $U(1)^2$ Gauge Flux Configurations

The maximal subgroups of $SU(6)$ which contain $G_{\rm std}$ and the consistent MSSM spectrum are as follows [15]:

$$SU(6) \supset SU(5) \times U(1) \supset G_{\rm std} \times U(1) \tag{2.79}$$

$$SU(6) \supset SU(2) \times SU(4) \times U(1) \supset G_{\rm std} \times U(1) \tag{2.80}$$

$$SU(6) \supset SU(3) \times SU(3) \times U(1) \supset G_{\rm std} \times U(1). \tag{2.81}$$

It follows from Eqs. (2.79)-(2.81) that up to linear combinations of the $U(1)$ charges in the breaking patterns, it is enough to analyze the case of $U(1)^2$ gauge fluxes which break $SU(6)$ via the sequence $SU(6) \to SU(5) \times U(1) \to G_{\rm std} \times U(1)$. The breaking pattern is as follows:

$$\begin{aligned} SU(6) &\to SU(3) \times SU(2) \times U(1)_1 \times U(1)_2 \\ \mathbf{35} &\to (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\bar{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{2})_{3,6} + (\mathbf{3},\mathbf{1})_{-2,6} + (\mathbf{1},\bar{\mathbf{2}})_{-3,-6} + (\bar{\mathbf{3}},\mathbf{1})_{2,-6}. \end{aligned} \tag{2.82}$$

Note that $U(1)_1$ is consistent with $U(1)_Y$ in the MSSM. Let L_1 and L_2 be non-trivial supersymmetric line bundles associated with $U(1)_1$ and $U(1)_2$, respectively, in the breaking pattern (2.82). The bulk zero modes are given by

$$(\mathbf{3},\mathbf{2})_{-5,0} \in H^0_{\bar\partial}(S, L_1^5)^\vee \oplus H^1_{\bar\partial}(S, L_1^{-5}) \oplus H^2_{\bar\partial}(S, L_1^5)^\vee \tag{2.83}$$

$$(\bar{\mathbf{3}},\mathbf{2})_{5,0} \in H^0_{\bar\partial}(S, L_1^{-5})^\vee \oplus H^1_{\bar\partial}(S, L_1^{5}) \oplus H^2_{\bar\partial}(S, L_1^{-5})^\vee \tag{2.84}$$

$$(\mathbf{1},\mathbf{2})_{3,6} \in H^0_{\bar\partial}(S, L_1^{-3}\otimes L_2^{-6})^\vee \oplus H^1_{\bar\partial}(S, L_1^{3}\otimes L_2^{6}) \oplus H^2_{\bar\partial}(S, L_1^{-3}\otimes L_2^{-6})^\vee \tag{2.85}$$

$$(\mathbf{1},\bar{\mathbf{2}})_{-3,-6} \in H^0_{\bar\partial}(S, L_1^{3}\otimes L_2^{6})^\vee \oplus H^1_{\bar\partial}(S, L_1^{-3}\otimes L_2^{-6}) \oplus H^2_{\bar\partial}(S, L_1^{3}\otimes L_2^{6})^\vee \tag{2.86}$$

$$(\mathbf{3},\mathbf{1})_{-2,6} \in H^0_{\bar\partial}(S, L_1^{2}\otimes L_2^{-6})^\vee \oplus H^1_{\bar\partial}(S, L_1^{-2}\otimes L_2^{6}) \oplus H^2_{\bar\partial}(S, L_1^{2}\otimes L_2^{-6})^\vee \tag{2.87}$$

$$(\bar{\mathbf{3}},\mathbf{1})_{2,-6} \in H^0_{\bar\partial}(S, L_1^{-2}\otimes L_2^{6})^\vee \oplus H^1_{\bar\partial}(S, L_1^{2}\otimes L_2^{-6}) \oplus H^2_{\bar\partial}(S, L_1^{-2}\otimes L_2^{6})^\vee. \tag{2.88}$$

Note that $(\mathbf{3},\mathbf{2})_{-5,0}$, $(\bar{\mathbf{3}},\mathbf{2})_{5,0}$, and $(\mathbf{3},\mathbf{1})_{-2,6}$ are exotic fields in the MSSM. To avoid these exotics, L_1^5 and $L_1^{-2}\otimes L_2^6$ need to be non-trivial line bundles. If $L_1^3\otimes L_2^6$ is trivial, it follows from Eq. (2.85) and Eq. (2.86) that $n_{(\mathbf{1},\mathbf{2})_{3,6}} = n_{(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}} = 1$. By the vanishing theorem (2.6), no exotic fields requires that

$$n_{(\mathbf{3},\mathbf{2})_{-5,0}} = -\chi(S, L_1^{-5}) = 0 \tag{2.89}$$

$$n_{(\bar{\mathbf{3}},\mathbf{2})_{5,0}} = -\chi(S, L_1^{5}) = 0 \tag{2.90}$$

$$n_{(\mathbf{3},\mathbf{1})_{-2,6}} = -\chi(S, L_1^{-2}\otimes L_2^6) = 0. \tag{2.91}$$

We define

$$n_{(\bar{\mathbf{3}},\mathbf{1})_{2,-6}} = -\chi(S, L_1^{2}\otimes L_2^{-6}) \equiv \alpha_3, \tag{2.92}$$

where $\alpha_3 \in \mathbb{Z}_{\geqslant 0}$. Note that since $L_1^3\otimes L_2^6$ is trivial, then $L_1^2\otimes L_2^{-6} \cong L_1^5$. It follows from Eq. (2.90) that $\alpha_3 = 0$[16]. Therefore, the non-trivial conditions are (2.89) and

[16]This case will be denoted by $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)^*$ later.

(2.90), namely $\chi(S, L_1^{\pm 5}) = 0$, which imply that $c_1(L_1{}^{\pm 5})^2 = -2$ and $c_1(L_1^{\pm 5}) \cdot K_S = 0$. Note that $c_1(L_1{}^{\pm 5}) \in H_2(S, \mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{h, e_i,\ i = 1, 2, 3, ...8\}$, where h and e_i are the hyperplane divisor and exceptional divisors in $S = dP_8$. Immediately we get a fractional line bundle [17] $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and then $L_2 = \mathcal{O}_S(e_i - e_j)^{1/10}$. It is clear that L_1 and L_2 satisfy the BPS condition (2.51). As a result, (L_1, L_2) is a supersymmetric $U(1)^2$ gauge flux configuration on the bulk. If $L_1^3 \otimes L_2^6$ is non-trivial, by the vanishing theorem (2.6), an exotic-free bulk spectrum requires that

$$n_{(\mathbf{3},\mathbf{2})_{-5,0}} = -\chi(S, L_1^{-5}) = 0 \tag{2.93}$$

$$n_{(\bar{\mathbf{3}},\mathbf{2})_{5,0}} = -\chi(S, L_1^{5}) = 0 \tag{2.94}$$

$$n_{(\mathbf{3},\mathbf{1})_{-2,6}} = -\chi(S, L_1^{-2} \otimes L_2^6) = 0. \tag{2.95}$$

We define

$$n_{(\mathbf{1},\mathbf{2})_{3,6}} = -\chi(S, L_1^{3} \otimes L_2^6) \equiv \alpha_1 \tag{2.96}$$

$$n_{(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}} = -\chi(S, L_1^{-3} \otimes L_2^{-6}) \equiv \alpha_2 \tag{2.97}$$

$$n_{(\bar{\mathbf{3}},\mathbf{1})_{2,-6}} = -\chi(S, L_1^{2} \otimes L_2^{-6}) \equiv \alpha_3, \tag{2.98}$$

where $\alpha_i \in \mathbb{Z}_{\geqslant 0},\ i = 1, 2, 3$. To simplify the notation, we define $C = L_1^{-5}$, and $D = L_1^3 \otimes L_2^6$. By Eqs. (2.93)-(2.98) and the Riemann-Roch theorem (2.5), we obtain the following equations:

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[17] Note that with $\alpha_3 = 0$, there is a symmetry $(L_1, L_2) \leftrightarrow (L_1^{-1}, L_2^{-1})$ in Eq. (2.89)-(2.92). Without loss of generality, we choose $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$.

$$\begin{cases} c_1(C)^2 = -2 \\ c_1(D)^2 = -\alpha_1 - \alpha_2 - 2 \\ c_1(C)\cdot c_1(D) = 1 + \frac{1}{2}(\alpha_1+\alpha_2-\alpha_3) \\ \alpha_3 = \alpha_2 - \alpha_1 \\ c_1(C)\cdot K_S = 0 \\ c_1(D)\cdot K_S = \alpha_1 - \alpha_2. \end{cases} \tag{2.99}$$

Note that C and D are required to be honest line bundles, in other words, $c_1(C)$, $c_1(D) \in H_2(S,\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{h, e_i,\ i = 1,2,3,...8\}$. Note that $(\bar{\mathbf{3}},\mathbf{1})_{2,-6}$ is a candidate for a matter field in the MSSM. Therefore, we shall restrict to the case of $\alpha_3 \leqslant 3$. In what follows, we shall demonstrate how to derive explicit expressions for $U(1)^2$ gauge fluxes from Eq. (2.99). For the case of $\alpha_3 = 0$, by the constraints in Eq. (2.99), we may assume $(\alpha_1,\alpha_2,\alpha_3) = (k,k,0)$ with $k \in \mathbb{Z}_{\geqslant 0}$. We shall show that there is no solution for $k \geqslant 4$. Note that in this case, Eq. (2.99) reduces to

$$c_1(C)^2 = -2,\ \ c_1(D)^2 = -2k-2,\ \ c_1(C)\cdot c_1(D) = 1+k, \tag{2.100}$$

with $c_1(C)\cdot K_S = c_1(D)\cdot K_S = 0$. From the conditions $c_1(C)^2 = -2$, $c_1(C)\cdot K_S = 0$, and BPS condition (2.51), it follows that $C = \mathcal{O}_S(e_i - e_j)$, which is the universal line bundle in the case of $G_S = SU(6)$ since these two conditions are independent of α_i, $i = 1,2,3$ and always appear in Eq. (2.99). Actually, the corresponding fractional line bundle L_1 of C is the $U(1)_Y$ hypercharge flux in the minimal $SU(5)$ GUT [14–16]. In what follows, we shall focus on the solutions for the line bundle D. By Eq. (2.100), we can obtain the upper bound of k. Write $D = \mathcal{O}_S(c_ie_i + c_je_j + \tilde{D})$,[18] where $\tilde{D}$ is a integral divisor containing no h, e_i, and e_j. Note that the repeat indices

[18]Due to the BPS condition (2.51), D contains no component h.

are not a summation, and c_i, $c_j \in \mathbb{Z}$. By Eq. (2.100), we get $-c_i + c_j = k + 1$ and $c_1^2 + c_2^2 - \tilde{D}^2 = 2k + 2$. Note that $\tilde{D}^2 \leqslant 0$ by the construction. Using the inequality[19] $c_1^2 + c_2^2 \geqslant \frac{1}{2}(c_1 - c_2)^2$ and the condition $k \in \mathbb{Z}_{\geqslant 0}$, we obtain $0 \leqslant k \leqslant 3$, which implies that there is no solution D for $k \geqslant 4$. Next we shall explicitly solve the configurations (L_1, L_2) satisfying Eq. (2.99) for the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $0 \leqslant k \leqslant 3$.

Let us start with the simplification of Eq. (2.99). Note that in Eq. (2.99), there are two conditions that are independent of α_i, namely,

$$c_1(C)^2 = -2, \quad c_1(C) \cdot K_S = 0, \tag{2.101}$$

which gives rise to the universal line bundle, $C = \mathcal{O}_S(e_i - e_j)$, as mentioned earlier. The remaining conditions are

$$\begin{cases} c_1(D)^2 = -\alpha_1 - \alpha_2 - 2 \\ c_1(C) \cdot c_1(D) = 1 + \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3) \\ \alpha_3 = \alpha_2 - \alpha_1 \\ c_1(D) \cdot K_S = \alpha_1 - \alpha_2. \end{cases} \tag{2.102}$$

Since C is universal, all we have to do is to solve the line bundles D in Eq. (2.102) for a given $(\alpha_1, \alpha_2, \alpha_3)$ and $C = \mathcal{O}_S(e_i - e_j)$. When $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$, Eq. (2.102) reduces to

$$c_1(D)^2 = -2, \quad c_1(C) \cdot c_1(D) = 1, \tag{2.103}$$

with $c_1(D) \cdot K_S = 0$. By Eq. (2.103), we have $D = \mathcal{O}_S(\pm e_l - e_i)$ or $\mathcal{O}_S(\pm e_l + e_j)$. The former gives rise to fractional line bundles $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and $L_2 = \mathcal{O}_S(\pm 5e_l - 2e_i - 3e_j)^{1/30}$. For the latter, we have $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and $L_2 = \mathcal{O}_S(\pm 5e_l + 3e_i + 2e_j)^{1/30}$. Recall that $K_S = -3h + \sum_{k=1}^{8} e_k$. To solve the condition $c_1(D) \cdot K_S = 0$, it

[19] In general, $(c_1(C)^2)(c_1(D)^2) \geqslant (c_1(C) \cdot c_1(D))^2$.

is clear that D has to be $\mathcal{O}_S(e_l - e_i)$ or $\mathcal{O}_S(-e_l + e_j)$. The corresponding fractional line bundle is $\mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$ or $\mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$. In addition to Eq. (2.103), these fractional line bundles need to satisfy the BPS condition (2.51). More precisely, for the case of $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and $L_2 = \mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$, BPS equation (2.51) reduces to

$$(e_i - e_j) \cdot \omega = 0, \;\; (5e_l - 2e_i - 3e_j) \cdot \omega = 0. \tag{2.104}$$

It is not difficult to see that[20] $\omega = Ah - (e_i + e_j + e_l + ...)$ solves Eq. (2.104). Similarly, for the case of $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and $L_2 = \mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$, L_1 and L_2 are also supersymmetric with respect to $\omega = Ah - (e_i + e_j + e_l + ...)$. As a result, for the case of $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$, we find two supersymmetric $U(1)^2$ gauge flux configurations (L_1, L_2).

When $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$, Eq. (2.102) reduces to

$$c_1(D)^2 = -4, \;\; c_1(C) \cdot c_1(D) = 2, \tag{2.105}$$

with $c_1(D) \cdot K_S = 0$. By Eq. (2.105), D can be $\mathcal{O}_S(2e_j)$, $\mathcal{O}_S(-2e_i)$ or $\mathcal{O}_S([e_l, e_m] - e_i + e_j)$, where the bracket is defined by $[A_1, A_2, ..A_k] = \{\pm A_1 \pm A_2 ... \pm A_k\}$. For later use, we also define $[A_1, A_2, ..A_k]' = \{\pm A_1 \pm A_2 ... \pm A_k\} \setminus (+A_1 + A_2 + ... + A_k)$, $[A_1, A_2, ..A_k]'' = \{\pm A_1 \pm A_2 ... \pm A_k\} \setminus \{(+A_1 + A_2 + ... + A_k), (-A_1 - A_2 - ... - A_k)\}$, and $[A_1, A_2, ..A_k]''' = \{(A_1 + A_2 ... + A_{k-1} - A_k), (A_1 + A_2 ... - A_{k-1} + A_k), ..., (-A_1 + A_2 ... + A_{k-1} + A_k)\}$. Note that $\mathcal{O}_S(2e_j)$, $\mathcal{O}_S(-2e_i)$, $\mathcal{O}_S(e_l + e_m - e_i + e_j)$, and $\mathcal{O}_S(-e_l - e_m - e_i + e_j)$ cannot solve the equation $c_1(D) \cdot K_S = 0$. As a result, $D = \mathcal{O}_S([e_l, e_m]'' - e_i + e_j)$, which correspond to the fractional bundles $L_2 = \mathcal{O}_S(5[e_l, e_m]'' - 2e_i + 2e_j)^{1/30}$. Clearly

[20]"..." in ω always stands for non-relevant terms for checking the BPS condition Eq. (2.51). Of course, those terms are relevant for the ampleness of ω and note that the choice of the polarizations is not unique.

L_1 and L_2 satisfy Eq. (2.51) with $\omega = Ah - (e_i + e_j + e_l + e_m + ...)$.

For the case of $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 0)$, Eq. (2.102) becomes

$$c_1(D)^2 = -6, \ \ c_1(C) \cdot c_1(D) = 3, \tag{2.106}$$

with $c_1(D) \cdot K_S = 0$. By Eq. (2.106), D can be $\mathcal{O}_S([e_l] - e_i + 2e_j)$ or $\mathcal{O}_S([e_l] - 2e_i + e_j)$. For the former, it is clear that $\mathcal{O}_S(e_l - e_i + 2e_j)$ does not satisfy the condition $c_1(D) \cdot K_S = 0$. Similarly, for the latter, $\mathcal{O}_S(-e_l - 2e_i + e_j)$ is not a solution as well. In this case, the solutions are $L_2 = \mathcal{O}_S(-5e_l - 2e_i + 7e_j)^{1/30}$ or $L_2 = \mathcal{O}_S(5e_l - 7e_i + 2e_j)^{1/30}$. It is easy to see that the solutions also satisfy the BPS condition (2.51). Note that for the case of $\alpha_3 = 0$, taking $\omega = Ah - (\sum_{k=1}^{8} e_k) = (-K_S) + (A-3)h$, the conditions $c_1(C) \cdot K_S = c_1(D) \cdot K_S = 0$ are equivalent to Eq. (2.51). Therefore, the solutions of Eq. (2.99) are all supersymmetric for the case of $\alpha_3 = 0$.

Next we consider the case of $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$. In this case, the line bundle D satisfies the following equations:

$$c_1(D)^2 = -8, \ \ c_1(C) \cdot c_1(D) = 4, \tag{2.107}$$

with $c_1(D) \cdot K_S = 0$. By Eq. (2.107), we obtain $D = \mathcal{O}_S(2e_j - 2e_i)$. The corresponding fractional line bundle is $L_2 = \mathcal{O}_S(e_j - e_i)^{7/30}$. Obviously, L_2 satisfies the condition $c_1(D) \cdot K_S = 0$, and Eq. (2.51) for $\omega = Ah - (e_i + e_j + ...)$.

Next we shall consider the case of $\alpha_3 = 1$. By the constraints of Eq. (2.102), we may assume that $(\alpha_1, \alpha_2, \alpha_3) = (m, m+1, 1)$, where $m \in \mathbb{Z}_{\geqslant 0}$. Then Eq. (2.102) becomes

$$c_1(D)^2 = -2m - 3, \ \ c_1(C) \cdot c_1(D) = 1 + m, \tag{2.108}$$

with $c_1(D) \cdot K_S = -1$. Again the first thing we need to do is to get the upper bound of m. Eq. (2.108) implies that $1 - \sqrt{6} \leqslant m \leqslant 1 + \sqrt{6}$. Since $m \in \mathbb{Z}_{\geqslant 0}$, we obtain $0 \leqslant m \leqslant$

3. Therefore, the possible configurations are $(\alpha_1, \alpha_2, \alpha_3) = (0,1,1), (1,2,1), (2,3,1)$ or $(3,4,1)$.

Let us look at the case of $(\alpha_1, \alpha_2, \alpha_3) = (0,1,1)$. In this case, Eq. (2.108) reduces to the following equations

$$c_1(D)^2 = -3, \;\; c_1(C) \cdot c_1(D) = 1. \tag{2.109}$$

It is easy to see that D can be $\mathcal{O}_S([e_l, e_m] - e_i)$ or $\mathcal{O}_S([e_l, e_m] + e_j)$. Note that $\mathcal{O}_S([e_l, e_m]'' - e_i)$, $\mathcal{O}_S(-e_l - e_m - e_i)$, $\mathcal{O}_S(e_l + e_m + e_j)$, and $\mathcal{O}_S(-e_l - e_m + e_j)$ do not satisfy the equation $c_1(D) \cdot K_S = -1$, so we have to eliminate these cases. It turns out that the resulting fractional line bundles are $\mathcal{O}_S(5(e_l + e_m) - 2e_i - 3e_j)^{1/30}$ and $\mathcal{O}_S(5[e_l, e_m]'' + 3e_i + 2e_j)^{1/30}$. In order to preserve supersymmetry, the solutions need to solve Eq. (2.51). For the case of $L_2 = \mathcal{O}_S(5(e_l + e_m) - 2e_i - 3e_j)^{1/30}$, Eq. (2.51) reduces to

$$(e_i - e_j) \cdot \omega = 0, \;\; [(e_l + e_m) - e_i] \cdot \omega = 0. \tag{2.110}$$

For another fractional line bundle $L_2 = \mathcal{O}_S(5[e_l, e_m]'' + 3e_i + 2e_j)^{1/30}$, Eq. (2.51) becomes

$$(e_i - e_j) \cdot \omega = 0, \;\; ([e_l, e_m]'' + e_i) \cdot \omega = 0 \tag{2.111}$$

It is clear that $\omega = Ah - (e_l + e_m + 2e_i + 2e_j + ...)$ solves Eq. (2.110) and $\omega = Ah - (2e_l + e_m + e_i + e_j + ...)$ solves Eq. (2.111) if $[e_l, e_m]'' = -e_l + e_m$. For the case of $[e_l, e_m]'' = e_l - e_m$, $\omega = Ah - (e_l + 2e_m + e_i + e_j + ...)$ is a solution of Eq. (2.111). Therefore, $\mathcal{O}_S(5(e_l + e_m) - 2e_i - 3e_j)^{1/30}$ and $\mathcal{O}_S(5[e_l, e_m]'' + 3e_i + 2e_j)^{1/30}$ are supersymmetric. In this case, the solutions of Eq. (2.109) and the equations, $c_1(C) \cdot K_S = 0, \; c_1(D) \cdot K_S = -1$ satisfy Eq. (2.51). It seems that for the case $\alpha_3 = 1$, the condition $c_1(C) \cdot K_S = 0, \; c_1(D) \cdot K_S = -1$ is stronger than BPS condition (2.51). For example, $D = \mathcal{O}_S(e_l - e_m - e_i)$ with corresponding fractional line bundle

	$(\alpha_1, \alpha_2, \alpha_3)$	L_2
1	$(1,1,0)^*$	$\mathcal{O}_S(e_i - e_j)^{1/10}$
2	$(0,0,0)$	$\mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$, $\mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$
3	$(1,1,0)$	$\mathcal{O}_S(5[e_l, e_m]'' - 2e_i + 2e_j)^{1/30}$
4	$(2,2,0)$	$\mathcal{O}_S(-5e_l - 2e_i + 7e_j)^{1/30}$, $\mathcal{O}_S(5e_l - 7e_i + 2e_j)^{1/30}$
5	$(3,3,0)$	$\mathcal{O}_S(e_j - e_i)^{7/30}$
6	$(0,1,1)$	$\mathcal{O}_S(5[e_l, e_m]'' + 3e_i + 2e_j)^{1/30}$, $\mathcal{O}_S(5(e_l + e_m) - 2e_i - 3e_j)^{1/30}$
7	$(1,2,1)$	$\mathcal{O}_S(-5e_l + 3e_i + 7e_j)^{1/30}$, $\mathcal{O}_S(5[e_l, e_m, e_k]''' - 2e_i + 2e_j)^{1/30}$
8	$(2,3,1)$	$\mathcal{O}_S(5[e_l, e_m]'' - 2e_i + 7e_j)^{1/30}$, $\mathcal{O}_S(5(e_l + e_m) - 7e_i + 2e_j)^{1/30}$
9	$(3,4,1)$	No Solution

Table VIII. Flux configurations for $G_S = SU(6)$ with $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and $\alpha_3 = 0, 1$.

$L_2 = \mathcal{O}_S(5e_l - 5e_m - 2e_i - 3e_j)^{1/30}$ is supersymmetric but does not satisfy the condition $c_1(D) \cdot K_S = -1$. Actually, we shall see that this is not the case in the next examples.

Let us turn to the case of $(\alpha_1, \alpha_2, \alpha_3) = (3,4,1)$. In this case, Eq. (2.108) reduces to

$$c_1(D)^2 = -9, \quad c_1(C) \cdot c_1(D) = 4. \tag{2.112}$$

It is not difficult to find that the solutions are $D = \mathcal{O}_S([e_l] - 2e_i + 2e_j)$ and the corresponding fractional line bundle are $L_2 = \mathcal{O}_S(5[e_l] - 7e_i + 7e_j)^{1/30}$. Note that only $D = \mathcal{O}_S(e_l - 2e_i + 2e_j)$ satisfies the condition $c_1(D) \cdot K_S = -1$. However, it is clear that it does not satisfy the BPS condition (2.51), which means that no configuration (L_1, L_2) for an exotic-free spectrum exists in this case. From this example, we know that for the case of $\alpha_3 = 1$, the solutions of Eq. (2.102) are not guaranteed to be supersymmetric and vice versa. Therefore, in general we need to check these two conditions for each solution in the case of $\alpha_3 \in \mathbb{Z}_{>0}$. Following a similar procedure, one can obtain all configurations (L_1, L_2) for the cases of $\alpha_3 = 1$. We summarize the results of $\alpha_3 = 0, 1$ in Table VIII in which all L_1 and L_2 satisfy the BPS condition (2.51) for suitable polarizations ω and the conditions $L_1^5 \neq \mathcal{O}_S$, $L_1^{-2} \otimes L_2^6 \neq \mathcal{O}_S$ and $L_1^3 \otimes L_1^6 \neq \mathcal{O}_S$.

Next we consider the case of $\alpha_3 = 2$. By the last constraint of Eq. (2.99), we may assume $(\alpha_1, \alpha_2, \alpha_3) = (l, l+2, 2)$, where $l \in \mathbb{Z}_{\geqslant 0}$. One can show that the necessary condition for existence of the solutions of Eq. (2.99) is $0 \leqslant l \leqslant 3$. Therefore, $(\alpha_1, \alpha_2, \alpha_3)$ can be $(0, 2, 2)$, $(1, 3, 2)$, $(2, 4, 2)$ or $(3, 5, 2)$. Following the previous procedure, one can obtain all configurations (L_1, L_2) for the case of $\alpha_3 = 2$.

For the case of $\alpha_3 = 3$, we may assume that $(\alpha_1, \alpha_2, \alpha_3) = (n, n+3, 3)$ with $n \in \mathbb{Z}_{\geqslant 0}$. The necessary condition for existence of the solutions of Eq. (2.99) is $0 \leqslant n \leqslant 4$, which implies that $(\alpha_1, \alpha_2, \alpha_3) = (0, 3, 3), (1, 4, 3), (2, 5, 3)$, $(3, 6, 3)$, or $(4, 7, 3)$. Following the previous procedure, one can obtain all configurations (L_1, L_2) for the case of $\alpha_3 = 3$. Let us look at the case of $(\alpha_1, \alpha_2, \alpha_3) = (3, 6, 3)$. In this case, Eq. (2.102) reduces to

$$c_1(D)^2 = -11, \quad c_1(C) \cdot c_1(D) = 4, \tag{2.113}$$

with $c_1(D) \cdot K_S = -3$. It follows from Eq. (2.113) that D can be $\mathcal{O}_S([e_l] - e_i + 3e_j)$, $\mathcal{O}_S([e_l] - 3e_i + e_j)$, or $\mathcal{O}_S([e_l, e_m, e_n] - 2e_i + 2e_j)$. When one takes the condition $c_1(D) \cdot K_S = -3$ into account, there are only two solutions, $D = \mathcal{O}_S(e_l - e_i + 3e_j)$ or $\mathcal{O}_S((e_l + e_m + e_n) - 2e_i + 2e_j)$, which corresponds to the fractional line bundles $\mathcal{O}_S(5e_l - 2e_i + 12e_j)^{1/30}$ and $\mathcal{O}_S(5(e_l + e_m + e_n) - 7e_i + 7e_j)^{1/30}$, respectively. However, these two solutions cannot satisfy Eq. (2.51). Therefore, in this case there do not

	$(\alpha_1, \alpha_2, \alpha_3)$	L_2
1	$(0,2,2)$	$\mathcal{O}_S(5(e_l+e_m+e_k)-2e_i-3e_j)^{1/30}$ $\mathcal{O}_S(5[e_l,e_m,e_k]'''+3e_i+2e_j)^{1/30}$
2	$(1,3,2)$	$\mathcal{O}_S(5[e_l,e_m]''+3e_i+7e_j)^{1/30}$ $\mathcal{O}_S(5[e_l,e_m,e_n,e_k]'''-2e_i+2e_j)^{1/30}$
3	$(2,4,2)$	$\mathcal{O}_S(5[e_l,e_m,e_k]'''-2e_i+7e_j)^{1/30}$ $\mathcal{O}_S(5(e_l+e_m+e_k)-7e_i+2e_j)^{1/30}$
4	$(3,5,2)$	No Solution
5	$(0,3,3)$	$\mathcal{O}_S(5(e_l+e_m+e_n+e_k)-2e_i-3e_j)^{1/30}$ $\mathcal{O}_S(5[e_l,e_m,e_n,e_k]'''+3e_i+2e_j)^{1/30}$
6	$(1,4,3)$	$\mathcal{O}_S(5[e_l,e_m,e_k]'''+3e_i+7e_j)^{1/30}$ $\mathcal{O}_S(5[e_l,e_m,e_n,e_k,e_p]'''-2e_i+2e_j)^{1/30}$
7	$(2,5,3)$	$\mathcal{O}_S(5[e_l,e_m,e_n,e_k]'''-2e_i+7e_j)^{1/30}$ $\mathcal{O}_S(5(e_l+e_m+e_n+e_k)-7e_i+2e_j)^{1/30}$
8	$(3,6,3)$	No Solution
9	$(4,7,3)$	No Solution

Table IX. Flux configurations for $G_S = SU(6)$ with $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and $\alpha_3 = 2, 3$.

exist any $U(1)^2$ gauge fluxes for an exotic-free spectrum. A similar situation occurs in the case of $(\alpha_1, \alpha_2, \alpha_3) = (4, 7, 3)$. In this case, D can be $\mathcal{O}_S(-3e_i + 2e_j)$ or $\mathcal{O}_S(-2e_i + 3e_j)$ by Eq. (2.102). However, they neither solve Eq. (2.51) nor satisfy the condition $c_1(D) \cdot K_S = -3$. As a result, there are no $U(1)^2$ gauge fluxes without producing exotics in this case. We summarize the results of $\alpha_3 = 2, 3$ in Table IX in which all L_1 and L_2 satisfy the BPS condition (2.51) for suitable polarizations ω and the conditions $L_1^5 \neq \mathcal{O}_S$, $L_1^{-2} \otimes L_2^6 \neq \mathcal{O}_S$ and $L_1^3 \otimes L_1^6 \neq \mathcal{O}_S$.

b. Spectrum from the Curves

With $G_S = SU(6)$, to obtain matter in $SU(5)$ GUT, it is required that $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. In this case, there are three kinds of intersecting curves, $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$ and Σ_{E_6} with enhanced gauge groups $SU(7)$, $SO(12)$, and E_6, respectively. The breaking patterns are as shown in Eqs. (2.45)-(2.47). To achieve doublet-triplet splitting and make contact with the spectrum in the MSSM, we consider $U(1)^2$ flux configurations (L_1, L_2) already solved in the previous section. In this section we shall study the spectrum from the curves and show that the doublet-triplet splitting and non-minimal spectrum of the MSSM can be achieved. A detailed example can be found in subsection 3.

In local F-theory models, the gauge group on the curve along which S intersects with S' will be enhanced at least by one rank. In the present case of $G_S = SU(6)$, the possible enhanced gauge groups are $SU(7)$, $SO(12)$ and E_6. The matter fields transform as fundamental representation **6**, anti-symmetric tensor representation of rank two **15**, and anti-symmetric tensor representation of rank three **20** in $SU(6)$ can be engineered to localize on the curves with gauge groups $SU(7)$, $SO(12)$, and E_6, respectively. In order to split doublet and triplet states in Higgs and obtain the spectrum of the MSSM, $L_{1\Sigma}$, $L_{2\Sigma}$ and L'_Σ have to be non-trivial, which breaks G_Σ into $G_{\rm std} \times U(1)^2$. The breaking patterns of $SU(7)$, $SO(12)$ and E_6 are as follows:

$$\begin{array}{ccccl} SU(7) & \to & SU(6)\times U(1)' & \to & SU(3)\times SU(2)\times U(1)'\times U(1)_1\times U(1)_2 \\ \mathbf{48} & \to & \mathbf{35}_0+\mathbf{1}_0 & \to & (\mathbf{8},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{3})_{0,0,0}+(\mathbf{3},\mathbf{2})_{0,-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{0,5,0} \\ & & +\mathbf{6}_{-7}+\bar{\mathbf{6}}_7 & & +(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{0,3,6}+(\mathbf{3},\mathbf{1})_{0,-2,6} \\ & & & & +(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{-7,3,1} \\ & & & & +(\mathbf{3},\mathbf{1})_{-7,-2,1}+(\mathbf{1},\mathbf{1})_{-7,0,-5}+(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1} \\ & & & & +(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}+(\mathbf{1},\mathbf{1})_{7,0,5} \end{array} \tag{2.114}$$

$$\begin{array}{ccccl} SO(12) & \to & SU(6)\times U(1)' & \to & SU(3)\times SU(2)\times U(1)'\times U(1)_1\times U(1)_2 \\ \mathbf{66} & \to & \mathbf{35}_0+\mathbf{1}_0 & \to & (\mathbf{8},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{3})_{0,0,0}+(\mathbf{3},\mathbf{2})_{0,-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{0,5,0} \\ & & +\mathbf{15}_2+\overline{\mathbf{15}}_{-2} & & +(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{0,3,6}+(\mathbf{3},\mathbf{1})_{0,-2,6} \\ & & & & +(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{2,3,-4} \\ & & & & +(\mathbf{3},\mathbf{1})_{2,-2,-4}+(\mathbf{1},\mathbf{1})_{2,6,2}+(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}+(\mathbf{3},\mathbf{2})_{2,1,2} \\ & & & & +(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}+(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}+(\mathbf{1},\mathbf{1})_{-2,-6,-2} \\ & & & & +(\mathbf{3},\mathbf{1})_{-2,4,-2}+(\bar{\mathbf{3}},\bar{\mathbf{2}})_{-2,-1,-2} \end{array} \tag{2.115}$$

$$\begin{array}{ccccl} E_6 & \to & SU(6)\times U(1)' & \to & SU(3)\times SU(2)\times U(1)'\times U(1)_1\times U(1)_2 \\ \mathbf{78} & \to & \mathbf{35}_0+\mathbf{1}_0+\mathbf{1}_{\pm 2} & \to & (\mathbf{8},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{3})_{0,0,0}+(\mathbf{3},\mathbf{2})_{0,-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{0,5,0} \\ & & +\mathbf{20}_1+\mathbf{20}_{-1} & & +(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{0,3,6}+(\mathbf{3},\mathbf{1})_{0,-2,6} \\ & & & & +(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{\pm 2,0,0} \\ & & & & +[(\mathbf{1},\mathbf{1})_{1,6,-3}+(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}+(\mathbf{3},\mathbf{2})_{1,1,-3}+c.c] \\ & & & & +[(\mathbf{1},\mathbf{1})_{-1,6,-3}+(\bar{\mathbf{3}},\mathbf{1})_{-1,-4,-3}+(\mathbf{3},\mathbf{2})_{-1,1,-3}+c.c]. \end{array} \tag{2.116}$$

Coupling	Representation	Configuration
$Qu^c\bar{h}$	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{SO(12)}\Sigma_{E_6}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}(\mathbf{1},\mathbf{2})_{2,3,-4}$	$\Sigma_{SO(12)}\Sigma_{SO(12)}\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}(\mathbf{1},\mathbf{2})_{0,3,6}$	$\Sigma_{E_6}\Sigma_{E_6}S$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}(\mathbf{1},\mathbf{2})_{0,3,6}$	$\Sigma_{E_6}\Sigma_{E_6}S$
Qd^ch	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SO(12)}\Sigma_{SU(7)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{SO(12)}S\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{E_6}\Sigma_{SU(7)}\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{E_6}\Sigma_{SU(7)}\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}$	$\Sigma_{SO(12)}\Sigma_{SO(12)}S$
Le^ch	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{2,6,2}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SU(7)}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SO(12)}\Sigma_{E_6}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{-1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SO(12)}\Sigma_{E_6}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{SU(7)}\Sigma_{E_6}\Sigma_{SO(12)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{-1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{SU(7)}\Sigma_{E_6}\Sigma_{SO(12)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{2,6,2}(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}$	$\Sigma_{SO(12)}\Sigma_{SO(12)}S$
$L\nu^c\bar{h}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{0,0,0}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{SU(7)}S\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{-7,0,-5}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{SO(12)}\Sigma_{SU(7)}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{7,0,5}(\mathbf{1},\mathbf{2})_{2,3,-4}$	$\Sigma_{SU(7)}\Sigma_{SU(7)}\Sigma_{SO(12)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{0,0,0}(\mathbf{1},\mathbf{2})_{2,3,-4}$	$\Sigma_{SU(7)}S\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{-7,0,-5}(\mathbf{1},\mathbf{2})_{0,3,6}$	$\Sigma_{SU(7)}\Sigma_{SU(7)}S$

Table X. The Yukawa couplings of the MSSM model from $G_S = SU(6)$.

	Q	u^c	d^c	e^c	L	$\bar{h}$	h
M_0	$(\mathbf{3},\mathbf{2})_{2,1,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}$	$(\mathbf{1},\mathbf{1})_{1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-2,3,-4}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$
M_1	$(\mathbf{3},\mathbf{2})_{2,1,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}$	$(\mathbf{1},\mathbf{1})_{2,6,2}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_2	$(\mathbf{3},\mathbf{2})_{1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{2,6,2}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_3	$(\mathbf{3},\mathbf{2})_{-1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{2,6,2}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_4	$(\mathbf{3},\mathbf{2})_{2,1,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}$	$(\mathbf{1},\mathbf{1})_{-1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_5	$(\mathbf{3},\mathbf{2})_{1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{-1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_6	$(\mathbf{3},\mathbf{2})_{-1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{-1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$

Table XI. Field content in the MSSM from $G_S = SU(6)$.

Due to non-trivial $U(1)^2$ flux configurations on the bulk S, the last two $U(1)$ charges of the fields on the curves should be conserved in each Yukawa coupling. From the breaking patterns, we list possible Yukawa couplings of type $\Sigma\Sigma S$ and $\Sigma\Sigma\Sigma$ in Table X. According to Table X, the possible field content is shown in Table XI. In what follows, we shall focus on the case of $\Sigma\Sigma\Sigma$-type couplings and find all possible field configurations supported by the curves $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$, and Σ_{E_6} with given $U(1)^2$ flux configuration (L_1, L_2).

Let us start with the case of $\Sigma_{SU(7)}$ and consider $(\alpha_1,\alpha_2,\alpha_3) = (k,k,0)$ with $k = 0,1,2,3$. When $(\alpha_1,\alpha_2,\alpha_3) = (0,0,0)$, which is the second case in Table VIII, it is clear that we have $L_2 = \mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$ or $L_2 = \mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$. We define $(n_1,n_2,n_3) = (n_{(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}}, n_{(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}}, n_{(\mathbf{1},\mathbf{1})_{7,0,5}})$. To avoid exotic fields, we require that $n_1 \in \mathbb{Z}_{\geqslant 0}$. Given field configurations (n_1,n_2,n_3) on the curve $\Sigma_{SU(7)}$, the necessary conditions[21] for the homological class of the curve $\Sigma_{SU(7)}$ are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{SU(7)} = n_2 - n_1 \\ (e_i - e_l) \cdot \Sigma_{SU(7)} = n_2 - n_3. \end{cases} \tag{2.117}$$

if $L_2 = \mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$. For the case of $L_2 = \mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$, the conditions are as follows:

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{SU(7)} = n_2 - n_1 \\ (e_i - e_l) \cdot \Sigma_{SU(7)} = n_3 - n_1. \end{cases} \tag{2.118}$$

Note that the first condition of Eq. (2.117) and Eq. (2.118) is universal since it comes from the restriction of the universal supersymmetric line bundle $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ to the curve $\Sigma_{SU(7)}$. Note that there are no further constraints for $n_i,\ i = 1, 2, 3$ except $n_1 \in \mathbb{Z}_{\geqslant 0}$, $n_1 \neq n_2$, $3n_1 + 2n_2 \neq 5n_3$ and $3n_1 + 2n_2 + n_3 \neq 0$. The last three constraints follow from the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Let us look at an example. Consider the case of $(n_1, n_2, n_3) = (0, 1, 0)$, Eq. (2.117) and Eq. (2.118) can be easily solved by $\Sigma = h - e_i - e_m$ and $\Sigma = h - e_i - e_l$, respectively. In this case, double and triplet states in the Higgs field $\bar{\mathbf{5}}_{7,-1}$ can be split without producing exotic fields. Let us look at one more case, $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$. It follows from Table VIII that $L_2 = \mathcal{O}_S(e_j - e_i)^{7/30}$. The conditions for the homological class of the curve $\Sigma_{SU(7)}$ to support the field configurations (n_1, n_2, n_3) are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{SU(7)} = n_2 - n_1 \\ 2n_1 = n_2 + n_3. \end{cases} \tag{2.119}$$

This time we get one more constraint, $2n_1 = n_2 + n_3$. It follows that when $(\bar{\mathbf{3}}, \mathbf{1})_{7,2,-1}$

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[21] $L_{1\Sigma_{SU(7)}} = \mathcal{O}_{\Sigma_{SU(7)}}(\frac{1}{5}(n_1 - n_2))$, $L_{2\Sigma_{SU(7)}} = \mathcal{O}_{\Sigma_{SU(7)}}(\frac{1}{30}(-3n_1 - 2n_2 + 5n_3))$, and $L'_{\Sigma_{SU(7)}} = \mathcal{O}_{\Sigma_{SU(7)}}(\frac{1}{42}(3n_1 + 2n_2 + n_3))$.

$(\alpha_1,\alpha_2,\alpha_3)$	Conditions	L_2
$(0,0,0)$	$(e_i-e_l)\cdot\Sigma_{SU(7)}=n_2-n_3$	$\mathcal{O}_S(5e_l-2e_i-3e_j)^{1/30}$
	$(e_i-e_l)\cdot\Sigma_{SU(7)}=n_3-n_1$	$\mathcal{O}_S(-5e_l+3e_i+2e_j)^{1/30}$
$(1,1,0)^*$	$n_2=n_3$	$\mathcal{O}_S(e_i-e_j)^{1/10}$
$(1,1,0)$	$([e_l,e_m]'')\cdot\Sigma_{SU(7)}=n_3-n_1$	$\mathcal{O}_S(5[e_l,e_m]''-2e_i+2e_j)^{1/30}$
$(2,2,0)$	$(-e_l+e_j)\cdot\Sigma_{SU(7)}=n_3-n_1$	$\mathcal{O}_S(-5e_l-2e_i+7e_j)^{1/30}$
	$(e_l-e_i)\cdot\Sigma_{SU(7)}=n_3-n_1$	$\mathcal{O}_S(5e_l-7e_i+2e_j)^{1/30}$
$(3,3,0)$	$2n_1=n_2+n_3$	$\mathcal{O}_S(e_j-e_i)^{7/30}$

Table XII. The conditions for $\Sigma_{SU(7)}$ supporting the field configurations (n_1,n_2,n_3) with $L_1=\mathcal{O}_S(e_j-e_i)^{1/5}$.

vanishes, the doublets always show up together with singlets. For the cases of $(\alpha_1,\alpha_2,\alpha_3)=(k,k,0)$ with $k=1,2$, we summarize the results[22] in Table XII

Similarly, we can extend the calculation to the curve $\Sigma_{SO(12)}$. Let us define $(s_1,s_2,s_3,s_4,s_5)=(n_{(\mathbf{3},\mathbf{2})_{2,1,2}},n_{(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}},n_{(\mathbf{3},\mathbf{1})_{2,-2,-4}},n_{(\mathbf{1},\mathbf{2})_{2,3,-4}},n_{(\mathbf{1},\mathbf{1})_{2,6,2}})$ and consider the case of $(\alpha_1,\alpha_2,\alpha_3)=(1,1,0)$, which is the third case in Table VIII. It is clear that we have $L_2=\mathcal{O}_S(5[e_l,e_m]''-2e_i+2e_j)^{1/30}$. The necessary conditions[23] for the homological class of the curve $\Sigma_{SO(12)}$ with field configurations (s_1,s_2,s_3,s_4,s_5) are

$$\begin{cases}(e_i-e_j)\cdot\Sigma_{SO(12)}=s_2-s_1\\ ([e_l,e_m]'')\cdot\Sigma_{SO(12)}=s_2-s_3,\end{cases}\tag{2.120}$$

[22]For simplicity, we are not going to show the universal conditions $(e_i-e_j)\cdot\Sigma=w_2-w_1,\ w\in\{n,s\}$ for $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$, respectively and $(e_i-e_j)\cdot\Sigma=p_3-p_1$ for Σ_{E_6} in Table XII, XIII, and XIV.

[23]$L_{1\Sigma_{SO(12)}}=\mathcal{O}_{\Sigma_{SO(12)}}(\frac{1}{5}(s_1-s_2))$, $L_{2\Sigma_{SO(12)}}=\mathcal{O}_{\Sigma_{SO(12)}}(\frac{1}{30}(2s_1+3s_2-5s_3))$, and $L'_{\Sigma_{SO(12)}}=\mathcal{O}_{\Sigma_{SO(12)}}(\frac{1}{6}(2s_1+s_3))$.

and

$$\begin{cases} s_4 = s_3 + s_1 - s_2 \\ s_5 = 2s_1 - s_2. \end{cases} \tag{2.121}$$

Note that Eq. (2.121) impose severe restrictions on the configurations $(s_1, s_2, s_3, s_4, s_5)$. For example, one cannot simply set $(s_1, s_2, s_3, s_4, s_5) = (0, 0, 0, m, 0)$ to achieve the doublet-triplet splitting of Higgs $\mathbf{5}_{2,-4}$; it is easy to see that m is forced to be zero by the constraints in Eq. (2.121). This will cause trouble when we attempt to engineer the Higgs on the curve $\Sigma_{SO(12)}$ with doublet-triplet splitting. Consider the case of $s_4 > 0$ and set $s_1 = 0$. From the constraints in Eq. (2.121), we obtain $s_2 + (-s_3) < 0$. Note that to avoid exotic fields from $\Sigma_{SO(12)}$, it is required that $s_1,\ s_2 \in \mathbb{Z}_{\geqslant 0}$ and $s_3 \in \mathbb{Z}_{\leqslant 0}$. It follows that $0 \leqslant s_2 + (-s_3) < 0$, which leads to a contradiction. As a result, the appearance of $(\mathbf{3}, \mathbf{2})_{2,1,2}$ cannot be avoided on the curve $\Sigma_{SO(12)}$ as $N_{(\mathbf{1},\mathbf{2})_{2,3,-4}} = s_4 > 0$. If $s_4 > 0$, actually the most general non-trivial configurations are $(s_1, s_2, s_3, s_4, s_5) = (l, l+n-m, n, m, l+m-n)$, where $m,\ l \in \mathbb{Z}_{>0}$ and $m-l \leqslant n \leqslant 0$. Note that $(\mathbf{3}, \mathbf{2})_{2,1,2}$ is treated as matter in the MSSM, which requires that[24] $l \leqslant 3$. It follows that $1 \leqslant m \leqslant 3$ and $m \leqslant l \leqslant 3$. It turns out that there are finitely many non-trivial configurations. More precisely, the field configurations are as follows:

$$(s_1, s_2, s_3, s_4, s_5) = \left\{ \begin{array}{l} (1,0,0,1,2), (2,1,0,1,3), (2,0,-1,1,4), \\ (3,2,0,1,4), (3,1,-1,1,5), (3,0,-2,1,6), \\ (2,0,0,2,4), (3,1,0,2,5), (3,0,-1,2,6), \\ (3,0,0,3,6) \end{array} \right\}. \tag{2.122}$$

If $-3 \leqslant s_4 \leqslant 0$, with $0 \leqslant s_1,\ s_2 \leqslant 3$ and $-3 \leqslant s_3 \leqslant 0$, we have another branch of the

[24]We allow the cases in which three copies of matter fields can be distributed over different matter curves.

configurations as follows:

$$(s_1,s_2,s_3,s_4,s_5)=\left\{\begin{array}{l}(0,1,-1,-2,-1),(0,1,-2,-3,-1),(0,2,-1,-3,-2),\\(1,0,-1,0,2),(1,0,-3,-2,2),(1,2,0,-1,0),\\(1,2,-1,-2,0),(1,3,0,-2,-1),(1,3,-1,-3,-1),\\(2,0,-2,0,4),(2,0,-3,-1,4),(2,1,-2,-1,3),\\(2,3,0,-1,1),(2,1,-3,-2,3),(2,3,-1,-2,1),\\(2,1,-1,0,3),(2,3,-2,-3,1),(3,0,-3,0,6),\\(3,1,-2,0,5),(3,1,-3,-1,5),(3,2,-1,0,4),\\(3,2,-2,-1,4),(3,2,-3,-2,4)\end{array}\right\}, \tag{2.123}$$

where all configurations[25] in (2.122) and (2.123) satisfy the conditions $L_{1\Sigma}\neq\mathcal{O}_\Sigma$, $L_{2\Sigma}\neq\mathcal{O}_\Sigma$, and $L'_\Sigma\neq\mathcal{O}_\Sigma$. With these configurations, one can solve the conditions for the intersection numbers, namely, the conditions in Eq. (2.120). Let us consider the case of $(s_1,s_2,s_3,s_4,s_5)=(1,0,0,1,2)$, it is clear that $\Sigma=2h-e_l-e_m-e_j$ is a solution. For a more complicated case, for example $(s_1,s_2,s_3,s_4,s_5)=(3,1,-1,1,5)$, the conditions can be solved by $\Sigma=4h+e_p-2e_j-2e_l$ if $[e_l,e_m]''=e_l-e_m$ and by $\Sigma=4h+e_p-2e_j-2e_m$ if $[e_l,e_m]''=e_m-e_l$.

Let us turn to another case. Consider the first case in Table VIII, namely $(\alpha_1,\alpha_2,\alpha_3)=(1,1,0)^*$. The supersymmetric fractional line bundle L_2 is $\mathcal{O}_S(e_i-e_j)^{1/10}$. The necessary conditions are

$$\left\{\begin{array}{l}(e_i-e_j)\cdot\Sigma_{SO(12)}=s_2-s_1\\ s_1=s_3,\end{array}\right. \tag{2.124}$$

and Eq. (2.121). Note that $(\bar{\mathbf{3}},\mathbf{2})_{-2,-1,-2}$ and $(\mathbf{3},\mathbf{1})_{2,-2,-4}$ are exotic fields in the

[25] $s_3<0$ represents $n_{(\mathbf{3},\mathbf{1})_{2,-2,-4}}=0$ and $n_{(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}}=-s_3$. The same rule can be applied to other s_i.

MSSM. The constraint, $s_1 = s_3$ in Eq. (2.124) and Eq. (2.121) imply that $s_1 = s_3 = 0$. If $s_4 \geqslant 0$, by the constraints in Eq. (2.121), we obtain $(s_1, s_2, s_3, s_4, s_5) = (0, 0, 0, 0, 0)$. If $s_4 < 0$, we have general configurations $(s_1, s_2, s_3, s_4, s_5) = (0, n, 0, -n, -n)$, where $1 \leqslant n \leqslant 3$. However, these configurations violate the condition $L'_\Sigma \neq \mathcal{O}_\Sigma$. As a check, using the configurations in (2.122), (2.123), and taking the condition $s_1 = s_3$ into account, one can see that there are no solutions in this case.

Next we consider the fifth case in Table VIII, namely $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$. In this case, L_2 is $\mathcal{O}_S(e_j - e_i)^{7/30}$. The necessary conditions are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{SO(12)} = s_2 - s_1 \\ 2s_2 = s_1 + s_3, \end{cases} \tag{2.125}$$

and Eq. (2.121). It is easy to see that $s_2 = s_4$. If $s_2 = 0$, we obtain the non-trivial configurations $(s_1, s_2, s_3, s_4, s_5) = (k, 0, -k, 0, 2k)$, where $1 \leqslant k \leqslant 3$. Note that these configurations satisfy the conditions, $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Let us turn to the case of $s_2 = m \in \mathbb{Z}_{>0}$. The general configurations are $(s_1, s_2, s_3, s_4, s_5) = (l, m, 2m - l, m, 2l - m)$ with $l \geqslant 2m > 0$. Note that $(\mathbf{3}, \mathbf{2})_{2,1,2}$ is treated as matter in the MSSM. As a result, we focus on the case of $l \leqslant 3$, which implies that $m = 1$ and $l = 2, 3$. It turns out that the allowed configurations are $(s_1, s_2, s_3, s_4, s_5) = \{(2, 1, 0, 1, 3), (3, 1, -1, 1, 5)\}$, where the configurations satisfy the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Putting these two branches together, we obtain

$$(s_1, s_2, s_3, s_4, s_5) = \left\{ \begin{array}{l} (1, 0, -1, 0, 2), (2, 0 - 2, 0, 4), (3, 0, -3, 0, 6), \\ (2, 1, 0, 1, 3), (3, 1, -1, 1, 5) \end{array} \right\}. \tag{2.126}$$

As a check, from the field configurations in (2.122), (2.123) and the constraint $2s_2 = s_1 + s_3$, one can find that there are exactly five solutions as shown in (2.126).

$(\alpha_1, \alpha_2, \alpha_3)$	Conditions	L_2
$(0,0,0)$	$(e_i - e_l)\cdot\Sigma_{SO(12)} = s_3 - s_1$	$\mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$
	$(e_i - e_l)\cdot\Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$
$(1,1,0)^*$	$s_1 = s_3$	$\mathcal{O}_S(e_i - e_j)^{1/10}$
$(1,1,0)$	$([e_l, e_m]'')\cdot\Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(5[e_l, e_m]'' - 2e_i + 2e_j)^{1/30}$
$(2,2,0)$	$(-e_l + e_j)\cdot\Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(-5e_l - 2e_i + 7e_j)^{1/30}$
	$(E_l - E_i)\cdot\Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(5e_l - 7e_i + 2e_j)^{1/30}$
$(3,3,0)$	$2s_2 = s_1 + s_3$	$\mathcal{O}_S(e_j - e_i)^{7/30}$

Table XIII. The conditions for $\Sigma_{SO(12)}$ supporting the field configurations $(s_1, s_2, s_3, s_4, s_5)$ with $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and constraints $2s_1 = s_2 + s_5$, $s_4 = s_3 + s_1 - s_2$.

Let us take a look at some solutions for the curve satisfying Eq. (2.125). For the the case of $(s_1, s_2, s_3, s_4, s_5) = (2, 1, 0, 1, 3)$, it is easy to see that $\Sigma = h - e_j - e_s$ solves the first equation in Eq. (2.125). For the case of $(s_1, s_2, s_3, s_4, s_5) = (2, 0, -2, 0, 4)$, $\Sigma = 3h - 2e_j - e_p$ can be a solution. From these examples, we expect that if we choose $\Sigma_{SO(12)}$ to house Higgs fields, it will be difficult to achieve doublet-triple splitting without introducing extra chiral fields. For other $U(1)^2$ flux configurations corresponding to the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $k = 0, 2$, the analysis is similar to the case of $k = 1$. We summarize the results in Table XIII.

In addition to doublet-triplet splitting problem, we also would like to study the matter spectrum. According to Table XI, the matter fields can come from the curves $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$, and Σ_{E_6}. The configurations of the fields and the conditions of the intersection numbers on the curves $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$ have been studied earlier in this section. Next we are going to analyze the case of Σ_{E_6}. Note that for the case of

M_0 in Table XI, to engineer $3 \times d_R$ on the bulk, it is required to set $\alpha_3 = 3$. However, it gives rise to exotic fields $(\mathbf{1}, \mathbf{2})_{3,6}$ and $(\mathbf{1}, \bar{\mathbf{2}})_{-3,-6}$ on the bulk. In what follows, we are going to focus on the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ on the bulk.

Let us start with the case of $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. It is clear that $L_2 = \mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$ or $L_2 = \mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$. We define $(p_1, p_2, p_3, p_4, p_5, p_6) = (n_{(\mathbf{3},\mathbf{2})_{1,1,-3}}, n_{(\mathbf{3},\mathbf{2})_{-1,1,-3}}, n_{(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}}, n_{(\bar{\mathbf{3}},\mathbf{1})_{-1,-4,-3}}, n_{(\mathbf{1},\mathbf{1})_{1,6,-3}}, n_{(\mathbf{1},\mathbf{1})_{-1,6,-3}})$. The necessary conditions[26] for the curve Σ_{E_6} are as follows:

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ (e_i - e_l) \cdot \Sigma_{E_6} = p_2 + p_3, \end{cases} \tag{2.127}$$

and

$$\begin{cases} p_4 = p_2 + p_3 - p_1 \\ p_5 = 2p_1 - p_3 \\ p_6 = p_1 + p_2 - p_3, \end{cases} \tag{2.128}$$

if $L_2 = \mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$. For the case of $L_2 = \mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$, the conditions are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ (e_i - e_l) \cdot \Sigma_{E_6} = -p_1 - p_2, \end{cases} \tag{2.129}$$

and Eq. (2.128), where $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ has been used. Note that the first condition in Eq. (2.127) and Eq. (2.128) are universal since they come from the restriction of the universal supersymmetric line bundle $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ to the curve Σ_{E_6} and from the consistency of the definition of $(p_1, p_2, p_3, p_4, p_5, p_6)$, respectively and that Eq. (2.128) impose severe restrictions on the configurations $(p_1, p_2, p_3, p_4, p_5, p_6)$. For example, one can simply set $(p_1, p_2, p_3, p_4, p_5, p_6) = (n, 0, 0, 0, 0, 0)$ to engineer n copies

[26] $L_{1\Sigma_{E_6}} = \mathcal{O}_{\Sigma_{E_6}}(\frac{1}{5}(p_1 - p_3))$, $L_{2\Sigma_{E_6}} = \mathcal{O}_{\Sigma_{E_6}}(-\frac{1}{30}(3p_1 + 5p_2 + 2p_3))$, and $L'_{\Sigma_{E_6}} = \mathcal{O}_{\Sigma_{E_6}}(\frac{1}{2}(p_1 - p_2))$.

of $(\mathbf{3},\mathbf{2})_{1,1,-3}$ on the curve Σ_{E_6}. Then by constraints in Eq. (2.128), n is forced to be vanishing in order to avoid the exotic fields. Let us look at some examples of the non-trivial configurations. It is easy to see that if $p_1 = p_3 = 0$, we obtain non-trivial configurations $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, l, 0, l, 0, l)$, where $l \in \mathbb{Z}_{>0}$. When $p_2 = p_4 = 0$, the non-trivial configurations are $(p_1, p_2, p_3, p_4, p_5, p_6) = (m, 0, m, 0, m, 0)$ with $m \in \mathbb{Z}_{>0}$. If $p_3 = p_4 = 0$, it follows that $(p_1, p_2, p_3, p_4, p_5, p_6) = (n, n, 0, 0, 2n, 2n)$, where $n \in \mathbb{Z}_{>0}$. However, these configurations violate the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Therefore, we need to find more general non-trivial configurations. For the matter fields in the MSSM, we require that the number of the matter field is equal to or less than three. As a result, we impose the conditions $1 \leqslant p_i \leqslant 3,\ i = 1, 2, 3, 4$ in this case. By the constraints in Eq. (2.128), we obtain the following configurations

$$(p_1, p_2, p_3, p_4, p_5, p_6) = \left\{ \begin{array}{l} (0, r, 1-r, 1, r-1, 2r-1), (1, r, 1-r, 0, r+1, 2r), \\ (0, q, 2-q, 2, q-2, 2q-2), (1, q, 2-q, 1, q, 2q-1), \\ (2, q, 2-q, 0, q+2, 2q), (0, v, 3-v, 3, v-3, 2v-3), \\ (1, v, 3-v, 2, v-1, 2v-2), (3, v, 3-v, 0, v+3, 2v), \\ (2, v, 3-v, 1, v+1, 2v-1), (1, t, 4-t, 3, t-2, 2t-3), \\ (2, t, 4-t, 2, t, 2t-2), (3, t, 4-t, 1, t+2, 2t-1), \\ (2, u, 5-u, 3, u-1, 2u-3), (3, 3, 3, 3, 3, 3), \\ (3, u, 5-u, 2, u+1, 2u-2) \end{array} \right\}, \tag{2.130}$$

where $r = 0, 1$, $q = 0, 1, 2$, $v = 0, 1, 2, 3$, $t = 1, 2, 3$, and $u = 2, 3$. Taking the conditions of $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$ into account, the resulting configurations are as follows:

$$(p_1, p_2, p_3, p_4, p_5, p_6) = \left\{ \begin{array}{l} (0,1,1,2,-1,0), (1,0,2,1,0,-1), (1,2,0,1,2,3), \\ (2,1,1,0,3,2), (0,1,2,3,-2,-1), (0,2,1,3,-1,1), \\ (1,3,0,2,2,4), (1,0,3,2,-1,-2), (2,0,3,1,1,-1), \\ (2,3,0,1,4,5), (3,1,2,0,4,2), (3,2,1,0,5,4), \\ (1,2,2,3,0,1), (2,1,3,2,1,0), (2,3,1,2,3,4), \\ (3,2,2,1,4,3) \end{array} \right\}. \tag{2.131}$$

Once we get allowed configurations, it is not difficult to calculate the homological classes of the curves, which satisfy Eq. (2.127) or Eq. (2.129). For example, consider the case of $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, 1, 1, 2, -1, 0)$, one can check that $\Sigma = 3h - e_i + e_l$ solves Eq. (2.127). Let us look at one more complicated example, $(p_1, p_2, p_3, p_4, p_5, p_6) = (3, 2, 2, 1, 4, 3)$. In this case, $\Sigma = 6h + 3e_i + 2e_j - 2e_l$ is a solution of Eq. (2.129). Next we consider the case of $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$. It is clear that we have $L_2 = \mathcal{O}_S(5[e_l, e_m]'' - 2e_i + 2e_j)^{1/30}$. The necessary conditions are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ ([e_l, e_m]'') \cdot \Sigma_{E_6} = -p_1 - p_2, \end{cases} \tag{2.132}$$

and Eq. (2.128). Note that the constraints are the same as the previous case, $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. As a result, the allowed configurations are the same as (2.131). Let us take a look at the classes of the curves, which solve Eq. (2.132). For simplicity, we focus on the case of $[e_l, e_m]'' = e_l - e_m$ and consider $(p_1, p_2, p_3, p_4, p_5, p_6) = (1, 0, 2, 1, 0, -1)$, it is not difficult to see that $\Sigma = h - e_i - e_m$ is a solution. For the case of $(p_1, p_2, p_3, p_4, p_5, p_6) = (2, 1, 1, 0, 3, 2)$, $\Sigma = 4h + 2e_l - e_j - e_m$ can solve Eq. (2.132).

Let us turn to the first case in Table VIII, namely $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)^*$. In this case, L_2 is $\mathcal{O}_S(e_i - e_j)^{1/10}$ and the necessary conditions for the homological class of Σ_{E_6} with given configurations $(p_1, p_2, p_3, p_4, p_5, p_6)$ are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ p_2 + p_3 = 0, \end{cases} \tag{2.133}$$

and Eq. (2.128). Note that to avoid exotic fields, we require that $p_1,\ p_2,\ p_3,\ p_4 \in \mathbb{Z}_{\geqslant 0}$. The constraint, $p_2 + p_3 = 0$ in Eq. (2.133) implies that $p_2 = p_3 = 0$. By the constraints in Eq. (2.128), we obtain $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, 0, 0, 0, 0, 0)$, which means that there are no non-trivial configurations in this case. As a check, by the configurations in (2.131) and the constraint $p_2 + p_3 = 0$, it is easy to see that there is indeed no solution, namely all configurations in (2.131) are completely ruled out by the constraint $p_2 + p_3 = 0$.

For the case of $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$, we have $L_2 = \mathcal{O}_S(e_j - e_i)^{7/30}$. Given the configuration $(p_1, p_2, p_3, p_4, p_5, p_6)$, the necessary conditions are

$$\begin{cases} (e_i - e_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ p_3 = 2p_1 + p_2, \end{cases} \tag{2.134}$$

and Eq. (2.128). Since $(\mathbf{3}, \mathbf{2})_{1,1,-3}$, $(\mathbf{3}, \mathbf{2})_{-1,1,-3}$, $(\bar{\mathbf{3}}, \mathbf{1})_{1,-4,-3}$, and $(\bar{\mathbf{3}}, \mathbf{1})_{1,-4,-3}$ are all matter in the MSSM, we require that $p_i \leqslant 3,\ i = 1, 2, 3, 4$. By the second condition in Eq. (2.134), we have $(p_1, p_2) = (1, 0),\ (0, 1),\ (0, 2),\ (0, 3)$, or $(1, 1)$. Since $p_4 \leqslant 3$, it follows that the allowed configurations are $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, 1, 1, 2, -1, 0)$, $(1, 0, 2, 1, 0, -1)$, and $(1, 1, 3, 3, -1, -1)$. Recall that in order to obtain matter in the MSSM, it is required that $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. As a result, the resulting configurations are

$$(p_1, p_2, p_3, p_4, p_5, p_6) = \Big\{\ (0, 1, 1, 2, -1, 0),\ (1, 0, 2, 1, 0, -1)\ \Big\}. \tag{2.135}$$

$(\alpha_1, \alpha_2, \alpha_3)$	Conditions	L_2
$(0,0,0)$	$(e_i - e_l) \cdot \Sigma = p_2 + p_3$	$\mathcal{O}_S(5e_l - 2e_i - 3e_j)^{1/30}$
	$(e_i - e_l) \cdot \Sigma = -p_1 - p_2$	$\mathcal{O}_S(-5e_l + 3e_i + 2e_j)^{1/30}$
$(1,1,0)^*$	$p_2 + p_3 = 0$	$\mathcal{O}_S(e_i - e_j)^{1/10}$
$(1,1,0)$	$([e_l, e_m]'') \cdot \Sigma = -p_1 - p_2$	$\mathcal{O}_S(5[e_l, e_m]'' - 2e_i + 2e_j)^{1/30}$
$(2,2,0)$	$(-e_l + e_j) \cdot \Sigma = -p_1 - p_2$	$\mathcal{O}_S(-5e_l - 2e_i + 7e_j)^{1/30}$
	$(e_l - e_i) \cdot \Sigma = -p_1 - p_2$	$\mathcal{O}_S(5e_l - 7e_i + 2e_j)^{1/30}$
$(3,3,0)$	$p_3 = 2p_1 + p_2$	$\mathcal{O}_S(e_j - e_i)^{7/30}$

Table XIV. The conditions for Σ_{E_6} supporting the field configurations $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $L_1 = \mathcal{O}_S(e_j - e_i)^{1/5}$ and constraints $p_4 = p_2 + p_3 - p_1$, $p_5 = 2p_1 - p_3$, and $p_6 = p_1 + p_2 - p_3$.

As a check, using the configurations in (2.131) and the constraint $p_3 = 2p_1 + p_2$, one can see that the resulting configurations are the same as that in (2.135). Now let us solve the classes of the curves satisfying Eq. (2.134). For these two configurations, the first condition in Eq. (2.134) can be solved by $\Sigma = h - e_i - e_l$. For other $U(1)^2$ flux configurations corresponding to the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $k = 2$, the analysis is similar to the case of $k = 0, 1$. We summarize the results in Table XIV.

After analyzing the spectrum from the curves, it is clear that we are unable to obtain a minimal spectrum of the MSSM, but non-minimal spectra with doublet-triplet splitting can be obtained. In the next section we will give examples of non-minimal spectra for the MSSM.

3. Non-minimal Spectrum for the MSSM: Examples

In the previous section we already analyzed the spectrum from the curves $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$, and Σ_{E_6}. With some physical requirements, we obtain all field configurations supported by the curves. In what follows, we shall give examples of the non-minimal MSSM spectra using the results shown in the previous subsection 2b.

In what follows, we shall focus on the case M_1 in Table XIII. In this case, Q and e^c are localized on the curves with $G_\Sigma = SO(12)$. u^c comes from Σ_{E_6} and $d^c, L, \bar{h}$ and h live on $\Sigma_{SU(7)}$. It is not difficult to see that in the examples considered, we are unable to get a minimal spectrum of the MSSM without exotic fields. However, it is possible to construct non-minimal spectra of the MSSM. One possible way is that we can make the exotic fields form trilinear couplings with conserved $U(1)$ charges so that they can decouple from the low-energy spectrum. According to the results in Table VIII, let us consider the $U(1)^2$ flux configuration $L_1 = \mathcal{O}_S(e_1 - e_2)^{1/5}$ and $L_2 = \mathcal{O}_S(5e_3 - 2e_2 - 3e_1)^{1/30}$, which corresponds to the case of $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$ on the bulk. To obtain three copies of Q and e^c, we engineer two curves $\Sigma^1_{SO(12)}$ and $\Sigma^2_{SO(12)}$ with field content $(2,0,-2,0,4)$ and $(1,0,-1,0,2)$, respectively. The exotic fields are $2 \times (\bar{\mathbf{3}}, \mathbf{1})_{-2,2,4}$ and one singlet on $\Sigma^1_{SO(12)}$. For the curve $\Sigma^2_{SO(12)}$, we get exotic fields $1 \times (\bar{\mathbf{3}}, \mathbf{1})_{-2,2,4}$ and two singlets. To get three copies of u^c, we arrange two curves, $\Sigma^1_{E_6}$ and $\Sigma^2_{E_6}$ with field content $(3,1,2,0,4,2)$ and $(2,1,1,0,3,2)$, respectively. We have exotic fields $3 \times (\mathbf{3}, \mathbf{2})_{1,1,-3}$, $1 \times (\mathbf{3}, \mathbf{2})_{-1,1,-3}$ and six singlets on $\Sigma^1_{E_6}$. On $\Sigma^2_{E_6}$, the exotic fields are $2 \times (\mathbf{3}, \mathbf{2})_{1,1,-3}$, $1 \times (\mathbf{3}, \mathbf{2})_{-1,1,-3}$ and five singlets. Since the rest of the fields in the case of M_1 come from the curves with $G_\Sigma = SU(7)$, we can easily engineer $3 \times d^c$, $3 \times L$, $1 \times \bar{h}$ and $1 \times h$ on individual curves, denoted respectively by $\Sigma^1_{SU(7)}$, $\Sigma^2_{SU(7)}$, $\Sigma^u_{SU(7)}$, and $\Sigma^d_{SU(7)}$. Note that $(\mathbf{3}, \mathbf{2})_{\pm 1,1,-3}$, $(\bar{\mathbf{3}}, \mathbf{1})_{-2,2,4}$, and $(\mathbf{1}, \bar{\mathbf{2}})_{7,-3,-1}$ can form trilinear couplings.

Multi.	Curve	Σ	g_Σ	$L_{1\Sigma}$	$L_{2\Sigma}$	L'_Σ
$2\times Q$ [27] $+3\times e^c$	$\Sigma^1_{SO(12)}$	$5h+2e_2-2e_3$ $-2e_4-2e_5$	0	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{2/5}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{7/15}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{1/3}$
$1\times Q$ [28]	$\Sigma^2_{SO(12)}$	$4h+e_2-e_3$ $-2e_4-2e_5$	0	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{7/30}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/6}$
$2\times u^c$ [29]	$\Sigma^1_{E_6}$	$5h+3e_3-e_1$	0	$\mathcal{O}_{\Sigma^1_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^1_{E_6}}(-1)^{3/5}$	$\mathcal{O}_{\Sigma^1_{E_6}}(1)$
$1\times u^c$ [30]	$\Sigma^2_{E_6}$	$4h+2e_3-e_1$	0	$\mathcal{O}_{\Sigma^2_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{E_6}}(-1)^{13/30}$	$\mathcal{O}_{\Sigma^2_{E_6}}(1)^{1/2}$
$3\times d^c$	$\Sigma^1_{SU(7)}$	$4h+e_2+e_3-2e_1$	0	$\mathcal{O}_{\Sigma^1_{SU(7)}}(1)^{3/5}$	$\mathcal{O}_{\Sigma^1_{SU(7)}}(-1)^{3/10}$	$\mathcal{O}_{\Sigma^1_{SU(7)}}(1)^{3/14}$
$3\times L$	$\Sigma^2_{SU(7)}$	$4h+e_3+e_1-2e_2$	0	$\mathcal{O}_{\Sigma^2_{SU(7)}}(-1)^{3/5}$	$\mathcal{O}_{\Sigma^2_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{SU(7)}}(1)^{1/7}$
$1\times \bar{h}$	$\Sigma^u_{SU(7)}$	$h-e_1-e_3$	0	$\mathcal{O}_{\Sigma^u_{SU(7)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^u_{SU(7)}}(1)^{1/15}$	$\mathcal{O}_{\Sigma^u_{SU(7)}}(-1)^{1/21}$
$1\times h$	$\Sigma^d_{SU(7)}$	$h-e_2-e_4$	0	$\mathcal{O}_{\Sigma^d_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^d_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\Sigma^d_{SU(7)}}(1)^{1/21}$
$1\times \Phi$	$\Sigma^\Phi_{SU(7)}$	$3h-e_1-e_3-2e_2$	0	$\mathcal{O}_{\Sigma^\Phi_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^\Phi_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\Sigma^\Phi_{SU(7)}}(1)^{1/21}$

Table XV. An example for a non-minimal MSSM spectrum from $G_S = SU(6)$ with the $U(1)^2$ gauge flux configuration $L_1 = \mathcal{O}_S(e_1-e_2)^{1/5}$ and $L_2 = \mathcal{O}_S(5e_3-2e_2-3e_1)^{1/30}$.

To make the exotic fields form the couplings, we introduce one extra curve $\Sigma^\Phi_{SU(7)}$ with $\Phi = (\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$. Now we arrange $\Sigma^1_{SO(12)}$ intersects $\Sigma^1_{E_6}$ and $\Sigma^2_{E_6}$, so does $\Sigma^2_{SO(12)}$. The curve $\Sigma^u_{SU(7)}$ passes through the intersection point of $\Sigma^1_{SO(12)}$ and $\Sigma^1_{E_6}$ and that of $\Sigma^2_{SO(12)}$ and $\Sigma^2_{E_6}$. The vertices of the triple intersections $(\Sigma^1_{SO(12)},\Sigma^1_{E_6},\Sigma^u_{SU(7)})$ and $(\Sigma^2_{SO(12)},\Sigma^2_{E_6},\Sigma^u_{SU(7)})$ represent the coupling $Qu^c\bar{h}$. Another two vertices are

[27] With one additional singlet.

[28] With two additional singlets.

[29] With six additional singlets.

[30] With five additional singlets.

formed by triple intersections $(\Sigma^1_{SO(12)}, \Sigma^2_{E_6}, \Sigma^{\Phi}_{SU(7)})$ and $(\Sigma^2_{SO(12)}, \Sigma^1_{E_6}, \Sigma^{\Phi}_{SU(7)})$, which represent the coupling $\Theta\Psi\Phi$ and $\widetilde{\Theta}\Psi\Phi$, where $\Theta = (\mathbf{3}, \mathbf{2})_{1,1,-3}$, $\widetilde{\Theta} = (\mathbf{3}, \mathbf{2})_{-1,1,-3}$, and $\Psi = (\bar{\mathbf{3}}, \mathbf{1})_{-2,2,4}$. When Φ gets a vev, the exotic fields are decoupled through the coupling, which means that at low energy, those fields will not show up in the spectrum. To obtain the coupling Qd^ch, one can arrange two curves $\Sigma^1_{SU(7)}$, and $\Sigma^d_{SU(7)}$ intersect $\Sigma^1_{SO(12)}$ at one point. For the coupling Le^ch, one can let the curve $\Sigma^2_{SU(7)}$ intersect $\Sigma^d_{SU(7)}$ at another point on $\Sigma^1_{SO(12)}$. The intersection point of $\Sigma^u_{SU(7)}$ and $\Sigma^2_{SU(7)}$ represents the coupling $L\nu^c\bar{h}$. To sum up, the superpotential is as follows:

$$\mathcal{W} \supset \mathcal{W}_{\text{MSSM}} + \Theta\Psi\Phi + \widetilde{\Theta}\Psi\Phi + \cdots . \tag{2.136}$$

As mentioned earlier, through the last two couplings in (2.136), we obtain a non-minimal MSSM spectrum at low energy. Note that in this case, $\bar{h}$ and h come from the curves $\Sigma^u_{SU(7)}$ and $\Sigma^d_{SU(7)}$, respectively. As shown in section F-2b, doublet-triplet splitting can be achieved by $U(1)^2$ gauge fluxes. Therefore, a non-minimal spectrum of the MSSM with doublet-triple splitting can be achieved in a local F-theory model where $G_S = SU(6)$ and with $U(1)^2$ gauge fluxes. As shown in section F-2b, given the field configurations, one can calculate the homological classes of the curves supporting the configurations. For the present example, we simply summarize the field content and the classes of the curves in Table XV. Note that in the previous example there are some exotic singlets. Following similar procedure, these singlets can be lifted via trilinear couplings. Let us consider the following example. To obtain three copies of Q and e^c, we engineer two curves $\widetilde{\Sigma}^1_{SO(12)}$ and $\widetilde{\Sigma}^2_{SO(12)}$ with field content $(2, 1, -2, -1, 3)$ and $(1, 2, -1, -2, 0)$, respectively. Clearly the exotic fields are $1 \times (\bar{\mathbf{3}}, \mathbf{1})_{2,-4,2}$, $2 \times (\bar{\mathbf{3}}, \mathbf{1})_{-2,2,4}$, and $1 \times (\mathbf{1}, \bar{\mathbf{2}})_{-2,-3,4}$ on $\widetilde{\Sigma}^1_{SO(12)}$. For the curve $\widetilde{\Sigma}^2_{SO(12)}$, we get exotic fields $2 \times (\bar{\mathbf{3}}, \mathbf{1})_{2,-4,2}$, $1 \times (\bar{\mathbf{3}}, \mathbf{1})_{-2,2,4}$, and $2 \times (\mathbf{1}, \bar{\mathbf{2}})_{-2,-3,4}$. To get three copies of u^c, we arrange two curves, $\widetilde{\Sigma}^1_{E_6}$ and $\widetilde{\Sigma}^2_{E_6}$ with field content $(2, 1, 1, 0, 3, 2)$ and

$(3,1,2,0,4,2)$, respectively. We have exotic fields $2\times(\mathbf{3},\mathbf{2})_{1,1,-3}$, $1\times(\mathbf{3},\mathbf{2})_{-1,1,-3}$, and five singlets on $\widetilde{\Sigma}^1_{E_6}$. On $\widetilde{\Sigma}^2_{E_6}$, the exotic fields are $3\times(\mathbf{3},\mathbf{2})_{1,1,-3}$, $1\times(\mathbf{3},\mathbf{2})_{-1,1,-3}$, and six singlets. Since the rest of the fields in the case of M_1 come from the curves with $G_\Sigma = SU(7)$, we can easily engineer $3\times d^c$, $3\times L$, $1\times\bar{h}$ and $1\times h$ on individual curves, denoted respectively by $\widetilde{\Sigma}^1_{SU(7)}$, $\widetilde{\Sigma}^2_{SU(7)}$, $\widetilde{\Sigma}^u_{SU(7)}$, and $\widetilde{\Sigma}^d_{SU(7)}$. Note that these exotic fields can form trilinear couplings with triplets on $\Sigma_{SU(7)}$. To make the exotic fields form the couplings, we introduce three extra curves $\Sigma^{\Upsilon_1}_{SU(7)}$, $\Sigma^{\bar{\Upsilon}_2}_{SU(7)}$, and $\Sigma^{\Upsilon_3'}_{SU(7)}$ with $\Upsilon_1 = (\mathbf{3},\mathbf{1})_{-7,-2,1}$, $\bar{\Upsilon}_2 = (\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}$, and $\Upsilon_3+\Lambda$, respectively, where $\Upsilon_3 = (\mathbf{3},\mathbf{1})_{-7,-2,1}$ and $\Lambda = (\mathbf{1},\mathbf{1})_{-7,0,-5}$. The superpotential is as follows:

$$\mathcal{W} \;\supset\; \mathcal{W}_{\text{MSSM}} + \Xi\Delta\Upsilon_1 + \Xi\widetilde{\Delta}\Upsilon_1 + \Theta\Pi\bar{\Upsilon}_2 + \widetilde{\Theta}\Pi\bar{\Upsilon}_2 + \Psi\Lambda\Upsilon_3 + \cdots\,, \tag{2.137}$$

where $\Xi = (\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$, $\Delta = (\mathbf{1},\mathbf{1})_{1,6,-3}$, $\widetilde{\Delta} = (\mathbf{1},\mathbf{1})_{-1,6,-3}$, and $\Pi = (\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$. When Υ_1, $\bar{\Upsilon}_2$, and Υ_3 get vevs, the exotic fields are decoupled via the couplings, which means that at low energy, those fields will not show up in the spectrum. For the couplings in $\mathcal{W}_{\text{MSSM}}$, the arrangement of the curves is similar to the previous example. We are not going to repeat that. In this example, we obtain a non-minimal MSSM spectrum at low energy. The field content and the classes of the curves are summarized in Table XVI.[31]

[31] In this example Q and u^c are localized on different curves. The Yukawa coupling $Qu^c\bar{h}$ descended from **10105** can be expressed as $[\widetilde{\Sigma}^1_{SO(12)}(1,2)+\widetilde{\Sigma}^2_{SO(12)}(3)][\widetilde{\Sigma}^1_{E_6}(1)+\widetilde{\Sigma}^2_{E_6}(2,3)][\widetilde{\Sigma}^u_{SU(7)}]$ generating nonvanishing diagonal elements in the Yukawa mass matrix, where the indices in the parenthesis represent the generations.

Multi.	Curve	Σ	g_Σ	$L_{1\Sigma}$	$L_{2\Sigma}$	L'_Σ
$2\times Q$ $+3\times e^c$	$\widetilde{\Sigma}^1_{SO(12)}$	$5h-e_1-4e_3$ $-e_5$	0	$\mathcal{O}_{\widetilde{\Sigma}^1_{SO(12)}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SO(12)}}(1)^{17/30}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SO(12)}}(1)^{1/3}$
$1\times Q$	$\widetilde{\Sigma}^2_{SO(12)}$	$4h+e_1-2e_3+e_6$	0	$\mathcal{O}_{\widetilde{\Sigma}^2_{SO(12)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SO(12)}}(1)^{13/30}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SO(12)}}(1)^{1/6}$
$1\times u^c$	$\widetilde{\Sigma}^1_{E_6}$	$4h+2e_3-e_1$	0	$\mathcal{O}_{\widetilde{\Sigma}^1_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{E_6}}(-1)^{13/30}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{E_6}}(1)^{1/2}$
$2\times u^c$	$\widetilde{\Sigma}^2_{E_6}$	$5h+3e_3-e_1$	0	$\mathcal{O}_{\widetilde{\Sigma}^2_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{E_6}}(-1)^{3/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{E_6}}(1)$
$3\times d^c$	$\widetilde{\Sigma}^1_{SU(7)}$	$4h+e_2+e_3-2e_1$	0	$\mathcal{O}_{\widetilde{\Sigma}^1_{SU(7)}}(1)^{3/5}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SU(7)}}(-1)^{3/10}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SU(7)}}(1)^{3/14}$
$3\times L$	$\widetilde{\Sigma}^2_{SU(7)}$	$4h+e_3+e_1-2e_2$	0	$\mathcal{O}_{\widetilde{\Sigma}^2_{SU(7)}}(-1)^{3/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SU(7)}}(1)^{1/7}$
$1\times \bar{h}$	$\widetilde{\Sigma}^u_{SU(7)}$	$3h+e_2-e_4$	0	$\mathcal{O}_{\widetilde{\Sigma}^u_{SU(7)}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^u_{SU(7)}}(1)^{1/15}$	$\mathcal{O}_{\widetilde{\Sigma}^u_{SU(7)}}(-1)^{1/21}$
$1\times h$	$\widetilde{\Sigma}^d_{SU(7)}$	$h-e_2-e_4$	0	$\mathcal{O}_{\widetilde{\Sigma}^d_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^d_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\widetilde{\Sigma}^d_{SU(7)}}(1)^{1/21}$
$1\times \Upsilon_1$	$\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}$	$h-e_2-e_3$	0	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}}(1)^{1/10}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}}(-1)^{1/14}$
$1\times \bar{\Upsilon}_2$	$\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}$	$2h-e_1-e_4-e_5$	0	$\mathcal{O}_{\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}}(-1)^{1/10}$	$\mathcal{O}_{\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}}(1)^{1/14}$
$1\times \Upsilon_3$ $+1\times \Lambda$	$\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}$	$h-e_2-e_4$	0	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}}(-1)^{2/21}$

Table XVI. An example for a non-minimal MSSM spectrum from $G_S = SU(6)$ with the $U(1)^2$ gauge flux configuration $L_1 = \mathcal{O}_S(e_1 - e_2)^{1/5}$ and $L_2 = \mathcal{O}_S(5e_3 - 2e_2 - 3e_1)^{1/30}$.

G. Conclusion

In this chapter we demonstrated how to obtain $U(1)^2$ gauge flux configurations (L_1, L_2) with an exotic-free bulk spectrum of the local F-theory model with $G_S = SU(6)$. In this case each configuration is constructed by two fractional line bundles, which are well-defined in the sense that up to a linear transformation of the $U(1)$ charges, an $U(1)^2$ flux configuration can be associated with a polystable bundle of rank two with structure group $U(1)^2$. Under physical assumptions, we obtained all flux configurations as shown in Table VIII and Table IX. For the case of $G_S = SO(10)$, as shown in [15] there is a no-go theorem which states that for an exotic-free spectrum, there are no solutions for $U(1)^2$ gauge fluxes.

To build a model of the MSSM, we studied the field configurations localized on the curves with non-trivial gauge fluxes induced from the restriction of the flux configurations on the bulk S. With non-trivial induced fluxes, the enhanced gauge group G_Σ will be broken into $G_{\text{std}} \times U(1)$. Under physical assumptions, we obtained all field configurations localized on the curves with $G_\Sigma = SU(7)$, $G_\Sigma = SO(12)$ and $G_\Sigma = E_6$. Form the breaking patterns, we knew that Higgs fields are localized on the curves $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$. On the curve $\Sigma_{SU(7)}$, we found that doublet-triplet splitting can be achieved. However, it is impossible to get the splitting on the curve $\Sigma_{SO(12)}$ without raising exotic fields, which means that when building models, we should engineer the Higgs fields on the curve $\Sigma_{SU(7)}$ instead of $\Sigma_{SO(12)}$. Unlike Higgs fields, matter fields in the MSSM are distributed over the curves $G_\Sigma = SU(7)$, $G_\Sigma = SO(12)$ and $G_\Sigma = E_6$. With the solved field configurations, it is clear that it is extremely difficult to get the minimal spectrum of the MSSM without exotic fields. However, if those exotic fields can form trilinear couplings with the doublets or triplets on the curves with $G_\Sigma = SU(7)$, the exotic fields can be lifted from the massless spectrum when these doublets or triplets get vevs. In order to achieve this, we introduced extra curves to support these doublets or triplets coupled to exotic fields. With this procedure, we can construct a non-minimal spectrum of the MSSM with doublet-triple splitting.

CHAPTER III

SEMI-LOCAL FLIPPED $SU(5)$ MODELS[32]

In this chapter we construct supersymmetric flipped $SU(5)$ GUTs from E_8 singularities in F-theory. We start from an $SO(10)$ singularity unfolded from an E_8 singularity by using an $SU(4)$ spectral cover. To obtain realistic models, we consider $(3,1)$ and $(2,2)$ factorizations of the $SU(4)$ cover. After turning on the massless $U(1)_X$ gauge flux, we obtain the $SU(5)\times U(1)_X$ gauge group. Based on the well-studied geometric backgrounds in the literature, we demonstrate several models and discuss their phenomenology.

A. ADE Singularities and Spectral Covers

1. Elliptically fibered Calabi-Yau Fourfolds and ADE Singularities

Let us consider an elliptically fibered Calabi-Yau fourfold $\pi : X_4 \to B_3$ with a section $\sigma_{B_3} : B_3 \to X_4$. Due to the presence of the section σ_{B_3}, X_4 can be described by the Weierstrass form:

$$y^2 = x^3 + fx + g, \tag{3.1}$$

where f and g are sections of suitable line bundles over B_3. More precisely, to maintain Calabi-Yau condition $c_1(X_4) = 0$, it is required that[33] $f \in \Gamma(K_{B_3}^{-4})$ and $g \in \Gamma(K_{B_3}^{-6})$, where K_{B_3} is the canonical bundle of B_3. Let $\Delta \equiv 4f^3 + 27g^2$ be the discriminant of the elliptic fibration Eq. (3.1) and S be one component of the locus $\{\Delta = 0\}$ where elliptic fibers degenerate. In the vicinity of S, one can regard X_4 as an ALE fibration

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[32]Reprinted from *Journal of High Energy Physics*, Vol 2011, Number 3, 49, Ching-Ming Chen and Yu-Chieh Chung, Flipped $SU(5)$ GUTs from E_8 Singularities in F-theory, Copyright 2011, with permission from SISSA.

[33]The symbol $\Gamma(L)$ stands for a set of global sections of the bundle L.

over the surface S. To construct $SO(10)$ and flipped $SU(5)$ GUT models, one can start with engineering a D_5 singularity corresponding to the gauge group $SO(10)$ in the following way. Let z be a section of the normal bundle N_{S/B_3} of S in B_3 and the zero section then represents the surface S. Since f and g are sections of some line bundles over B_3, one can locally expand f and g in terms of z as follows:

$$f = 3\sum_{k=0}^{4} f_k(u,v)z^k, \qquad g = 2\sum_{l=0}^{6} g_l(u,v)z^l, \tag{3.2}$$

where (u,v) are coordinates of S and the prefactors 2 and 3 are just for convenience. Then the Weierstrass form Eq. (3.1),

$$y^2 = x^3 + 3\sum_{k=0}^{4} f_k(u,v)z^k x + 2\sum_{l=0}^{6} g_l(u,v)z^l, \tag{3.3}$$

describes an ALE fibration over S, where $f_k \in \Gamma(K_{B_3}^{-4} \otimes \mathcal{O}_{B_3}(-kS))$ and $g_l \in \Gamma(K_{B_3}^{-6} \otimes \mathcal{O}_{B_3}(-lS))$.[34] According to the Kodaira classification of singular elliptic fibers, one can classify the singularity of an elliptic fibration by the vanishing order of f, g, and Δ, denoted by $\mathrm{ord}(f)$, $\mathrm{ord}(g)$, and $\mathrm{ord}(\Delta)$, respectively. We summarize the relevant ADE classification and corresponding gauge groups in Table XVII. The detailed list can be found in [17]. According to Table XVII, a D_5 singularity corresponds to the case of $(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (\geqslant 2, 3, 7)$ or $(2, \geqslant 3, 7)$. Recall that S is the locus $\{z = 0\}$. To obtain a D_5 singularity, the vanishing orders of f and g at $z = 0$ are required to be two and three, respectively[35]. Let us consider the sections f and g to

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[34]By adjunction formula, $K_S = K_{B_3} \otimes N_{S/B_3}|_S$, we have $f_k \in \Gamma(K_S^{-4} \otimes N_{S/B_3}^{4-k})$ and $g_l \in \Gamma(K_S^{-6} \otimes N_{S/B_3}^{6-l})$, where K_S is the canonical bundle of S.

[35]One can show that in this case the only consistent triplet vanishing orders for a D_5 singularity is $(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (2, 3, 7)$. The higher order terms are irrelevant to the singularity. However, they may change the monodromy group [76].

Singularity	ord(f)	ord(g)	ord(Δ)	Gauge Group
A_n	0	0	$n+1$	$SU(n+1)$
D_{n+4}	$\geqslant 2$	3	$n+6$	$SO(2n+8)$
D_{n+4}	2	$\geqslant 3$	$n+6$	$SO(2n+8)$
E_6	$\geqslant 3$	4	8	E_6
E_7	3	$\geqslant 5$	9	E_7
E_8	$\geqslant 4$	5	10	E_8

Table XVII. ADE singularities and corresponding gauge groups.

be

$$f = 3(f_2 z^2 + f_3 z^3), \qquad g = 2(g_3 z^3 + g_4 z^4 + g_5 z^5). \tag{3.4}$$

Then the corresponding discriminant is given by

$$\begin{aligned} \Delta &= cz^6[(f_2^3 + g_3^2) + (3f_2^2 f_3 + 2g_3 g_4)z + (3f_2 f_3^2 + g_4^2 + 2g_3 g_5)z^2 \\ &+ (f_3^3 + 2g_4 g_5)z^3 + \mathcal{O}(z^4)], \end{aligned} \tag{3.5}$$

where $c = 4 \cdot 27$. To obtain $\text{ord}(\Delta) = 7$, let us set $f_2 = -h^2$ and $g_3 = h^3$, where $h \in \Gamma(K_{B_3}^{-2} \otimes \mathcal{O}_{B_3}(-S))$. Then the discriminant is reduced to

$$\Delta = cz^7[(3h^4 f_3 + 2h^3 g_4) + (-3h^2 f_3^2 + g_4^2 + 2h^3 g_5)z + (f_3^3 + 2g_4 g_5)z^2 + \mathcal{O}(z^3)]. \tag{3.6}$$

The singularity of ALE fibration is now characterized by the sections $\{h, f_3, g_4, g_5\}$. When $h = 0$, one can find that $(\text{ord}(f), \text{ord}(g), \text{ord}(\Delta)) = (3, 4, 8)$ at the locus $\{z = 0\} \cap \{h = 0\}$. It follows from the Kodaira classification that the singularity is enhanced to E_6. When $3hf_3 + 2g_4 = 0$, the triplet vanishing orders becomes $(2, 3, 8)$, which implies that the singularity at the locus $\{z = 0\} \cap \{3hf_3 + 2g_4 = 0\}$ is D_6 and that

G_S	$(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta))$	Locus
$SO(10)$	$(2, 3, 7)$	$\{z = 0\}$
E_6	$(3, 4, 8)$	$\{z = 0\} \cap \{h = 0\}$
$SO(12)$	$(2, 3, 8)$	$\{z = 0\} \cap \{3hf_3 + 2g_4 = 0\}$
E_7	$(3, 5, 9)$	$\{z = 0\} \cap \{h = 0\} \cap \{g_4 = 0\}$
$SO(14)$	$(2, 3, 9)$	$\{z = 0\} \cap \{3hf_3 + 2g_4 = 0\} \cap \{3f_3^2 - 8hg_5 = 0\}$

Table XVIII. Gauge enhancements and corresponding loci.

the corresponding enhanced gauge group is $SO(12)$. In a similar manner, one can find the codimension two singularities corresponding to E_7 and $SO(14)$ in S. We summarize the results in Table XVIII.

For later use, it is convenient to introduce the Tate form of the fibration:

$$y^2 = x^3 + \mathbf{b}_4 x^2 z + \mathbf{b}_3 y z^2 + \mathbf{b}_2 x z^3 + \mathbf{b}_0 z^5, \tag{3.7}$$

where $\mathbf{b}_m \in \Gamma(K_S^{m-6} \otimes N_{S/B_3})$. Actually, Eq. (3.7) is nothing more than the unfolding of an E_8 singularity to a singularity of $SO(10)$. Notice that by comparing Eq. (3.7) with Eqs. (3.3) and (3.4), one can obtain the relations between $\{f_2, f_3, g_3, g_4, g_5\}$ and $\{\mathbf{b}_0, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ as follows:

$$\begin{cases} f_2 = -\frac{1}{9}\mathbf{b}_4^2 \\ f_3 = \frac{1}{3}\mathbf{b}_2 \\ g_3 = \frac{1}{27}\mathbf{b}_4^3 \\ g_4 = \frac{1}{8}\mathbf{b}_3^2 - \frac{1}{6}\mathbf{b}_2\mathbf{b}_4 \\ g_5 = \frac{1}{2}\mathbf{b}_0. \end{cases} \tag{3.8}$$

G_S	Locus	Object
$SO(10)$	$\{z=0\}$	GUT Seven-branes
E_6	$\{z=0\}\cap\{\mathbf{b}_4=0\}$	Matter **16**
$SO(12)$	$\{z=0\}\cap\{\mathbf{b}_3=0\}$	Matter **10**
E_7	$\{z=0\}\cap\{\mathbf{b}_3=0\}\cap\{\mathbf{b}_4=0\}$	Yukawa Coupling **16 16 10**
$SO(14)$	$\{z=0\}\cap\{\mathbf{b}_3=0\}\cap\{\mathbf{b}_2^2-4\mathbf{b}_0\mathbf{b}_4=0\}$	Extra Coupling

Table XIX. Gauge enhancements in $SO(10)$ GUT geometry.

With the relations in Eq. (3.8), the discriminant Eq. (3.6) becomes

$$\begin{aligned}\Delta &= \widetilde{c}z^7\{16\mathbf{b}_3^2\mathbf{b}_4^3+[27\mathbf{b}_3^4-72\mathbf{b}_2\mathbf{b}_3^2\mathbf{b}_4-16\mathbf{b}_4^2(\mathbf{b}_2^2-4\mathbf{b}_0\mathbf{b}_4)]z\\ &+ [16\mathbf{b}_2(4\mathbf{b}_2^2-18\mathbf{b}_0\mathbf{b}_4)+216\mathbf{b}_0\mathbf{b}_3^2]z^2+\mathcal{O}(z^3)\},\end{aligned} \tag{3.9}$$

where $\widetilde{c}=\frac{1}{16}$. It follows from Eq. (3.8) that the codimension one loci $\{z=0\}\cap\{h=0\}$ and $\{z=0\}\cap\{3hf_3+2g_4\}$ in S can be equivalently expressed as $\{z=0\}\cap\{\mathbf{b}_4=0\}$ and $\{z=0\}\cap\{\mathbf{b}_3=0\}$, respectively. Due to the gauge enhancements, matter **16** and **10** are localized at the loci of E_6 and $SO(12)$ singularities, respectively. One can also find that the loci of codimension two singularities E_7 and $SO(14)$ in S are $\{z=0\}\cap\{\mathbf{b}_3=0\}\cap\{\mathbf{b}_4=0\}$ and $\{z=0\}\cap\{\mathbf{b}_3=0\}\cap\{\mathbf{b}_2^2-4\mathbf{b}_0\mathbf{b}_4=0\}$, respectively. At these loci, the corresponding gauge groups are enhanced to E_7 and $SO(14)$, respectively[36]. In particular, the Yukawa coupling **16 16 10** can be realized at the points with E_7 singularities. We summarize the results in Table XIX.

[36]One can also use Tate's algorithm to determine the singularity type of the Tate form Eq. (3.7) [12].

2. $SU(4)$ Spectral Cover

To engineer the $SO(10)$ gauge group from an E_8 singularity, let us consider the following decomposition

$$\begin{aligned} E_8 &\rightarrow SO(10) \times SU(4) \\ \mathbf{248} &\rightarrow (\mathbf{1},\mathbf{15}) + (\mathbf{45},\mathbf{1}) + (\mathbf{10},\mathbf{6}) + (\mathbf{16},\mathbf{4}) + (\overline{\mathbf{16}},\bar{\mathbf{4}}). \end{aligned} \tag{3.10}$$

and the Tate form of the fibration,

$$y^2 = x^3 + \mathbf{b}_4 x^2 z + \mathbf{b}_3 y z^2 + \mathbf{b}_2 x z^3 + \mathbf{b}_0 z^5. \tag{3.11}$$

For simplicity, let us define $c_1 \equiv c_1(S)$ and $t \equiv -c_1(N_{S/B_3})$, then the homological classes of the sections x, y, z, and b_m can be expressed as

$$[x] = 3(c_1 - t),\ [y] = 2(c_1 - t),\ [z] = -t,\ [\mathbf{b}_m] = (6-m)c_1 - t \equiv \eta - mc_1. \tag{3.12}$$

Recall that locally X_4 can be described by an ALE fibration over S. Pick a point $p \in S$ and the fiber is an ALE space denoted by ALE_p. One can construct an ALE space by resolving an orbifold $\mathbb{C}^2/\Gamma_{ADE}$, where Γ_{ADE} is a discrete subgroup of $SU(2)$ [112], for more information, see [113–117]. It was shown that the intersection matrix of the exceptional 2-cycles corresponds to the Cartan matrix of ADE types. In this chapter we will focus on engineering the $SO(10)$ gauge group by unfolding an E_8 singularity. To this end, let us consider $\alpha_i \in H_2(\mathrm{ALE}_p, \mathbb{Z})$, $i = 1, 2, ..., 8$ to be the roots[37] of E_8. The extended E_8 Dynkin diagram with roots and Dynkin indices are shown in Fig 1. Notice that $\alpha_{-\theta}$ is the highest root and satisfies the condition $\alpha_{-\theta} + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0$. To obtain $SO(10)$, we

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[37] By abuse of notation, the corresponding exceptional 2-cycles are also denoted by α_i

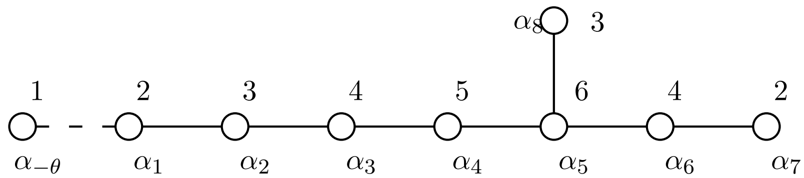

Fig. 1. The extended E_8 Dynkin diagram and indices.

keep the volume of the cycles $\{\alpha_4, \alpha_5, ..., \alpha_8\}$ vanishing and then $SU(4)$ is generated by $\{\alpha_1, \alpha_2, \alpha_3\}$. An enhancement to E_6 happens when α_3 or any of its image under the Weyl permutation shrinks to zero size. Let $\{\lambda_1, ..., \lambda_4\}$ be the periods of these 2-cycles. As described in [16, 47], the information of theses λ_i can be encoded in the coefficients $\mathbf{b}_m$ in Eq. (3.11) via the following relations:

$$\begin{cases} \sum_i \lambda_i = \frac{b_1}{b_0} = 0 \\ \sum_{i<j} \lambda_i\lambda_j = \frac{b_2}{b_0} \\ \sum_{i<j<k} \lambda_i\lambda_j\lambda_k = \frac{b_3}{b_0} \\ \prod_l \lambda_l = \frac{b_4}{b_0}, \end{cases} \tag{3.13}$$

where $b_m \equiv \mathbf{b}_m|_{z=0}$. Equivalently, $\{\lambda_1, ..., \lambda_4\}$ can be regarded as the roots of the equation

$$b_0 \prod_k (s + \lambda_k) = b_0 s^4 + b_2 s^2 + b_3 s + b_4 = 0. \tag{3.14}$$

When $p \in S$ varies along S, Eq. (3.14) defines a fourfold cover over S, called the fundamental $SU(4)$ spectral cover. This cover is a section of the canonical bundle $K_S \to S$. When λ_i vanish, $\prod_i \lambda_i = b_4 = 0$ in which the gauge group is enhanced to E_6 and matter **16** is localized. According to the decomposition (3.10), matter **10** corresponds to the anti-symmetric representation **6** of $SU(4)$, associated to a sixfold cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ over S. This associated cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ can be constructed as follows:

$$b_0^2 \prod_{i<j}(s+\lambda_i+\lambda_j) = b_0^2 s^6 + 2b_0 b_2 s^4 + (b_2^2 - 4b_0 b_4)s^2 - b_3^2 = 0. \tag{3.15}$$

Since matter **10** corresponds to $\lambda_i + \lambda_j = 0,\ i \neq j$, it follows from Eq. (3.15) that $b_3 = 0$, which means that matter **10** is localized at the locus $\{b_3 = 0\}$ as shown in Table XIX. It is not difficult to see that the spectral covers indeed encode the information of singularities and gauge group enhancements. However, the power of spectral cover is more than that. With the spectral cover, we can construct a Higgs bundle to calculate the chirality of matter **16** and **10** by switching on a line bundle on the cover.

Let us define X to be the total space of the canonical bundle K_S over S. Note that X is a local Calabi-Yau threefold. However, X is non-compact. To obtain a compact space, one can compactify X to the total space $\bar{X}$ of the projective bundle over S, *i.e.*

$$\bar{X} = \mathbb{P}(\mathcal{O}_S \oplus K_S), \tag{3.16}$$

with a map $\pi : \bar{X} \to S$, where $\mathcal{O}_S$ is the trivial bundle over S. Notice that $\bar{X}$ is compact but no longer a Calabi-Yau threefold. Let $\mathcal{O}(1)$ be a hyperplane section of $\mathbb{P}^1$ fiber and denote its first Chern class by σ_∞. We define the homogeneous coordinates of the fiber by $[U : V]$. Note that $\{U = 0\}$ and $\{V = 0\}$ are sections of $\mathcal{O}(1) \otimes K_S$ and $\mathcal{O}(1)$, while the class of $\{U = 0\}$ and $\{V = 0\}$ are $\sigma \equiv \sigma_\infty - \pi^* c_1(S)$ and σ_∞, respectively. By the emptiness of intersection of $\{U = 0\}$ and $\{V = 0\}$, one can obtain $\sigma \cdot \sigma = -\sigma \cdot \pi^* c_1$. The affine coordinate s is defined by $s = U/V$. In $\bar{X}$, the $SU(4)$ cover Eq. (3.14) is homogenized as

$$\mathcal{C}^{(4)}: \quad b_0 U^4 + b_2 U^2 V^2 + b_3 U V^3 + b_4 V^4 = 0 \tag{3.17}$$

with induced map $p_4 : \mathcal{C}^{(4)} \to S$. It is not difficult to see that the homological class $[\mathcal{C}^{(4)}]$ of the cover $\mathcal{C}^{(4)}$ is given by $[\mathcal{C}^{(4)}] = 4\sigma + \pi^*\eta$. One can calculate the locus of matter **16** curve by intersection of $[\mathcal{C}^{(4)}]$ with σ

$$[\mathcal{C}^{(4)}] \cap \sigma = (4\sigma + \pi^*\eta) \cdot \sigma = \sigma \cdot \pi^*(\eta - 4c_1), \tag{3.18}$$

which implies that $[\Sigma_{\mathbf{16}}] = \eta - 4c_1$ in S. Alternatively, it can be followed from the fact that the locus of $\Sigma_{\mathbf{16}}$ in S is $\{b_4 = 0\}$. On the other hand, it follows from Eq. (3.15) that the homological class of the cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ is given by

$$[\mathcal{C}^{(6)}_{\wedge^2 V}] = 6\sigma + 2\pi^*\eta \tag{3.19}$$

Notice that $\mathcal{C}^{(6)}_{\wedge^2 V}$ is generically singular. To solve this problem, one can consider intersection $\tau\mathcal{C} \cap \mathcal{C}$ and define [58, 121]

$$[D] = [\mathcal{C}^{(4)}] \cap [\mathcal{C}^{(4)}] - [\mathcal{C}^{(4)}] \cap \sigma - [\mathcal{C}^{(4)}] \cap 3\sigma_\infty \tag{3.20}$$

where τ is a $\mathbb{Z}_2$ involution $V \to -V$ acting on the spectral cover[38]. The **10** curve can then be evaluated by

$$[D]|_\sigma = 4(\eta - 3c_1), \tag{3.21}$$

which implies that $[\Sigma_{\mathbf{10}}] = 2\eta - 6c_1$ in S.

To obtain chiral spectrum, we turn on a spectral line bundle $\mathcal{L}$ on the cover $\mathcal{C}^{(4)}$. The corresponding Higgs bundle is given by $V = p_{4*}\mathcal{L}$. For an $SU(n)$ bundle, it is

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[38] Note that there are double points on $\Sigma_{\mathbf{10}}$. One can resolve these double points by blowing-up and then obtain resolved $\tilde{\Sigma}_{\mathbf{10}}$ with a mapping $\tilde{\pi}_D : D \to \tilde{\Sigma}_{\mathbf{10}}$ of degree 4 and $[\tilde{\Sigma}_{\mathbf{10}}] = \eta - 3c_1$ [52].

required that $c_1(V) = 0$. It follows that

$$c_1(p_{4*}\mathcal{L}) = p_{4*}c_1(\mathcal{L}) - \frac{1}{2}p_{4*}r, \tag{3.22}$$

where r is the ramification divisor given by $r = p_{4*}c_1 - c_1(\mathcal{C}^{(4)})$. It is convenient to define the cover flux γ by

$$c_1(\mathcal{L}) = \lambda\gamma + \frac{1}{2}r, \tag{3.23}$$

where λ is a parameter used to compensate the non-integral class $\frac{1}{2}r$. The traceless condition $c_1(p_{4*}\mathcal{L}) = 0$ is then equivalent to the condition $p_{4*}\gamma = 0$. One can show that

$$\gamma = (4 - p_4^* p_{4*})(\mathcal{C}^{(4)} \cdot \sigma) \tag{3.24}$$

satisfies the traceless condition. Since the first Chern class of a line bundle must be integral, it follows that λ and γ have to obey the following quantization condition

$$\lambda\gamma + \frac{1}{2}[p_4^* c_1 - c_1(\mathcal{C}^{(4)})] \in H_4(\bar{X}, \mathbb{Z}). \tag{3.25}$$

With the given cover flux γ, the net chirality of matter **16** is calculated by [47, 52]

$$N_{\mathbf{16}} = (\mathcal{C}^{(4)} \cdot \sigma) \cdot \lambda\gamma = -\lambda\eta \cdot (\eta - 4c_1) \tag{3.26}$$

On the other hand, the matter **10** corresponds to the anti-symmetric representation **6** in $SU(4)$, associated to the spectral cover $\mathcal{C}^{(6)}_{\wedge^2 V}$. It turns out that for the $SU(4)$ cover, the net chirality of matter **10** is given by [52]

$$N_{\mathbf{10}} = D \cdot \gamma = 0. \tag{3.27}$$

It follows from Eqs. (3.26) and (3.27) that one obtain an $SO(10)$ model with $-\lambda\eta \cdot (\eta - 4c_1)$ copies of matter on the **16** curve and nothing on the **10** curve. The flux γ does not have many degrees of freedom to tune and the candidate of **10** Higgs is absent. Therefore, in search of realistic models, we shall consider factorization of the $SU(4)$ cover $\mathcal{C}^{(4)}$ to enrich the configuration, along the line of the $SU(5)$ cover studied in [53, 57, 65, 66]. In the next section, we shall focus on the construction of $(3,1)$ and $(2,2)$ factorizations of the cover $\mathcal{C}^{(4)}$.

B. $SU(4)$ Cover Factorization

1. $(3,1)$ Factorization

We consider the $(3,1)$ factorization, $C^{(4)} \to \mathcal{C}^{(a)} \times \mathcal{C}^{(b)}$ corresponding to the factorization of Eq. (3.17) as follows:

$$\mathcal{C}^{(a)} \times \mathcal{C}^{(b)}: \quad (a_0U^3 + a_1U^2V + a_2UV^2 + a_3V^3)(d_0U + d_1V) = 0. \tag{3.28}$$

By comparing with Eq. (3.17), one can obtain the following relations:

$$b_0 = a_0d_0, \quad b_1 = a_1d_0 + a_0d_1, \quad b_2 = a_2d_0 + a_1d_1, \quad b_3 = a_3d_0 + a_2d_1, \quad b_4 = a_3d_1. \tag{3.29}$$

Let ξ_1 be the homological class $[d_1]$ of d_1 and write

$$[d_0] = c_1 + \xi_1, \qquad [a_k] = \eta - (k+1)c_1 - \xi_1, \quad k = 0, 1, 2, 3. \tag{3.30}$$

To solve the traceless condition $b_1 = a_1d_0 + a_0d_1 = 0$, we impose the ansatz $a_0 = \alpha d_0,\ a_1 = -\alpha d_1$ where $[\alpha] = \eta - 2c_1 - 2\xi_1$. It is easy to see that the homological classes of $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ in $\bar{X}$ are

$$[\mathcal{C}^{(a)}] = 3\sigma + \pi^*(\eta - c_1 - \xi_1), \quad [\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi_1). \tag{3.31}$$

With the classes given in Eq. (3.31), the homological classes of factorized matter curves $\Sigma_{\mathbf{16}^{(a)}}$ and $\Sigma_{\mathbf{16}^{(b)}}$ in S are given by

$$[\Sigma_{\mathbf{16}^{(a)}}] = [\mathcal{C}^{(a)}]|_\sigma = \eta - 4c_1 - \xi_1, \quad [\Sigma_{\mathbf{16}^{(b)}}] = [\mathcal{C}^{(b)}]|_\sigma = \xi_1. \tag{3.32}$$

	$[\mathcal{C}^{(b)(b)}]$	$2[\mathcal{C}^{(a)(b)}]$	$[\mathcal{C}^{(a)(a)}]$
16	$\sigma \cdot \pi^*\xi_1$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi_1)$
10	-	$2[\sigma + \pi^*(c_1 + \xi_1)]$ $\cdot \pi^*(\eta - 3c_1 - \xi_1) + 2\sigma \cdot \pi^*\xi_1$	$[2\sigma + \pi^*(\eta - 2c_1 - \xi_1)]$ $\cdot \pi^*(\eta - 3c_1 - \xi_1)$ $+2(\sigma + \pi^*c_1) \cdot \pi^*\xi_1$
∞	$\sigma_\infty \cdot \pi^*(c_1 + \xi_1)$	$4\sigma_\infty \cdot \pi^*(c_1 + \xi_1)$	$\sigma_\infty \cdot \pi^*(\eta - c_1 - \xi_1)$ $+2\sigma_\infty \cdot \pi^*(\eta - 2c_1 - 2\xi_1)$

Table XX. The homological classes of the matter curves in the $(3, 1)$ factorization.

To obtain the factorized **10** curves, we follow the method proposed in [57,65,66,121] to calculate the intersection $\mathcal{C}^{(4)} \cap \tau\mathcal{C}^{(4)}$, where τ is the $\mathbb{Z}_2$ involution $\tau : V \to -V$ acting on the spectral cover. Since the calculation is straightforward, we omit the detailed calculation here and only summarize the results in Table XX.[39][40]

It follows from Table XX that the relevant classes in $\bar{X}$ for **10** curves are

$$[\mathcal{C}^{(a)(a)}] = [2\sigma + \pi^*(\eta - 2c_1 - \xi_1)] \cdot \ \pi^*(\eta - 3c_1 - \xi_1) + 2(\sigma + \pi^*c_1) \cdot \pi^*\xi_1, \tag{3.33}$$

$$[\mathcal{C}^{(a)(b)}] = [\sigma + \pi^*(c_1 + \xi_1)] \cdot \ \pi^*(\eta - 3c_1 - \xi_1) + \sigma \cdot \pi^*\xi_1, \tag{3.34}$$

which give rise to the **10** curve

$$[\Sigma_{\mathbf{10}^{(a)(a)}}] = [\Sigma_{\mathbf{10}^{(a)(b)}}] = \eta - 3c_1. \tag{3.35}$$

[39]To simplify notations, we denote $\mathcal{C}^{(k)} \cap \tau\mathcal{C}^{(l)}$ by $\mathcal{C}^{(k)(l)}$ and notice that $[\mathcal{C}^{(k)(l)}] = [\mathcal{C}^{(l)(k)}]$.

[40]To avoid a singularity of non-Kodaira type, we impose the condition $\xi_1 \cdot_S (c_1 + \xi_1) = 0$. Therefore, $[\mathcal{C}^{(b)(b)}]|_{\mathbf{10}} = \pi^*\xi_1 \cdot \pi^*(c_1 + \xi_1)$=0.

2. (2,2) Factorization

In the (2,2) factorization, the cover is split as $\mathcal{C}^{(4)} \to \mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}$. More precisely, the cover defined in Eq. (3.17) is factorized into the following form:

$$\mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}: \quad (e_0U^2 + e_1UV + e_2V^2)(f_0U^2 + f_1UV + f_2V^2) = 0. \tag{3.36}$$

By comparing the coefficients with Eq. (3.17), one obtains

$$b_0 = e_0f_0, \quad b_1 = e_0f_1 + e_1f_0, \quad b_2 = e_0f_2 + e_1f_1 + e_2f_0, \quad b_3 = e_1f_2 + e_2f_1, \quad b_4 = e_2f_2. \tag{3.37}$$

Let ξ_2 be the homological class of f_2 and then the homological classes of other sections can be written as

$$[f_1] = c_1 + \xi_2, \quad [f_0] = 2c_1 + \xi_2, \quad [e_m] = \eta - (m+2)c_1 - \xi_2, \quad m = 0,1,2. \tag{3.38}$$

To solve the traceless condition $b_1 = e_0f_1 + e_1f_0 = 0$, we impose the ansatz $e_0 = \beta f_0,\ e_1 = -\beta f_1$ where $[\beta] = \eta - 4c_1 - 2\xi_2$. In this case, the homological classes of $\mathcal{C}^{(d_1)}$ and $\mathcal{C}^{(d_2)}$ are given by

$$[\mathcal{C}^{(d_1)}] = 2\sigma + \pi^*(\eta - 2c_1 - \xi_2), \quad [\mathcal{C}^{(d_2)}] = 2\sigma + \pi^*(2c_1 + \xi_2). \tag{3.39}$$

The homological classes of the corresponding matter curves $\Sigma_{\mathbf{16}^{(d_1)}}$ and $\Sigma_{\mathbf{16}^{(d_2)}}$ are then computed as

$$[\Sigma_{\mathbf{16}^{(d_1)}}] = [\mathcal{C}^{(d_1)}]|_\sigma = \eta - 4c_1 - \xi_2, \quad [\Sigma_{\mathbf{16}^{(d_2)}}] = [\mathcal{C}^{(d_2)}]|_\sigma = \xi_2, \tag{3.40}$$

respectively. To calculate the homological classes of the factorized **10** curves, we again follow the method proposed in [57,65,66,121] to calculate the intersection $\mathcal{C}^{(4)} \cap \tau\mathcal{C}^{(4)}$. We omit the detailed calculation here and only summarize the results in Table XXI.

It follows from Table XXI that the classes in $\bar{X}$ for the factorized **10** curves are

	$[\mathcal{C}^{(d_2)(d_2)}]$	$2[\mathcal{C}^{(d_1)(d_2)}]$	$[\mathcal{C}^{(d_1)(d_1)}]$
16	$\sigma \cdot \pi^* \xi_2$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi_2)$
10	$[2\sigma + \pi^*(2c_1 + \xi_2)]$ $\cdot\ \pi^*(c_1 + \xi_2)$	$2[2\sigma + \pi^*(2c_1 + \xi_2)]$ $\cdot\ \pi^*(\eta - 4c_1 - \xi_2)$	$\pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2)$ $+2(\sigma + \pi^* c_1) \cdot \pi^*(c_1 + \xi_2)$
∞	$\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$	$4\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$	$\sigma_\infty \cdot \pi^*(\eta - 2c_1 - \xi_2)$ $+2\sigma_\infty \cdot \pi^*(\eta - 4c_1 - 2\xi_2)$

Table XXI. The homological classes of the matter curves in the $(2,2)$ factorization.

as follows:

$$[\mathcal{C}^{(d_1)(d_1)}] = 2(\sigma + \pi^* c_1) \cdot \pi^*(c_1 + \xi_2) + \pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2), \quad (3.41)$$

$$[\mathcal{C}^{(d_1)(d_2)}] = (2\sigma + \pi^*(2c_1 + \xi_2)) \cdot\ \pi^*(\eta - 4c_1 - \xi_2), \quad (3.42)$$

$$[\mathcal{C}^{(d_2)(d_2)}] = (2\sigma + \pi^*(2c_1 + \xi_2)) \cdot\ \pi^*(c_1 + \xi_2). \quad (3.43)$$

With the classes $[\mathcal{C}^{(d_1)(d_1)}]$, $[\mathcal{C}^{(d_1)(d_2)}]$, and $[\mathcal{C}^{(d_2)(d_2)}]$, one can calculate the classes of the corresponding **10** curves in S as follows:

$$[\Sigma_{\mathbf{10}^{(d_1)(d_1)}}] = [\Sigma_{\mathbf{10}^{(d_2)(d_2)}}] = c_1 + \xi_2, \quad [\Sigma_{\mathbf{10}^{(d_1)(d_2)}}] = 2\eta - 8c_1 - 2\xi_2. \quad (3.44)$$

C. Spectral Cover Fluxes

Let us consider the case of the cover factorization $\mathcal{C}^{(n)} \to \mathcal{C}^{(l)} \times \mathcal{C}^{(m)}$. To obtain well-defined cover fluxes and maintain supersymmetry, we impose the following constraints [66]:

$$c_1(p_{l*}\mathcal{L}^{(l)}) + c_1(p_{m*}\mathcal{L}^{(m)}) = 0, \quad (3.45)$$

$$c_1(\mathcal{L}^{(k)}) \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \quad (3.46)$$

$$[c_1(p_{l*}\mathcal{L}^{(l)}) - c_1(p_{m*}\mathcal{L}^{(m)})] \cdot_S [\omega_S] = 0, \quad (3.47)$$

where p_k denotes the projection map from the cover $\mathcal{C}^{(k)}$ to S, $p_k:\ \mathcal{C}^{(k)} \to S$, $\mathcal{L}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$ and $[\omega_S]$ is an ample divisor dual to a Kähler form of S. The first constraint Eq. (3.45) is the traceless condition for the induced Higgs bundle[41]. The second constraint Eq. (3.46) requires that the first Chern class of a well-defined line bundle $\mathcal{L}^{(k)}$ over $\mathcal{C}^{(k)}$ must be integral. The third constraint states that the 2-cycle $c_1(p_{l*}\mathcal{L}^{(l)}) - c_1(p_{m*}\mathcal{L}^{(m)})$ in S has to be supersymmetic. Note that Eq. (3.45) can be expressed as

$$p_{l*}c_1(\mathcal{L}^{(l)}) - \frac{1}{2}p_{l*}r^{(l)} + p_{m*}c_1(\mathcal{L}^{(m)}) - \frac{1}{2}p_{m*}r^{(m)} = 0, \tag{3.48}$$

where $r^{(l)}$ and $r^{(m)}$ are the ramification divisors for the maps p_l and p_m, respectively. Recall that the ramification divisors $r^{(k)}$ are defined by

$$r^{(k)} = p_k^* c_1 - c_1(\mathcal{C}^{(k)}), \quad k = l, m. \tag{3.49}$$

The term $c_1(\mathcal{C}^{(k)})$ in Eq. (3.49) can be calculated by the adjuction formula [109,110],

$$c_1(\mathcal{C}^{(k)}) = (c_1(\bar{X}) - [\mathcal{C}^{(k)}]) \cdot [\mathcal{C}^{(k)}]. \tag{3.50}$$

It is convenient to define cover fluxes $\gamma^{(k)}$ as

$$c_1(\mathcal{L}^{(k)}) = \gamma^{(k)} + \frac{1}{2}r^{(k)}, \quad k = l, m. \tag{3.51}$$

With Eq. (3.51), the traceless condition Eq. (3.45) can be expressed as $p_{l*}\gamma^{(l)} + p_{m*}\gamma^{(m)} = 0$. By using Eq. (3.49) and Eq. (3.51), we can recast the quantization condition Eq. (3.46) by $\gamma^{(k)} + \frac{1}{2}[p_k^* c_1 - c_1(\mathcal{C}^{(k)})] \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m$. Finally, the supersymmetry condition Eq. (3.47) is reduced to $p_{k*}\gamma^{(k)} \cdot_S [\omega_S] = 0$. We summarize

[41] One may think of Eq. (3.45) as the traceless condition of an $SU(4)$ bundle V_4 over S split into $V_3 \oplus L$ with $V_3 = p_{a*}\mathcal{L}^{(a)}$ and $L = p_{b*}\mathcal{L}^{(b)}$. Then the traceless condition of V_4 can be expressed by $c_1(V_4) = c_1(p_{a*}\mathcal{L}^{(a)}) + c_1(p_{b*}\mathcal{L}^{(b)}) = 0$.

the constraints as follows:

$$p_{l*}\gamma^{(l)} + p_{m*}\gamma^{(m)} = 0, \tag{3.52}$$

$$\gamma^{(k)} + \frac{1}{2}[p_k^* c_1 - c_1(\mathcal{C}^{(k)})] \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \tag{3.53}$$

$$p_{k*}\gamma^{(k)} \cdot_S [\omega_S] = 0, \quad k = l, m. \tag{3.54}$$

In the next section, we shall explicitly construct the cover fluxes $\gamma^{(k)}$ satisfying Eq. (3.52), (3.53), and (3.54) for the $(3,1)$ and $(2,2)$ factorizations. We also calculate the restrictions of the fluxes to each matter curve.

1. (3,1) Factorization

In the $(3,1)$ factorization, the ramification divisors for the spectral covers $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ are given by

$$r^{(a)} = [\mathcal{C}^{(a)}] \cdot [\sigma + \pi^*(\eta - 2c_1 - \xi_1)], \tag{3.55}$$

$$r^{(b)} = [\mathcal{C}^{(b)}] \cdot (-\sigma + \pi^*\xi_1), \tag{3.56}$$

respectively. We define traceless fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ by

$$\gamma_0^{(a)} = (3 - p_a^* p_{a*})\gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [3\sigma - \pi^*(\eta - 4c_1 - \xi_1)], \tag{3.57}$$

$$\gamma_0^{(b)} = (1 - p_b^* p_{b*})\gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot (\sigma - \pi^*\xi_1)\,, \tag{3.58}$$

where $\gamma^{(a)}$ and $\gamma^{(b)}$ are non-traceless fluxes and defined as

$$\gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot \sigma, \quad \gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot \sigma. \tag{3.59}$$

Then we can calculate the restriction of fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ to each matter curve. We omit the calculation here and only summarize the results in Table XXII. Due to the

	$\gamma_0^{(b)}$	$\gamma_0^{(a)}$
$\mathbf{16}^{(b)}$	$-\xi_1 \cdot_S (c_1 + \xi_1)$	0
$\mathbf{16}^{(a)}$	0	$-(\eta - c_1 - \xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(b)}$	$-\xi_1 \cdot_S (c_1 + \xi_1)$	$-(\eta - 3c_1 - 3\xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(a)}$	0	$(\eta - 3c_1 - 3\xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$

Table XXII. Chirality induced by the fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$.

	$\delta^{(b)}$	$\delta^{(a)}$	$\widetilde{\rho}$
$\mathbf{16}^{(b)}$	$-3c_1 \cdot_S \xi_1$	$-\xi_1 \cdot_S (\eta - 4c_1 - \xi_1)$	$3\rho \cdot_S \xi_1$
$\mathbf{16}^{(a)}$	$-\xi_1 \cdot_S (\eta - 4c_1 - \xi_1)$	$-c_1 \cdot_S (\eta - 4c_1 - \xi_1)$	$-\rho \cdot_S (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(b)}$	$\xi_1 \cdot_S (2\eta - 9c_1 - 3\xi_1)$	$-(\eta - 3c_1 - \xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$	$2\rho \cdot_S (\eta - 3c_1)$
$\mathbf{10}^{(a)(a)}$	$-2\xi_1 \cdot_S (\eta - 3c_1)$	$(\eta - 3c_1 - \xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$	$-2\rho \cdot_S (\eta - 3c_1)$

Table XXIII. Chirality induced by the fluxes $\delta^{(a)}$, $\delta^{(b)}$, and $\widetilde{\rho}$.

factorization, one also can define additional fluxes $\delta^{(a)}$ and $\delta^{(b)}$ by

$$\begin{aligned}\delta^{(a)} &= (1 - p_b^* p_{a*})\gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot \sigma - [\mathcal{C}^{(b)}] \cdot \pi^*(\eta - 4c_1 - \xi_1) \\ \delta^{(b)} &= (3 - p_a^* p_{b*})\gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot 3\sigma - [\mathcal{C}^{(a)}] \cdot \pi^* \xi_1.\end{aligned} \tag{3.60}$$

Another flux one can include is [66]

$$\tilde{\rho} = (3p_b^* - p_a^*)\rho, \tag{3.61}$$

for any $\rho \in H_2(S, \mathbb{R})$. We summarize the restriction of fluxes $\delta^{(a)}$, $\delta^{(b)}$ and $\tilde{\rho}$ to each matter curve in Table XXIII.

With Eqs. (3.58), (3.60), and (3.61), we define the universal cover flux Γ to be [66]

$$\Gamma = k_a \gamma_0^{(a)} + k_b \gamma_0^{(b)} + m_a \delta^{(a)} + m_b \delta^{(b)} + \tilde{\rho} \equiv \Gamma^{(a)} + \Gamma^{(b)}, \tag{3.62}$$

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are given by

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(3k_a + m_a)\sigma - \pi^*(k_a(\eta - 4c_1 - \xi_1) + m_b\xi_1 + \rho)]\,, \tag{3.63}$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 3m_b)\sigma - \pi^*(k_b\xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho)]\,. \tag{3.64}$$

Note that

$$p_{a*}\Gamma^{(a)} = -3m_b\xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho, \tag{3.65}$$

$$p_{b*}\Gamma^{(b)} = 3m_b\xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho. \tag{3.66}$$

Clearly, $\Gamma^{(a)}$ and $\Gamma^{(b)}$ obey the traceless condition $p_{a*}\Gamma^{(a)} + p_{b*}\Gamma^{(b)} = 0$. Besides, the quantization condition in this case becomes

$$(3k_a + m_a + \frac{1}{2})\sigma - \pi^*[k_a(\eta - 4c_1 - \xi_1) + m_b\xi_1 + \rho - \frac{1}{2}(\eta - 2c_1 - \xi_1)] \in H_4(\bar{X}, \mathbb{Z}), \tag{3.67}$$

$$(k_b + 3m_b - \frac{1}{2})\sigma - \pi^*[k_b\xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho - \frac{1}{2}\xi_1] \in H_4(\bar{X}, \mathbb{Z}). \tag{3.68}$$

The supersymmetry condition is given by

$$[3m_b\xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho] \cdot_S [\omega_S] = 0. \tag{3.69}$$

2. (2,2) Factorization

We can calculate the ramification divisors $r^{(d_1)}$ and $r^{(d_2)}$ for the $(2,2)$ factorization and obtain

$$r^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot \pi^*(\eta - 3c_1 - \xi_2), \tag{3.70}$$

$$r^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot \pi^*(c_1 + \xi_2). \tag{3.71}$$

	$\gamma_0^{(d_2)}$	$\gamma_0^{(d_1)}$
$\mathbf{16}^{(d_2)}$	$-\xi_2 \cdot_S (2c_1 + \xi_2)$	0
$\mathbf{16}^{(d_1)}$	0	$-(\eta - 2c_1 - \xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	0	0
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(\eta - 4c_1 - 2\xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$
$\mathbf{10}^{(d_1)(d_1)}$	0	$2(\eta - 4c_1 - 2\xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$

Table XXIV. Chirality induced by the fluxes $\gamma_0^{(d_1)}$ and $\gamma_0^{(d_2)}$.

We then define traceless cover fluxes $\gamma_0^{(d_1)}$ and $\gamma_0^{(d_2)}$ by

$$\gamma_0^{(d_1)} = (2 - p^*_{d_1} p_{d_1 *})\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot [2\sigma - \pi^*(\eta - 4c_1 - \xi_2)]\,, \tag{3.72}$$

$$\gamma_0^{(d_2)} = (2 - p^*_{d_2} p_{d_2 *})\gamma^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot (2\sigma - \pi^*\xi_2)\,, \tag{3.73}$$

where $\gamma^{(d_1)}$ and $\gamma^{(d_21)}$ are non-traceless fluxes and given by

$$\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot \sigma, \quad \gamma^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot \sigma. \tag{3.74}$$

We summarize the restriction of the fluxes to each factorized curve in Table XXIV. Due to the factorization, one also can define following fluxes [66]

$$\begin{aligned} \delta^{(d_1)} &= (2 - p^*_{d_2} p_{d_1 *})\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot 2\sigma - [\mathcal{C}^{(d_2)}] \cdot \pi^*(\eta - 4c_1 - \xi_2), \\ \delta^{(d_2)} &= (2 - p^*_{d_1} p_{d_2 *})\gamma^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot 2\sigma - [\mathcal{C}^{(d_1)}] \cdot \pi^*\xi_2, \end{aligned} \tag{3.75}$$

and

$$\widehat{\rho} = (p^*_{d_2} - p^*_{d_1})\rho, \tag{3.76}$$

for any $\rho \in H_2(S, \mathbb{R})$. We summarize the restriction of the fluxes $\delta^{(d_1)}$, $\delta^{(d_2)}$, and $\widehat{\rho}$ to each factorized curve in Table XXV.

	$\delta^{(d_2)}$	$\delta^{(d_1)}$	$\widehat{\rho}$
$\mathbf{16}^{(d_2)}$	$-2c_1 \cdot_S \xi_2$	$-\xi_2 \cdot_S (\eta - 4c_1 - \xi_2)$	$\rho \cdot_S \xi_2$
$\mathbf{16}^{(d_1)}$	$-\xi_2 \cdot_S (\eta - 4c_1 - \xi_2)$	$-2c_1 \cdot_S (\eta - 4c_1 - \xi_2)$	$-\rho \cdot_S (\eta - 4c_1 - \xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	$2\xi_2 \cdot_S (c_1 + \xi_2)$	$-2(c_1 + \xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$	$2\rho \cdot_S (c_1 + \xi_2)$
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(\eta - 4c_1 - 2\xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$	0
$\mathbf{10}^{(d_1)(d_1)}$	$-2\xi_2 \cdot_S (c_1 + \xi_2)$	$2(\eta - 3c_1 - \xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$	$-2\rho \cdot_S (c_1 + \xi_2)$

Table XXV. Chirality induced by the fluxes $\delta^{(d_1)}$, $\delta^{(d_2)}$, and $\widehat{\rho}$.

In this case the universal cover flux is defined by

$$\Gamma = k_{d_1}\gamma_0^{(d_1)} + k_{d_2}\gamma_0^{(d_2)} + m_{d_1}\delta^{(d_1)} + m_{d_2}\delta^{(d_2)} + \widehat{\rho} = \Gamma^{(d_1)} + \Gamma^{(d_2)}, \tag{3.77}$$

where

$$\Gamma^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot \{2(k_{d_1} + m_{d_1})\sigma - \pi^*[k_{d_1}(\eta - 4c_1 - \xi_2) + m_{d_2}\xi_2 + \rho]\},$$

$$\Gamma^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot \{2(k_{d_2} + m_{d_2})\sigma - \pi^*[k_{d_2}\xi_2 + m_{d_1}(\eta - 4c_1 - \xi_2) - \rho]\}. \tag{3.78}$$

Note that

$$p_{d_1*}\Gamma^{(d_1)} = -2m_{d_2}\xi_2 + 2m_{d_1}(\eta - 4c_1 - \xi_2) - 2\rho, \tag{3.79}$$

$$p_{d_2*}\Gamma^{(d_2)} = 2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho. \tag{3.80}$$

It is easy to see that $\Gamma^{(d_1)}$ and $\Gamma^{(d_2)}$ satisfy the traceless condition $p_{d_1*}\Gamma^{(d_1)} + p_{d_2*}\Gamma^{(d_2)} = 0$. In addition, the quantization condition in this case becomes

$$2(k_{d_1} + m_{d_1})\sigma - \pi^*[k_{d_1}(\eta - 4c_1 - \xi_2) + m_{d_2}\xi_2 + \rho - \frac{1}{2}(\eta - 3c_1 - \xi_2)] \in H_4(\bar{X}, \mathbb{Z}), \tag{3.81}$$

$$2(k_{d_2} + m_{d_2})\sigma - \pi^*[k_{d_2}\xi_2 + m_{d_1}(\eta - 4c_1 - \xi_2) - \rho - \frac{1}{2}(c_1 + \xi_2)] \in H_4(\bar{X}, \mathbb{Z}). \tag{3.82}$$

The supersymmetry condition is then given by

$$[2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho] \cdot_S [\omega_S] = 0. \tag{3.83}$$

D. $D3$-brane Tadpole Cancellation

The cancellation of tadpoles is crucial for consistent compactifications. In general, there are induced tadpoles from 7-brane, 5-brane, and 3-brane charges in F-theory. It is well-known that 7-brane tadpole cancellation in F-theory is automatically satisfied since X_4 is a Calabi-Yau manifold. In spectral cover models, the cancellation of the $D5$-brane tadpole follows from the topological condition that the overall first Chern class of the Higgs bundle vanishes. Therefore, the non-trivial tadpole cancellation needed to be satisfied is the $D3$-brane tadpole. The $D3$-brane tadpole can be calculated by the Euler characteristic $\chi(X_4)$. The cancellation condition is of the form [122]

$$N_{D3} = \frac{\chi(X_4)}{24} - \frac{1}{2}\int_{X_4} G \wedge G, \tag{3.84}$$

where N_{D3} is the number of $D3$-branes and G is the four-form flux on X_4. For a non-singular elliptically fibered Calabi-Yau manifold, it was shown in [122] that the Euler characteristic $\chi(X_4)$ can be expressed as

$$\chi(X_4) = 12\int_{B_3} c_1(B_3)[c_2(B_3) + 30c_1(B_3)^2], \tag{3.85}$$

where $c_k(B_3)$ are the Chern classes of B_3. It follows from Eq. (4.108) that $\chi(X_4)/24$ is at least half-integral[42]. When X_4 admits non-abelian singularities, the Euler characteristic of X_4 is replaced by the refined Euler characteristic, the Euler characteristic of the smooth fourfold obtained from a suitable resolution of X_4. On the other hand,

[42]For a generic Calabi-Yau manifold, it was shown in [122] that $\chi(X_4)/6 \in \mathbb{Z}$, which implies that $\chi(X_4)/24$ takes value in $\mathbb{Z}_4$.

G-flux encodes the two-form gauge fluxes on 7-branes. It was shown in [123] that

$$\frac{1}{2}\int_{X_4} G \wedge G = -\frac{1}{2}\Gamma^2, \tag{3.86}$$

where Γ is the universal cover flux defined in section C and Γ^2 is the self-intersection number of Γ inside the spectral cover[43]. It is a challenge to find compactifications with non-vanishing G-flux and non-negative N_{D_3} to satisfy the tadpole cancellation condition Eq. (4.107). In the next two subsections, we shall derive the formulae of refined Euler characteristic $\chi(X_4)$ and the self-intersection of universal cover fluxes Γ^2 for $(3,1)$ and $(2,2)$ factorizations.

1. Geometric Contribution

In the presence of non-abelian singularities, X_4 becomes singular and the Euler characteristic $\chi(X_4)$ is modified by resolving the singularities. To be more concrete, let us consider X_4 with an elliptic fibration which degenerates over S to a non-abelian singularity corresponding to gauge group H and define G to be the complement of H in E_8. The Euler characteristic is modified to

$$\chi(X_4) = \chi^*(X_4) + \chi_G - \chi_{E_8}, \tag{3.87}$$

where $\chi^*(X_4)$ is the Euler characteristic for a smooth fibration over B_3 given by Eq. (4.108). The characteristic χ_{E_8} is given by [53, 123, 124]

$$\chi_{E_8} = 120\int_S (3\eta^2 - 27\eta c_1 + 62c_1^2). \tag{3.88}$$

[43]Eq. (3.86) originates from the spectral cover construction in heterotic string compactifications [123]. This equation holds for F-theory compactified on elliptically fibered fourfolds possessing a heterotic dual by heterotic/F-theory duality. However, since X_4 is not a global fibration over S, we assume that Eq. (3.86) is valid for F-theory models without heterotic dual, and the fluxes can correctly described by spectral covers.

For the case of $G = SU(n)$, the characteristic $\chi_{SU(n)}$ is given by[44]

$$\chi_{SU(n)} = \int_S (n^3 - n)c_1^2 + 3n\eta(\eta - nc_1). \tag{3.89}$$

When G splits into a product of two groups G_1 and G_1, χ_G in Eq. (4.110) is then replaced by $\chi^{(k)}_{G_1} + \chi^{(l)}_{G_2}$ in which η is replaced by the class $\eta^{(m)}$ in the spectral cover $\mathcal{C}^{(m)}$ for $m = k, l$. For the case of $(3, 1)$ factorization, the refined Euler characteristic is then calculated by

$$\begin{aligned}\chi(X_4) &= \chi^*(X_4) + \chi^{(a)}_{SU(3)} + \chi^{(b)}_{SU(1)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 3[c_1(38c_1 - 21t - 20\xi_1) + (3t^2 + 6t\xi_1 + 4\xi_1^2)] - \chi_{E_8}. \end{aligned} \tag{3.90}$$

In the $(2, 2)$ factorization, the refined Euler characteristic[45] is

$$\begin{aligned}\chi(X_4) &= \chi^*(X_4) + \chi^{(d_1)}_{SU(2)} + \chi^{(d_2)}_{SU(2)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 6[c_1(10c_1 - 6t - 4\xi_2) + (t^2 + 2t\xi_2 + 2\xi_2^2)] - \chi_{E_8}. \end{aligned} \tag{3.91}$$

2. Cover Flux Contribution

It follows from Eqs. (4.107) and (3.86) that

$$N_{D3} = \frac{\chi(X_4)}{24} + \frac{1}{2}\Gamma^2. \tag{3.92}$$

---

[44] Eqs. (4.110)-(3.89) initially were derived in heterotic string compactifications [123,124]. A priori, these formulae are valid only for F-theory models with a heterotic dual. It was observed in [53] that these formulae also hold for some F-theory models which do not admit a heterotic dual. However, this match fails in other examples observed in [80]. In these examples, extra gauge groups appear in regions away from S and cannot be described by spectral covers. We assume that Eqs. (4.110)-(3.89) hold for our models.

[45] For the $(3, 1)$ factorization, $\eta^{(a)} = (\eta - c_1 - \xi_1)$ and $\eta^{(b)} = (c_1 + \xi_1)$. For the $(2, 2)$ factorization, $\eta^{(d_1)} = (\eta - 2c_1 - \xi_2)$ and $\eta^{(d_2)} = (2c_1 + \xi_2)$.

In the previous subsection, we discussed the first term on the right hand side of Eq. (3.92). To calculate N_{D3}, it is necessary to compute the self-intersection Γ^2 of the universal cover flux Γ. Recall that in section C, the universal cover flux was defined by

$$\Gamma = \sum_k \Gamma^{(k)}, \tag{3.93}$$

where $\Gamma^{(k)}$ are cover fluxes satisfying the traceless condition,

$$\sum_k p_{k*}\Gamma^{(k)} = 0. \tag{3.94}$$

In what follows, we will compute Γ^2 for both the $(3,1)$ and $(2,2)$ factorizations.

a. $(3,1)$ Factorization

Recall that for the case of $(3,1)$ factorization, the universal cover flux is given by

$$\Gamma = k_a\gamma_0^{(a)} + k_b\gamma_0^{(b)} + m_a\delta^{(a)} + m_b\delta^{(b)} + \tilde{\rho} = \Gamma^{(a)} + \Gamma^{(b)}, \tag{3.95}$$

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(3k_a + m_a)\sigma - \pi^*(k_a[a_3] + m_b[d_1] + \rho)] \equiv [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}], \tag{3.96}$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 3m_b)\sigma - \pi^*(k_b[d_1] + m_a[a_3] - 3\rho)] \equiv [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \tag{3.97}$$

Then the self-intersection of the cover flux Γ is calculated by [66]

$$\Gamma^2 = [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] + [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \tag{3.98}$$

In the $(3,1)$ factorization, $[\mathcal{C}^{(a)}] = 3\sigma + \pi^*(\eta - c_1 - \xi_1)$ and $[\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi_1)$. By Eqs. (4.117) and (4.118), one can obtain

$$\begin{aligned} [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] &= -(3k_a + m_a)^2([a_3] \cdot_S c_1) - k_a(3k_a + 2m_a)[a_3]^2 + 3m_b^2[d_1]^2 \\ &- 2m_b m_a([a_3] \cdot_S [d_1]) - 2(m_a[a_3] - 3m_b[d_1]) \cdot_S \rho \\ &+ 3(\rho \cdot_S \rho), \end{aligned} \tag{3.99}$$

and

$$\begin{aligned} [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] &= -(k_b + 3m_b)^2([d_1] \cdot_S c_1) - k_b(k_b + 6m_b)[d_1]^2 + m_a^2[a_3]^2 \\ &- 6m_b m_a([a_3] \cdot_S [d_1]) - 6(m_a[a_3] - 3m_b[d_1]) \cdot_S \rho \\ &+ 9(\rho \cdot_S \rho). \end{aligned} \tag{3.100}$$

Putting everything together, one obtains

$$\Gamma^2 = -\frac{1}{3}(3k_a + m_a)^2([a_0] \cdot_S [a_3]) - (k_b + 3m_b)^2([d_0] \cdot_S [d_1]) + \frac{4}{3}(m_a[a_3] - 3m_b[d_1] - 3\rho)^2. \tag{3.101}$$

b. $(2,2)$ Factorization

Recall that in the $(2,2)$ factorization, the universal flux is given by

$$\Gamma = k_{d_1}\gamma_0^{(d_1)} + k_{d_2}\gamma_0^{(d_2)} + m_{d_1}\delta^{(d_1)} + m_{d_2}\delta^{(d_2)} + \widehat{\rho} \equiv \Gamma^{(d_1)} + \Gamma^{(d_2)}, \tag{3.102}$$

where $\Gamma^{(d_1)}$ and $\Gamma^{(d_2)}$ are

$$\Gamma^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot [2(k_{d_1} + m_{d_1})\sigma - \pi^*(k_{d_1}[e_2] + m_{d_2}[f_2] + \rho)] \equiv [\mathcal{C}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}], \tag{3.103}$$

$$\Gamma^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot [2(k_{d_2} + m_{d_2})\sigma - \pi^*(k_{d_2}[f_2] + m_{d_1}[e_2] - \rho)] \equiv [\mathcal{C}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}]. \tag{3.104}$$

Then the self-intersection Γ^2 can be computed as

$$\Gamma^2 = [\mathcal{C}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] + [\mathcal{C}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}]. \tag{3.105}$$

Notice that $[\mathcal{C}^{(d_1)}] = 2\sigma + \pi^*(\eta - 2c_1 - \xi_2)$ and $[\mathcal{C}^{(d_2)}] = 2\sigma + \pi^*(2c_1 + \xi_2)$ in the $(2,1)$ factorization. It follows from Eqs. (3.103) and (3.104) that

$$\begin{aligned} [\mathcal{C}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] &= -4(k_{d_1} + m_{d_1})^2([e_2] \cdot_S c_1) - 2k_{d_1}(k_{d_1} + 2m_{d_1})[e_2]^2 \\ &+ 2m_{d_2}^2[f_2]^2 - 4m_{d_1}m_{d_2}([e_2] \cdot_S [f_2]) - 4(m_{d_1}[e_2] \\ &- m_{d_2}[f_2]) \cdot_S \rho + 2(\rho \cdot_S \rho), \end{aligned} \tag{3.106}$$

and

$$\begin{aligned} [\mathcal{C}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}] &= -4(k_{d_2} + m_{d_2})^2([f_2] \cdot_S c_1) - 2k_{d_2}(k_{d_2} + 2m_{d_2})[f_2]^2 \\ &+ 2m_{d_1}^2[e_2]^2 - 4m_{d_1}m_{d_2}([f_2] \cdot_S [e_2]) \\ &- 4(m_{d_1}[e_2] - m_{d_2}[f_2]) \cdot_S \rho + 2(\rho \cdot_S \rho). \end{aligned} \tag{3.107}$$

Therefore, Γ^2 is given by

$$\Gamma^2 = -2(k_{d_1} + m_{d_1})^2([e_0] \cdot_S [e_2]) - 2(k_{d_2} + m_{d_2})^2([f_0] \cdot_S [f_2]) + 4(m_{d_1}[e_2] - m_{d_2}[f_2] - \rho)^2. \tag{3.108}$$

E. Models

1. $U(1)_X$ Flux and Spectrum

Let us start with the $(3,1)$ factorization. Consider the breaking pattern as follows:

$$\begin{aligned} SU(4)_{\perp} &\rightarrow SU(3)\times U(1) \\ \mathbf{15} &\rightarrow \mathbf{8}_0+\mathbf{3}_{-4}+\bar{\mathbf{3}}_4+\mathbf{1}_0 \\ \mathbf{6} &\rightarrow \mathbf{3}_2+\bar{\mathbf{3}}_{-2} \\ \mathbf{4} &\rightarrow \mathbf{3}_{-1}+\mathbf{1}_3 \end{aligned} \tag{3.109}$$

Then the representations $(\mathbf{16},\mathbf{4})$ and $(\mathbf{10},\mathbf{6})$ in Eq. (3.10) are decomposed as

$$(\mathbf{16},\mathbf{4})\rightarrow(\mathbf{16}_{-1},\mathbf{3})+(\mathbf{16}_3,\mathbf{1}),\quad(\mathbf{10},\mathbf{6})\rightarrow(\mathbf{10}_2,\mathbf{3})+(\mathbf{10}_{-2},\bar{\mathbf{3}}) \tag{3.110}$$

On the other hand, we can further break $SO(10)$ in Eq. (3.10) by $U(1)_X$ flux as follows:

$$\begin{aligned} SO(10) &\rightarrow SU(5)\times U(1)_X \\ \mathbf{16} &\rightarrow \mathbf{10}_{-1}+\bar{\mathbf{5}}_3+\mathbf{1}_{-5} \\ \mathbf{10} &\rightarrow \mathbf{5}_2+\bar{\mathbf{5}}_{-2} \end{aligned} \tag{3.111}$$

We suppose that $V_{\mathbf{16}}\otimes L_X^{-1}$ has restriction of degree M_k to $\Sigma_{\mathbf{16}^{(k)}}$ while L_X^4 has restriction of degree N_k. Similarly, we define $V_{\mathbf{10}}\otimes L_X^{-2}$ has restriction of degree M_{kl} to $\Sigma_{\mathbf{10}^{(k)(l)}}$ while L_X^4 has restriction of degree N_{kl}. We summarize the chirality on each matter curve in Table XXVI. For the $(2,2)$ factorization, the analysis is similar to the case of the $(3,1)$ factorization. We summarize the chirality induced from the cover and $U(1)_X$ fluxes in Table XXVII.

Curve	Matter	Bundle	Chirality
$\mathbf{16}^{(a)}_{-1}$	$\mathbf{10}_{-1,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(a)}_{\mathbf{16}}}$	M_a
	$\bar{\mathbf{5}}_{-1,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(a)}_{\mathbf{16}}}$	$M_a + N_a$
	$\mathbf{1}_{-1,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(a)}_{\mathbf{16}}}$	$M_a - N_a$
$\mathbf{16}^{(b)}_{3}$	$\mathbf{10}_{3,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(b)}_{\mathbf{16}}}$	M_b
	$\bar{\mathbf{5}}_{3,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(b)}_{\mathbf{16}}}$	$M_b + N_b$
	$\mathbf{1}_{3,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(b)}_{\mathbf{16}}}$	$M_b - N_b$
$\mathbf{10}^{(a)(a)}_{-2}$	$\mathbf{5}_{-2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(a)(a)}_{\mathbf{10}}}$	$M_{aa} + N_{aa}$
	$\bar{\mathbf{5}}_{-2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(a)(a)}_{\mathbf{10}}}$	M_{aa}
$\mathbf{10}^{(a)(b)}_{2}$	$\mathbf{5}_{2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(a)(b)}_{\mathbf{10}}}$	$M_{ab} + N_{ab}$
	$\bar{\mathbf{5}}_{2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(a)(b)}_{\mathbf{10}}}$	M_{ab}

Table XXVI. Chirality of matter localized on matter curves **16** and **10** in the (3,1) factorization.

2. (3,1) Factorization and CY_4 with a dP_2 Surface

In this section, we shall explicitly realize models in specific geometries. We first consider the Calabi-Yau fourfold constructed in [64] to be our X_4. This Calabi-Yau fourfold contains a dP_2 surface embedded into the base B_3. For the detailed geometry of this Calabi-Yau fourfold, we refer readers to [64]. Here we only collect the relevant geometric data for calculation. The basic geometric data of X_4 is

$$c_1 = 3h - e_1 - e_2, \quad t = -c_1(N_{S/B_3}) = h, \quad \chi^*(X_4) = 13968. \tag{3.112}$$

From Eq. (3.112), we can conclude $\eta = 17h - 6e_1 - 6e_2$, $\eta^2 = 217$, $c_1 \cdot \eta = 39$, and $c_1^2 = 7$. For the (3,1) factorization, it follows from Eq. (3.90) that the refined Euler

Curve	Matter	Bundle	Chirality
$\mathbf{16}^{(d_2)}_{-1}$	$\mathbf{10}_{-1,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(d_2)}_{\mathbf{16}}}$	M_{d_2}
	$\bar{\mathbf{5}}_{-1,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(d_2)}_{\mathbf{16}}}$	$M_{d_2} + N_{d_2}$
	$\mathbf{1}_{-1,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(d_2)}_{\mathbf{16}}}$	$M_{d_2} - N_{d_2}$
$\mathbf{16}^{(d_1)}_{1}$	$\mathbf{10}_{1,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(d_1)}_{\mathbf{16}}}$	M_{d_1}
	$\bar{\mathbf{5}}_{1,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(d_1)}_{\mathbf{16}}}$	$M_{d_1} + N_{d_1}$
	$\mathbf{1}_{1,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(d_1)}_{\mathbf{16}}}$	$M_{d_1} - N_{d_1}$
$\mathbf{10}^{(d_2)(d_2)}_{-2}$	$\mathbf{5}_{-2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(d_2)(d_2)}_{\mathbf{10}}}$	$M_{d_2d_2} + N_{d_2d_2}$
	$\bar{\mathbf{5}}_{-2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(d_2)(d_2)}_{\mathbf{10}}}$	$M_{d_2d_2}$
$\mathbf{10}^{(d_1)(d_2)}_{0}$	$\mathbf{5}_{0,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(d_1)(d_2)}_{\mathbf{10}}}$	$M_{d_1d_2} + N_{d_1d_2}$
	$\bar{\mathbf{5}}_{0,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(d_1)(d_2)}_{\mathbf{10}}}$	$M_{d_1d_2}$
$\mathbf{10}^{(d_1)(d_1)}_{2}$	$\mathbf{5}_{2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(d_1)(d_1)}_{\mathbf{10}}}$	$M_{d_1d_1} + N_{d_1d_1}$
	$\bar{\mathbf{5}}_{2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(d_1)(d_1)}_{\mathbf{10}}}$	$M_{d_1d_1}$

Table XXVII. Chirality of matter localized on matter curves **16** and **10** in the (2,2) factorization.

characteristic is

$$\chi(X_4) = 10746 + (12\xi_1^2 - 18\xi_1\eta + 48\xi_1 c_1). \tag{3.113}$$

The self-intersection of the cover flux Γ is then given by

$$\begin{aligned} \Gamma^2 \;=\; & -(3k_a^2 + 2k_a m_a)(50 + \xi_1^2 - 2\xi_1\eta + 5\xi_1 c_1) + m_a^2(6 + \xi_1^2 - 2\xi_1\eta + 9\xi_1 c_1) \\ & -(k_b + 3m_b)^2(\xi_1^2 + \xi_1 c_1) + 12m_b^2\xi_1^2 + 8m_a m_b(\xi_1^2 - \xi_1\eta + 4\xi_1 c_1) \\ & +12\rho^2 - 8m_a(\rho\eta - \rho\xi_1 - 4\rho c_1) + 24m_b\rho\xi_1, \end{aligned} \tag{3.114}$$

and the number of generations for matter **16** and **10** on the curves are

$$N_{\mathbf{16}^{(b)}} = (m_a - k_b)\xi_1^2 - m_a\xi_1\eta + (4m_a - k_b - 3m_b)\xi_1 c_1 + 3\rho\xi_1, \qquad (3.115)$$

$$N_{\mathbf{16}^{(a)}} = -(50k_a + 11m_a) + (m_b - k_a)\xi_1^2 + (2k_a - m_b)\xi_1\eta$$
$$+(4m_b - 5k_a + m_a)\xi_1 c_1 - \rho\eta + 4\rho c_1 + \rho\xi_1, \qquad (3.116)$$

$$N_{\mathbf{10}^{(a)(b)}} = -28(k_a + m_a) - (k_b + 3k_a + m_a + 3m_b)\xi_1^2 + (4k_a + 2m_a + 2m_b)\xi_1\eta$$
$$-(k_b + 15k_a + 7m_a + 9m_b)\xi_1 c_1 + 2\rho\eta - 6\rho c_1, \qquad (3.117)$$

$$N_{\mathbf{10}^{(a)(a)}} = 28(k_a + m_a) + (3k_a + m_a)\xi_1^2 - (4k_a + 2m_a + 2m_b)\xi_1\eta$$
$$+(15k_a + 7m_a + 6m_b)\xi_1 c_1 - 2\rho\eta + 6\rho c_1. \qquad (3.118)$$

In this case, the supersymmetric condition Eq. (3.54) reduces to

$$[(3m_b + m_a)\xi_1 - m_a(\eta - 4c_1) + 3\rho] \cdot_S [\omega_{dP_2}], \qquad (3.119)$$

where we choose $[\omega_{dP_2}] = \alpha(e_1 + e_2) + \beta(h - e_1 - e_2),\ 2\alpha > \beta > \alpha > 0$ to be an ample divisor in dP_2. In the (3,1) factorization, one more constraint that we may impose is that the ramification of the degree-one cover should be trivial. In other words, we impose the following constraint:

$$(c_1 + \xi_1) \cdot_S \xi_1 = 0. \qquad (3.120)$$

In what follows, we show three examples based on this geometry. We find that there are only finite number of solutions for parameters.

a. Model 1

In this model we represent a three-generation example. The numerical parameters are listed in Table XXVIII. The matter content and the corresponding classes are listed in Table XXIX. By using Eqs. (3.113) and (3.114), we obtain $\chi(X_4) = 10674$

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	-0.5	-2	1	$h+3e_1+e_2$	e_2	9	11

Table XXVIII. Parameters of Model 1 of the (3,1) factorization in dP_2.

Matter	Class in S	Class with fixed ξ_1	Generation	Restr. of $[F_X]$
$\mathbf{16}^{(b)}$	ξ_1	e_2	0	1
$\mathbf{16}^{(a)}$	$\eta-4c_1-\xi_1$	$5h-2e_1-3e_2$	3	-1
$\mathbf{10}^{(a)(b)}$	$\eta-3c_1$	$8h-3e_1-3e_2$	14	0
$\mathbf{10}^{(a)(a)}$	$\eta-3c_1$	$8h-3e_1-3e_2$	-14	0

Table XXIX. Model 1 matter content with $[F_X]=e_1-e_2$. It is a three-generation model with non-trivial flux restrictions.

and $\Gamma^2=-159.5$. It follows from Eq. (3.92) that $N_{D3}=365$.

b. Model 2

Model 2 is another example of a three-generation model with $\chi(X_4)=10674$, $\Gamma^2=-159.5$, and $N_{D3}=365$. The construction is similar to the model 1. We list the numerical parameters in Table XXX. The matter content and the corresponding classes are shown in Table XXXI.

c. Model 3

In this model we demonstrate a four-generation model in $SO(10)$. The reason why we would like to discuss such a case is that the only choice for the $U(1)_X$ flux on dP_2 is $[F_X]=\pm(e_1-e_2)$, and then the restrictions of $[F_X]$ to the **16** curves are always non-zero, which results in the variation of the chirality numbers of the $SU(5)$ matter

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	0.5	-2	-2	$-4h+4e_1+5e_2$	e_1	9	11

Table XXX. Parameters of Model 2 of the (3,1) factorization in dP_2.

Matter	Class in S	Class with fixed ξ_1	Generation	Restr. of $[F_X]$
$\mathbf{16}^{(b)}$	ξ_1	e_1	0	1
$\mathbf{16}^{(a)}$	$\eta-4c_1-\xi_1$	$5h-3e_1-2e_2$	3	-1
$\mathbf{10}^{(a)(b)}$	$\eta-3c_1$	$8h-3e_1-3e_2$	14	0
$\mathbf{10}^{(a)(a)}$	$\eta-3c_1$	$8h-3e_1-3e_2$	-14	0

Table XXXI. Model 2 matter content with $[F_X]=e_1-e_2$.

descended from the **16** curves. The two examples shown above only make sense for an three-generation $SO(10)$ model, and they are no longer three-generation models after gauge breaking. Since we expect to build a three-generation model at $SU(5)$ level, we slightly increase the generation number at the $SO(10)$ level to prevent the chirality being too small. The numerical parameters are listed in Table XXXII. In this model, it is straightforward to obtain $\chi(X_4)=10674$ and $\Gamma^2=-355.5$. It turns out that $N_{D3}=267$ is a positive integer. The matter content and the corresponding classes are listed in Table XXXIII.

d. Discussion

Model 1 and Model 2 of (3,1) factorization have the $SO(10)$ structure shown in Table XXXIV, where $U(1)_C$ is from the cover and is of the $U(1)^3$ Cartan subalgebra of $SU(4)$ that is not removed from the monodromy. The Yukawa coupling is filtered by the conservation of this $U(1)_C$. Before turning on the $U(1)_X$ flux, this spectrum can

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	-0.5	-2	1	$5e_1+e_2$	e_2	12	17

Table XXXII. Parameters of Model 3 of the (3,1) factorization in dP_2.

Matter	Class in S	Class with fixed ξ_1	Generation	Restr. of $[F_X]$
$\mathbf{16}^{(b)}$	ξ_1	e_2	0	1
$\mathbf{16}^{(a)}$	$\eta-4c_1-\xi_1$	$5h-2e_1-3e_2$	4	-1
$\mathbf{10}^{(a)(b)}$	$\eta-3c_1$	$8h-3e_1-3e_2$	10	0
$\mathbf{10}^{(a)(a)}$	$\eta-3c_1$	$8h-3e_1-3e_2$	-10	0

Table XXXIII. Model 3 matter content with $[F_X]=e_1-e_2$. There are four generations on the $\mathbf{16}^{(a)}$ curve.

fit the minimum requirement by forming the Yukawa coupling $\mathbf{16}^{(a)}_{-1}\mathbf{16}^{(a)}_{-1}\mathbf{10}^{(a)(b)}_{2}$ of the $SO(10)$ GUT with some exotic **10**s. However, when $U(1)_X$ flux is turned on, the non-vanishing restrictions of the flux to two **16** curves change the chirality on these two curves, while the chirality on the **10** curves remain untouched. The analysis in Table XXVI suggests that a three-generation model may be descended from a four-generation $SO(10)$ model after the gauge group is broken to $SU(5)\times U(1)_X$ by $[F_X]=e_1-e_2$. Here we try to explain Model 3 as a flipped $SU(5)$ model with its spectrum presented in Table XXXV.

In this case, the Yukawa couplings are

$$\begin{aligned}\mathcal{W} \;\supset\;& \mathbf{10}_{-1,-1M}\mathbf{10}_{-1,-1M}\mathbf{5}_{2,2h}+\mathbf{10}_{-1,-1M}\bar{\mathbf{5}}_{-1,3M}\bar{\mathbf{5}}_{2,-2h}+\bar{\mathbf{5}}_{-1,3M}\mathbf{1}_{-1,-5M}\mathbf{5}_{2,2h}\\ +\;& \mathbf{10}_{-1,-1H}\mathbf{10}_{-1,-1H}\mathbf{5}_{2,2h}+\overline{\mathbf{10}}_{-1,1H}\overline{\mathbf{10}}_{-1,1H}\bar{\mathbf{5}}_{2,-2h}+\ldots. \end{aligned} \tag{3.121}$$

We may identify the flipped $SU(5)$ superheavy Higgs fields with one of the $\mathbf{10}+\overline{\mathbf{10}}$

Maatter	Copy	$U(1)_C$
$\mathbf{16}^{(b)}$	0	-3
$\mathbf{16}^{(a)}$	3	1
$\mathbf{10}^{(a)(b)}$	14	-2
$\mathbf{10}^{(a)(a)}$	-14	2

Table XXXIV. Matter spectrum for $(3,1)$ factorization.

vector-like pairs on the $\mathbf{16}^{(a)}$ curve, which is not obvious from this configuration. Since the restrictions of the flux to the curves change the chirality, there are unavoidable exotic fermions, like the examples studied in [66]. In the following subsection, we will study models from a different geometric backgrounds to see if it is possible to retain the chirality unchanged while the flux F_X is turned on.

3. (3,1) Factorization and CY_4 with a dP_7 Surface

Although dP_2 surface is elegant, it does not possess enough degrees of freedom in the number of exceptional divisors for model building. Therefore, we turn to the the geometry of the compact Calabi-Yau fourfold realized as complete intersections of two hypersurfaces with an embedded dP_7 surface[46]. The detailed construction can be found in [53]. Again here we only collect relevant geometric data for calculation. The basic geometric data is as follows:

$$\begin{aligned} c_1 &= 3h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7, \\ t &= 2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6, \\ \eta &= 16h - 5e_1 - 5e_2 - 5e_3 - 5e_4 - 5e_5 - 5e_6 - 6e_7. \end{aligned} \tag{3.122}$$

[46]By abuse of notation, we also denote this Calabi-Yau fourfold by X_4.

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{-1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{-1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{-1,-5}$	3
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{-1,-1} + \overline{\mathbf{10}}_{-1,1}$	1
$\mathbf{5}_h$	$\mathbf{5}_{2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{2,-2}$	1
$\mathbf{10}$	$\mathbf{10}_{-1,-1}$	1
$\bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{3,3}$	1
$\mathbf{1}$	$\mathbf{1}_{-1,-5}$	2
$\mathbf{1}$	$\mathbf{1}_{3,5}$	1
$\mathbf{5} + \bar{\mathbf{5}}$ exotics	$\mathbf{5}_{-2,2} + \bar{\mathbf{5}}_{-2,-2}$	9
	$\mathbf{5}_{2,2} + \bar{\mathbf{5}}_{2,-2}$	-10

Table XXXV. Flipped $SU(5)$ spectrum of Model 3.

with $\chi^*(X_4) = 1728$. From Eq. (3.122), we have $\eta^2 = 70$, $\eta \cdot c_1 = 12$, and $c_1^2 = 2$. The refined Euler characteristic is given by

$$\chi(X_4) = 738 + (12\xi_1^2 - 18\xi_1\eta + 48\xi_1 c_1), \tag{3.123}$$

and the self-intersection of the cover flux Γ is

$$\begin{aligned}\Gamma^2 \;=\; & -(3k_a^2 + 2k_a m_a)(18 + \xi_1^2 - 2\xi_1\eta + 5\xi_1 c_1) + m_a^2(2 + \xi_1^2 - 2\xi_1\eta + 9\xi_1 c_1) \\ & -(k_b + 3m_b)^2(\xi_1^2 + \xi_1 c_1) + 12m_b^2\xi_1^2 + 8m_a m_b(\xi_1^2 - \xi_1\eta + 4\xi_1 c_1) \\ & +12\rho^2 - 8m_a(\rho\eta - \rho\xi_1 - 4\rho c_1) + 24m_b\rho\xi_1. \end{aligned} \tag{3.124}$$

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	-1	0	1.5	$\frac{1}{2}(2e_1+2e_2+e_4)$	$2h-e_1-e_2-e_3-e_5-e_6$	3	2

Table XXXVI. Parameters of the (3,1) factorization model in dP_7.

Again we summarize the generation number on each curve as follows:

$$N_{\mathbf{16}^{(b)}} = (m_a-k_b)\xi_1^2 - m_a\xi_1\eta + (4m_a-k_b-3m_b)\xi_1 c_1 + 3\rho\xi_1, \tag{3.125}$$

$$\begin{aligned} N_{\mathbf{16}^{(a)}} &= -(18k_a+4m_a)+(m_b-k_a)\xi_1^2+(2k_a-m_b)\xi_1\eta \\ &\quad +(4m_b-5k_a+m_a)\xi_1 c_1 - \rho\eta + 4\rho c_1 + \rho\xi_1, \end{aligned} \tag{3.126}$$

$$\begin{aligned} N_{\mathbf{10}^{(a)(b)}} &= -10(k_a+m_a)-(k_b+3k_a+m_a+3m_b)\xi_1^2+(4k_a+2m_a+2m_b)\xi_1\eta \\ &\quad -(k_b+15k_a+7m_a+9m_b)\xi_1 c_1 + 2\rho\eta - 6\rho c_1, \end{aligned} \tag{3.127}$$

$$\begin{aligned} N_{\mathbf{10}^{(a)(a)}} &= 10(k_a+m_a)+(3k_a+m_a)\xi_1^2-(4k_a+2m_a+2m_b)\xi_1\eta \\ &\quad +(15k_a+7m_a+6m_b)\xi_1 c_1 - 2\rho\eta + 6\rho c_1. \end{aligned} \tag{3.128}$$

The supersymmetry condition is then

$$[(3m_b+m_a)\xi_1 - m_a(\eta-4c_1)+3\rho]\cdot_S [\omega_{dP_7}] = 0, \tag{3.129}$$

where $[\omega_{dP_7}]$ is an ample divisor dual to a Kähler form of dP_7. For simplicity, we choose $[\omega_{dP_7}]$ to be

$$[\omega_{dP_7}] = 14\beta h - (5\beta-\alpha)\sum_{i=1}^{7} e_i, \tag{3.130}$$

with constraints $5\beta > \alpha > 0$.

In what follows, we present one example based on this geometry. This model is three-generation with vanishing restrictions of the $U(1)_X$ flux to the **16** curves.

Matter	Class in S	Class with fixed ξ_1	Generation
$\mathbf{16}^{(b)}$	ξ_1	$2h - e_1 - e_2 - e_3 - e_5 - e_6$	0
$\mathbf{16}^{(a)}$	$\eta - 4c_1 - \xi_1$	$2h - e_4 - 2e_7$	3
$\mathbf{10}^{(a)(b)}$	$\eta - 3c_1$	$7h - 2\sum_{i=1}^{6} e_i - 3e_7$	1
$\mathbf{10}^{(a)(a)}$	$\eta - 3c_1$	$7h - 2\sum_{i=1}^{6} e_i - 3e_7$	-1

Table XXXVII. The dP_7 model matter content. Since it is a three-generation model, the flux is chosen to have trivial restriction. For example, $[F_X] = e_5 - e_6$.

a. Model

We present a three-generation model in this example. The numerical result of the parameters is listed in Table XXXVI. The matter content and the corresponding classes are listed in Table XXXVII. With data in Table XXXVI and Table XXXVII, one can obtain $\chi(X_4) = 648$ and $\Gamma^2 = -42$ by using Eqs. (3.123) and (3.124). It follows from Eq. (3.92) that $N_{D3} = 6$.

b. Discussion

In this example we tune $[F_X] = e_4 - e_5$ to obtain trivial restrictions on all the curves, so the chirality on each curve remains unchanged. By the analysis of Table XXVI, we can create a flipped $SU(5)$ spectrum as shown in Table XXXVIII. The Yukawa couplings turn out to be

$$\begin{aligned}\mathcal{W} \supset\ & \mathbf{10}_{-1,-1M}\mathbf{10}_{-1,-1M}\mathbf{5}_{2,2h} + \mathbf{10}_{-1,-1M}\bar{\mathbf{5}}_{-1,3M}\bar{\mathbf{5}}_{2,-2h} + \bar{\mathbf{5}}_{-1,3M}\mathbf{1}_{-1,-5M}\mathbf{5}_{2,2h} \\ + \ & \mathbf{10}_{-1,-1H}\mathbf{10}_{-1,-1H}\mathbf{5}_{2,2h} + \overline{\mathbf{10}}_{-1,1H}\overline{\mathbf{10}}_{-1,1H}\bar{\mathbf{5}}_{2,-2h} + \cdots \,. \end{aligned} \quad (3.131)$$

[47]There is one $(\mathbf{5}, \bar{\mathbf{5}})$ on the $\mathbf{10}^{(a)(a)}$ curve.

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{-1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{-1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{-1,-5}$	3
$\mathbf{5}_h$	$\mathbf{5}_{2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{2,-2}$	1
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{-1,-1} + \overline{\mathbf{10}}_{-1,1}$	1
$\mathbf{5} + \bar{\mathbf{5}}$ exotics^{47}		

Table XXXVIII. Flipped $SU(5)$ spectrum with vanishing restrictions of $[F_X]$ on the curves in (3,1) factorization in dP_7.

This spectrum looks standard, and the advantage is that there are no exotic fermions and the quantum numbers(charges) of the matter are typical. We again assume that the superheavy Higgses $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ come from one of the vector-like $\mathbf{10} + \overline{\mathbf{10}}$ pairs on the $\mathbf{16}^{(a)}$ curve. It is not obvious to calculate the number of such pairs. For simplicity, we just extract one pair for phenomenology purposes.

4. (2,2) Factorization and CY_4 with a dP_2 Surface

Let us consider the $(2,2)$ factorization with the geometric background in Eq. (3.112) [64]. In this case, the refined Euler characteristic turns out to be

$$\chi(X_4) = 10446 + (12\xi_2^2 - 12\xi_2\eta + 48\xi_2 c_1). \tag{3.132}$$

The self-intersection of the cover flux Γ is

$$\begin{aligned}\Gamma^2 &= -2(k_{d_1}+m_{d_1})^2(39+\xi_2^2-2\xi_2\eta+6\xi_2c_1)+4m_{d_1}^2(17+\xi_2^2-2\xi_2\eta+8\xi_2c_1)\\ &\quad -2(k_{d_2}+m_{d_2})^2(\xi_2^2+2\xi_2c_1)+4m_{d_2}^2\xi_2^2+8m_{d_1}m_{d_2}(\xi_2^2-\xi_2\eta+4\xi_2c_1)\\ &\quad +4\rho^2-8m_{d_1}(\rho\eta-\rho\xi_2-4\rho c_1)+8m_{d_2}\rho\xi_2. \end{aligned} \tag{3.133}$$

In this case, we can find models with integral N_{D3}. However, to have more degrees of freedom for model building, we shall focus on the geometry of the CY_4 with an embedded dP_7 surface [53] in the next subsection.

5. (2,2) Factorization and CY_4 with a dP_7 Surface

We again consider the geometric background in Eq. (3.122)and the (2,2) factorization. In this case, the refined Euler characteristic is given by

$$\chi(X_4)=636+(12\xi_2^2-12\xi_2\eta+48\xi_2c_1). \tag{3.134}$$

The self-intersection of the cover flux Γ is

$$\begin{aligned}\Gamma^2 &= -2(k_{d_1}+m_{d_1})^2(14+\xi_2^2-2\xi_2\eta+6\xi_2c_1)+4m_{d_1}^2(6+\xi_2^2-2\xi_2\eta+8\xi_2c_1)\\ &\quad -2(k_{d_2}+m_{d_2})^2(\xi_2^2+2\xi_2c_1)+4m_{d_2}^2\xi_2^2+8m_{d_1}m_{d_2}(\xi_2^2-\xi_2\eta+4\xi_2c_1)\\ &\quad +4\rho^2-8m_{d_1}(\rho\eta-\rho\xi_2-4\rho c_1)+8m_{d_2}\rho\xi_2. \end{aligned} \tag{3.135}$$

The generations of matter on the curves are

$$
\begin{aligned}
N_{\mathbf{16}^{(d_2)}} &= (m_{d_1} - k_{d_2})\xi_2^2 - m_{d_1}\xi_2\eta + (4m_{d_1} - 2k_{d_2} - 2m_{d_2})\xi_2 c_1 + \rho\xi_2, && (3.136)\\
N_{\mathbf{16}^{(d_1)}} &= -(14k_{d_1} + 8m_{d_1}) + (m_{d_2} - k_{d_1})\xi_2^2 + (2k_{d_1} - m_{d_2})\xi_2\eta \\
&\quad +(4m_{d_2} - 6k_{d_1} + 2m_{d_1})\xi_2 c_1 - \rho\eta + 4\rho c_1 + \rho\xi_2, && (3.137)\\
N_{\mathbf{10}^{(d_2)(d_2)}} &= -8m_{d_1} + 2(m_{d_1} + m_{d_2})\xi_2^2 + 2(m_{d_2} + 5m_{d_1})\xi_2 c_1 - 2m_{d_1}\xi_2\eta \\
&\quad +2\rho c_1 + 2\rho\xi_2, && (3.138)\\
N_{\mathbf{10}^{(d_1)(d_2)}} &= -2(k_{d_1} + m_{d_1})(6 + 2\xi_2^2 - 3\xi_2\eta + 12\xi_2 c_1), && (3.139)\\
N_{\mathbf{10}^{(d_1)(d_1)}} &= (12k_{d_1} + 20m_{d_1}) + (4k_{d_1} + 2m_{d_1} - 2m_{d_2})\xi_2^2 - 2(3k_{d_1} + 2m_{d_1})\xi_2\eta \\
&\quad +(24k_{d_1} - 2m_{d_2} + 14m_{d_1})\xi_2 c_1 - 2\rho c_1 - 2\rho\xi_2. && (3.140)
\end{aligned}
$$

The supersymmetry condition is then

$$[2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho] \cdot_S [\omega_{dP_2}] = 0, \tag{3.141}$$

where $[\omega_{dP_2}]$ is an ample divisor dual to a Kähler form of dP_7. For simplicity, we choose $[\omega_{dP_2}]$ to be

$$[\omega_{dP_2}] = 14\beta h - (5\beta - \alpha)\sum_{i=1}^{7} e_i, \tag{3.142}$$

with constraints $5\beta > \alpha > 0$.

In the (2,2) factorization of the $SU(4)$ cover, we expect the matter spectrum for an $SO(10)$ model to be Table XXXIX. The $U(1)_C$ is of the $U(1)^3$ Cartan subalgebra of $SU(4)$ that is not removed from the monodromy. The Yukawa coupling is filtered by the conservation of this $U(1)_C$. The possible Yukawa couplings for constructing a minimum $SO(10)$ GUT are then $\mathbf{16}^{(d_1)}\mathbf{16}^{(d_1)}\mathbf{10}^{(d_2)(d_2)}$ and $\mathbf{16}^{(d_2)}\mathbf{16}^{(d_2)}\mathbf{10}^{(d_1)(d_1)}$. We will demonstrate examples of the flipped $SU(5)$ GUT model from the following models.

Maatter	Copy	$U(1)_C$
$\mathbf{16}^{(d_2)}$	0/3	-1
$\mathbf{16}^{(d_1)}$	3/0	1
$\mathbf{10}^{(d_2)(d_2)}$	n_1	-2
$\mathbf{10}^{(d_1)(d_2)}$	n_2	0
$\mathbf{10}^{(d_1)(d_1)}$	n_3	2

Table XXXIX. Matter spectrum for $(2,2)$ factorization.

k_{d_2}	k_{d_1}	m_{d_2}	m_{d_1}	ρ	ξ_2	α	β
-1	0	1.5	-0.5	$-\frac{1}{2}(h-2e_1+2e_2+2e_3+2e_4+e_7)$	$h-e_1$	1	3

Table XL. Parameters of Model 1 of the (2,2) Factorization in dP_7.

a. Model 1

In this example we demonstrate a three-generation model. The numerical parameters are shown in Table XL, and the matter content and the corresponding classes with the flux $[F_X]=e_2-e_3$ are listed in Table XLI. By using Eqs. (3.134) and (3.135), we obtain $\chi(X_4)=600$ and $\Gamma^2=-18$ which gives rise to $N_{D3}=16$.

b. Model 2

In this model, we show a four-generation example with non-zero restrictions of F_X on the matter curves. The spectrum can maintain a three-generation model after the gauge is broken to $SU(5)\times U(1)_X$ by F_X. The parameters are presented in Table XLII, while the matter content and the corresponding classes with the flux $[F_X]=e_3-e_4$ are listed in Table XLIII. In this model, we have $\chi(X_4)=600$ and $\Gamma^2=-26$ which gives rise to $N_{D3}=12$.

Matter	Class in S	Class with fixed ξ_2	Generation	Restr. of F_X
$\mathbf{16}^{(d_2)}$	ξ_2	$h - e_1$	0	0
$\mathbf{16}^{(d_1)}$	$\eta - 4c_1 - \xi_2$	$3h - \sum_{i=2}^{6} e_i - 2e_7$	3	0
$\mathbf{10}^{(d_2)(d_2)}$	$c_1 + \xi_2$	$4h - 2e_1 - \sum_{i=2}^{6} e_i - 2e_7$	4	0
$\mathbf{10}^{(d_1)(d_2)}$	$2\eta - 8c_1 - 2\xi_2$	$6h - 2\sum_{i=2}^{6} e_i - 4e_7$	-3	0
$\mathbf{10}^{(d_1)(d_1)}$	$c_1 + \xi_2$	$4h - 2e_1 - \sum_{i=2}^{6} e_i - 2e_7$	-1	0

Table XLI. The Matter content of Model 1. The flux is tuned that the restriction is zero on each curve.

k_{d_2}	k_{d_1}	m_{d_2}	m_{d_1}	ρ	ξ_2	α	β
1	0	-0.5	-0.5	$-\frac{1}{2}(h - 2e_1 + 2e_2 - 2e_3 - e_7)$	$2h - e_1 - e_2 - e_3 - e_7$	1	3

Table XLII. Parameters of Model 2 of the (2,2) Factorization in dP_7.

c. Discussion

The number of (-2) 2-cycles in dP_7 is large enough that it is possible to remain the chirality unchanged by tuning F_X with vanishing restrictions on all the curves. An example is presented in Model 1, and the corresponding flipped $SU(5)$ spectrum can be found in Table XLIV.

The Yukawa couplings of the flipped $SU(5)$ model from Model 1 then are

$$\begin{aligned} \mathcal{W} \supset\ & \mathbf{10}_{1,-1M}\mathbf{10}_{1,-1M}\mathbf{5}_{-2,2h} + \mathbf{10}_{1,-1M}\bar{\mathbf{5}}_{1,3M}\bar{\mathbf{5}}_{-2,-2h} + \bar{\mathbf{5}}_{1,3M}\mathbf{1}_{1,-5M}\mathbf{5}_{-2,2h} \\ + \ & \mathbf{10}_{1,-1H}\mathbf{10}_{1,-1H}\mathbf{5}_{-2,2h} + \overline{\mathbf{10}}_{1,1H}\overline{\mathbf{10}}_{1,1H}\bar{\mathbf{5}}_{-2,-2h} + \ldots . \end{aligned} \quad (3.143)$$

Similar to the examples with trivial restriction of F_X in the previous models, the spectrum in this model is standard in the sense that there are no exotic chiral fermions, and the quantum numbers of the matter are typical. We claim that the superheavy Higgses $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ come from a vector-like pair on the $\mathbf{16}^{(d_1)}$ curve,

Matter	Class in S	Class with fixed ξ_2	Gen.	Restr. of F_X
$\mathbf{16}^{(d_2)}$	ξ_2	$2h - e_1 - e_2 - e_3 - e_7$	0	1
$\mathbf{16}^{(d_1)}$	$\eta - 4c_1 - \xi_2$	$2h - e_4 - e_5 - e_6 - e_7$	4	-1
$\mathbf{10}^{(d_2)(d_2)}$	$c_1 + \xi_2$	$5h - 2e_1 - 2e_2 - 2e_3 - \sum_{i=4}^{6} e_i - 2e_7$	4	1
$\mathbf{10}^{(d_1)(d_2)}$	$2\eta - 8c_1 - 2\xi_2$	$4h - 2e_4 - 2e_5 - 2e_6 - 2e_7$	-3	-2
$\mathbf{10}^{(d_1)(d_1)}$	$c_1 + \xi_2$	$5h - 2e_1 - 2e_2 - 2e_3 - \sum_{i=4}^{6} e_i - 2e_7$	-1	1

Table XLIII. Matter content of Model 2. The flux $[F_X] = e_3 - e_4$ has restrictions on the curves.

however again it is not obvious and we are not able to fix the number of such pairs. In addition, there exist a few exotic **5** fields from the **10** curves.

On the other hand, the restrictions of the flux F_X on the curves in Model 2 are non-vanishing, thus they contribute to the chirality on the curves. From the information in Table XXVII we can interpret the matter content to fit the flipped $SU(5)$ GUT spectrum in Table XLV.

In this case, the Yukawa couplings for flipped $SU(5)$ are the same:

$$\begin{aligned} \mathcal{W} \supset\ & \mathbf{10}_{-1,-1M}\mathbf{10}_{-1,-1M}\mathbf{5}_{2,2h} + \mathbf{10}_{-1,-1M}\bar{\mathbf{5}}_{1,3M}\bar{\mathbf{5}}_{0,-2h'} + \bar{\mathbf{5}}_{1,3M}\mathbf{1}_{-1,-5M}\mathbf{5}_{0,2h'} \\ +\ & \mathbf{10}_{-1,-1H}\mathbf{10}_{-1,-1H}\mathbf{5}_{2,2h} + \overline{\mathbf{10}}_{1,1H}\overline{\mathbf{10}}_{1,1H}\bar{\mathbf{5}}_{-2,-2h} + \dots . \end{aligned} \tag{3.144}$$

The $\mathbf{10} + \overline{\mathbf{10}}$ superheavey Higgses are identified as a vector-like pair from the **16** curve. In this model there are a few unavoidable exotic fields descended from both **16** and **10** curves.

d. The Singlet Higgs

In the flipped $SU(5)$ model, the matter singlet is the right-handed electron, while it is the right-handed neutrino in the Georgi-Glashow $SU(5)$ GUT. Different from

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{1,-5}$	3
$\mathbf{5}_h$	$\mathbf{5}_{-2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,-2}$	1
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{1,-1} + \overline{\mathbf{10}}_{1,1}$	1
$\mathbf{5} + \bar{\mathbf{5}}$ exotics	$\mathbf{5}_{-2,2} + \bar{\mathbf{5}}_{-2,-2}$	3
	$\mathbf{5}_{0,2} + \bar{\mathbf{5}}_{0,-2}$	3
	$\mathbf{5}_{2,2} + \bar{\mathbf{5}}_{2,-2}$	-1

Table XLIV. Flipped $SU(5)$ spectrum of Model 1 of the (2,2) factorization in dP_7.

the $SU(5)$ spectral cover construction, the flipped $SU(5)$ matter singlet is naturally embedded into the **16** representation of $SO(10)$ in the $SU(4)$ spectral cover configuration. Thus there is no need of additional effort to identify it in the spectrum.

Moreover, in flipped $SU(5)$ models, a Yukawa coupling needed to explain neutrino masses with the seesaw mechanism is [125, 126]

$$\mathbf{10}_{1M}\overline{\mathbf{10}}_{-1H}\mathbf{1}_{0\phi}. \tag{3.145}$$

This singlet $\mathbf{1}_0$ is an $SO(10)$ object and descends neither from the **16** nor from the **10** curves. Naively, one might think that it can be captured by the spectral cover associated to the adjoint representation in $SU(4)$ and the matter curve corresponds to $\pm(\lambda_i - \lambda_j) = 0$ with $i \neq j$. The locus would then be given by [66]

$$b_0^5 \prod_{i<j}^{4} (\lambda_i - \lambda_j)^2 = -4b_2^3 b_3^2 - 27 b_0 b_3^4 + 16 b_2^4 b_4 + 144 b_0 b_2 b_3^2 b_4 - 128 b_0 b_2^2 b_4^2 + 256 b_0^2 b_4^3 = 0.$$

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{1,-5}$	3
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{1,-1} + \overline{\mathbf{10}}_{1,1}$	1
$\mathbf{5}_h$	$\mathbf{5}_{-2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,-2}$	1
$\bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{-1,3}$	1
$\mathbf{1}$	$\mathbf{1}_{-1,5}$	1
$\mathbf{1}$	$\mathbf{1}_{1,-5}$	2
$\mathbf{5} + \bar{\mathbf{5}}$ exotics from the **10** curves[48]		

Table XLV. Flipped $SU(5)$ spectrum of Model 2 of the (2,2) factorization in dP_7.

However, this is not the case. In fact, this singlet matter curve lives in the base B_3 instead of the surface S and can not be described by the spectral cover. To calculate the matter chirality on this singlet matter curve, we need the information of global geometry transverse to the surface S. In other words, we need to go beyond the spectral cover construction[49]. In the future, we hope there will be a global understanding of this singlet curve [66]. Therefore, we just assume this singlet exists and can provide the above Yukawa coupling.

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[48]The $(\mathbf{5}, \bar{\mathbf{5}})$ exotics from the **10** curves of $SO(10)$ can be obtained from Table XXVII.

[49]Recently this singlet has been discussed in [127] for the $SU(5)$ GUT, and it is possible to apply the same idea in this case. We leave this topic for our future work.

F. Conclusion

In this chapter we built flipped $SU(5)$ models from the $SO(10)$ singularity by the $SU(4)$ spectral cover construction in F-theory. The **10** curve in the $SU(4)$ spectral cover configuration forms a double curve, and there are codimension two singularities on this curve [52]. It was also shown that the net chirality on the **10** curve vanishes [52]. In order to obtain more degrees of freedom and non-zero generation number on the **10** curve, we split the $SU(4)$ cover into two factorizations. In the (3,1) factorization there are two **16** curves and two **10** curves on S, while in the (2,2) factorization there are two **16** curves and three **10** curves. The fluxes are also spread over the curves, providing additional parameters for model building.

We started model building from setting up appropriate $SO(10)$ spectrum on the **16** and **10** curves. Some Higgs fields, such as **210**, **120**, and $\mathbf{126}+\overline{\mathbf{126}}$ breaking the $SO(10)$ gauge group are absent in this construction. Therefore, we introduced a $U(1)_X$ flux to break $SO(10)$ to $SU(5)\times U(1)_X$. We interpreted the resulting spectrum as a flipped $SU(5)$ model. The flux may have non-vanishing restrictions on the curves such that the corresponding chiralities may be modified. The superheavy Higgs fields $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ needed for breaking the gauge group to the MSSM are not obvious from the spectrum. We assumed that they are a vector-like pair from the **16** curve including the fermion representations, but we are not able to fix the number of such pairs.

In the (3,1) factorization, we discussed first the construction on the geometry of the Calabi-Yau fourfold with an embedded dP_2 surface constructed in [64]. We demonstrated three examples. Two of them have three-generation, minimal $SO(10)$ GUT matter spectra. The $U(1)_X$ flux has always non-vanishing restrictions on the **16** curves, while it generically has vanishing restrictions on the **10** curves. Therefore, on a **16** curve, the chiralities of the **10**, **5**, and **1** representations are modified in the factor of the $U(1)_X$ charges, and the model no longer has three generations after the $SO(10)$ gauge symmetry is broken. To solve this problem, we constructed a four-generation model such that its corresponding flipped $SU(5)$ spectrum can possess at least three generations after the $U(1)_X$ flux is turned on. On the other hand, the $U(1)_X$ flux in the case of dP_7 geometry background [53] can be tuned to have trivial restrictions on the **16** curves so the chiralities remain untouched. We presented one three-generation example of the (3,1) factorization based on this geometry.

In the (2,2) factorization, to have more degrees of freedom for model building, we focused only on the geometry of the Calabi-Yau fourfold with an embedded dP_7 surface [53] and presented two examples. The first was a three-generation flipped $SU(5)$ model from the $SO(10)$ gauge group broken by the flux with trivial restrictions on all the matter curves. The second example, however, starts from a four-generation $SO(10)$ model whose gauge group is broken to $SU(5) \times U(1)_X$ by the flux with non-trivial restrictions on the matter curves. The resulting chiralities are modified by the flux restrictions to achieve the spectrum of a three-generation flipped $SU(5)$ model. Generically, the flipped $SU(5)$ models from a four-generation $SO(10)$ setup with non-vanishing flux restrictions to the **16** curves results in exotic fields from the **16** curves.

CHAPTER IV

SEMI-LOCAL E_6 MODELS[50]

In this chapter we approach the MSSM from an E_6 GUT by using the spectral cover construction and non-abelian gauge fluxes in F-theory. We start with an E_6 singularity unfolded from an E_8 singularity, and obtain E_6 GUTs by using an $SU(3)$ spectral cover. By turning on $SU(2)\times U(1)^2$ gauge fluxes, we obtain a rank 5 model with the gauge group $SU(3)\times SU(2)\times U(1)^2$. Based on the well-studied geometric backgrounds in the literature, we demonstrate several models and discuss their phenomenology.

A. $SU(3)$ Spectral Cover

Let X_4 be an elliptically fibered Calabi-Yau fourfold $\pi_{X_4}: X_4 \to B_3$ with a section $\sigma_{B_3}: B_3 \to X_4$, and let S be one component of the discriminant locus of X_4 with a projection $\widetilde{\pi}: X_4 \to S$, where X_4 develops an E_6 singularity[51]. To describe X_4, let us consider the Tate model [12]:

$$y^2 = x^3 + \mathbf{b}_3 yz^2 + \mathbf{b}_2 xz^3 + \mathbf{b}_0 z^5, \tag{4.1}$$

where x, y are the coordinates of the fibration and z is the coordinate of the normal direction of S in B_3. Note that the coefficients $\mathbf{b}_k$ generically depend on the coordinate z and that Eq. (4.1) can be regarded as unfolding of an E_8 singularity[52] into an E_6

[50]Reprinted from *Journal of High Energy Physics*, Volume 2011, Number 3, 129, Ching-Ming Chen and Yu-Chieh Chung, On F-theory E_6 GUTs, Copyright 2011, with permission from SISSA.

[51]In this chapter, S will be assumed to be a del Pezzo surface unless otherwise stated [49, 50].

[52]If $\mathbf{b}_3 = \mathbf{b}_2 = 0$, the elliptic fibration $y^2 = x^3 + \mathbf{b}_0 z^5$ possess an E_8 singularity at z=0.

singularity. For convenience, we define the shorthand notations $c_1(S) \equiv c_1$, $t \equiv -c_1(N_{S/B_3})$, and $\eta \equiv 6c_1 - t$ where c_1 is the first Chern class and N_{S/B_3} is the normal bundle of S in B_3. To maintain the Calabi-Yau condition $c_1(X_4) = 0$, it is required that x and y in Eq. (4.1) are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively. It follows that the homological classes $[\mathbf{b}_k]$ are $\eta - kc_1$. Note that the fiber $\widetilde{\pi}^{-1}(b)$ for $b \in S$ is an ALE space [112–117]. The singularity of the fiber over S is determined by the volumes λ_k of (-2) 2-cycles of the ALE space. So unfolding a singularity corresponds to giving some of these 2-cycles finite volumes. In the Tate model Eq. (4.1), the fibration singularity is determined by the coefficients $\mathbf{b}_k$. Indeed, the coefficients $\mathbf{b}_k$ encode the information of the volumes λ_k. In what follows, we shall introduce the spectral cover construction making the relation between the coefficients $\mathbf{b}_k$ in Eq. (4.1) and the volumes λ_k of (-2) 2-cycles manifest[53].

Let us consider the eight-dimensional $\mathcal{N} = 1$ gauge theory compactified on S. To obtain unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions, it was shown [14, 17, 62] that the bosonic fields, a gauge connection A and an adjoint Higgs field Φ, have to satisfy the BPS equations (2.10). To solve BPS equations, one may take V as a holomorphic vector bundle over S with the connection A and Φ being holomorphic. The simplest solution for (A, Φ) is that Φ is diagonal and V is a stable bundle. In particular, let us consider a 3×3 case as follows:

$$\Phi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \sum_{k=1}^{3} \lambda_k = 0, \tag{4.2}$$

where λ_k is holomorphic for $k = 1, 2, 3$. In this case $[\Phi^\dagger, \Phi] = 0$, and Eq. (2.10) is then reduced to the Hermitian Yang-Mills equations (2.11). The low-energy spec-

[53]For more details, please see [47] and references therein.

trum is therefore decoupled to Φ and only depends on the Hermitian Yang-Mills connection A. The eigenvalues λ_k characterize the locations of intersecting seven-branes. Alternatively, the information of intersecting seven-branes can be encoded in the characteristic polynomial $P_\Phi(s) = \det(sI - \Phi)$ associated with a spectral cover over S. For generically diagonal Φ, the polynomial equation $P_\Phi(s) = 0$ has distinct roots and the associated spectral cover is smooth. In what follows, we shall focus on the case of Eq. (4.2) and its associated spectral cover. Notice that the polynomial equation

$$b_0 \det(sI - \Phi) = b_0 s^3 + b_2 s + b_3 = 0 \tag{4.3}$$

defines a three-sheeted cover of S inside the total space of the canonical bundle $K_S \to S$, a local Calabi-Yau threefold, where $b_k \equiv \mathbf{b}_k|_{z=0},\ k = 0, 2, 3$. However, this threefold is non-compact. For well-defined intersection numbers, one can compactify the non-compact threefold to the total space of projective bundle $\mathbb{P}(\mathcal{O}_S \oplus K_S)$ over S. Let us define X as the total space of the projective bundle with two sections U, V and with a projection map $\pi : X \to S$. The homological classes of zero sections $\{U = 0\}$ and $\{V = 0\}$ are σ and $\sigma + c_1$, respectively. In compact threefold X, the spectral cover Eq. (4.3) can be expressed as a homogeneous polynomial as follows:

$$\mathcal{C}^{(3)} :\ b_0 U^3 + b_2 U V^2 + b_3 V^3 \equiv b_0 \prod_{k=1}^{3} (U + \lambda_k V) = 0, \tag{4.4}$$

with a projection map $p_3 : \mathcal{C}^{(3)} \to S$ where $b_k \equiv \mathbf{b}_k|_{z=0},\ k = 0, 2, 3$. The homological class of $\mathcal{C}^{(3)}$ is given by $[\mathcal{C}^{(3)}] = 3\sigma + \pi^* \eta$. The singularities get enhanced at some loci of S. Let us consider the following breaking pattern

$$\begin{aligned} E_8 &\ \to\ E_6 \times SU(3) \\ \mathbf{248} &\ \to\ (\mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \bar{\mathbf{3}}). \end{aligned} \tag{4.5}$$

The matter $\mathbf{27}$ is localized on the curve $\Sigma_{\mathbf{27}}$ given by the locus of $\{b_3 = 0\}$ where the singularity E_6 is enhanced to E_7, so it implies the homological class of $[\Sigma_{\mathbf{27}}]$ is $\eta - 3c_1$ in S. Alternatively, it follows from $\lambda_i = 0$ in Eq. (4.4) that the homological class of $[\Sigma_{\mathbf{27}}]$ can be also computed by $[\mathcal{C}^{(3)}] \cdot \sigma|_\sigma = \eta - 3c_1$. With a spectral cover $\mathcal{C}^{(3)}$, one can obtain a Higgs bundle $p_{3*}\mathcal{L}$ on S by the pushforward of a line bundle $\mathcal{L}$ on $\mathcal{C}^{(3)}$. To maintain the traceless condition $c_1(p_{3*}\mathcal{L}) = 0$, it is required that $p_{3*}\gamma^{(3)} = 0$ where $c_1(\mathcal{L}) \equiv \gamma^{(3)} + \frac{1}{2}r^{(3)} \in H_4(X, \mathbb{Z})$ and $r^{(3)}$ is the ramification divisor of the projection map $p_3 : \mathcal{C}^{(3)} \to S$. Up to a constant, the unique solution of the traceless condition $p_{3*}\gamma^{(3)} = 0$ is $\gamma^{(3)} = (3 - p_3^* p_{3*})[\mathcal{C}^{(3)}] \cdot \sigma$, and one can calculate the chiral spectrum by turning on the traceless flux $\gamma^{(3)}$. More precisely, the net chirality $N_{\mathbf{27}}$ of the matter field **27** can be computed as

$$N_{\mathbf{27}} = \gamma^{(3)} \cdot \Sigma_{\mathbf{27}} = -\eta \cdot_S (\eta - 3c_1). \tag{4.6}$$

To obtain three generations for **27**, it is required that $(6c_1 - t) \cdot_S (3c_1 - t) = -3$ which is a non-trivial constraint on embedding of S into the Calabi-Yau fourfold X_4. On the other hand, the irreducible cover $\mathcal{C}^{(3)}$ only provides a single matter curve, so we need more matter curves and more degrees of freedom on the cover flux to build promising realistic models. Therefore we shall study the factorizations of the spectral cover $\mathcal{C}^{(3)}$ in what follows.

1. (2,1) Factorization

Let us consider the factorization $\mathcal{C}^{(3)} \to \mathcal{C}^{(a)} \times \mathcal{C}^{(b)}$:

$$b_0 U^3 + b_2 U V^2 + b_3 V^3 = (a_0 U^2 + a_1 U V + a_2 V^2)(d_0 U + d_1 V) \tag{4.7}$$

with projection maps $p_a : \mathcal{C}^{(a)} \to S$ and $p_b : \mathcal{C}^{(b)} \to S$, respectively. Let $[d_1] \equiv \xi$. One can write the homological class of remaining sections as

$$[a_n] = \eta - (n+1)c_1 - \xi,\ n = 0, 1, 2, \qquad [d_0] = c_1 + \xi. \tag{4.8}$$

It follows from Eqs. (4.7) and (4.8) that the homological classes of the covers $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ are given by

$$[\mathcal{C}^{(a)}] = 2\sigma + \pi^*(\eta - \xi - c_1), \qquad [\mathcal{C}^{(b)}] = \sigma + \pi^*(\xi + c_1). \tag{4.9}$$

With the homological classes $[\mathcal{C}^{(a)}]$ and $[\mathcal{C}^{(b)}]$, one can compute the homological classes of matter curves $\Sigma^{(a)}_{\mathbf{27}}$ and $\Sigma^{(b)}_{\mathbf{27}}$ as

$$[\Sigma^{(a)}_{\mathbf{27}}] = [\mathcal{C}^{(a)}] \cdot \sigma|_\sigma = \eta - 3c_1 - \xi, \quad [\Sigma^{(b)}_{\mathbf{27}}] = [\mathcal{C}^{(b)}] \cdot \sigma|_\sigma = \xi. \tag{4.10}$$

The ramification divisors of the maps $p_a : \mathcal{C}^{(a)} \to S$ and $p_b : \mathcal{C}^{(b)} \to S$ are given by

$$r^{(a)} = [\mathcal{C}^{(a)}] \cdot \pi^*(\eta - 2c_1 - \xi), \quad r^{(b)} = [\mathcal{C}^{(b)}] \cdot (-\sigma + \pi^*\xi). \tag{4.11}$$

The traceless fluxes $\gamma^{(a)}_0$ and $\gamma^{(b)}_0$ is defined as $(2 - p^*_a p_{a*})[\mathcal{C}^{(a)}] \cdot \sigma$ and $(1 - p^*_b p_{b*})[\mathcal{C}^{(b)}] \cdot \sigma$, respectively, where $p_{a*}\gamma^{(a)}_0 = 0$ and $p_{b*}\gamma^{(b)}_0 = 0$. The explicit forms of the traceless fluxes $\gamma^{(a)}_0$ and $\gamma^{(b)}_0$ are given by

$$\gamma^{(a)}_0 = [\mathcal{C}^{(a)}] \cdot (2\sigma - \pi^*(\eta - 3c_1 - \xi)), \quad \gamma^{(b)}_0 = [\mathcal{C}^{(b)}] \cdot (\sigma - \pi^*\xi). \tag{4.12}$$

The chirality of matter **27** on each matter curve due to the fluxes $\gamma^{(a)}_0$ and $\gamma^{(b)}_0$ is then shown in Table XLVI.

Due to the factorization, one can introduce the additional fluxes $\delta^{(a)} = (1 - p^*_b p_{a*})[\mathcal{C}^{(a)}] \cdot \sigma$ and $\delta^{(b)} = (2 - p^*_a p_{b*})[\mathcal{C}^{(b)}] \cdot \sigma$. It is not difficult to obtain [66]:

$$\delta^{(a)} = [\mathcal{C}^{(a)}] \cdot \sigma - [\mathcal{C}^{(b)}] \cdot \pi^*(\eta - 3c_1 - \xi), \quad \delta^{(b)} = [\mathcal{C}^{(b)}] \cdot 2\sigma - [\mathcal{C}^{(a)}] \cdot \pi^*\xi. \tag{4.13}$$

	$\gamma_0^{(a)}$	$\gamma_0^{(b)}$
$\mathbf{27}^{(a)}$	$-(\eta - c_1 - \xi) \cdot_S (\eta - 3c_1 - \xi)$	0
$\mathbf{27}^{(b)}$	0	$-\xi \cdot_S (c_1 + \xi)$

Table XLVI. Chirality induced by the fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$.

	$\delta^{(a)}$	$\delta^{(b)}$	$\tilde{\rho}$
$\mathbf{27}^{(a)}$	$-c_1 \cdot_S (\eta - 3c_1 - \xi)$	$-\xi \cdot_S (\eta - 3c_1 - \xi)$	$-\rho \cdot_S (\eta - 3c_1 - \xi)$
$\mathbf{27}^{(b)}$	$-\xi \cdot_S (\eta - 3c_1 - \xi)$	$-2c_1 \cdot_S \xi$	$2\rho \cdot_S \xi$

Table XLVII. Chirality induced by the fluxes $\delta^{(a)}$, $\delta^{(b)}$, and $\tilde{\rho}$.

Also for any $\rho \in H_2(S, \mathbb{R})$, one can define a non-trivial flux $\tilde{\rho}$ as

$$\tilde{\rho} = (2p_b^* - p_a^*)\rho, \tag{4.14}$$

then the chirality induced by these additional fluxes on each matter curve is summarized in Table XLVII.

The total flux Γ is then a linear combination of the fluxes above:

$$\Gamma = k_a \gamma_0^{(a)} + k_b \gamma_0^{(b)} + m_a \delta^{(a)} + m_b \delta^{(b)} + \tilde{\rho} \equiv \Gamma^{(a)} + \Gamma^{(b)}, \tag{4.15}$$

where

$$\Gamma^{(a)} \equiv [\mathcal{C}^{(a)}] \cdot [\tilde{\mathcal{C}}^{(a)}] = [\mathcal{C}^{(a)}] \cdot [(2k_a + m_a)\sigma - \pi^*(k_a(\eta - 3c_1 - \xi) + m_b \xi + \rho)], \tag{4.16}$$

$$\Gamma^{(b)} \equiv [\mathcal{C}^{(b)}] \cdot [\tilde{\mathcal{C}}^{(b)}] = [\mathcal{C}^{(b)}] \cdot [(k_b + 2m_b)\sigma - \pi^*(k_b \xi + m_a(\eta - 3c_1 - \xi) - 2\rho)]. \tag{4.17}$$

The parameters k_a, k_b, m_a, m_b will be determined later by physical and consistency conditions. In addition, by

$$p_{a*}\Gamma^{(a)} = m_a(\eta - 3c_1 - \xi) - 2m_b\xi - 2\rho, \tag{4.18}$$

$$p_{b*}\Gamma^{(b)} = -m_a(\eta - 3c_1 - \xi) + 2m_b\xi + 2\rho, \tag{4.19}$$

we find that $\Gamma^{(a)}$ and $\Gamma^{(a)}$ indeed satisfy the traceless condition $p_{a*}\Gamma^{(a)} + p_{b*}\Gamma^{(b)} = 0$. In the $(2, 1)$ factorization, the quantization conditions are then given by

$$(2k_a + m_a)\sigma - \pi^*(k_a(\eta - 3c_1 - \xi) + m_b\xi + \rho - \frac{1}{2}(\eta - 2c_1 - \xi)) \in H_4(X, \mathbb{Z}), \tag{4.20}$$

$$(k_b + 2m_b - \frac{1}{2})\sigma - \pi^*(k_b\xi + m_a(\eta - 3c_1 - \xi) - 2\rho - \frac{1}{2}\xi) \in H_4(X, \mathbb{Z}). \tag{4.21}$$

In addition, the supersymmetry condition is

$$[m_a(\eta - 3c_1 - \xi) - 2m_b\xi - 2\rho] \cdot_S [\omega_S] = 0, \tag{4.22}$$

where $[\omega_S]$ is an ample divisor dual to a Kähler form of S.

2. (1,1,1) Factorization

Let us consider the factorization $\mathcal{C}^{(3)} \to \mathcal{C}^{(l_1)} \times \mathcal{C}^{(l_2)} \times \mathcal{C}^{(l_3)}$:

$$b_0U^3 + b_2UV^2 + b_3V^3 = (f_0U + f_1V)(g_0U + g_1V)(h_0U + h_1V), \tag{4.23}$$

with the projection maps $p_{l_1} : \mathcal{C}^{(l_1)} \to S$, $p_{l_2} : \mathcal{C}^{(l_2)} \to S$, and $p_{l_3} : \mathcal{C}^{(l_3)} \to S$. Let $[g_1] \equiv \xi_1$ and $[h_1] \equiv \xi_2$, the homological classes of the remaining sections are

$$[f_m] = \eta - (m + 2)c_1 - \xi_1 - \xi_2,\ m = 0, 1. \quad [g_0] = c_1 + \xi_1,\ [h_0] = c_1 + \xi_2. \tag{4.24}$$

It follows from Eqs. (4.23) and (4.24) that the homological classes of the covers $\mathcal{C}^{(l_1)}$, $\mathcal{C}^{(l_2)}$, and $\mathcal{C}^{(l_3)}$ are given by

$$[\mathcal{C}^{(l_1)}] = \sigma + \pi^*(\eta - 2c_1 - \xi_1 - \xi_2), \quad [\mathcal{C}^{(l_2)}] = \sigma + \pi^*(\xi_1 + c_1), \quad [\mathcal{C}^{(l_3)}] = \sigma + \pi^*(\xi_2 + c_1). \tag{4.25}$$

The homological classes of the matter curves can be obtained from the intersection $[\mathcal{C}^{(l_i)}] \cdot \sigma|_\sigma$:

$$[\Sigma^{(l_1)}_{\mathbf{27}}] = \eta - 3c_1 - \xi_1 - \xi_2, \quad [\Sigma^{(l_2)}_{\mathbf{27}}] = \xi_1, \quad [\Sigma^{(l_3)}_{\mathbf{27}}] = \xi_2. \tag{4.26}$$

In the $(1,1,1)$ factorization, the ramification divisors are given by

$$r_{l_1} = [\mathcal{C}^{(l_1)}]\cdot[-\sigma+\pi^*(\eta-3c_1-\xi_1-\xi_2)], \;\; r_{l_2} = [\mathcal{C}^{(l_2)}]\cdot(-\sigma+\pi^*\xi_1), \;\; r_{l_3} = [\mathcal{C}^{(l_3)}]\cdot(-\sigma+\pi^*\xi_2). \tag{4.27}$$

For general fluxes $\gamma^{(i)} = [\mathcal{C}^{(i)}] \cdot \sigma$, we define the traceless fluxes $\gamma_0^{(i)}$ as

$$\gamma_0^{(l_1)} = \;\; (1 - p^*_{l_1} p_{l_1 *})\gamma^{(l_1)} = [\mathcal{C}^{(l_1)}] \cdot [\sigma - \pi^*(\eta - 3c_1 - \xi_1 - \xi_2)], \tag{4.28}$$

$$\gamma_0^{(l_2)} = \;\; (1 - p^*_{l_2} p_{l_2 *})\gamma^{(l_2)} = [\mathcal{C}^{(l_2)}] \cdot (\sigma - \pi^*\xi_1), \tag{4.29}$$

$$\gamma_0^{(l_3)} = \;\; (1 - p^*_{l_3} p_{l_3 *})\gamma^{(l_3)} = [\mathcal{C}^{(l_3)}] \cdot (\sigma - \pi^*\xi_2). \tag{4.30}$$

It is easy to see that $\gamma_0^{(i)}$ satisfies the condition $p_{i*}\gamma_0^{(i)} = 0$ for all i. The chirality induced by the fluxes $\gamma_0^{(l_1)}$, $\gamma_0^{(l_2)}$, and $\gamma_0^{(l_3)}$ is summarized in Table XLVIII.

	$\gamma_0^{(l_1)}$	$\gamma_0^{(l_2)}$	$\gamma_0^{(l_3)}$
$\mathbf{27}^{(l_1)}$	$-(\eta-2c_1-\xi_1-\xi_2)\cdot_S(\eta-3c_1-\xi_1-\xi_2)$	0	0
$\mathbf{27}^{(l_2)}$	0	$-\xi_1\cdot_S(c_1+\xi_1)$	0
$\mathbf{27}^{(l_3)}$	0	0	$-\xi_2\cdot_S(c_1+\xi_2)$

Table XLVIII. Chirality induced by the fluxes $\gamma_0^{(l_1)}$, $\gamma_0^{(l_2)}$, and $\gamma_0^{(l_3)}$.

There are many choices of the additional fluxes, for simplicity, we consider

$$\begin{aligned}
\delta^{(l_1)} &= [(1-p_{l_2}^*p_{l_1*})+(1-p_{l_3}^*p_{l_1*})]\gamma^{(l_1)} \\
&= [\mathcal{C}^{(l_1)}]\cdot 2\sigma-([\mathcal{C}^{(l_2)}]+[\mathcal{C}^{(l_3)}])\cdot\pi^*(\eta-3c_1-\xi_1-\xi_2), \qquad (4.31)\\
\delta^{(l_2)} &= [(1-p_{l_1}^*p_{l_2*})+(1-p_{l_3}^*p_{l_2*})]\gamma^{(l_2)} \\
&= [\mathcal{C}^{(l_2)}]\cdot 2\sigma-[\mathcal{C}^{(l_1)}]\cdot\pi^*\xi_1-[\mathcal{C}^{(l_3)}]\cdot\pi^*\xi_1, \qquad (4.32)\\
\delta^{(l_3)} &= [(1-p_{l_1}^*p_{l_3*})+(1-p_{l_2}^*p_{l_3*})]\gamma^{(l_3)} \\
&= [\mathcal{C}^{(l_3)}]\cdot 2\sigma-[\mathcal{C}^{(l_1)}]\cdot\pi^*\xi_2-[\mathcal{C}^{(l_2)}]\cdot\pi^*\xi_2. \qquad (4.33)\\
\widehat{\rho} &= (p_{l_2}^*-p_{l_1}^*)\rho_1+(p_{l_3}^*-p_{l_2}^*)\rho_2+(p_{l_1}^*-p_{l_3}^*)\rho_3, \qquad (4.34)
\end{aligned}$$

where $\rho_i\in H_2(S,\mathbb{R})$, $\forall i$. The chirality induced by these additional fluxes on each matter curve is summarized in Table XLIX.

The total flux Γ with the parameters k_{l_1}, k_{l_2} k_{l_3}, m_{l_1}, m_{l_2}, and m_{l_3} is [66]

$$\Gamma=k_{l_1}\gamma_0^{(l_1)}+k_{l_2}\gamma_0^{(l_2)}+k_{l_3}\gamma_0^{(l_3)}+m_{l_1}\delta^{(l_1)}+m_{l_2}\delta^{(l_2)}+m_{l_3}\delta^{(l_3)}+\widehat{\rho}\equiv\Gamma^{(l_1)}+\Gamma^{(l_2)}+\Gamma^{(l_3)}, \qquad (4.35)$$

	$\delta^{(l_1)}$	$\delta^{(l_2)}$	$\delta^{(l_3)}$	$\widehat{\rho}$
$\mathbf{27}^{(l_1)}$	$-2c_1 \cdot_S [f_1]$	$-\xi_1 \cdot_S [f_1]$	$-\xi_2 \cdot_S [f_1]$	$(\rho_3 - \rho_1) \cdot_S [f_1]$
$\mathbf{27}^{(l_2)}$	$-\xi_1 \cdot_S [f_1]$	$-2c_1 \cdot_S \xi_1$	$-\xi_1 \cdot_S \xi_2$	$(\rho_1 - \rho_2) \cdot_S \xi_1$
$\mathbf{27}^{(l_3)}$	$-\xi_2 \cdot_S [f_1]$	$-\xi_1 \cdot_S \xi_2$	$-2c_1 \cdot_S \xi_2$	$(\rho_2 - \rho_3) \cdot_S \xi_2$

Table XLIX. Chirality induced by the fluxes $\delta^{(l_1)}$, $\delta^{(l_2)}$, $\delta^{(l_3)}$ and $\widehat{\rho}$.

where

$$\Gamma^{(l_1)} = [\mathcal{C}^{(l_1)}] \cdot [(k_{l_1} + 2m_{l_1})\sigma - \pi^*(k_{l_1}[f_1] + m_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_1 - \rho_3)], \quad (4.36)$$

$$\Gamma^{(l_2)} = [\mathcal{C}^{(l_2)}] \cdot [(k_{l_2} + 2m_{l_2})\sigma - \pi^*(m_{l_1}[f_1] + k_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_2 - \rho_1)], \quad (4.37)$$

$$\Gamma^{(l_3)} = [\mathcal{C}^{(l_3)}] \cdot [(k_{l_3} + 2m_{l_3})\sigma - \pi^*(m_{l_1}[f_1] + m_{l_2}\xi_1 + k_{l_3}\xi_2 + \rho_3 - \rho_2)]. \quad (4.38)$$

It is then straightforward to compute

$$p_{l_1*}\Gamma^{(l_1)} = 2m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_1 + \rho_3, \quad (4.39)$$

$$p_{l_2*}\Gamma^{(l_2)} = -m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) + 2m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_2 + \rho_1, \quad (4.40)$$

$$p_{l_3*}\Gamma^{(l_3)} = -m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 + 2m_{l_3}\xi_2 - \rho_3 + \rho_2. \quad (4.41)$$

The sum is zero, as it should be for the traceless condition. In this case, the quantization conditions are given by

$$(k_{l_1} + 2m_{l_1} - \frac{1}{2})\sigma - \pi^*\{(k_{l_1} - \frac{1}{2})[f_1] + m_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_1 - \rho_3\} \in H_4(X, \mathbb{Z}), (4.42)$$

$$(k_{l_2} + 2m_{l_2} - \frac{1}{2})\sigma - \pi^*\{m_{l_1}[f_1] + (k_{l_2} - \frac{1}{2})\xi_1 + m_{l_3}\xi_2 + \rho_2 - \rho_1\} \in H_4(X, \mathbb{Z}), (4.43)$$

$$(k_{l_3} + 2m_{l_3} - \frac{1}{2})\sigma - \pi^*\{m_{l_1}[f_1] + m_{l_2}\xi_1 + (k_{l_3} - \frac{1}{2})\xi_2 + \rho_3 - \rho_2\} \in H_4(X, \mathbb{Z}), (4.44)$$

and the supersymmetry conditions are as follows:

$$[2m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_1 + \rho_3] \cdot_S [\omega_S] = 0, \quad (4.45)$$

$$[-m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) + 2m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_2 + \rho_1] \cdot_S [\omega_S] = 0, \quad (4.46)$$

$$[-m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 + 2m_{l_3}\xi_2 - \rho_3 + \rho_2] \cdot_S [\omega_S] = 0. \quad (4.47)$$

B. Breaking E_6

The MSSM fermion and electroweak Higgs fields can be included in the same **27** multiplet of a three-family E_6 GUT model. On the other hand, it is possible to assign the Higgs fields to a different $\mathbf{27}_H$ multiplet where only the Higgs doublets and singlets obtain the electroweak scale energy. The Yukawa coupling for these two cases can be written as

$$\mathcal{W} \supset \mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27} \text{ (Case A)} \quad \text{or} \quad \mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27}_H \text{ (Case B)}. \quad (4.48)$$

The Yukawa coupling of Case A is either a triple-intersection of one **27** curve or an intersection of three different curves in F-theory model building. It is difficult to obtain a three family model from a single curve and the geometry of a triple-intersection is generally complicated. On other hand, it is not easy to achieve the mass hierarchy of the third generation in the three-curve model. Therefore, we do not consider Case A in this paper. In case B, there are two possible constructions from spectral cover factorizations. In the $(2,1)$ factorization, the fermions are assigned to $\mathbf{27}^{(a)}$ curve and the Higgs fields come from the other $\mathbf{27}^{(b)}$ curve. The Yukawa coupling then turns out

$$\mathcal{W}^{(2,1)} \supset \mathbf{27}^{(a)} \cdot \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)}. \quad (4.49)$$

In the $(1,1,1)$ factorization, the matter fields are assigned to curve $\mathbf{27}^{(a)}$ and $\mathbf{27}^{(b)}$ while the Higgs fields come from the $\mathbf{27}^{(c)}$ curve. In this case the Yukawa coupling is then

$$\mathcal{W}^{(1,1,1)} \supset \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)} \cdot \mathbf{27}^{(c)}. \tag{4.50}$$

In order to realize the MSSM in the E_6 GUT models, it is useful to study the subgroups of E_6. In our F-theory model building we consider the picture that the E_6 gauge group is broken by the $SU(2) \times U(1)^2$ flux on the seven-branes. This flux may tilt the chirality of the matter on the curve after E_6 is broken.

1. Subgroups of E_6

The subgroups of E_6 including the Standard Model gauge group can be denoted $E_6 \supset SU(3) \times SU(2)_L \times G_c$. Here G_c marks a rank 3 group which is a product of $U(1)$ or $SU(2)$. It has been shown (for example, [90,94,100,128]) that by suitable assignments of the hypercharge of the SM and the $B - L$ symmetry, these E_6 subgroups with different G_c are equivalent to different matter content arrangements. This property would be useful for the analysis of the non-abelian fluxes of type G_c. In this section we will briefly review the subgroups of E_6.

Let us consider the following breaking patterns of E_6:

$$(1a) \quad E_6 \to SO(10) \times U(1) \to SU(5) \times U(1)^2, \tag{4.51}$$

$$(1b) \quad E_6 \to SO(10) \times U(1) \to SU(4) \times SU(2) \times SU(2) \times U(1), \tag{4.52}$$

$$(2a) \quad E_6 \to SU(6) \times SU(2) \to SU(5) \times U(1) \times SU(2), \tag{4.53}$$

$$(2b) \quad E_6 \to SU(6) \times SU(2) \to SU(4) \times SU(2) \times U(1) \times SU(2), \tag{4.54}$$

$$(2c) \quad E_6 \to SU(6) \times SU(2) \to SU(3) \times SU(3) \times U(1) \times SU(2), \tag{4.55}$$

$$(3) \quad E_6 \to SU(3) \times SU(3) \times SU(3). \tag{4.56}$$

In all of these cases, there are two possible outcomes when E_6 is broken down to the subgroups containing the Standard Model group. Case ($1a$) turns out to be

$$E_6 \to SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_\chi \times U(1)_\psi, \tag{4.57}$$

and the other cases become

$$E_6 \to SU(3) \times SU(2) \times SU(2) \times U(1)_U \times U(1)_W. \tag{4.58}$$

Note that the assignments of $U(1)_U$ and $U(1)_W$ groups of the cases ($1b$), ($2a$), ($2b$), ($2c$) and (3) are different, but they are equivalent up to linear transformations. Take case (3) as an example, the breaking is through a trinification model, therefore we can write

$$E_6 \supset SU(3) \times SU(2)_L \times SU(2)_{(R)} \times U(1)_{Y_L} \times U(1)_{Y_{(R)}}. \tag{4.59}$$

The parenthesis on R in $SU(2)_{(R)}$ indicates that it has three different assignments denoted by $SU(2)_R$, $SU(2)_{R'}$, and $SU(2)_E$ [128]. The third component $I_{3(R)}$ of $SU(2)_{(R)}$ along with the quantum numbers of $U(1)_{Y_L}$ and $U(1)_{Y_{(R)}}$ can have a linear relation to the quantum numbers of $U(1)_Y$, $U(1)_\chi$ and $U(1)_\psi$ of case ($1a$) in (4.57), i.e.,

$$Y = a_1 Y_L + a_2 Y_{(R)} + a_3 I_{3(R)},\ \chi = b_1 Y_L + b_2 Y_{(R)} + b_3 I_{3(R)},\ \psi = c_1 Y_L + c_2 Y_{(R)} + c_3 I_{3(R)}, \tag{4.60}$$

where a_i, b_i and c_i are coefficients of the transformation. These three different kinds of $SU(2)_{(R)}$ assignments also confine the three different embedding of SM matter representations into the $SU(5)$ multiplets belonging to **27** of E_6, as well as the corresponding assignments of the hypercharge. The three assignments of $U(1)_Y$ should be orthogonal to the three $SU(2)_{(R)}$, respectively.

The $U(1)_{B-L}$ symmetry is conserved in SUSY E_6 models, which is not difficult to see from the gauge breaking via the Pati-Salam gauge group. $U(1)_{B-L}$ has a linear relation with $U(1)_{Y_L}$, $U(1)_{Y_{(R)}}$, and the third component of $SU(2)_{(R)}$. There are also three $U(1)_{B-L}$ assignments orthogonal to the three $SU(2)_{(R)}$, respectively. For consistency with the SM structure, $U(1)_{B-L}$ and $U(1)_Y$ are not orthogonal to the same $SU(2)_{(R)}$. Therefore, there are six totally different charge assignments of the SM multiplets - six different embedding of SM multiplets in **27** of E_6. For the detailed analysis, we refer readers to [128].

The E_6 subgroups listed in Eqs. (4.57) and (4.58) are rank 6. In heterotic string compactifications, E_6 can be broken by a non-abelian flux down to a rank 5 subgroup [91, 93–95]:

$$E_6 \to SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_\eta. \tag{4.61}$$

This model is usually marked as the η-model. Rank 6 models [96, 98, 99] have more symmetries, but it is common practice to give a large VEV to one $U(1)$ gauge group to reduce them to the so called *effective* rank 5 models. For instance, from Eq. (4.57) the remaining abelian gauge group $U(1)_\theta$ is a reduction

$$U(1)_\theta = \cos\theta U(1)_\chi + \sin\theta U(1)_\psi. \tag{4.62}$$

Particularly, the rank 5 η-model can be regarded as a special case of this setup by

$$U(1)_\eta = \sqrt{\frac{3}{8}} U(1)_\chi - \sqrt{\frac{5}{8}} U(1)_\psi. \tag{4.63}$$

In our F-theory models, a non-abelian flux $SU(2) \times U(1)^2$ is turned on to break the E_6 gauge group to $SU(3) \times SU(2) \times U(1)^2$ taken to be the η-model. However, since $U(1)_\eta$ is only determined by the two $U(1)$s while the $SU(2)$ is integrated out, the η-model does not possess the degrees of freedom from the mixing angle θ preserving some symmetries such as the $B-L$ symmetry [100]. The corresponding phenomenology of the F-theory rank 5 model will basically follow the properties of the η-model.

The particle content of the E_6 model we will consider is conventional. It includes three copies of **27**-plets, each copy includes an SM ordinary family, two Higgs-type doublets, two SM singlets, and two exotic $SU(2)$-singlet quarks. The **27** matter content of the $SU(3) \times SU(2) \times U(1)_Y \times U(1)_\eta$ model with the corresponding charges are

$$\begin{aligned} \mathbf{27} \;&\to Q(\mathbf{3},\mathbf{2})_{\frac{1}{3},2} + u^c(\bar{\mathbf{3}},\mathbf{1})_{-\frac{4}{3},2} + e^c(\mathbf{1},\mathbf{1})_{2,2} \\ &+\; L(\mathbf{1},\mathbf{2})_{-1,-1} + d^c(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3},-1} + \nu^c(\mathbf{1},\mathbf{1})_{0,5} \\ &+\; \bar{D}(\mathbf{3},\mathbf{1})_{-\frac{2}{3},-4} + \bar{h}(\mathbf{1},\mathbf{2})_{1,-4} \\ &+\; D(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3},-1} + h(\mathbf{1},\mathbf{2})_{-1,-1} + S(\mathbf{1},\mathbf{1})_{0,5}, \end{aligned} \tag{4.64}$$

where the first subscript denotes the $U(1)_Y$ charge and the second indicates the $U(1)_\eta$ charge. The superpotential for the $\mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27}$ coupling can be expanded as

$$\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \cdots , \tag{4.65}$$

$$\mathcal{W}_0 = \lambda_1 \bar{h} Q u^c + \lambda_2 h Q d^c + \lambda_3 h L e^c + \lambda_4 h \bar{h} S + \lambda_5 D \bar{D} S, \tag{4.66}$$

$$\mathcal{W}_1 = \lambda_6 \bar{D} u^c e^c + \lambda_7 D Q L + \lambda_8 \bar{D} \nu^c d^c, \tag{4.67}$$

$$\mathcal{W}_2 = \lambda_9 \bar{D} Q Q + \lambda_{10} D u^c d^c, \tag{4.68}$$

$$\mathcal{W}_3 = \lambda_{11} \bar{h} L \nu^c. \tag{4.69}$$

Additional symmetries should be considered to avoid the terms that may cause serious phenomenological problems. The exotic fields are only confined by the charge, isospin, and hypercharge assignments while their baryon and lepton numbers remain unspecified. By assigning baryon and lepton numbers to D, it is possible to forbid some of the interactions in $\mathcal{W}$ by the conservation of baryon and lepton numbers. For example, if the baryon number $B(D) = \frac{1}{3}$ and the lepton number $L(D) = 1$, $\mathcal{W}_2 = 0$; if $B(D) = -\frac{2}{3}$ and $L(D) = 0$, then $\mathcal{W}_1 = 0$. In the case $B(D) = \frac{1}{3}$ and $L(D) = 0$, D is regarded as a conventional quark - able to mix with the d-quarks - then decaying via flavor changing neutral currents (FCNC) or charged currents (CC) [100]. By setting

$B(h,\bar{h}) = L(h,\bar{h}) = 0$ and $B(S) = L(S) = 0$, h and $\bar{h}$ are the usual MSSM Higgs doublets, and the VEV of S provides a mass for D. See [100] for a detailed review.

Another possibility is considering the MSSM Higgs fields coming from a different $\mathbf{27}_H$ (or $\overline{\mathbf{27}}_H$). In this case the exotics of the matter $\mathbf{27}$-plet are taken as the ordinary quarks and leptons, $B(D) = \frac{1}{3}$ and $L(D) = 0$, as well as $B(h,\bar{h},\nu^c,S) = 0$ and $L(h,\bar{h},\nu^c,S) = \pm 1$. The doublets $H_1(\mathbf{1},\mathbf{2})_{-1,-1}$, $H_2(\mathbf{1},\mathbf{2})_{-1,-1}$ and $\bar{H}_2(\mathbf{1},\mathbf{2})_{1,-4}$, and the singlets $H_3(\mathbf{1},\mathbf{1})_{0,5}$ and $H_4(\mathbf{1},\mathbf{1})_{0,5}$ of $\mathbf{27}_H$ develop VEVs so that the superpotential takes the form

$$\begin{aligned} \mathcal{W}' \supset\ & \bar{H}_2 Q u^c + H_2 Q d^c + H_2 L e^c + H_1 h e^c + \bar{h} h H_4 \\ +\ & \bar{H}_2 h S + H_2 \bar{h} S + \bar{D} D H_4 + H_1 Q D + H_3 \bar{D} d^c + \bar{H}_2 L \nu^c + \cdots . \end{aligned} \quad (4.70)$$

We can see the mixings between the ordinary fermions and their corresponding exotic fields. These kinds of mixings allow the exotics to decay via FCNC or CC [100].

There can be one or more additional Higgs-like doublets from $(\mathbf{27} + \overline{\mathbf{27}})$ vector-like pairs preserving the gauge unification without introducing anomalies. In summary, with the picture of electroweak Higgs fields from a different $\mathbf{27}_H$, the minimum spectrum at low energy is

$$3 \times \mathbf{27} + (\mathbf{27}_H) + (\mathbf{27} + \overline{\mathbf{27}}). \quad (4.71)$$

2. Non-abelian Gauge Fluxes

In what follows, we shall analyze the effects on the chirality after the $SU(2) \times U(1)^2$ flux is turned on. We choose the breaking chain $(1b)$ in Eq. (4.52) via $SO(10)$ and $SU(4) \times SU(2) \times SU(2)$. When the flux is turned on, the matter on the bulk decomposes as

$$
\begin{aligned}
E_6 &\xrightarrow[U(1)_a]{} SO(10)\times[U(1)_a]\\
&\xrightarrow[SU(2)]{} SU(4)\times SU(2)_1\times[SU(2)_2\times U(1)_a]\\
&\xrightarrow[U(1)_b]{} SU(3)\times SU(2)_1\times[SU(2)_2\times U(1)_a\times U(1)_b]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{78} \quad &\rightarrow \quad \mathbf{45}_0+\mathbf{1}_0+\mathbf{16}_{-3}+\overline{\mathbf{16}}_3\\
&\rightarrow \quad (\mathbf{15},\mathbf{1},\mathbf{1})_0+(\mathbf{6},\mathbf{2},\mathbf{2})_0+(\mathbf{1},\mathbf{3},\mathbf{1})_0+(\mathbf{1},\mathbf{1},\mathbf{3})_0+(\mathbf{1},\mathbf{1},\mathbf{1})_0\\
&\qquad [(\mathbf{4},\mathbf{2},\mathbf{1})_{-3}+(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{-3}+c.c.]\\
&\rightarrow \quad (\mathbf{8},\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{3},\mathbf{1},\mathbf{1})_{0,-4}+(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{0,4}+(\mathbf{1},\mathbf{1},\mathbf{1})_{0,0}\\
&\qquad +(\mathbf{3},\mathbf{2},\mathbf{2})_{0,2}+(\bar{\mathbf{3}},\mathbf{2},\mathbf{2})_{0,-2}+(\mathbf{1},\mathbf{3},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{1},\mathbf{3})_{0,0}+(\mathbf{1},\mathbf{1},\mathbf{1})_{0,0}\\
&\qquad +[(\mathbf{3},\mathbf{2},\mathbf{1})_{-3,-1}+(\mathbf{1},\mathbf{2},\mathbf{1})_{-3,3}+(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{-3,1}+(\mathbf{1},\mathbf{1},\mathbf{2})_{-3,-3}+c.c.].
\end{aligned}
\tag{4.72}
$$

The SM hypercharge is defined as

$$
U(1)_Y=\frac{1}{2}[U(1)_a+\frac{1}{3}U(1)_b]. \tag{4.73}
$$

Under the breaking pattern (4.72), the gauge group E_6 can be broken down to $SU(3)\times SU(2)_1\times U(1)_a\times U(1)_b$ by turning on a gauge bundle on S with the structure group $SU(2)_2\times U(1)_a\times U(1)_b$. Let us define L_1 and L_2 to be the line bundles associated with $U(1)_a$ and $U(1)_b$, respectively. V_2 is defined as a vector bundle of rank two with the structure group $SU(2)$. To preserve supersymmtry, the connection of the gauge bundle $W=V_2\oplus L_1\oplus L_2$ has to satisfy the Hermitian Yang-Mills equations (2.11)[54] It was shown in [118, 119] that the bundle W has to be poly-stable with $\mu_{[\omega]}(V_2)=\mu_{[\omega]}(L_1)=\mu_{[\omega]}(L_2)=0$, where slope $\mu_{[\omega]}(E)$ of a bundle E on S is defined by $\mu_{[\omega]}(E)=\frac{1}{\mathrm{rank}(E)}c_1(E)\cdot_S[\omega]$ and $[\omega]$ is an ample divisor of S. The poly-

[54]More precisely, L_1 and L_2 are fractional line bundles [14–17].

stability also requires that V_2 is a $[\omega]$-stable bundle. Since S is a del Pezzo surface, it was shown in [14] that for any non-trivial holomorphic vector bundle E satisfies Eq. (2.11), $h^0(S,E)=h^2(S,E)=0$. This vanishing theorem dramatically simplifies the calculation of the chiral spectrum. It turns out that the matter spectrum can be calculated by the holomorphic Euler characteristic [109, 110]. By the decomposition Eq. (4.72) and the vanishing theorem, the spectrum is given by

$$n_{(\mathbf{3},\mathbf{1},\mathbf{1})_{0,-4}} = -\chi(S, G^{-1}) \equiv \gamma_1, \tag{4.74}$$

$$n_{(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{0,4}} = -\chi(S, G) \equiv \gamma_2, \tag{4.75}$$

$$n_{(\mathbf{3},\mathbf{2},\mathbf{2})_{0,2}} = -\chi(S, U_2) \equiv \gamma_3, \tag{4.76}$$

$$n_{(\bar{\mathbf{3}},\mathbf{2},\mathbf{2})_{0,-2}} = -\chi(S, U_2^\vee) \equiv \gamma_4, \tag{4.77}$$

$$n_{(\mathbf{3},\mathbf{2},\mathbf{1})_{-3,-1}} = -\chi(S, F) \equiv \gamma_5, \tag{4.78}$$

$$n_{(\bar{\mathbf{3}},\mathbf{2},\mathbf{1})_{3,1}} = -\chi(S, F^{-1}) \equiv \gamma_6, \tag{4.79}$$

$$n_{(\mathbf{3},\mathbf{1},\mathbf{2})_{3,-1}} = -\chi(S, U_2^\vee \otimes F^{-1}) \equiv \gamma_7, \tag{4.80}$$

$$n_{(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{-3,1}} = -\chi(S, U_2 \otimes F) \equiv \gamma_8, \tag{4.81}$$

$$n_{(\mathbf{1},\mathbf{1},\mathbf{2})_{-3,-3}} = -\chi(S, U_2^\vee \otimes F) \equiv \delta_1, \tag{4.82}$$

$$n_{(\mathbf{1},\mathbf{1},\mathbf{2})_{3,3}} = -\chi(S, U_2 \otimes F^{-1}) \equiv \delta_2, \tag{4.83}$$

$$n_{(\mathbf{1},\mathbf{2},\mathbf{1})_{-3,3}} = -\chi(S, G \otimes F) \equiv \delta_3, \tag{4.84}$$

$$n_{(\mathbf{1},\mathbf{2},\mathbf{1})_{3,-3}} = -\chi(S, G^{-1} \otimes F^{-1}) \equiv \delta_4, \tag{4.85}$$

where $\vee$ stands for the dual bundle, χ is the holomorphic Euler characteristic defined by $\chi(S,E)=\sum_i h^{0,i}(S,E)$, $U_2=V_2\otimes L_2^2$, $F=L_1^{-3}\otimes L_2^{-1}$, $G=L_2^4$, and $\gamma_i,\ \delta_i\in\mathbb{Z}_{\geqslant 0}$.

After some algebra, Eqs. (4.74)-(4.85) can be recast as

$$c_1(G)^2 = -2 - \gamma_1 - \gamma_2, \tag{4.86}$$

$$c_1(F)^2 = -2 - \gamma_5 - \gamma_6, \tag{4.87}$$

$$c_1(S) \cdot c_1(G) = \gamma_1 - \gamma_2, \tag{4.88}$$

$$c_1(S) \cdot c_1(F) = \gamma_6 - \gamma_5, \tag{4.89}$$

$$c_2(V_2) = \frac{1}{4}(6 - \gamma_1 - \gamma_2 + 2\gamma_3 + 2\gamma_4), \tag{4.90}$$

$$c_1(G) \cdot c_1(F) = \frac{1}{2}(4 + \gamma_3 + \gamma_4 + 2\gamma_5 + 2\gamma_6 - \gamma_7 - \gamma_8), \tag{4.91}$$

$$\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0, \tag{4.92}$$

$$\gamma_1 - \gamma_2 - 2\gamma_5 + 2\gamma_6 - \gamma_7 + \gamma_8 = 0, \tag{4.93}$$

$$\delta_1 = \frac{1}{2}(8 + \gamma_1 - \gamma_2 + 2\gamma_3 + 2\gamma_4 + 6\gamma_5 + 2\gamma_6 - \gamma_7 - \gamma_8), \tag{4.94}$$

$$\delta_2 = \frac{1}{2}(8 - \gamma_1 + \gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_6 - \gamma_7 - \gamma_8), \tag{4.95}$$

$$\delta_3 = -\frac{1}{2}(2 - 2\gamma_2 + \gamma_3 + \gamma_4 + 2\gamma_6 - \gamma_7 - \gamma_8), \tag{4.96}$$

$$\delta_4 = -\frac{1}{2}(2 - 2\gamma_1 + \gamma_3 + \gamma_4 + 2\gamma_5 - \gamma_7 - \gamma_8). \tag{4.97}$$

Note that given $\gamma_k,\ k = 1, 2, ..., 8$ satisfying the constraints Eqs. (4.92) and (4.93), (F, G, V_2) are constrained by Eqs. (4.86)-(4.91) and $(\delta_1, \delta_2, \delta_3, \delta_4)$ are then given by Eqs. (4.94)-(4.97). In particular, we are interested in the configurations of the vector-like pairs, namely $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \delta_1, \delta_2, \delta_3, \delta_4) = (a, a, b, b, c, c, d, d, e, e, f, f)$, where a, b, c, d, e are all non-negative integers. Then Eqs. (4.86)-(4.97) reduce to

$$\begin{cases} c_1(G)^2 = -2-2a \\ c_1(F)^2 = -2-2c \\ c_1(S)\cdot c_1(G) = 0 \\ c_1(S)\cdot c_1(F) = 0 \\ c_2(V_2) = \frac{1}{2}(3+2b-a) \\ c_1(G)\cdot c_1(F) = 2+b+2c-d \\ e = 4+2b+4c-d \\ f = -1+a-b-c+d. \end{cases} \tag{4.98}$$

It was proven in [129] that for an algebraic surface S with a given $n \geqslant 4([h^0(S,K_S)/2]+1)$, there exists a $[\omega_S]$-stable bundle V of rank two with $c_1(V)=0$ and $c_2(V)=n$. When S is a del Pezzo surface, $h^0(S,K_S)=0$ and this theorem implies that for any given number $m \geqslant 4$, there exists a $[\omega_S]$-stable bundle of rank two with $c_1(V)=0$ and $c_2(V)=m$. To apply this theorem to our case, we require that $c_2(V_2)\geqslant 4$. In general, $c_1(V)$ and $c_2(V)$ of a stable bundle V over a compact Kähler surface S with $c_1(S)>0$ satisfy the inequality $2rc_2(V)-(r-1)c_1(V)^2 \geqslant (r^2-1)$, where r is the rank of V [111]. When $r=2$ and $c_1(V)=0$, one can obtain the lower bound $c_2(V)\geqslant 2$. It is possible to obtain a $[\omega_S]$-stable bundle V of rank two with $c_1(V)=0$ and $c_2(V)\leqslant 4$ for S being a del Pezzo surface. One can start with V defined by the following extension:

$$0 \to L \to V \to M \to 0. \tag{4.99}$$

To obtain vanishing $c_1(V)$, one can set $M=L^{-1}$ and compute $c_2(V)=-c_1(L)^2$. The extension is classified by $\mathrm{Ext}^1(L,M)=H^1(S,L\otimes M^*)$. When $M=L^{-1}$, the obstruction of the non-trivial extension is $h^1(S,L^2)\neq 0$. Let L be a non-trivial line bundle and S be a del Pezzo surface. By the vanishing theorem, one can obtain

$$h^1(S,L^2) = -1 - c_1(S)\cdot c_1(L) - 2c_1(L)^2. \tag{4.100}$$

If $c_1(S)\cdot c_1(L) = 0$ with negative $c_1(L)^2$, it is easy to see that $h^1(S,L^2) \geqslant 1$. The simple example for such a line bundle is $L = \mathcal{O}_S(e_i - e_j)$, $i \neq j$, where $\{e_1, ..., e_8\}$ is a set of the exceptional divisors of S. With non-trivial extensions, one may construct a $[\omega_S]$-stable bundle V with $(r, c_1(V), c_2(V)) = (2,0,2)$ and with the structure group $SU(2)$. In what follows, we shall focus on the case of $c_2(V_2) \geqslant 4$. We summarize the constraints for (a,b,c,d) as follows:

$$\begin{cases} 2b + 4c - d \geqslant -4 \\ a - b - c + d \geqslant 1 \\ a - 2b \leqslant -5 \\ a,b,c,d \in \mathbb{Z}_{\geqslant 0}. \end{cases} \tag{4.101}$$

Note that a must be odd otherwise $c_2(V_2)$ cannot be integral. It follows from the condition $c_2(V_2) \geqslant 4$ that $b \geqslant 3$. Let us consider the case $(a,b,c) = (1,3,0)$. Then Eq. (4.98) becomes

$$\begin{cases} c_1(G)^2 = -4 \\ c_1(F)^2 = -2 \\ c_1(S)\cdot c_1(G) = 0 \\ c_1(S)\cdot c_1(F) = 0 \\ c_2(V_2) = 4 \\ c_1(G)\cdot c_1(F) = 5 - d \\ e = 10 - d \\ f = -3 + d. \end{cases} \tag{4.102}$$

Note that for the case $(a, b, c) = (1, 3, 0)$, the necessary condition for d is $3 \leqslant d \leqslant 10$. From the conditions $c_1(G)^2 = -4$ and $c_1(F)^2 = -2$, we set $G = \mathcal{O}_S(e_i - e_j + e_k - e_l),\ i \neq j \neq k \neq l$ and $F = \mathcal{O}_S(e_m - e_n),\ m \neq n$. Clearly, G and F also satisfy the conditions $c_1(S) \cdot c_1(G) = 0$ and $c_1(S) \cdot c_1(F) = 0$. We shall not attempt to explore all solutions (G, F) and only list some solutions as follows:

$$(G, F) = \begin{cases} (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_i - e_j)),\ (d, e, f) = (7, 3, 4) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_m - e_j)),\ (d, e, f) = (6, 4, 3) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_i - e_k)),\ (d, e, f) = (5, 5, 2) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_j - e_n)),\ (d, e, f) = (4, 6, 1) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_j - e_k)),\ (d, e, f) = (3, 7, 0). \end{cases} \tag{4.103}$$

Let us consider another example, $(a, b, c) = (3, 4, 0)$. In this case Eq. (4.98) reduces to

$$\begin{cases} c_1(G)^2 = -8 \\ c_1(F)^2 = -2 \\ c_1(S) \cdot c_1(G) = 0 \\ c_1(S) \cdot c_1(F) = 0 \\ c_2(V_2) = 4 \\ c_1(G) \cdot c_1(F) = 6 - d \\ e = 12 - d \\ f = -2 + d. \end{cases} \tag{4.104}$$

When $(a, b, c) = (3, 4, 0)$, it follows from Eq. (4.104) that the necessary condition for d is $2 \leqslant d \leqslant 12$. From the conditions $c_1(G)^2 = -8$ and $c_1(F)^2 = -2$, we set $G = \mathcal{O}_S(2e_i - 2e_j),\ i \neq j$ and $F = \mathcal{O}_S(e_m - e_n),\ m \neq n$. It is not difficult to see that G and F satisfy the conditions $c_1(S) \cdot c_1(G) = 0$ and $c_1(S) \cdot c_1(F) = 0$. Some solutions of (G, F) are as follows:

$$(G,F) = \begin{cases} (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_i - e_j)),\ (d,e,f) = (10,2,8) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_m - e_j)),\ (d,e,f) = (8,4,6) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_m - e_n)),\ (d,e,f) = (6,6,4) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_m - e_i)),\ (d,e,f) = (4,8,2) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_j - e_i)),\ (d,e,f) = (2,10,0). \end{cases} \tag{4.105}$$

Let us turn to the chiral spectrum on the matter curves. The breaking pattern of the presentation $\mathbf{27}$ is

$$\begin{aligned} E_6 &\rightarrow SU(3) \times SU(2)_1 \times [SU(2)_2 \times U(1)_a \times U(1)_b] \\ \mathbf{27} &\rightarrow (\mathbf{3},\mathbf{2},\mathbf{1})_{1,-1} + (\mathbf{1},\mathbf{2},\mathbf{1})_{1,3} + (\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{1,1} + (\mathbf{1},\mathbf{1},\mathbf{2})_{1,-3} \\ &\quad +(\mathbf{3},\mathbf{1},\mathbf{1})_{-2,2} + (\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{-2,-2} + (\mathbf{1},\mathbf{2},\mathbf{2})_{-2,0} + (\mathbf{1},\mathbf{1},\mathbf{1})_{4,0}. \end{aligned} \tag{4.106}$$

Let us define $V_{\mathbf{27}} \otimes L_1^4|_{\Sigma_{\mathbf{27}}^{(k)}} = \Gamma|_{\Sigma_{\mathbf{27}}^{(k)}} = M^{(k)}$, $F|_{\Sigma_{\mathbf{27}}^{(k)}} = N_1^{(k)}$, and $G|_{\Sigma_{\mathbf{27}}^{(k)}} = N_2^{(k)}$. The chirality of matter localized on matter curves $\Sigma_{\mathbf{27}}^{(k)}$ is determined by the restrictions of the cover flux Γ and gauge fluxes to the curves. The spectrum induced by the cover flux and gauge fluxes is summarized in Table L.

C. Tadpole Cancellation

The cancellation of tadpoles is crucial for consistent compactifications. In general, there are induced tadpoles from 7-brane, 5-brane, and 3-brane charges in F-theory. The 7-brane tadpole cancellation in F-theory is automatically satisfied since X_4 is a Calabi-Yau manifold. The cancellation of the $D5$-brane tadpole in the spectral cover construction follows from the topological condition that the overall first Chern class of the Higgs bundle vanishes. Therefore, the non-trivial tadpole cancellation in

Curve	Matter	Bundle	Chirality
$\mathbf{27}^{(k)}$	$(\mathbf{3},\mathbf{2},\mathbf{1})_{1,-1}$	$V_{\mathbf{27}}\otimes L_1\otimes L_2^{-1}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+N_1^{(k)}$
	$(\mathbf{1},\mathbf{2},\mathbf{1})_{1,3}$	$V_{\mathbf{27}}\otimes L_1\otimes L_2^{3}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+N_1^{(k)}+N_2^{(k)}$
	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{1,1}$	$V_{\mathbf{27}}\otimes V_2\otimes L_1\otimes L_2\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$2(M^{(k)}+N_1^{(k)})+N_2^{(k)}$
	$(\mathbf{1},\mathbf{1},\mathbf{2})_{1,-3}$	$V_{\mathbf{27}}\otimes V_2\otimes L_1\otimes L_2^{-3}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$2(M^{(k)}+N_1^{(k)})-N_2^{(k)}$
	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2,2}$	$V_{\mathbf{27}}\otimes L_1^{-2}\otimes L_2^{2}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+2N_1^{(k)}+N_2^{(k)}$
	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{-2,-2}$	$V_{\mathbf{27}}\otimes L_1^{-2}\otimes L_2^{-2}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+2N_1^{(k)}$
	$(\mathbf{1},\mathbf{2},\mathbf{2})_{-2,0}$	$V_{\mathbf{27}}\otimes V_2\otimes L_1^{-2}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$2(M^{(k)}+2N_1^{(k)})+N_2^{(k)}$
	$(\mathbf{1},\mathbf{1},\mathbf{1})_{4,0}$	$V_{\mathbf{27}}\otimes L_1^{4}\vert_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}$

Table L. Chirality of matter localized on matter curve $\mathbf{27}^{(k)}$.

F-theory needed to be satisfied is the $D3$-brane tadpole which can be calculated by the Euler characteristic $\chi(X_4)$. The cancellation condition is of the form [122]

$$N_{D3}=\frac{\chi(X_4)}{24}-\frac{1}{2}\int_{X_4}G\wedge G, \tag{4.107}$$

where N_{D3} is the number of $D3$-branes and G is the four-form flux on X_4. For a non-singular elliptically fibered Calabi-Yau fourfold X_4, it was shown in [122] that the Euler characteristic $\chi(X_4)$ can be expressed as

$$\chi(X_4)=12\int_{B_3}c_1(B_3)[c_2(B_3)+30c_1(B_3)^2], \tag{4.108}$$

where $c_k(B_3)$ are the Chern classes of B_3. It follows from Eq. (4.108) that $\chi(X_4)/24$ is at least half-integral[55]. When X_4 admits non-abelian singularities, the Euler characteristic of X_4 is replaced by a refined Euler characteristic, the Euler characteristic of the smooth fourfold obtained from a suitable resolution of X_4. On the other hand, G-flux encodes the two-form gauge fluxes on the 7-branes. It was shown in [123] that

$$\int_{X_4} G \wedge G = -\Gamma^2, \tag{4.109}$$

where Γ is the universal cover flux defined in section A and Γ^2 is defined as the self-intersection number of Γ inside the spectral cover. It is a challenge to find compactifications with non-vanishing G-flux and non-negative N_{D_3} to satisfy the tadpole cancellation condition (4.107). In the next two subsections, we shall derive the formulae of the refined Euler characteristic $\chi(X_4)$ and the self-intersection of the universal cover fluxes Γ^2 for the $(2,1)$ and $(1,1,1)$ factorizations.

1. Geometric Contribution

In the presence of non-abelian singularities, X_4 becomes singular and the Euler characteristic $\chi(X_4)$ needs to be modified by resolving the singularities. To be more concrete, let us define H to be the gauge group corresponding to the non-abelian singularity over S and G to be the complement of H in E_8. Then the Euler characteristic is modified to

$$\chi(X_4) = \chi^*(X_4) + \chi_G - \chi_{E_8}, \tag{4.110}$$

[55] For a generic Calabi-Yau manifold X_4, $\chi(X_4)/24$ takes value in $\mathbb{Z}_4$ [122].

where $\chi^*(X_4)$ is the Euler characteristic for a smooth fibration over B_3 given by Eq. (4.108) and the characteristic χ_{E_8} is given by [53, 123, 124]

$$\chi_{E_8} = 120 \int_S (3\eta^2 - 27\eta c_1 + 62c_1^2). \tag{4.111}$$

For the case of $G = SU(n)$, the characteristic $\chi_{SU(n)}$ is computed as

$$\chi_{SU(n)} = \int_S (n^3 - n)c_1^2 + 3n\eta(\eta - nc_1). \tag{4.112}$$

When the group G splits into a product of two groups G_1 and G_1, χ_G in Eq. (4.110) is then replaced by $\chi^{(k)}_{G_1} + \chi^{(l)}_{G_2}$, where η in χ_G is split into the classes $\eta^{(m)}$ as shown in the footnote below. It turns out that the refined Euler characteristic of the $(2,1)$ factorization is given by

$$\begin{aligned} \chi(X_4) &= \chi^*(X_4) + \chi^{(a)}_{SU(2)} + \chi^{(b)}_{SU(1)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 3[c_1(32c_1 - 16t - 15\xi) + (2t^2 + 4t\xi + 3\xi^2)] - \chi_{E_8}. \end{aligned} \tag{4.113}$$

In the $(1,1,1)$ factorization, the refined Euler characteristic[56] is

$$\begin{aligned} \chi(X_4) &= \chi^*(X_4) + \chi^{(l_1)}_{SU(1)} + \chi^{(l_2)}_{SU(1)} + \chi^{(l_3)}_{SU(1)} - \chi_{E_8} \\ &= \chi^*(X_4) + 3\int_S c_1\left[12c_1 - 7t - 6(\xi_1 + \xi_2)\right] \\ &+ 3\int_S \left[t^2 + 2t(\xi_1 + \xi_2) + 2(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)\right] - \chi_{E_8}. \end{aligned} \tag{4.114}$$

2. Cover Flux Contribution

Under cover factorizations, the universal cover flux is of the form

$$\Gamma = \sum_k \Gamma^{(k)}, \tag{4.115}$$

[56]For the $(2,1)$ factorization, $\eta^{(a)} = (\eta - c_1 - \xi)$ and $\eta^{(b)} = (c_1 + \xi)$. For the $(1,1,1)$ factorization, $\eta^{(l_1)} = (\eta - 2c_1 - \xi_1 - \xi_2)$, $\eta^{(l_2)} = (c_1 + \xi_1)$, and $\eta^{(l_3)} = (c_1 + \xi_2)$.

where the fluxes $\Gamma^{(k)}$ satisfy the traceless condition $\sum_k p_{k*}\Gamma^{(k)} = 0$. In what follows, we shall compute the self-intersection Γ^2 of the universal fluxes for the (2,1) and (1,1,1) factorizations.

a. $(2,1)$ Factorization

Let us recall that in the $(2,1)$ factorization, the universal cover flux is given by

$$\Gamma = k_a\gamma_0^{(a)} + k_b\gamma_0^{(b)} + m_a\delta^{(a)} + m_b\delta^{(b)} + \tilde{\rho} = \Gamma^{(a)} + \Gamma^{(b)}, \tag{4.116}$$

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(2k_a + m_a)\sigma - \pi^*(k_a[a_2] + m_b[d_1] + \rho)] \equiv [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}], \tag{4.117}$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 2m_b)\sigma - \pi^*(k_b[d_1] + m_a[a_2] - 2\rho)] \equiv [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \tag{4.118}$$

Then the self-intersection Γ^2 is calculated by [66]

$$\Gamma^2 = [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] + [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \tag{4.119}$$

In the $(2,1)$ factorization, $[\mathcal{C}^{(a)}] = 2\sigma + \pi^*(\eta - c_1 - \xi)$ and $[\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi)$. With Eqs. (4.117) and (4.118), it is straightforward to compute

$$\begin{aligned} \Gamma^2 &= [\mathcal{C}_2^{(a)}] \cdot [\tilde{\mathcal{C}}_2^{(a)}]^2 + [\mathcal{C}_1^{(b)}] \cdot [\tilde{\mathcal{C}}_1^{(b)}]^2 \\ &= -\frac{1}{2}(2k_a + m_a)^2[a_2] \cdot [a_0] - (k_b + 2m_b)^2[d_1] \cdot [d_0] \\ &+\frac{3}{2}(m_a[a_2] - 2m_b[d_1] - 2\rho)^2. \end{aligned} \tag{4.120}$$

b. (1, 1, 1) Factorization

In the (1, 1, 1) factorization, the universal flux is given by

$$\Gamma = k_{l_1}\gamma_0^{(l_1)} + k_{l_2}\gamma_0^{(l_2)} + k_{l_3}\gamma_0^{(l_3)} + m_{l_1}\delta^{(l_1)} + m_{l_2}\delta^{(l_2)} + m_{l_3}\delta^{(l_3)} + \tilde{\rho} \equiv \Gamma^{(l_1)} + \Gamma^{(l_2)} + \Gamma^{(l_3)}, \tag{4.121}$$

where $\Gamma^{(l_1)}$, $\Gamma^{(l_2)}$, and $\Gamma^{(l_3)}$ are

$$\Gamma^{(l_1)} = [\mathcal{C}^{(l_1)}] \cdot [(k_{l_1} + 2m_{l_1})\sigma - \pi^*(k_{l_1}[f_1] + m_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_1 - \rho_3)], \tag{4.122}$$

$$\Gamma^{(l_2)} = [\mathcal{C}^{(l_2)}] \cdot [(k_{l_2} + 2m_{l_2})\sigma - \pi^*(m_{l_1}[f_1] + k_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_2 - \rho_1)], \tag{4.123}$$

$$\Gamma^{(l_3)} = [\mathcal{C}^{(l_3)}] \cdot [(k_{l_3} + 2m_{l_3})\sigma - \pi^*(m_{l_1}[f_1] + m_{l_2}\xi_1 + k_{l_3}\xi_2 + \rho_3 - \rho_2)]. \tag{4.124}$$

In this case the self-intersection Γ^2 is computed as

$$\Gamma^2 = [\mathcal{C}^{(l_1)}] \cdot [\widetilde{\mathcal{C}}^{(l_1)}] \cdot [\widetilde{\mathcal{C}}^{(l_1)}] + [\mathcal{C}^{(l_2)}] \cdot [\widetilde{\mathcal{C}}^{(l_2)}] \cdot [\widetilde{\mathcal{C}}^{(l_2)}] + [\mathcal{C}^{(l_2)}] \cdot [\widetilde{\mathcal{C}}^{(l_3)}] \cdot [\widetilde{\mathcal{C}}^{(l_3)}]. \tag{4.125}$$

Recall that $[\mathcal{C}^{(l_1)}] = \sigma + \pi^*(\eta - 2c_1 - \xi_1 - \xi_2)$, $[\mathcal{C}^{(l_2)}] = \sigma + \pi^*(c_1 + \xi_1)$, and $[\mathcal{C}^{(l_3)}] = \sigma + \pi^*(c_1 + \xi_2)$. It follows from Eqs. (4.122)-(4.124) that

$$\begin{aligned}
\Gamma^2 &= [\mathcal{C}^{(l_1)}] \cdot [\tilde{\mathcal{C}}^{(l_2)}]^2 + [\mathcal{C}^{(l_2)}] \cdot [\tilde{\mathcal{C}}^{(l_2)}]^2 + [\mathcal{C}^{(l_3)}] \cdot [\tilde{\mathcal{C}}^{(l_3)}]^2 \\
&= -(k_{l_1} + 2m_{l_1})^2[f_1] \cdot [f_0] - (k_{l_2} + 2m_{l_2})^2[g_1] \cdot [g_0] - (k_{l_3} + 2m_{l_3})^2[h_1] \cdot [h_0] \\
&\quad +(\rho_1 - \rho_3 - 2m_{l_1}[f_1] + m_{l_2}[g_1] + m_{l_3}[h_1])^2 \\
&\quad +(\rho_2 - \rho_1 + m_{l_1}[f_1] - 2m_{l_2}[g_1] + m_{l_3}[h_1])^2 \\
&\quad +(\rho_3 - \rho_2 + m_{l_1}[f_1] + m_{l_2}[g_1] - 2m_{l_3}[h_1])^2.
\end{aligned} \tag{4.126}$$

D. Models

In this section we give some numerical examples in the geometric backgrounds dP_2 studied in [64] and dP_7 in [53]. The basic geometric data of dP_2 in X_4 is

$$c_1 = 3h - e_1 - e_2, \quad t = h, \quad \eta = 17h - 6e_1 - 6e_2. \tag{4.127}$$

It follows from Eqs. (4.113) and (4.114) that the refined Euler characteristic $\chi(X_4)$ for the $(2,1)$ and $(1,1,1)$ factorizations are

$$\chi(X_4)_{(2,1)} = 10662 + \int_S 3[-15\xi c_1 + 4t\xi + 3\xi^2], \tag{4.128}$$

$$\chi(X_4)_{(1,1,1)} = 10320 + \int_S 6\left[(t - 3c_1)(\xi_1 + \xi_2) + (\xi_1^2 + \xi_1\xi_2 + \xi_2^2)\right], \tag{4.129}$$

where $\chi^*(X_4) = 13968$ has been used. The ample divisor $[\omega_{dP_2}]$ is chosen to be

$$[\omega_{dP_2}] = \alpha(e_1 + e_2) + \beta(h - e_1 - e_2), \quad 2\alpha > \beta > \alpha > 0. \tag{4.130}$$

For the dP_7 studied in [53], the basic geometric data is

$$\begin{aligned} c_1 &= 3h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7, \\ t &= 2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6, \\ \eta &= 16h - 5e_1 - 5e_2 - 5e_3 - 5e_4 - 5e_5 - 5e_6 - 6e_7. \end{aligned} \tag{4.131}$$

with $\chi^*(X_4) = 1728$. By Eqs. (4.113) and (4.114), the refined Euler characteristic $\chi(X_4)$ for the $(2,1)$ and $(1,1,1)$ factorizations are

$$\chi(X_4)_{(2,1)} = 708 + \int_S 3[-15\xi c_1 + 4t\xi + 3\xi^2], \tag{4.132}$$

$$\chi(X_4)_{(1,1,1)} = 594 + \int_S 6\left[(t - 3c_1)(\xi_1 + \xi_2) + (\xi_1^2 + \xi_1\xi_2 + \xi_2^2)\right]. \tag{4.133}$$

In this case we choose the ample divisor $[\omega_{dP_7}]$ to be

$$[\omega_{dP_7}] = 14\beta h - (5\beta - \alpha)\sum_{i=1}^{7} e_i, \quad 5\beta > \alpha > 0. \tag{4.134}$$

We shall discuss the models of the (2,1) and (1,1,1) factorizations. In each case the trivial and non-trivial restrictions of the $U(1)$ fluxes to the matter curves will be discussed. Non-trivial restriction leads to the modification of the chirality of each matter on the curve after E_6 is broken according to the calculation in section B. In addition, there could exist vector-like pairs on each curve since we only know the net chirality. The Higgs vector-like pair $(\mathbf{27} + \overline{\mathbf{27}})$ needed for the gauge unification is therefore assigned to one of these pairs, though the machinery to calculate the exact number of these vector-like fields is not clear yet.

1. Examples of the $(2,1)$ Factorization

In the (2,1) factorization the matter fields are assigned to $\mathbf{27}^{(a)}$ curve and the Higgs fields come from the other $\mathbf{27}^{(b)}$ curve. The Yukawa coupling then turns out to be

$$\mathcal{W} \supset \mathbf{27}^{(a)} \cdot \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)}. \tag{4.135}$$

Since the fermion and Higgs fields are not on the same $\mathbf{27}$ curve, the exotic fields in $\mathbf{27}^{(a)}$ can be taken as exotic quarks and leptons which are able to mix with the ordinary ones by suitable discrete symmetries and to decay via mechanisms such as FCNC after E_6 is broken mentioned in section B.

a. A three-family E_6 model in dP_2

The parameters of the model are listed in Table LI. These parameters give the spectrum $N_{\mathbf{27}^{(a)}} = 3$ and $N_{\mathbf{27}^{(b)}} = 3$ with $N_{D3} = 415$ as shown in Table LII. The dP_2 surface is probably too limited for the fluxes to break the E_6 gauge group. Therefore,

k_a	k_b	m_a	m_b	ρ	ξ	α	β
0.5	-1.5	-1	-1	$-\frac{5}{2}h+\frac{3}{2}e_1-\frac{3}{2}e_2$	e_1	2	3

Table LI. Parameters of an example of a three-generation E_6 GUT.

Curve	Class	Gen.
$\mathbf{27}^{(a)}$	$8h-4e_1-3e_2$	3
$\mathbf{27}^{(b)}$	e_1	3

Table LII. The **27** curves of the three-generation E_6 example in dP_2.

we stop at a three-generation E_6 GUT model in this example.

b. An example of three-generation without flux restriction in dP_7

The parameters of the model with $N_{D3}=12$ are listed in the Table LIII. The matter contents on the curves are listed in Table LIV. If the line bundles G and F associated to $SU(2)\times U(1)_a\times U(1)_b$ flux are chosen to have trivial restrictions[57] to both matter **27** curves, for example, $F=\mathcal{O}_S(e_5-e_6)$ and $G=\mathcal{O}_S(e_1-e_2+e_3-e_4)$ [58], then the chirality on each matter curve remains the same after E_6 is broken down to $SU(3)\times SU(2)\times U(1)_a\times U(1)_b$. After suitably transforming the $U(1)$ gauge groups, the corresponding matter content and phenomenology at low energy is a conventional rank 5 model discussed in section B.

[57]To avoid receiving a Green-Schwarz mass, it is required that $[H]\cdot_S c_1=0$ and $[H]\cdot_S \eta=0$, for $H=F,\ G$ [14–17, 65].

[58]G can be chosen also as $G=\mathcal{O}_S(2(e_3-e_4))$ from Eq. (4.105).

k_a	k_b	m_a	m_b	ρ	ξ	α	β
-0.5	1.5	0	-0.5	$\frac{1}{2}(3e_1+e_2+e_3+e_4)$	$h-e_5-e_6+e_7$	3	1

Table LIII. Parameters of an example of the (2,1) factorization in dP_7.

Curve	Class	Gen.
$\mathbf{27}^{(a)}$	$6h-2e_1-2e_2-2e_3-2e_4-e_5-e_6-4e_7$	3
$\mathbf{27}^{(b)}$	$h-e_5-e_6+e_7$	2

Table LIV. The **27** curves of the example of the (2,1) factorization without flux restrictions in dP_7.

c. An example with non-trivial flux restrictions in dP_7

In this example we consider a model with non-trivial flux restrictions to the matter curves in dP_7. From the chirality formulae discussed in section B and listed in Table L, we find that it is unavoidable to have exotic fields under this construction. To maintain at least three copies for the MSSM matter after the gauge group E_6 is broken, we may have to start from a model with more chirality on the **27** curves. The parameters of an example of this scenario are listed in Table LV.

It follows from Eq. (4.107) and the parameters in Table LV that $N_{D3}=14$. We choose chirality-three curve for the matter fields and a chirality-four curve for the Higgs fields to make sure that there are enough MSSM matter after the gauge group E_6 is broken. From Eq. (4.103), we can turn on the fluxes $F=\mathcal{O}_S(e_1-e_2)$ and $G=\mathcal{O}_S(e_2-e_3+e_4-e_5)$ in dP_7 [59]. The detailed information of the curves and the

[59] G can be chosen also as $G=\mathcal{O}_S(2(e_4-e_5))$ from Eq. (4.105).

k_a	k_b	m_a	m_b	ρ	ξ	α	β
0.5	-0.5	-1	-0.5	$-h+\frac{1}{2}(e_1-2e_2+e_3+e_4+e_6)$	$h-e_2+e_5-e_7$	13	11

Table LV. Parameters of an example with non-trivial flux restrictions in dP_7.

Curve	Class	M	N_1	N_2
$\mathbf{27}^{(a)}$	$6h-2e_1-e_2-2e_3-2e_4-3e_5-2e_6-2e_7$	3	1	-2
$\mathbf{27}^{(b)}$	$h-e_2+e_5-e_7$	4	-1	2

Table LVI. The **27** curves with non-trivial flux restrictions in dP_7.

restrictions of fluxes to each curve are listed in Table LVI.

The low energy spectrum is listed in Table LVII. One can see that there are exotic fields from non-trivial restrictions of fluxes to the curves.

2. Examples of the $(1,1,1)$ Factorization

The Yukawa coupling of the **27** curves in the $(1,1,1)$ factorization is $\mathbf{27}^{(l_1)}\mathbf{27}^{(l_2)}\mathbf{27}^{(l_3)}$. The fermions are assigned to the two **27** curves while the Higgs fields are located on the third **27** curve. For instance,

$$\mathcal{W} \supset \mathbf{27}_M^{(l_1)} \cdot \mathbf{27}_M^{(l_2)} \cdot \mathbf{27}_H^{(l_3)}. \tag{4.136}$$

In this scenario the fermions are separated on different matter curves and the sum of the generations should accomplish a three-family model, for example, two families on $\mathbf{27}^{(l_1)}$ and one family on $\mathbf{27}^{(l_2)}$, or vice versa. However, this construction generally has some problems with the mass matrices. With the assistance from the flux restrictions, the method studied in [27] can be applied to obtain a more reasonable Yukawa

Rep.	Gen. on $\mathbf{27}^{(a)}$	Gen. on $\mathbf{27}^{(b)}$
$(\mathbf{3},\mathbf{2})_{1,-1}$	$3\times Q+1\times(\mathbf{3},\mathbf{2})_{1,-1}$	3
$(\bar{\mathbf{3}},\mathbf{1})_{-2,-2}$	$3\times u^c+2\times(\bar{\mathbf{3}},\mathbf{1})_{-2,-2}$	2
$(\bar{\mathbf{3}},\mathbf{1})_{1,1}$	$3\times d^c+3\times D$	4+4
$(\mathbf{1},\mathbf{2})_{-2,0}$	$3\times L+5\times h$	$3\times(H_1+H_2)$
$(\mathbf{1},\mathbf{1})_{4,0}$	$3\times e^c$	4
$(\mathbf{1},\mathbf{1})_{1,-3}$	$3\times\nu^c+7\times S$	$2\times(H_3+H_4)$
$(\mathbf{3},\mathbf{1})_{-2,2}$	$3\times\bar{D}$	4
$(\mathbf{1},\mathbf{2})_{1,3}$	$2\times\bar{h}$	$5\times\bar{H}_2$

Table LVII. The MSSM spectrum of the $(2,1)$ factorization in dP_7.

k_{l_1}	k_{l_2}	k_{l_3}	m_{l_1}	m_{l_2}	m_{l_3}	ρ_1	ξ_1	ξ_2	α	β
-1.5	-0.5	1.5	0	0	0	$-h+e_1+2e_2$	e_1	$2h-2e_1-e_2+e_3-e_7$	1	3

Table LVIII. Parameters of a three family model in dP_7 with $\rho_2=2\rho_1$ and $\rho_3=0$.

structure. However, from the chirality given in Table L we expect exotic fields to remain in the spectrum after this mechanism. In what follows, we demonstrate one example for each case in the $(1,1,1)$ factorization.

a. An example of three-generation without flux restriction in dP_7

The parameters of the model are listed in Table LVIII. These parameters give the spectrum shown in Table LIX with $N_{D3}=10$. Let us choose the line bundles to

Curve	Class	Gen.	Matter
$\mathbf{27}^{(l_1)}$	$5h - e_1 - e_2 - 3e_3 - 2e_4 - 2e_5 - 2e_6 - 2e_7$	2	Fermion
$\mathbf{27}^{(l_2)}$	e_1	1	Fermion
$\mathbf{27}^{(l_3)}$	$2h - 2e_1 - e_2 + e_3 - e_7$	4	Higgs

Table LIX. The spectrum of the three-generation model in dP_7.

k_{l_1}	k_{l_2}	k_{l_3}	m_{l_1}	m_{l_2}	m_{l_3}	ρ_1	ξ_1	ξ_2	α	β
-0.5	-0.5	-0.5	0	0	-1	e_2	$2h - 2e_1 - e_3 - e_7$	$h - e_1 - e_2$	1	3

Table LX. Parameters of a three family model in dP_7 with $\rho_2 = 2\rho_1$ and $\rho_3 = 0$.

be $F = \mathcal{O}_S(e_5 - e_6)$ and $G = \mathcal{O}_S(e_2 - e_4 + e_3 - e_6)$ [60] having trivial restrictions to each **27** curve. Then the chirality remains the same after E_6 is broken down to $SU(3)\times SU(2)\times U(1)_a\times U(1)_b$. After suitably transforming the $U(1)$ charges, the corresponding matter content and phenomenology at low energy is again a conventional rank 5 model.

b. An Example of non-trivial flux restrictions in dP_7

The parameters of the model are listed in Table LX. These parameters confine the spectrum of E_6 shown in Table LXI with $N_{D3} = 10$. If the line bundles associated to $SU(2)\times U(1)_a\times U(1)_b$ flux are chosen as $F = \mathcal{O}_S(e_3 - e_5)$ and $G = \mathcal{O}_S(e_1 - e_2 + e_4 - e_6)$ [61], then the chirality of MSSM matter after E_6 is broken will be modified by numbers N_1 and N_2 shown in Table LXI.

[60] G can be chosen also as $G = \mathcal{O}_S(2(e_4 - e_5))$ from Eq. (4.105).
[61] G can be chosen also as $G = \mathcal{O}_S(2(e_3 - e_4))$ from Eq. (4.105).

Curve	Class	M	N_1	N_2	Matter
$\mathbf{27}^{(l_1)}$	$4h + e_1 - e_2 - e_3 - 2e_4 - 2e_5 - 2e_6 - 2e_7$	3	-1	-2	Fermion
$\mathbf{27}^{(l^2)}$	$2h - 2e_1 - e_3 - e_7$	0	1	2	Fermion
$\mathbf{27}^{(l_3)}$	$h - e_1 - e_2$	4	0	0	Higgs

Table LXI. The spectrum of the three-generation model in dP_7.

Originally, there is no chirality on curve $\mathbf{27}^{(l_2)}$ so it does not look realistic before the E_6 gauge group is broken. However after the fluxes are turned on, the chirality is "reshuffled" and shared between curves $\mathbf{27}^{(l_1)}$ and $\mathbf{27}^{(l_2)}$. Therefore, we can interpret the model in the way studied in [27] that is able to give a rich structure to the mass matrices via the Yukawa couplings. We demonstrate the corresponding MSSM spectrum in Table LXII.

E. Conclusion

In this chapter we discussed E_6 GUT models where the gauge group is broken by the non-abelian flux $SU(2) \times U(1)^2$ in F-theory. The non-abelian part $SU(2)$ of the flux is not commutative with E_6 so the gauge group after breaking is $SU(3) \times SU(2)_L \times U(1)_a \times U(1)_b$ which is equivalent to a rank-5 model with $SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_\eta$. We start building models from the $SU(3)$ spectral cover and then factorize it into $(2,1)$ and $(1,1,1)$ structures to obtain enough curves and degrees of freedom to construct models with MSSM matter content. The restrictions of the line bundles associated with two $U(1)$ gauge groups to matter curves can modify the chirality of matter localized on matter curves. This modification generally results in plenty of exotic fields that may cause troubles in the phenomenological interpretation of the

Rep.	Gen. on $\mathbf{27}^{(l_1)}$	Gen. on $\mathbf{27}^{(l_2)}$	Gen. on $\mathbf{27}^{(l_3)}$
$(\mathbf{3},\mathbf{2})_{1,-1}$	$2\times Q$	$1\times Q$	4
$(\bar{\mathbf{3}},\mathbf{1})_{-2,-2}$	$1\times u^c$	$2\times u^c$	4
$(\bar{\mathbf{3}},\mathbf{1})_{1,1}$	$1\times d^c+1\times D$	$2\times d^c+2\times D$	8
$(\mathbf{1},\mathbf{2})_{-2,0}$	0	$3\times L+3\times h$	$4\times(H_1+H_2)$
$(\mathbf{1},\mathbf{1})_{4,0}$	$3\times e^c$	0	4
$(\mathbf{1},\mathbf{1})_{1,-3}$	$3\times\nu^c+3\times S$	0	$4\times(H_3+H_4)$
$(\mathbf{3},\mathbf{1})_{-2,2}$	$1\times(\bar{\mathbf{3}},\mathbf{1})_{2,-2}$	$3\times\bar{D}+1\times(\mathbf{3},\mathbf{1})_{-2,2}$	4
$(\mathbf{1},\mathbf{2})_{1,3}$	0	$3\times\bar{h}$	$4\times\bar{H}_2$

Table LXII. The MSSM matter shared by two curves in dP_7.

models.

One way to arrange the matter content in the conventional E_6 GUT model building is that all the MSSM matter and Higgs fields are included in the same $\mathbf{27}$-plet with three copies and the Yukawa coupling is $\mathbf{27}\cdot\mathbf{27}\cdot\mathbf{27}$. Such kind of interaction implies a structure of either one curve intersecting itself twice or three curves intersecting, which causes difficulties in geometry or the mass hierarchy structure in F-theory model building. Therefore, we adopt an alternate way that the weak scale Higgs particles are assigned to another $\mathbf{27}$ curve while the representations of their original assignments in the matter $\mathbf{27}$ curve are taken as exotic leptons. By using additional symmetries such as baryon and lepton numbers, we can rule out the undesired interactions coupled to the exotic fields. The $(2,1)$ factorization providing two curves $\mathbf{27}^{(a)}$ and $\mathbf{27}^{(b)}$ with the interaction $\mathbf{27}^{(a)}\cdot\mathbf{27}^{(a)}\cdot\mathbf{27}^{(b)}$ satisfies the basic requirements of this picture. One the other hand, the $(1,1,1)$ factorization confines three curves to the interaction $\mathbf{27}^{(l_1)} \cdot \mathbf{27}^{(l_2)} \cdot \mathbf{27}^{(l_3)}$. In this case we have to distribute the MSSM matter to both $\mathbf{27}^{(l_1)}$ and $\mathbf{27}^{(l_2)}$ curves while the electroweak Higgs fields are assigned on the third curve. The fermion mass matrices are generally not able to admit the hierarchical structures except they are tuned by appropriate flux restrictions. As mentioned before, the additional one or more $(\mathbf{27}+\overline{\mathbf{27}})$ pairs can be included to make sure that the gauge unification occurs. These vector-like pairs generically exist on the curves in F-theory and can be assigned to the same curve containing the electroweak Higgs fields. However, the exact number of the vector-like pairs on a matter curve is still unclear in the present construction, so we assume that there exits at least one pair.

We demonstrated several models both in the $(2,1)$ and $(1,1,1)$ factorizations with geometric backgrounds dP_2 and dP_7 studied in [64] and [53], respectively. We also discuss the cases that the restrictions of the line bundles associated with $U(1)$s to the curves are trivial or non-trivial. Due to the chirality constraints to the fields on the bulk, it is hard to construct consistent $U(1)$ fluxes in dP_2. Therefore, we only demonstrated a three-family E_6 GUT model without gauge breaking in the dP_2 geometry. On the other hand, the dP_7 geometry has more degrees of freedom for the parameters to build realistic models. We therefore showed in the $(2,1)$ case an example of a three-generation model without $U(1)$ flux restrictions, and an example with non-trivial $U(1)$ flux restrictions which gives rise to exotic particles. In the $(1,1,1)$ factorization, we also presented an example of three-family model without flux restriction. In that case there are two flavors on one matter curve and the third flavor on the other. In the model with non-trivial flux restrictions, we adjusted the parameters so that the total chirality of each representation on the two matter curves remain three while the hierarchies of the mass matrices can be maintained.

Regardless of the exotic fields, the matter contents of our examples are conventional and the corresponding phenomenology has been discussed in the literature.

CHAPTER V

SUMMARY

The Grand Unified Theory is a natural framework to unified gauge symmetries in particle physics. F-theory is a twelve-dimensional geometric version of string theory which is so far the most promising candidate for a fundamental unified theory. How to realize GUTs in F-theory is an natural and important question one might ask. In this dissertation, we studied supersymmetric F-theory GUT models. In particular, we focused on local and semi-local model GUT building. In chapter II, to obtain non-minimal local $SU(5)$ GUTs with doublet-triplet splitting, we considered supersymmetric $U(1)^2$ gauge fluxes associated with polystable bundles of rank two over a del Pezzo surface. We explicitly solved all $U(1)^2$ flux configurations for the requirements of an exotic-free bulk spectrum and supersymmetry. We also constructed examples of a non-minimal spectrum of the MSSM with doublet-triplet splitting. We then considered semi-local GUT models in F-theory. In chapter III we constructed semi-local flipped $SU(5)$ models in F-theory by using the spectral cover construction. We started with an E_8 singularity and unfolded it into a D_5 singularity controlled by an $SU(4)$ spectral cover. We calculated the spectra induced by cover fluxes and by $U(1)_X$ gauge fluxes. We constructed three-generation models satisfying the tadpole cancellation condition and discussed their phenomenology. In addition to flipped $SU(5)$ models, we also studied semi-local E_6 GUTs in chapter IV. We started with an E_8 singularity and unfolded it into a singularity of type E_6. This unfolding can be described by an $SU(3)$ spectral cover. We broke the gauge group E_6 down to $SU(3) \times SU(2) \times U(1)_Y \times U(1)_\eta$ by turning on non-abelian fluxes and found three-generation models satisfying all constraints including the tadpole cancellation condi-

tion. It is becoming clear that F-theory open a new window for model building and provides a powerful framework to study four-dimensional particle phenomenology.

There remain interesting directions for future research. Here are two: First, we could construct GUT models by using spectral covers associated with non-diagonalizable Higgs fields. The heart of the spectral cover construction are Hitchin's equations governing the dynamics of the resulting four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory. The spectral covers used to construct semi-local GUT models are only a special class of spectral covers. In this special class, the adjoint Higgs field parameterizing the normal motion of a seven-brane stack in the ambient space is diagonal and the spectral cover is then given by the characteristic polynomial of the Higgs field. In this case the low-energy spectrum is decoupled to the Higgs field, and the eigenvalues of the Higgs field characterize the locations of intersecting seven-branes. It would be interesting to construct GUT models by using spectral covers associated with non-diagonalizable Higgs fields, for example, nilpotent Higgs fields [130, 131]. Second, model building in M-theory has been studied for a long time, but computation of the matter spectrum has been deemphasized due to the difficulty of carrying out the procedure in conventional approaches. One can obtain four-dimensional $\mathcal{N} = 1$ theory by compactifying M-theory on a seven-dimensional G_2 manifold. To obtain interesting physics in four dimensions, this G_2 manifold is required to admit particular kinds of singularities where matter fields are localized [132][62]. Similar to F-theory model building, one may use a bottom-up approach to construct GUT models in M-theory. It was conjectured [134] that locally a G_2 manifold can be described by an ALE fibration over a three-dimensional manifold Q_3. With the bottom-up approach, one can only focus on a seven-dimensional super-Yang-Mills theory on $\mathbb{R}^{3,1} \times Q_3$.

[62]For a nice review, see [133].

The ALE fibration admits the singularities of A_{n-1}, D_k, and E_m types corresponding to the gauge symmetries of $SU(n)$, $SO(2k)$, and E_m types, respectively. In this framework, chiral matter is localized at critical points of the Morse functions of Q_3, and the four-dimensional physics is governed by Hitchin's equations arising from the compactification of the seven-dimensional super-Yang-Mills theory on Q_3 [134]. It would be interesting to study semi-local GUTs in M-theory by using the spectral cover construction or Hitchin's equations.

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VITA

Yu-Chieh Chung entered the National Tsing Hua University in September 1994, received his Bachelor of Science in chemistry in June 1998, and received his Master of Science in chemistry in July 2000. In August 2005, he entered Texas A&M University and completed the requirements for the Doctor of Philosophy degree in physics in May 2011. His permanent address is Department of Physics & Astronomy, Texas A&M University 4242 TAMU, College Station, TX 77843-4242, USA.