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Stability of the Enhanced Area Law of the Entanglement Entropy

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Abstract. We consider a multi-dimensional continuum Schrödinger operator which is given by a perturbation of the negative Laplacian by a compactly supported potential. We establish both an upper bound and a lower bound on the bipartite entanglement entropy of the ground state of the corresponding quasi-free Fermi gas. The bounds prove that the scaling behaviour of the entanglement entropy remains a logarithmically enhanced area law as in the unperturbed case of the free Fermi gas. The central idea for the upper bound is to use a limiting absorption principle for such kinds of Schrödinger operators.

1. Introduction and Result

Entanglement properties of the ground state of quasi-free Fermi gases have received considerable attention over the last two decades, see, for example, [1, 2, 8–10, 15, 16, 18, 20–22, 24, 25, 27, 37]. Here, entanglement is understood with respect to a spatial bipartition of the system into a subsystem of linear size proportional to L and the complement. Entanglement entropies are a common measure for entanglement. Often, the von Neumann entropy of the reduced ground state of the Fermi gas is considered. Its investigations give rise to non-trivial mathematical questions and to answers that are of physical relevance. This is true even for the simplest case of a quasi-free Fermi gas, namely the free Fermi gas with (single-particle) Hamiltonian $H_0 := -\Delta$ given by the Laplacian in $d \in \mathbb{N}$ space dimensions. Its entanglement entropy was suggested [14–16, 37] to obey a *logarithmically enhanced area law*,

$$S_E(H_0, \Lambda_L) = \Sigma_0\, L^{d-1} \ln L + o\big(L^{d-1} \ln L\big), \tag{1.1}$$

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as $L \to \infty$. Here, $E > 0$ stands for the Fermi energy, which characterises the ground state, and $\Lambda_L := L \cdot \Lambda$ is the scaled version of some "nice" bounded subset $\Lambda \subset \mathbb{R}^d$, which is specified in Assumption 1.1(i). The leading-order coefficient

$$\Sigma_0 \equiv \Sigma_0(\Lambda, E) := \frac{E^{(d-1)/2}\,|\partial\Lambda|}{3 \cdot 2^d \pi^{(d-1)/2}\Gamma\big((d+1)/2\big)}, \tag{1.2}$$

where $|\partial\Lambda|$ denotes the surface area of the boundary $\partial\Lambda$ of Λ, was expected [14–16] to be determined by Widom's conjecture [35]. This was finally proved in [20] based on celebrated works by Sobolev [32,33]. The occurrence of the logarithm $\ln L$ in the leading term of (1.1) is attributed to the delocalisation or transport properties of the Laplacian dynamics. It leads to long-range correlations in the ground state of the Fermi gas across the surface of the subsystem in Λ_L. If a periodic potential is added to H_0, and the Fermi energy falls into a spectral band, the logarithmically enhanced area law (1.1) is still valid, as was proven in [27] for $d = 1$.

If H_0 is replaced by another Schrödinger operator H with a mobility gap in the spectrum and if the Fermi energy falls into the mobility gap, then the $\ln L$-factor is expected to be absent in the leading asymptotic term of the entanglement entropy. Such a phenomenon is referred to as an *area law*, namely $S_E(H, \Lambda_L) \sim L^{d-1}$ as $L \to \infty$. It was first observed by Bekenstein [6,7] in a toy model for the Hawking entropy of black holes. An area law also holds if H models a particle in a constant magnetic field [9,22]. Area laws are proven to occur for random Schrödinger operators and Fermi energies in the region of dynamical localisation [10,25,26]. The proofs rely on the exponential decay in space of the Fermi projection for E in the region of complete localisation. It should be pointed out that spectral localisation alone is not sufficient for the validity of an area law. This has been recently demonstrated [24] for the random dimer model if the Fermi energy coincides with one of the critical energies where the localisation length diverges and dynamical delocalisation takes over.

Due to the complexity of the problem, there does not exist a mathematical approach which allows to determine the leading behaviour of the entanglement entropy for general Schrödinger operators H. All that is known is what happens for the examples discussed above. The experts in the field have conjectured for a decade that given H with a "reasonable" potential, a possibly occurring enhancement to the area law for $S_E(H, \Lambda_L)$ should not be stronger than logarithmic. Even though no counterexamples are known so far, proving the conjecture turned out to be a very difficult task which has not been solved yet. As an aside, we mention that for interacting quantum systems, stronger enhancements to area laws than logarithmic are known in peculiar cases. In fact, spin chains ($d = 1$) can be designed in such a way as to realise any growth rate up to L [23,28].

In this paper, we undertake a first step towards a proof of the conjecture. We establish an upper bound on the entanglement entropy corresponding to $H = -\Delta + V$ which grows like $L^{d-1} \ln L$ as $L \to \infty$, provided the potential V is bounded and has compact support. Compactness of the support is the crucial restriction of our result. It could be relaxed to having a sufficiently fast decay at infinity, but we have chosen not to focus on this for reasons of simplicity. The main technical input in our analysis is a limiting absorption principle for H. Since H has absolutely continuous spectrum filling the nonnegative real half-line, one expects $S_E(H,\Lambda_L)$ to obey an enhanced area law for Fermi energies $E>0$. Therefore, a corresponding lower bound, which grows also like $L^{d-1}\ln L$ as $L\to\infty$, is of interest, too. These findings are summarised in Theorem 1.3, which is our main result. The proof of the upper bound is much more involved than that of the lower bound. Both bounds require the representation of the Fermi projection as a Riesz projection with the integration contour cutting through the continuous spectrum. Such a representation may be of independent interest. We prove it in Appendix A in a more general setting for operators for which a limiting absorption principle holds.

Let $H:=-\Delta+V$ be a densely defined Schrödinger operator in the Hilbert space $L^2(\mathbb{R}^d)$ with bounded potential $V\in L^\infty(\mathbb{R}^d)$. Although the entanglement entropy is a many-body quantity, in case of a quasi-free Fermi gas, it can be solely expressed in terms of single-particle quantities [19]. We take this result as our definition for the entanglement entropy

$$S_E(H,\Omega):=\operatorname{tr}\left\{h(1_\Omega 1_{<E}(H)1_\Omega)\right\}. \tag{1.3}$$

Here, $\Omega\subset\mathbb{R}^d$ is any bounded Borel set, we write 1_A for the indicator function of a set A and, in abuse of notation, $1_{<E}:=1_{]-\infty,E[}$ for the Fermi function with Fermi energy $E\in\mathbb{R}$. We also introduced the entanglement entropy function $h:\ [0,1]\to[0,1]$,

$$h(\lambda):=-\lambda\log_2\lambda-(1-\lambda)\log_2(1-\lambda), \tag{1.4}$$

and use the convention $0\log_2 0:=0$ for the binary logarithm.

Assumption 1.1. We consider a bounded Borel set $\Lambda\subset\mathbb{R}^d$ such that

(i) it is a Lipschitz domain with, if $d\geqslant 2$, a piecewise C^1-boundary,
(ii) the origin $0\in\mathbb{R}^d$ is an interior point of Λ.

Remark 1.2. Assumption 1.1(i) is taken from [20] and guarantees the validity of the enhanced area law (1.1) for the free Fermi gas which is proven there, see also [21, Cond. 3.1] for the notion of a Lipschitz domain. Assumption 1.1(ii) does not impose any restriction because it can always be achieved by a translation of the potential V in Theorem 1.3.

We recall that $\Lambda_L=L\cdot\Lambda$. The main result of this paper is summarised in

Theorem 1.3. *Let $\Lambda\subset\mathbb{R}^d$ be as in Assumption 1.1, and let $V\in L^\infty(\mathbb{R}^d)$ have compact support. Then, for every Fermi energy $E>0$ there exist constants $\Sigma_l\equiv\Sigma_l(\Lambda,E)\in\,]0,\infty[$ and $\Sigma_u\equiv\Sigma_u(\Lambda,E,V)\in\,]0,\infty[$ such that*

$$\Sigma_l\leqslant\liminf_{L\to\infty}\frac{S_E(H,\Lambda_L)}{L^{d-1}\ln L}\leqslant\limsup_{L\to\infty}\frac{S_E(H,\Lambda_L)}{L^{d-1}\ln L}\leqslant\Sigma_u. \tag{1.5}$$

Remark 1.4. (i) The constant Σ_l can be expressed in terms of the coefficient Σ_0 in the leading term of the unperturbed entanglement entropy $S_E(H_0, \Lambda_L)$ for large L, cf. (1.1) and (1.2). The explicit form

$$\Sigma_l = \frac{6}{\pi^2}\,\Sigma_0 \tag{1.6}$$

is derived in (2.71).

(ii) If $d > 1$, the constant Σ_u can also be expressed in terms of Σ_0. According to (2.64) and (2.68), we have

$$\Sigma_u = 1672\Sigma_0. \tag{1.7}$$

In particular, this constant is independent of V. The numerical prefactor in (1.7) can be improved by using the alternative approach described in Remark 2.6. In $d = 1$ dimension, however, we only obtain a constant Σ_u which also depends on V, because there is an additional contribution from (2.68).

(iii) Pfirsch and Sobolev [27] proved that the coefficient of the leading-order term of the enhanced area law is not altered by adding a periodic potential in $d = 1$. Therefore, we expect the V-dependence of Σ_u in $d = 1$ to be an artefact of our method.

(iv) At negative energies, there is at most discrete spectrum of H. Thus, if $E < 0$, the Fermi function can be smoothed out without changing the operator $1_{<E}(H)$. Therefore, the operator kernel of $1_{<E}(H)$ has fast polynomial decay, and $S_E(H, \Lambda_L) = \mathcal{O}(L^{d-1})$ follows as in [10,25]. In other words, the growth of the entanglement entropy is at most an area law. The same holds at $E = 0$ because eigenvalues cannot accumulate from below at 0 due to the boundedness of V and its compact support.

(v) The stability analysis we perform in this paper requires only that the spatial domain Λ is a bounded measurable subset of $\mathbb{R}^d$ which has an interior point. The stronger assumptions we make are to ensure the validity of Widom's formula for the unperturbed system as proven in [20].

2. Proof of Theorem 1.3

We prove the upper bound of Theorem 1.3 in Sect. 2.2 and the lower bound in Sect. 2.3. Section 2.1 contains results needed for both bounds.

2.1. Preliminaries

Our strategy is a perturbation approach which bounds the entanglement entropy of H in terms of that of H_0 for large volumes. We estimate the function h in (1.3) according to

$$4g \leqslant h \leqslant -2g \log_2 g, \tag{2.1}$$

where

$$g : [0,1] \to [0,1],\ \lambda \mapsto \lambda(1-\lambda), \tag{2.2}$$

see Lemma A.2 for a proof of the well-known lower bound in (2.1) and Lemma A.3 for a proof of the upper bound. Thus, we will be concerned with the operator

$$g\big(1_{\Lambda_L} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big) = \big|1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big|^2, \tag{2.3}$$

where $|A|^2 := A^*A$ for any bounded operator A, and the superscript ^{c} indicates the complement of a set. Throughout this paper, we use the notation $H_{(0)}$ as a placeholder for either H or H_0. The observation in (2.3) leads us to consider von Neumann–Schatten norms of operator differences $1_{\Lambda_L^c}[1_{<E}(H_0) - 1_{<E}(H)]1_{\Lambda_L}$, which is done in Lemmas 2.3 and 2.5. Lemma 2.3 allows to deduce the lower bound in Theorem 1.3, whereas the upper bound requires more work due to the presence of the additional logarithm. Lemma 2.7 will tackle this issue.

In order to show the crucial Lemma 2.3, we need two preparatory results. The first one is about the decay in space of the free resolvent in Lemma 2.1. For $z \in \mathbb{C}\backslash\mathbb{R}$, let $G_0(\,\cdot\,,\,\cdot\,;z): \ \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be the kernel of the resolvent $\frac{1}{H_0 - z}$. The explicit formula for $G_0(\,\cdot\,,\,\cdot\,;z)$ is well known. Likewise, there exists an estimate for $G_0(\cdot,\cdot;z)$ evaluated for large arguments, i.e. there exists $R \equiv R(d) > 0$ and $C \equiv C(d) > 0$ such that for all $x, y \in \mathbb{R}^d$ with Euclidean distance $|x-y| \geqslant R/|z|^{1/2}$, we have

$$|G_0(x,y;z)| \leqslant C|z|^{(d-3)/4}\,\frac{\mathrm{e}^{-|\operatorname{Im}\sqrt{z}||x-y|}}{|x-y|^{(d-1)/2}}. \tag{2.4}$$

For a reference, see [30] and [3, Chap. 9.2] for $d \geqslant 2$ and [5, Chap. I.3.1] for $d = 1$. Here, $\sqrt{\cdot}$ denotes the principal branch of the square root.

We write $\Gamma_l := l + [0,1]^d$ for the closed unit cube translated by $l \in \mathbb{Z}^d$.

Lemma 2.1. *Let $V \in L^\infty(\mathbb{R}^d)$ with compact support in $[-R_V, R_V]^d$ for some $R_V > 0$. Given $z \in \mathbb{C}\setminus\mathbb{R}$, let $\ell_0 \equiv \ell_0(d,V,z) := 2\sqrt{d}(R_V+1) + R(d)/|z|^{1/2}$.*

Then, there exists a constant $C_1 \equiv C_1(d,V) > 0$ such that for any $z \in \mathbb{C}\setminus\mathbb{R}$ and any $n \in \mathbb{Z}^d \setminus\,]-\ell_0,\ell_0[^{\,d}$, we have

$$\Big\| |V|^{1/2}\frac{1}{H_0 - z}1_{\Gamma_n}\Big\|_4 \leqslant C_1|z|^{(d-3)/4}\,\frac{\mathrm{e}^{-|\operatorname{Im}\sqrt{z}||n|/2}}{|n|^{(d-1)/2}}. \tag{2.5}$$

Here, $\|\cdot\|_p$ denotes the von Neumann–Schatten norm for $p \in [1,\infty[$.

Proof. Let $z \in \mathbb{C}\setminus\mathbb{R}$. Since the Hilbert–Schmidt norm of an operator can be computed in terms of the integral kernel, we get

$$\begin{aligned}\Big\| |V|^{1/2}\frac{1}{H_0 - z}1_{\Gamma_n}\Big\|_4^4 &= \Big\|1_{\Gamma_n}\frac{1}{H_0-\overline{z}}\,|V|\,\frac{1}{H_0-z}1_{\Gamma_n}\Big\|_2^2 \\ &= \int_{\Gamma_n}\mathrm{d}x\int_{\Gamma_n}\mathrm{d}y\,\bigg|\int_{\mathbb{R}^d}\mathrm{d}\xi\, G_0(x,\xi;\overline{z})\,|V(\xi)|\,G_0(\xi,y;z)\bigg|^2.\end{aligned} \tag{2.6}$$

For every $n \in \mathbb{Z}^d\setminus\,]-\ell_0,\ell_0[^{\,d}$, every $x \in \Gamma_n$ and every $\xi \in \operatorname{supp}V$, we infer that $|x-\xi| \geqslant R(d)/|z|^{1/2}$. Therefore, the Green's function estimate (2.4) yields

$$|G_0(x,\xi;z)| \leqslant 2^{(d-1)/2}C(d)|z|^{(d-3)/4}\,\frac{\mathrm{e}^{-|\operatorname{Im}\sqrt{z}||n|/2}}{|n|^{(d-1)/2}} \tag{2.7}$$

because

$$|x-\xi| \geqslant |x| - \sqrt{d}R_V \geqslant |n| - \sqrt{d}(R_V+1) \geqslant \frac{|n|}{2}. \tag{2.8}$$

This implies the lemma. □

As a second preparatory result for one of our central bounds, we require

Lemma 2.2. *Let $V \in L^\infty(\mathbb{R}^d)$ with compact support. We fix an energy $E>0$ and consider two compact subsets $\Gamma, \Gamma' \subset \mathbb{R}^d$. Then, we have the representation*

$$1_\Gamma 1_{<E}(H_{(0)}) 1_{\Gamma'} = -\frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}z \, 1_\Gamma \frac{1}{H_{(0)}-z} 1_{\Gamma'}. \tag{2.9}$$

The right-hand side of (2.9) exists as a Bochner integral with respect to the operator norm, and the integration contour γ is a closed curve in the complex plane $\mathbb{C}$ which traces the boundary of the rectangle $\{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant E, \operatorname{Re} z \in [-1+\inf \sigma(H), E]\}$ once in the counterclockwise direction.

Proof. The lemma follows from the corresponding abstract result in Theorem A.1 in Appendix A. Indeed, according to [4, Thm. 4.2], see also, for example, [17], both H and H_0 fulfil a limiting absorption principle at any $E>0$,

$$\sup_{z \in \mathbb{C}:\, \operatorname{Re} z = E,\, \operatorname{Im} z \neq 0} \left\| \langle X \rangle^{-1} \frac{1}{H_{(0)}-z} \Pi_c(H_{(0)}) \langle X \rangle^{-1} \right\| < \infty \tag{2.10}$$

with X being the position operator, $\langle\,\cdot\,\rangle := \sqrt{1+|\cdot|^2}$ the Japanese bracket and $\Pi_c(H_{(0)})$ the projection onto the continuous spectral subspace of $H_{(0)}$. Also, $\sigma_{pp}(H) \subset\,]-\infty, 0]$ because the potential V is bounded and compactly supported [29, Cor. on p. 230]. □

The statement of the next lemma is a crucial estimate that will be needed for both the upper bound and the lower bound in Theorem 1.3.

Lemma 2.3. *Let $\Lambda \subset \mathbb{R}^d$ satisfy Assumption 1.1(ii), and let $V \in L^\infty(\mathbb{R}^d)$ have compact support in $[-R_V, R_V]^d$ for some $R_V > 0$. Then, for every Fermi energy $E>0$ there exists a constant $C_2 \equiv C_2(\Lambda, V, E) > 0$ such that for all $L>0$, we have the bound*

$$\left\| 1_{\Lambda_L^c} \big(1_{<E}(H_0) - 1_{<E}(H)\big) 1_{\Lambda_L} \right\|_2 \leqslant C_2. \tag{2.11}$$

Proof. We fix $E>0$. To estimate the difference between the perturbed and the unperturbed Fermi projections, we express them in terms of a contour integral as stated in Lemma 2.2. We set

$$\ell_1 \equiv \ell_1(d,V,E) := \max_{z \in \operatorname{img}(\gamma)} \{\ell_0(d,V,z)\} < \infty, \tag{2.12}$$

where ℓ_0 is defined in Lemma 2.1 and $\operatorname{img}(\gamma)$ denotes the image of the curve γ in Lemma 2.2. We obtain for all $m,n \in \mathbb{Z}^d \setminus\,]-\ell_1, \ell_1[^{\,d}$

$$1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m} = -\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{d}z \, 1_{\Gamma_n} \Big(\frac{1}{H_0 - z} - \frac{1}{H-z}\Big) 1_{\Gamma_m}. \tag{2.13}$$

The Bochner integral exists even with respect to the Hilbert–Schmidt norm, as will follow from estimates (2.17) and (2.23). We point out that (2.23) relies again on the limiting absorption principle (2.10).

In order to estimate the integral in (2.13), we apply the resolvent identity twice to the integrand. The integrand then reads

$$1_{\Gamma_n}\Big(\frac{1}{H_0-z}V\frac{1}{H_0-z}-\frac{1}{H_0-z}V\frac{1}{H-z}V\frac{1}{H_0-z}\Big)1_{\Gamma_m}. \tag{2.14}$$

This implies the Hilbert–Schmidt-norm estimate

$$\begin{aligned}&\Big\|1_{\Gamma_n}\Big(\frac{1}{H_0-z}-\frac{1}{H-z}\Big)1_{\Gamma_m}\Big\|_2\\ &\quad\leqslant\Big\|1_{\Gamma_n}\frac{1}{H_0-z}|V|^{1/2}\Big\|_4\Big(1+\Big\||V|^{1/2}\frac{1}{H-z}|V|^{1/2}\Big\|\Big)\\ &\qquad\times\Big\||V|^{1/2}\frac{1}{H_0-z}1_{\Gamma_m}\Big\|_4.\end{aligned} \tag{2.15}$$

Lemma 2.1 already provides bounds for the first and third factor on the right-hand side of (2.15). To estimate the second factor, we employ two different methods, depending on the location of z on the contour. Therefore, we split the curve γ into two parts. We denote by γ_1 the right vertical part of γ with image $\mathrm{img}(\gamma_1)=\big\{z\in\mathbb{C}:\ \mathrm{Re}\,z=E,\ |\mathrm{Im}\,z|\leqslant\min\{E,1\}\big\}$. The remaining part of the curve γ is denoted by γ_2.

Let us first consider the curve γ_2. We observe

$$\mathrm{dist}\big(z,\sigma(H_{(0)})\big)\geqslant\min\{1,E\}\quad\text{for all } z\in\mathrm{img}(\gamma_2). \tag{2.16}$$

Therefore, the middle factor in the second line of (2.15) is bounded from above by $(1+\|V\|_\infty/\min\{1,E\})$. Since the curve γ_2 does not intersect $[0,\infty[$, there exists $\zeta_2\equiv\zeta_2(V,E)>0$ such that $|\mathrm{Im}\sqrt{z}|/2\geqslant\zeta_2$ for all $z\in\mathrm{img}(\gamma_2)\setminus\mathbb{R}$. Hence, according to Lemma 2.1, we estimate (2.15) by

$$\begin{aligned}\Big\|1_{\Gamma_n}\Big(\frac{1}{H_0-z}-\frac{1}{H-z}\Big)1_{\Gamma_m}\Big\|_2&\leqslant\frac{c_2\ \mathrm{e}^{-\zeta_2(|n|+|m|)}}{(|n||m|)^{(d-1)/2}}\\ &\leqslant\frac{c_2/\zeta_2}{(|n||m|)^{(d-1)/2}(|n|+|m|)}\end{aligned} \tag{2.17}$$

for all $z\in\mathrm{img}(\gamma_2)\setminus\mathbb{R}$ with

$$c_2\equiv c_2(d,V,E):=C_1^2\Big(\max_{z\in\mathrm{img}(\gamma_2)}|z|^{(d-3)/2}\Big)\Big(1+\frac{\|V\|_\infty}{\min\{1,E\}}\Big)<\infty. \tag{2.18}$$

We now turn our attention to γ_1, the part of the contour that intersects the continuous spectrum of H. Writing $1=\Pi_{pp}(H)+\Pi_c(H)$ and recalling $\sigma_{pp}(H)\subset\,]-\infty,0]$, see the end of the proof of Lemma 2.2, we infer

$$\Big\||V|^{1/2}\frac{1}{H-z}|V|^{1/2}\Big\|\leqslant\frac{\|V\|_\infty}{E}+\Big\||V|^{1/2}\frac{1}{H-z}\Pi_c(H)|V|^{1/2}\Big\| \tag{2.19}$$

for every $z \in \operatorname{img}(\gamma_1) \setminus \mathbb{R}$. The second term on the right-hand side admits the uniform upper bound

$$\|\langle X\rangle|V|^{1/2}\|^2 \sup_{\substack{z\in\mathbb{C}:\,\mathrm{Re}\, z=E,\\ \mathrm{Im}\, z\neq 0}} \Big\|\langle X\rangle^{-1}\frac{1}{H-z}\Pi_c(H)\langle X\rangle^{-1}\Big\| \leqslant (1+dR_V^2)\|V\|_\infty\, C_{LA}. \tag{2.20}$$

Here, we used the compact support of V and introduced the abbreviation $C_{LA} \equiv C_{LA}(d,E,V) < \infty$ for the supremum on the left-hand side of (2.20). It is finite because of the limiting absorption principle (2.10) for H.

In addition, we need a lower bound for the decay rate of the exponential in (2.5) along the curve γ_1. We write $\operatorname{img}(\gamma_1) \ni z = E + i\eta$ with $|\eta| \leqslant \min\{1,E\}$. Then,

$$|\operatorname{Im}\sqrt{z}| = \sqrt[4]{E^2+\eta^2}\;\alpha(|\eta|/E) \geqslant \sqrt{E}\;\alpha(|\eta|/E), \tag{2.21}$$

with $\alpha:\ [0,\infty[\ \to [0,1]$, $x \mapsto \sin\big(\frac{1}{2}\arctan x\big)$. We note that $\sin y \geqslant y(1-y^2/6)$ for all $y \geqslant 0$, $\arctan x \leqslant \pi/2$ and $\arctan x \geqslant x/2$ for all $x \in [0,1]$. Therefore, we infer the existence of a constant $\zeta_1 \equiv \zeta_1(E) > 0$ such that

$$|\operatorname{Im}\sqrt{z}|/2 \geqslant \zeta_1|\eta| \qquad \text{for all } z = E + \mathrm{i}\eta \in \operatorname{img}(\gamma_1). \tag{2.22}$$

By applying Lemma 2.1 together with (2.22), as well as (2.19) and (2.20), we get the estimate

$$\Big\|1_{\Gamma_n}\Big(\frac{1}{H_0-z} - \frac{1}{H-z}\Big)1_{\Gamma_m}\Big\|_2 \leqslant \frac{c_1\,\mathrm{e}^{-\zeta_1|\eta|(|n|+|m|)}}{(|n||m|)^{(d-1)/2}} \tag{2.23}$$

from (2.15) and any $\operatorname{img}(\gamma_1) \ni z = E + \mathrm{i}\eta$ with $|\eta| \leqslant \min\{1,E\}$. Here, we introduced the constant

$$c_1 \equiv c_1(d,V,E) := C_1^2\Big(\max_{z\in\operatorname{img}(\gamma_1)} |z|^{(d-3)/2}\Big)\Big[1 + \big(E^{-1} + (1+dR_V^2)C_{LA}\big)\|V\|_\infty\Big]. \tag{2.24}$$

We are now able to estimate the contour integral in (2.13) with the help of bounds (2.17) and (2.23)

$$\begin{aligned}\big\|1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m}\big\|_2 &\leqslant \frac{\tilde{c}_2}{(|n||m|)^{(d-1)/2}(|n|+|m|)} \\ &\quad + \int_{-1}^{1}\mathrm{d}\eta\,\frac{c_1\,\mathrm{e}^{-\zeta_1|\eta|(|n|+|m|)}}{2\pi(|n||m|)^{(d-1)/2}} \\ &= \frac{\tilde{c}}{(|n||m|)^{(d-1)/2}(|n|+|m|)}\end{aligned} \tag{2.25}$$

for all $m,n \in \mathbb{Z}^d \setminus\,]-\ell_1,\ell_1[^{\,d}$, where

$$\tilde{c}_2 \equiv \tilde{c}_2(d,V,E) := \frac{c_2(E+\|V\|_\infty+2)}{\pi\zeta_2} \quad \text{and} \quad \tilde{c} \equiv \tilde{c}(d,V,E) := \frac{c_1}{\pi\zeta_1} + \tilde{c}_2. \tag{2.26}$$

In order to prove the lemma for any $L > 0$, we introduce a length $L_0 > 0$, which will be determined below, and first consider the case of $L \in\,]0,L_0]$. In this case, we have

$$\big\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\big\|_2^2 \leqslant \big\|\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_{L_0}}\big\|_2^2. \tag{2.27}$$

Following [31, Thm. B.9.2 and its proof], we infer the existence of a constant $C_S \equiv C_S(d, V, E)$ such that

$$\big\|1_{<E}\big(H_{(0)}\big)1_{\Gamma_m}\big\|_1 \leqslant C_S \tag{2.28}$$

holds uniformly in $m \in \mathbb{Z}^d$. By applying the binomial inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$ and the inequality $\|A\|_2^2 \leqslant \|A\|_1$ for any trace-class operator A with $\|A\| \leqslant 1$, we estimate the right-hand side of (2.27) by

$$\begin{aligned} &2\Big(\big\|1_{<E}(H_0)1_{\Lambda_{L_0}}\big\|_2^2 + \big\|1_{<E}(H)1_{\Lambda_{L_0}}\big\|_2^2\Big) \\ &\quad\leqslant \sum_{m\in\Xi_{L_0}} 2\big(\big\|1_{<E}(H_0)1_{\Gamma_m}\big\|_1 + \big\|1_{<E}(H)1_{\Gamma_m}\big\|_1\big) \\ &\quad\leqslant 4C_S|\tilde{\Lambda}_{L_0}| < \infty, \end{aligned} \tag{2.29}$$

where we introduced the "coarse-grained box domains"

$$\tilde{\Lambda}_\ell^{(\text{ext})} := \bigcup_{m\in\Xi_\ell^{(\text{ext})}} \Gamma_m \quad \text{with} \quad \Xi_\ell^{(\text{ext})} := \big\{m \in \mathbb{Z}^d : \Gamma_m \cap \Lambda_\ell^{(c)} \neq \varnothing\big\} \tag{2.30}$$

for $\ell > 0$. We note that $\tilde{\Lambda}_\ell^{\text{ext}}$ is not the complement of $\tilde{\Lambda}_\ell$. It will be needed below.

In order to tackle the other case of $L > L_0$, we first determine a suitable value for L_0 as follows: we recall that the origin is an interior point of the bounded domain Λ, whence there exists a length $L_0 \equiv L_0(\Lambda, V, E) > 0$ such that for all $L \geqslant L_0$

$$\tilde{\Lambda}_L^{\text{ext}} \subset \mathbb{R}^d \setminus]-\ell_1, \ell_1[^{\,d}. \tag{2.31}$$

Now, we cover Λ_L^c and $\Lambda_L \setminus \Lambda_{L_0}$ by unit cubes. Hence, we have

$$\begin{aligned} &\big\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\big\|_2^2 \\ &\quad\leqslant \big\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_{L_0}}\big\|_2^2 \\ &\qquad + \sum_{\substack{n\in\Xi_L^{\text{ext}} \\ m\in\Xi_L\cap\Xi_{L_0}^{\text{ext}}}} \big\|1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m}\big\|_2^2. \end{aligned} \tag{2.32}$$

The first term on the right-hand side of (2.32) is estimated by (2.27) and (2.29). To bound the double sum in (2.32) from above, we use (2.25), which is applicable due to definition (2.31) of L_0, and obtain

$$\begin{aligned} \big\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\big\|_2^2 &\leqslant 4C_S|\tilde{\Lambda}_{L_0}| \\ &\quad + \sum_{\substack{n\in\Xi_L^{\text{ext}} \\ m\in\Xi_L\cap\Xi_{L_0}^{\text{ext}}}} \frac{\tilde{c}^2}{(|n||m|)^{d-1}|n|^2}. \end{aligned} \tag{2.33}$$

We conclude from the definition of ℓ_1 that $|l| \geqslant |u| - \sqrt{d} \geqslant |u|/2$ for every $l \in \Xi_L^{\text{ext}} \cup (\Xi_L \cap \Xi_{L_0}^{\text{ext}})$ and every $u \in \Gamma_l \subseteq \mathbb{R}^d \setminus]-\ell_1, \ell_1[^{\,d}$. Therefore, we infer that the double sum in (2.33) is upper bounded by the double integral

$$\int_{\tilde{\Lambda}_L} \mathrm{d}x \int_{\tilde{\Lambda}_L^{\mathrm{ext}}} \mathrm{d}y \, \frac{(2^d \tilde{c})^2}{(|x||y|)^{d-1}|y|^2} = (2^d \tilde{c})^2 \int_{\frac{L_0}{L}\tilde{\Lambda}_L} \frac{\mathrm{d}x}{|x|^{d-1}} \int_{\frac{L_0}{L}\tilde{\Lambda}_L^{\mathrm{ext}}} \frac{\mathrm{d}y}{|y|^{d+1}}. \quad (2.34)$$

But $\tilde{\Lambda}_L^{(\mathrm{ext})} \subseteq \bigcup_{x\in\Lambda_L^{(c)}} (x + [-1,1]^d)$ so that the scaled domains satisfy

$$\frac{L_0}{L}\, \tilde{\Lambda}_L^{(\mathrm{ext})} \subseteq \bigcup_{x\in\Lambda_{L_0}^{(c)}} \Big(x + \frac{L_0}{L}[-1,1]^d \Big) \subseteq \bigcup_{x\in\Lambda_{L_0}^{(c)}} \big(x + [-1,1]^d \big) =: K_{L_0}^{(\mathrm{ext})} \quad (2.35)$$

for any $L \geqslant L_0$. Clearly, K_{L_0} is bounded. Furthermore, we ensure that $K_{L_0}^{\mathrm{ext}}$ has a positive distance to the origin. This relies on the origin being an interior point of Λ and may require an enlargement of L_0, which can always be done. It follows that the right-hand side of (2.34) is bounded from above by some constant $c_3 \equiv c_3(\Lambda, V, E) < \infty$, uniformly in $L \geqslant L_0$. Combining this with (2.27), (2.29), (2.33) and (2.34), we arrive at the final estimate

$$\sup_{L>0} \big\| 1_{\Lambda_L^c} \big(1_{<E}(H_0) - 1_{<E}(H) \big) 1_{\Lambda_L} \big\|_2^2 \leqslant 4C_S |\tilde{\Lambda}_{L_0}| + c_3 =: C_2^2. \quad (2.36)$$

□

Remark 2.4. The limiting absorption principle has been used in Lemmas 2.2 and 2.3. In the latter case, it serves to estimate the difference of the perturbed and unperturbed Fermi projection. A limiting absorption principle was used in a similar way in [11–13] to estimate differences of functions of the Laplacian and of a perturbation thereof.

2.2. Proof of the upper bound

We begin with an interpolation result.

Lemma 2.5. *Let $\Lambda \subset \mathbb{R}^d$ be as in Assumption 1.1*(ii)*, let $V \in L^\infty(\mathbb{R}^d)$ have compact support and fix $E > 0$. Then, there exists a constant $C_3 \equiv C_3(\Lambda, V, E) > 0$ such that for all $s \in]1/2, 1[$ and all $L \geqslant 1$, we have*

$$\big\| 1_{\Lambda_L^c} \big(1_{<E}(H) - 1_{<E}(H_0) \big) 1_{\Lambda_L} \big\|_{2s}^{2s} \leqslant C_3 L^{2d(1-s)}. \quad (2.37)$$

Proof. Given a trace-class operator A and $s \in]1/2, 1[$, we conclude from the interpolation inequality, see, for example, [34, Lemma 1.11.5],

$$\|A\|_{2s}^{2s} \leqslant \|A\|_1^{2(1-s)} \|A\|_2^{2(2s-1)}. \quad (2.38)$$

Due to the boundedness of Λ, there exists a length $r \equiv r(\Lambda) \in [1, \infty[$ such that $\Lambda \subseteq [-r, r]^d$. Estimate (2.28) implies that the operator

$$A_L := 1_{\Lambda_L^c} \big(1_{<E}(H) - 1_{<E}(H_0) \big) 1_{\Lambda_L} \quad (2.39)$$

is trace class for all $L \geqslant 1$ with norm $\|A_L\|_1 \leqslant 2(2\lceil rL \rceil)^d C_S \leqslant 2(4rL)^d C_S$. Here, we used that $\lceil x \rceil \leqslant 2x$ for every $x \geqslant 1$, where $\lceil x \rceil$ denotes the smallest integer larger or equal to $x \in \mathbb{R}$. Moreover, $\|A_L\|_2^2 \leqslant C_2^2$ for all $L \geqslant 1$ by Lemma 2.3. This proves the claim with

$$\big(2(4r)^d C_S\big)^{2(1-s)} C_2^{2(2s-1)} \leqslant 2^{2d+1} r^d (C_S+1)(C_2^2+1) =: C_3 \equiv C_3(\Lambda, V, E). \tag{2.40}$$

□

Remark 2.6. Lemma 2.5 allows for a quick proof of the upper bound in Theorem 1.3, if we restrict ourselves to the case $d \geqslant 2$. First, we apply the estimate $h(\lambda) \leqslant \frac{6}{1-s}\big(g(\lambda)\big)^s$ for all $\lambda \in [0,1]$ and $s \in]0,1[$, see Lemma A.2 for a proof, to the entanglement entropy and rewrite it with (2.3) to obtain

$$\begin{aligned} S_E(H,\Lambda_L) &\leqslant \frac{6}{1-s}\big\|1_{\Lambda_L^c} 1_{<E}(H) 1_{\Lambda_L}\big\|_{2s}^{2s} \\ &\leqslant \frac{12}{1-s}\Big(\big\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\big\|_{2s}^{2s} + \|A_L\|_{2s}^{2s}\Big). \end{aligned} \tag{2.41}$$

Here, A_L is defined in (2.39). The first term on the right-hand side scales like $\mathcal{O}(L^{d-1}\ln L)$ according to the lemma and subsequent remarks in [20]. The second term is of order $\mathcal{O}\big(L^{2d(1-s)}\big)$ according to Lemma 2.5. If we choose $s \equiv s(d,\varepsilon) := 1 - \varepsilon(2d)^{-1}$ for any $\varepsilon \in [0,1]$, the second term is of the order $\mathcal{O}(L^{\varepsilon})$, thus subleading as compared to the first term in all but one dimensions.

Unfortunately, there is no choice for s which yields only a logarithmic growth in $d=1$. To appropriately bound the term $(1-s)^{-1}\mathcal{O}\big(L^{2d(1-s)}\big)$ in (2.41) requires an L-dependent choice of s with $s \equiv s(L) \to 1$ as $L \to \infty$. However, such a choice of s leads to an additional diverging prefactor $(1-s)^{-1}$ multiplying the asymptotics $\mathcal{O}(L^{d-1}\ln L)$ from the first term.

We now present an approach, which yields the optimal upper bound of order $\mathcal{O}(L^{d-1}\ln L)$ for all dimensions.

Lemma 2.7. *Let A and B be two compact operators with $\|A\|, \|B\| \leqslant \mathrm{e}^{-1/2}/3$ and consider the function*

$$f : [0,\infty[\to [0,1], \; x \mapsto -1_{[0,1]}(x)\, x^2 \log_2(x^2). \tag{2.42}$$

Then, we have

$$\operatorname{tr}\{f(|A|)\} \leqslant 4\operatorname{tr}\{f(|B|)\} + 4\operatorname{tr}\{f(|A-B|)\}. \tag{2.43}$$

For any compact operator A, let $\big(a_n(A)\big)_{n\in\mathbb{N}} \subseteq [0,\infty[$ denote the non-increasing sequence of its singular values. They coincide with the eigenvalues of the self-adjoint operator $|A|$.

Proof of Lemma 2.7. By assumption, we have $0 \leqslant a_{2n}(A) \leqslant a_{2n-1}(A) \leqslant \mathrm{e}^{-1/2}/3$ for all $n \in \mathbb{N}$. Since the function f is monotonously increasing on $[0, \mathrm{e}^{-1/2}]$, we deduce

$$\operatorname{tr}\{f(|A|)\} = \sum_{n\in\mathbb{N}} f\big(a_n(A)\big) \leqslant 2\sum_{n\in\mathbb{N}} f\big(a_{2n-1}(A)\big). \tag{2.44}$$

The singular values of any compact operators A and B satisfy the inequality

$$a_{n+m-1}(A) \leqslant a_n(B) + a_m(A-B) \tag{2.45}$$

for all $n, m \in \mathbb{N}$ [36, Prop. 2 in Sect. III.G]. We point out that the right-hand side of (2.45) does not exceed the upper bound $\mathrm{e}^{-1/2}$ because of $\|A-B\| \leqslant$

$\|A\| + \|B\| \leqslant (2/3)\,\mathrm{e}^{-1/2}$. Together with the monotonicity of f, we conclude from (2.44) that

$$\operatorname{tr}\{f(|A|)\} \leqslant 2 \sum_{n\in\mathbb{N}} f\big(a_n(B) + a_n(A-B)\big). \tag{2.46}$$

Next, we claim that

$$f(x+y) \leqslant -2(x^2+y^2)\log_2[(x+y)^2] \leqslant 2f(x) + 2f(y) \tag{2.47}$$

for all $x, y \geqslant 0$ with $x + y < 1$. The first estimate follows from the binomial inequality together with $-\log_2[(x+y)^2] \geqslant 0$ for $x+y < 1$, the second estimate from $(x+y)^2 \geqslant x^2$, respectively, $(x+y)^2 \geqslant y^2$, and the fact that $-\log_2$ is monotonously decreasing. Combining (2.46) and (2.47), we arrive at

$$\operatorname{tr}\{f(|A|)\} \leqslant 4 \sum_{n\in\mathbb{N}} \big[f\big(a_n(B)\big) + f\big(a_n(A-B)\big)\big]. \tag{2.48}$$

□

Proof of the upper bound in Theorem 1.3. Let $L \geqslant 1$ and $E > 0$. Lemma A.3 and (2.3) yield

$$S_E(H, \Lambda_L) \leqslant 2 \sum_{n=1}^{\infty} f\big(a_n(1_{\Lambda_L^c} 1_{<E}(H) 1_{\Lambda_L})\big), \tag{2.49}$$

where f was defined in Lemma 2.7. In order to apply Lemma 2.7, we will decompose the compact operator $1_{\Lambda_L^c} 1_{<E}\big(H_{(0)}\big) 1_{\Lambda_L}$ into a part bounded by $\mathrm{e}^{-1/2}/3$ in norm and a finite-rank operator. To this end, we introduce

$$\begin{aligned} N_{(0)} &\equiv N_{(0)}(\Lambda, V, E, L) \\ &:= \min\Big\{n \in \mathbb{N} :\ a_n\big(1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big) \leqslant \mathrm{e}^{-1/2}/3\Big\} - 1, \end{aligned} \tag{2.50}$$

the number of singular values of $1_{\Lambda_L^c} 1_{<E}\big(H_{(0)}\big) 1_{\Lambda_L}$ which are larger than $\mathrm{e}^{-1/2}/3$. We define $F_{(0)}$ as the contribution from the first $N_{(0)}$ singular values in the singular value decomposition of $1_{\Lambda_L^c} 1_{<E}\big(H_{(0)}\big) 1_{\Lambda_L}$, whence $\operatorname{rank}(F_{(0)}) = N_{(0)}$ and $\|F_{(0)}\| \leqslant 1$. The remainder

$$Q_{(0)} := 1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L} - F_{(0)} \tag{2.51}$$

fulfils $\|Q_{(0)}\| \leqslant \mathrm{e}^{-1/2}/3$ by definition of $N_{(0)}$. We note the upper bound

$$N_{(0)} \leqslant 9\,\mathrm{e} \sum_{n=1}^{N_{(0)}} \Big(a_n\big(1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big)\Big)^2 \leqslant 9\,\mathrm{e}\, \big\|1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big\|_2^2. \tag{2.52}$$

Using Lemma 2.3, we further estimate N in terms of unperturbed quantities

$$N \leqslant 18\,\mathrm{e}\, \big\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\big\|_2^2 + 18\,\mathrm{e}\, C_2^2. \tag{2.53}$$

Identity (2.3) and the lower bound in (A.10) imply $\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\|_2^2 \leqslant S_E(H_0, \Lambda_L)$ so that we obtain

$$N_0 \leqslant 9\,\mathrm{e}\, S_E(H_0, \Lambda_L) \quad \text{and} \quad N \leqslant 18\,\mathrm{e}\, S_E(H_0, \Lambda_L) + 18\,\mathrm{e}\, C_2^2 \tag{2.54}$$

for later usage.

We deduce from (2.45) and $\operatorname{rank}(F) = N$ that for all $n \in \mathbb{N}$

$$a_{n+N}(Q+F) \leqslant a_n(Q) + a_{N+1}(F) = a_n(Q) \leqslant \mathrm{e}^{-1/2}/3. \tag{2.55}$$

Hence, (2.49) implies that

$$S_E(H,\Lambda_L) \leqslant 2\sum_{n=1}^{N} f\big(a_n(Q+F)\big)+2\sum_{n=1}^{\infty} f\big(a_n(Q)\big) \leqslant 2N+2\operatorname{tr}\{f(|Q|)\}, \tag{2.56}$$

where the monotonicity of f on $[0,\mathrm{e}^{-1/2}]$ and $f \leqslant 1$ is used. Now, Lemma 2.7 allows to estimate (2.56) so that

$$S_E(H,\Lambda_L) \leqslant 2N + 8\operatorname{tr}\{f(|Q_0|)\} + 8\operatorname{tr}\{f(|\delta Q|)\}, \tag{2.57}$$

where $\delta Q := Q - Q_0$. The rank of $\delta F := F - F_0$ obeys

$$\delta N \equiv \delta N(\Lambda, V, E, L) := \operatorname{rank}(\delta F) \leqslant N + N_0. \tag{2.58}$$

We deduce again from (2.45) and from the definition of δN that for all $n \in \mathbb{N}$

$$a_{n+2\delta N}(\delta Q) = a_{(n+\delta N)+(\delta N+1)-1}(\delta Q) \leqslant a_{n+\delta N}(\delta Q + \delta F). \tag{2.59}$$

Yet another application of (2.45) and the definition of δN yield for all $n \in \mathbb{N}$

$$a_{n+\delta N}(\delta Q + \delta F) \leqslant a_n(\delta Q) \leqslant \|\delta Q\| \leqslant 2\,\mathrm{e}^{-1/2}/3. \tag{2.60}$$

Therefore, the singular values in (2.59) lie in the range where the function f is monotonously increasing. Hence, we obtain

$$\begin{aligned}\operatorname{tr}\{f(|\delta Q|)\} &\leqslant \sum_{n=1}^{2\delta N} f\big(a_n(\delta Q)\big) + \sum_{n\in\mathbb{N}} f\big(a_{\delta N+n}(\delta Q+\delta F)\big)\\ &\leqslant 2\,\delta N + \sum_{n\in\mathbb{N}} f\big(a_n(\delta Q+\delta F)\big),\end{aligned} \tag{2.61}$$

where the second line follows from $0 \leqslant f \leqslant 1$.

Now, we repeat the arguments from (2.59) to (2.61) for Q_0 instead of δQ, F_0 instead of δF and N_0 instead of δN. This implies

$$\operatorname{tr}\{f(|Q_0|)\} \leqslant 2N_0 + \sum_{n\in\mathbb{N}} f\big(a_n(Q_0+F_0)\big). \tag{2.62}$$

The sum in (2.62) is bounded from above by the unperturbed entanglement entropy, which follows from (2.51), the definition of f, (2.3) and the lower bound in Lemma A.3, whence

$$\operatorname{tr}\{f(|Q_0|)\} \leqslant 2N_0 + S_E(H_0,\Lambda_L). \tag{2.63}$$

Next, we combine (2.57), (2.54), (2.61), (2.58) and (2.63) to obtain

$$S_E(H,\Lambda_L) \leqslant 1672\, S_E(H_0,\Lambda_L) + 882\, C_2^2 + 8\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q+\delta F)\big). \tag{2.64}$$

In order to estimate the sum in (2.64), we appeal to the definitions of δQ and δF, (2.51), the definition of f and (A.9) to deduce

$$\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q+\delta F)\big) \leqslant \frac{1}{1-s}\,\big\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\big\|_{2s}^{2s} \tag{2.65}$$

for any $s \in]0,1[$. Restricting ourselves to $s \in]1/2,1[$ allows us to apply Lemma 2.5 so that

$$\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q + \delta F)\big) \leqslant \frac{C_3}{1-s}\, L^{2d(1-s)}, \tag{2.66}$$

where $C_3 = C_3(\Lambda, V, E) > 0$ is given in Lemma 2.5 and independent of s. Assuming $L \geqslant 8$, we choose the L-dependent exponent

$$s \equiv s(L) := 1 - \frac{1}{\ln L} \in]1/2, 1[\tag{2.67}$$

which implies

$$\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q + \delta F)\big) \leqslant C_3\, \mathrm{e}^{2d} \ln L. \tag{2.68}$$

The entanglement entropy of a free Fermi gas exhibits an enhanced area law, $S_E(H_0, \Lambda_L) = \mathcal{O}(L^{d-1} \ln L)$ [20, Theorem], so that the claim follows from (2.64) together with (2.68). □

2.3. Proof of the lower bound

Proof of the lower bound in Theorem 1.3. We fix $L > 0$ and $E > 0$. The lower bound in (A.10), identity (2.3) and the elementary inequality $(a-b)^2 \geqslant a^2/2 - b^2$ for $a, b \in \mathbb{R}$ imply

$$\begin{aligned} S_E(H, \Lambda_L) &\geqslant 4 \operatorname{tr}\big\{g\big(1_{\Lambda_L} 1_{<E}(H) 1_{\Lambda_L}\big)\big\} = 4\, \|1_{\Lambda_L^c} 1_{<E}(H) 1_{\Lambda_L}\|_2^2 \\ &\geqslant 2 \big\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\big\|_2^2 \\ &\quad - 4 \big\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big) 1_{\Lambda_L}\big\|_2^2. \end{aligned} \tag{2.69}$$

The second term on the right-hand side is uniformly bounded in L according to Lemma 2.3. For the first term, it was shown in [20, Eq. (7)] that the leading behaviour of the asymptotic expansion in L is of order $L^{d-1} \ln L$. Hence,

$$\liminf_{L\to\infty} \frac{S_E(H, \Lambda_L)}{L^{d-1} \ln L} \geqslant 2 \lim_{L\to\infty} \frac{\operatorname{tr}\big\{g\big(1_{\Lambda_L} 1_{<E}(H_0) 1_{\Lambda_L}\big)\big\}}{L^{d-1} \ln L} =: \Sigma_l. \tag{2.70}$$

Finally, Eqs. (1), (4), (7) and (8) in [20] and (1.1) imply

$$\Sigma_l = \frac{6}{\pi^2}\, \Sigma_0. \tag{2.71}$$

□

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Appendix

A. Auxiliary results

Representation (A.2) of the Fermi projection in terms of a Riesz projection with the integration contour cutting through the continuous spectrum may be of independent interest.

Theorem A.1. *Let K be a densely defined self-adjoint operator in a Hilbert space $\mathcal{H}$, which is bounded below and satisfies a limiting absorption principle at $E \in \mathbb{R}$ in the sense that there exists a bounded operator B on $\mathcal{H}$ with inverse B^{-1}, which is possibly only densely defined and unbounded, such that*

$$\mathcal{S}_E := \sup_{z\in\mathbb{C}\colon \mathrm{Re}\, z=E,\, \mathrm{Im}\, z\neq 0} \Big\| B \frac{1}{K-z} \Pi_c(K) B \Big\| < \infty. \tag{A.1}$$

Here, $\Pi_c(K)$ denotes the projection onto the continuous spectral subspace of K. Let A_1, A_2 be two bounded operators on $\mathcal{H}$ such that $\|A_1 B^{-1}\| < \infty$ and $\|B^{-1} A_2\| < \infty$. Finally, we assume that there are no eigenvalues of K near E, i.e. $\mathrm{dist}\big(\sigma_{pp}(K), E\big) > 0$. *Then, we have the representation*

$$A_1 1_{<E}(K) A_2 = -\frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}z\, A_1 \frac{1}{K-z} A_2. \tag{A.2}$$

The right-hand side of (A.2) exists as a Bochner integral with respect to the operator norm $\|\cdot\|$, and the integration contour γ is a closed curve in the complex plane $\mathbb{C}$ which, for $s>0$, traces the boundary of the rectangle $\big\{z \in \mathbb{C} : \ |\mathrm{Im}\, z| \leqslant s,\ \mathrm{Re}\, z \in [-1+\inf\sigma(K), E]\big\}$ once in the counterclockwise direction.

We remark that the projection $\Pi_c(K)$ in (A.1) can be omitted because we also assume $\mathrm{dist}\big(\sigma_{pp}(K), E\big) > 0$ in the theorem. Theorem A.1 readily generalises from Fermi projections to spectral projections of more general intervals.

Proof of Theorem A.1. Let $\varepsilon > 0$, and let γ_ε be the curve γ without the vertical line segment from $E - \mathrm{i}\varepsilon$ to $E + \mathrm{i}\varepsilon$. Since $\|(K-z)^{-1}\|$ is uniformly bounded for z in the image of γ_ε, it suffices to verify that

$$\int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \Big\| A_1 \, \frac{1}{K - E - \mathrm{i}\eta} \, A_2 \Big\| < \infty \tag{A.3}$$

in order to show the existence of the right-hand side of (A.2) as a Bochner integral with respect to the operator norm. But

$$\begin{aligned} \Big\| A_1 \, \frac{1}{K - E - \mathrm{i}\eta} \, A_2 \Big\| &\leqslant \Big\| A_1 \, \frac{1}{K - E - \mathrm{i}\eta} \, \Pi_{pp}(K) \, A_2 \Big\| \\ &\quad + \|A_1 B^{-1}\| \|B^{-1} A_2\| \Big\| B \frac{1}{K - E - \mathrm{i}\eta} \, \Pi_c(K) B \Big\| \\ &\leqslant \frac{\|A_1\| \|A_2\|}{\operatorname{dist}\big(\sigma_{pp}(K), E\big)} + \|A_1 B^{-1}\| \|B^{-1} A_2\| \, \mathcal{S}_E \end{aligned} \tag{A.4}$$

uniformly in $\eta \in [-\varepsilon, \varepsilon]$, and estimate (A.3) holds.

It remains to prove the equality in (A.2). Let $\varphi, \psi \in \mathcal{H}$. Since the contour integral along γ exists in the Bochner sense with respect to the operator norm, we equate

$$\begin{aligned} \Big\langle \varphi, \Big(\oint_\gamma \mathrm{d}z \, A_1 \, \frac{1}{K - z} \, A_2 \Big) \psi \Big\rangle &= \lim_{\varepsilon \searrow 0} \int_{\gamma_\varepsilon} \mathrm{d}z \, \langle \varphi, A_1 \, \frac{1}{K - z} \, A_2 \psi \rangle \\ &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^* \varphi), (A_2 \psi)}(\lambda) \, \int_{\gamma_\varepsilon} \mathrm{d}z \, \frac{1}{\lambda - z}, \end{aligned} \tag{A.5}$$

where we introduced the complex spectral measure $\mu_{\varphi,\psi} := \langle \varphi, 1_\bullet(K) \psi \rangle$ of K and used Fubini in the last step. On the other hand, we apply the spectral theorem and Cauchy's integral formula to conclude

$$\begin{aligned} \langle \varphi, A_1 1_{<E}(K) A_2 \psi \rangle &= \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^* \varphi), (A_2 \psi)}(\lambda) \, 1_{<E}(\lambda) \\ &= -\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^* \varphi), (A_2 \psi)}(\lambda) \, \int_\gamma \mathrm{d}z \, \frac{1}{\lambda - z}, \end{aligned} \tag{A.6}$$

which is justified because E is not an eigenvalue of K. Up to the prefactor $-1/(2\pi\mathrm{i})$, the right-hand side of (A.6) equals

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^* \varphi), (A_2 \psi)}(\lambda) \, \int_{\gamma_\varepsilon} \mathrm{d}z \, \frac{1}{\lambda - z} \\ &+\mathrm{i} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^* \varphi), (A_2 \psi)}(\lambda) \, \int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \frac{1}{\lambda - E - \mathrm{i}\eta}. \end{aligned} \tag{A.7}$$

The explicit computation, using the fact that the imaginary part of the integrand is an odd function, gives

$$\int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \frac{1}{\lambda - E - \mathrm{i}\eta} = \int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \frac{\lambda - E}{(\lambda - E)^2 + \eta^2} = 2 \arctan\Big(\frac{\varepsilon}{\lambda - E} \Big) \tag{A.8}$$

for every real $\lambda \neq E$. Therefore, dominated convergence implies that the second limit in (A.7) vanishes. Here, we used again that E is not an eigenvalue of K. Since φ and ψ are arbitrary, the theorem follows from (A.5) to (A.7). $\square$

In the remaining part, we prove some elementary estimates.

Lemma A.2. *For all* $s \in]0,1[$ *and all* $x \in [0,1]$, *we have*

$$-x\log_2 x \leqslant \frac{x^s}{1-s}. \tag{A.9}$$

and

$$4g(x) \leqslant h(x) \leqslant \frac{6}{1-s}\big(g(x)\big)^s, \tag{A.10}$$

where g *was defined in* (2.2).

Proof. We introduce the continuous function $\varphi : [0,1] \to [0,\infty[,\ x \mapsto -x^{1-s}\log_2 x$. The first claim follows from the observation

$$0 \leqslant \varphi \leqslant \frac{1}{1-s}, \tag{A.11}$$

which holds true because $\varphi(1) = \varphi(0) = 0$ and φ has a unique maximum at $\mathrm{e}^{-1/(1-s)}$.

Due to the symmetry $h(x) = h(1-x)$ and $g(x) = g(1-x)$ for all $x \in [0,1]$, it is sufficient to prove (A.10) for all $x \in [0,1/2]$ only. As for the upper bound in (A.10), we note that with $\psi : [0,1/2] \to [0,\infty[,\ x \mapsto -(1-x)\log_2(1-x)$, we have

$$\psi(x) \leqslant \frac{x}{\ln 2} \leqslant \frac{x^s}{\ln 2} \quad \text{for all } x \in [0,1/2], \tag{A.12}$$

because $\psi(0) = 0$ and $\psi' \leqslant 1/\ln 2$. This and (A.11) imply

$$h(x) = x^s\varphi(x) + \psi(x) \leqslant x^s\Big(\frac{1}{\ln 2} + \frac{1}{1-s}\Big) \leqslant \frac{6}{1-s}\big(x(1-x)\big)^s \tag{A.13}$$

for all $x \in [0,1/2]$.

The lower bound is well known in the literature, see, for example, [25, Eq. (8)] and references therein. But as we could not find a proof, we briefly sketch the argument here. Again, we consider only $x \in [0,1/2]$ and solve the relation $1-y := 4g(x)$ for x. The lower bound in (A.10) is therefore equivalent to

$$0 \leqslant y - 1 + h\big((1-\sqrt{y})/2\big) =: \xi(y) \qquad \text{for } y \in [0,1]. \tag{A.14}$$

We observe that $\xi(0) = 0 = \xi(1)$ and that the derivative

$$\xi'(y) = 1 - \frac{1}{4\sqrt{y}}\log_2\frac{1+\sqrt{y}}{1-\sqrt{y}} = 1 - \frac{1}{2\ln 2}\sum_{k=0}^{\infty}\frac{y^k}{2k+1} \tag{A.15}$$

is strictly decreasing for $y \in]0,1[$. Thus, ξ is strictly concave on $[0,1]$, and inequality (A.14) holds. $\square$

Lemma A.3. *For every* $x \in [0,1]$, *we have*

$$-g(x)\log_2 g(x) \leqslant h(x) \leqslant -2g(x)\log_2 g(x). \tag{A.16}$$

Proof. Since $g(x) \leqslant \min\{x, 1-x\}$ for all $x \in [0,1]$, the left inequality of the claim follows from

$$-g(x)\log_2 g(x) = -g(x)\big(\log_2 x + \log_2(1-x)\big) \leqslant h(x). \tag{A.17}$$

For the right inequality, we consider only $x \in [0, 1/2]$, which suffices by symmetry. We rewrite

$$-2g(x)\log_2 g(x) - h(x) = (1-2x)\varphi(x) \tag{A.18}$$

with $\varphi(x) := -x\log_2 x + (1-x)\log_2(1-x)$. We observe that $\varphi(x) \geqslant 0$ for all $x \in [0, 1/2]$ because $\varphi(0) = 0 = \varphi(1/2)$, it is twice differentiable on $]0, 1/2[$ and $\varphi''(x) < 0$ for all $x \in]0, 1/2[$, hence concave. Thus, the claim follows from (A.18). □

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