Monoid Extensions, Relaxed Actions and Cohomology

Peter Fredrick Faul

This thesis is submitted for the degree of Doctor of Philosophy

Peterhouse
University of Cambridge

2020

Declaration of originality

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared below.

This thesis is the synthesis of the six papers [15, 13, 14, 12, 16, 18]. Three of these papers were done in collaboration [15, 16, 18] in which the work was split evenly and no finer distinctions can be drawn. These six papers are reproduced more or less in full with some editorial changes made so as to make them appropriate for this thesis.

The most major departure is that the papers [12, 16] were combined into a single chapter. Consequently the first half of the corresponding chapter is completely original, whereas the second half was done in collaboration. A full discussion is given in the outline of the introduction. The introduction is essentially new, though the outlines of each chapter are adaptations of each papers original abstract.

It is not substantially the same as any work that has already been submitted before for any degree or other qualification.

Acknowledgements

I would like to thank my supervisor Peter Johnstone for allowing me full academic freedom to pursue my research interests as well as supplying answers to my many questions over the years. I also thank my advisor Martin Hyland for the extensive, helpful guidance he provided throughout my time at Cambridge.

I would also like to thank Andrea Montoli for both the early interest he took in my work and his many helpful suggestions. Much of this thesis would not have happened without his encouragement. I am also grateful to Victoria Gould for introducing me to inverse semigroups and more specifically λ-semidirect products, as well as for answering the many questions I posed to her on the topic.

My friend Graham Manuell also deserves special mention for our daily discussions, many of which led on to the papers comprising this thesis.

Finally, I would like to thank my parents for their constant support during this endeavour.

Contents

Chapter 1

Introduction

In category theory it is hard not to be aware of the many connections that exist between algebra and topology. After all, it was Mac Lane and Eilenberg's investigations into homology that spawned the field itself. Intuitively, homology and cohomology involve the association of algebraic invariants to topological spaces. This allows for a new perspective in which spaces are studied through the lens of algebra. (Co)homological ideas are not restricted purely to topology, with applications to graphs, toposes and even groups themselves. This latter case is of particular interest to us.

There are a number of ways to think about group cohomology. Given a group H we may consider the H-modules N. Each H-module N may naturally be viewed as a topological space. One may then compute the singular cohomology of this space in order to associate a chain of cohomology groups to N itself. These cohomology groups measure decidedly algebraic properties of the H-module N.

For instance, if α is the action of H on N associated to the H-module N, then the second cohomology group 'measures' the extensions $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ with associated action α. Here the group's identity is given by the semidirect product associated to α.

The starting point of this thesis studies a different link between topology and algebra. All the results stem from the simple observation that a certain operation between topological spaces, the Artin glueing, has a very algebraic flavour. Surprisingly, by the end of this exploration, we will have looped back to the very beginning and have applied these results to give a new theory of monoid cohomology. The basic idea is as follows.

Artin glueings between frames have much in common with semidirect products of groups. If H and N are groups (frames) then the process of constructing semidirect products (Artin glueings) provides a means of finding all objects in which N embeds as a normal (closed) subobject and H embeds as its complement. They are both characterised by functions out of H, with semidirect products given by group homomorphisms out of H into the group of automorphisms on N and Artin glueings constructed from finite-limit preserving maps from H into N.

It is natural to wonder if there is a certain sense in which these functions are doing 'the same thing'. They are not. In the group case, the function is best thought of as an action of H on N, whereas in the frame case the function is best thought of as supplying the data for an H-indexed equivalence relation on N. As it turns out, these are two orthogonal approaches to a generalised notion of semidirect product between monoids which we call a *relaxed semidirect product* of monoids.

Briefly, a relaxed semidirect product of monoids H and N is determined by an H-indexed equivalence relation E on N (satisfying certain reasonable axioms) and a function $\alpha\colon H \times N \to N$ which behaves like an action with respect to E. We call the pair (E, α) a *relaxed action* and these constitute the fundamental object of study in this thesis.

From this perspective the semidirect product of groups is a relaxed semidirect product in which the associated equivalence relation is trivial, whereas an Artin glueing of frames is a relaxed semidirect product with trivial action. A natural example of a 'mixed' relaxed semidirect product is given by λ-semidirect products of inverse semigroups where both the equivalence relation and the action are non-trivial.

We study relaxed semidirect products primarily from the point of view of extension theory. Just as semidirect products of groups correspond to split extensions, relaxed semidirect products naturally correspond to a certain class of monoid extensions, the *weakly Schreier split extensions*. These constitute a natural generalization of *Schreier split extensions* (Bourn, Martins-Ferreira, Montoli, and Sobral [8]).

At this point the theory bifurcates with two distinct paths. The one route involves developing an associated cohomology theory while the other is to generalise these results to the setting of Artin glueings of toposes. Both paths are explored in this thesis.

Outline

Artin glueings of frames In this chapter we show that Artin glueings of frames can be described as split extensions satisfying the weakly Schreier condition in the category of frames with finite-meet preserving maps. We then show that these extensions arrange into an extension bifunctor and conclude with a discussion of Baer sums and the induced order structure on extensions.

This is joint work with Graham Manuell and this chapter is a modified version of [15].

Weakly Schreier split extensions Motivated by the Artin glueing example above, in this chapter we study weakly Schreier extensions in more detail. We provide a complete characterization of the weakly Schreier extensions of H by N in terms of 'relaxed actions'. In addition, we demonstrate the failure of the split short lemma in this setting and provide a full characterization of the morphisms that occur between weakly Schreier extensions. The category is shown to be a preorder from which we may induce an order structure on the set of relaxed actions. Finally, we use the characterization to construct some new classes of examples of weakly Schreier extensions.

This chapter is a modified version of [13].

λ-semidirect products as weakly Schreier split extensions In this chapter we show that the λ-semidirect products of inverse monoids are also examples of weakly Schreier split extensions. The characterization of weakly Schreier extensions sheds some light on the structure of λ-semidirect products. The poset of relaxed actions induces an order on the set of λ-semidirect products between two inverse monoids and we then study this order structure. We show that Artin glueings are in fact λ-semidirect products and inspired by this, identify a class of Artin-like λ-semidirect products. We show that joins exist for this special class of λ-semidirect product in the aforementioned order.

This chapter is a modified version of [14].

Cosetal extensions and Baer sums In this chapter we begin constructing a monoid cohomology theory related to the theory of relaxed actions. We consider cosetal extensions $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ which are related to weakly Schreier split extensions and satisfy that whenever $e(g) = e(g')$, then there exists a (not necessarily unique) $n \in N$ such that $g = k(n)g'$.

These extensions generalise the notion of special Schreier extensions. Just as in the group case where an action could be associated to each extension with abelian kernel, we show that to each cosetal extension with abelian group kernel, we can uniquely associate a relaxed action. We may then consider the concept of a relaxed factor set, which may in turn be used to completely characterise cosetal extensions. Moreover, they may be shown to arrange into an abelian group reminiscent of a second cohomology group. This induces a Baer sum on the set of (equivalence classes of) cosetal extensions.

We then explore the non-trivial poset structure on relaxed actions and its interplay with the second cohomology groups. We find that the mapping is indeed functorial and use this result to construct an analogue of the first cohomology group.

This chapter is a concatenation of the two papers [12] and [16]. The first is solo work and the second paper is joint work with Graham Manuell beginning from Section 5.6.

Artin glueings of toposes as adjoint split extensions In this chapter we extend the results of Artin glueings of frames to the setting of toposes and show that Artin glueings of toposes correspond to a 2-categorical notion of adjoint split extensions in the 2-category of toposes, finite-limit-preserving functors and natural transformations. A notion of morphism between these split extensions is defined, allowing the category $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ to be constructed. We show that $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ is equivalent to $\mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}}$, and moreover, that this can be extended to a 2-natural contravariant equivalence between the Hom 2-functor and a naturally defined Ext 2-functor.

This work was done in collaboration with Graham Manuell and José Siqueira and the chapter is a modified version of [18].

Chapter 2

Artin glueings of frames

In this chapter we explore the similarities between Artin glueings of frames and semidirect products of groups.

2.1 Introduction

Let $H = (|H|, \mathcal{O}H)$ and $N = (|N|, \mathcal{O}N)$ be topological spaces. We might ask which topological spaces G have H as an open subspace and N as its closed complement. This is solved by the so-called *Artin glueing* construction which may be found in Chapter 9 of SGA 4 [1] and in (Wraith [45]). Let us briefly describe the intuition behind this construction.

In such a situation it is clear that $|G| = |N| \sqcup |H|$. Furthermore, each open U in G corresponds to a pair (U_N, U_H) where $U_N = U \cap N \in \mathcal{O}N$ and $U_H = U \cap H \in \mathcal{O}H$. This gives an alternative description of $\mathcal{O}G$ as the frame L_G of such pairs with the meet and join operations corresponding to componentwise intersection and union respectively.

Suppose G is such a topological space. Since H is an open subspace, we have an element $(\emptyset, U) \in L_G$ for each $U \in \mathcal{O}H$. Let V_U be the largest open in N such that $(V_U, U) \in L_G$. Such an element exists, because we can take the join of all opens V' with $(V', U) \in L_G$. Consider the function $\alpha\colon \mathcal{O}H \to \mathcal{O}N$ which sends each $U \in \mathcal{O}H$ to V_U as defined above. This map is order preserving, because if $U \subseteq W$, we can consider the join $(\alpha(U), U) \vee (\emptyset, W) = (\alpha(U), W) \leq (\alpha(W), W)$. Furthermore, if $U, W \in \mathcal{O}H$ then $(\alpha(U), U) \wedge (\alpha(W), W) = (\alpha(U) \cap \alpha(W), U \cap W)$ which implies that $\alpha(U) \cap \alpha(W) \subseteq \alpha(U \cap W)$. These two facts taken together imply that α preserves binary meets. In fact, α preserves finite meets, because it clearly preserves the top element.

The topology of G can be recovered from the meet-preserving map α. A pair (V, U) belongs to L_G if and only if $V \subseteq \alpha(U)$. The forward direction is trivial. For the backward direction, consider $V \in \mathcal{O}N$ with $V \subseteq \alpha(U)$. Since N is a subspace of G, we have $V = W \cap N$ for some $W \in \mathcal{O}G$. Setting $U' = W \cap H$, we obtain a pair $(V, U') \in L_G$. Now $(V, U' \cap U) = (\alpha(U), U) \wedge (V, U')$ is an element of L_G and hence so is $(V, U) = (V, U' \cap U) \vee (\emptyset, U)$.

Therefore, the topological spaces G obtained by glueing H and N are completely determined by finite-meet-preserving maps $\alpha \colon \mathcal{O}H \to \mathcal{O}N$ by setting $(V, U) \in L_G$ if and only if $V \subseteq \alpha(U)$. We call resulting space the Artin glueing of α.

The above argument deals only with the lattices of open sets of the topological spaces and so the construction works equally well for frames. The aim of this chapter is to explore the commonalities between this construction and the semidirect product of groups.

Let us recall some basic properties of semidirect products. Given a group homomorphism $\alpha \colon H \to \mathrm{Aut}(N)$, we can construct a group G satisfying:

i) $H \leq G$ and $N \lhd G$,

ii) $H \vee N = G$,

iii) $H \cap N = \{e\}$.

We see here a vague analogy between the semidirect products of groups and the Artin glueings of frames. In both cases we have objects H and N which we want to embed as complemented 'subobjects' (sublocales in the frame case) of some other object, with N normal in the group case and closed in the frame case. In both cases these constructions are entirely determined by certain structure-preserving maps involving N and H.

In order to make this analogy precise, we look at the characterisation of semidirect products of groups as the solutions to the split extension problem. A split extension of groups is a diagram of the form

$$N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H,$$

where k is the kernel of e, e is the cokernel of k, and s is a section of e. Here G will always be a semidirect product of H and N, and the maps k and s will be the appropriate inclusions into the semidirect product. (Note that throughout this thesis $\rightarrowtail$ denotes a normal monomorphism and $\twoheadrightarrow$ denotes a normal epimorphism).

Furthermore, for each group N there is a functor $\mathrm{SplExt}(-, N)\colon \mathrm{Grp}^{\mathrm{op}} \to \mathrm{Set}$ which sends a group H to the set of split extensions of H by N. This functor is naturally isomorphic to $\mathrm{Hom}(-, \mathrm{Aut}(N))$. For more details on this functor, see (Borceux, Janelidze, and Kelly [4]) and (Borceux, Janelidze, and Kelly [5]).

Split extensions of groups are very well behaved and the notion of *pointed protomodular category* provides a general setting in which they can be studied (Bourn and Janelidze [7]) and (Borceux and Bourn [3]). Sometimes, however, only some of the split extensions in a category are well behaved and this motivates the more general idea of $\mathcal{S}$-protomodularity (Bourn, Martins-Ferreira, Montoli, and Sobral [8]), where $\mathcal{S}$ can be thought of as a collection of split extensions.

It is in this vein that we study Artin glueings. It is not possible to talk about extensions in the usual category of frames (or locales), as without zero morphisms it does not make sense to talk about kernels and cokernels. Instead we move to the category $\mathrm{Frm}_\wedge$ which has frames as objects and finite-meet preserving maps as morphisms.

The relationship between Frm and $\mathrm{Frm}_\wedge$ is similar to the relationship between the category Set of sets and functions and the category Rel of sets and relations. In particular, $\mathrm{Frm}_\wedge$ is order-enriched and we can find the frame homomorphisms inside it as the left adjoints. In this way $\mathrm{Frm}_\wedge$ provides a proarrow equipment for Frm. This category has been used alongside glueings in (Niefield [37]). The category $\mathrm{Frm}_\wedge$ can also be thought of as the category of injective meet-semilattices (Bruns and Lakser [9]), though we are less sure of the implications of this.

We concern ourselves with the collection of split extensions of the form described above, but where k and s are required to satisfy a ‘Schreier’-type condition (Martins-Ferreira, Montoli, and Sobral [33]) (or equivalently, where s is required to be right adjoint to e). We find that G will always be an Artin glueing of H and N determined by the map k^*s. As one might expect, there is a family of functors $\mathrm{AdjExt}(-, N)$ here too, but now these extend to a bifunctor AdjExt, which is naturally isomorphic to $\mathrm{Hom}\colon \mathrm{Frm}_\wedge^{\mathrm{op}} \times \mathrm{Frm}_\wedge \to \mathrm{Set}$. The fact that hom-sets have a natural meet-semilattice structure gives a notion of Baer sum of the extensions. We also study the induced order structure on the extensions.

2.2 Background

A frame is an algebraic structure that captures the lattice of open sets of a topological space. A frame has finite meet operations capturing finite intersections of open sets, and arbitrary joins corresponding to arbitrary unions of opens. Finally we require meets distribute over arbitrary joins. For a more comprehensive look at frames, see (Picado and Pultr [41]).

Definition 2.2.1. A *frame* L is a poset with finite meets and arbitrary joins such that finite meets distribute over joins. △

For any two elements x and y in a frame there exists the exponential x^y.

We treat frames as algebraic structures and so the morphisms are just the maps preserving this structure.

Definition 2.2.2. A *morphism* $f\colon L \to M$ *of frames* satisfies

i) $f(0) = 0$,

ii) $f(1) = 1$,

iii) $f(a \wedge b) = f(a) \wedge f(b)$,

iv) $f(\bigvee S) = \bigvee f(S)$. △

Let Frm be the category of frames and frame homomorphisms.

Given a continuous map between two topological spaces, we know that the preimage sends opens to opens and from set theoretic properties of the preimage, preserves the empty set, the whole space, finite intersections and arbitrary unions. That is, the preimage is a frame homomorphism between the corresponding lattices of open sets.

This idea gives rise to a contravariant functor from the category of topological spaces to the category of frames. Furthermore, the category of frames is seen to be a subcategory of the category of monoids, where a frame $(L, \wedge, \vee, 1, 0)$ is thought of as the monoid $(L, \wedge, 1)$. (To see that this is injective on objects note that the finite meet operation determines the order structure and consequently the joins). Note that it is not a full subcategory, as general monoid homomorphisms need not preserve the joins. Thus, we obtain a functor from the category of topological spaces into the category of monoids. This will be important when we generalise these ideas in later chapters.

It is often convenient to consider the opposite category Frm^{op} as in it the morphisms will go in the same direction as in the category of topological spaces. Object of this category, while still frames, are called locales.

In $\mathrm{Frm}^{\mathrm{op}}$ we call the regular monomorphisms sublocales and they correspond intuitively to subspaces in the topological setting. Just as there are open and closed subspaces, there is a concept of open and closed sublocales. For our purposes it is enough to note that if u is an element of a frame then the associated closed sublocale is given by the right inclusion of $\uparrow u$ whereas the open sublocale is given by the map $s\colon \downarrow u \to G$ sending x to x^u.

2.3 An extension problem in $\mathrm{Frm}_\wedge$

2.3.1 Adjoint extensions

In all that follows N, G and H denote frames unless otherwise stated.

The category $\mathrm{Frm}_\wedge$ of frames and finite-meet preserving maps is enriched over meet-semilattices and so between any two frames L and M there is a largest meet-preserving map. This map is the constant 1 map, which sends each element of L to the top element of M. It is apparent that composing with this map on either side again yields a constant 1 map and so we see that these maps are the zero morphisms of our category. Due to this somewhat unfortunate conflict of terminology we use $\top_{L,M}$ to refer to the zero morphism between L and M or just $\top$ when its meaning is unambiguous.

We can now define the *kernel* of a morphism f as the equaliser of f and $\top$ and the *cokernel* of f as the coequaliser of f and $\top$.

Definition 2.3.1. A diagram of the form

$$N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$$

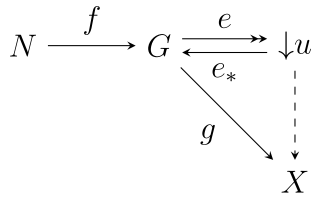
is called a *split extension* if k is the kernel of e, e is the cokernel of k and s is a section of e. It is called an *adjoint extension* if furthermore, s is right adjoint to e.△

Kernels do not always exist in this category; however, cokernels always do exist in the form described below.

Proposition 2.3.2. *Let $f\colon N \to G$ be a morphism in $\mathrm{Frm}_\wedge$ and let $u = f(0)$. The cokernel of f is given by $e\colon G \twoheadrightarrow {\downarrow}u$, where $e(x) = x \wedge u$. Furthermore, e has a right adjoint section given by $e_*(y) = y^u$.*

Proof. The right adjoint to e exists by well-known properties of frames. Since e is surjective, e_* splits e by general properties of adjoints.

Clearly e composes with f to give $\top$ and so we need only check that it satisfies the universal property. Suppose $g\colon G \to X$ composes with f to give $\top_{N,X}$. In order to show that e is the cokernel of f we must show that g factors through e to give a unique map. Uniqueness is automatic, because e is epic.



To show that the meet-semilattice homomorphism g factors through the surjection e, it is enough to show that $g(x) = g(y)$ whenever $e(x) = e(y)$. But if $e(x) = e(y)$, then $x \wedge f(0) = y \wedge f(0)$ and so $g(x) = g(x) \wedge 1 = g(x) \wedge g(f(0)) = g(x \wedge f(0)) = g(y \wedge f(0))$, which equals $g(y)$ by running the same argument in reverse. $\square$

Definition 2.3.3. We say a morphism is a *normal epimorphism* if it occurs as the cokernel of some morphism. Dually, a monomorphism is a *normal monomorphism* if it occurs as the kernel of some morphism. $\triangle$

Proposition 2.3.2 shows that every normal epimorphism is of the form $- \wedge u \colon G \twoheadrightarrow {\downarrow} u$. Conversely, such a morphism is always a normal epimorphism as it is clearly seen to be the cokernel of the inclusion of ${\uparrow} u \subseteq G$. Note that normal epimorphisms in $\mathrm{Frm}_\wedge$ are precisely the open frame quotients, which accords well with the idea that H should be an open sublocale of the glueing.

While $\mathrm{Frm}_\wedge$ does not possess all kernels, this is not a problem for working with extensions, since kernels of normal epimorphisms always do exist.

Proposition 2.3.4. *Let $e \colon G \twoheadrightarrow {\downarrow} u$ be a normal epimorphism in* $\mathrm{Frm}_\wedge$*. The kernel of e is given by $k \colon {\uparrow} u \rightarrowtail G$, where $k(x) = x$. The map k has a left adjoint $k^*(x) = x \vee u$ which preserves finite meets.*

Proof. Let $f \colon X \to G$ compose with e to give $\top_{X,H}$. The only elements sent by e to 1 lie in ${\uparrow} u$ and so the image of f is contained in ${\uparrow} u$. The restriction of f to ${\uparrow} u$ shows the existence condition for the universal property, while uniqueness follows since k is monic. It is then easy to see that $x \mapsto x \vee u$ provides a left adjoint to k. $\square$

It is well known that every normal monomorphism is the kernel of its cokernel. Thus the previous result implies that a map is a normal monomorphism if and only if it is of the form ${\uparrow} u \hookrightarrow G$. In other words, the normal monomorphisms in $\mathrm{Frm}_\wedge$ are precisely the right adjoints of closed frame quotients.

We now have a good understanding of both the kernel and cokernel maps in a split extension. It is only the splitting s that remains mysterious. We will need to impose some conditions on the splitting in order to obtain a well-behaved theory.

Notice that for a split extension of groups, every element of G is of the form $k(n)s(h)$ for some $n \in N$ and $h \in H$. For split extensions of monoids this condition does not hold automatically, but when assumed explicitly gives rise to the class of *weakly Schreier* extensions (Bourn [6]). This is the condition we will impose — that is, we assume that every element of the frame G is of the form $k(n) \wedge s(h)$. In fact, as we will see below, there is a *canonical* choice of n and h, since h is uniquely determined and we may choose n to be as large as possible. This actually resembles the *Schreier* condition, a stronger condition on split extensions of monoids that requires n (and h) to be unique (Bourn, Martins-Ferreira, Montoli, and Sobral [8]).

The following result shows that under our Schreier-type condition, the splitting is uniquely determined by e.

Proposition 2.3.5. *A split extension* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *in* $\mathrm{Frm}_\wedge$ *is weakly Schreier if and only if it is an adjoint extension.*

Proof. By Propositions 2.3.2 and 2.3.4 we may take N to be of the form $\uparrow u$ and H to be $\downarrow u$ for some element $u \in G$. Then $k(x) = x$ and $e(x) = x \wedge u$.

Suppose the extension is weakly Schreier and consider an element $g \in G$. By assumption, $g = k(n) \wedge s(h)$ for some $n \in N$ and some $h \in H$. Then $e(g) = ek(n) \wedge es(h) = 1 \wedge h = h$. Thus, $g = k(n) \wedge se(g)$. In particular, $g \leq se(g)$ and so $\mathrm{id}_G \leq se$. But $es = \mathrm{id}_H \leq \mathrm{id}_H$ and so s is right adjoint to e as required.

For the other direction suppose s is the right adjoint of e so that $s(x) = e_*(x) = x^u$. We must show that each element of $g \in G$ can be expressed as $k(n) \wedge e_*(h)$ for some $n \in N$ and $h \in H$. By the above we may take $h = e(g)$, while $k^*(g)$ is the most natural candidate for n. Taking the meet yields $kk^*(g) \wedge e_*e(g) = (g \vee u) \wedge (g \wedge u)^u = (g \wedge (g \wedge u)^u) \vee (u \wedge (g \wedge u)^u)$. Since $g \leq (g \wedge u)^u$, we have that $g \wedge (g \wedge u)^u = g$. Furthermore, we have $u \wedge (g \wedge u)^u \leq g \wedge u$. Thus, $kk^*(g) \wedge e_*e(g) = g$ as required. □

This means that the information of weakly Schreier split extensions $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is contained in the normal epi e, since k can be recovered as its kernel and s as its right adjoint. Since every normal epi gives rise to a weakly Schreier split extension, we have a complete classification of the weakly Schreier split extensions in $\mathrm{Frm}_\wedge$. From now on, we will refer to this class of split extensions as the *adjoint extensions*.

Let $\mathcal{S}$ be the class of normal epimorphisms $\mathrm{Frm}_\wedge$ equipped with their left adjoints. It is clear that all isomorphisms belong to $\mathcal{S}$ and we will show $\mathcal{S}$ is stable under pullback in Proposition 2.4.1. Finally, for any such map, the adjoint and the kernel are jointly extremally epic by Proposition 2.3.5. Thus, except for the requirement of finite-completeness, the pointed category $\mathrm{Frm}_\wedge$ is $\mathcal{S}$-protomodular in the sense of (Bourn [6]).

Notice also that any *extension* in $\mathrm{Frm}_\wedge$ — that is, a diagram $N \xrightarrow{k} G \xrightarrow{e} H$ in which e is the cokernel of k, and k is the kernel of e — gives rise to a unique adjoint extension and vice versa. Thus the adjoint extension problem coincides with the extension problem in $\mathrm{Frm}_\wedge$.

2.3.2 Artin glueings

In the introduction we described the Artin glueing construction in the context of topological spaces. We now explore this construction in the current context.

Definition 2.3.6. The *Artin glueing* of two frames N and H along a finite-meet preserving map $\alpha\colon H \to N$ is given by the frame

$$\mathrm{Gl}(\alpha) = \{(n, h) \in N \times H \mid n \leq \alpha(h)\}$$

equipped with projections $\pi_1\colon \mathrm{Gl}(\alpha) \to N$ and $\pi_2\colon \mathrm{Gl}(\alpha) \to H$. Here the finite meet and arbitrary join operations in $\mathrm{Gl}(\alpha)$ are taken componentwise. △

Since meets in $\mathrm{Gl}(\alpha)$ are computed componentwise, we see that π_1 and π_2 preserve finite meets and so are morphisms in $\mathrm{Frm}_\wedge$. In fact, they both have right adjoints in $\mathrm{Frm}_\wedge$. The right adjoint of π_1 is given by $\pi_{1*}(n) = (n, 1)$ and the right adjoint of π_2 is given by $\pi_{2*}(h) = (\alpha(h), h)$.

With these right adjoints, Artin glueings give rise to adjoint extensions.

Proposition 2.3.7. *Let $\alpha\colon H \to N$ be a finite-meet preserving map. The diagram $N \xrightarrow{\pi_{1*}} \mathrm{Gl}(\alpha) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} H$ is an adjoint extension.*

Proof. It is enough to show that π_2 is a normal epi and that π_{1*} is its kernel. Observe that H is isomorphic to ${\downarrow}(0, 1) \subseteq \mathrm{Gl}(\alpha)$ via the map $h \mapsto (0, h)$ and that this makes the following diagram commute.

$$\begin{array}{ccc} \mathrm{Gl}(\alpha) & \xrightarrow{\pi_2} & H \\ & {\scriptstyle - \wedge (0,1)} \searrow & \downarrow \wr \\ & & {\downarrow}(0, 1) \end{array}$$

The map $- \wedge (0,1)$ is a normal epi and hence so is π_2. Furthermore, the kernel of $- \wedge (0,1)$ is $\uparrow(0,1) \hookrightarrow \mathrm{Gl}(\alpha)$, which is clearly isomorphic to $\pi_{1*} \colon N \to \mathrm{Gl}(\alpha)$. $\square$

Notice that α can be recovered from the glueing by considering the composite $\pi_1 \pi_{2*} = \alpha$. In fact, any extension $N \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} H$ gives rise a finite-meet preserving map $k^* e_*$ and we may glue along this map to obtain another extension as above.

In order to compare the original adjoint extension to the one given by the glueing of $k^* e_*$ we define a morphism of adjoint extensions to be a meet-preserving map $f \colon G \to G'$ making the three squares in the following diagram commute.

$$\begin{array}{ccccc} N & \overset{k}{\rightarrowtail} & G & \underset{e_*}{\overset{e}{\rightleftarrows}} & H \\ \Vert & & \downarrow f & & \Vert \\ N & \overset{k'}{\rightarrowtail} & G' & \underset{e'_*}{\overset{e'}{\rightleftarrows}} & H \end{array}$$

Explicitly, we require $fk = k'$, $e'f = e$ and $fe_* = e'_*$. It is apparent that isomorphisms of adjoint extensions are those for which the meet-preserving map is an isomorphism.

Theorem 2.3.8. *For frames N, G and H, the adjoint extension $N \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} H$ is isomorphic to the Artin glueing extension $N \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(k^* e_*) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} H$.*

Proof. Without loss of generality we may assume $N = \uparrow u$, $H = \downarrow u$, $k(x) = x$, $e(x) = x \wedge u$ and $e_*(x) = x^u$.

Consider the maps $f \colon G \to \mathrm{Gl}(k^* e_*)$ and $f' \colon \mathrm{Gl}(k^* e_*) \to G$ given by $f(g) = (k^*(g), e(g))$ and $f'(n,h) = k(n) \wedge e_*(h)$. Notice that the map f is well defined because $g \leq e_* e(g)$ and so $k^*(g) \leq k^* e_* e(g)$. Clearly both f and f' preserve finite meets.

$$\begin{array}{ccccc} N & \overset{k}{\rightarrowtail} & G & \underset{e_*}{\overset{e}{\rightleftarrows}} & H \\ \Vert & & f' \uparrow\downarrow f & & \Vert \\ N & \overset{\pi_{1*}}{\rightarrowtail} & \mathrm{Gl}(k^* e_*) & \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} & H \end{array}$$

We claim these maps are inverses. First note $f'f(g) = kk^*(g) \wedge e_* e(g) = g$, where the final equality follows as in the proof of Proposition 2.3.5. Next we have $ff'(n,h) = (k^*(k(n) \wedge e_*(h)), e(k(n) \wedge e_*(h))) = (k^* k(n) \wedge k^* e_*(h), ek(n) \wedge ee_*(h))$. Notice that $ek(n) = 1$ and $ee_*(h) = h$ and so the second component is h as required. Next observe that $k^*k(n) = n$ and $n \leq k^*e_*(h)$, since $(n, h) \in \mathrm{Gl}(k^*e_*)$. This gives that the meet in the first component is n as required. Thus G and $\mathrm{Gl}(k^*e_*)$ are isomorphic in $\mathrm{Frm}_\wedge$.

It remains to show that f makes the appropriate squares commute. Firstly, $fk(n) = (k^*k(n), ek(n)) = (n, 1) = \pi_{1*}(n)$ as required. Next notice $\pi_2 f(g) = \pi_2(k^*(g), e(g)) = e(g)$ as required. Finally, $fe_*(h) = (k^*e_*(h), ee_*(h)) = (k^*e_*(h), h) = \pi_{2*}(h)$ as required. □

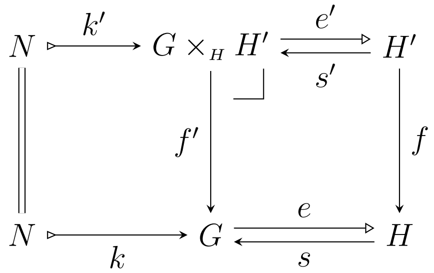
So similarly to the case of groups where the split extensions could be identified with maps $\alpha\colon H \to \mathrm{Aut}(N)$, we see that the adjoint extensions between H and N correspond to finite-meet preserving maps $\beta\colon H \to N$.

2.4 Extension functors

2.4.1 Functoriality of adjoint extensions

In the category of groups (or any protomodular category with semidirect products — see (Bourn and Janelidze [7]) and (Borceux, Janelidze, and Kelly [4])) there is a functor $\mathrm{SplExt}(-, N)$ for each object N which sends an object H to the set of isomorphism classes of split extensions of H by N. This functor acts on morphisms by pullback as described below.

Let $f\colon H' \to H$ be a morphism and suppose $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ is a split extension of groups. Consider the pullback e' of e along f.



The kernel k' of e' has domain N and together with e' forms an extension. The universal property of the pullback then gives a map s' as a canonical choice of splitting of e'. We define $\mathrm{SplExt}(f, N)\colon \mathrm{SplExt}(H, N) \to \mathrm{SplExt}(H', N)$ to be the function sending the original extension to the one described above.

We now show that the adjoint extensions in $\mathrm{Frm}_\wedge$ have associated functors functors $\mathrm{AdjExt}(-, N)$ in a similar way.

Proposition 2.4.1. *Let $e\colon G \twoheadrightarrow H$ be a normal epimorphism and k its kernel. The pullback of e along an arbitrary morphism $f\colon H' \to H$ exists and is a normal epimorphism of the form $\pi_2'\colon \mathrm{Gl}(k^*e_*f) \twoheadrightarrow H'$.*

Proof. We may compute the pullback in the category SLat of meet-semilattices and to observe that it is a frame. By Theorem 2.3.8 we can take the cokernel e to be $\pi_2\colon \mathrm{Gl}(k^*e_*) \to H$. Since SLat is algebraic, the pullback of π_2 and f is given as usual by $\mathrm{Gl}(k^*e_*) \times_H H' = \{((n,h),h') \in \mathrm{Gl}(k^*e_*) \times H' : \pi_2(n,h) = f(h')\}$ with projections p_1 and p_2.

$$\begin{array}{ccc} \mathrm{Gl}(k^*e_*) \times_H H' & \xrightarrow{p_2} & H' \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathrm{Gl}(k^*e_*) & \xrightarrow[\pi_2]{} & H \end{array}$$

Because $\pi_2(n,h) = h$, the element $((n,h),h')$ belongs to the pullback if and only if $f(h') = h$. Combining this with the fact that $(n,h) \in \mathrm{Gl}(k^*e_*)$ we have that $n \le k^*e_*f(h')$. It is then easy to see that $((n,h),h')$ belongs to the pullback if and only if $(n,h') \in \mathrm{Gl}(k^*e_*f)$. This induces an obvious isomorphism giving that $\pi_2'\colon \mathrm{Gl}(k^*e_*f) \to H'$ is the pullback of e along f. This map is a normal epi by Proposition 2.3.7. □

This result allows us to define a family of functors $(\mathrm{AdjExt}(-,N))_{N\in\mathrm{Frm}_\wedge}$, where $\mathrm{AdjExt}(H,N)$ is the set of isomorphism classes of adjoint extensions of H by N and $\mathrm{AdjExt}(f,N)\colon \mathrm{AdjExt}(H,N) \to \mathrm{AdjExt}(H',N)$ sends the adjoint extension $N \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} H$ to $N \overset{\pi_{1*}'}{\rightarrowtail} \mathrm{Gl}(k^*e_*f) \underset{\pi_{2*}'}{\overset{\pi_2'}{\rightleftarrows}} H'$.

Recall that in the case of groups $\mathrm{SplExt}(-,N)$ is representable with $\mathrm{Aut}(N)$ as the representing object. Proposition 2.4.1 and Theorem 2.3.8 together give a similar result in our setting.

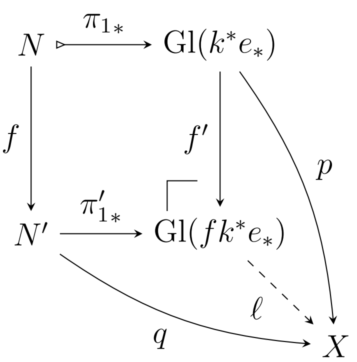
Corollary 2.4.2. *For each frame N, the functor $\mathrm{AdjExt}(-,N)\colon \mathrm{Frm}_\wedge^{\mathrm{op}} \to \mathrm{Set}$ is naturally isomorphic to $\mathrm{Hom}(-,N)\colon \mathrm{Frm}_\wedge^{\mathrm{op}} \to \mathrm{Set}$.*

In fact, things work even better in our situation than in the group case. It is natural to ask whether the family of functors $(\mathrm{SplExt}(-,N))_{N\in\mathrm{Grp}}$ assemble into a bifunctor $\mathrm{SplExt}\colon \mathrm{Grp}^{\mathrm{op}} \times \mathrm{Grp} \to \mathrm{Set}$. In particular, we could ask if there is a functor $\mathrm{SplExt}(H,-)$ for each group H. The functor $\mathrm{SplExt}(-,N)$ is computed by taking a pullback, so we might expect $\mathrm{SplExt}(H,-)$ is given by a pushout. However, this does not work as smoothly as in the former case. In fact, $(\mathrm{SplExt}(-, N))_{N \in \mathrm{Grp}}$ cannot be extended to a bifunctor at all, as can be seen from the isomorphism $\mathrm{SplExt}(-, N) \cong \mathrm{Hom}(-, \mathrm{Aut}(N))$, the Yoneda lemma and the fact that $\mathrm{Aut}(-)$ cannot be extended to a functor.

However, in the frame case the isomorphism $\mathrm{AdjExt}(-, N) \cong \mathrm{Hom}(-, N)$ shows that the family $(\mathrm{AdjExt}(-, N))_{N \in \mathrm{Frm}_\wedge}$ does extend to a bifunctor in an obvious way. Below we show that unlike in the group case the pushout construction for $\mathrm{AdjExt}(H, -)$ succeeds.

Proposition 2.4.3. *Let $k \colon N \rightarrowtail G$ be a normal monomorphism and $e \colon G \twoheadrightarrow H$ its cokernel. Let $f \colon N \to N'$ be a morphism. The pushout of k along f exists and is a normal monomorphism of the form $\pi'_{1*} \colon N' \rightarrowtail \mathrm{Gl}(fk^*e_*)$.*

Proof. By Theorem 2.3.8 we may take k to be the map $\pi_{1*} \colon N \to \mathrm{Gl}(k^*e_*)$. We claim that the pushout of π_{1*} and f is given by $\mathrm{Gl}(fk^*e_*)$ with the injections $\pi'_{1*} \colon N' \to \mathrm{Gl}(fk^*e_*)$ and $f' \colon \mathrm{Gl}(k^*e_*) \to \mathrm{Gl}(fk^*e_*)$, where $f'(n,h) = (f(n), h)$, as in the diagram below.



Firstly note that f' is well defined, since if $n \leq k^*e_*(h)$, then $f(n) \leq fk^*e_*(h)$. Next notice that $f'\pi_{1*}(n) = f'(n, 1) = (f(n), 1) = \pi'_{1*}f(n)$ and so the diagram commutes.

We will show the uniqueness condition of the universal property first. Suppose $p \colon \mathrm{Gl}(k^*e_*) \to X$ and $q \colon N' \to X$ together form a cocone and that there is a morphism $\ell \colon \mathrm{Gl}(fk^*e_*) \to X$ such that $\ell f' = p$ and $\ell\pi'_{1*} = q$. This means that $\ell(f(n), h) = p(n, h)$ and $\ell(n, 1) = q(n)$. Notice that each element $(n, h) \in \mathrm{Gl}(fk^*e_*)$ can be written as $(n, h) = (fk^*e_*(h), h) \wedge (n, 1)$ and so ℓ is uniquely determined by $\ell(n, h) = \ell(fk^*e_*(h), h) \wedge \ell(n, 1) = p(k^*e_*(h), h) \wedge q(n)$.

Note that the map ℓ as defined above does indeed preserve finite meets. So we need only show that p and q factor through ℓ as required. For q we have the simple equality $\ell\pi'_{1*}(n) = \ell(n, 1) = p(1, 1) \wedge q(n) = q(n)$.

For p we must show that $\ell(f(n), h) = p(n, h)$. We already know that $\ell(f(n), h) =$

$p(fk^*e_*(h),h) \wedge q(f(n))$. Because p and q form a cocone, we have that $q(f(n)) = p\pi_{1*}(n) = p(n,1)$. Substituting this in we get $\ell(f(n),h) = p(fk^*e_*(h),h) \wedge p(n,1) = p(fk^*e_*(h),h) \wedge (n,1)) = p(n,h)$ as required. □

This allows us to define a functor $\mathrm{AdjExt}(H,-)$ for each frame H. For a map $f\colon N \to N'$, the resulting map $\mathrm{AdjExt}(H,f)$ sends the adjoint extension $N \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} H$ to $N \overset{\pi'_{1*}}{\rightarrowtail} \mathrm{Gl}(k^*e_*f) \underset{\pi'_{2*}}{\overset{\pi'_2}{\rightleftarrows}} H'$. We then have the following corollary as above.

Corollary 2.4.4. *The functor* $\mathrm{AdjExt}(H,-)\colon \mathrm{Frm}_\wedge \to \mathrm{Set}$ *is naturally isomorphic to* $\mathrm{Hom}(H,-)\colon \mathrm{Frm}_\wedge^{\mathrm{op}} \to \mathrm{Set}$.

To show the families $(\mathrm{AdjExt}(-,N))_{N\in\mathrm{Frm}_\wedge}$ and $(\mathrm{AdjExt}(H,-))_{H\in\mathrm{Frm}_\wedge}$ yield a bifunctor, we only need that $\mathrm{AdjExt}(H',g)\mathrm{AdjExt}(f,N) = \mathrm{AdjExt}(f,N')\mathrm{AdjExt}(H,g)$ and to set $\mathrm{AdjExt}(f,g)$ to be their common value. By Corollaries 2.4.2 and 2.4.4 we know that each family is isomorphic to hom functors, which clearly satisfy the condition and hence AdjExt is a bifunctor naturally isomorphic to Hom. We record this in the following theorem.

Theorem 2.4.5. *The bifunctor* $\mathrm{AdjExt}\colon \mathrm{Frm}_\wedge^{\mathrm{op}} \times \mathrm{Frm}_\wedge \to \mathrm{Set}$ *where* $\mathrm{AdjExt}(H,N)$ *is the set of isomorphism classes of adjoint extensions of H by N and* $\mathrm{AdjExt}(f,N)$ *is given by pullback along f and* $\mathrm{AdjExt}(H,g)$ *is given by pushout along g as above is naturally isomorphic to* $\mathrm{Hom}\colon \mathrm{Frm}_\wedge^{\mathrm{op}} \times \mathrm{Frm}_\wedge \to \mathrm{Set}$.

2.4.2 The enriched Ext functor

Recall that an extension is a diagram $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ in which k is the kernel of e and e is the cokernel of k. As discussed at the end of Section 2.3.1, every extension in $\mathrm{Frm}_\wedge$ admits a unique splitting that turns it into an adjoint extension. Thus far we have mainly focused on describing the elements of $\mathrm{AdjExt}(H,N)$ as isomorphism classes of adjoint extensions in order to make an analogy with split extensions of groups. But describing it in terms of *extensions* instead allows us to make a different analogy — this time to extensions of *abelian* groups. With this in mind we write Ext for the bifunctor isomorphic to AdjExt, but which returns isomorphism classes of extensions instead of adjoint extensions.

In an abelian category, the Ext functor admits a natural abelian group structure. Here the binary operation is called the *Baer sum* of extensions. More generally, in any category with biproducts every object has a unique commutative monoid structure and so any product-preserving Set-valued functor on such a category factors through the category of commutative monoids. When applied to the Ext functor of an abelian category, this yields the Baer sum operation. In our situation, Ext preserves finite products (as it is isomorphic to Hom) and this construction endows $\mathrm{Ext}(H, N)$ with the structure of a meet-semilattice.

Of course, we can obtain the same result more directly by applying the isomorphism $\mathrm{Ext} \cong \mathrm{Hom}$ and using the natural meet-semilattice structure on the hom-sets. Explicitly, this gives that $N \overset{\pi_{1*}}{\rightarrowtail} N \times H \overset{\pi_2}{\twoheadrightarrow} H$ is the top element and the meet of extensions $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$ and $N \overset{k_2}{\rightarrowtail} G_2 \overset{e_2}{\twoheadrightarrow} H$ is given by $N \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(k_1^* e_{1*} \wedge k_2^* e_{2*}) \overset{\pi_2}{\twoheadrightarrow} H$. We can then apply the following proposition to give a particularly concrete description of the meet.

Proposition 2.4.6. *Let $\alpha, \beta \colon H \to N$ be meet-semilattice homomorphisms. Then $\mathrm{Gl}(\alpha \wedge \beta)$ is given by the intersection of $\mathrm{Gl}(\alpha)$ and $\mathrm{Gl}(\beta)$ as sub-meet-semilattices of $N \times H$ in the obvious way.*

Proof. Simply observe that

$$\begin{aligned} \mathrm{Gl}(\alpha \wedge \beta) &= \{(n, h) \in N \times H \mid n \leq \alpha(h) \wedge \beta(h)\} \\ &= \{(n, h) \in N \times H \mid n \leq \alpha(h)\} \cap \{(n, h) \in N \times H \mid n \leq \beta(h)\} \\ &= \mathrm{Gl}(\alpha) \cap \mathrm{Gl}(\beta), \end{aligned}$$

where the final intersection is taken in $N \times H$. □

It is also interesting to consider the ordering of extensions induced by this meet-semilattice structure.

Corollary 2.4.7. *If $\alpha \leq \beta$, then there is an obvious inclusion $\mathrm{Gl}(\alpha) \hookrightarrow \mathrm{Gl}(\beta)$.*

So whenever an extension $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$ is less than or equal to another extension $N \overset{k_2}{\rightarrowtail} G_2 \overset{e_2}{\twoheadrightarrow} H$ in $\mathrm{Ext}(N, H)$, we can view G_1 as a subset of G_2. In fact, we obtain a morphism of extension in the following sense.

Definition 2.4.8. The category $\overline{\mathrm{Ext}}(H, N)$ has extensions of H by N as objects and commutative diagrams of the form

$$\begin{array}{ccccc} N & \overset{k}{\rightarrowtail} & G & \overset{e}{\twoheadrightarrow} & H \\ \| & & \downarrow{\scriptstyle f} & & \| \\ N & \underset{k'}{\rightarrowtail} & G' & \underset{e'}{\twoheadrightarrow} & H \end{array}$$

as morphisms. The meet-semilattice structure on $\mathrm{Hom}(G, G')$ induces a meet-semilattice structure on the hom-sets of $\overline{\mathrm{Ext}}(H, N)$ and so, in particular, the category is order-enriched. △

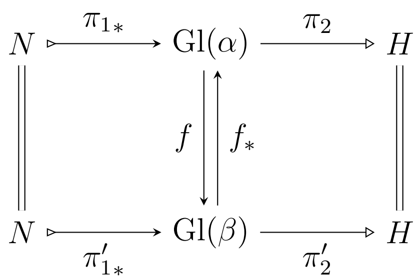
The following proposition describes precisely which morphisms of extensions are induced by the order on $\mathrm{Ext}(H, N)$.

Theorem 2.4.9. *There is an equivalence of categories between the underlying poset of* $\mathrm{Ext}(H, N)$ *and the category of adjunctions of* $\overline{\mathrm{Ext}}(H, N)$ *(where we take the morphisms to be in the direction of the left adjoint).*

Proof. Let $\alpha, \beta \colon H \to N$ be meet-semilattice homomorphisms such that $\alpha \le \beta$. Then there is a corresponding inclusion $i \colon \mathrm{Gl}(\alpha) \hookrightarrow \mathrm{Gl}(\beta)$ as in Corollary 2.4.7. It is easy to see that this is a morphism of extensions. It is also a frame homomorphism and thus has a right adjoint i_*. This adjoint sends (n, h) to $(n \wedge \alpha(h), h)$ and we quickly see that this is also a morphism of extensions. Thus, i is a left adjoint in the 2-category $\overline{\mathrm{Ext}}(H, N)$.

This procedure defines the action on morphisms of a functor from $\mathrm{Ext}(H, N)$ to the category of adjunctions of $\overline{\mathrm{Ext}}(H, N)$. This functor is essentially surjective by Theorem 2.3.8 and it is automatically faithful, since $\mathrm{Ext}(H, N)$ is a poset. We now show it is full.

Suppose $f \colon \mathrm{Gl}(\alpha) \to \mathrm{Gl}(\beta)$ is a morphism of extensions and that f has a right adjoint which is also a morphism of extensions, as shown in the following diagram.



We must show that f is the necessary inclusion. From the diagram $\pi'_2 f = \pi_2$ and $f_* \pi'_{1*} = \pi_{1*}$. Taking left adjoints of the latter condition gives $\pi'_1 f = \pi_1$. Then this together with the former condition implies $f(n, h) = (n, h)$, as required. □

Notice that we can construct the 'inverse' equivalence in the above proof without using the axiom of choice, since every equivalence class of extensions has a canonical representative in the form of a glueing.

Theorem 2.4.9 shows that there are often a number of non-trivial morphisms between extensions in $\mathrm{Frm}_\wedge$. This is in contrast to the situation with groups where these are forbidden by the short five lemma.

Chapter 3

Weakly Schreier split extensions

As discussed in the previous chapter, Artin glueings of frames may naturally be thought of as the class of weakly Schreier split extensions. In this chapter we give a treatment of weakly Schreier split extensions in the category of monoids, providing a full characterization in terms of *relaxed actions*. We then study some properties of these relaxed actions.

3.1 Introduction

It is well understood that for groups H and N, the semidirect product construction provides an equivalence between actions of H on N and split extensions of H by N. The same cannot be said when H and N are replaced with monoids; however, monoid actions do correspond naturally to a certain class of split extensions of monoids: the Schreier split extensions. These split extensions of monoids were first alluded to in (Patchkoria [40]) and were first studied explicitly in (Martins-Ferreira, Montoli, and Sobral [33]), where their relationship to actions was established. We briefly sketch one direction of this relationship below.

A *Schreier split extension* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is a split extension in which for all $g \in G$, there exist unique $n \in N$ such that $g = k(n) \cdot se(g)$. When n is not required to be unique, we call the split extension *weakly Schreier*. These were first considered in (Bourn [6]) and are equivalent to the quasi-decompositions studied by Kohler in (Köhler [25]).

Given a Schreier split extension $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$, we can associate to it a set-theoretic map q which satisfies that for all $g \in G$, $g = kq(g) \cdot se(g)$. From q we can construct an action $\alpha\colon H \times N \to N$ where $\alpha(h, n) = q(s(h)k(n))$, which can then be used to define a multiplication on the set $N \times H$ given by $(n, h) \cdot (n', h') = (n \cdot \alpha(h, n'), hh')$. The result is a monoid $(N \times H, \cdot, (1, 1))$ isomorphic to G via $\varphi\colon (N \times H, \cdot, (1, 1)) \to G$, where (n, h) is sent to $k(n) \cdot s(h)$.

S-protomodularity

Pointed protomodular categories (see (Bourn and Janelidze [7]) and (Borceux and Bourn [3])) may be thought of as categories with well behaved split extensions. The study of Schreier split extensions motivated a more relaxed notion, that of pointed $\mathcal{S}$-protomodularity, where only a restricted class S of split extensions need be well behaved (Bourn, Martins-Ferreira, Montoli, and Sobral [8]).

Just what properties this class S must satisfy has not been firmly established. When inspiration is taken from Schreier split epimorphisms it is required that, in addition to other properties, S be closed under the taking of finite limits. However in (Bourn [6]), a situation is considered in which this property is relaxed. The latter situation captures the case of weakly Schreier split extensions, whereas the former does not.

Outline

In this chapter we provide a complete classification of the weakly Schreier split extensions of H by N proving them equivalent to certain quotients of $N \times H$, equipped with something that behaves like an action relative to the quotient. Further, we demonstrate the failure of the split short five lemma and provide a complete classification of the morphisms that occur between two weakly Schreier split extensions. Finally, we provide some techniques for constructing weakly Schreier split extensions, first by generalising the Artin glueing construction and then by considering the coarsest quotient compatible with our construction.

3.2 Weak semidirect products

In all that follows N, G and H denote monoids unless otherwise stated.

Inspired by the semidirect product construction for Schreier split extensions, we consider a related construction in the weakly Schreier setting.

Definition 3.2.1. A diagram $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is a *split extension* (in the category of monoids) if

i) k is the kernel of e,

ii) e is the cokernel of k,

iii) $es = 1_H$. △

Definition 3.2.2. The category $\mathrm{SplExt}(H, N)$ has split extensions of H by N as objects and, as morphisms, monoid maps $f: G_1 \to G_2$ making the three squares in the following diagram commute.

$$\begin{array}{ccccc} N & \overset{k_1}{\rightarrowtail} & G_1 & \underset{s_1}{\overset{e_1}{\rightleftarrows}} & H \\ \| & & \downarrow{\scriptstyle f} & & \| \\ N & \overset{k_2}{\rightarrowtail} & G_2 & \underset{s_2}{\overset{e_2}{\rightleftarrows}} & H \end{array}$$

△

Definition 3.2.3. A *split extension* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is called weakly Schreier when every element of $g \in G$ can be written as $g = k(n) \cdot se(g)$ for some $n \in N$. △

Recall that if n is required to be unique, then the split extension is called Schreier.

Similar to Schreier split extensions, at least under the assumption of the axiom of choice, the definition can be reframed in terms of a set-theoretic map q. However, in the weakly Schreier setting, this map will not in general be unique.

Proposition 3.2.4. *Under the assumption of the axiom of choice, an extension* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *is weakly Schreier if and only if there exists a set theoretic map* $q: G \to N$ *satisfying that for all* $g \in G$, $g = kq(g) \cdot se(g)$.

Inspired by (Bourn, Martins-Ferreira, Montoli, and Sobral [8]) we call such a map q, an *associated Schreier retraction* of the extension $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$. We now prove some basic results.

Proposition 3.2.5. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be weakly Schreier and let* q *be an associated Schreier retraction. Then the following properties hold:*

i) $qk = 1_N$,

ii) $q(1) = 1$,

iii) $kq(s(h)k(n)) \cdot s(h) = s(h)k(n)$.

Proof. (i) Per the definition of q, for each $n \in N$ we can write $k(n) = kqk(n) \cdot sek(n) = kqk(n)$. Since k is injective we find that $n = qk(n)$, and so q is a retraction of k.

(ii) We know that $1 = q(1) \cdot se(1) = q(1)$.

(iii) Notice that $e(s(h)k(n)) = h$. Thus we can write $s(h)k(n) = kq(s(h)k(n) \cdot se(s(h)k(n)) = kq(s(h)k(n)) \cdot s(h)$. $\square$

We will make extensive use of Proposition 3.2.4 in Section 3.4. For the remainder of this section, as well as in Section 3.3 we will work choice free. It is likely that, with some thought, the results in Section 3.4 can be presented in a choice free manner too.

Definition 3.2.6. The category $\mathrm{WSExt}(H, N)$ denotes the full subcategory of $\mathrm{SplExt}(H, N)$ consisting of the weakly Schreier split extensions. $\triangle$

3.2.1 Canonical quotients

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a weakly Schreier split extension and consider the set-function $\varphi \colon N \times H \to G$ sending (n, h) to $k(n) \cdot s(h)$. The weakly Schreier condition gives that φ is surjective and so we can quotient $N \times H$ by φ.

Definition 3.2.7. Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a weakly Schreier split extension and let $\varphi \colon N \times H \to G$ denote the surjective map sending (n, h) to $k(n) \cdot s(h)$. Then let $E(e, s)$ denote the equivalence relation on $N \times H$ induced by φ where $(n, h) \sim (n', h')$ if and only if $k(n) \cdot s(h) = k(n') \cdot s(h')$. $\triangle$

Thus we can consider the map $\ell \colon N \times H \to (N \times H)/E(e, s)$, where ℓ send (n, h) to $[n, h]$, the equivalence class of (n, h) with respect to $E(e, s)$. Naturally we have a bijection $\overline{\varphi} \colon (N \times H)/E(e, s) \to G$ such that $\overline{\varphi}\ell = \varphi$.

Before we equip $(N \times H)/E(e, s)$ with a multiplication let us study some properties of $E(e, s)$.

Proposition 3.2.8. *Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a weakly Schreier split extension. If $(n_1, h_1) \sim (n_2, h_2)$ in $E(e, s)$, then $h_1 = h_2$.*

Proof. Let $(n_1, h_1) \sim (n_2, h_2)$. Then we have that $k(n_1) \cdot s(h_1) = k(n_2) \cdot s(h_2)$. Applying e to both sides yields $h_1 = h_2$ as required. $\square$

Since two pairs can be related only if their second components agree, this means that we can view this equivalence relation instead as an H-indexed equivalence relation on N. Let $\sim^h_{E(e,s)}$ denote the equivalence relation corresponding to $h \in H$. Then we say $n \sim^h_{E(e,s)} n'$ if and only if $k(n)s(h) = k(n')s(h)$. When it is unambiguous to do so, we omit the subscript and write $n \sim^h n'$.

Proposition 3.2.9. *The coproduct $\bigsqcup_{h \in H} N/{\sim^h}$ is isomorphic to $N \times H/E(e, s)$.*

Proof. The elements of $\bigsqcup_{h\in H} N/{\sim^h}$ are pairs $([n],h)$ where $[n] \in N/{\sim^h}$. It is not hard to see that the map sending $([n],h)$ to $[n,h]$ is a bijection. □

This is the perspective we will use for the remainder of the thesis, and so $E(e,s)$ will henceforth refer to the associated H-indexed equivalence relation.

Proposition 3.2.10. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension. If* $n_1 \sim^1 n_2$ *in* $E(e,s)$*, then* $n_1 = n_2$*.*

Proof. Let $n_1 \sim^1 n_2$. Then we have that $k(n_1)\cdot s(1) = k(n_2)\cdot s(1)$. Since $s(1) = 1$ this gives that $k(n_1) = k(n_2)$ which further implies that $n_1 = n_2$, as k is injective. □

Combining Proposition 3.2.8 and Proposition 3.2.10 we get that for each $n \in N$, for $([n],1)$ the equivalence class $[n]$ is a singleton.

Proposition 3.2.11. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension. If* $n_1 \sim^h n_2$ *in* $E(e,s)$ *then for all* $n \in N$ *we have* $nn_1 \sim^h nn_2$*, and for all* $h' \in H$ *we have* $n_1 \sim^{hh'} n_2$*.*

Proof. Let $n_1 \sim^h n_2$. Then we have that $k(n_1)\cdot s(h) = k(n_2)\cdot s(h)$. Hence $k(n)\cdot k(n_1)\cdot s(h) = k(n)\cdot k(n_2)\cdot s(h)$ and $k(n_1)\cdot s(h)\cdot s(h') = k(n_2)\cdot s(h)\cdot s(h')$ which gives that $nn_1 \sim^h nn_2$ and that $n_1 \sim^{hh'} n_2$. □

Now let us discuss the monoid structure of $\bigsqcup_{h\in H} N/{\sim^h}$. It inherits its multiplication and identity from G through $\overline{\varphi}$ — that is, we define $([n],h)\cdot([n'],h') = \overline{\varphi}^{-1}(\overline{\varphi}([n],h)\overline{\varphi}([n'],h'))$. The identity is readily seen to be $([1],1)$. By construction $\overline{\varphi}$ preserves multiplication and so we see that $(\bigsqcup_{h\in H} N/{\sim^h}, \cdot, ([1],1))$ is isomorphic to G. In Section 3.4 we will give an explicit description of this multiplication, making use of Proposition 3.2.4.

For convenience, we now let $\bigsqcup_{h\in H} N/{\sim^h}$ denote the monoid $(\bigsqcup_{h\in H} N/{\sim^h}, \cdot, ([1],1))$ and we might think of it as a weak semidirect product. We call this construction a *relaxed semidirect product* of H by N. This choice of terminology will be fully justified in Section 3.4 after the full characterization of weakly Schreier split extensions.

In fact $\bigsqcup_{h\in H} N/{\sim^h_{E(e,s)}}$ is part of a split extension isomorphic to $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$. Consider

$$N \overset{k'}{\rightarrowtail} \bigsqcup_{h\in H} N/{\sim^h_{E(e,s)}} \underset{s'}{\overset{e'}{\rightleftarrows}} H,$$

where $k'(n) = ([n],1)$, $e'([n],h) = h$ and $s'(h) = ([1],h)$. Note that $([n],1)\cdot([1],h) = ([n],h)$ and so this split extension is unsurprisingly weakly Schreier.

We can now conclude the following.

Proposition 3.2.12. *The map $\overline{\varphi}$ is an isomorphism of split extensions between* $N \overset{k}{\longrightarrow} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *and* $N \overset{k'}{\longrightarrow} \bigsqcup_{h\in H} N/{\sim^h_{E(e,s)}} \underset{s'}{\overset{e'}{\rightleftarrows}} H$.

Example 3.2.13. (Schreier split extensions) Let $N \overset{k}{\longrightarrow} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a Schreier split extension and $q\colon G \to N$ the associated Schreier retraction. Recall that the semidirect product construction, as in (Martins-Ferreira, Montoli, and Sobral [33]), applied to the above Schreier split extension, gives $(N \times H, \cdot, (1,1))$ where $(n,h)\cdot(n',h') = (n \cdot q(s(h)k(n')), hh')$. Let us show that this agrees with our weak semidirect product construction on weakly Schreier split extensions.

Let $N \overset{k}{\longrightarrow} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a Schreier split extension, let φ be defined as above and consider $\bigsqcup_{h\in H} N/{\sim^h}$. Observe that the Schreier condition gives that $n \sim^h n'$ if and only if $n = n'$. Thus $\bigsqcup_{h\in H} N/{\sim^h}$ as a set may just be thought of as the product $N \times H$. This agrees with the semidirect product construction and so we must just show that the two constructions agree on multiplication.

For our relaxed semidirect product we define $(n,h)\cdot(n',h') = \varphi^{-1}(\varphi(n,h)\varphi(n',h'))$ which is the unique element in $N \times H$ which φ sends to $k(n)\cdot s(h)\cdot k(n')\cdot s(h')$. Thus we need only show that $\varphi(n \cdot q(s(h)k(n')), hh') = k(n)\cdot s(h)\cdot k(n')\cdot s(h')$.

Per the definition of q we see that $kq(s(h)k(n))\cdot s(h) = s(h)\cdot k(n)$. Thus

$$\begin{aligned}\varphi(n \cdot q(s(h)k(n')), hh') &= k(n)\cdot kq(s(h)k(n'))\cdot s(h)\cdot s(h') \\ &= k(n)\cdot s(h)\cdot k(n')\cdot s(h').\end{aligned}$$

Thus we see that the weak semidirect product construction agrees with the semidirect product construction on Schreier split extensions. △

Example 3.2.14. (Artin glueings)

Artin glueings are our primary examples of weakly Schreier split extensions that are not Schreier. Artin glueings are usually considered as subobjects of the product and not quotients. For frames N and H and finite meet preserving map $f\colon H \to N$, the Artin glueing $\mathrm{Gl}(f)$ is the frame of pairs (n,h) where $n \le f(h)$, with componentwise meets and joins. In this example we discuss how an Artin glueing can also be viewed as a quotient of the product.

Let N, G and H be frames considered as monoids with multiplication given by meet and consider a weakly Schreier split extension $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$. As shown in Chapter 2, k has a left adjoint k^* which is an associated Schreier retraction and, in fact, a monoid map. Furthermore we find that s must be the right adjoint e_* of e.

Now let $\varphi\colon N \times H \to G$ be defined in the usual way and consider the equivalence classes it generates. We know that $kk^*(g) \wedge e_*e(g) = g$ and thus the equivalence class corresponding to each g has a canonical choice of representative given by $(k^*(g), e(g))$. This allows us to represent the inverse $\overline{\varphi}^{-1}(g) = ([k^*(g)], e(g))$.

Starting with the fact that for all g we have that $g \leq e_*e(g)$, we apply k^* to both sides and arrive at $k^*(g) \leq k^*e_*e(g)$. This means that all of the canonical elements $(k^*(g), e(g))$ are elements of $\mathrm{Gl}(k^*e_*)$. Furthermore, if $(n, h) \in \mathrm{Gl}(k^*e_*)$ then we have $k^*(k(n) \wedge e_*(h)) = k^*k(n) \wedge k^*e_*(h) = n \wedge k^*e_*(h) = n$. Thus $(n, h) = (k^*(k(n) \wedge e_*(h)), e(k(n) \wedge e_*(h)))$ and so we conclude that the canonical representatives are precisely the pairs in $\mathrm{Gl}(k^*e_*)$.

Looking at the prescribed multiplication on $N \times H/E(e, s)$ we see

$$\begin{aligned}[k^*(g), e(g)] \cdot [k^*(g'), e(g')] &= \overline{\varphi}^{-1}(g \wedge g') \\ &= [k^*(g \wedge g'), e(g \wedge g')] \\ &= [k^*(g) \wedge k^*(g'), e(g) \wedge e(g')].\end{aligned}$$

This means that when we are dealing with the canonical representations of each class, multiplication is just taking the meet componentwise. This then justifies the usual interpretation of the Artin glueing as a subobject of the product.

Taken together we see that $(N \times H)/E(e, s)$ is isomorphic to $\mathrm{Gl}(k^*e_*)$ as required.△

3.3 Failure of the Split Short Five Lemma

For any $\mathcal{S}$-protomodular category in the sense of (Bourn, Martins-Ferreira, Montoli, and Sobral [8]), it was shown in that same paper that the split short five lemma holds. This lemma says that when given two split extension $N \overset{k_1}{\rightarrowtail} G_1 \underset{s_1}{\overset{e_1}{\leftrightarrows}} H$ and $N \overset{k_2}{\rightarrowtail} G_2 \underset{s_2}{\overset{e_2}{\leftrightarrows}} H$, if $\psi\colon G_1 \to G_2$ is a morphism of split extensions, then ψ is an isomorphism. Since weakly Schreier split extensions are only $\mathcal{S}$-protomodular in a weaker sense (Bourn [6]), the split short five lemma need not hold and in fact does not. In this section we study the morphisms of $\mathrm{WSExt}(H, N)$ before providing a complete characterization of them in Section 3.4.

Theorem 3.3.1. *Let* $N \overset{k_1}{\rightarrowtail} G_1 \underset{s_1}{\overset{e_1}{\rightleftarrows}} H$ *and* $N \overset{k_2}{\rightarrowtail} G_2 \underset{s_2}{\overset{e_2}{\rightleftarrows}} H$ *be two weakly Schreier split extensions in the category of monoids, let* $E_1 = E(e_1, s_1)$ *and* $E_2 = E(e_2, s_2)$ *and finally let* $\psi \colon \bigsqcup_{h\in H} N/{\sim}^h_{E_1} \to \bigsqcup_{h\in H} N/{\sim}^h_{E_2}$ *be a morphism of the following weakly Schreier split extensions.*

$$\begin{array}{ccccc}
N & \overset{k'_1}{\rightarrowtail} & \bigsqcup_{h\in H} N/{\sim}^h_{E_1} & \underset{s'_1}{\overset{e'_1}{\rightleftarrows}} & H \\
\| & & \downarrow \psi & & \| \\
N & \overset{k'_2}{\rightarrowtail} & \bigsqcup_{h\in H} N/{\sim}^h_{E_2} & \underset{s'_2}{\overset{e'_2}{\rightleftarrows}} & H
\end{array}$$

Then $\psi([n], h) = ([n], h)$.

Proof. Since $\psi k(n) = k'(n)$, we find that $\psi([n], 1) = ([n], 1)$. Similarly we find that $\psi([1], h) = ([1], h)$. Now observe that

$$\begin{aligned}
\psi([n], h) &= \psi(([n], 1)([1], h)) \\
&= \psi([n], 1) \cdot \psi([1], h) \\
&= ([n], 1)([1], h) \\
&= ([n], h).
\end{aligned}$$

This completes the proof. □

Notice that this is in agreement with the Schreier case, as there the equivalence classes are all singletons and so Theorem 3.3.1 implies that any morphism between Schreier split extensions must be the identity.

From Theorem 3.3.1 we can conclude that any morphism between weakly Schreier split extensions must be unique. We thus arrive at the following corollary.

Corollary 3.3.2. $\mathrm{WSExt}(H, N)$ *is a preorder category for all monoids* H *and* N*.*

In fact this result can be generalised to any $\mathcal{S}$-protomodular category in the sense of (Bourn [6]), as all that is required is that for each extension $N \overset{e}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ in S, (k, s) is jointly extremally epic.

Of course Theorem 3.3.1 does not by itself demonstrate that the split short five lemma fails in the case of weakly Schreier split extensions. In the following section we will completely determine the conditions which yield a morphism between two weakly Schreier split extensions. For now we will demonstrate the failure of the lemma by exhibiting a non-isomorphism between two extensions of Artin glueings.

Example 3.3.3. (Artin glueings)

Let N and H be frames, $f, g\colon H \to N$ meet preserving maps and $\psi\colon \operatorname{Gl}(f) \to \operatorname{Gl}(g)$ a morphism of the following extensions.

$$\begin{array}{ccccc} N & \overset{k_1}{\rightarrowtail} & \operatorname{Gl}(f) & \underset{s_1}{\overset{e_1}{\rightleftarrows}} & H \\ \| & & \downarrow{\scriptstyle \psi} & & \| \\ N & \overset{k_2}{\rightarrowtail} & \operatorname{Gl}(g) & \underset{s_2}{\overset{e_2}{\rightleftarrows}} & H \end{array}$$

Applying our results in Chapter 2 to the above diagram we find that $k_1(n) = (n, 1)$, $k_2(n) = (n, 1)$, $s_1(h) = (f(h), h)$ and $s_2(h) = (g(h), h)$.

Since ψ is a morphism of split extensions we have that $\psi(n, 1) = (n, 1)$ and $\psi(f(h), h) = (g(h), h)$. This is enough to completely determine ψ as we have

$$\begin{aligned} \psi(n, h) &= \psi((n, 1) \wedge (f(h), h)) \\ &= \psi(n, 1) \wedge \psi(f(h), h) \\ &= (n, 1) \wedge (g(h), h) \\ &= (n \wedge g(h), h). \end{aligned}$$

However to ensure consistency we require that $(n, 1) = \psi(n, 1) = (n \wedge g(1), 1)$ and that $(g(h), h) = \psi(f(h), h) = (f(h) \wedge g(h), h)$. The former expression will always be true, but the latter is true if and only if $g \leq f$.

Thus whenever $g \leq f$, we have that ψ as described above will be a morphism of weakly Schreier split extensions. Whenever g is strictly less than f, it is evident that this ψ is not an isomorphism. For instance let N be a non-trivial frame and let $N = H$. Then take f to be the constant 1 map and g to be the identity. △

3.4 Characterizing weakly Schreier extensions

In the Schreier case, split extensions correspond to actions $\alpha\colon H \times N \to N$, which are used to construct a multiplication on the set $N \times H$. If we are to do something similar in the weakly Schreier case, must define a multiplication on the coproduct

$\bigsqcup_{h\in H} N/{\sim^h}$ for some (H-indexed) equivalence relation. The question is then: what are the appropriate equivalence relations to consider and how can we induce the appropriate monoid operations? In what proceeds we will make use of the axiom of choice in the form of Proposition 3.2.4.

3.4.1 The H-indexed equivalence relation

Let us tackle the question of the equivalence relation first. We do so by considering the properties of an equivalence relation constructed from a weakly Schreier split extension as in Section 3.2.

Here we take inspiration from Propositions 3.2.8, 3.2.10 and 3.2.11 and consider only the H-indexed equivalence relations on N whose corresponding equivalence relation satisfies the following conditions.

Definition 3.4.1. Let E be an H-indexed equivalence relation on N. We say E is an *admissible equivalence relation* if it satisfies the following conditions.

i) $n_1 \sim^1 n_2$ implies $n_1 = n_2$,

ii) for all $n \in N$, $n_1 \sim^h n_2$ implies $nn_1 \sim^h nn_2$ and

iii) for all $h' \in H$, $n_1 \sim^h n_2$ implies $n_1 \sim^{hh'} n_2$. △

Notice that when h has a right inverse, condition (i) and (iii) together imply the following.

Proposition 3.4.2. *Let E be an admissible equivalence relation on N. If $h \in H$ has a right inverse, then $n \sim^h n'$ implies $n = n'$.*

In particular this means that for groups, the only admissible equivalence relation is the discrete one. This is consistent with the observation that all split extensions of groups are Schreier.

We now consider an action of N on $\bigsqcup_{h\in H} N/{\sim^h}$ and an action of H on $\bigsqcup_{h\in H} N/{\sim^h}$, which will be well-defined thanks to conditions (3) and (4) above. For each $n' \in N$ let $n' * ([n], h) = ([n'n], h)$ and for each $h' \in H$ let $([n], h) * h' = ([n], hh')$. Equipped with these actions we can consider $\bigsqcup_{h\in H} N/{\sim^h}$ to be similar in character to a bi-module.

3.4.2 The multiplication

Suppose we have a weakly Schreier split extension $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ and let $q\colon G \to N$ be an associated Schreier retraction. Let us construct the associated relaxed semidirect product $\bigsqcup_{h \in H} N/{\sim^h}$ as in Section 3.2 and examine the multiplication in more detail. In this section we make extensive use of (3) in Proposition 3.2.5, which says that $kq(s(h)k(n)) \cdot s(h) = s(h)k(n)$.

Proposition 3.4.3. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ *be a weakly Schreier split extension and let* q *be an associated Schreier retraction. For* $([n_1], h_1), ([n_2], h_2) \in \bigsqcup_{h \in H} N/{\sim^h}$ *we can equivalently express the multiplication as*

$$([n_1], h_1) \cdot ([n_2], h_2) = ([n_1 q(s(h_1)k(n_2)), h_1 h_2).$$

Proof. We must show that $([n_1 q(s(h_1)k(n_2)), h_1 h_2)$ is sent by $\overline{\varphi}$ to $k(n_1) \cdot s(h_1) \cdot k(n_2) \cdot s(h_2)$.

We know that

$$\overline{\varphi}([n_1 \cdot q(s(h_1)k(n_2))], h_1 h_2) = k(n_1) \cdot kq(s(h_1)k(n_2)) \cdot s(h_1) \cdot s(h_2).$$

Since $kq(s(h)k(n)) \cdot s(h) = s(h) \cdot k(n)$, the above expression simplifies to $k(n_1) \cdot s(h_1) \cdot k(n_2) \cdot s(h_2)$ as required. ☐

This presentation of the multiplication suggests that something resembling the actions of the Schreier case will play an equally crucial role in defining the multiplication on $\bigsqcup_{h \in H} N/{\sim^h}$.

Note that since the multiplication of $\bigsqcup_{h \in H} N/{\sim^h}$ was defined without any reference to Schreier retractions, it must be that all Schreier retractions induce the same multiplication.

Corollary 3.4.4. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ *be a weakly Schreier split extension and let* q *and* q' *be Schreier retractions. Then we have that* $([n_1 q(s(h_1)k(n_2))], h_1 h_2) = ([n_1 q'(s(h_1)k(n_2))], h_1 h_2)$ *for all* $n_1, n_2 \in N$ *and* $h_1, h_2 \in H$.

Now let $\alpha \colon H \times N \to N$ send (h, n) to $q(s(h)k(n))$ and let us study its properties.

Proposition 3.4.5. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ *be a weakly Schreier split extension,* q *an associated Schreier retraction and let* $\alpha(b, a) = q(s(b)k(a))$*. If* $n_1 \sim^h n_2$*, then* $([n_1 \alpha(h, n)], h) = ([n_2 \alpha(h, n)], h)$.

Proof. Suppose that $n_1 \sim^h n_2$ and consider the following calculation.

$$\begin{aligned}\overline{\varphi}([n_1\alpha(h,n)],h) &= k(n_1)\cdot k\alpha(h,n)\cdot s(h)\\ &= k(n_1)\cdot s(h)\cdot k(n)\\ &= k(n_2)\cdot s(h)\cdot k(n)\\ &= k(n_2)\cdot k\alpha(h,n)\cdot s(h)\\ &= \overline{\varphi}([n_2\alpha(h,n)],h)\end{aligned}$$

Since $\overline{\varphi}$ is injective, we must have that $[n_1\alpha(h,n),h] = [n_2\alpha(h,n),h]$. □

Similarly we have the following proposition.

Proposition 3.4.6. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension,* q *an associated Schreier retraction and let* $\alpha(b,a) = q(s(b)k(a))$*. If* $n \sim^y n'$*, then* $([\alpha(h,n)],hy] = ([\alpha(h,n')],hy)$*.*

Given an admissible equivalence relation E, any maps $\alpha\colon H\times N \to N$ satisfying Proposition 3.4.5 and Proposition 3.4.6 we call *pre-actions compatible with* E.

Next we show that α satisfies conditions analogous to being an action in the Schreier case.

Proposition 3.4.7. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension,* q *an associated Schreier retraction and let* $\alpha(b,a) = q(s(b)k(a))$*. Then* $([\alpha(h,nn')],h) = ([\alpha(h,n)\cdot\alpha(h,n')],h)$*.*

Proof. In order to prove that these classes are equal we show that $\overline{\varphi}$ maps them to the same element of G. Thus consider

$$\begin{aligned}\overline{\varphi}([\alpha(h,nn')],h) &= kq(s(h)k(nn'))\cdot s(h)\\ &= s(h)\cdot k(n)\cdot k(n').\end{aligned}$$

We also have

$$\begin{aligned}\overline{\varphi}([\alpha(h,n)\alpha(h,n')],h) &= kq(s(h)k(n))\cdot kq(s(h)k(n'))\cdot s(h)\\ &= kq(s(h)k(n))\cdot s(h)\cdot k(n')\\ &= s(h)\cdot k(n)\cdot k(n').\end{aligned}$$

As discussed above, this gives that $([\alpha(h,nn')],h) = ([\alpha(h,n)\cdot\alpha(h,n')],h)$. □

Proposition 3.4.8. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension, q an associated Schreier retraction and let $\alpha(b,a) = q(s(b)k(a))$. Then* $([\alpha(hh',n)],hh') = ([\alpha(h,\alpha(h',n))],hh')$.

Proof. We need only show that $\overline{\varphi}$ maps each class to the same element. We have

$$\begin{aligned}\overline{\varphi}([\alpha(hh',n)],hh') &= kq(s(hh')k(n))\cdot s(hh')\\ &= s(h)\cdot s(h')\cdot k(n).\end{aligned}$$

Compare it to the following.

$$\begin{aligned}\overline{\varphi}([\alpha(h,\alpha(h',n))],hh') &= kq(s(h)kq(s(h')k(n)))\cdot s(h)\cdot s(h')\\ &= s(h)\cdot kq(s(h')k(n))\cdot s(h')\\ &= s(h)\cdot s(h')\cdot k(n)\end{aligned}$$

Thus $\overline{\varphi}$ sends $([\alpha(hh',n)],hh')$ and $([\alpha(h,\alpha(h',n))],hh')$ to the same element and so they are equal. □

Proposition 3.4.9. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension, q an associated Schreier retraction and let $\alpha(b,a) = q(s(b)k(a))$. Then* $([\alpha(h,1)],h) = ([1],h)$.

Proof. Just consider $\overline{\varphi}([\alpha(h,1)],h) = kq(s(h))s(h) = s(h) = \overline{\varphi}([1],h)$ and observe that $\overline{\varphi}$ is injective and sends each class to the same element. □

The following result is proved similarly.

Proposition 3.4.10. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension, q an associated Schreier retraction and let $\alpha(b,a) = q(s(b)k(a))$. Then* $([\alpha(1,n)],1) = ([n],1)$.

We now say an action is a *compatible pre-action* if it satisfies the above properties.

Definition 3.4.11. A function $\alpha\colon H\times N\to N$ is an *action* with respect to an admissible quotient Q of $N\times H$ if it satisfies the following conditions:

i) $n_1 \sim^h n_2$ implies $n_1\alpha(h,n) \sim^h n_2\alpha(h,n)$ for all $n\in N$,

ii) $n \sim^y n'$ implies $\alpha(h,n) \sim^{hy} \alpha(h,n')$ for all $h\in H$,

iii) $\alpha(h,nn') \sim^h \alpha(h,n)\cdot\alpha(h,n')$,

iv) $\alpha(hh', n) \sim^{hh'} \alpha(h, \alpha(h', n))$,

v) $\alpha(h, 1) \sim^{h} 1$,

vi) $\alpha(1, n) \sim^{1} n$. △

Let us again draw attention to this definition as it applies to groups. By Proposition 3.4.2, the only admissible equivalence relation is discrete and so conditions (1) and (2) are immediately satisfied. The remaining four conditions then reduce to requiring that α be an action in the traditional sense. This same argument gives that α must be an action in the Schreier setting.

Let Act_E denote the set of actions with respect to an admissible equivalence relation E.

Definition 3.4.12. Let E be an admissible H-indexed equivalence relation on N and let $\alpha \in \mathrm{Act}_E$. Then we call the pair (E, α) a *relaxed action* of H on N. △

Given a relaxed action (E, α) of H on N, we can equip $\bigsqcup_{h \in H} N/{\sim^h}$ with a multiplication as follows.

$$([n], h) \cdot ([n'], h') = ([n\alpha(h, n')], hh').$$

Let us verify that this is indeed well defined. Suppose that $([a], h) = ([n], h)$ and that $([a'], h') = ([n'], h')$. Then by the first condition of Definition 3.4.11 we have that $([a], h)([n'], h') = ([n], h)([n'], h')$. Then applying the second condition of Definition 3.4.11 we get $([a], h)([a'], h') = ([a], h)([n'], h')$. Thus this operation is well defined and it remains to prove it is associative and has an identity.

First we prove a lemma that shall be used extensively throughout the remainder of this thesis.

Lemma 3.4.13. *Let E be an H-indexed equivalence relation on N. Then if $n \sim^h n'$ we have that $([xn], hy) = ([xn'], hy)$ for all $x \in N$ and $y \in H$.*

Proof. We simply apply the axioms of an admissible equivalence relation. If $n \sim^h n'$ then $xn \sim^h xn'$ which then further implies that $xn \sim^{hy} xn'$. □

Proposition 3.4.14. *Let (E, α) be a relaxed action of H on N. Then we have $([n], h)([n'], h') = ([n\alpha(h, n')], hh')$ makes $\bigsqcup_{h \in H} N/{\sim^h}$ a monoid with identity $([1], 1)$.*

Proof. For the identity simply observe $([n], h)([1], 1) = ([n\alpha(h, 1)], h) = ([n], h)$ and then $([1], 1)([n], h) = ([\alpha(1, n)], 1 \cdot h] = ([n], h)$.

For associativity consider

$$\begin{aligned}(([n_1],h_1)([n_2],h_2))([n_3],h_3) &= ([n_1\alpha(h_1,n_2)],h_1h_2)([n_3],h_3)\\ &= ([n_1\alpha(h_1,n_2)\cdot\alpha(h_1h_2,n_3)],h_1h_2h_3)\\ &= ([n_1\alpha(h_1,n_2)\alpha(h_1,\alpha(h_2,n_3))],h_1h_2h_3),\end{aligned}$$

and compare it to

$$\begin{aligned}([n_1],h_1)(([n_2],h_2)([n_3],h_3)) &= ([n_1],h_1)([n_2\alpha(h_2,n_3)],h_2h_3)\\ &= ([n_1\alpha(h_1,n_2\alpha(h_2,n_3))],h_1h_2h_3)\\ &= ([n_1\alpha(h_1,n_2)\alpha(h_1,\alpha(h_2,n_3))],h_1h_2h_3).\end{aligned}$$

Thus this operation is indeed associative and so $\bigsqcup_{h\in H} N/{\sim^h}$ becomes a monoid. □

Let us call the resulting monoid the *relaxed semidirect product* and write it $N \rtimes_{E,\alpha} H$.

We may now state the result that justifies our terminology by making the analogy to the group case concrete.

Theorem 3.4.15. *Let N and H be monoids and (E,α) a relaxed action of H on N. The diagram $N \overset{k}{\rightarrowtail} N \rtimes_{E,\alpha} H \underset{s}{\overset{e}{\rightleftarrows}} H$ where $k(n) = ([n],1)$, $e([n],h) = h$ and $s(h) = ([1],h)$, is a weakly Schreier split extension.*

Proof. Observe that for every element $([n],h)$ we have $k(n)s(h) = ([n],1)([1],h) = ([n\alpha(1,1)],h) = [n,h]$. So it satisfies the weakly Schreier condition. Thus it remains only to show that k is the kernel of e, and e the cokernel of k.

It is clear that $ek = 0$ and further that the image of k is precisely the submonoid sent to 1. Thus for any $t\colon X \to G$ satisfying $et = 0$, it must map into the image k. It is clear then that t factors uniquely through k.

Next suppose we have a map $t\colon Q \to X$ satisfying $tk = 0$. Consider $t([n],h) = t(([n],1)([1],h)) = t([1],h)$. Thus where t sends a pair is entirely determined by where it sends $([1],h)$. So define $t'\colon H \to X$ which sends h to $t([1],h)$. It is clear that $t = t'e$ and is unique as e is epi. □

Given two relaxed actions (E,α) and (E,α') with the same first component, note that they induce a multiplication on the same underlying set $\bigsqcup_{h\in H} N/{\sim^h_E}$ determined by E. It is possible for two relaxed actions to give the same relaxed semidirect product.

Proposition 3.4.16. *Two relaxed actions (E, α) and (E, α') induce the same relaxed semidirect product if and only if for all $(h, n) \in H \times N$ we have $\alpha(h, n) \sim^h \alpha'(h, n)$.*

Proof. Suppose that α and α' satisfy that $\alpha(h, n) \sim^h \alpha'(h, n)$ and let $([n], h) \cdot_\alpha ([n'], h')$ and $([n], h) \cdot_{\alpha'} ([n'], h')$ denote the multiplication in $N \rtimes_{E,\alpha} H$ and $N \rtimes_{E,\alpha} H$ respectively. Then observe that $([n], h) \cdot_\alpha ([n'], h') = ([n\alpha(h, n')], hh') = ([n\alpha'(h, n')], hh') = ([n], h) \cdot_{\alpha'} ([n'], h')$.

For the other direction suppose there exists a pair $(h, n) \in H \times N$ such that $\alpha(h, n) \nsim^h \alpha'(h, n)$ and consider the associated weakly Schreier split extensions $N \overset{k}{\rightarrowtail} N \rtimes_{E,\alpha} H \underset{s}{\overset{e}{\leftrightarrows}} H$ and $N \overset{k'}{\rightarrowtail} N \rtimes_{E,\alpha'} H \underset{s'}{\overset{e'}{\leftrightarrows}} H$. Suppose we have an isomorphism of extensions $\psi \colon N \rtimes_{E,\alpha} H \to N \rtimes_{E,\alpha'} H$. By Theorem 3.3.1 we know that $\psi([n], h) = ([n], h)$, but observe that $\psi([\alpha(h, n)], h) = \psi(([1], h) \cdot_\alpha ([n], 1)) = ([1], h) \cdot_{\alpha'} ([n], 1) = ([\alpha'(h, n)], h) \neq ([\alpha(h, n)], h)$. This yields a contradiction. □

We want to view relaxed actions that yield the same relaxed semidirect product as being essentially the same. Thus we may accordingly quotient the set of relaxed actions, to yield the set $\mathrm{RAct}(H, N)$. As an abuse of notation we also call the elements of $\mathrm{RAct}(H, N)$ relaxed actions.

Since two relaxed actions will be related only if they agree on the admissible equivalence relation, we can again view this as a particular indexed equivalence relation. Let us thus quotient the set Act_E by the equivalence relation given by $\alpha \sim \alpha'$ if and only if $\alpha(h, n) \sim^h_E \alpha'(h, n)$ for all $n \in N$ and $h \in H$. Call this set $\mathrm{Act}_E/\sim$. It is not hard to see that the set of pairs $(E, [\alpha])$ where $[\alpha] \in \mathrm{Act}_E/\sim$ is isomorphic to $\mathrm{RAct}(H, N)$.

Proposition 3.4.17. *Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ be weakly Schreier and let q and q' be associated Schreier retractions. Then if $\alpha(h, n) = q(s(h)k(n))$ and $\alpha'(h, n) = q'(s(h)k(n))$ we have that $\alpha \sim \alpha'$ in $\mathrm{Act}_Q/\sim$.*

Proof. This follows from Corollary 3.4.4, as we have $([\alpha(h, n), h) = ([1], h)([n], 1) = ([\alpha'(h, n)], h)$ in $N \rtimes_{E,\alpha} H = N \rtimes_{E,\alpha'} H$. □

In order to show that the above forms a complete characterization of the weakly Schreier split extensions between H and N and to simultaneously characterise the morphisms of $\mathrm{WSExt}(H, N)$, we introduce the following preorder.

Definition 3.4.18. Let $\mathrm{RAct}(H, N)$ be the preorder whose objects are equivalence classes of relaxed actions. We say that $(E, [\alpha]) \leq (E', [\alpha'])$ if and only if $n \sim^h_E n'$ implies that $n \sim^h_{E'} n'$ and $\alpha(h, n) \sim^h_{Q'} \alpha'(h, n)$. △

The last condition implies that every inequality can be written of the form $(E, [\alpha]) \leq (E', [\alpha])$ and the first condition says that E' is a larger equivalence relation.

Theorem 3.4.19. *The categories* $\mathrm{RAct}(H, N)$ *and* $\mathrm{WSExt}(H, N)$ *are equivalent.*

Proof. Let $S\colon \mathrm{RAct}(H, N) \to \mathrm{WSExt}(H, N)$ send the relaxed action $(E, [\alpha])$ to $N \overset{k}{\rightarrowtail} N \rtimes_{E,\alpha} H \underset{s}{\overset{e}{\rightleftarrows}} H$, as described in Theorem 3.4.15. Let us begin by demonstrating that S preserves the order.

Suppose that $(E, [\alpha]) \leq (E', [\alpha])$. Theorem 3.3.1 tells us that any morphism of split extensions between $S(E, [\alpha])$ and $S(E', [\alpha])$ must be a map $\psi\colon N \rtimes_{E,\alpha} H \to N \rtimes_{E',\alpha} H$ sending $([n], h)$ to $([n], h)$. Let us show that indeed ψ is a well-defined morphism of split extensions.

Suppose that $n_1 \sim^h_E n_2$. We know that $(E, [\alpha]) \leq (E', [\alpha])$, and so we have $n_1 \sim^h_{E'} n_2$. Thus $\psi(([n_1], h) = ([n_1], h) = ([n_2], h) = \psi([n_2], h)$, which proves that our description of ψ is well-defined.

It is apparent that the multiplication is preserved as each relaxed semidirect product is defined using the same α. Furthermore it is apparent that ψ makes the required diagram commute and so is a morphism of split extensions. Thus S preserves the order as required.

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a weakly Schreier split extension, $E = E(e, s)$ and let q be an associated Schreier retraction. Then let $T\colon \mathrm{WSExt}(H, N) \to \mathrm{RAct}(H, N)$ send

$$N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$$

to $(E, [\alpha])$ where $\alpha(h, n) = q(s(h)k(n))$. By Proposition 3.4.17 we have that T is well defined. We must show that T respects the preorder structure.

Suppose we have a morphism $\psi\colon G_1 \to G_2$ of weakly Schreier split extensions as in the following diagram.

$$\begin{array}{ccccc}
N & \overset{k_1}{\rightarrowtail} & G_1 & \underset{s_1}{\overset{e_1}{\rightleftarrows}} & H \\
\| & & \downarrow{\scriptstyle \psi} & & \| \\
N & \overset{k_2}{\rightarrowtail} & G_2 & \underset{s_2}{\overset{e_2}{\rightleftarrows}} & H
\end{array}$$

Let q_1 be an associated Schreier retraction of $N \overset{k_1}{\rightarrowtail} G_1 \underset{s_1}{\overset{e_1}{\leftrightarrows}} H$ and q_2 an associated Schreier retraction of $N \overset{k_2}{\rightarrowtail} G_2 \underset{s_2}{\overset{e_2}{\leftrightarrows}} H$. Further let $E_1 = E(e_1, s_2)$ and $E_2 = E(e_2, s_2)$.

Then in order to show that T is order preserving we must show that $E_1, [\alpha_1]) \leq (E_2, [\alpha_2])$. This requires us to show that if $n, \sim^h_{E_1} n'$ then $n \sim^h_{E_2} n'$ and finally that $\alpha_1(h, n) \sim^h_{E_2} \alpha_2(h, n)$.

Suppose that $n \sim^h_{E_1} n'$. This means that $k_1(n) \cdot s_1(h) = k_1(n') \cdot s_1(h)$. In order to show that n and n' are h-related in E_2, we must show that $k_2(n) \cdot s_2(h) = k_2(n') \cdot s_2(h)$. Consider

$$\begin{aligned} k_2(n) \cdot s_2(h) &= \psi k_1(n) \cdot \psi s_1(h) \\ &= \psi(k_1(n) s_1(h)) \\ &= \psi(k_1(n') s_1(h)) \\ &= k_2(n') \cdot s_2(h). \end{aligned}$$

In order to show that the second condition holds consider the following calculation.

$$\begin{aligned} k_2 \alpha_1(h, n) \cdot s_2(h) &= k_2 q_1(s_1(h) k_1(n)) \cdot s_2(h) \\ &= \psi k_1 q_1(s_1(h) k_1(n)) \cdot \psi s_1(h) \\ &= \psi(k_1 q_1(s_1(h) k_1(n)) s_1(h)) \\ &= \psi(s_1(h) k_1(n)) \\ &= s_2(h) k_2(n) \\ &= k_2 q_2(s_2(h) k_2(n)) \cdot s_2(h) \\ &= k_2 \alpha_2(h, n) \cdot s_2(h) \end{aligned}$$

Thus indeed $(E_1, [\alpha_1]) \leq (E_2, [\alpha_2])$ and so T preserves the order. All that remains is to show that the functors T and S form an equivalence of categories. Proposition 3.2.12 and Proposition 3.4.3 together give us that ST is equivalent to the identity and so we can shift our attention to TS.

Suppose we apply S to the relaxed action $(E, [\alpha])$ and generate the extension $N \overset{k}{\rightarrowtail} N \rtimes_{E,\alpha} H \underset{s}{\overset{e}{\leftrightarrows}} H$. Let q be an associated Schreier retraction. Let us thus define a map $\alpha'(h, n) = q(s(h)k(n))$. So $TS(E, [\alpha])$ yields the relaxed action $(E(e, s), [\alpha'])$. We will now show that $(E, [\alpha]) = (E(e, s), [\alpha'])$.

We have that $n \sim_E^h n'$ if and only if

$$\begin{aligned} k(n) \cdot s(h) &= ([n], 1) \cdot ([1], h) \\ &= ([n], h) \\ &= ([n'], h) \\ &= k(n') \cdot s(h). \end{aligned}$$

But notice that this is precisely the condition for n to be h-related to n' in $E(e, s)$. Thus we get that $E = E(e, s)$.

Per our definition of α' we must show that $\alpha'(h, n) \sim_{E(e,s)}^h \alpha(h, n), h$ — that is, $kq(s(h)k(n)) \cdot s(h) = k\alpha(h, n) \cdot s(h)$. To see this consider the following calculation.

$$\begin{aligned} kq(s(h)k(n)) \cdot s(h) &= s(h) \cdot k(n) \\ &= ([\alpha(h, n)], 1) \cdot ([1], h) \\ &= k\alpha(h, n) \cdot s(h). \end{aligned}$$

Thus $[\alpha] = [\alpha']$, which then finally gives that TS is the identity. □

We thus have a full characterization of all weakly Schreier split extensions in the category of monoids, as well as a characterization of the morphisms between them given by the following corollary.

Corollary 3.4.20. *Let* $N \overset{k_1}{\rightarrowtail} G_1 \underset{s_1}{\overset{e_1}{\rightleftarrows}} H$ *and* $N \overset{k_2}{\rightarrowtail} G_2 \underset{s_2}{\overset{e_2}{\rightleftarrows}} H$ *be weakly Schreier split extensions,* q_1 *and* q_2 *respective associated Schreier retractions and let* $E_1 = E(e_1, s_1)$ *and* $E_2 = E(e_2, s_2)$*. Then a morphism* $\psi \colon G_1 \to G_2$ *of split extensions exists if and only if for all* $n \in N$ *and* $h \in H$ *we have* $n \sim_{E_1}^h n'$ *implies that* $n \sim_{E_2}^h n'$ *and* $q_1(s(h)k(n) \sim_{E_2}^h q_2(s(h)k(n))$*.*

From this characterization, we can deduce the following results about weakly Schreier split extensions in the full subcategories of commutative monoids and abelian groups.

Proposition 3.4.21. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension in which* N*,* G *and* H *are commutative. Then if* $(E, [\alpha])$ *corresponds to the extension* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *it must be that* $\alpha(h, n) \sim^h n$*.*

Proof. If $(E, [\alpha])$ corresponds to $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ then $N \rtimes_{E,\alpha} H$ is isomorphic to G.

Thus $N \rtimes_{E,\alpha} H$ is commutative and so $([n], h) = ([n], 1) \cdot ([1], h) = ([1], h) \cdot ([n], 1) = ([\alpha(h, n)], h)$. $\square$

Thus α is equivalent to the trivial action and so multiplication in $N \rtimes_{E,\alpha}$ is given by $([n], h) \cdot ([n'], h') = ([nn'], hh')$. This is of course in agreement with our understanding of the Artin glueing case and is analgous to the case of abelian groups.

In the subcategory of abelian groups where all admissible quotients must be discrete we then find that the only weakly Schreier split extension is the direct product.

Remark 3.4.22. Originally in place of the H-indexed equivalence relation, this characterization was taken to be an equivalence relation on $N \times H$. The perspective in terms of H-indexed equivalence relations resulted from conversations with Graham Manuell, and it was not long before we realised that the axioms for the H-indexed equivalence relation could be represented in the following way: E should be an H-indexed right-congruence, which preserves order with respect to left divisibility of H, as well as the bottom elements. This new perspective allowed me to find a link in the literature that was not obviously related before.

In (Köhler [25]), Kohler characterised quasi-decompositions of monoids with his characterization of the form just described above. It is not hard to see that quasi-decompositions correspond to weakly Schreier split extensions and vice versa. $\triangle$

3.5 Constructing examples

In this section we concern ourselves with the practicalities of constructing weakly Schreier split extensions.

3.5.1 Generalising the Artin glueing

We present a construction reminiscent of Artin glueing, for weakly Schreier split extensions of H by N, where N is commutative.

Proposition 3.5.1. *Let N be a commutative monoid and $f\colon H \to N$ a monoid homomorphism. Then the H-indexed equivalence relation E on N given by $n \sim^h n'$ if and only if $n \cdot f(h) = n' \cdot f(h)$ is admissible.*

Proof. By definition we have that $n \sim^1 n'$ implies that $n = n'$. If $n_1 \sim^h n_2$ then we get $n_1 \cdot f(h) = n_2 \cdot f(h)$. Multiplying both sides on the left by n yields $n \cdot n_1 \cdot f(h) = n \cdot n_2 \cdot f(h)$ which then gives that $nn_1 \sim^h nn_2$ for all $n \in N$. If instead we multiplied both sides of the equation on the right by $f(h')$ and use that f is a monoid homomorphism, we see that $n_1 \sim^{hh'} n_2$ for all h'. Thus E is admissible. $\square$

Proposition 3.5.2. *Let N be a commutative monoid. Then if we have the trivial action $\alpha(h,n) = n$, we have that (E,α) is a relaxed action where E is taken from Proposition 3.5.1.*

Proof. Consider the function $\alpha(h,n) = n$ for all $h \in H$ and $n \in N$. Were (E,α) a relaxed action it would yield multiplication $([n],h)([n'],h') = ([n \cdot \alpha(h,n')],hh') = ([nn'],hh')$. Let us now show that α is a compatible action which entails showing that it satisfies the six condition in Definition 3.4.11. However only the first two need to be checked as the others follow from α being an action.

First suppose that $n_1 \sim^h n_2$ and consider $n_1 \cdot \alpha(h,n)$ and $n_2 \cdot \alpha(h,n))$. In order to show that they are h-related in E we consider the following.

$$\begin{aligned} n_1 \cdot \alpha(h,n) \cdot f(h) &= n_1 n \cdot f(h) \\ &= n n_1 \cdot f(h) \\ &= n n_2 \cdot f(h) \\ &= n_2 \alpha(h,n) \cdot f(h) \end{aligned}$$

Similarly if we let $n \sim^y n'$ we can consider $([\alpha(h,n)],hy)$ and $([\alpha(h,n')],hy)$. We perform a similar calculation

$$\begin{aligned} \alpha(h,n) \cdot f(h) \cdot f(h') &= n \cdot f(h) \cdot f(h') \\ &= n \cdot f(h') \cdot f(h) \\ &= n' \cdot f(h') \cdot f(h) \\ &= \alpha(h,n') \cdot f(h) \cdot f(h'), \end{aligned}$$

and see that indeed $\alpha(h,n) \sim^{hy} \alpha(h,n')$. Thus α is indeed compatible with E. $\square$

Notice that two monoid homomorphisms $f,g \colon H \to N$ may yield the same weakly Schreier split extension. In fact when N is right-cancelative, each homomorphism $f \colon H \to N$ yields the usual product $N \times H$.

Example 3.5.3. Let N be a meet-semilattice and $f \colon H \to N$ a monoid map. Then $N \overset{k}{\rightarrowtail} \mathrm{Gl}(f) \underset{s}{\overset{e}{\rightleftarrows}} H$ is a weakly Schreier split extension, where $\mathrm{Gl}(f)$ is the set of pairs (n,h) in which $n \le f(h)$ with $(n,h) \cdot (n',h') = (n \wedge n', h \cdot h')$, $k(n) = (n,1)$, $e(n,h) = h$ and $s(h) = (f(h),h)$.

To see this consider the admissible quotient given by E as in Proposition 3.5.1. Notice that for each element $([n], h)$ there exists a smallest element in $[n]$ given by $n \wedge f(h)$. Picking this as a canonical element we easily see that the set of these representatives is $\mathrm{Gl}(f)$.

Proposition 3.5.2 gives that the action $\alpha(h, n) = n$ for all $n \in N$ and $h \in H$ is compatible with E. This induces multiplication $([n], h) \cdot ([n'], h') = ([n \wedge n'], h \cdot h')$. If $(n, h), (n', h') \in \mathrm{Gl}(f)$ then $(n \wedge n', h \cdot h') \in \mathrm{Gl}(f)$. This gives that α induced componentwise multiplication on $\mathrm{Gl}(f)$ and so we are done. △

3.5.2 The coarsest admissible quotient

Schreier split extensions of H by N may be thought of as the weakly Schreier split extensions with the finest admissible quotient on $N \times H$. As discussed above, (Martins-Ferreira, Montoli, and Sobral [33]) provides a complete characterization of all actions compatible with this discrete quotient. Dual to this problem might be considering the coarsest admissible quotient and characterizing the actions compatible with it. In this section we give a partial answer to this question. We will again return to this question in Chapter 5, where a new perspective allows us to give a more detailed answer.

As discussed in Proposition 3.4.2, if $h \in H$ has a right inverse, then $n \sim^h n'$ implies that $n = n'$. Taking this as our only constraint, we can consider the equivalence relation generated by the condition that for $n \neq n'$, $n \sim^h n'$ if and only if h has no right inverse. If admissible, this would be the coarsest admissible equivalence relation on N.

Let $L(H) \subseteq H$ be the submonoid of right invertible elements — that is, elements h with right inverses and let $\overline{L(H)}$ be the set of elements which are not right invertible.

Proposition 3.5.4. *Let $(E, [\alpha]) \in \mathrm{RAct}(H, N)$. Then $\alpha\,|_{L(H) \times N}$ is an action of $L(H)$ on N.*

Proof. Proposition 3.4.2 gives us that for all $x \in L(H)$, $n \sim^x n'$ implies that $n = n'$. Thus $\alpha(x, nn') \sim^x \alpha(x, n)\alpha(x, n')$ implies that $\alpha(x, nn') = \alpha(x, n)\alpha(x, n')$.

Similarly if $x, x' \in L(H)$ we get that $\alpha(xx', n) \sim^{xx'} \alpha(x, \alpha(x', n))$ implies $\alpha(xx', n) = \alpha(x, \alpha(x', n))$, as $xx' \in L(H)$.

Finally it is easy to see that these same arguments give that $\alpha(1, n) = n$ and $\alpha(x, 1) = 1$. □

This result tells us that in general there are some maps $\alpha\colon H \times N \to N$ such that no admissible quotient makes it an action.

We also get the following corollary.

Corollary 3.5.5. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension and* $N \rtimes H$ *the associated weak semidirect product. Then the submonoid of pairs whose second component lies in* $L(H)$ *is itself part of a Schreier split extension of* $L(H)$ *by* N *given by* $(N \times L(H), [\alpha|_{L(H)\times N}])$.

Thus in each weakly Schreier split extension we understand how a particular submonoid behaves. Let us now study how the rest of the monoid behaves.

Proposition 3.5.6. *Let* H *be a monoid. The subset* $\overline{L(H)} \subseteq H$ *is a monoid right-ideal — that is, if* $y \in \overline{L(H)}$ *and* $x \in H$ *then* $yx \in \overline{L(H)}$.

Proof. Suppose $y \in \overline{L(H)}$ and let $x \in H$. Then if $yx(yx)^* = 1$, that would imply that $x(yx)^*$ was a right inverse for y, contradicting the fact that $y \in \overline{L(H)}$. □

Corollary 3.5.7. *Let* $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ *be a weakly Schreier split extension and* $N \rtimes H$ *the associated weak semidirect product. Then the subset of* $N \rtimes H$ *comprising the elements* $([n], y)$ *where* $y \in \overline{L(H)}$, *is a right ideal of* $N \rtimes H$.

Definition 3.5.8. Let E be the equivalence relation given by the following

i) $n \sim^h n'$ implies $n = n'$ whenever $h \in L(H)$ and,

ii) $n \sim^h n'$ for all $n, n' \in N$ whenever $h \in \overline{L(H)}$.

We call E the *coarse equivalence relation* on N. △

Proposition 3.5.9. *The coarse quotient on* N *is admissible.*

Proof. Since $1 \in L(H)$ we immediately get that $n \sim^1 n'$ implies that $n = n'$.

Now suppose $n_1 \sim^h n_2$. If $h \in L(H)$ then $n_1 = n_2$ and so for all $n \in N$ and $h' \in H$ we have $nn_1 \sim^h nn_2$ and $n_1 \sim^{hh'} n_2$ by virtue of their equality.

If instead $h \in \overline{L(H)}$ it is immediate that for all $n \in N$, $nn_1 \sim^h nn_2$. For $h' \in H$ we have that $hh' \in \overline{L(H)}$ as $\overline{L(H)}$ is a right ideal and so $n_1 \sim^{hh'} n_2$. □

Proposition 3.5.10. *Suppose that* $\overline{L(H)}$ *is a two-sided ideal. Each map* $\alpha\colon H \times N \to N$ *in which* $\alpha|_{L(H)\times N}$ *is an action of* $L(H)$ *on* N, *is compatible with the coarse equivalence relation. If* $\overline{L(H)}$ *is not a two-sided ideal and* $N \neq \{1\}$ *then no function* $\alpha\colon H \times N \to N$ *is compatible with the coarse equivalence relation.*

Proof. Suppose $\overline{L(H)}$ is an ideal and suppose $\alpha|_{L(H)\times N}$ is an action of $L(H)$ on N.

The first compatibility condition in Definition 3.4.11 says that whenever $n_1 \sim^h n_2$ that $n_1\alpha(h,n) \sim^h n_2\alpha(h,n)$ for all $n \in N$. If $h \in L(H)$ then $n_1 = n_2$ and so this holds. If $h \in \overline{L(H)}$ then all elements are related and so this also holds.

The second compatibility condition requires that if $n_1 \sim^y n_2$ then $\alpha(h,n_1) \sim^{hy} \alpha(h,n_2)$ for all $h \in H$. Now again if $y \in L(H)$, $n_1 = n_2$ and we have that the condition is satisfied. If $y \in \overline{L(H)}$ then $hy \in \overline{L(H)}$ as $\overline{L(H)}$ is a two-sided ideal. Thus $\alpha(h,n_1) \sim^{hy} \alpha(h,n_2)$, as all such elements are related.

We know that $\alpha\,|_{L(H)\times N}$ is an action of $L(H)$ on N and so we have that for all $x \in L(H)$ and $n,n' \in N$:

i) $\alpha(x,nn') \sim^x \alpha(x,n)\alpha(x,n')$,

ii) $\alpha(1,n) \sim^1 n$,

iii) $\alpha(x,1) \sim^x 1$.

By definition of the coarse quotient we have that for all $y \in L(H)$ and $n,n' \in N$:

i) $\alpha(y,nn') \sim^y \alpha(y,n)\alpha(y,n')$,

ii) $\alpha(y,1) \sim^y 1$.

Since $L(H)$ and $\overline{L(H)}$ are complements this means that the only condition remaining is to check that for all $h,h' \in H$ and $n \in N$ we have that $\alpha(hh',n) \sim^{hh'} \alpha(h,\alpha(h',n))$. Now if either $h \in \overline{L(H)}$ or $h' \in \overline{L(H)}$ we will have $hh' \in \overline{L(H)}$ which will immediately give equality. If neither h nor h' are elements of $\overline{L(H)}$ then they both belong to $L(H)$ and so using the fact that $\alpha|_{L(H)\times N}$ is an action of $L(H)$ on N we have that $\alpha(hh',n) \sim^{hh'} \alpha(h,\alpha(h',n))$. Thus α is compatible with the coarse quotient Q.

If $N \neq \{1\}$ and $\overline{L(H)}$ is not an ideal then there exist elements $y \in \overline{L(H)}$ and $x \in L(H)$ such that $xy \in L(H)$. Now given some function $\alpha\colon H \times N \to N$, for the second compatibility condition to hold we need in particular that $n_1 \sim^y n_2$ implies that $\alpha(x,n_1) \sim^{xy} \alpha(x,n_2)$ which must then imply that $\alpha(x,n_1) = \alpha(x,n_2)$ for all $n_1,n_2 \in N$. But we also know that $\alpha(x,1) = 1$ and so $\alpha(x,n) = 1$ for all n. Now let x^* be the left inverse of x. We know that for all $n \in N$ we have

$$\begin{aligned} n &\sim^1 \alpha(1,n) \\ &= \alpha(xx^*,n) \\ &\sim^1 \alpha(x,\alpha(x^*,n)) \\ &= 1. \end{aligned}$$

This is a contradiction and so α cannot be compatible with the coarse equivalence relation. $\square$

Whenever H is finite, commutative, a group or has no (non-trivial) inverses at all, then this right ideal $\overline{L(H)}$ will be a two-sided ideal. Similarly if H is a monoid of $n \times n$ matrices over a field, then this same property holds.

Example 3.5.11. Let H and N both be the monoid of $n \times n$ matrices with entries from some field K. Matrices with right inverses always have two-sided inverses and so have non-zero determinant. Thus the coarsest equivalence relation E on N gives that when $\det(B) = 0$ we have that $A \sim^B A'$ for all $A, A' \in N$ and when $\det(B) \neq 0$ we have $A \sim^B A'$ if and only if $A = A'$. Observe that the submonoid $L(H)$ is actually the group $\mathrm{GL}(n, K)$. Thus we can consider the map $\alpha\colon H \times N \to N$ where $\alpha(B, A) = BAB^{-1}$ whenever $\det(B) \neq 0$ and $\alpha(B, A) = A$ otherwise. This is clearly an action of $\mathrm{GL}(n, K)$ on N.

Note that the right ideal $\overline{L(H)}$ is a two-sided ideal due to the multiplicative nature of the determinant. Thus this map α is compatible with the coarsest quotient Q.$\triangle$

Example 3.5.12. Suppose that besides the identity H has no elements with right inverses. Then the disjoint union $N \sqcup (H - \{1\})$ can be equipped with the following multiplication and made a monoid. Let $n, n' \in N$ and $h, h' \in H - \{1\}$. Then

i) $n \cdot n' = n \cdot_N n'$,

ii) $h \cdot h' = h \cdot_H h'$,

iii) $n \cdot h = h = h \cdot n$.

The extension $N \overset{k}{\rightarrowtail} N \sqcup (H - \{1\}) \underset{s}{\overset{e}{\rightleftarrows}} H$ is weakly Schreier as it is isomorphic to the weakly Schreier split extension given by $(E, [\alpha])$ where E is the coarse quotient on N and $\alpha(h, n) = n$. The isomorphism is given by sending n to $([n], 1)$ and h to $([1], h)$. $\triangle$

Crucial in the proof of these last few results is that we have partitioned the elements of H into a submonoid and an ideal. Ideals whose complements are submonoids are called prime. We can generalise the above results for any prime ideal Y. Replace the coarse quotient with a new quotient Q in which for $y \in Y$ we have that $n \sim^y n'$ for all $n, n' \in N$ and for $x \in H - Y$ we have that $n \sim^x n'$ implies that $n = n'$. Crucially a non-trivial prime ideal can never contain a right invertible element, for if it did it would contain the identity. Thus E is seen to be admissible and the above results can be seen to equally apply to this construction.

Chapter 4

λ-semidirect products

With our characterization of weakly Schreier split extensions, we spend a chapter studying a new example of one, the λ-semidirect product of two inverse semigroups from the new perspective offered.

4.1 Introduction

The ideas underlying the semidirect product of groups can be adapted to a number of structures. One such example is the context of semigroups wherein an action of semigroups $\alpha\colon H \times N \to N$ gives rise to a semidirect product $N \rtimes_\alpha H$ defined just as in the group case. These semidirect products have found much use in semigroup theory, for instance they provide some insight into the structure of inverse semigroups (McAlister [36]). However, when applied naively to two inverse semigroups, this semidirect product does not in general yield an inverse semigroup. To remedy this Billhardt introduced a related notion called a λ-semidirect product (Billhardt [2]). Given inverse semigroups N and H the idea is to use an action of H on N to equip a certain *subset* of $N \times H$ (determined by the action) with a multiplication turning it into an inverse semigroup.

When we restrict this construction to inverse monoids we find that just as with groups, these λ-semidirect products naturally form a split extension. Here a split extension is a diagram $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ in which k is the kernel of e, e is the cokernel of k and s is a section of e. Our restriction in this chapter to inverse monoids may not be strictly necessary as there does exist a notion of a split extensions between general semigroups. These split extensions play a role in the structure theory of regular semigroups, analogous to the role λ-semidirect products play in the structure theory of inverse semigroups (McAlister and Blyth [35]).

Outline

In this chapter we will show that the λ-semidirect products of inverse monoids are also examples of weakly Schreier extensions. In fact this subsumes our previous example since we show that Artin glueings are examples of λ-semidirect products. The characterization of weakly Schreier extensions sheds some light on λ-semidirect products. The set of weakly Schreier extensions between two monoids comes with a natural poset structure, which induces an order on the λ-semidirect products between two inverse monoids. The Artin glueing leads us to define a class of Artin-like λ-semidirect products. We show that this class is closed under binary joins.

4.2 Background

Definition 4.2.1. A semigroup S is called an *inverse semigroup* when for each $x \in S$ there exists a unique element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. △

Note that when S is a group this notion of inverse will agree with the usual group inverse. Given an inverse semigroup S and an element $x \in S$ it can be easily seen that xx^{-1} and $x^{-1}x$ are idempotents. It is also not hard to see that if x^{-1} is the inverse of x, that x is the inverse of x^{-1} and so $(x^{-1})^{-1} = x$.

Proposition 4.2.2. *Let X be an inverse semigroup. The subset of idempotents is a commutative sub-inverse semigroup.*

Proof. First note that if x is an idempotent then $x^{-1} = x$. Now by the uniqueness of inverses it may be verified that $(xy)^{-1} = y(xy)^{-1}x$ which is readily seen to be idempotent. Hence $(xy)^{-1}$ is idempotent and hence $xy = (xy)^{-1}$ is idempotent. Now we simply chain these results together to get $xy = (xy)^{-1} = y(xy)^{-1}x = yxyx$. But we know that the product of idempotents is idempotent, hence yx is idempotent. Thus the expression reduces to $xy = yx$. □

Proposition 4.2.3. *Let X be an inverse semigroup. For any two elements $x, y \in X$ we have that $(xy)^{-1} = y^{-1}x^{-1}$.*

Proof. The one direction follows from the following simple calculation.

$$\begin{aligned} xy(y^{-1}x^{-1})xy &= x(yy^{-1})(x^{-1}x)y \\ &= x(x^{-1}x)(yy^{-1})y \\ &= xy \end{aligned}$$

Where yy^{-1} commutes past $x^{-1}x$ since they are idempotents.

The other direction is similar. □

For more elementary facts see the first few chapters of (Lawson [27]).

As discussed above, the standard semigroup semidirect product construction, when applied to two inverse semigroups, will not in general return an inverse semigroup. Thus, we study Billhardt's λ-*semidirect product* (Billhardt [2]). The idea is to consider an algebraic structure on a subset of the product of two inverse semigroups.

Definition 4.2.4. Let N and H be inverse semigroups and let $\alpha\colon H \times N \to N$ be a function which we write as $\alpha(h, n) = h \cdot n$. Then α is an *action of inverse semigroups* if the following conditions are satisfied for all $h, h' \in H$ and $n, n' \in N$.

i) $h \cdot (nn') = (h \cdot n)(h \cdot n')$,

ii) $hh' \cdot n = h \cdot (h' \cdot n)$. △

An action could, of course, equivalently be defined as a homomorphism from H into the endomorphisms of N.

Definition 4.2.5. Let N and H be inverse semigroups and let H act on N. Then the λ-semidirect product associated to this action has as underlying set

$$\{(n, h) \in N \times H : hh^{-1} \cdot n = n\}$$

and multiplication defined by

$$(n_1, h_1)(n_2, h_2) = \Big(((h_1h_2)(h_1h_2)^{-1} \cdot n_1)(h_1 \cdot n_2), h_1h_2\Big) . \qquad \triangle$$

This multiplication resembles the multiplication of the standard semidirect product in a number of ways. The only disagreement is that instead of $(h \cdot n_2)$ being multiplied on the left by n_1, it is being multiplied on the left by $(h_1h_2)(h_1h_2)^{-1} \cdot n_1$.

4.3 λ-semidirect products of inverse monoids

In all that follows N, G and H denote inverse monoids unless otherwise stated.

In order to relate the λ-semidirect product to weakly Schreier extensions of monoids, we must work inside the category of monoids. Thus, in this section we consider only *inverse monoids* — that is, inverse semigroups with a unit.

In order to consider λ-semidirect products in this context there is one standard modification that is made to the theory, relating to the definition of an action.

Definition 4.3.1. Let N and H be inverse monoids and let $\alpha\colon H \times N \to N$ be a function with application written $\alpha(h, n) = h \cdot n$. Then α is an *action of inverse monoids* if it is an action of inverse semigroups and satisfies that for all $n \in N$

$$1 \cdot n = n. \qquad \triangle$$

Notably, it is not required that $h \cdot 1 = 1$. Thus, the action can equivalently be thought of as a monoid homomorphism into the monoid of semigroup endomorphisms of N.

The λ-semidirect products we consider in this context are only taken with respect to actions of inverse monoids, as these are precisely the actions for which the associated λ-semidirect product is a monoid. (The pair $(1, 1)$ acts as identity.)

Proposition 4.3.2. *Let N and H be inverse monoids and let $\alpha\colon H \times N \to N$ be an action of inverse monoids. If $N \rtimes_\alpha H$ is the associated λ-semidirect product, then the following functions are monoid homomorphisms.*

i) $k\colon N \to N \rtimes_\alpha H$, where $k(n) = (n, 1)$,

ii) $e\colon N \rtimes_\alpha H \to H$, where $e(n, h) = h$,

iii) $s\colon H \to N \rtimes_\alpha H$, where $s(h) = (hh^{-1} \cdot 1, h)$.

Proof. (i) We begin by proving that the function is well defined. This entails showing that $1(1^{-1}) \cdot n = n$. Since the inverse of 1 is 1 we use the fact that α is an action of inverse monoids.

Next observe that

$$\begin{aligned} k(n_1)k(n_2) &= (n_1, 1)(n_2, 1) \\ &= ((1(1^{-1}) \cdot n_1)(1 \cdot n_2), 1) \\ &= (n_1 n_2, 1) \\ &= k(n_1 n_2). \end{aligned}$$

It is clear the unit is preserved.

(ii) The function is automatically well defined and it is very easy to see that it preserves the multiplication and unit.

(iii) Again we begin by proving it is well defined. We must show that $(hh^{-1}) \cdot (hh^{-1} \cdot 1) = hh^{-1} \cdot 1$. This follows from the fact that α is action of semigroups and that hh^{-1} is an idempotent.

Finally observe the following calculation.

$$\begin{aligned}
s(h_1)s(h_2) &= (h_1h_1^{-1}\cdot 1, h_1)(h_2h_2^{-1}\cdot 1, h_2)\\
&= (((h_1h_2)(h_1h_2)^{-1}\cdot h_1h_1^{-1}\cdot 1)(h_1\cdot h_2h_2^{-1}\cdot 1), h_1h_2)\\
&= ((h_1h_2h_2^{-1}h_1^{-1}h_1h_1^{-1}\cdot 1)(h_1h_2h_2^{-1}\cdot 1), h_1h_2)\\
&= (h_1h_2h_2^{-1}\cdot ((h_1^{-1}h_1h_1^{-1}\cdot 1)(1)), h_1h_2)\\
&= (h_1h_2h_2^{-1}h_1^{-1}\cdot 1, h_1h_2)\\
&= s(h_1h_2).
\end{aligned}$$

Where the transition from the third line to the fourth involves factoring out $h_1h_2h_2^{-1}$ from both terms.

Finally, note that $s(1) = (1(1^{-1})\cdot 1, 1) = (1\cdot 1, 1) = (1, 1)$, the identity. □

It is apparent that k is the kernel of e and that s splits e. Below we show that this diagram is indeed a weakly Schreier extension.

Proposition 4.3.3. *Let N and H be inverse monoids, $\alpha\colon H\times N\to N$ an action of inverse monoids, $N\rtimes_\alpha H$ the associated λ-semidirect product and let k, e and s be as in Proposition 4.3.2. Then $N \overset{k}{\rightarrowtail} N\rtimes_\alpha H \underset{s}{\overset{e}{\rightleftarrows}} H$ is a weakly Schreier extension.*

Proof. As discussed, it is apparent that k is the kernel and s is the splitting of e. Thus, we must only show that e is the cokernel of k and that the weakly Schreier condition holds. We begin with the latter. Let $(n,h)\in N\rtimes_\alpha H$ and consider

$$\begin{aligned}
k(n)s(h) &= (n,1)(hh^{-1}\cdot 1, h)\\
&= ((hh^{-1}\cdot n)(1\cdot hh^{-1}\cdot 1), h)\\
&= (hh^{-1}\cdot n, h)\\
&= (n,h).
\end{aligned}$$

Here the last line follows because (n,h) was assumed to belong to $S\rtimes_\alpha T$.

To see that e is the cokernel consider a map $t\colon N\rtimes_\alpha H\to X$ such that tk is the zero morphism. We must show that there is a unique map $\ell\colon H\to X$ such that $t=\ell e$.

By the above $t(n,h) = t(k(n)s(h)) = ts(h)$. We then need only observe that for $\ell = ts$ we have $\ell e(n,h) = ts(h)$, as required. Since e has a splitting, it is epic and consequently the map $\ell = ts$ must be unique. □

Since λ-semidirect products of inverse monoids N and H are weakly Schreier extensions, we can view them instead as relaxed semidirect products, corresponding to particular relaxed actions.

4.3.1 The relaxed action interpretation

Let α be an action of inverse monoids of H on N and let $N \overset{k}{\rightarrowtail} N \rtimes_\alpha H \underset{s}{\overset{e}{\leftrightarrows}} H$ be the weakly Schreier extension corresponding to the associated λ-semidirect product. Then n_1 and n_2 will be h-related if and only if $k(n_1)s(h) = k(n_2)s(h)$. This amounts to requiring that $hh^{-1} \cdot n_1 = hh^{-1} \cdot n_2$.

Proposition 4.3.4. *Let $N \overset{k}{\rightarrowtail} N \rtimes_\alpha H \underset{s}{\overset{e}{\leftrightarrows}} H$ be the weakly Schreier extension corresponding to a λ-semidirect product. Then $n \sim^h hh^{-1} \cdot n$ in the associated admissible equivalence relation.*

Proof. We need only verify that $hh^{-1} \cdot n = hh^{-1} \cdot hh^{-1} \cdot n$. This follows from α being an action of semigroups and from the idempotence of hh^{-1}. □

This means that each equivalence class $[n]_h$ of h-related elements has a canonical representative $hh^{-1} \cdot n$. The set of these representatives is easily seen to be the underlying set of $S \rtimes_\alpha T$.

In order to determine a compatible action we first consider the associated Schreier retraction. It is easy to see that the first projection $\pi_1 \colon N \rtimes_\alpha H \to N$ is such a map. (Recall that Schreier retractions need not be monoid homomorphisms.) Given this Schreier retraction the compatible action is thus $\beta \colon H \times N \to N$ where

$$\begin{aligned}
\beta(h, n) &= \pi_1(s(h)k(n)) \\
&= \pi_1((hh^{-1} \cdot 1, h)(n, 1)) \\
&= \pi_1((hh^{-1} \cdot hh^{-1} \cdot 1)(h \cdot n), h) \\
&= (hh^{-1} \cdot 1)(h \cdot n) \\
&= (hh^{-1} \cdot 1)(hh^{-1}h \cdot n) \\
&= hh^{-1} \cdot (1(h \cdot n)) \\
&= h \cdot n.
\end{aligned}$$

Thus, the compatible action β is just the original action α.

Recall that from the weakly Schreier perspective the multiplication is given by

$$([n_1], h_1)([n_2], h_2) = ([n_1(h_1 \cdot n_1)], h_1 h_2)$$

The element $n_1(h_1 \cdot n_2)$ will not in general be the canonical element of its equivalence class of h_1h_2 related elements. Thus, we pass to the canonical element and arrive at

$$\begin{aligned} h_1h_2(h_1h_2)^{-1} \cdot (n_1(h_1 \cdot n_2) &= (h_1h_2(h_1h_2)^{-1} \cdot n_1)(h_1h_2(h_1h_2)^{-1} \cdot h_1 \cdot n_2) \\ &= (h_1h_2(h_1h_2)^{-1} \cdot n_1)(h_1 \cdot (h_2h_2^{-1} \cdot n_2)). \end{aligned}$$

Note that if $(n_2, h_2) \in N \rtimes_\alpha H$, then the expression reduces to

$$(h_1h_2(h_1h_2)^{-1} \cdot n_1)(h_1 \cdot n_2),$$

which corresponds precisely with the multiplication in $N \rtimes_\alpha H$.

Notice that the monoid actions in this setting completely specify the relaxed action as they determine *both* the admissible equivalence relation as well as the compatible action.

4.4 The order on λ-semidirect products

Since the set of weakly Schreier extensions between monoids N and H has a natural preorder structure, we can now ask what order this induces on the set of λ-semidirect products when we take N and H to be inverse monoids.

It will be convenient to think in terms of the actions of inverse monoids instead of the λ-semidirect products themselves. Thus, we consider the preorder induced on the set of actions by the function sending an action to its associated weakly Schreier extension.

This function is not injective as two distinct actions can be mapped to isomorphic weakly Schreier extensions.

Example 4.4.1. Let N be an inverse monoid with at least two distinct idempotents u and u' and let H be an inverse semigroup satisfying that $h_1h_2 = 1$ implies $h_1 = 1 = h_2$.

Consider the function $\alpha_u \colon H \times N \to N$ where $\alpha_u(h, n) = u$ whenever $h \neq 1$ and $\alpha_u(1, n) = n$. Because $h_1h_2 = 1$ implies $h_1 = 1 = h_2$ we have that α_u is an action of inverse monoids.

Similarly, consider the action $\alpha_{u'} \colon H \times N \to N$ where $\alpha_{u'}(h, n) = u'$ whenever $h \neq 1$ and $\alpha_{u'}(1, n) = n$.

It is apparent that $\alpha_u \neq \alpha_{u'}$. Furthermore, both actions result in an equivalence relation in which $n_1 \sim^h n_2$ for all $n_1, n_2 \in N$ and $h \in H - \{1\}$, and $n_1 \sim^1 n_2$ if and only if $n_1 = n_2$. The multiplications agree as required, as in both equivalence relations we have that $\alpha_u(h,n) \sim^h \alpha_{u'}(h,n)$. △

Proposition 4.4.2. *Let N and H be inverse monoids and let $\alpha\colon H \times N \to N$ and $\beta\colon H \times N \to N$ be actions of inverse monoids. Then $\alpha \leq \beta$ if and only if for all $n \in N$ and $h \in H$, $\beta(hh^{-1}, \alpha(h,n)) = \beta(h,n)$.*

Proof. ($\Rightarrow$) Suppose that $\alpha \leq \beta$.

Let E_α and E_β be the admissible equivalence relations corresponding to α and β respectively. By assumption $\alpha(h,n) \sim^h_{E_\beta} \beta(h,n)$. Unwinding this gives

$$\begin{aligned}\beta(hh^{-1}, \alpha(h,n)) &= \beta(hh^{-1}, \beta(h,n)) \\ &= \beta(h,n).\end{aligned}$$

($\Leftarrow$) Suppose that for all $n \in N$ and $h \in H$, $\beta(hh^{-1}, \alpha(h,n)) = \beta(h,n)$.

First we show that $n_1 \sim^h_{E_\alpha} n_2$ implies $n_1 \sim^h_{E_\beta} n_2$. Suppose that $n_1 \sim^h_{E_\alpha} n_2$. This means that $\alpha(hh^{-1}, n_1) = \alpha(hh^{-1}, n_2)$. Thus, making use of our assumption we find

$$\begin{aligned}\beta(hh^{-1}, n_1) &= \beta(hh^{-1}(hh^{-1})^{-1}, \alpha(hh^{-1}, n_1)) \\ &= \beta(hh^{-1}, \alpha(hh^{-1}, n_1)) \\ &= \beta(hh^{-1}, \alpha(hh^{-1}, n_2)) \\ &= \beta(hh^{-1}, n_2).\end{aligned}$$

Now we must show that $\alpha(h,n) \sim^h_{E_\beta} \beta(h,n)$. For these to be related we need that $\beta(hh^{-1}, \alpha(h,n)) = \beta(hh^{-1}, \beta(h,n))$. By assumption $\beta(hh^{-1}, \alpha(h,n)) = \beta(h,n)$ and combined with the fact that $\beta(h,n) = \beta(hh^{-1}, \beta(h,n))$, we obtain the desired equality. □

4.5 Artin-glueing-like actions

Given an order structure on the set of λ-semidirect products, it is natural to consider if meets and joins exist. In the section we show that joins exist for a natural class of λ-semidirect products, reminiscent of Artin glueings of frames.

As alluded to in the introduction, Artin glueings of frames are nothing more than a certain class of λ-semidirect products between certain meet-semilattices.

Proposition 4.5.1. *Let N and H be frames considered in the category of monoids* *and let* $f\colon H \to N$ *be a monoid homomorphism. Then the Artin glueing* $\mathrm{Gl}(f)$ *is a* λ*-semidirect product of* N *by* H*.*

Proof. The action corresponding to $\mathrm{Gl}(f)$ is given by $\alpha(h,n) = f(h) \wedge n$. Let us begin by confirming that this is an action of inverse monoids.

It is clear that $\alpha(1,n) = n$ as f preserves the identity. Next observe following.

$$\begin{aligned}\alpha(h, n \wedge n') &= f(h) \wedge n \wedge n' \\ &= f(h) \wedge n \wedge f(h) \wedge n' \\ &= \alpha(h,n) \wedge \alpha(h,n').\end{aligned}$$

Finally consider the calculation below.

$$\begin{aligned}\alpha(h \wedge h', n) &= f(h \wedge h') \wedge n \\ &= f(h) \wedge f(h') \wedge n \\ &= \alpha(h, f(h') \wedge n) \\ &= \alpha(h, \alpha(h', n)).\end{aligned}$$

Thus, it remains only to show that $N \rtimes_\alpha H = \mathrm{Gl}(f)$.

Since the inverse of an element in a meet semilattice is itself and because of idempotence, we have that the elements of $N \rtimes_\alpha H$ are those pairs (n,h) in which $n = f(h) \wedge n$. These are precisely the pairs in which $n \le f(h)$ and so $N \rtimes_\alpha H$ and $\mathrm{Gl}(f)$ agree on elements.

Using the same properties of meet-semilattices we see that the multiplication in $N \rtimes_\alpha H$ is given by

$$\begin{aligned}(n,h)(n',h') &= ((f(h) \wedge f(h') \wedge n) \wedge (f(h) \wedge n'), h \wedge h') \\ &= (f(h) \wedge n \wedge f(h') \wedge n', h \wedge h') \\ &= (n \wedge n', h \wedge h').\end{aligned}$$

This coincides with the multiplication of $\mathrm{Gl}(f)$ and so we are done. □

As discussed in Chapter 2, if N and H are frames and $f, g\colon H \to N$ are monoid homomorphisms, then $\mathrm{Gl}(f \wedge g)$ is the join of $\mathrm{Gl}(f)$ and $\mathrm{Gl}(g)$ in the order structure on Artin glueings. In fact, as we shall see, $\mathrm{Gl}(f \wedge g)$ is the join of $\mathrm{Gl}(f)$ and $\mathrm{Gl}(g)$ in $\mathrm{WSExt}(H,N)$.

Inspired by the above, we would like to consider actions α of inverse monoids such that $\alpha(h, n) = f(h) \cdot n$ where f is some function from H into N. The condition that α be an action precludes many functions f from serving this purpose. It is sufficient for f to factor through the central idempotents of S.

Proposition 4.5.2. *Let H and N be inverse monoids and let $f \colon H \to E(N) \cap Z(N)$ be a monoid homomorphism into the central idempotents of N, where $E(N)$ denotes the idempotents of N and $Z(N)$ the central elements. Then $\alpha(h, n) = f(h) \cdot n$ is an action of inverse monoids.*

Proof. For $\alpha(h, n_1 n_2)$ we have the following.

$$\begin{aligned} \alpha(h, n_1 n_2) &= f(h) \cdot n_1 n_2 \\ &= f(h) f(h) \cdot n_1 n_2 \\ &= f(h) n_1 \cdot f(h) n_2 \\ &= \alpha(h, n_1) \alpha(h, n_2), \end{aligned}$$

The second equality makes use of the fact that $f(h)$ is an idempotent, and the third makes use of the fact that $f(h)$ is central.

Next we must check that $\alpha(h_1 h_2, n) = \alpha(h_1, \alpha(h_2, n))$. Here we consider

$$\begin{aligned} \alpha(h_1 h_2, n) &= f(h_1 h_2) \cdot n \\ &= f(h_1) f(h_2) \cdot n \\ &= f(h_1) \cdot \alpha(h_2, n) \\ &= \alpha(h_1, \alpha(h_2, n)). \end{aligned}$$

The final condition follows easily with $\alpha(1, n) = f(1) \cdot n = 1 \cdot n = n$. □

Definition 4.5.3. Let H and N be inverse monoids and $f \colon H \to E(N) \cap Z(N)$ a monoid homomorphism into the central idempotents of N. Then we call the action $\alpha_f(h, n) = f(h) \cdot n$ the *Artin-like-action* corresponding to f. △

Notice that in the associated relaxed action, the action part may always be taken to be the trivial action.

The λ-semidirect products resulting from Artin-like-actions have many nice properties. For instance, when interpreted as weakly Schreier extensions, the canonical element of each equivalence class can be easily seen to be the smallest element in each class.

Furthermore, just as in the frame setting, we can combine two actions of this form in a natural way.

Proposition 4.5.4. *Let N and H be inverse semigroups and let α_f, α_g be Artin-like-actions corresponding to the maps $f, g\colon H \to E(N) \cap Z(N)$ respectively. Then the action $\gamma\colon H \times N \to N$ given by $\gamma(h, n) = f(h)g(h)n$, is an Artin-like-action.*

Proof. We claim that γ corresponds to $\alpha_{f\cdot g}$, where $f \cdot g(h) = f(h)g(h)$. It is clear that $f \cdot g$ preserves the identity. To see that it preserves multiplication we make use of the fact that both f and g map into the centre of N. Thus we have

$$\begin{aligned} f \cdot g(h_1h_2) &= f(h_1h_2)g(h_1h_2) \\ &= f(h_1)f(h_2)g(h_1)g(h_2) \\ &= f(h_1)g(h_1)f(h_2)g(h_2) \\ &= f \cdot g(h_1)f \cdot g(h_2) \end{aligned}$$

and can conclude that $f \cdot g$ is a monoid homomorphism as required.

We then invoke Proposition 4.5.2 and we are done. □

Proposition 4.5.5. *Let N and H be inverse monoids and let $f, g\colon H \to E(N) \cap Z(N)$ be monoid homomorphisms into the central idempotents of N. Then the join of α_f and α_g exists in* $\mathrm{WSExt}(H, N)$ *and is equal to $\alpha_{f\cdot g}$.*

Proof. Let E_f, E_g and $E_{f\cdot g}$ be the admissible equivalence relations corresponding to f, g and $f \cdot g$ respectively.

First we show that $\alpha_{f\cdot g}$ is larger than α_f and α_g in $\mathrm{WSExt}(H, N)$.

If $n_1 \sim^h_{E_f} n_2$ then $f(h)n_1 = f(h)n_2$. Thus, $g(h)f(h)n_1 = g(h)f(h)n_2$ and since $g(h)$ is central, we have $fg(h)n_1 = fg(h)n_2$. This means that $n_1 \sim^h_{E_{f\cdot g}} n_2$ as required. This same argument gives that $n_1 \sim^h_{E_g} n_2$ implies that $n_1 \sim^h_{E_{f\cdot g}} n_2$.

The final condition to check is that $g(h)n \sim^h_{E_{f\cdot g}} fg(h)n \sim^h_{E_{f\cdot g}} f(h)n$. This follows because $f(h)$ and $g(h)$ are both central and idempotent.

To show that $\alpha_{f\cdot g}$ is the join suppose we have a weakly Schreier extension (E, β) larger than α_f and α_g, but smaller than $\alpha_{f\cdot g}$. Since (E, β) is smaller than $\alpha_{f\cdot g}$, we have that if $n_1 \sim^h_E n_2$ then $n_1 \sim^h_{E_{f\cdot g}} n_2$. We will show that (E, β) being larger than α_f and α_g means that $n_1 \sim^h_{E_{f\cdot g}} n_2$ implies that $n_1 \sim^h_E n_2$.

We know that $g(h)n \sim^h_E n \sim^h_E f(h)n$ for all $n \in N$ and $h \in H$. Now suppose that

$n_1 \sim^h_{E_{f\cdot g}} n_2$. This means that $f(h)g(h)n_1 = f(h)g(h)n_2$. Now simply consider

$$\begin{aligned}([n],h) &= ([f(h)n_1],h)\\ &= ([f(h)],1)([n_1],h)\\ &= ([f(h)],1)([g(h)n_1],h)\\ &= ([f(h)g(h)n_1],h)\\ &= ([f(h)g(h)n_2],h)\\ &= ([n_2],h).\end{aligned}$$

Thus, the equivalence relations are equal and so $(E,\beta) = \alpha_{f\cdot g}$. $\square$

Notice that this gives that $\mathrm{Gl}(f \wedge g) = \mathrm{Gl}(f) \vee \mathrm{Gl}(g)$ in $\mathrm{WSExt}(H,N)$. This is clearly reminiscent of a Baer sum.

Chapter 5

Cosetal extensions and Baer sums

Just as for groups the second cohomology groups parameterised by actions classify extensions, we now construct a related theory parameterised by relaxed actions. The associated extensions generalise much of monoid extension theory, and the cohomology group induces a Baer sum on this class.

5.1 Introduction

Group cohomology

That the second cohomology group classifies extensions of groups with abelian kernel is a classical piece of mathematics. We associate to each such extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ an action φ of H on N. We do so by noting that, since N is normal, it is closed under conjugation by G. This conjugation gives an action $\alpha\colon G \to \mathrm{Aut}(N)$ and since N is abelian, αk is the zero morphism. As e is the cokernel of k, we then have that α uniquely extends to a map $\varphi\colon H \to \mathrm{Aut}(N)$ — the desired action of H on N.

We can then collect all isomorphism classes of extensions with the same action together in a set $\mathrm{Ext}(H, N, \varphi)$ and show that this set is isomorphic in a natural way to the abelian group of factor sets quotiented by inner factor sets. This allows $\mathrm{Ext}(H, N, \varphi)$ to inherit an abelian group structure called the Baer sum. For more on this, see Chapter 4 of (Mac Lane [29]).

Monoid cohomology

Generalising this to the setting of extensions of monoids presents some difficulties. Notably, in the above we made crucial use of conjugation, which is not something available in the monoid setting.

Much work has been done to get around this problem. In (Rédei [42]), Schreier extensions of monoids were introduced. An extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is Schreier if in each fibre $e^{-1}(h)$ there exists an element u_h such that for all $g \in e^{-1}(h)$ there exists a unique $n \in N$ such that $g = k(n)u_h$. This means that the fibre $e^{-1}(h)$ is equal to the coset Nu_h.

Although closer to the structure of a group extension, this setting is not quite enough to adapt our original argument and extract an action. However, if an action is supplied — that is, if Schreier extensions of a monoid H by an H-module N are considered — then such extensions are classified by a cohomology group, as seen in (Tuyen [44]). This is further generalised to cohomology groups for extensions of H by H-semimodules in (Patchkoria [39]) and (Patchkoria [38]).

In (Martins-Ferreira, Montoli, and Sobral [34]), a class of extensions are considered which have enough in common with the group setting that an action can be extracted from the extension itself. The idea behind these *special Schreier* extensions is as follows.

An extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is special Schreier when the kernel equivalence split extension of e is a Schreier split extension. Translating this into familiar terms, an extension is special Schreier if and only if for each $e(g) = e(g')$ there exists a unique element $n \in N$ such that $k(n)g' = g$. It is clear that special Schreier extensions are Schreier, but that the converse is not in general true.

To extract the action we observe that $e(g) = e(gk(n))$ and apply the special Schreier property, which says that there is a unique element $\alpha(g, n)$ such that $k\alpha(g, n) \cdot g = g \cdot k(n)$. Notice that if we were in the group setting we would have that $\alpha(g, n) = g \cdot k(n) \cdot g^{-1}$ and so this action generalises the one from the group case. This action then extends as before to one of H on N.

The authors of (Martins-Ferreira, Montoli, and Sobral [34]) then consider isomorphism classes $\mathrm{SExt}(H, N, \varphi)$ of extensions associated to the action φ and are able to classify these extensions using a cohomology group corresponding to a generalised notion of factor sets, and thus imbue $\mathrm{SExt}(H, N, \varphi)$ with a Baer sum.

In Chapter 3, weakly Schreier split extensions were characterized in a way that suggested the possibility of defining a cohomology derived from the analogous special weakly Schreier extensions. We will demonstrate that this approach succeeds and a coholomogy group can be associated to this class of extensions.

Outline

In this chapter we generalise the notion of a special Schreier extension, doing away with the uniqueness requirements. We call these extensions *cosetal* because of their relation to cosets. Cosetal extensions are shown to be in one to one correspondence with extensions whose associated kernel equivalence split extension is weakly Schreier.

It is shown that analogous to the characterization of weakly Schreier split extensions in terms of an admissible quotient and a compatible action, such data can be uniquely associated to a cosetal extension.

We then consider isomorphism classes of extensions with the same associated data and characterize them using a cohomology group defined in terms of a natural weakening of factor sets in our setting. This naturally yields a Baer sum.

We provide a survey of monoid extension theory and see how cosetal extensions compare. We find that they subsume most of the existing classes studied.

We recall that relaxed actions have a non-trivial poset structure and so the assignment of relaxed actions to cohomology groups is not automatically functorial. However, we manage to show that it is functorial and moreover that these arguments may be adapted to provide a full classification of the morphisms between cosetal extensions. In doing so we introduce a relaxed analogue of the first cohomology group.

Finally, we study another way of parameterising the cohomology groups, by actions alone. This results in an inverse semigroup and generalises results in the literature.

5.2 Background

The idea of group cohomology is to study groups through their associated modules. If H is a group and N an H-module then this is in a sense equivalent to the triple (H, N, φ), where now we treat N as just an abelian group and where φ is an action of H on N 'containing' the H-module structure. It is then usual to associate to each triple (N, H, φ) a corresponding cohomology group. In this chapter, it is the second cohomology group that is of interest.

Here we flesh out the argument provided in the introduction. For an extension of groups $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ note that $e(gk(n)g^{-1}) = 1$ and so there exists an element $\theta(g, n)$ such that $k\theta(g, n) = gk(n)g^{-1}$. This map θ is an action of G on N. Now for any set-theoretic section s of e we can define $\varphi\colon H \times N \to N$ by $\varphi(h, n) = \theta(s(h), n)$. It turns out that each choice of splitting s (which we always assume preserves the unit) gives the same map φ, and furthermore, that φ is a group action of H on N. This action satisfies the important identity that $k\varphi(h, n)s(h) = s(h)k(n)$.

One can show that if $e(g) = e(g')$, there is a unique $n \in N$ such that $g = k(n)g'$. Now notice that $e(s(hh')) = e(s(h)s(h'))$, so that there exists an $x \in N$ such that $k(x)s(hh') = s(h)s(h')$. Define a map $g_s \colon H \times H \to N$ that chooses these elements so that $kg_s(h, h')s(hh') = s(h)s(h')$. We call g_s the *associated factor set* of s.

Observe that $g_s(h, 1) = 1 = g_s(1, h)$. Furthermore, we have that $g_s(x, y)g_s(xy, z) = \varphi(x, g_s(y, z))g_s(x, yz)$. Any map g satisfying the above equations is called a *factor set*. It is not hard to see that the set of factor sets is an abelian group under pointwise multiplication.

Now let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be an extension and s an arbitrary set-theoretic section of e. Let φ be the associated action and g_s the factor set associated to s. We now define $N \rtimes_{\varphi}^{g_s} H$, a generalisation of the semidirect product whose underlying set is $N \times H$ and with multiplication given by

$$(n, h)(n', h') = (n\varphi(h, n')g_s(h, h'), hh').$$

The function $f \colon N \rtimes_{\varphi}^{g_s} H \to G$ given by $f(n, h) = k(n)s(h)$ can be easily seen to be a bijection. Furthermore, the following calculation shows that it preserves the multiplication.

$$\begin{aligned} f((n, h)(n', h')) &= f(n\varphi(h, n')g(h, h'), hh') \\ &= k(n)k\varphi(h, n')kg(h, h')s(hh') \\ &= k(n)\varphi(h, n')s(h)s(h') \\ &= k(n)s(h)k(n')s(h') \\ &= f(n, h)f(n', h') \end{aligned}$$

If φ is an action of H on N and g is a factor set, then $N \rtimes_{\varphi}^{g} H$ can always be made into an extension with associated action φ. Thus, it seems likely that extensions can be characterised by actions and factor sets. The only hurdle is that our above calculation made use of an arbitrary splitting s. If instead we chose a different splitting s', we would get a different factor set $g_{s'}$ which would give back the same extension. Therefore, we need to take a quotient of the abelian group of factor sets.

The appropriate subgroup to quotient by is the subgroup of *inner factor sets*: factor sets of the form $\delta t(h, h') = \varphi(h, t(h'))t(hh')^{-1}t(h)$ where $t \colon H \to N$ is any identity preserving function. Intuition for this will be provided in a more general context in Section 5.4.

The resulting quotient group is the second cohomology group $\mathcal{H}^2(H, N, \varphi)$, which by the above arguments can be easily seen to correspond to the set of isomorphism classes of extensions with action φ. The bijection endows this set with a natural abelian group structure.

5.3 Cosetal extensions

In all that follows N, G and H denote monoids unless otherwise stated. We now consider a class of extensions we call cosetal extensions, which have much in common with extensions of groups, specifically pertaining to their relationship with cosets of the kernel.

Definition 5.3.1. An extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is *cosetal* if for all $g, g' \in G$ in which $e(g) = e(g')$, there exists an $n \in N$ such that $k(n)g' = g$. △

Proposition 5.3.2. *An extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is cosetal if and only if $Ng = Ng'$ whenever $e(g) = e(g')$. Furthermore in this case the monoid of cosets is isomorphic to H.*

Proof. Suppose the extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is cosetal.

Suppose $e(g) = e(g')$ and consider $x \in Ng$. Notice that $e(x) = e(g) = e(g')$ thus there exists an $n \in N$ such that $x = k(n)g'$. Thus $x \in Ng'$ and so $Ng \subseteq Ng'$. By a symmetric argument we get that $Ng' \subseteq Ng$, which gives the desired result.

Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be an extension and suppose $Ng = Ng'$ whenever $e(g) = e(g')$.

This means that $g \in Ng'$ which in turn means that there exists an $n \in N$ such that $g = k(n)g'$, giving us that the extension is cosetal.

If G/N is the monoid of cosets then the map sending Ng to $e(g)$ can easily be seen to be an isomorphism. □

It is not hard to see that the kernel of cosetal extensions will always be groups. Simply note that for $n \in N$ we have that $e(n) = 1 = e(1)$ and use the cosetal property.

The following lemma follows immediately from the definition.

Lemma 5.3.3. *Let* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *be cosetal and let* s *and* s' *be (set-theoretic) sections of* e*. Then there exists a function* $t\colon H \to N$ *such that* $s(h) = kt(h) \cdot s'(h)$ *for all* $h \in H$*.*

There is a connection between cosetal extensions and weakly Schreier extensions of monoids involving the kernel equivalence.

If $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is an extension, then the *kernel equivalence split extension* of e is the diagram

$$N \overset{(k,0)}{\rightarrowtail} \mathrm{Eq}(e) \underset{(1_G,1_G)}{\overset{\pi_2}{\rightleftarrows}} G$$

where $\mathrm{Eq}(e)$ is the monoid of all pairs (g, g') in which $e(g) = e(g')$, $(k, 0)(n) = (k(n), 1)$, $\pi_2(g, g') = g'$ and $(1_G, 1_G)(g) = (g, g)$.

Proposition 5.3.4. *An extension* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *is cosetal if and only if the associated kernel equivalence split extension is weakly Schreier.*

Proof. Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be an extension and consider the kernel equivalence split extension

$$N \overset{(k,0)}{\rightarrowtail} \mathrm{Eq}(e) \underset{(1_G,1_G)}{\overset{\pi_2}{\rightleftarrows}} G.$$

For it to be weakly Schreier we require that for all $(g, g') \in \mathrm{Eq}(e)$ there exists an $n \in N$ such that $(g, g') = (k, 0)(n) \cdot (1_G, 1_G)\pi_2(g, g') = (k(n)g', g')$. Thus, we see that this property will hold for all pairs if and only if whenever $e(g) = e(g')$ there exists an $n \in N$ such that $k(n)g' = g$, which is precisely the cosetal condition. □

5.4 The cosetal extension problem

Associating relaxed actions

Since we are interested in generalising the work done on group extensions to this new setting, we shall henceforth assume that the kernel N is always an abelian group.

Despite a cosetal extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ not in general being a split extension, there is a version of the weakly Schreier condition that holds for all set theoretic splittings of e. For convenience we assume that all set theoretic sections s of e which we consider, preserve the identity.

Proposition 5.4.1. *Let* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *be cosetal and let* s *be a section of* e*. Then for all* $g \in G$ *there exists an* $n \in N$*, such that* $g = k(n)se(g)$*.*

Proof. Simply observe that $e(g) = ese(g)$ and apply the cosetal property to g and $se(g)$. □

In (Martins-Ferreira [32]), a class of extensions more general than weakly Schreier extensions, called *semi-biproducts*, are considered. These extensions $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ have as additional data a set theoretic section s of e and also a set theoretic retraction q of k. Together they satisfy the weakly Schreier condition that for all $g \in G$, $g = kq(g)se(g)$. It is clear from Proposition 5.4.1, that cosetal extensions can be equipped with q and s turning them into semi-biproducts.

It was shown (albeit in a different, but equivalent form) that the characterization of weakly Schreier extensions in Chapter 3 generalises naturally to semi-biproducts. When N is an abelian group and $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is assumed to be cosetal, we obtain a characterization which even more closely resembles the weakly Schreier characterization.

Proposition 5.4.2. *Let* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *be a cosetal extension and let* s *be a section of* e*. The equivalence relation* E_s*, defined by* $n \sim^h n'$ *if and only if* $k(n)s(h) = k(n')s(h')$*, is admissible.*

Proof. Notice that if $n \sim^1 n'$, then $k(n) = k(n')$, since s preserves the unit. This implies that $n = n'$ as required.

If $k(n)s(h) = k(n')s(h)$ then of course $k(x)k(n)s(h) = k(x)k(n')s(h)$. Since k is a monoid homomorphism, this gives that $n \sim^h n'$ implies that $xn \sim^h xn'$ for all $x \in N$.

Finally, suppose that $k(n)s(h) = k(n')s(h)$ and consider $k(n)s(hx)$ and $k(n')s(hx)$. Notice that $e(s(h)s(x)) = es(hx)$ and so, since our extension is cosetal, we have that there exists an $a \in N$ such that $k(a)s(hx) = s(h)s(x)$. Now consider the following calculation.

$$\begin{aligned} k(a)k(n)s(hx) &= k(n)k(a)s(hx) \\ &= k(n)s(h)s(x) \\ &= k(n')s(h)s(x) \\ &= k(a)k(n')s(hx). \end{aligned}$$

Here the first equality holds because N is an abelian group. Now since a is invertible it follows that $k(n)s(hx) = k(n')s(hx)$. This shows that for all $x \in H$, $n \sim^h n'$ implies $n \sim^{hx} n'$, and hence that E is admissible. $\square$

The above result required an arbitrary choice of splitting. The following proposition demonstrates that the choice of splitting does not matter.

Proposition 5.4.3. *Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be a cosetal extension and let s and s' be sections of e. Then the associated equivalence relations E_s and $E_{s'}$ are equal.*

Proof. Without loss of generality, it is sufficient to show that $E_s \subseteq E_{s'}$. By Lemma 5.3.3 there exists a function $t\colon H \to N$ such that $kt(h)s(h) = s'(h)$.

Suppose that $n \sim^h_{E_s} n'$. This means that $k(n)s(h) = k(n')s(h)$. We now have

$$\begin{aligned} k(n)s'(h) &= k(n)kt(h)s(h) \\ &= kt(h)k(n)s(h) \\ &= kt(h)k(n')s(h) \\ &= k(n')kt(h)s(h) \\ &= k(n')s'(h). \end{aligned}$$

Hence $n \sim^h_{E'_s} n'$ as required. $\square$

For admissible equivalence relations, it makes sense to consider the following two operations.

i) $n' * ([n], h) = ([n'n], h)$ and

ii) $([n], h) * h' = ([n], hh')$.

We also find that each cosetal extension

$$N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$$

has a unique equivalence class of actions compatible with the admissible equivalence relation. The idea is to consider the kernel equivalence split extension

$$N \overset{(k,0)}{\rightarrowtail} \mathrm{Eq}(e) \underset{(1_G,1_G)}{\overset{\pi_2}{\rightleftarrows}} G$$

which we know to be weakly Schreier and to take one of the compatible actions $\alpha\colon G \times N \to N$ associated to it. Then we simply define the 'action' $\varphi\colon H \times N \to N$ as $\alpha(s \times 1_N)$ for some section s. Before we can show this action is compatible, we prove the following useful lemma.

Lemma 5.4.4. *Let* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *be cosetal and consider its associated weakly Schreier kernel equivalence split extension* $N \overset{(k,0)}{\rightarrowtail} \mathrm{Eq}(e) \underset{(1_G,1_G)}{\overset{\pi_2}{\leftrightarrows}} G$. *Then if* $\alpha \colon G \times N \to N$ *is a compatible action, we have that* $k\alpha(g,n)g = gk(n)$.

Proof. Recall that all compatible actions α come from particular Schreier retractions. Let q be a Schreier retraction associated to the kernel equivalence split extension and let us define α as follows.

$$\begin{aligned} \alpha(g,n) &= q((1_G,1_G)(g)\cdot(k,0)(n)) \\ &= q(gk(n),g). \end{aligned}$$

Notice that we have

$$\begin{aligned} (gk(n),g) &= (k,0)q(gk(n),g)\cdot(1_G,1_G)\pi_2(gk(n),g) \\ &= (k,0)\alpha(g,n)\cdot(1_G,1_G)\pi_2(gk(n),g) \\ &= (k\alpha(g,n),1)\cdot(g,g) \\ &= (k\alpha(g,n)g,g). \end{aligned}$$

Thus we can deduce that $k\alpha(g,n)g = gk(n)$ as required. □

Proposition 5.4.5. *Let* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *be cosetal, let* s *be a section of* e *and let* $\alpha \colon G \times N \to N$ *be a compatible action associated to its (weakly Schreier) kernel equivalence split extension. Then the map* $\varphi = \alpha(s \times 1_N)$ *is compatible with the associated admissible equivalence relation* E.

Proof. We begin by showing that $n \sim^h n'$ implies that $n\varphi(h,x) \sim^h n'\varphi(h,x)$ for all $x \in N$.

Consider $k(n)k\varphi(h,x)s(h)$. Using Lemma 5.4.4 and the fact that $\varphi(h,x) = \alpha(s(h),x)$ we get

$$\begin{aligned} k(n)k\varphi(h,x)s(h) &= k(n)s(h)k(x) \\ &= k(n')s(h)k(x) \\ &= k(n')k\varphi(h,x). \end{aligned}$$

This gives the desired result.

Now let us show that $n \sim^h n'$ implies that $\varphi(x, n) \sim^{xh} \varphi(x, n')$.

Let $a \in N$ be such that $k(a)s(xh) = s(x)s(h)$ and consider

$$\begin{aligned} k(a)k\varphi(x,n)s(xh) &= k\varphi(x,n)s(x)s(h) \\ &= s(x)k(n)s(h) \\ &= s(x)k(n')s(h) \\ &= k(a)k\varphi(x,n')s(xh). \end{aligned}$$

Again, since a is invertible we get that $k\varphi(x, n)s(xh) = k\varphi(x, n')s(xh)$ as required.

Next we show that $\varphi(h, nn') \sim^h \varphi(h, n)\varphi(h, n')$.

Observe the following calculation.

$$\begin{aligned} k\varphi(h,nn')s(h) &= s(h)k(n)k(n') \\ &= k\varphi(h,n)s(h)k(n') \\ &= k\varphi(h,n)k\varphi(h,n')s(h). \end{aligned}$$

This gives the desired result.

Next we show that $\varphi(hh', n) \sim^{hh'} \varphi(h, \varphi(h', n))$.

Let $a \in N$ be such that $k(a)s(hh') = s(h)s(h')$ and consider the following.

$$\begin{aligned} k(a)k\varphi(hh',n)s(hh') &= k(a)s(hh')k(n) \\ &= s(h)s(h')k(n) \\ &= s(h)k\varphi(h',n)s(h') \\ &= k\varphi(h,\varphi(h',n))s(h)s(h') \\ &= k(a)k\varphi(h,\varphi(h',n))s(hh'). \end{aligned}$$

This gives that $k\varphi(hh', n)s(hh') = k\varphi(h, \varphi(h', n))s(hh')$, which in turn yields our desired result.

Finally, we must show that $\varphi(h, 1) \sim^h 1$ and that $\varphi(1, n) \sim^1 n$.

For the first observe that $k\varphi(h, 1)s(h) = s(h)k(1) = s(h)$ and for the second that $k\varphi(1, n)s(1) = k(n)$. Notice that the latter case in fact implies that $\varphi(1, n) = n$.

Thus, we have shown that each of the six necessary conditions are satisfied and so φ is compatible with E. $\square$

Our construction of φ required an arbitrary choice of α. We now show this choice does not matter.

Proposition 5.4.6. *Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be cosetal, let s be a section of e and let $\alpha\colon G \times N \to N$ and $\alpha'\colon G \times N \to N$ be compatible actions associated to its kernel equivalence split extension. Then the maps $\varphi = \alpha(s \times 1_N)$ and $\varphi' = \alpha'(s \times 1_N)$ are equivalent compatible actions with respect to the admissible equivalence relation E.*

Proof. We must show that $\varphi(h,n) \sim^h \varphi'(h,n)$ for all $n \in N$ and $h \in H$. This follows immediately from Lemma 5.4.4 applied to α and α' as $k\varphi(h,n)s(h) = s(h)k(n) = k\varphi'(h,n)s(h)$. □

In fact, the choice of splitting does not matter either.

Proposition 5.4.7. *Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be cosetal, let s and s' be sections of e and let $\alpha\colon G \times N \to N$ be a compatible action associated to its kernel equivalence split extension. Then the maps $\varphi = \alpha(s \times 1_N)$ and $\varphi' = \alpha(s' \times 1_N)$ are equivalent with respect to the associated admissible equivalence relation E.*

Proof. We must show that $\varphi(h,n) \sim^h \varphi'(h,n)$. By Lemma 5.3.3, we have a function $t\colon H \to N$ such that $kt(h)s'(h) = s(h)$. Now consider

$$\begin{aligned} k\varphi'(h,n)s(h) &= k\varphi'(h,n)kt(h)s'(h) \\ &= kt(h)k\varphi'(h,n)s'(h) \\ &= kt(h)s'(h)k(n) \\ &= s(h)k(n) \\ &= k\varphi(h,n)s(h). \end{aligned}$$

This completes the proof. □

So given a cosetal extension

$$N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H,$$

we can associate a unique relaxed action to it.

Factor sets and the Baer sum

We can now partition the set of isomorphism classes of cosetal extensions, parameterised by a relaxed actions.

Definition 5.4.8. Let $\mathrm{CExt}(H, N, E, [\varphi])$ be the set of isomorphism classes of cosetal extensions

$$N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H,$$

such that $(E, [\varphi])$ is the associated relaxed action. △

As in the case of extensions groups or special Schreier extensions of monoids, the extensions in $\mathrm{CExt}(H, N, E, \varphi)$ correspond to some notion of factor sets.

Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be a cosetal extension and let s be a section of e. Recall that $e(s(h)s(h')) = hh' = e(s(hh'))$ and so there exists an $x \in N$ such that $xs(hh') = s(h)s(h')$. Let $g\colon H \times H \to N$ be a function such that

$$g(h, h')s(hh') = s(h)s(h').$$

Notice that we may always choose g such that $g(x, 1) = 1 = g(1, x)$.

Definition 5.4.9. Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be a cosetal extension and let s be a section of e. Then an *associated factor set* is a function $g_s\colon H \times H \to N$ for which

i) $g_s(x, 1) = 1 = g_s(1, x)$,

ii) $g_s(h, h')s(hh') = s(h)s(h')$ for all $h, h' \in H$. △

The following result will motivate our definition of a general factor set below.

Proposition 5.4.10. *Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be a cosetal extension, s be a section of e, g_s an associated factor set and $(E, [\varphi])$ the associated relaxed action. Then*

$$g(x, y)g(xy, z) \sim^{xyz} \varphi(x, g(y, z))g(x, yz).$$

Proof. We must check that

$$kg(x, y)kg(xy, z)s(xyz) = k\varphi(x, g(y, z))kg(x, yz)s(xyz).$$

The left hand side gives

$$\begin{aligned} kg(x, y)kg(xy, z)s(xyz) &= kg(x, y)s(xy)s(z) \\ &= s(x)s(y)s(z). \end{aligned}$$

The right side similarly gives

$$\begin{aligned} k\varphi(x, g(y,z))kg(x,yz)s(xyz) &= k\varphi(x, g(y,z))s(x)s(yz) \\ &= s(x)kg(y,z)s(yz) \\ &= s(x)s(y)s(z). \end{aligned}$$

Thus it follows that these two pairs are equivalent. □

Definition 5.4.11. A map $g\colon H \times H \to N$ is a *factor set* with respect to a relaxed action $(E, [\varphi])$ if $g(x,1) \sim^x 1 \sim^x g(1,x)$ and

$$g(x,y)g(xy,z) \sim^{xyz} \varphi(x, g(y,z))g(x,yz). \qquad \triangle$$

Notice that this is just the obvious weakening of a factor set to the context of relaxed actions.

Given an abelian group N and a monoid H with the additional data of an admissible equivalence relation E on $N \times H$, a compatible action φ and a factor set g, we can construct an extension.

Lemma 5.4.12. *Let E be an admissible H-indexed equivalence on N with N an abelian group. Then if $([n], h) = ([n'], h)$, we have $([xny], hz) = ([xn'y], hz)$ for all $x, y \in N$ and $z \in H$.*

Proof. Suppose $([n], h) = ([n'], h)$. Then consider

$$\begin{aligned} ([xny], hz) &= xy * ([n], h) * z \\ &= xy * ([n'], h) * z \\ &= ([xn'y], hz). \end{aligned}$$

This completes the proof. □

Proposition 5.4.13. *Let N be an abelian group, H a monoid, $(E, [\varphi])$ a relaxed action and g a factor set. Then $\bigsqcup_{h \in H} N/{\sim_h}$ can be equipped with a multiplication*

$$([n], h)([n'], h') = ([n\varphi(h, n')g(h, h')], hh'),$$

which makes it into a monoid with identity $([1], 1)$. We call this monoid $N \rtimes^g_{E,\varphi} H$.

Proof. For the identity we have $([1], 1)([n], h) = ([\varphi(1, n)g(1, h)], h) = ([n], h)$ and $([n], h)([1], 1) = ([n\varphi(h, 1)g(h, 1)], h) = ([n], h)$.

Thus, it remains to show that the multiplication is associative. First we consider

$$\begin{aligned}
\Big(([n_1], h_1)([n_2],h_2)\Big)([n_3], h_3) \\
&= ([n_1\varphi(h_1, n_2)g(h_1, h_2)], h_1h_2)([n_3], h_3) \\
&= ([n_1\varphi(h_1, n_2)g(h_1, h_2)\varphi(h_1h_2, n_3)g(h_1h_2, h_3)], h_1h_2h_3) \\
&= n_1\varphi(h_1, n_2)\varphi(h_1h_2, n_3) * ([g(h_1, h_2)g(h_1h_2, h_3)], h_1h_2h_3).
\end{aligned}$$

Compare this to

$$\begin{aligned}
([n_1], h_1)\Big(([n_2],h_2)([n_3], h_3)\Big) \\
&= ([n_1], h_1)([n_2\varphi(h_2, n_3)g(h_2, h_3)], h_2h_3) \\
&= ([n_1\varphi(h_1, n_2\varphi(h_2, n_3)g(h_2, h_3))g(h_1, h_2, h_3)], h_1h_2h_3) \\
&= ([n_1\varphi(h_1, n_2)\varphi(h_1, \varphi(h_2, n_3))\varphi(h_1, g(h_2, h_3))g(h_1, h_2h_3)], h_1h_2h_3) \\
&= n_1\varphi(h_1, n_2)\varphi(h_1h_2, n_3) * ([\varphi(h_1, g(h_2, h_3))g(h_1, h_2h_3)], h_1h_2h_3) \\
&= n_1\varphi(h_1, n_2)\varphi(h_1h_2, n_3) * ([g(h_1, h_2)g(h_1h_2, h_3)], h_1h_2h_3),
\end{aligned}$$

which gives us our result. □

Proposition 5.4.14. *Let N be an abelian group, H a monoid, $(E, [\varphi])$ a relaxed action and g a factor set. Then $N \overset{k}{\rightarrowtail} N \rtimes^g_{E,\varphi} H \overset{e}{\twoheadrightarrow} H$ is a cosetal extension, where $k(n) = ([n], 1)$ and $e([n], h) = h$.*

Proof. It is apparent that k and e are well defined monoid homomorphisms. It is also not hard to see that k is the kernel of e. Thus, we must just demonstrate that e is the cokernel of k and that the extension is cosetal.

Let $f \colon N \rtimes^g_{E,\varphi} H \to M$ be a monoid homomorphism in which $fk = 0$. It is easy to see that

$$([n], h) = ([n], 1)([1], h)$$

and so we have

$$\begin{aligned}
f([n], h) &= f(([n], 1])([1], h)) \\
&= f(k(n)([1], h)) \\
&= f(k(n))f([1], h) \\
&= f([1], h).
\end{aligned}$$

We have a map $\ell \colon H \to M$ such that $\ell(h) = f([1], h)$. It is clear that $\ell e = f$ and since e is surjective we must just check that ℓ is a homomorphism. We have

$$\begin{aligned}\ell(h)\ell(h') &= f([1],h)f([1],h') \\ &= f(([1],h)([1],h')) \\ &= f([g(h,h')],hh') \\ &= f([(g(h,h')],1)([1],hh')) \\ &= f([1],hh') \\ &= \ell(hh'),\end{aligned}$$

which demonstrates that e is the cokernel.

Now we must show that $N \overset{k}{\rightarrowtail} (N \times H)/E_g^\varphi \overset{e}{\twoheadrightarrow} H$ is cosetal. This entails demonstrating that for two elements $([n],h)$ and $([n'],h)$, there exists an $x \in N$ such that $([x],1)([n],h) = ([n'],h)$. Choosing $x = n'n^{-1}$ suffices. This completes the proof. □

We know how to extract from a cosetal extension the data $(E,[\varphi],g)$, where $(E,[\varphi])$ is a relaxed action and g a factor set associated to some section s of e.

We also know how to take data $(E,[\varphi],g)$ of the same type and generate a cosetal extension

$$N \overset{k}{\rightarrowtail} (N \times H)/E_g^\varphi \overset{e}{\twoheadrightarrow} H.$$

We now relate these two processes to one another.

Fixing a relaxed action $(E,[\varphi])$ we can define the set of associated factor sets $\mathcal{F}^*(H,N,E,[\varphi])$. This has a natural abelian group structure given by pointwise multiplication.

Proposition 5.4.15. $\mathcal{F}^*(H,N,E,[\varphi])$ *is an abelian group where* $(g \cdot g')(h,h') = g(h,h') \cdot g(h,h')$.

Proof. It is clear that the constant 1 map is a factor set and that this will behave as an identity.

If g and g' are factor sets, then using commutativity and Lemma 5.4.12 we can show that

$$(g \cdot g')(x,y)(g \cdot g')(xy,z) \sim^{xyz} \varphi(x,(g \cdot g')(y,z))(g \cdot g')(x,yz).$$

and also

Finally, we claim that if g is a factor set, then the map g^{-1} with $g^{-1}(h,h') = g(h,h')^{-1}$

is a factor set. Of course $(g(x,y)g(x,yz))(g^{-1}(x,y)g^{-1}(x,yz)) = 1$. Now consider

$$\begin{aligned} g(x,y)g(x,yz)\varphi(x,g^{-1}(y,z))g^{-1}(x,yz) \\ &\sim^{xyz} \varphi(x,g(y,z))g(x,yz)\varphi(x,g^{-1}(y,z))g^{-1}(x,yz) \\ &\sim^{xyz} 1 \end{aligned}$$

Multiplying on the left by $g^{-1}(x,y)g^{-1}(x,yz)$ gives the desired result. □

From Proposition 5.4.14 we have a map $\rho\colon \mathcal{F}^*(H,N,E,[\varphi]) \to \mathrm{CExt}(H,N,E,[\varphi])$. We do not have a canonical map

$$\zeta\colon \mathrm{CExt}(H,N,E,[\varphi]) \to \mathcal{F}^*(H,N,E,[\varphi]),$$

as in general there are many factor sets associated to each cosetal extension. We thus would like to quotient $\mathcal{F}^*(H,N,E,[\varphi])$ so that all such factor sets are equivalent.

In classical group cohomology and in (Martins-Ferreira, Montoli, and Sobral [34]) this is a matter of defining the subgroup of inner factor sets. The idea is that if factor sets g and g' correspond to different splittings of the same extension, that they differ by an inner factor set.

Here our situation is slightly more complicated. It is possible to have two factor sets g and g' corresponding to the same splitting of a particular extension. So before we turn to inner factor sets, let us resolve this issue first.

Proposition 5.4.16. *The equivalence relation F on $\mathcal{F}^*(H,N,E,[\varphi])$ defined by $g \sim g'$ if and only if $g(h,h') \sim^{hh'} g'(h,h')$ is a congruence.*

Proof. Suppose $g \sim g'$ and $r \sim r'$ and consider the two terms $([g(h,h')r(h,h')],hh')$ and $([g'(h,h')r'(h,h')],hh')$. Using Lemma 5.4.12 we may easily verify that these two terms are equal. □

Intuitively, this is the correct equivalence relation as it gives $kg(h,h')s(hh') = kg'(h,h')s(hh')$ for all splittings s.

Now define $\mathcal{F}(H,N,E,[\varphi]) = \mathcal{F}^*(H,N,E,[\varphi])/F$ where F is the equivalence relation above. We can now consider the generalisation of inner factor sets, which will allow us to take the desired quotient.

Definition 5.4.17. A factor set $g \in \mathcal{F}^*(H,N,E,[\varphi])$ is an *inner factor set* if and only if for some identity preserving $t\colon H \to N$ we have $g = \delta t$ where $\delta t(h,h') = \varphi(h,t(h'))t(hh')^{-1}t(h)$. △

We begin by showing that if $\rho(g) = \rho(g')$, then the relaxed factor sets g and g' differ by an inner factor set.

Proposition 5.4.18. *Let $g, g' \in \mathcal{F}^*(H, N, E, [\varphi])$ and let $\rho(g) = \rho(g')$. Then there exists an inner factor set δt such that $g' \sim_F \delta t \cdot g$.*

Proof. Let $N \overset{k}{\rightarrowtail} (N \times H)/E_g^{\varphi} \overset{e}{\twoheadrightarrow} H$ and $N \overset{k'}{\rightarrowtail} (N \times H)/E_{g'}^{\varphi} \overset{e'}{\twoheadrightarrow} H$ be the associated cosetal extensions and let $s \colon H \to N \rtimes_{E,\varphi}^{g} H$ be such that $s(h) = ([1], h)$ and $s' \colon H \to N \rtimes_{E,\varphi}^{g'} H$ be such that $s'(h) = ([1], h)$.

Since $\rho(g) = \rho(g')$ there is an isomorphism of extensions $f \colon N \rtimes_{E,\varphi}^{g} H \to N \rtimes_{E,\varphi}^{g'} H$. Now observe that we have the following.

$$\begin{aligned} f([n], h) &= f(([n], 1)([1], h)) \\ &= f([n], 1) f([1], h) \\ &= ([n], 1) f([1], h) \\ &= ([n], 1)([f^*(h)], h). \end{aligned}$$

Here $f^* \colon H \to N$ is a function which preserves identity and for which $f([1], h) = ([f^*(h)], h)$. Observe then that $f([n], h) = ([f^*(h)n], h)$. We can then define $s^* = fs$ and notice that for $t(h) = f^*(h)^{-1}$ we have that $s'(h) = kt(h)s^*(h)$. It is also not hard to see that $k'g(h, h')s^*(hh') = s^*(h)s^*(h')$.

We must show that $\delta t \cdot g(h, h') \sim^{hh'} g'(h, h')$. We already have that $k'g'(h, h')s'(hh') = s'(h)s'(h')$ and so a single calculation remains.

$$\begin{aligned} k'(\delta t \cdot g)(h, h')s'(hh') &= k'\varphi(h, t(h'))k't(hh')^{-1}k't(h)k'g(h, h')s'(hh') \\ &= k'\varphi(h, t(h'))k't(hh')^{-1}k't(h)k'g(h, h')k't(hh')s^*(hh') \\ &= k'\varphi(h, t(h'))k't(h)k'g(h, h')s^*(hh') \\ &= k'\varphi(h, t(h'))k't(h)s^*(h)s^*(h') \\ &= k'\varphi(h, t(h'))s'(h)s^*(h') \\ &= s'(h)k't(h')s^*(h') \\ &= s'(h)s'(h'). \end{aligned}$$

This completes the proof. ∎

In order to show that equivalence classes of inner factor sets are the appropriate subgroup to quotient by, there is one final result to check. That a factor set g belongs to the same class as $\delta t \cdot g$.

Proposition 5.4.19. *Let $g \in \mathcal{F}^*(H, N, E, [\varphi])$ and let δt be an inner factor set. Then $\rho(g) = \rho(\delta t \cdot g)$.*

Proof. Let $N \xrightarrow{k} N \rtimes^{g}_{E,\varphi} H \xrightarrow{e} H$ and $N \xrightarrow{k'} N \rtimes^{\delta t \cdot g}_{E,\varphi} H \xrightarrow{e'} H$ be the associated cosetal extensions and let $s \colon H \to N \rtimes^{g}_{E,\varphi} H$ be such that $s(h) = ([1], h)$ and $s' \colon H \to N \rtimes^{\delta t \cdot g}_{E,\varphi} H$ be such that $s'(h) = ([1], h)$.

Now inspired by the proof of Proposition 5.4.18 we define a function $f \colon N \rtimes^{g}_{E,\varphi} H \to N \rtimes^{\delta t \cdot g}_{E,\varphi} H$ such that $f([n], h) = ([t(h)^{-1}n], h)$. Since $t(h)^{-1}$ is invertible, it is clear that f is bijective.

Furthermore we have $fk(n) = f([n], 1) = ([n], 1) = k'(n)$ and $e'f([n], h)) = h = e([n], h)$. It is also clear that f preserves the identity and so all that remains is to show that f preserves multiplication.

As before we define $s^* = fs$ and we see that $k't(h)s^*(h) = s'(h)$.

First we look at $f([n], h)f([n'], h')$. Notice that

$$\begin{aligned} f([n], h)f([n'], h') &= ([t(h)^{-1}n], h)([t(h')^{-1}n'], h') \\ &= ([n], 1)([t(h)^{-1}], h)([n'], 1)([t(h')^{-1}], h') \\ &= k'(n)s^*(h)k'(n')s^*(h'). \end{aligned}$$

Next we consider $f(([n], h)([n'], h))$. We have the following.

$$\begin{aligned} f(([n], h)([n'], h)) &= ([t(hh')^{-1}n\varphi(h, n')g(h, h')], hh') \\ &= k't(hh')^{-1}k'(n)k'\varphi(h, n')k'g(h, h')s'(hh') \\ &= k't(hh')^{-1}k'(n)k'\varphi(h, n')k'g(h, h')k't(hh')s^*(hh') \\ &= k'(n)k'\varphi(h, n')k'g(h, h')s^*(hh') \\ &= k'(n)k'\varphi(h, n')s^*(h)s^*(h') \\ &= k'(n)k'\varphi(h, n')k't(h)^{-1}s'(h)s^*(h') \\ &= k'(n)k't(h)^{-1}k'\varphi(h, n')s'(h)s^*(h') \\ &= k'(n)k't(h)^{-1}s'(h)k'(n')s^*(h) \\ &= k'(n)s^*(h)k'(n')s^*(h'). \end{aligned}$$

This completes the proof. □

Let $\mathcal{IF}^*(H, N, E, [\varphi])$ be the subgroup of inner factor sets and then define the subgroup

$$\mathcal{IF}(H, N, E, [\varphi]) = \{[\delta t] : \delta t \in \mathcal{IF}^*(H, N, E, [\varphi])\}.$$

This then allows us to define $\mathcal{H}^2(H, N, E, [\varphi]) = \mathcal{F}(H, N, E, [\varphi])/\mathcal{IF}(H, N, E, [\varphi])$ and the map

$$\zeta \colon \mathrm{CExt}(H, N, E, [\varphi]) \to \mathcal{H}^2(H, N, E, [\varphi])$$

in which an isomorphism class of extensions is sent to the equivalence class of factor sets which generate it.

It is clear that $\zeta\rho$ is the identity. We now show that the reverse also holds true.

Proposition 5.4.20. *Let $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ be a cosetal extension, $(E, [\varphi])$ the associated relaxed action and g the factor set corresponding to a splitting s. Then we have an isomorphism*

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{k'} & N \rtimes^{g}_{E,\varphi} H & \xrightarrow{e'} & \mathcal{H} \\ \| & & \downarrow \wr & & \| \\ \mathcal{N} & \xrightarrow[k]{} & G & \xrightarrow{e} & \mathcal{H} \end{array}$$

which gives that $\rho\zeta$ is the identity.

Proof. Let s be a section of $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ and consider the map $f \colon N \rtimes^{g}_{E,\varphi} H \to G$ where $f([n], h) = k(n)s(h)$. It is clear that this is a well-defined bijective map and preserves the identity. Let us show that it preserves the multiplication.

$$\begin{aligned} f(([n], h)([n'], h)) &= f([n\varphi(h, n')g(h, h')], hh') \\ &= k(n)k\varphi(h, n')kg(h, h')s(hh') \\ &= k(n)k\varphi(h, n')s(h)s(h') \\ &= k(n)s(h)k(n')s(h') \\ &= f([n], h)f([n'], h'). \end{aligned}$$

Now it only remains to show $fk' = k$ and $ef = e'$. For the first consider $fk(n) = f([n], 1) = k(n)s(1) = k(n)$. For the second $ef([n], h) = e(k(n)s(h)) = h$. □

Thus, putting this together we obtain our main result.

Theorem 5.4.21. *The maps ρ and ζ form part of an isomorphism between the set* $\mathrm{CExt}(H, N, E, [\varphi])$ *and the abelian group* $\mathcal{H}(H, N, E, [\varphi])$.

Naturally, $\mathrm{CExt}(H, N, E, [\varphi])$ inherits a multiplication through this isomorphism. It is this that we call the *Baer sum*.

5.5 Relation to other monoid extension theories

Let us discuss the relationship between cosetal extensions and other notions of monoid extension found in the literature. In (Leech [28]) Leech considers extensions of groups by monoids $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ in which N is a normal subgroup of G in the sense that N is a subgroup of the group of units of G and $gN = Ng$ for all $g \in G$. It is easy to see that every such extension is cosetal. However, the following example demonstrates that not all cosetal extensions are Leech extensions, even when the kernel is an abelian group.

Example 5.5.1. Let $\mathbb{2} = \{\top, \bot\}$ denote the two-element meet-semilattice and consider the action $\alpha\colon \mathbb{2} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\alpha(\top, n) = n$ and $\alpha(\bot, n) = 0$. We may construct the semidirect product $\mathbb{Z} \rtimes_\alpha \mathbb{2}$ and the following extension

$$\mathbb{Z} \overset{k}{\rightarrowtail} \mathbb{Z} \rtimes_\alpha \mathbb{2} \overset{e}{\twoheadrightarrow} \mathbb{2}.$$

Here $k(n) = (n, \top)$ and $e(n, x) = x$.

Explicitly, the multiplication in $\mathbb{Z} \rtimes_\alpha \mathbb{2}$ is given by $(n, \top) \cdot (n', h) = (n + n', h)$ and $(n, \bot) \cdot (n', h) = (n, \bot)$. The right coset $\mathbb{Z} \cdot (0, \bot)$ is then $\mathbb{Z} \times \{\bot\}$, while the left coset $(0, \bot) \cdot \mathbb{Z}$ is $\{(0, \bot)\}$. Thus this extension is not a Leech extension, but it is cosetal as it is a (weakly) Schreier split extension with group kernel. △

The special Schreier extensions of (Martins-Ferreira, Montoli, and Sobral [34]) are extensions $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ such that whenever $e(g) = e(g')$ there exists a *unique* element $n \in N$ such that $k(n)g' = g$. Of course, every special Schreier extension is cosetal. The following example exhibits a Leech extension, and hence a cosetal extension, which is not special Schreier. Thus, cosetal extensions constitute a non-trivial simultaneous generalization of Leech's extensions of groups by monoids and special Schreier extensions of monoids.

Example 5.5.2. Let $\mathbb{Z}_\infty$ be the monoid obtained by adjoining an absorbing element

∞ to the integers under addition. Consider the following extension

$$\mathbb{Z} \overset{k}{\rightarrowtail} \mathbb{Z}_\infty \overset{e}{\twoheadrightarrow} \mathbb{2}$$

in which $k(n) = n$ and $e(n) = \top$ for $n \in \mathbb{Z}$ and $e(\infty) = \bot$. Because Z_∞ is commutative, it is clear that this is a Leech extension. However, it is not special Schreier as $n + \infty = \infty$ for all $n \in \mathbb{Z}$ and so uniqueness fails. △

In (Fulp and Stepp [20]) a form of central monoid extension is considered. With respect to these structures they construct an inverse semigroup operation, reminiscent of a generalised Baer sum. These extensions are cosetal, and a more general form of their 'semigroup of semigroups' is considered in Section 5.7.

In (Fleischer [19]) a class of extensions strictly more general than cosetal extensions is considered. A rather complicated characterization is given, though ultimately it proves too unwieldy to provide a good notion of factor set or Baer sum.

Finally, on a more speculative note, in (Grillet [22]) Grillet defines a left coset extension of a monoid H as a functor $F: H \to \text{Grp}$, where H has the left divisibilty preorder. We know that each cosetal extension has an associated relaxed action (E, φ). By assumption the H-indexed equivalence relation is functorial and moreover for each h, $\sim^h$ is a right congruence. Thus when we restrict to abelian group kernel each $\sim^h$ is in fact a congruence and so may instead be viewed as a group. It is likely that this connection serves as a bridge between my and Grillet's work, but a thorough study has not yet been done.

5.6 Natural morphisms between the second cohomology groups

Throughout this section, the kernel N will be assumed to be an abelian group. In the setting of special Schreier extensions there is an assignment of monoid actions φ of H on N to abelian groups $\mathcal{H}^2(H, N, \varphi)$. Since the set of actions $\text{Act}(H, N)$ can be thought of as a discrete category, this assignment gives rise immediately to a functor $L_{H,N} : \text{Act}(H, N) \to \mathbf{Ab}$.

For the theory of cosetal extensions we get an assignment of relaxed actions $(E, [\varphi])$ to the abelian groups $\mathcal{H}^2(H, N, E, [\varphi])$. But in this case $\text{RAct}(H, N)$ is a preorder and so functoriality of this assignment is a non-trivial property. In this section we demonstrate that it is indeed functorial and use it investigate the morphisms of cosetal extensions.

5.6.1 The functor

Recall that if two relaxed actions satisfy $(E, [\varphi]) \leq (E', [\varphi'])$, then $(E, [\varphi']) = (E, [\varphi])$ and so inequalities may always be expressed of the form $(E, [\varphi]) \leq (E', [\varphi])$.

Suppose that $(E, [\varphi]) \leq (E', [\varphi])$. We would like to find a relationship between $\mathcal{H}^2(H, N, E, [\varphi])$ and $\mathcal{H}^2(H, N, E', [\varphi])$.

Proposition 5.6.1. *If $(E, [\varphi]) \leq (E', [\varphi])$ then the map $\ell\colon \mathcal{H}^2(H, N, E, [\varphi]) \to \mathcal{H}^2(H, N, E', [\varphi])$ sending $[g]_E$ to $[g]_{E'}$ is a group homomorphism.*

Proof. Let us begin by showing that ℓ is well-defined.

As g is a factor set, we have $g(x, y)g(xy, z) \sim_E^{xyz} \varphi(x, g(y, z))g(x, yz)$. Now since $E \subseteq E'$, we see that this identity is automatically satisfied with respect to E' and so g is also a factor set with respect to E'.

Recall that two factor sets are equivalent in $\mathcal{H}^2(H, N, E, [\varphi])$ if there exists an inner factor set δt such that $g(h, h') \sim_E^{hh'} (\delta t + g')(h, h')$ for all $h, h' \in H$. But since $E \subseteq E'$, this is also satisfied with respect to E'. Thus, $g \equiv_{E'} g'$ and the mapping is well defined.

It is clear from the definition that the mapping preserves the identity and addition. □

It is not hard to see that this makes the assignment of relaxed actions to the cohomology groups into a functor.

Definition 5.6.2. We obtain a functor $L_{H,N}\colon \mathrm{RAct}(H, N) \to \mathbf{Ab}$ that sends $(E, [\varphi])$ to $\mathcal{H}^2(H, N, E, [\varphi])$ and yields the morphism $\ell\colon \mathcal{H}^2(H, N, E, [\varphi]) \to \mathcal{H}^2(H, N, E', [\varphi])$ described above for $(E, [\varphi]) \leq (E', [\varphi])$. △

5.6.2 Morphisms of cosetal extensions

There is the immediate question of whether the extension corresponding to a cohomology class $[g]_E \in \mathcal{H}^2(H, N, E, [\varphi])$ is in any way related to the extension corresponding to $\ell([g]) \in \mathcal{H}^2(H, N, E', [\varphi])$.

Proposition 5.6.3. *Suppose $(E, [\varphi]) \leq (E', [\varphi])$ and take $[g] \in \mathcal{H}^2(H, N, E, [\varphi])$. Consider the extensions $N \overset{k'}{\rightarrowtail} N \rtimes_{E,\varphi}^g H \overset{e'}{\twoheadrightarrow} H$ and $N \overset{k}{\rightarrowtail} N \rtimes_{E',\varphi}^g H \overset{e}{\twoheadrightarrow} H$ associated $[g]$ and $\ell([g])$ respectively. Then the map $\lambda\colon N \rtimes_{E,\varphi}^g H \to N \rtimes_{E',\varphi}^g H$ defined by $\lambda([n], h) = ([n], h)$ is a morphism of extensions.*

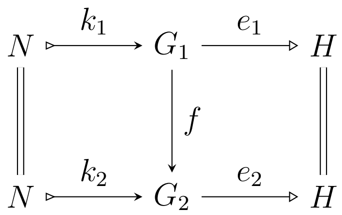
Proof. This map is well-defined, since $E \subseteq E'$, and it is immediate that the identity $([1], 1)$ is preserved. Preservation of multiplication follows from the following trivial calculation.

$$\begin{aligned}\lambda(([n], h)([n'], h')) &= \lambda([n\varphi(h, n')g(h, h')], hh') \\ &= [n\varphi(h, n')g(h, h')], hh') \\ &= ([n], h)([n'], h') \\ &= \lambda([n], h)\lambda([n'], h')\end{aligned}$$

We then clearly have $\lambda k = k'$ and that $e'\lambda = e$, which completes the proof. □

We can now ask a converse to the above question: if f is a morphism between cosetal extensions, is there any way to relate their associated cohomology classes?

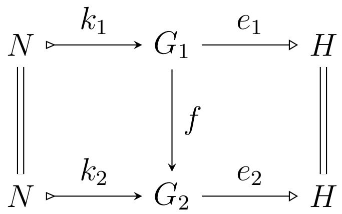
Consider the following morphism of extensions.



To find the associated H-indexed equivalence relation for $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$, we take an arbitrary set-theoretic splitting s of e_1 and define $n \sim^h n'$ if and only if $k_1(n)s(h) = k_1(n')s(h)$.

Notice that if s is such a splitting, then $e_2 fs = e_1 s = 1$ and so fs is a splitting of e_2. This observation allows us to prove many of the results that follow.

Lemma 5.6.4. *Let f be a morphism of extensions as in the following diagram.*



If E_1 and E_2 are the equivalence relations associated to $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$ and $N \overset{k_2}{\rightarrowtail} G_2 \overset{e_2}{\twoheadrightarrow} H$ respectively, then $E_1 \subseteq E_2$.

Proof. Let s be a splitting of e_1 and suppose that $n \sim^h_{E_1} n'$. This means that $k_1(n)s(h) = k_1(n')s(h)$. Now recall that fs splits e_2 and consider the following sequence of equalities.

$$\begin{aligned} k_2(n)fs(h) &= fk_1(n)fs(h) \\ &= f(k_1(n)s(h)) \\ &= f(k_1(n')s(h)) \\ &= k_2(n')fs(h). \end{aligned}$$

Thus, $n \sim^h_{E_2} n'$ and hence $E_1 \subseteq E_2$. □

Using similar ideas we can show that the same action can be extracted from both extensions.

Lemma 5.6.5. *Let f be a morphism of extensions as in the following diagram.*

$$\begin{array}{ccccc} N & \xrightarrow{k_1} & G_1 & \xrightarrow{e_1} & H \\ \| & & \big\downarrow f & & \| \\ N & \xrightarrow{k_2} & G_2 & \xrightarrow{e_2} & H \end{array}$$

Then there exists a φ such that $N \xrightarrow{k_1} G_1 \xrightarrow{e_1} H$ belongs to $\mathrm{CExt}(H, N, E_1, [\varphi])$ *and $N \xrightarrow{k_2} G_2 \xrightarrow{e_2} H$ belongs to* $\mathrm{CExt}(H, N, E_2, [\varphi])$.

Proof. Suppose that s splits e_1 and that (E_1, φ) is the relaxed action associated to $N \xrightarrow{k_1} G_1 \xrightarrow{e_1} H$. This means that $k_1\varphi(h, n)s(h) = s(h)k_1(n)$. Applying f to both sides yields $fk_1\varphi(h, n)fs(h) = fs(h)fk_1(h)$ and hence $k_2\varphi(h, n)fs(h) = fs(h)k_2(n)$. This shows that φ is compatible with $N \xrightarrow{k_2} G_2 \xrightarrow{e_2} H$ as required. □

Thus, if f is a morphism of extensions, we know that the extensions must correspond to cohomology classes in $\mathcal{H}^2(H, N, E_1, [\varphi])$ and $\mathcal{H}^2(H, N, E_2, [\varphi])$ with $(E_1, [\varphi]) \leq (E_2, [\varphi])$. But in this situation, Proposition 5.6.1 supplies us with a map $\ell \colon \mathcal{H}^2(H, N, E_1, [\varphi]) \to \mathcal{H}^2(H, N, E_2, [\varphi])$. We now show that ℓ must map the one extension to the other.

Lemma 5.6.6. *Let f be a morphism of extensions as in the following diagram.*

$$\begin{array}{ccccc} N & \xrightarrow{k_1} & G_1 & \xrightarrow{e_1} & H \\ \| & & \big\downarrow f & & \| \\ N & \xrightarrow{k_2} & G_2 & \xrightarrow{e_2} & H \end{array}$$

If $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$ *corresponds to* $[g]_E \in \mathcal{H}^2(H, N, E_1, [\varphi])$*, then the extension* $N \overset{k_2}{\rightarrowtail} G_2 \overset{e_2}{\twoheadrightarrow} H$ *corresponds to* $\ell([g]_E) \in \mathcal{H}^2(H, N, E_2, [\varphi])$*.*

Proof. Suppose that s is a splitting of e_1 and then let g_s be the associated factor set. We will show that g_s is also a factor set associated to the splitting fs of e_2.

We know that $k_1 g_s(h, h')s(hh') = s(h)s(h')$. Now simply apply f to both sides to get $fk_1 g_s(h, h')fs(hh') = fs(h)fs(h')$. This of course gives $k_2 g_s(h, h')fs(hh') = fs(h)fs(h')$. This shows that g_s is also an associated factor set of fs, which completes the proof. □

Together these results give the following theorem.

Theorem 5.6.7. *Let* $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$ *and* $N \overset{k_2}{\rightarrowtail} G_2 \overset{e_2}{\twoheadrightarrow} H$ *be cosetal extension in the category of monoids corresponding to* $[g_1]_{E_1} \in \mathcal{H}^2(H, N, E_1, [\varphi_1])$ *and* $[g_2]_{E_2} \in \mathcal{H}^2(H, N, E_2, [\varphi_2])$ *respectively. Then there exists a morphism* f *of extensions as in the following diagram*

$$\begin{array}{ccccc} N & \overset{k_1}{\rightarrowtail} & G_1 & \overset{e_1}{\twoheadrightarrow} & H \\ \Vert & & \downarrow{\scriptstyle f} & & \Vert \\ N & \overset{k_2}{\rightarrowtail} & G_2 & \overset{e_2}{\twoheadrightarrow} & H \end{array}$$

if and only if $(E_1, [\varphi_1]) \leq (E_2, [\varphi_2])$ *and if* $[g_1]_{E_2} = [g_2]_{E_2}$*.*

Notice that this implies that the only morphisms between extensions belonging to $\mathrm{CExt}(H, N, E, [\varphi])$ are endomorphisms. In fact, we can show that they are all automorphisms.

Proposition 5.6.8. *Let* $A\colon N \overset{k}{\rightarrowtail} N \rtimes^g_{E,\varphi} H \overset{e}{\twoheadrightarrow} H$ *be a cosetal extension. If* f *is an endomorphism on* A*, then* f *is an automorphism.*

Proof. Let s be a splitting of e. Since f is an endomorphism of extensions, we have $fk = k$ and $ef = e$. Note that $f([n], h) = f(k(n)s(h)) = fk(n)fs(h) = k(n)fs(h)$. Because $ef = e$, we know $efs(h) = h$ and so $fs(h) = [f^*(h), h]$ for some $f^*(h) \in N$. Thus, $f([n], h) = ([f^*(h)n], h)$.

It is now easily seen that $([n], h) \mapsto ([f^*(h)^{-1}n], h)$ provides an inverse function. □

In particular, we arrive at the following parameterised version of the short five lemma.

Theorem 5.6.9. *If* f *is a morphism of extensions as in the following diagram*

$$\begin{array}{ccccc} N & \overset{k_1}{\rightarrowtail} & G_1 & \overset{e_1}{\twoheadrightarrow} & H \\ \Vert & & \downarrow{\scriptstyle f} & & \Vert \\ N & \overset{k_2}{\rightarrowtail} & G_2 & \overset{e_2}{\twoheadrightarrow} & H \end{array}$$

where $N \overset{k_1}{\rightarrowtail} G_1 \overset{e_1}{\twoheadrightarrow} H$ *and* $N \overset{k_2}{\rightarrowtail} G_2 \overset{e_2}{\twoheadrightarrow} H$ *are cosetal extensions in the category of monoids and belong to* $\mathrm{CExt}(H, N, E, [\varphi])$*, then* f *is an isomorphism.*

We can extend the ideas in Proposition 5.6.8 to give a full characterisation of the automorphisms of extensions in terms of an obvious generalisation of *crossed homomorphisms* from the theory of group extensions.

Let $N \overset{k}{\rightarrowtail} N \rtimes^{g}_{E,\varphi} H \overset{e}{\twoheadrightarrow} H$ belong to $\mathrm{CExt}(H, N, E, [\varphi])$. Notice that any automorphism f must send $([n], h)$ to $([f^*(h)n], h)$, where $f^*(h) \colon H \to N$. We then ask for which functions $t^* \colon H \to N$ is $t([n], h) = ([t^*(h)n], h)$ a homomorphism. For t to preserve the unit we need that $t^*(1) = 1$. For it to preserve the multiplication we need for the expressions $t(([n], h)([n'], h')) = ([t^*(hh')n\varphi(h, n')g(h, h')], hh')$ and $t([n], h)t([n'], h') = ([t^*(h)n\varphi(h, t^*(h'))\varphi(h, n')g(h, h')], hh')$ to be equal. This happens when

$$t^*(hh')n\varphi(h, n')g(h, h') \sim^{hh'} t^*(h)n\varphi(h, t^*(h'))\varphi(h, n')g(h, h').$$

Multiplying on the left by $n^{-1}\varphi(h, n')^{-1}g(h, h')^{-1}$ gives $t^*(hh') \sim^{hh'} t^*(h)\varphi(h, t^*(h'))$, which is the natural weakening of a crossed homomorphism to our setting.

Definition 5.6.10. A function $t^* \colon H \to N$ is a *crossed homomorphism* relative to (E, φ) when

$$t^*(hh') \sim^{hh'} t^*(h)\varphi(h, t^*(h')). \qquad \triangle$$

We have shown above that crossed homomorphisms give rise to automorphisms of extensions and that every endomorphism arises in this way. However, it is possible for two crossed homomorphisms to yield the same endomorphism. To remedy this we quotient the set of crossed homomorphisms with the equivalence relation defined by $t_1^* \sim t_2^* \iff t_1^*(h) \sim^h t_2^*(h)$.

Definition 5.6.11. Let $\mathcal{Z}^1(H, N, E, [\varphi])$ denote the set of equivalence classes of crossed homomorphisms. This inherits an abelian groups structure from the pointwise multiplication of crossed homomorphisms. $\triangle$

Theorem 5.6.12. *If* Γ *is a cosetal extension* $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ *in the category of* *monoids belonging to* $\mathcal{H}^2(H, N, E, [\varphi])$*, then there is a bijection between* $\mathrm{End}(\Gamma)$ *and* $\mathcal{Z}^1(H, N, E, [\varphi])]$*. Moreover, this an isomorphism of monoids.*

The same approach allows us to characterise arbitrary morphisms. Let f be a morphism of extensions as in the following diagram.

$$\begin{array}{lccccc}
\Gamma\colon & N & \overset{k_1}{\rightarrowtail} & N \rtimes^g_{E,\varphi} H & \overset{e_1}{\twoheadrightarrow} & H \\
& \| & & \downarrow f & & \| \\
\Psi\colon & N & \overset{k_2}{\rightarrowtail} & N \rtimes^g_{E',\varphi} H & \overset{e_2}{\twoheadrightarrow} & H
\end{array}$$

Then by a similar argument we have that $f([n], h) = ([t^*(h)n], h)$ where $t^* \colon H \to N$ is a crossed homomorphism relative to the admissible H-indexed equivalence relation E'. Thus, we arrive at the following result.

Theorem 5.6.13. *Let* Γ *and* Ψ *be cosetal extensions as above. Then* $\mathrm{Hom}(\Gamma, \Psi)$ *is bijective to* $\mathcal{Z}^1(H, N, E', \varphi)$ *whenever* Γ *and* Ψ *satisfy the condition outlined in Theorem 5.6.7. Here an element of* $\mathcal{Z}^1(H, N, E', \varphi)$ *is sent to the composition of the corresponding automorphism on* Ψ *and the map* $\lambda\colon \Gamma \to \Psi$ *as defined in Proposition 5.6.3. In particular,* λ *corresponds to the identity of* $\mathcal{Z}^1(H, N, E', \varphi)$*.*

Remark 5.6.14. Composition with an automorphism in the codomain corresponds to translation in the group $\mathcal{Z}^1(H, N, E', \varphi)$ by the associated crossed homomorphism. A crossed homomorphism from the domain naturally gives one in the codomain and composition with an automorphism in the *domain* corresponds to translation by this corresponding crossed homomorphism. $\triangle$

5.7 Parameterisation by actions alone

In Chapter 3 and in the earlier parts of this chapter weakly Schreier and cosetal extensions are characterised in terms of admissible equivalence relations, compatible actions, and in the latter case, cohomology classes. Instead of starting with an admissible equivalence relation and then specifying an action compatible with it, it is also possible to start by specifying a candidate action and then choosing a compatible equivalence relation. By Theorem 5.6.7 we know that different extensions admit morphisms between them only when they have the same action and cohomology class and so it can be helpful to consider all the compatible equivalence relations corresponding to a given candidate action. To start we will not require N to be an abelian group.

Proposition 5.7.1. *Let N and H be monoids and let $\alpha\colon H\times N \to N$ be an arbitrary function. The equivalence relations compatible with α are closed under inhabited pointwise intersections.*

Proof. All of the conditions for α to be a compatible action and the second and third conditions of an admissible equivalence relation are implications from one of the equivalence relations to another and so they are clearly closed under arbitrary pointwise intersections. Furthermore, the first condition of an admissible equivalence relation is clearly downwards closed and hence closed under inhabited pointwise intersections. ☐

The equivalence relations compatible with a given map $\alpha\colon H \times N \to N$ are not closed under arbitrary meets in general, since it might happen that no such equivalence relation exists at all. For instance, it is shown in Chapter 3 that when H is a group, only true actions are compatible with any equivalence relation.

Definition 5.7.2. We say a function $\alpha\colon H \times N \to N$ is *valid* if it is compatible with some H-indexed equivalence relation. △

Validity is in fact the *only* obstruction to the compatible equivalence relations forming a complete lattice. To show this, we will prove that when the set is nonempty, it has a largest element. We start by finding an indexed equivalence relation which contains every indexed equivalence relation compatible with α.

Proposition 5.7.3. *Let $\alpha\colon H\times N \to N$ be compatible with an H-indexed equivalence relation E. Then whenever $n \sim_E^h n'$, we have that the following always holds.*

$$\forall x, y \in H.\ [xhy = 1 \implies \alpha(x, n) = \alpha(x, n')] \,.$$

Proof. Suppose that $n \sim_E^h n'$ and consider $x, y \in H$ such that $xhy = 1$. Since $n \sim_E^h n'$, we have that $n \sim_E^{hy} n'$. Now applying the second compatibility condition, we obtain $\alpha(x, n) \sim_E^{xhy} \alpha(x, n')$. But $xhy = 1$ and so $\alpha(x, n) = \alpha(x, n')$ as required.☐

This suggests the following definition.

Definition 5.7.4. Given a function $\alpha\colon H \times N \to N$, we call the associated H-indexed equivalence relation E_α defined by

$$n \sim_\alpha^h n' \iff \forall x, y \in H.\ [xhy = 1 \implies \alpha(x, n) = \alpha(x, n')]$$

the associated *coarse equivalence relation.* △

Note that this is indeed an H-indexed equivalence relation.

Theorem 5.7.5. *If α is compatible with any H-indexed equivalence relation E, then it is compatible with the coarse equivalence relation E_α.*

Proof. Let us begin by showing that E_α is admissible. Suppose $n \sim_\alpha^1 n'$ and observe that $x \cdot 1 \cdot y = 1$ when $x = 1 = y$. Thus, $\alpha(1, n) = \alpha(1, n')$. Since α is compatible with E, we get that $n \sim_E^1 n'$ and hence $n = n'$.

Next we must show that if $n \sim_\alpha^h n'$ then $an \sim_\alpha^h an'$. Now suppose $n \sim_\alpha^h n'$ and $xhy = 1$. We know that $\alpha(x, n) = \alpha(x, n')$. Consider

$$\begin{aligned}\alpha(x, an) &\sim_E^x \alpha(x, a)\alpha(x, n)\\ &\sim_E^x \alpha(x, a)\alpha(x, n')\\ &\sim_E^x \alpha(x, an').\end{aligned}$$

It follows that

$$\alpha(x, an) \sim_E^{xhy} \alpha(x, an').$$

But $xhy = 1$ and so we find $\alpha(x, an) = \alpha(x, an')$.

Finally, we must show that if $n \sim_\alpha^h n'$ then $n \sim^h h'n'$. Suppose $n \sim_\alpha^h n'$ and $xhh'y' = 1$. Then for $y = h'y'$ we have that $xhy = 1$. Thus, by assumption we get that

$$\alpha(x, n) = \alpha(x, n')$$

and hence E_α is admissible.

Now we show that α is compatible with E_α. Notice that because $E \subseteq E_\alpha$ and α is compatible with E, we immediately have that conditions (iii)–(vi) hold. So we need only check conditions (i) and (ii).

To prove condition (i) we must show that if $n \sim_\alpha^h n'$ then $n\alpha(h, a) \sim_\alpha^h n'\alpha(h, a)$ for all $a \in N$. Suppose $xhy = 1$. By assumption we have $\alpha(x, n) = \alpha(x, n')$. We must show that $\alpha(x, n\alpha(h, a)) = \alpha(x, n'\alpha(h, a))$. Using E we have

$$\begin{aligned}\alpha(x, n\alpha(h, a)) &\sim_E^x \alpha(x, n) \cdot \alpha(x, \alpha(h, a))\\ &\sim_E^x \alpha(x, n') \cdot \alpha(x, \alpha(h, a))\\ &\sim_E^x \alpha(x, n'\alpha(h, a)).\end{aligned}$$

As above, it follows that $\alpha(x, n\alpha(h, a)) = \alpha(x, n'\alpha(h, a))$ since $x(hy) = 1$.

Finally, for condition (i) we must show that if $n \sim_\alpha^h n'$ then $\alpha(b, n) \sim_\alpha^{bh} \alpha(b, n')$ for all $b \in H$. Suppose $n \sim_\alpha^h n'$ and $xbhy = 1$. We must show $\alpha(x, \alpha(b, n)) = \alpha(x, \alpha(b, n'))$.

Notice that by assumption $\alpha(xb, n) = \alpha(xb, n')$. Now consider

$$\begin{aligned}\alpha(x, \alpha(b, n)) &\sim_E^{xb} \alpha(xb, n) \\ &\sim_E^{xb} \alpha(xb, n') \\ &\sim_E^{xb} \alpha(x, \alpha(b, n')).\end{aligned}$$

Because $xb(hy) = 1$, we then find $\alpha(x, \alpha(b, n)) = \alpha(x, \alpha(b, n'))$, as required. □

Corollary 5.7.6. *The equivalence relations compatible with any valid map $\alpha\colon H \times N \to N$ form a complete lattice.*

The existence of the coarsest compatible equivalence relation was proved in Chapter 3 under the assumption that whenever $xh \in H$ is right-invertible, so is h. In this case, every valid α has the same coarsest equivalence relation given by $n \sim_E^h n'$, we have $\forall y \in H.\ [hy = 1 \implies n = n']$. The following example gives a coarsest compatible equivalence relation, which does not satisfy these conditions.

Example 5.7.7. Let H be the bicyclic monoid $B = \langle p, q \mid pq = 1\rangle$, N the group $\mathbb{Z}^\omega$ of integer sequences under addition and α the true action $\alpha\colon H \times N \to N$ defined by

$$\alpha(q, s)_n = \begin{cases} s_{n-1} & n > 0 \\ 0 & n = 0 \end{cases}$$

and $\alpha(p, s)_n = s_{n+1}$.

We can show that $x(q^a p^b)y = 1$ if and only if $x = p^{a+i}$ and $y = q^{b+i}$ for some $i \geq 0$ and hence the coarse equivalence relation is given by $s \sim_\alpha^{q^a p^b} s' \iff \forall n \geq a.\ s_n = s'_n$.

The resulting weak semidirect product can be expressed as the set of pairs of the form $(s, q^a p^b)$ where $s\colon \mathbb{N} \cap [a, \infty) \to \mathbb{Z}$ with unit $(0, 1)$ and multiplication given by

$$(s, q^a p^b) \cdot (s', q^{a'} p^{b'}) = (n \mapsto s_n + s'_{n+b-a}, q^{a+a'-\min(b,a')} p^{b+b'-\min(b,a')}).$$

The kernel sends s to $(s, 1)$ and the cokernel sends (s, x) to x. △

By Corollary 5.7.6 the equivalence relations compatible with a valid map $\alpha\colon H \times N \to N$ are closed under *joins*. These correspond to meets in the order of the associated quotients. In Chapter 2 we showed that for Artin glueings of frames this meet operation can be interpreted as a kind of Baer sum. Of course, there is a different notion of Baer sum for cosetal extensions with abelian group kernel, in which case the equivalence relation is fixed beforehand. We might attempt to gain some insight into the interaction of the equivalence relations and cohomology classes by combining them into a single structure. From here on we again assume N is always an abelian group.

Definition 5.7.8. Let N be an abelian group and fix a valid action $\varphi\colon H \times N \to N$. We define $\widetilde{H}^2_\varphi(H,N)$ to be the set of pairs $(E,[g])$ where E is an H-index equivalence relation compatible with α and $[g] \in \mathcal{H}^2(H,N,E,[\varphi])$.

We define an operation $(E_1,[g_1])+(E_2,[g_2]) = (E_1 \vee E_2, \ell_1([g_1])+\ell_2([g_2]))$ on $\widetilde{H}^2_\varphi(H,N)$ where the maps $\ell_{1,2}\colon \mathcal{H}^2(H,N,E_{1,2},[\varphi]) \to \mathcal{H}^2(H,N,E_1 \vee E_2,[\varphi])$ are defined as in Proposition 5.6.1 and we define a constant $0 = (\perp, 0) \in \widetilde{H}_\varphi(H,N)$ where $\perp$ is the finest equivalence relation compatible with φ. △

Theorem 5.7.9. *Let H and N be monoids. The algebra $\widetilde{\mathcal{H}}^2_\varphi(H,N)$ is an inverse monoid where $(E,[-g])$ is the inverse of $(E,[g])$.*

Proof. The axioms are all routine calculations. □

Remark 5.7.10. The addition of this monoid appears to be related to a notion of Baer sum considered in (Fulp and Stepp [21]) and (Fulp and Stepp [20]) in the special case of central extensions. △

This inverse monoid allows us to understand the relationship between all cosetal extensions of H by N with a given fixed valid action φ. Of course, the choice of φ for a given extension is not unique, since it is only defined up to a quotient and so, unlike the cohomology groups $\mathcal{H}^2(H,N,E,[\varphi])$, the objects $\widetilde{\mathcal{H}}^2_\varphi(H,N)$ no longer partition the extensions, though if the sum of two elements is taken in different monoids, the results will be compatible.

Furthermore, suppose we are given extensions whose corresponding relaxed actions satisfy $(E_1,\varphi_1),(E_2,\varphi_2) \le (E_3,\varphi_3)$, but where φ_1 and φ_2 are distinct with respect to E_1 and E_2. In this case, we could analyse the relationship between the first and third extensions in $\widetilde{\mathcal{H}}^2_{\varphi_1}(H,N)$ and between the second and the third in $\widetilde{\mathcal{H}}^2_{\varphi_2}(H,N)$, but there is no inverse monoid that would allow us to analyse all three.

One way to address this problem is to define the following *category*.

Definition 5.7.11. We define a category $\widetilde{\mathcal{H}}^2(H,N)$ whose objects are given by valid actions of H on N and where the morphisms from φ to φ' are given by pairs of the form $(E,[g])$ where E is an H-indexed equivalence relation compatible with φ and φ' and with respect to which φ and φ' are equivalent and $[g] \in \mathcal{H}^2(H,N,E,[\varphi])$. Composition is given by multiplication in $\widetilde{\mathcal{H}}^2_\varphi(H,N)$. △

The inverse monoids $\widetilde{\mathcal{H}}^2_\varphi(H,N)$ are then simply the endomorphism monoids in this category, but it also allows us to move from one of these inverse monoids to another as necessary. The category $\widetilde{\mathcal{H}}^2(H,N)$ is an example of what is known as an *inverse category*.

Definition 5.7.12. An inverse category is a category for which every morphism $f\colon X \to Y$ has a unique 'inverse' $g\colon Y \to X$ such that $fgf = f$ and $gfg = g$. △

Any inverse category can be encoded as a certain kind of ordered groupoid as in Chapter 4, Section 2 of (DeWolf [11]). Applying this construction in our case, we obtain the disjoint union over all relaxed actions (E,φ) of the groups $\mathcal{H}^2(H,N,E,[\varphi])$ considered as one-object groupoids. The order on objects is given by $(E_1,\varphi_1) \preceq (E_2,\varphi_2)$ if and only if $E_1 \supseteq E_2$ and $\varphi_1 = \varphi_2$ and the order on morphisms is induced by the order on their (co)domains.

This order is somewhat too strict, since it distinguishes between relaxed actions with equivalent actions. This motivates the following definition.

Definition 5.7.13. The ordered groupoid $\widehat{\mathcal{H}}^2(H,N)$ with relaxed actions $(E,[\varphi])$ as objects with the reverse of the usual order. The morphisms from $(E_1,[\varphi_1])$ to $(E_2,[\varphi_2])$ exist when $E_1 = E_2$ and φ_1 is equivalent to φ_2 and are given by elements of $\mathcal{H}^2(H,N,E_1,[\varphi_1])$. Composition is given in the obvious way and the morphisms can be ordered according to their (co)domains. △

Alternatively, we can apply the Grothendieck construction (see [23, B.1.3.1] for details) to the functor $L_{H,N}\colon \mathrm{RAct}(H,N) \to \mathbf{Ab}$ from Section 5.6.1 (composed with the inclusion $\mathbf{Ab} \hookrightarrow \mathrm{Cat}$) to obtain a category $\int L_{H,N}$ consisting of relaxed actions $(E,[\varphi])$ as objects and with morphisms from $(E_1,[\varphi])$ to $(E_2,[\varphi])$ given by elements of $\mathcal{H}^2(H,N,E_2,[\varphi])$ for $E_1 \subseteq E_2$. The underlying groupoid of $\widehat{\mathcal{H}}^2(H,N)$ is then simply the *core* (i.e. the groupoid of invertible morphisms) of $\int L_{H,N}$ equipped with the order induced by the order on $\mathrm{RAct}(H,N)$.

Chapter 6

Artin glueings of toposes as adjoint split extensions

We now make a break with monoids and consider another direction in which to generalise the results of Chapter 2. In this chapter we generalise the results involving Artin glueings of frames to Artin glueings of toposes. This motivates a more general Schreier-like extension theory in the 2-category of monoidal categories.

6.1 Introduction

Artin glueings of toposes were introduced in Chapter 9 of (Artin, Grothendieck, and Verdier [1]) and provide a way to view a topos $\mathcal{G}$ as a combination of an open subtopos $\mathcal{G}_{\mathfrak{o}(U)}$ and its closed complement $\mathcal{G}_{\mathfrak{c}(U)}$. This situation may be described as the 'internal' view, but we might instead look at it externally. Here we have that Artin glueings of two toposes $\mathcal{H}$ and $\mathcal{N}$ correspond to solutions to the problem of which toposes $\mathcal{G}$ does $\mathcal{H}$ embed in as an open subtopos and $\mathcal{N}$ as its closed complement.

As previously discussed, there is an analogy to be made with semidirect products of groups. There we may either, internally, view a group as being generated in a natural way from two complemented subgroups (one of which is normal), or externally, view a semidirect product as a solution to the problem of how to embed groups H and N as complemented subobjects so that N is normal. Of particular importance to us is that semidirect products precisely correspond to split extensions of groups.

Artin glueings of Grothendieck toposes decategorify to the setting of frames which we studied extensively in Chapter 2. While our results were proved for frames, it is not hard to see that the arguments carry over to Heyting algebras. It is this view that we now extend back to the elementary topos setting. We now recall the main results of Chapter 2.

In the category of frames with finite-meet-preserving maps, there exist zero morphisms given by the constant 'top' maps. This allows us to consider kernels and cokernels. Cokernels always exist and the cokernel of $f\colon N \to G$ is given by $e\colon G \to {\downarrow} f(0)$ where $e(g) = f(0) \wedge g$. This map has a right adjoint splitting e_* sending h to $h^{f(0)}$. Kernels do not always exist, but kernels of cokernels always do, and the kernel of $e\colon G \to {\downarrow}u$ is the inclusion of ${\uparrow}u \subseteq G$. The cokernel is readily seen to be the open sublocale corresponding to u and the kernel the corresponding closed sublocale. This immediately gives that the split extensions whose splittings are adjoint to the cokernel correspond to Artin glueings.

With this correspondence established, the corresponding Ext functor was shown to be naturally isomorphic to the Hom functor. Each hom-set $\mathrm{Hom}(H, N)$ has an order structure and this order structure was shown to correspond contravariantly in $\mathrm{Ext}(H, N)$ to morphisms of split extensions. Finally, it was demonstrated how the meet operation in $\mathrm{Hom}(H, N)$ naturally induces a kind of 'Baer sum' in $\mathrm{Ext}(H, N)$.

In this chapter all of the above results find natural generalisation to the topos setting after we provide definitions for the analogous 2-categorical concepts.

6.2 Background

6.2.1 2-categorical preliminaries

There are a number of 2-categories and 2-categorical constructions considered in this chapter and so we provide a brief overview of these here. All the 2-categories we consider shall be strict.

Briefly, a (strict) 2-category consists of objects, 1-morphisms between objects and 2-morphisms between 1-morphisms. Phrased another way, instead of hom-sets between objects as is the case with 1-categories, for any two objects A and B we have an associated hom-category $\mathrm{Hom}(A, B)$.

Both 1-morphisms and 2-morphisms may be composed under the right conditions. If $F\colon A \to B$ and $G\colon B \to C$ are 1-morphisms, then we may compose them to yield $GF\colon A \to C$. We usually represent this with juxtaposition, though if an expression is particularly complicated we may use $G \circ F$. As with natural transformations, there are two ways to compose 2-morphisms — vertically and horizontally. We may compose $\alpha\colon F \to G$ and $\beta\colon G \to H$ vertically to give $\beta\alpha\colon F \to H$, sometimes written

$\beta \circ \alpha$. Orthogonally, if we have $F_2F_1 \colon A \to C$, $G_2G_1 \colon A \to C$, $\alpha \colon F_1 \to G_1$ and $\beta \colon F_2 \to G_2$, then we may compose α and β horizontally to form $\beta * \alpha \colon F_2F_1 \to G_2G_1$. Vertical and horizontal composition are related by the so-called interchange law. For each object A there exists an identity 1-morphism id_A and for each 1-morphism F there exists an identity 2-morphism id_F.

Just as one can reverse the arrows of a category $\mathcal{B}$ to give $\mathcal{B}^{\mathrm{op}}$, one can reverse the 1-morphisms of a 2-category $\mathcal{C}$ to give $\mathcal{C}^{\mathrm{op}}$. It is also possible to reverse the directions of the 2-morphisms yielding $\mathcal{C}^{\mathrm{co}}$ and when both the 1-morphisms and 2-morphisms are reversed we obtain $\mathcal{C}^{\mathrm{co\,op}}$.

In this chapter we will make extensive use of string diagrams. For an introduction to string diagrams for 2-categories see (Marsden [31]). We will use the convention that vertical composition is read from bottom to top and horizontal composition runs diagrammatically from left to right.

We consider 2-functors between 2-categories defined as follows. (We follow the convention that 2-functors are not necessarily strict.)

Definition 6.2.1. A *2-functor* $\mathcal{F}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$ consists of a function $\mathcal{F}$ sending objects C in $\mathcal{C}$ to objects $\mathcal{F}(C)$ in $\mathcal{D}$ and for each pair of objects C_1 and C_2 in $\mathcal{C}$ a functor $\mathcal{F}_{C_1,C_2} \colon \mathrm{Hom}(C_1, C_2) \to \mathrm{Hom}(\mathcal{F}(C_1), \mathcal{F}(C_2))$, for which we use the same name. Additionally, for each pair of composable 1-morphisms (F, G) we have an invertible 2-morphism $\omega_{G,F} \colon \mathcal{F}(G) \circ \mathcal{F}(F) \to \mathcal{F}(G \circ F)$ called the compositor, and for each object C in $\mathcal{C}$ we have an invertible 2-morphism $\kappa_A \colon \mathrm{Id}_{\mathcal{F}(A)} \to \mathcal{F}(\mathrm{Id}_A)$ called the unitor. This data satisfies the following constraints.

i) Let $\alpha \colon F_1 \to F_2$ and $\beta \colon G_1 \to G_2$ be horizontally composable 2-morphisms. Then the compositors must satisfy that $\omega_{G_2,F_2}\,(P(\beta) * P(\alpha)) = P(\beta * \alpha)\,\omega_{G_1,F_1}$.

ii) The compositors must be associative in the sense that if $F \colon X \to Y$, $G \colon Y \to Z$ and $H \colon Z \to W$ are 1-morphisms, then $\omega_{H,GF}(\mathrm{id}_{P(H)} * \omega_{G,F}) = \omega_{HG,F}(\omega_{H,G} * \mathrm{id}_{P(F)})$.

iii) If $F \colon X \to Y$ is a 1-morphism, then we have the unit axiom $\omega_{F,\mathrm{Id}_X}(\mathrm{id}_{P(F)} * \kappa_X) = \mathrm{id}_{P(F)} = \omega_{\mathrm{Id}_Y,F}(\kappa_Y * \mathrm{id}_{P(F)})$. △

There is a notion of 2-natural transformation between 2-functors defined as follows.

Definition 6.2.2. Let $(\mathcal{F}_1, \omega_1, \kappa_1), (\mathcal{F}_2, \omega_2, \kappa_2) \colon \mathcal{X} \to \mathcal{Y}$ be two 2-functors. A 2-natural transformation $\rho \colon \mathcal{F}_1 \to \mathcal{F}_2$ is given by two families:

i) A 1-morphism $\rho_X \colon \mathcal{F}_1(X) \to \mathcal{F}_2(X)$ for each object X in $\mathcal{X}$.

ii) An invertible 2-morphism $\rho_F\colon \mathcal{F}_2(F)\rho_X \to \rho_Y \mathcal{F}_1(F)$ for each 1-morphism $F\colon X \to Y$ in $\mathcal{X}$.

They must satisfy the following coherence conditions. First if $F\colon X \to Y$ and $G\colon Y \to Z$ are 1-morphisms in $\mathcal{X}$, then ρ must respect composition, so that the following diagram commutes.

$$\begin{array}{ccccc}
\mathcal{F}_2(G)\circ\mathcal{F}_2(F)\circ\rho_X & \xrightarrow{\mathcal{F}_2(G)\rho_F} & \mathcal{F}_2(G)\circ\rho_Y\circ\mathcal{F}_1(F) & \xrightarrow{\rho_G\mathcal{F}_1(F)} & \rho_Z\circ\mathcal{F}_1(G)\circ\mathcal{F}_1(F) \\
\downarrow{\scriptstyle \kappa_2\rho_X} & & & & \downarrow{\scriptstyle \rho_Z\kappa_1} \\
\mathcal{F}_2(GF)\circ\rho_X & & \xrightarrow[\rho_{GF}]{} & & \rho_Z\circ\mathcal{F}_1(GF)
\end{array}$$

Next for each object $X \in \mathcal{X}$, ρ must respect the identity Id_X. For this we need the following diagram to commute.

$$\begin{array}{ccc}
\mathrm{Id}_{\mathcal{F}_2(X)}\circ\rho_X & = & \rho_X\circ\mathrm{Id}_{\mathcal{F}_1(X)} \\
\downarrow{\scriptstyle \omega_2\rho_X} & & \downarrow{\scriptstyle \rho_X\omega_1} \\
\mathcal{F}_2(\mathrm{Id}_X)\circ\rho_X & \xrightarrow[\rho_{\mathrm{Id}_X}]{} & \rho_X\circ\mathcal{F}_1(\mathrm{Id}_X)
\end{array}$$

Finally, they must satisfy the following 'naturality' condition for $F, F'\colon X \to Y$ and $\alpha\colon F \to F'$.

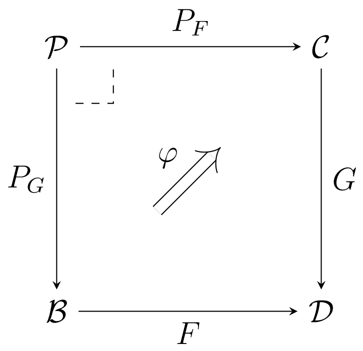
$$\begin{array}{ccc}
\mathcal{F}_2(F)\circ\rho_X & \xrightarrow{\rho_F} & \rho_Y\circ\mathcal{F}_1(F) \\
\downarrow{\scriptstyle \mathcal{F}_2(\alpha)\rho_X} & & \downarrow{\scriptstyle \rho_Y\mathcal{F}_1(\alpha)} \\
\mathcal{F}_2(F')\circ\rho_X & \xrightarrow[\rho_{F'}]{} & \rho_Y\circ\mathcal{F}_1(F')
\end{array}$$

When each component ρ_X is an equivalence, we call ρ a *2-natural equivalence.* $\triangle$

One 2-functor of note is the 2-functor $\mathrm{Op}\colon \mathrm{Cat}^{\mathrm{co}} \to \mathrm{Cat}$ which sends a category $\mathcal{C}$ to its opposite category $\mathcal{C}^{\mathrm{op}}$. We will use this 2-functor in Section 6.5 to help compare two 2-functors of different variances.

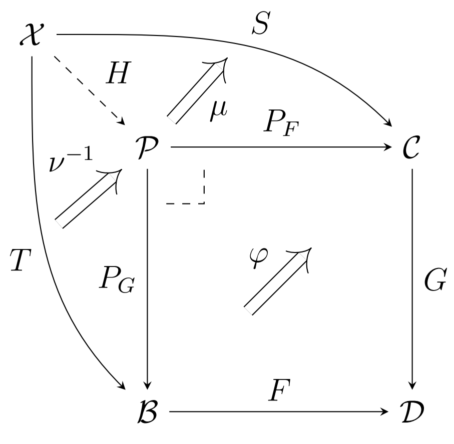
Limits and colimits have 2-categorical analogues, which will be used extensively throughout this chapter. A more complete introduction to these concepts can be found in (Lack [26]). In particular, we will make use of 2-pullbacks and 2-pushouts, as well as comma and cocomma objects, which we describe concretely below.

Definition 6.2.3. Given two 1-morphisms $F\colon \mathcal{B} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ their *comma object* is shown in the following diagram



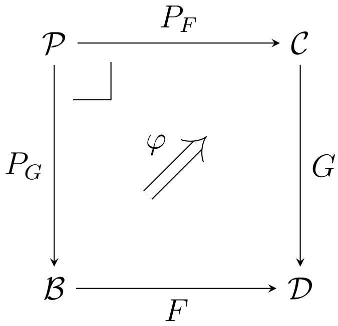
and satisfies the following two conditions.

i) Let $T\colon \mathcal{X} \to \mathcal{B}$ and $S\colon \mathcal{X} \to \mathcal{C}$ be 1-morphisms and let $\psi\colon FT \to GS$ be a 2-morphism. Then there exists a 1-morphism $H\colon \mathcal{X} \to \mathcal{P}$ and invertible 2-morphisms $\nu\colon P_G H \to T$ and $\mu\colon P_F H \to S$ satisfying that $G\mu \circ \varphi H \circ F\nu^{-1} = \psi$.



ii) If $H, K\colon \mathcal{X} \to \mathcal{P}$ are 1-morphisms and $\alpha\colon P_G H \to P_G K$ and $\beta\colon P_F H \to P_F K$ are 2-morphisms satisfying that $\varphi K \circ F\alpha = G\beta \circ \varphi H$, then there exists a unique 2-morphism $\gamma\colon H \to K$ such that $P_G\gamma = \alpha$ and $P_F\gamma = \beta$.

A 2-pullback is defined similarly, except both φ and ψ are required to be invertible, and is represented as follows.



Cocomma objects and 2-pushouts may be defined dually. △

We now recall the definition of a fibration of categories.

Definition 6.2.4. Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a functor. A morphism $\overline{f}\colon \overline{A} \to \overline{B}$ in $\mathcal{X}$ is *cartesian* with respect to F if for any $\overline{g}\colon \overline{C} \to \overline{B}$ in $\mathcal{X}$ and $h\colon F(\overline{C}) \to F(\overline{B})$ in $\mathcal{Y}$ with $F(\overline{g}) = F(\overline{f})h$, there exists a unique map $\overline{h}\colon \overline{C} \to \overline{A}$ with $F(\overline{h}) = h$ and $\overline{f}\,\overline{h} = \overline{g}$. We say $F\colon \mathcal{X} \to \mathcal{Y}$ is a *(Street) fibration* in Cat if for any morphism $f\colon A \to F(\overline{B})$ in $\mathcal{Y}$ there exists a cartesian lifting $\overline{f}\colon \overline{A} \to \overline{B}$ and an isomorphism $j\colon F(\overline{A}) \cong A$ with $F(\overline{f}) = fj$. △

In fact, we will also need the notion of a fibration in other 2-categories, such as the 2-category $\mathrm{Cat}_{\mathrm{lex}}$ of finitely-complete categories and finite-limit-preserving functors. The general definitions of fibrations, morphisms of fibrations and 2-morphisms of fibrations can be found, for example, in [10, Definitions 3.4.3–3.4.5]. However, it is not hard to see that the fibrations in $\mathrm{Cat}_{\mathrm{lex}}$ are simply the finite-limit-preserving functors which are fibrations in Cat.

6.2.2 Elementary toposes

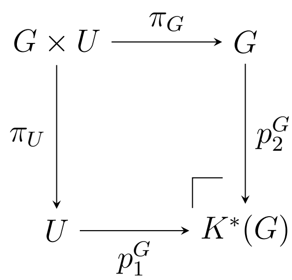
By a *topos* we mean an *elementary topos* — that is, a cartesian-closed category admitting finite limits and containing a subobject classifier. The usual 2-category of toposes has 1-morphisms given by *geometric morphisms* and 2-morphisms given by natural transformations. For an introduction see (Mac Lane and Moerdijk [30]).

For Grothendieck toposes, the subobjects of the terminal object may be imbued with the structure of a frame. Moreover, a geometric morphism between two Grothendieck toposes induces a locale homomorphism between their locales of subterminal objects. This induces a functor from the category of Grothendieck toposes into the category of locales, which is in fact a reflector.

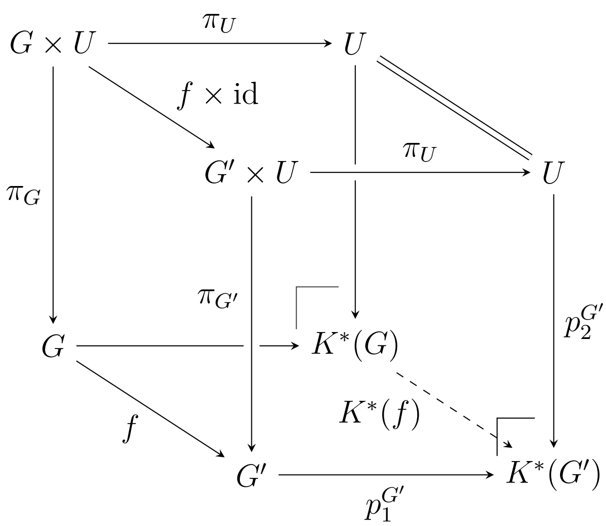
A *subtopos* of a topos $\mathcal{E}$ is a fully faithful geometric morphism $i\colon \mathcal{S} \hookrightarrow \mathcal{E}$. Subtoposes when acted upon by the localic reflection may sometimes be sent to open or closed sublocales. Those sent to open sublocales we call open subtoposes and those sent to closed sublocales we call closed subtoposes. Since open (or closed) sublocales correspond to elements of the frame, it follows that any open (or closed) subtopos corresponds to a particular subterminal object U. This suggests a similar notion of open/closed subtopos corresponding to a particular subterminal object U even in the setting of elementary toposes.

The open subtopos $\mathcal{E}_{\mathfrak{o}(U)}$ corresponding to a subterminal object U has a reflector given by the exponential functor $(-)^U$. It is not hard to see that this topos is equivalent to the slice topos $\mathcal{E}/U$, which in turn can be thought as the full subcategory of the objects in $\mathcal{E}$ admitting a map into U. From this point of view, the reflector $E\colon \mathcal{E} \to \mathcal{E}/U$ maps an object X to the product $X \times U$. We denote its right adjoint by $E_* = (-)^U$ and write θ and ε for the unit and counit respectively. Note that $E_*E(G) = (G \times U)^U \cong G^U$. In addition to a right adjoint, E also has a left adjoint $E_!$, which is simply the inclusion of $\mathcal{E}/U$ into $\mathcal{E}$.

The closed subtopos $\mathcal{E}_{\mathfrak{c}(U)}$ has reflector $K^*\colon \mathcal{E} \to \mathcal{E}_{\mathfrak{c}(U)}$ given on objects by the following pushout.



On morphisms $f\colon G \to G'$ in $\mathcal{E}$, $K^*(f)$ is given by the universal property of the pushout in the following diagram.



Here the left, front and top faces commute and so a diagram chase determines that $p_1^{G'} f$ and $p_2^{G'} \mathrm{id}_U$ indeed form a cocone.

We denote the right adjoint of K^* by K and write ζ and δ for the unit and counit respectively.

As expected, $\mathcal{E}_{\mathfrak{o}(U)}$ and $\mathcal{E}_{\mathfrak{c}(U)}$ are complemented subobjects.

6.2.3 Artin glueings

Given toposes $\mathcal{H}$ and $\mathcal{N}$ we can ask for which toposes $\mathcal{G}$ may $\mathcal{H}$ be embedded as an open subtopos and $\mathcal{N}$ as its closed complement. This is solved completely by the Artin glueing construction. For any finite-limit-preserving functor $F \colon \mathcal{H} \to \mathcal{N}$ we may construct the category $\mathrm{Gl}(F)$ whose objects are triples (N, H, ℓ) in which $N \in \mathcal{N}$, $H \in \mathcal{H}$ and $\ell \colon N \to F(H)$ and whose morphisms are pairs (f, g) making the following diagram commute.

$$\begin{array}{ccc} N & \xrightarrow{\ \ell\ } & F(H) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle F(g)} \\ N' & \xrightarrow[\ \ell'\ ]{} & F(H') \end{array}$$

The category $\mathrm{Gl}(F)$ is a topos as is shown for elementary toposes in (Wraith [45]). We provide a sketch of a direct proof of this fact.

Theorem 6.2.5. *If $F \colon \mathcal{H} \to \mathcal{N}$ is a finite-limit preserving functor between toposes, then* $\mathrm{Gl}(F)$ *is a topos.*

Proof. It is easy to see that finite limits always exist in $\mathrm{Gl}(F)$ and are simply computed componentwise.

It is also not hard to verify that the subobject classifier is $(\Omega_N \times F(\Omega_H), \Omega_H, \pi_2)$ where Ω_N and Ω_H are the respective subobject classifiers of N and H and the true map is given by $(true_{\Omega_N} \times F(true_{\Omega_H}), true_{\Omega_H})$.

For exponentials we take inspiration from the proof that the category of coalgebras on a topos is always a topos appearing in Theorem 4 of Section 8 in (Mac Lane and Moerdijk [30]). We provide a translation of the construction to Artin glueings as well as provide intuition for why it works.

We would like to compute $(N_Z, H_Z, \ell_Z)^{(N_Y, H_Y, \ell_Y)}$. As a starting point consider

$(N_Z^{N_Y} \times F(H_Z^{H_Y}), H_Z^{H_Y}, \pi_2)$ whose points are of the form $((f, x), g)$ where $F(g) = x$. Since x is completely determined by g, we see that it 'contains' all pairs of morphisms (f, g). Morphisms in the glueing should satisfy that $\ell_Z f = F(g)\ell_Y$, hence intuitively we would like the subobject of $(N_Z^{N_Y} \times F(H_Z^{H_Y}), H_Z^{H_Y}, \pi_2)$ containing all such pairs.

This may be computed via the equaliser N_e of the following (non-commuting) diagram in $\mathcal{N}$.

$$\begin{array}{ccc}
(N_Z^{N_Y}\times F(H_Z^{H_Y}) & \xrightarrow{(\mathrm{id},\ell_Z)^{N_Y}\times F(H_Z^{H_Y})} & (N_Z\times F(H_Z))^{N_Y}\times F(H_Z^{H_Y}) \\
{\scriptstyle N_Z^{N_Y}\times\Delta}\Big\downarrow & & \Big\uparrow{\scriptstyle N_Z\times(F(H_Z))^{(id,\ell_Y)}} \\
N_Z^{N_Y}\times F(H_Z^{H_Y})\times F(H_Z^{H_Y}) & \xrightarrow{p\times F(H_Z^{H_Y})} & (N_Z\times F(H_Z))^{N_Y\times F(H_Y)}\times F(H_Z^{H_Y})
\end{array}$$

Here (s, t) denotes the map given by the universal property of the product, the map $\Delta\colon F(H_Z^{H_Y}) \to F(H_Z^{H_Y}) \times F(H_Z^{H_Y})$ is the diagonal map and $p\colon N_Z^{N_Y} \times F(H_Z^{H_Y}) \to (N_Z \times F(H_Z))^{N_Y \times F(H_Y)}$ is the map which in the internal logic sends $(f, F(g))$ to $f \times F(g)$. The top path sends $(f, F(g))$ to $((f, \ell_Z f), F(g))$, whereas the bottom path sends $(f, F(g))$ to $(f, F(g)\ell_Y), F(g))$. Hence the elements of N_e are precisely those pairs $(f, F(g))$ satisfying the required condition. Intuitively the points of $(N_e, H_Z^{H_Y}, \pi_2)$ are then given by (f, g) satisfying $\ell_Z f = F(g)\ell_Y$. We omit the proof that this object is the desired exponential. $\square$

The obvious projections $\pi_1\colon \mathrm{Gl}(F) \to \mathcal{N}$ and $\pi_2\colon \mathrm{Gl}(F) \to \mathcal{H}$ are finite-limit preserving. The projection π_2 has a right adjoint π_{2*} sending objects H to the triple $(F(H), H, \mathrm{id}_{F(H)})$ and morphisms f to $(F(f), f)$. This map π_{2*} is a geometric morphism and, in particular, an open subtopos inclusion. Similarly, π_1 has a right adjoint sending objects N to $(N, 1, !)$ and morphisms f to $(f, !)$ where $!\colon N \to 1$ is the unique map to the terminal object. This is itself a geometric morphism, and indeed, a closed subtopos inclusion.

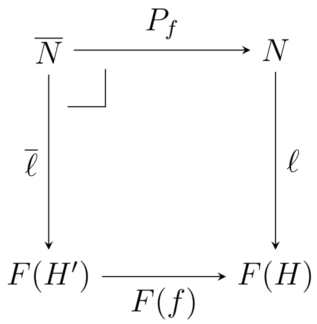
Remarkably, Artin glueings may be viewed as both comma and cocomma objects in the category of toposes with finite-limit-preserving functors. We provide a proof of the latter in Section 6.4.

One sees immediately that $\pi_1\pi_{2*} = F$. This suggests a way to view any open or closed subtopos as corresponding to one in glueing form. If $K\colon \mathcal{G}_{\mathfrak{c}(U)} \to \mathcal{G}$ and $E_*\colon \mathcal{G}/U \to \mathcal{G}$ are respectively the inclusions of open and closed subtoposes, then there is a natural sense in which these maps correspond to $\pi_{1*}\colon \mathcal{G}_{\mathfrak{c}(U)} \to \mathrm{Gl}(K^*E_*)$ and $\pi_{2*}\colon \mathcal{G}/U \to \mathrm{Gl}(K^*E_*)$ respectively. This fact is well known, though a new proof will be provided in Section 6.3.2.

We now provide proofs that the maps $\pi_1\colon\ \mathrm{Gl}(F) \to \mathcal{N}$ and $\pi_2\colon\ \mathrm{Gl}(F) \to \mathcal{H}$ are fibrations. By the above argument, these results apply equally to the inverse image maps of open and closed subtoposes. This is likely well known, though we were unable to find proofs in the literature.

Proposition 6.2.6. *Let $F\colon \mathcal{H} \to \mathcal{N}$ be a finite-limit-preserving map between toposes. Then the projection $\pi_2\colon\ \mathrm{Gl}(F) \to \mathcal{H}$ is a fibration.*

Proof. We must show that if $f\colon H' \to H$ is a morphism in $\mathcal{H}$, then for every object (H, N, ℓ) in $\mathrm{Gl}(F)$ there exists a cartesian lifting. This lifting is given by $(P_f, f)\colon (\overline{N}, H', \overline{\ell}) \to (N, H, \ell)$ where $\overline{N}$, $\overline{\ell}$ and P_f are defined by the following pullback.



The cartesian property of (P_f, f) follows from the universal property of the pullback.□

Proposition 6.2.7. *Let $F\colon \mathcal{H} \to \mathcal{N}$ be a finite-limit-preserving map between toposes. Then the projection $\pi_1\colon\ \mathrm{Gl}(F) \to \mathcal{N}$ is a fibration.*

Proof. We must show that if $f\colon N' \to N$ is a morphism in $\mathcal{N}$ then for every object (N, H, ℓ) in $\mathrm{Gl}(F)$ there exists a cartesian lifting. This map is given by $(f, \mathrm{id}_H)\colon (N', H, \ell f) \to (N, H, \ell)$.

To see that this map satisfies the universal property, suppose that we have a morphism $(g_1, g_2)\colon (A, B, k) \to (N, H, \ell)$ which is mapped by π_1 to fh. We must show there is a unique map $\overline{h}$ such that $(g_1, g_2) = (f, \mathrm{id}_H)\overline{h}$ and $\pi_1(\overline{h}) = h$. These constraints imply that $\overline{h} = (h, g_2)$.

To see that (h, g_2) is a morphism in $\mathrm{Gl}(F)$, we consider the following diagram. The left-hand square commutes as $\pi_1(g_1, g_2) = fh$ and the right-hand square commutes since (g_1, g_2) is a morphism.

$$\begin{array}{ccccc} A & = & A & \xrightarrow{k} & F(B) \\ {\scriptstyle h}\downarrow & & {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle F(g_2)} \\ N & \xrightarrow[f]{} & N' & \xrightarrow[\ell]{} & F(H) \end{array}$$

Finally, we immediately see that $(f, \mathrm{id}_H)(h, g_2) = (g_1, g_2)$, as required. $\square$

6.3 Adjoint extensions

In all that follows $\mathcal{H}$ and $\mathcal{N}$ are assumed to be toposes unless otherwise stated.

In generalising the frame results to the topos setting, it is clear that the appropriate 2-category to consider is $\mathrm{Top}_{\mathrm{lex}}$, the 2-category of toposes, finite-limit-preserving functors and natural transformations. For convenience, we will assume that 1 always refers to a distinguished terminal object in a topos, and 0 a distinguished initial object.

We will now introduce the necessary concepts in order to discuss extensions of toposes and show how Artin glueings can be viewed as adjoint extensions. In particular, the definition of extension will require notions of kernel and cokernel.

6.3.1 Zero morphisms, kernels and cokernels

The definition of extensions requires a notion of zero morphisms. Let us now define these in the 2-categorical context.

Definition 6.3.1. A *pointed 2-category* is a 2-category equipped with a class $\mathcal{Z}$ of 1-morphisms (called *zero morphisms)* satisfying the following conditions:

- $\mathcal{Z}$ contains an object of each hom-category,
- $\mathcal{Z}$ is an ideal with respect to composition — that is, $g \in \mathcal{Z} \implies fgh \in \mathcal{Z}$,
- $\mathcal{Z}$ is closed under 2-isomorphism in the sense that if $f \in \mathcal{Z}$ and $f' \cong f$ then $f' \in \mathcal{Z}$,
- for any parallel pair f_1, f_2 of morphisms in $\mathcal{Z}$, there is a unique 2-morphism $\xi\colon f_1 \to f_2$. $\triangle$

Definition 6.3.2. A *zero object* in a 2-category is an object which is both 2-initial and 2-terminal. $\triangle$

Definition 6.3.3. The *2-cokernel* of a morphism $f\colon A \to B$ in a pointed 2-category is an object C equipped with a morphism $c\colon B \to C$ such that cf is a zero morphism and which is the universal such in the following sense.

i) If $t\colon B \to X$ is such that tf is a zero morphism, then there exists a morphism $h\colon C \to X$ such that hc is isomorphic to t.

ii) Given $h, h'\colon C \to X$ and $\alpha\colon hc \to h'c$, there is a unique $\gamma\colon h \to h'$ such that $\gamma c = \alpha$.

The *2-kernel* of a morphism in a pointed 2-category $\mathcal{C}$ is simply the 2-cokernel in $\mathcal{C}^{\mathrm{op}}$. △

Note that these may also be defined in terms of 2-pushouts or 2-coequalisers involving the zero morphism.

Remark 6.3.4. Note that the condition (ii) for 2-kernels is simply the statement that the 2-kernel map is a fully faithful 1-morphism. Moreover, since an adjoint of a fully faithful morphism is fully faithful in the opposite 2-category, we have that when a putative 2-cokernel $c\colon B \to C$ has a (left or right) adjoint d, then condition (ii) for 2-cokernels is equivalent to d being fully faithful. △

We can now consider how these concepts behave in our case of interest. Note that $\mathrm{Top}_{\mathrm{lex}}$ has a zero object, the trivial topos. Then zero morphisms in $\mathrm{Top}_{\mathrm{lex}}$ are precisely those functors which send every object to a terminal object.

In $\mathrm{Top}_{\mathrm{lex}}$, 2-cokernels of morphisms $F\colon \mathcal{N} \to \mathcal{G}$ always exist and are given by the open subtopos corresponding to $F(0)$.

Proposition 6.3.5. *The 2-cokernel of $F\colon \mathcal{N} \to \mathcal{G}$ is given by $E\colon \mathcal{G} \to \mathcal{G}/F(0)$ sending objects G to $G \times F(0)$ and morphisms $f\colon G \to G'$ to $(f, id_{F(0)})\colon G \times F(0) \to G' \times F(0)$.*

Proof. We know that E lies in $\mathrm{Top}_{\mathrm{lex}}$ and so we begin by showing that EF is a zero morphism. The terminal object in $\mathcal{G}/F(0)$ is $F(0)$ and so consider the following calculation.

$$\begin{aligned} EF(N) &= F(N) \times F(0) \\ &\cong F(N \times 0) \\ &\cong F(0). \end{aligned}$$

Next suppose that $T\colon \mathcal{G} \to \mathcal{X}$ is such that TF is a zero morphism. We claim that $TE_*\colon \mathcal{H} \to \mathcal{X}$ when composed with E is naturally isomorphic to T.

Observe that

$$\begin{aligned} T(G) &\cong T(G) \times 1 \\ &\cong T(G) \times TF(0) \\ &\cong T(G \times F(0)) \\ &\cong T(E_! E(G)) \end{aligned}$$

where each isomorphism is natural in G. Hence, $T \cong TE_!E \cong (TE_!E)E_*E \cong TE_*E$ where the central isomorphism comes from $E\theta\colon E \xrightarrow{\sim} EE_*E$.

The final condition of the 2-cokernel holds immediately because E has a full and faithful adjoint. □

Unfortunately, 2-kernels do not always exist in $\mathrm{Top}_{\mathrm{lex}}$. However, they do exist in the larger 2-category $\mathrm{Cat}_{\mathrm{lex}}$ of finitely-complete categories and finite-limit-preserving functors.

Proposition 6.3.6. *Let $F\colon \mathcal{G} \to \mathcal{H}$ be a morphism in $\mathrm{Cat}_{\mathrm{lex}}$. The kernel of F, which we write as $\mathrm{Ker}(F)$, is given by the inclusion into $\mathcal{G}$ of the full subcategory of objects sent by F to a terminal object.*

Proof. Since F preserves finite limits and sends each object in $\mathrm{Ker}(F)$ to a terminal object, it is clear that $\mathrm{Ker}(F)$ is closed under finite limits. Naturally, the inclusion is a finite-limit-preserving functor.

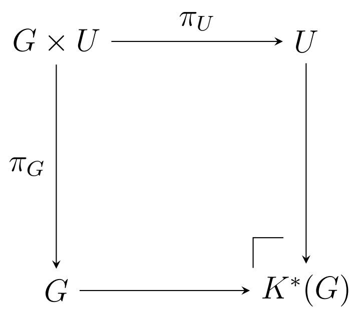
It is clear that FK is a zero morphism. We must check that if $T\colon \mathcal{X} \to \mathcal{G}$ is such that FT is a zero morphism, then it factors through $\mathrm{Ker}(F)$. Note that since FT is a zero morphism, all objects (and morphisms) in its image lie in $\mathrm{Ker}(F)$. Thus, it is easy to see that T factors through $\mathrm{Ker}(F)$. The uniqueness condition is immediate, as the inclusion of $\mathrm{Ker}(F)$ is full and faithful. □

We will only be concerned with 2-kernels of 2-cokernels. The following proposition shows that these do always exist in $\mathrm{Top}_{\mathrm{lex}}$.

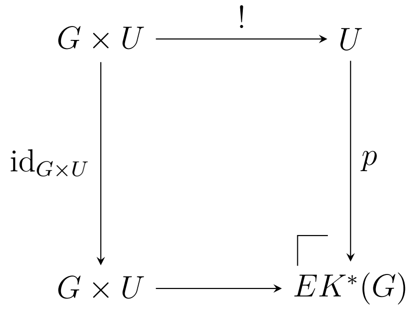
Proposition 6.3.7. *Let U be a subterminal object of a topos $\mathcal{G}$ and consider $E\colon \mathcal{G} \to \mathcal{G}/U$ defined as in Proposition 6.3.5. Then the kernel of E is given by $K\colon \mathcal{G}_{\mathfrak{c}(U)} \hookrightarrow \mathcal{G}$, the inclusion of the closed subtopos corresponding to U.*

Proof. Since $\mathrm{Top}_{\mathrm{lex}}$ is a full sub-2-category of $\mathrm{Cat}_{\mathrm{lex}}$, it suffices to show that the closed subtopos $\mathcal{G}_{\mathfrak{c}(U)}$ is equivalent to $\mathrm{Ker}(E)$, the full subcategory of objects sent by E to a terminal object.

As discussed in the background, the reflector $K^*\colon \mathcal{G} \to \mathcal{G}_{\mathfrak{c}(U)}$ sends an object G to the following pushout.



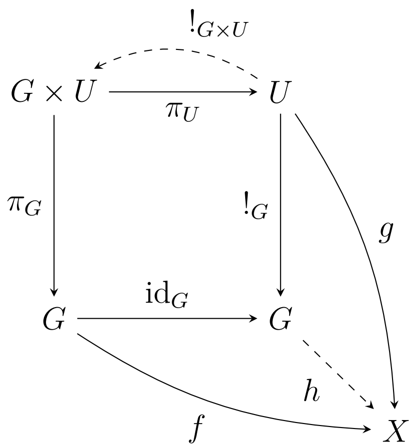
First we show that $K^*(G)$ lies in $\mathrm{Ker}(E)$. We know that E preserves colimits and so we obtain the following pushout in $\mathcal{G}/U$.



But note that p is an isomorphism, since it is the pushout of an identity morphism and thus $EK^*(G) \cong U$ and $K^*(G)$ lies in $\mathrm{Ker}(E)$.

Finally, we must show that K^* fixes the objects of $\mathrm{Ker}(E)$. First observe that U is the initial object in $\mathrm{Ker}(E)$, since if X is an object in $\mathrm{Ker}(E)$ then we have $\mathrm{Hom}_{\mathcal{G}}(U, X) = \mathrm{Hom}_{\mathcal{G}}(E_!(U), X) \cong \mathrm{Hom}_{\mathcal{G}/U}(U, E(X)) \cong \mathrm{Hom}_{\mathcal{G}/U}(U, U)$. There is precisely one morphism in $\mathrm{Hom}_{\mathcal{G}/U}(U, U)$, since U is the terminal object in $\mathcal{G}/U$.

Now consider the following candidate pushout diagram where G lies in $\mathrm{Ker}(E)$.



To see that the square commutes, note that by assumption G lies in $\mathrm{Ker}(E)$ and so $G \times U \cong U$. Therefore, $G \times U$ is initial in $\mathrm{Ker}(E)$ and there is a unique map into G.

Now suppose (f, g) is a cocone in $\mathcal{G}$. It is clear that the candidate morphism $h\colon G \to X$ must equal f and so we must just show that $f \circ !_G = g$. Since $G \times U$ and U are both initial in $\mathrm{Ker}(E)$, π_U has an inverse $!_{G\times U}\colon U \to G \times U$.

We now have

$$\begin{aligned} f \circ !_G &= f!_G\pi_U!_{G\times U} \\ &= f\pi_G!_{G\times U} \\ &= g\pi_U!_{G\times U} \\ &= g. \end{aligned}$$

This gives that G is the pushout and hence fixed by K^*. □

6.3.2 Adjoint extensions and Artin glueings

We are now in a position to define our main object of study: adjoint split extensions.

Definition 6.3.8. A diagram in $\mathrm{Top}_{\mathrm{lex}}$ of the form

$$\mathcal{N} \xrightarrow{K} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$$

equipped with a natural isomorphism $\varepsilon\colon EE_* \to \mathrm{Id}_{\mathcal{H}}$ is called an *adjoint split extension* if K is the 2-kernel of E, E is the 2-cokernel of K, E_* is the right adjoint of E and ε is the counit of the adjunction. △

Combining Propositions 6.3.5 and 6.3.7 yields the following theorem.

Theorem 6.3.9. *Every adjoint split extension is equivalent to an extension arising from a closed subtopos and its open complement in the sense that relevant squares in the following diagram commute up to coherent isomorphism. Here $U = K(0)$ is a subterminal object in $\mathcal{G}$.*

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{K} & \mathcal{G} & \underset{E_*}{\overset{E}{\rightleftarrows}} & \mathcal{H} \\ \downarrow\wr & & \| & & \downarrow\wr \\ \mathcal{G}_{\mathfrak{c}(U)} & \hookrightarrow & \mathcal{G} & \underset{(-)^U}{\overset{(-)\times U}{\rightleftarrows}} & \mathcal{G}/U \end{array}$$

The above situation is precisely the setting in which Artin glueings are studied and it is well known that in this case $\mathcal{G}$ is equivalent to an Artin glueing $\mathrm{Gl}(K^*E_*)$. We will present an alternative proof of this result from the perspective of extensions.

We begin by showing that Artin glueings can be viewed as adjoint split extensions in a natural way.

Proposition 6.3.10. *Let $F\colon \mathcal{H} \to \mathcal{N}$ be a finite-limit-preserving functor. Then the diagram*

$$\mathcal{N} \xrightarrow{\pi_{1*}} \mathrm{Gl}(F) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$$

is an adjoint split extension in $\mathrm{Top}_{\mathrm{lex}}$.

Proof. We first note that $\mathrm{Gl}(F)$ is a topos. This is a fundamental result in the theory of Artin glueings of toposes and a proof can be found in (Wraith [45]).

By Proposition 6.3.6, it is immediate that π_{1*} is the 2-kernel of π_2. To see that π_2 is the 2-cokernel of π_{1*}, we first observe that the slice category of $\mathrm{Gl}(F)$ by the subterminal object $(0,1,!) = \pi_{1*}(0)$ is equivalent to $\mathcal{H}$. The objects of $\mathrm{Gl}(F)/(0,1,!)$ are isomorphic to those the form $(0,H,!)$ (since every morphism into an initial object in $\mathcal{N}$ is an isomorphism) and its morphisms of the form $(!,f)$. If

$$L\colon\ \mathrm{Gl}(F)/(0,1,!) \to \mathcal{H}$$

is this isomorphism sending $(0,H,!)$ to H and $(!,f)$ to f and

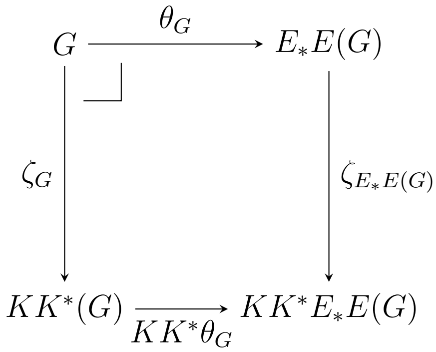
$$E\colon\ \mathrm{Gl}(F) \to \mathrm{Gl}(F)/(0,1,!)$$

is the cokernel map, then it is clear that $LE \cong \pi_2$. □

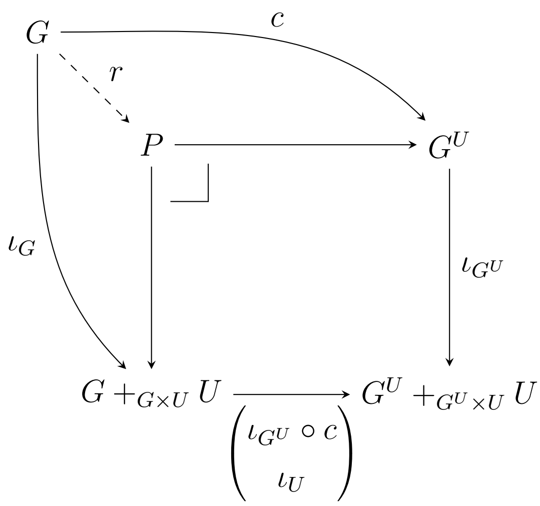
The following proposition is shown for Grothendieck toposes in Proposition 9.3.3b and 9.5.6 in (Artin, Grothendieck, and Verdier [1]), but deserves to be more well known. Here we prove it for general elementary toposes. (It also follows easily from the theory of Artin glueings, but here we will use it to develop that theory.)

Recall that we use θ for the unit of the open subtopos adjunction $E \dashv E_*$ and ζ for the unit of the closed subtopos adjunction $K^* \dashv K$.

Proposition 6.3.11. *Let $\mathcal{G}$ be a topos and consider an open subtopos $E_*\colon \mathcal{H} \hookrightarrow \mathcal{G}$ with closed complement $K\colon \mathcal{N} \hookrightarrow \mathcal{G}$. Then each object G in $\mathcal{G}$ can be expressed as the following pullback in $\mathcal{G}$ of objects from $\mathcal{N}$ and $\mathcal{H}$.*



Proof. First note that the diagram commutes by the naturality of ζ. Recall that, setting $U = KK^*(0) \cong K(0)$, we have $E_*E(G) = (G \times U)^U \cong G^U$ and that $KK^*(G)$ is the pushout of G and U along the projections $\pi_1 \colon G \times U \to G$ and $\pi_2 \colon G \times U \to U$. Now we can rewrite the relevant pullback diagram as follows.



Here the ι maps are injections into the pushout and c is the unit of the exponential adjunction, which intuitively maps elements of G to their associated constant functions.

Let us express P in the internal logic. We have

$$P = \{(f, [g]) \mid f \sim c(g)\} \cup \{(f, [*]) \mid f \sim *\} \subseteq G^U \times (G + U)/\sim$$

where $\sim$ denotes the equivalence relation generated by $f \sim *$ for $* \in U$. Explicitly, we find that $f \sim f' \iff f = f' \vee * \in U$. Thus, we find

$$\begin{aligned} P &= \{(f, [g]) \mid f = c(g) \vee * \in U\} \cup \{(f, [*]) \mid f \in G^U, * \in U\} \\ &= \{(c(g), [g]) \mid g \in G\} \cup \{(f, [g]) \mid * \in U\} \cup \{(f, [*]) \mid * \in U\}. \end{aligned}$$

Now observe that if $* \in U$, then $[g] = [*]$ and hence $\{(f, [g]) \mid * \in U\} \subseteq \{(f, [*]) \mid$

$* \in U\}$. Finally, commuting the subobject and the quotient we arrive at

$$P = (\{(c(g), g) \mid g \in G\} \sqcup \{(f, *) \mid * \in U\}) \,/\, \sim$$

where the equivalence relation is generated by $(f, g) \sim (f, *)$ for $* \in U$. Note that the union is now disjoint.

The map $r\colon G \to P$ sends $g \in G$ to $[(c(g), g)]$. We can define a candidate inverse by $s\colon P \to G$ by $(f, g) \mapsto g$ and $(f, *) \mapsto f(*)$, which can seen to be well-defined, since if (f, g) and $(f', *)$ are elements of the disjoint union with $f = f'$ and $* \in U$, then $f'(*) = c(g)(*) = g$.

We clearly have $sr = \mathrm{id}_G$. Now if $[(c(g), g)] \in P$ then

$$rs([(c(g), g)]) = r(g) = [(c(g), g)].$$

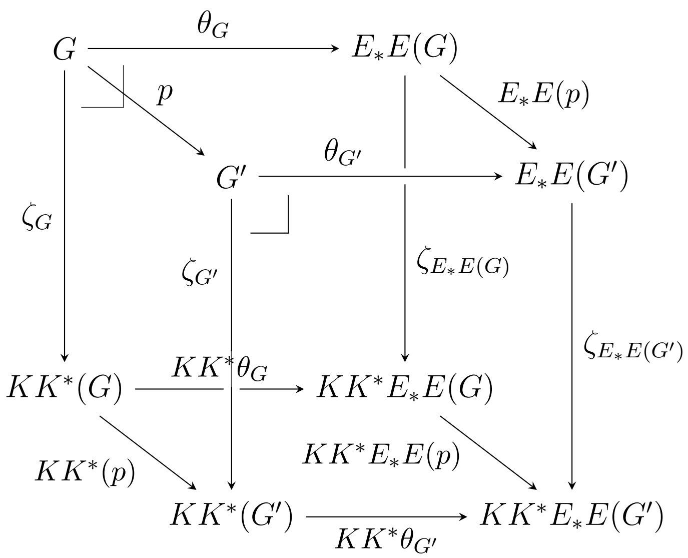
On the other hand, if $[(f, *)] \in P$ then $* \in U$ and $rs([f, *]) = r(f(*)) = [(c(f(*)), *)]$, which equals $[(f, *)]$ since $f(*) = c(f(*))(*)$ so that $f = c(f(*))$. Thus, r and s are inverses as required. □

Remark 6.3.12. The non-classical logic in the above proof can be hard to make sense of. It can help to consider the cases where $U = 0$ and $U = 1$. In the former case, G^U contains no information and $G +_{G \times U} U \cong G$, while in the latter case the opposite is true. The general case 'interpolates' between these. △

Proposition 6.3.13. *Let $\mathcal{G}$ be a topos and consider an open subtopos $E_*\colon \mathcal{H} \hookrightarrow \mathcal{G}$ with closed complement $K\colon \mathcal{N} \hookrightarrow \mathcal{G}$. We have the following pullback in the category of (finite-limit-preserving) endofunctors on $\mathcal{G}$.*

$$\begin{array}{ccc}
\mathrm{Id}_{\mathcal{G}} & \xrightarrow{\quad\theta\quad} & E_*E \\
{\scriptstyle\zeta}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\zeta E_*E} \\
KK^* & \xrightarrow[\quad KK^*\theta\quad]{} & KK^*E_*E
\end{array}$$

Proof. We proved this on objects in Proposition 6.3.11. Now consider a morphism $p\colon G \to G'$ in $\mathcal{G}$. We obtain the following commutative cube.



An enjoyable diagram chase around the cube shows that p is the unique morphism making the cube commute by the pullback property of the front face. Since pullbacks in the functor category are computed pointwise, this yields the desired result. □

Remark 6.3.14. It is remarked in Chapter 2 that adjoint extensions of frames can be viewed as weakly Schreier split extensions of monoids as defined in (Bourn [6]). Propositions 6.3.11 and 6.3.13 can be viewed as a categorified version of the weakly Schreier condition for the topos setting, though it is as yet unclear how the weakly Schreier condition might be categorified more generally. It would also be interesting to see how the general theory of ($\mathcal{S}$-)protomodular categories might be categorified (Borceux and Bourn [3]) and (Bourn, Martins-Ferreira, Montoli, and Sobral [8]). Another potential example of 2-dimensional protomodularity can be found in (Kasangian, Metere, and Vitale [24]). △

We can now prove the main result of this section. For now we will treat isomorphisms of adjoint extensions in a somewhat informal manner. We will discuss morphisms of extensions in detail in Section 6.4.

Theorem 6.3.15. *Let* $\mathcal{N} \overset{K}{\rightarrowtail} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ *be an adjoint split extension of toposes. Then it is equivalent to the split extension* $\mathcal{N} \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(K^*E_*) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$.

Proof. We denote the unit of $\pi_2 \dashv \pi_{2*}$ by $\theta' \colon \mathrm{Id}_{\mathrm{Gl}(K^*E_*)} \to \pi_{2*}\pi_2$ and observe that its component at (N, H, ℓ) can be given explicitly by $(\ell, \mathrm{id}_H) \colon (N, H, \ell) \to (K^*E_*(H), H, \mathrm{id}_H)$. We denote the unit of $\pi_1 \dashv \pi_{1*}$ by $\zeta' \colon \mathrm{Id}_{\mathrm{Gl}(K^*E_*)} \to \pi_{1*}\pi_1$ and its components are given by $(\mathrm{id}_N, !) \colon (N, H, \ell) \to (N, 1, !)$.

Consider the functor $\Phi \colon \mathcal{G} \to \mathrm{Gl}(K^*E_*)$ which sends objects G to the triple

$(K^*(G), E(G), K^*\theta_G)$ and morphisms $f\colon G \to G'$ to $(K^*(f), E(f))$. This is a morphism of extensions in the sense that we have isomorphisms $\alpha\colon \pi_{1*} \to \Phi K$, $\beta\colon \pi_2\Phi \to E$ and $\gamma\colon \pi_{2*} \to \Phi E_*$ given by $\alpha = (\mathrm{id}, !) \circ \pi_{1*}\delta^{-1}$ (where $\delta\colon K^*K \to \mathrm{Id}_{\mathcal{N}}$ is the counit of $K^* \dashv K$ and $(\mathrm{id}, !)\colon \pi_{1*}K^*K \cong \Phi K$ is simply to ensure the different choices of terminal object agree), $\beta = \mathrm{id}$ (using $\pi_2\Phi = E$) and $\gamma = (\mathrm{id}_{K^*E_*}, \varepsilon^{-1})$ (where $\varepsilon\colon EE_* \to \mathrm{Id}_{\mathcal{H}}$ is the counit of $E \dashv E_*$ and this is a morphism in $\mathrm{Gl}(K^*E_*)$ by the triangle identity). This can be seen to be a morphism of extensions as defined in Definition 6.4.1.

We must show that Φ is an equivalence. We claim that the following pullback in $\mathrm{Hom}(\mathrm{Gl}(K^*E_*), \mathcal{G})$ is the inverse of Φ. (Here the bottom equality comes from $K^*E_* = \pi_1\pi_{2*}$.)

$$\begin{array}{ccccc}
\Phi' & & \longrightarrow & & E_*\pi_2 \\
\downarrow & & & & \downarrow{\scriptstyle \zeta E_*\pi_2} \\
K\pi_1 & \xrightarrow[K\pi_1\theta']{} & K\pi_1\pi_{2*}\pi_2 & = & KK^*E_*\pi_2
\end{array}$$

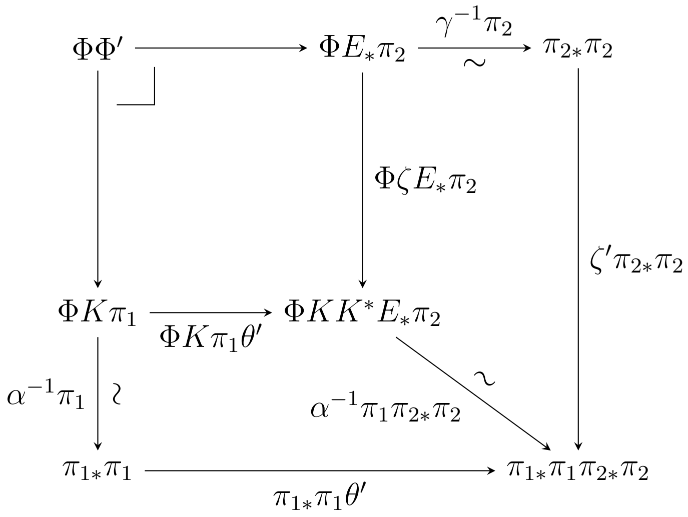
We shall make extensive use of Proposition 6.3.13 in order to prove this.

To see that $\Phi'\Phi \cong \mathrm{id}_{\mathcal{G}}$ note that composition with Φ on the right preserves limits and thus can be represented as the following pullback in $\mathrm{Hom}(\mathcal{G}, \mathcal{G})$.

$$\begin{array}{ccc}
\Phi'\Phi & \longrightarrow & E_*\pi_2\Phi \\
\downarrow & & \downarrow{\scriptstyle \zeta E_*\pi_2\Phi} \\
K\pi_1\Phi & \xrightarrow[K\pi_1\theta'\Phi]{} & KK^*E_*\pi_2\Phi
\end{array}$$

Note that $\pi_1\Phi = K^*$, $\pi_2\Phi = E$ and that $\theta'_{\Phi(G)} = (K^*\theta_G, \mathrm{id}_{E(G)})$ which of course gives that $K\pi_1\theta'\Phi = KK^*\theta$. After making these substitutions into the diagram above, we have the pullback square occurring in Proposition 6.3.13, which by the universal property gives that $\Phi\Phi'$ is naturally isomorphic to $\mathrm{Id}_{\mathcal{G}}$.

The same idea works for $\Phi\Phi'$. We consider the following pullback square in the category of endofunctors on $\mathrm{Gl}(K^*E_*)$.



The bottom trapezium commutes by naturality of α^{-1} (and using $K^*E_* = \pi_1\pi_{2*}$), whereas both paths around the right-hand trapezium can be seen to compose to $(\mathrm{id}_{K^*E_*(H)}, !)$. Thus this pullback diagram has the same form as that in Proposition 6.3.13 and so Proposition 6.3.10 allows us to deduce that $\Phi\Phi'$ is naturally isomorphic to the identity, completing the proof. $\square$

Together with Proposition 6.3.10 this shows that adjoint split extensions and Artin glueings are essentially the same. The equivalence Φ so defined is natural in a sense that will become clear later in Section 6.4.2.

6.4 The category of extensions

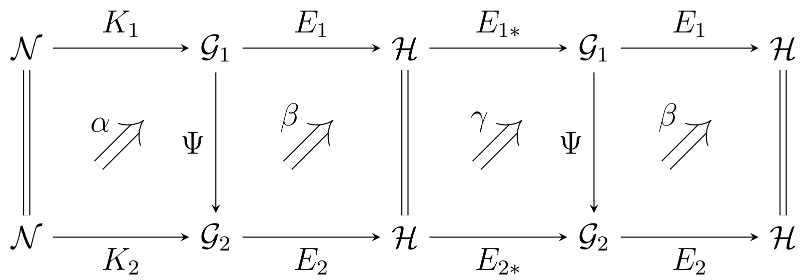
It follows from Theorem 6.3.15 that the set $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ of isomorphism classes of adjoint extensions is in bijection with $\mathrm{Hom}(\mathcal{H}, \mathcal{N})$. However, we could instead consider $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ whose objects are adjoint split extensions and with morphisms defined as follows.

6.4.1 Morphisms of extensions

Definition 6.4.1. Suppose we have two adjoint extensions,

$$\mathcal{N} \overset{K_1}{\rightarrowtail} \mathcal{G}_1 \underset{E_{1*}}{\overset{E_1}{\rightleftarrows}} \mathcal{H} \quad \text{and} \quad \mathcal{N} \overset{K_2}{\rightarrowtail} \mathcal{G}_2 \underset{E_{2*}}{\overset{E_2}{\rightleftarrows}} \mathcal{H},$$

with the same kernel and cokernel objects and associated isomorphisms ε_1 and ε_2 respectively. Consider the following diagram where α, β and γ are natural isomorphisms and Ψ is a finite-limit-preserving functor.



We say the functor $\Psi\colon \mathcal{G}_1 \to \mathcal{G}_2$ together with α, β and γ is a *morphism of adjoint extensions (of $\mathcal{H}$ by $\mathcal{N}$)* if $\varepsilon_2 = \varepsilon_1(\beta E_{1*})(E_2\gamma)$.

Given two such morphisms $(\Psi, \alpha, \beta, \gamma)$ and $(\Psi', \alpha', \beta', \gamma')$, a *2-morphism of adjoint extensions* is a natural transformation $\tau\colon \Psi \to \Psi'$ such that $\alpha' = (\tau K_1)\alpha$, $\beta = \beta'(E_2\tau)$ and $\gamma' = (\tau E_{1*})\gamma$.

The morphisms compose in the obvious way: by composing the functors and pasting the natural transformations together by juxtaposing the squares from the diagram above. Horizontal and vertical composition of 2-morphisms is given by the corresponding operations of the natural transformations τ. It is not hard to see that this gives a strict 2-category $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$. △

Remark 6.4.2. Note that the 'isomorphism of extensions' $(\Phi, \alpha, \beta, \gamma)$ defined in Theorem 6.3.15 is indeed a morphism of extensions in the above sense, since there ε_2 and β are identities and $\varepsilon(\pi_2\gamma) = \varepsilon\varepsilon^{-1} = \mathrm{id}$, as required. △

Lemma 6.4.3. *Given two adjoint extensions as above, a functor $\Psi\colon \mathcal{G}_1 \to \mathcal{G}_2$ and natural transformations $\alpha\colon K_2 \to \Psi K_1$ and $\gamma\colon E_{2*} \to \Psi E_{1*}$, there is a unique natural isomorphism $\beta\colon E_2\Psi \to E_1$ making $(\Psi, \alpha, \beta, \gamma)$ a morphism of adjoint extensions.*

Furthermore, given two morphisms of adjoint extensions as above, a natural transformation $\tau\colon \Psi \to \Psi'$ is a 2-morphism of adjoint extensions if and only if $\alpha' = (\tau K_1)\alpha$ and $\gamma' = (\tau E_{1*})\gamma$.

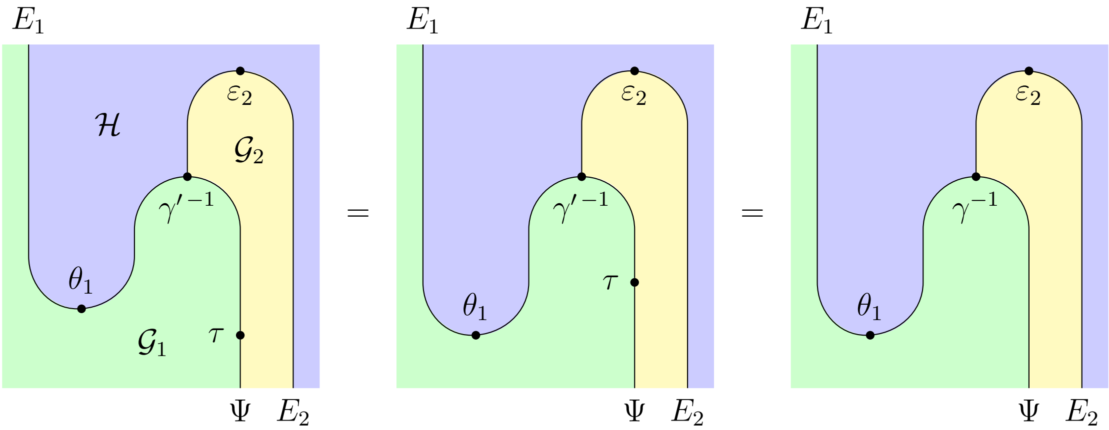
Proof. Any such β must satisfy $\varepsilon_2 = \varepsilon_1(\beta E_{1*})(E_2\gamma)$. But this can be rewritten as $\varepsilon_2(E_2\gamma^{-1}) = \varepsilon_1(\beta E_{1*})$, which shows that β and γ^{-1} are mates with respect to the adjunctions $E_1 \dashv E_{1*}$ and $E_2 \dashv E_{2*}$ and hence determine each other.

We now show that β so defined is an isomorphism. As the mate of γ^{-1}, we can express β as $(\varepsilon_2 E_1)(E_2\gamma^{-1}E_1)(E_2\Psi\theta_1)$. Now since ε_2 and γ^{-1} are isomorphisms, we need only show $E_2\Psi\theta_1$ is an isomorphism. This map occurs in the pullback obtained by applying $E_2\Psi$ to the pullback square of Proposition 6.3.13.

$$\begin{array}{ccc} E_2\Psi & \xrightarrow{E_2\Psi\theta_1} & E_2\Psi E_{1*}E_1 \\ {\scriptstyle E_2\Psi\zeta_1}\downarrow & \lrcorner & \downarrow{\scriptstyle E_2\Psi\zeta_1 E_{1*}E_1} \\ E_2\Psi K_1K_1^* & \xrightarrow[E_2\Psi K_1K_1^*\theta_1]{} & E_2\Psi K_1K_1^*E_{1*}E_1 \end{array}$$

Now observe that $E_2\Psi K_1 \cong E_2K_2 \cong 1$ is a zero morphism and hence so are $E_2\Psi K_1K_1^*$ and $E_2\Psi K_1K_1^*E_{1*}E_1$. Therefore, the bottom arrow of the above diagram is an isomorphism, and as the pullback of an isomorphism, $E_2\Psi\theta_1$ is an isomorphism too.

Finally, we show that the condition on β for 2-morphisms of extensions is automatic. Simply observe the following string diagrams.

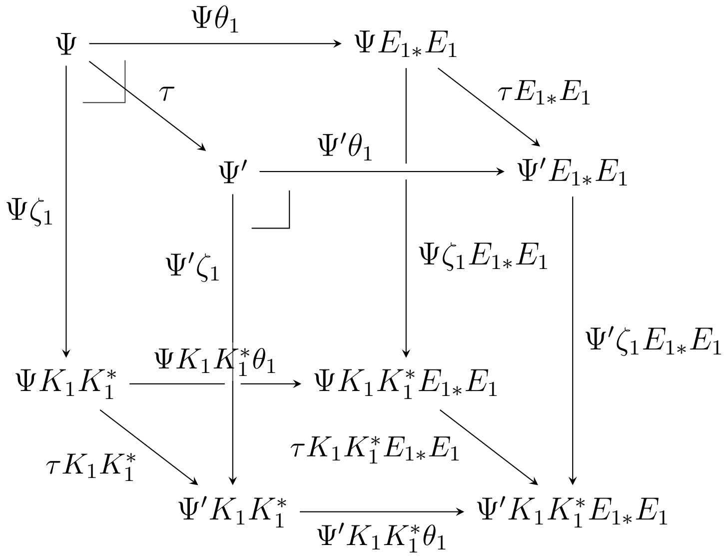


Here the first diagram represents $\beta'(E_2\tau)$ and the last diagram represents β. In moving from the first diagram to the second we shift τ above θ_1 and to move from the second diagram to the third we use $\gamma' = (\tau E_{1*})\gamma$. □

Lemma 6.4.4. *Suppose* $(\Psi, \alpha, \beta, \gamma)$ *and* $(\Psi', \alpha', \beta', \gamma')$ *are parallel morphisms of extensions. Then any 2-morphism* τ *between them is unique and invertible.*

Moreover, such a 2-morphism exists if and only if $\alpha'\alpha^{-1}K_1^*E_{1*} \circ \Psi\zeta_1E_{1*} = \Psi'\zeta_1E_{1*} \circ \gamma'\gamma^{-1}$.

Proof. Suppose $\tau\colon \Psi \to \Psi'$ is a 2-morphism of extensions. Then we have $\tau K_1 = \alpha'\alpha^{-1}$ and $\tau E_{1*} = \gamma'\gamma^{-1}$. Now by composing the pullback square of Proposition 6.3.13 with Ψ and Ψ' and using the naturality of τ we have the following commutative cube in $\mathrm{Hom}_{\mathrm{Top}_{\mathrm{lex}}}(\mathcal{G}_1, \mathcal{G}_2)$.



The universal property of the pullback on the front face then gives that τ is uniquely determined by $\tau K_1 K_1^*$ and $\tau E_{1*} E_1$, and hence by $\tau K_1 = \alpha' \alpha^{-1}$ and $\tau E_{1*} = \gamma' \gamma^{-1}$. Thus, the morphism τ is unique if it exists.

We can also attempt to use a similar cube to construct τ without assuming it exists a priori by replacing τK_1 with $\alpha' \alpha^{-1}$ and $\tau E_{1*} = \gamma' \gamma^{-1}$ in the above diagram. However, in order to obtain a map τ from the universal property of the pullback, we require that the right-hand face commutes.

$$\begin{array}{ccc} \Psi E_{1*} E_1 & \xrightarrow{\gamma' \gamma^{-1} E_1} & \Psi' E_{1*} E_1 \\ {\scriptstyle \Psi \zeta_1 E_{1*} E_1} \downarrow & & \downarrow {\scriptstyle \Psi' \zeta_1 E_{1*} E_1} \\ \Psi K_1 K_1^* E_{1*} E_1 & \xrightarrow[\alpha' \alpha^{-1} K_1^* E_{1*} E_1]{} & \Psi' K_1 K_1^* E_{1*} E_1 \end{array}$$

Note that this square commutes if and only if the similar square obtained by inverting the horizontal morphisms commutes. But this latter square is precisely the square we need to commute to obtain a 2-morphism in the opposite direction. Uniqueness then shows that these two 2-morphisms compose to give identities.

Finally, observe that commutativity of this square is the required equality stated above whiskered with E_1. This is equivalent to the desired condition, since E_1 is essentially surjective. □

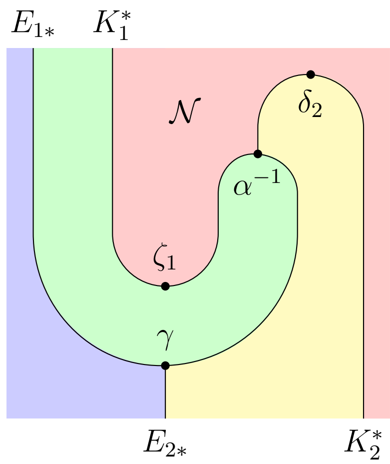
Corollary 6.4.5. *The 2-category of adjoint extensions is equivalent to the category of adjoint extensions and isomorphism classes of morphisms (with trivial 2-morphisms).*

This will justify treating $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ as a 1-category going forward.

Lemma 6.4.6. *From a morphism of extensions $(\Psi, \alpha, \beta, \gamma)$ we can form an associated natural transformation from $K_2^* E_{2*}$ to $K_1^* E_{1*}$ given by*

$$(\delta_2 K_1^* E_{1*})(K_2^* \alpha^{-1} K_1^* E_{1*})(K_2^* \Psi \zeta_1 E_{1*})(K_2^* \gamma)$$

where δ_2 is the counit of the $K_2^* \dashv K_2$ adjunction and which is depicted below. Two parallel morphisms of extensions are isomorphic if and only if their corresponding natural transformations are equal.

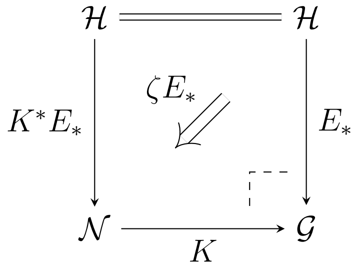


Proof. There is a (necessarily invertible) 2-morphism from $(\Psi, \alpha, \beta, \gamma)$ to $(\Psi', \alpha', \beta', \gamma')$ if and only if $(\alpha' \alpha^{-1} K_1^* E_{1*})(\Psi \zeta_1 E_{1*}) = (\Psi' \zeta_1 E_{1*}) \gamma' \gamma^{-1}$. We can now move all the unprimed variables to the left and primed variables to the right by multiplying both sides of this equation on the left by $\alpha'^{-1} K_1^* E_{1*}$ and on the right by γ to obtain $(\alpha^{-1} K_1^* E_{1*})(\Psi \zeta_1 E_{1*})\gamma = (\alpha'^{-1} K_1^* E_{1*})(\Psi' \zeta_1 E_{1*})\gamma'$. These are the mates of the desired natural transformations with respect to the adjunction $K_2^* \dashv K_2$. □

6.4.2 The equivalence of categories

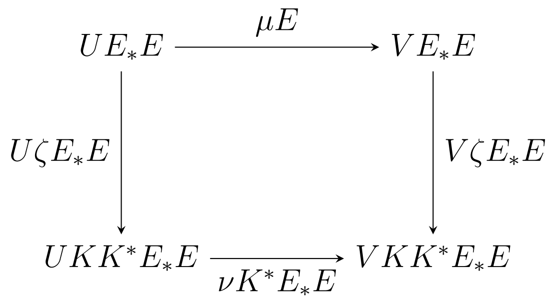
In this section we show that the categories $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ and $\mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}}$ are equivalent. This requires showing that isomorphism classes of morphisms of extensions correspond to natural transformations. We have already seen that each isomorphism class has an associated natural transformation. We will now further explore this relationship, making use of the following folklore result.

Proposition 6.4.7. *Let $\mathcal{N} \overset{K}{\rightarrowtail} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ be an adjoint extension. Then the following diagram is a cocomma square.*



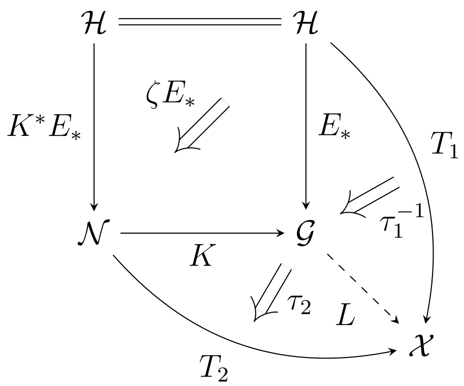
Proof. We first check the 2-categorical condition. Consider two finite-limit-preserving functors $U, V : \mathcal{G} \to \mathcal{X}$ and natural transformations $\mu\colon UE_* \to VE_*$ and $\nu\colon UK \to VK$ such that $(V\zeta E_*)\mu = (\nu K^* E_*)(U\zeta E_*)$. We must find a unique $\omega\colon U \to V$ such that $\omega E_* = \mu$ and $\omega K = \nu$.

We use Proposition 6.3.13 to express U and V as pullbacks and then as in Lemma 6.4.4 we find that there is a unique map $\omega\colon U \to V$ with $\omega E_* = \mu$ and $\omega K = \nu$ as long as the following diagram commutes.



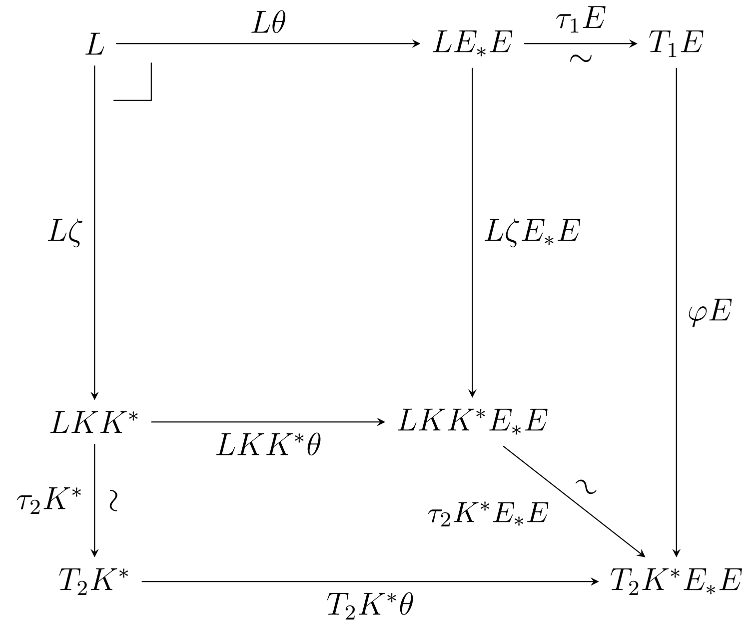
But commutativity of this diagram is simply the assumed condition whiskered with E on the right.

Now we show the 1-categorical condition. Suppose we have finite-limit-preserving functors $T_1\colon \mathcal{H} \to \mathcal{X}$ and $T_2\colon \mathcal{N} \to \mathcal{X}$ and a natural transformation $\varphi\colon T_1 \to T_2 K^* E_*$. Consider the following diagram.



We must construct a finite-limit-preserving functor $L\colon \mathcal{G} \to \mathcal{X}$ and natural isomorphisms $\tau_1\colon LE_* \to T_1$ and $\tau_2\colon LK \to T_2$ such that $\varphi = \tau_2 K^* E_* \circ L\zeta E_* \circ \tau_1^{-1}$.

Suppose we are given such a functor L and natural isomorphisms τ_1, τ_2 and consider the following pullback diagram.



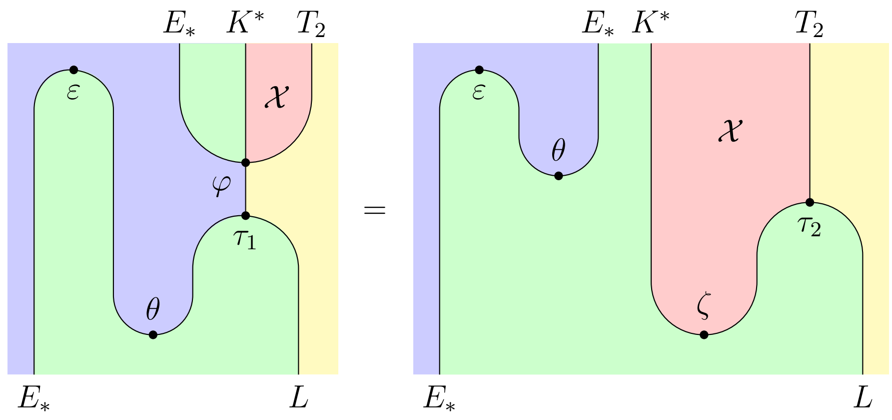
Here the bottom trapezium commutes by the naturality of $\tau_2 K^*$ and the right trapezium commutes since $(\varphi\tau_1)E = (\tau_2 K^* E_* \circ L\zeta E_*)E$ by assumption. Note that the left edge of the large square is the mate of τ_2 with respect to $K^* \dashv K$ and the top edge is the mate of τ_1 with respect to $E \dashv E_*$.

Now without assuming L exists to start with, we can use the outer pullback diagram to define it and we may recover τ_1 and τ_2 as the mates of the resulting pullback projections.

Observe that precomposing the pullback with K turns the right-hand edge into an isomorphism between zero morphisms. Hence the left-hand morphism $(\tau_2 K^*)(L\zeta)K$ is an isomorphism as well. Since τ_2 is given by composing this with the isomorphism $T_2\delta$, we find that τ_2 is an isomorphism.

On the other hand, precomposing the pullback with E_* turns the bottom edge into an isomorphism (as $\mathcal{N} \xhookrightarrow{E_*} \mathcal{G}$ is a reflective subcategory). It follows that τ_1 is also an isomorphism.

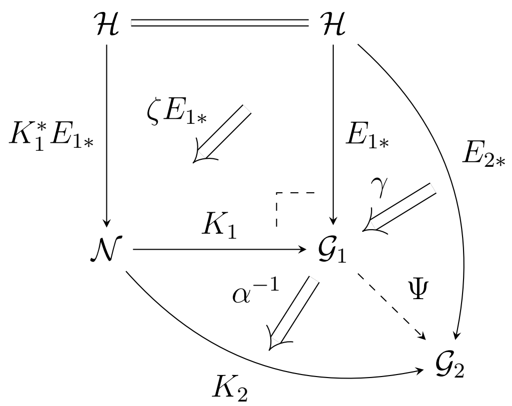
Finally, we show that φ be can recovered in the appropriate way. The commutativity of the pullback square gives $\varphi E \circ \tau_1 E \circ L\theta = T_2 K^*\theta \circ \tau_2 K^* \circ L\zeta$. The result of whiskering this with E_* on the right and composing with $T_2 K^* E_* \varepsilon$ is depicted in the string diagram below.



The desired equality follows after using the triangle identities to 'pull the wires straight'. ☐

Proposition 6.4.8. *Let A and B denote adjoint extensions* $\mathcal{N} \overset{K_1}{\rightarrowtail} \mathcal{G}_1 \underset{E_{1*}}{\overset{E_1}{\rightleftarrows}} \mathcal{H}$ *and* $\mathcal{N} \overset{K_2}{\rightarrowtail} \mathcal{G}_2 \underset{E_{2*}}{\overset{E_2}{\rightleftarrows}} \mathcal{H}$ *respectively. There is a bijection between the associated hom sets* $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(K_2^* E_{2*}, K_1^* E_{1*})$.

Proof. Let $\psi \colon K_2^* E_{2*} \to K_1^* E_{1*}$ be a natural transformation and consider its mate natural transformation $\overline{\psi} \colon E_{2*} \to K_2 K_1^* E_{1*}$ with respect to the adjunction $K_2^* \dashv K_2$. Notice that $\overline{\psi}$ determines a 2-cocone in the following cocomma object diagram.



By the universal property of the cocomma, we get a map $\Psi \colon \mathcal{G}_1 \to \mathcal{G}_2$ and natural isomorphisms $\alpha \colon K_2 \to \Psi K_1$ and $\gamma \colon E_{2*} \to \Psi E_{1*}$. By Lemma 6.4.3 we can derive a unique natural isomorphism $\beta \colon E_2 \Psi \to E_1$ such that $(\Psi, \alpha, \beta, \gamma)$ is a morphism of extensions.

For the other direction we begin with a morphism of extensions $(\Psi, \alpha, \beta, \gamma)$ and form the pasting diagram above. We may consider the composite natural transformation

$$\overline{\psi} = \alpha^{-1} K_1^* E_{1*} \circ \Psi \zeta E_{1*} \circ \gamma \colon E_{2*} \to K_2 K_1^* E_{1*}.$$

Again we may use the adjunction $K_2^* \dashv K_2$ to arrive at the natural transformation

$$\delta_2 K_1^* E_{1*} \circ K_2^* \alpha^{-1} K_1^* E_{1*} \circ K_2^* \Psi \zeta E_{1*} \circ K_2^* \gamma \colon K_2^* E_{2*} \to K_1^* E_{1*},$$

where $\delta_2 \colon K_2^* K_2 \to \mathrm{Id}_{\mathcal{N}}$ is the counit of the adjunction.

It is clear that these processes are inverses by the uniqueness of the universal property (bearing in mind that the morphisms in $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ are isomorphism classes). □

Remark 6.4.9. Notice that the natural transformation associated to a morphism of extensions $(\Psi, \alpha, \beta, \gamma)$ in the above proof is precisely the one described in Lemma 6.4.6. △

Corollary 6.4.10. *Let $F_1, F_2 \colon \mathcal{H} \to \mathcal{N}$ be finite-limit-preserving functors and let*

$$\Gamma_1 = \mathcal{N} \xrightarrow{\pi_{1*}^{F_1}} \mathrm{Gl}(F_1) \underset{\pi_{2*}^{F_1}}{\overset{\pi_2^{F_1}}{\rightleftarrows}} \mathcal{H} \quad \textit{and} \quad \Gamma_2 = \mathcal{N} \xrightarrow{\pi_{1*}^{F_2}} \mathrm{Gl}(F_2) \underset{\pi_{2*}^{F_2}}{\overset{\pi_2^{F_2}}{\rightleftarrows}} \mathcal{H}$$

be the corresponding glueing extensions. Then $\mathrm{Hom}(F_2, F_1) \cong \mathrm{Hom}(\Gamma_1, \Gamma_2)$.

Proof. Since $\pi_1^{F_1} \pi_{2*}^{F_1} = F_1$ and $\pi_1^{F_2} \pi_{2*}^{F_2} = F_2$, the above proposition implies that $\mathrm{Hom}(\Gamma_1, \Gamma_2) \cong \mathrm{Hom}(F_2, F_1)$, as required. □

We are now ready to show that the categories $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ and $\mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}}$ are equivalent. (A similar result can also be obtained from the results of (Rosebrugh and Wood [43]).)

Definition 6.4.11. Let $\Gamma_{\mathcal{H},\mathcal{N}} \colon \mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}} \to \mathrm{Ext}(\mathcal{H}, \mathcal{N})$ be the functor sending $F \colon \mathcal{H} \to \mathcal{N}$ to the extension $\mathcal{N} \xrightarrow{\pi_{1*}} \mathrm{Gl}(F) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$ and sending natural transformations to the associated morphism of extensions described in Corollary 6.4.10. △

Theorem 6.4.12. *Let $\mathcal{H}$ and $\mathcal{N}$ be toposes. The functor $\Gamma_{\mathcal{H},\mathcal{N}} \colon \mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}} \to \mathrm{Ext}(\mathcal{H}, \mathcal{N})$ is a part of an equivalence.*

An inverse $\Gamma_{\mathcal{H},\mathcal{N}}^{-1}$ sends an extension $\mathcal{N} \xrightarrow{K} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ to $K^* E_*$ and a morphism of extensions to the natural transformation described in Lemma 6.4.6 and Proposition 6.4.8. For the adjunction $\Gamma_{\mathcal{H},\mathcal{N}}^{-1} \dashv \Gamma_{\mathcal{H},\mathcal{N}}$ we take the counit to be the identity and the unit to be given by the isomorphisms described in Theorem 6.3.15.

Proof. Note that $\Gamma_{\mathcal{H},\mathcal{N}}^{-1} \Gamma_{\mathcal{H},\mathcal{N}} = \mathrm{Id}_{\mathrm{Hom}(\mathcal{H},\mathcal{N})^{\mathrm{op}}}$. We see that $\Gamma_{\mathcal{H},\mathcal{N}}^{-1} \dashv \Gamma_{\mathcal{H},\mathcal{N}}$ with the identity as the counit, since for each natural transformation $\psi \colon F \to \Gamma^{-1}(A)$, there is a unique map $\Psi \colon A \to \Gamma_{\mathcal{H},\mathcal{N}}(F)$ such that $\Gamma_{\mathcal{H},\mathcal{N}}^{-1}(\Psi) = \psi$, namely the image of ψ

under the inverse of the bijection

$$\mathrm{Hom}(A, \Gamma_{\mathcal{H},\mathcal{N}}(F)) \cong \mathrm{Hom}(\Gamma^{-1}_{\mathcal{H},\mathcal{N}}\Gamma_{\mathcal{H},\mathcal{N}}(F), \Gamma^{-1}_{\mathcal{H},\mathcal{N}}(A)) = \mathrm{Hom}(F, \Gamma^{-1}_{\mathcal{H},\mathcal{N}}(A))$$

from Proposition 6.4.8.

Let A denote the extension $\mathcal{N} \overset{K}{\rightarrowtail} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$. It remains to show that the isomorphism of extensions $\Phi = (\Phi, \alpha, \beta, \gamma)\colon A \to \Gamma_{\mathcal{H},\mathcal{N}}(K^* E_*)$ described in Theorem 6.3.15 is the component of the unit at A.

It suffices to show that the isomorphism

$$\mathrm{Hom}(A, \Gamma_{\mathcal{H},\mathcal{N}}\Gamma^{-1}_{\mathcal{H},\mathcal{N}}(A)) \cong \mathrm{Hom}(\Gamma^{-1}_{\mathcal{H},\mathcal{N}}(A), \Gamma^{-1}_{\mathcal{H},\mathcal{N}}(A))$$

maps Φ to the identity — that is, that $\Gamma^{-1}_{\mathcal{H},\mathcal{N}}(\Phi) = \mathrm{id}_{K^*E_*}$. As in Lemma 6.4.6 we have $\Gamma^{-1}_{\mathcal{H},\mathcal{N}}(\Phi) = \delta' K^* E_* \circ \pi_1 \alpha^{-1} K^* E_* \circ \pi_1 \Phi \zeta E_* \circ \pi_1 \gamma$. Now recall that $\delta' \colon \pi_1 \pi_{1*} \to \mathrm{Id}_{\mathcal{N}}$ is the identity, $\alpha = (\mathrm{id}, !) \circ \pi_{1*}\delta^{-1}$ and $\gamma = (\mathrm{id}_{K^*E_*}, \varepsilon^{-1})$. Thus the expression can be seen to simplify to $\delta K^* E_* \circ K^* \zeta E_*$, which in turn is $\mathrm{id}_{K^*E_*}$ by the triangle identity for $K^* \dashv K$.

Finally, $\Gamma_{\mathcal{H},\mathcal{N}}$ and $\Gamma^{-1}_{\mathcal{H},\mathcal{N}}$ form an equivalence since the unit and counit are isomorphisms. □

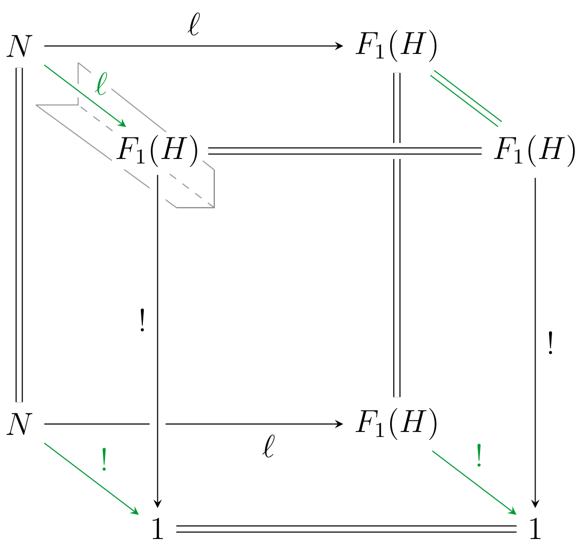
With this in mind we may now consider the full subcategory of $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ whose objects are only those extensions of the form $\mathcal{N} \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(F) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$ for some $F\colon \mathcal{H} \to \mathcal{N}$. It is evident that this full subcategory is equivalent to $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ and for the remainder of the chapter we will choose to perform calculations in this subcategory for simplicity. We will discuss how this can be done coherently when we investigate the Ext 2-functor in Section 6.5.

We can now give a concrete description of the behaviour of morphisms of extensions. Suppose that $(\Psi, \alpha, \beta, \gamma)\colon \Gamma_{\mathcal{H},\mathcal{N}}(F_1) \to \Gamma_{\mathcal{H},\mathcal{N}}(F_2)$ is a morphism of adjoint extensions as in the following diagram.

$$\begin{array}{ccccc}
\mathcal{N} & \xrightarrow{\pi_{1*}^{F_1}} & \mathrm{Gl}(F_1) & \underset{\pi_{2*}^{F_1}}{\overset{\pi_2^{F_1}}{\rightleftarrows}} & \mathcal{H} \\
\Vert & & \Big\downarrow \Psi & & \Vert \\
\mathcal{N} & \xrightarrow[\pi_{1*}^{F_2}]{} & \mathrm{Gl}(F_2) & \underset{\pi_{2*}^{F_2}}{\overset{\pi_2^{F_2}}{\rightleftarrows}} & \mathcal{H}
\end{array}$$

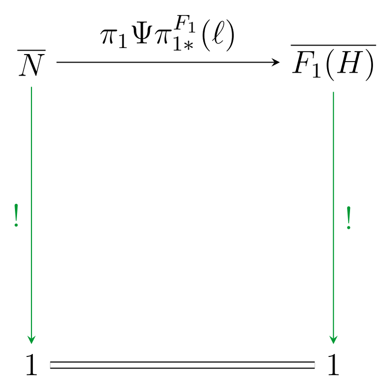
Since $\alpha_N\colon (N,1,!) \to \Psi(N,1,!)$ is an isomorphism, we have $\Psi(N,1,!) = (\overline{N},\overline{1},!)$ for some $\overline{N} \cong N$ and some terminal object $\overline{1}$. For simplicity, we will assume $\overline{1} = 1$ without any loss of generality. Similarly $\gamma_H\colon (F_2(H), H, \mathrm{id}) \to \Psi(F_1(H), H, \mathrm{id})$ is an isomorphism. So if $\Psi(F_1(H), H, \mathrm{id}) = (N_H, \overline{H}, t_H)$ then $t_H\colon N_H \to F_2(\overline{H})$ is an isomorphism.

Since Ψ preserves finite limits, we can use Proposition 6.3.11 to completely determine the behaviour of Ψ. We note that every object in $\mathrm{Gl}(F_1)$ can be written as the pullback diagram depicted below, where objects in the category are represented by the green arrows pointing out of the page and the pullback symbol has elongated into a wedge.

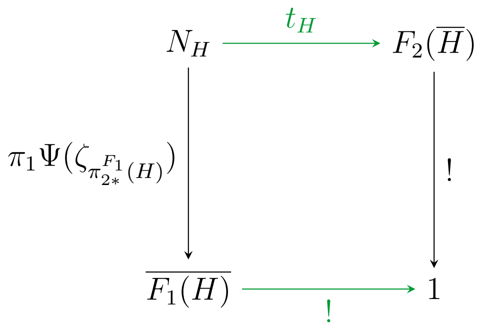


Note that the front and back faces are pullback squares in $\mathcal{N}$ and the other faces correspond to morphisms in $\mathrm{Gl}(F_1)$.

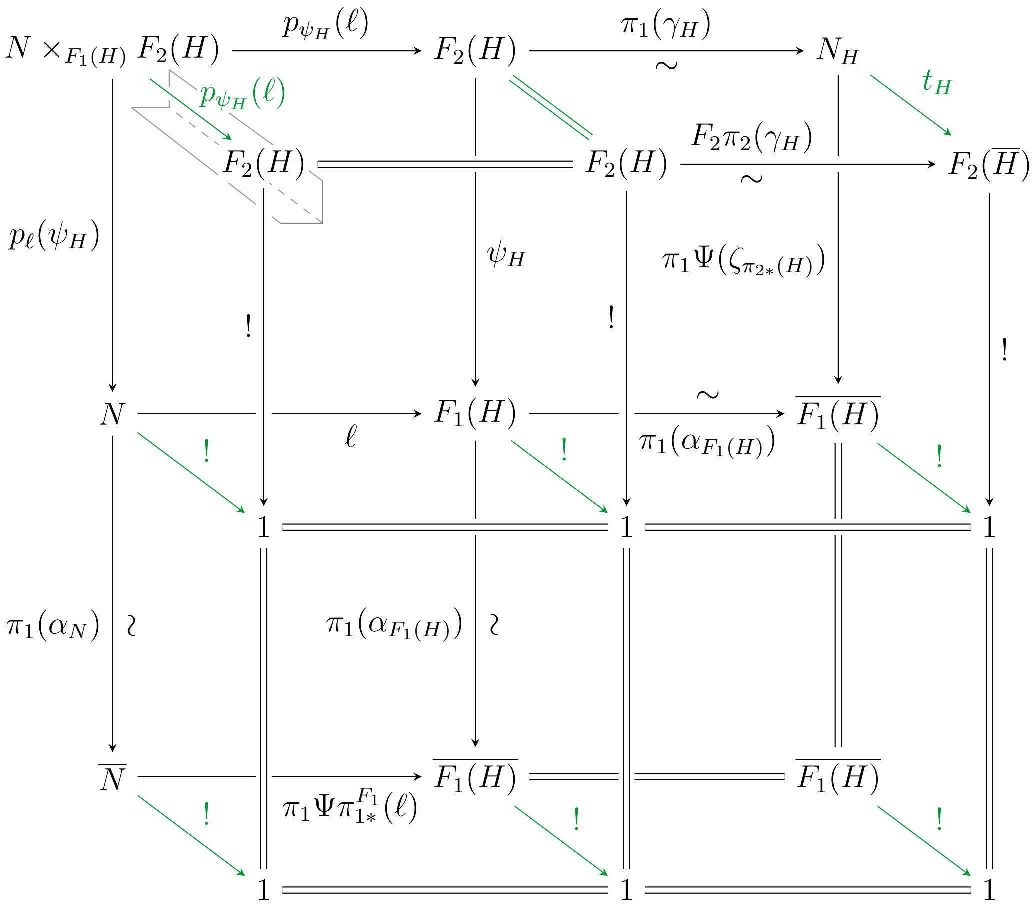
We may now study how Ψ acts on this pullback diagram. Observe that the bottom face corresponds to the morphism $\pi_{1*}^{F_1}(\ell)$ where $\ell\colon N \to F_1(H)$. It is then sent by Ψ to the morphism represented in the diagram below.



The right face is the unit $\zeta_{\pi_{2*}^{F_1}(H)} \colon \pi_{2*}^{F_1}(H) \to \pi_{1*}^{F_1} F_1(H)$. Thus we have that Ψ sends this face to the following commutative square.



The pullback of these two faces will then give the image of (N, H, ℓ) under Ψ. The pullback diagram is given by the large cuboid in the diagram below. Here we have factored this pullback as in the similar pullback diagram in the proof of Proposition 6.4.7.



The bottom face of the bottom left cube and the right face of the top right cube are the commutative squares considered above. These have also been extended by the identity maps in the bottom right cube, so that the bottom and right-hand faces of the full cuboid are as required for the pullback in question.

The bottom left cube commutes by the naturality of α, while the top right cube commutes by the definition of $\psi = \Gamma^{-1}_{\mathcal{H},\mathcal{N}}(\Psi)$ as in Proposition 6.4.8. Since α and γ are isomorphisms, the top left cube is also a pullback. Recall that the front and back faces are then also pullbacks. Since the top face of the top left cube must commute, we find that the green arrow we seek is given by $p_{\psi_H(\ell)}$. Hence, $\Psi(N, H, \ell)$ is isomorphic to $(N \times_{F_1(H)} F_2(H), H, p_{\psi_H}(\ell))$.

Of course, every natural transformation $\psi \colon F_2 \to F_1$ yields a morphism of extensions defined by such a pullback. For the associated natural isomorphisms we may take β to be the identity and α to be $(\widehat{\alpha}, \mathrm{id})$ where $\widehat{\alpha}$ is defined by the diagram below.

$$\begin{array}{ccc}
\overline{N} & \xrightarrow{\;!\;} & F_2(1) \\
{\scriptstyle \widehat{\alpha}_N^{-1}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \psi_1} \\
N & \xrightarrow[\;!\;]{} & F_1(1)
\end{array}$$

Finally, we take $\gamma = (\widehat{\gamma}, \mathrm{id})$ where $\widehat{\gamma}$ is specified by the diagram below.

$$\begin{array}{ccc}
\overline{F_1(H)} & \xrightarrow{\;\widehat{\gamma}_H^{-1}\;} & F_2(H) \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \psi_H} \\
F_1(H) & = & F_1(H)
\end{array}$$

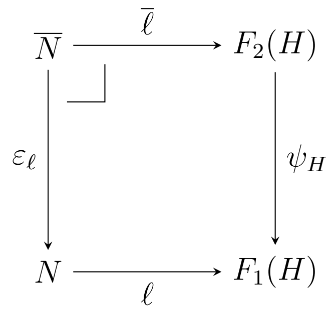
It is easy to see that $\varepsilon_2 = \varepsilon_1(\beta\pi^{F_1}_{2*})(\pi^{F_2}_2\gamma)$ as each factor is just the identity.

We now end this section with what is perhaps a surprising result about morphisms of extensions.

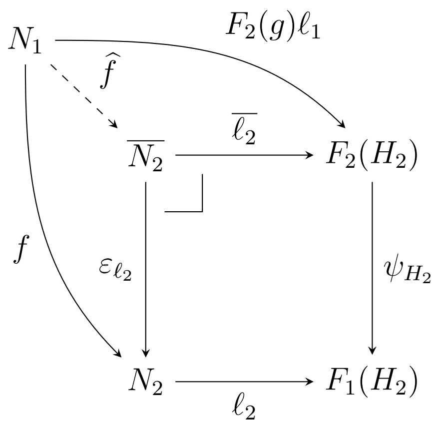
Proposition 6.4.13. *If* $(\Psi, \alpha, \beta, \gamma) : \Gamma_{\mathcal{H},\mathcal{N}}(F_1) \to \Gamma_{\mathcal{H},\mathcal{N}}(F_2)$ *is a morphism of adjoint extensions, then* $\Psi \colon \mathrm{Gl}(F_1) \to \mathrm{Gl}(F_2)$ *is a geometric morphism of toposes.*

Proof. Let $\psi \colon F_2 \to F_1$ be the natural transformation associated to Ψ. We can construct a functor $\Psi^* \colon \mathrm{Gl}(F_2) \to \mathrm{Gl}(F_1)$ which sends (N, H, ℓ) to $(N, H, \psi_H \ell)$ and leaves morphisms 'fixed' in the sense that $(f, g) \colon (N_1, H_1, \ell_1) \to (N_2, H_2, \ell_2)$ is sent to $(f, g) \colon (N_1, H_1, \psi_{H_1}\ell_1) \to (N_2, H_2, \psi_{H_2}\ell_2)$, which may be seen to be a morphism in $\mathrm{Gl}(F_1)$ using the naturality of ψ.

We claim that Ψ^* is left adjoint to Ψ. To see this we consider the candidate counit $\varepsilon_{N,H,\ell} = (\varepsilon_\ell, \mathrm{id}_H)$, where ε_ℓ is defined as in the following pullback diagram.



We must show that given a morphism $(f, g)\colon (N_1, H_1, \psi_{H_1}\ell_1) = \Psi^*(N_1, H_1, \ell_1) \to (N_2, H_2, \ell_2)$ there exists a unique morphism $(\widehat{f}, \widehat{g})\colon (N_1, H_1, \ell_1) \to \Psi(N_2, H_2, \ell_2) = (\overline{N_2}, H_2, \overline{\ell_2})$ such that $(\varepsilon_{\ell_2}, \mathrm{id}_{H_2}) \circ \Psi^*(\widehat{f}, \widehat{g}) = (f, g)$. We will construct this map using the following diagram.



Here the maps out of N_1 form a cone as we have $\psi_{H_2}F_2(g)\ell_1 = F_1(g)\psi_{H_1}\ell_1 = \ell_2 f$, where the first equality follows from naturality of ψ and the second from the fact that (f, g) is a morphism in $\mathrm{Gl}(F_1)$.

By the universal property we have that $\overline{l_2}\widehat{f} = F_2(g)\ell_1$, which means that $(\widehat{f}, g)$ is a morphism from (N_1, H_1, ℓ_1) to $(\overline{N_2}, H_2, \overline{\ell_2})$ in $\mathrm{Gl}(F_2)$. It is immediate from the diagram that $(\varepsilon_{\ell_2}, \mathrm{id}_{H_2}) \circ (\widehat{f}, g) = (f, g)$ and it is also not hard to see that this is the unique such morphism. Thus, Ψ^* is indeed left adjoint to Ψ.

Finally, we must show that Ψ^* preserves finite limits. This follows immediately from the fact that finite limits in the glueing may be computed componentwise. □

Remark 6.4.14. Notice that Ψ^* is in fact a morphism of *non-split* extensions in the sense that it commutes with the kernel and cokernel maps up to isomorphism.

However, it does not commute with the splittings unless Ψ is the identity. $\triangle$

6.4.3 Baer colimits

In Chapter 2 it was shown for frames H and N that there was something akin to a Baer sum of extensions in $\mathrm{Ext}(H, N)$. It is natural to ask if something analogous occurs in the category $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$. Indeed, it is not hard to see via the equivalence with $\mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}}$ that $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ has all finite colimits. The following proposition helps us compute these colimits.

Proposition 6.4.15. *Let $M\colon \mathrm{Ext}(\mathcal{H}, \mathcal{N})^{\mathrm{op}} \to \mathrm{Cat}/(\mathcal{N} \times H)$ be the functor sending extensions $\mathcal{N} \xrightarrow{K} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ to $!^*_{K^*E_*}\colon \mathrm{Gl}(K^*E_*) \to \mathcal{N} \times H$ where $!^*_{K^*E_*}$ is left adjoint to the universal map $!_{K^*E_*}$ in $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ out of the 2-initial object $\mathcal{N} \xrightarrow{\pi_{1*}} \mathcal{N} \times \mathcal{H} \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$. Explicitly, this adjoint sends (N, H, ℓ) to (N, H) and (f, g) to (f, g).*

Let $(\Psi, \alpha, \beta, \gamma)$ be a morphism of extensions and let $\psi\colon K_2^*E_{2*} \to K_1^*E_{1*}$ be the corresponding natural transformation.

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{K_1} & \mathcal{G}_1 & \underset{E_{1*}}{\overset{E_1}{\rightleftarrows}} & \mathcal{H} \\ \Vert & & \downarrow{\scriptstyle \Psi} & & \Vert \\ \mathcal{N} & \xrightarrow[K_2]{} & \mathcal{G}_2 & \underset{E_{2*}}{\overset{E_2}{\rightleftarrows}} & \mathcal{H} \end{array}$$

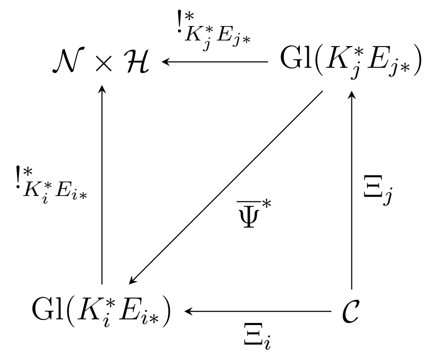
Then M maps Ψ to the morphism $\overline{\Psi}^*\colon \mathrm{Gl}(K_2^*E_{2*}) \to \mathrm{Gl}(K_1^*E_{1*})$, which sends (N, H, ℓ) to $(N, H, \psi_H \ell)$ and (f, g) to (f, g). It is immediate that the necessary diagram for this to be a morphism in the slice category commutes.

The functor M creates (and preserves) finite limits. Thus, finite colimits in $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ can be computed from limits in $\mathrm{Cat}/(\mathcal{N} \times \mathcal{H})$.

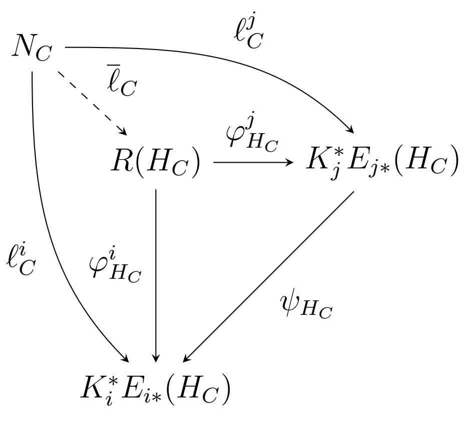
Proof. Let $D\colon \mathcal{J} \to \mathrm{Ext}(\mathcal{H}, \mathcal{N})$ be a diagram functor with finite domain $\mathcal{J}$. To compute the colimit of D we may compose D with $\Gamma^{-1}_{\mathcal{H},\mathcal{N}}$ and compute the limit in $\mathrm{Hom}(\mathcal{H}, \mathcal{N})$. Let $R\colon \mathcal{H} \to \mathcal{N}$ be the resulting limit in $\mathrm{Hom}(\mathcal{H}, \mathcal{N})$ and $(\varphi^i\colon R \to \Gamma^{-1}_{\mathcal{H},\mathcal{N}} D(i))_{i \in \mathcal{J}}$ be the corresponding projections. Then $\mathcal{N} \xrightarrow{\pi_{1*}} \mathrm{Gl}(R) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$ is the colimit of D, where the morphisms of the colimiting cone are given in the obvious way.

If we consider the diagram functor $MD\colon \mathcal{J} \to \mathrm{Cat}/(\mathcal{N} \times \mathcal{H})$, then we may again compute the limit with the assistance of the calculation in $\mathrm{Hom}(\mathcal{H}, \mathcal{N})$. We claim that $!^*_R\colon \mathrm{Gl}(R) \to \mathcal{N} \times \mathcal{H}$ is the required limit with the morphisms of the limiting cone given in the expected way — that is, if φ is a morphism of the limiting cone in $\mathrm{Hom}(\mathcal{H}, \mathcal{N})$ then $\Gamma_{\mathcal{H},\mathcal{N}}(\varphi)^* = \overline{\Phi}^*$ is the associated morphism in $\mathrm{Cat}/(\mathcal{N} \times \mathcal{H})$.

We must demonstrate that this cone satisfies the universal property. Suppose we have some other cone $(\Xi_i\colon \mathcal{C} \to \mathrm{Gl}(K_i^* E_{i*}))_{i \in \mathcal{J}}$ and consider the following diagram in Cat where $\Psi = D(f)$ for some morphism $f\colon i \to j$ in $\mathcal{J}$.



Since each Ξ_i is a morphism in $\mathrm{Cat}/(\mathcal{N} \times \mathcal{H})$ we have that it commutes with the ! maps. This means that the Ξ maps all agree on the first two components. If we assume that $\Xi_k(C) = (N_C, H_C, \ell^k_C)$, then $\Xi_i = \overline{\Psi}^* \Xi_j$ gives $\ell^i_C = \psi_{H_C} \ell^j_C$. Now consider the following diagram in $\mathcal{N}$ where we make use of the aforementioned limiting cone in $\mathrm{Hom}(\mathcal{H}, \mathcal{N})$.



Here we use the universal property of R componentwise at H_C to produce the map $\overline{\ell}_C$. This allows us to construct a map $S\colon \mathcal{C} \to \mathrm{Gl}(R)$ with $S(C) = (N_C, H_C, \overline{\ell}_C)$. As for morphisms, now note that each Ξ_k sends $f\colon C \to C'$ to the 'same' pair (f_1, f_2) and we define S to act on morphisms in the same way. The pair (f_1, f_2) can be seen to be a morphism in $\mathrm{Gl}(R)$ from $S(C)$ to $S(C')$ by considering the above diagram in the functor category and then using the naturality of $\overline{\ell}\colon \pi_1 \Xi_k \to R \pi_2 \Xi_k$. This morphism S is the desired map and is easily seen to be unique.

From the above it is clear that M preserves limits and that every limiting cone of MD is isomorphic to one of the form $(M\Gamma_{\mathcal{H},\mathcal{N}}(\varphi^i)\colon M\Gamma_{\mathcal{H},\mathcal{N}}(R) \to MD(i))_{i\in\mathcal{J}}$, where $\varphi^i\colon R \to \Gamma^{-1}_{\mathcal{H},\mathcal{N}}D(i)$ is the limiting cone in $\mathrm{Hom}(\mathcal{H},\mathcal{N})$. For M to create limits, it remains to show that every cone of D which maps to a limiting cone of MD is isomorphic to one of the above form. This follows since M is faithful and essentially injective on objects. $\square$

Notice that the limit diagram was embedded into the slice category so that each Ξ in the proof would agree on the first two components. If the limit diagram is connected, this will happen automatically and so we obtain the following corollary.

Corollary 6.4.16. *The functor* $M\colon \mathrm{Ext}(\mathcal{H},\mathcal{N}) \to \mathrm{Cat}$ *which sends extensions* $\mathcal{N} \xrightarrow{K} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ *to* $\mathrm{Gl}(K^*E_*)$ *and acting on morphisms as in Proposition 6.4.15 creates finite connected limits.*

A disconnected (co)limit is the subject of the following example.

Example 6.4.17. Let us consider the coproduct of extensions $\mathcal{N} \xrightarrow{K_1} \mathcal{G}_1 \underset{E_1*}{\overset{E_1}{\rightleftarrows}} \mathcal{H}$ and $\mathcal{N} \xrightarrow{K_2} \mathcal{G}_2 \underset{E_2*}{\overset{E_2}{\rightleftarrows}} \mathcal{H}$. Since products in a slice category correspond to pullbacks, we may construct this coproduct using the following pullback in Cat.

$$\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathrm{Gl}(K_2^*E_{2*}) \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle !^*_{K_2^*E_{2*}}} \\
\mathrm{Gl}(K_1^*E_{1*}) & \xrightarrow[!^*_{K_1^*E_{1*}}]{} & \mathcal{N}\times\mathcal{H}
\end{array}$$

If $!_{\mathcal{P}}$ is the composite morphism from $\mathcal{P}$ to $\mathcal{N}\times\mathcal{H}$, then the coproduct extension may be recovered as $\mathcal{N} \xrightarrow{(\pi_1 !_{\mathcal{P}})_*} \mathcal{P} \underset{(\pi_2 !_{\mathcal{P}})_*}{\overset{\pi_2 !_{\mathcal{P}}}{\rightleftarrows}} \mathcal{H}$. $\triangle$

6.5 The extension functor

Given that we have established that $\mathrm{Ext}(\mathcal{H},\mathcal{N})$ is equivalent to $\mathrm{Hom}(\mathcal{H},\mathcal{N})^{\mathrm{op}}$, it is natural to ask if Ext can be extended to a 2-bifunctor and if Ext and the $\mathrm{Hom}_{\mathrm{op}}$ will then be 2-naturally equivalent (where $\mathrm{Hom_{op}} = \mathrm{Op} \circ \mathrm{Hom}^{\mathrm{co}}$ and Op is the opposite category 2-functor).

The answer is of course "yes", for if $T\colon \mathcal{H}' \to \mathcal{H}$ and $S\colon \mathcal{N} \to \mathcal{N}'$, all we need do is define $\mathrm{Ext}(T, S) = \Gamma_{\mathcal{H}',\mathcal{N}'} \circ \mathrm{Hom_{op}}(T, S) \circ \Gamma^{-1}_{\mathcal{H},\mathcal{N}}$ (and similarly for natural transformations). However this is unsatisfactory, as there is already established behaviour for how an Ext functor ought to act on objects and morphisms. In this section, we show that the above definition conforms with the usual expectations of an Ext functor.

We consider each component of our Ext functor separately and begin by describing $\mathrm{Ext}(-, \mathcal{N})$. In other contexts, for instance, see (Borceux, Janelidze, and Kelly [4]), the extension functor can be obtained from a fibration. In the protomodular setting, we start from the 'fibration of points' sending split epimorphisms to their codomain. In the more general setting of $\mathcal{S}$-protomodularity, see (Bourn, Martins-Ferreira, Montoli, and Sobral [8]), we consider only a certain subclass of split epimorphisms. This suggests we consider a 2-fibration sending open subtopos adjunctions to the codomain of their inverse image functors.

A categorification of the Grothendieck construction, given in (Buckley [10]), gives that 2-fibrations correspond to 3-functors into 2Cat. Fortunately, aside from motivation, we will largely be able to avoid 3-functors for the same reasons that $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ is essentially a 1-category (Corollary 6.4.5).

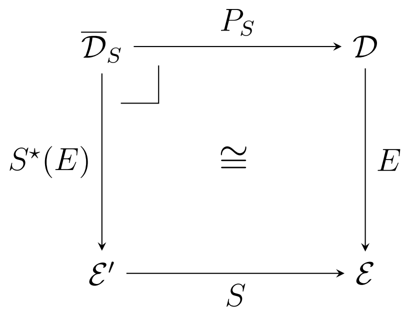
While the paradigmatic example of a fibration is the codomain fibration, which maps from the whole arrow category to the base category, the domain of the analogous 2-fibration is restricted to the category of fibrations. See (Buckley [10]) for more details on 2-fibrations.

The fibre 3-functor $\mathrm{Top}_{\mathrm{lex}}^{\mathrm{co\,op}} \to \mathrm{2Cat}$ corresponding to the 2-fibration $\mathrm{Cod}\colon \mathrm{Fib}_{\mathrm{Top}_{\mathrm{lex}}} \to \mathrm{Top}_{\mathrm{lex}}$ can be described as follows (omitting the description of the coherence data for simplicity):

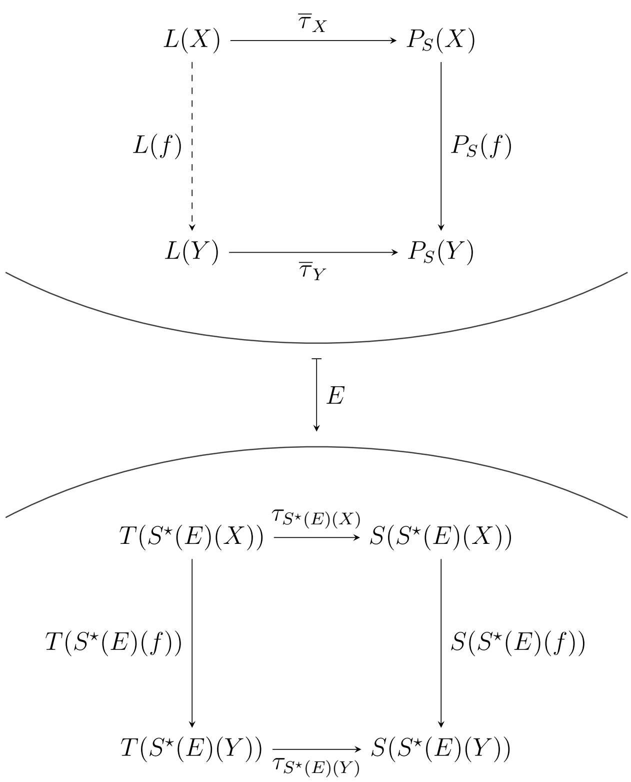
- On objects it sends a topos $\mathcal{E}$ to the slice 2-category of finite-limit-preserving fibrations from toposes to $\mathcal{E}$.
- On 1-morphisms it sends a finite-limit-preserving functor $T\colon \mathcal{E}' \to \mathcal{E}$ to the 2-functor $T^\star$ corresponding to pulling back along T.
- On 2-morphisms it sends a natural transformation $\tau\colon T \to S$ to a 2-natural transformation $\tau^\star$ from $S^\star$ to $T^\star$. The component $\tau^\star_E\colon S^\star(E) \to T^\star(E)$ indexed by the finite-limit-preserving fibration $E\colon \mathcal{D} \to \mathcal{E}$ can be constructed in three

steps.

- First define the morphism of fibrations $(P_S, S)\colon S^\star(E) \to E$ given by the pullback projection.

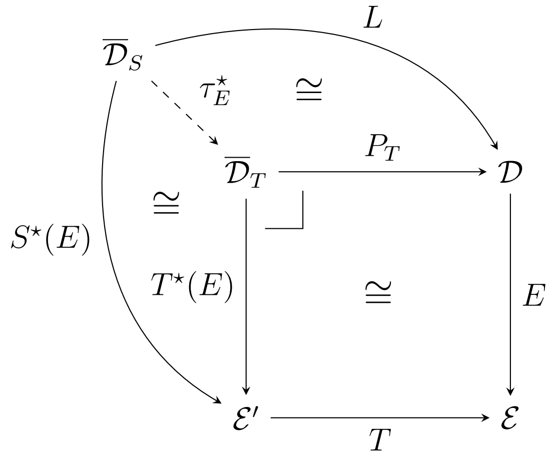


- Then we may use the fibration property of E to lift the natural transformation $\tau S^\star(E)$ to a natural transformation into P_S. Explicitly, for an object $X \in \overline{\mathcal{D}}_S$, the morphism $\tau_{S^\star(E)(X)}\colon T(S^\star(E)(X)) \to S(S^\star(E)(X))$ can be lifted to a morphism in $\mathcal{D}$ with codomain $P_S(X)$. These lifted morphisms assemble into a natural transformation $\overline{\tau}$ from a new functor $L\colon \overline{\mathcal{D}}_S \to \mathcal{D}$ to P_S. This functor sends a morphism $f\colon X \to Y$ to the morphism obtained by factoring $P_S(f)\overline{\tau}_X$ through $\overline{\tau}_Y$ as shown in the diagram below.



- Finally, consider the 2-pullback of E and T and note that the maps

$L\colon \overline{\mathcal{D}}_S \to \mathcal{D}$ and $S^\star(E)\colon \overline{\mathcal{D}}_S \to \mathcal{E}'$ form a cone as shown below.

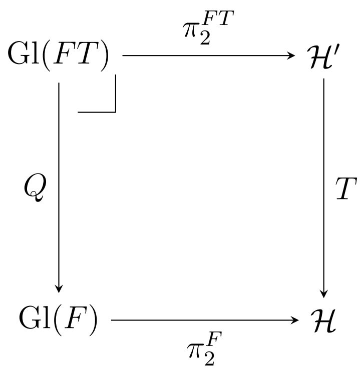


Thus, we may factor these through P_T and $T^\star(E)$ respectively to obtain a functor from $\overline{\mathcal{D}}_S$ to $\overline{\mathcal{D}}_T$. This is the desired functor $\tau_E^\star\colon S^\star(E) \to T^\star(E)$.

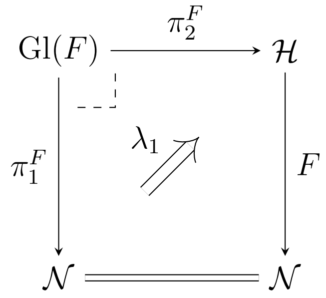
The coherent set of 2-isomorphisms for the 2-natural transformation can also be obtained by the cartesian property of the lifted maps and universal property of the 2-pullback.

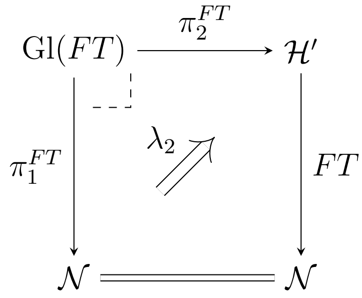
We can then easily modify this to describe the fibre 3-functor for the 2-fibration of (open) points. Moreover, to obtain $\mathrm{Ext}(-,\mathcal{N})\colon \mathrm{Top}_{\mathrm{lex}}^{\mathrm{co\,op}} \to 2\mathrm{Cat}$ we restrict to inverse image functors of open subtoposes (equipped with right adjoint splittings) with fixed kernel object $\mathcal{N}$. The above discussion restricts easily to this case, since these functors are stable under pullback along finite-limit-preserving functors and the relevant morphisms can be shown to be morphisms of extensions. To see this we will use the following folklore result, which we prove here for completeness.

Proposition 6.5.1. *Let $\mathcal{C}$ be a 2-category, $F\colon \mathcal{H} \to \mathcal{N}$, $T\colon \mathcal{H}' \to \mathcal{H}$ 1-morphisms. Then the comma object* $\mathrm{Gl}(FT)$ *can be represented as a (strict) 2-pullback (where we draw the 2-cokernels of the extension horizontally).*



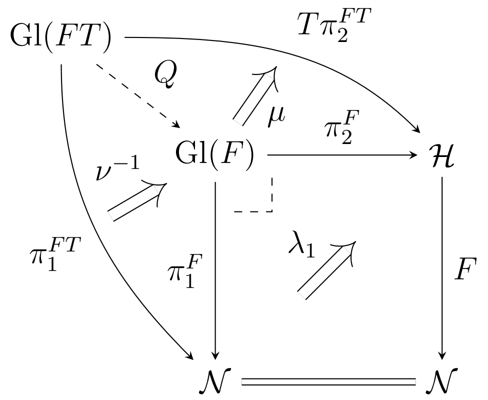
Proof. We must first describe Q. Consider the following comma object diagrams.





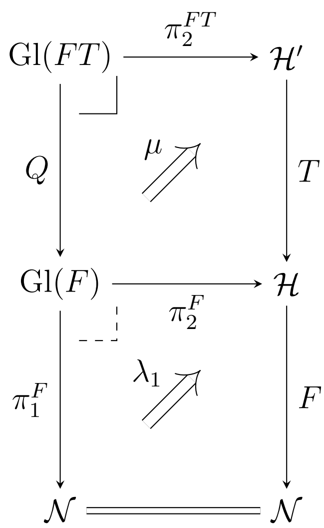
Here $\lambda_1 \colon \pi_1^F \to F\pi_2^F$ and $\lambda_2 \colon \pi_1^{FT} \to FT\pi_2^{FT}$ are the universal (not necessarily invertible) 2-morphisms. Explicitly, we have $\lambda_1 = \pi_1^F \theta^F$ and $\lambda_2 = \pi_1^{FT}\theta^{FT}$.

Now the map Q is given by the universal property of the comma object $\mathrm{Gl}(F)$ applied to the 2-cone given by $\lambda_2 \colon \pi_1^{FT} \to FT\pi_2^{FT}$.



Here ν and μ are 2-isomorphisms and satisfy that $F\mu \circ \lambda_1 Q \circ \nu^{-1} = \lambda_2$. Concretely, we have that $Q(N, H, \ell) = (N, T(H), \ell)$ and $Q(f, g) = (f, T(g))$ and that μ and ν are both the identity.

Now we paste our candidate 2-pullback square to our other comma object diagrams as follows.



Since $\pi_1^F Q = \pi_1^{FT}$, we see that the pasting diagram above is just the comma object diagram corresponding to $\mathrm{Gl}(FT)$. It follows that the upper square is a 2-pullback in a similar manner to (the converse direction of) the pullback lemma. $\square$

We will now describe the Ext 'functor' explicitly and at the same time demonstrate its relationship to Hom.

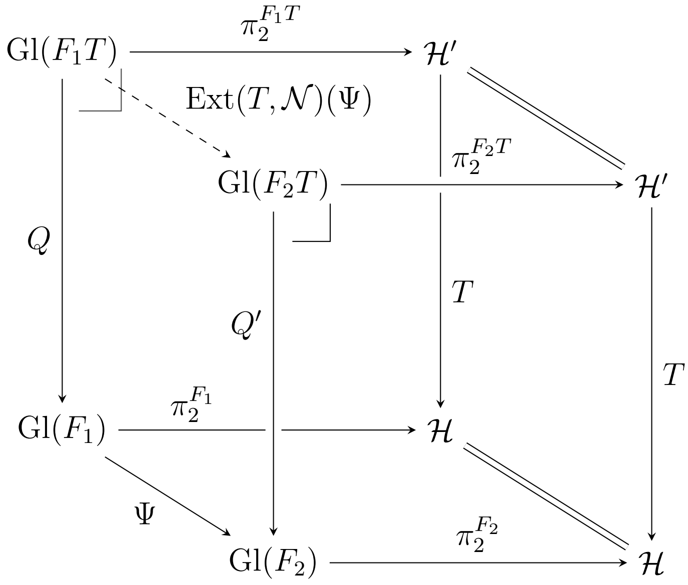
The 3-functor $\mathrm{Ext}(-,\mathcal{N})$ sends a topos $\mathcal{H}$ to the 2-category of extensions $\mathrm{Ext}(\mathcal{H},\mathcal{N})$ as defined above. In fact, since $\mathrm{Ext}(\mathcal{H},\mathcal{N})$ is (equivalent to) a 1-category, it turns out that $\mathrm{Ext}(-,\mathcal{N})$ factors through $\mathrm{Cat} \hookrightarrow 2\mathrm{Cat}$ and so can be treated as a 2-functor $\mathrm{Ext}(-,\mathcal{N})\colon \mathrm{Top}_{\mathrm{lex}}^{\mathrm{co\,op}} \to \mathrm{Cat}$.

The following computations will all be performed in the equivalent full subcategory of $\mathrm{Ext}(\mathcal{H},\mathcal{N})$ whose objects are of the form $\mathcal{N} \xrightarrow{\pi_{1*}} \mathrm{Gl}(F) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$. The fact that this may be done coherently is due to the 2-categorical analogue of the result which says that if $F\colon \mathcal{B} \to \mathcal{C}$ is a functor and for each B we have an isomorphism $F(B) \cong C_B$, then there is a functor F' such that $F'(B) = C_B$ (and which acts on morphisms by conjugating the result of F by the appropriate isomorphisms).

We can use Proposition 6.5.1 to show how $\mathrm{Ext}(-,\mathcal{N})$ acts on 1-morphisms. If $T\colon \mathcal{H}' \to \mathcal{H}$ is a finite-limit-preserving functor, then it is enough to describe how $\mathrm{Ext}(T,\mathcal{N})\colon \mathrm{Ext}(\mathcal{H},\mathcal{N}) \to \mathrm{Ext}(\mathcal{H}',\mathcal{N})$ acts on extensions in Artin-glueing form. It should 'take the 2-pullback along T' and hence it sends the object $\mathcal{N} \xrightarrow{\pi_{1*}^F} \mathrm{Gl}(F) \underset{\pi_{2*}^F}{\overset{\pi_2^F}{\rightleftarrows}} \mathcal{H}$ to $\mathcal{N} \xrightarrow{\pi_{1*}^{FT}} \mathrm{Gl}(FT) \underset{\pi_{2*}^{FT}}{\overset{\pi_2^{FT}}{\rightleftarrows}} \mathcal{H}'$ as in the following diagram.

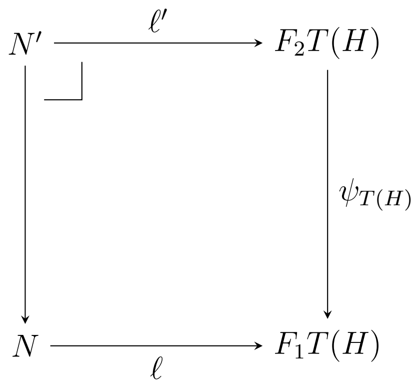
$$\begin{array}{ccccc}
\mathcal{N} & \xrightarrow{\pi_{1*}^{FT}} & \mathrm{Gl}(FT) & \underset{\pi_{2*}^{FT}}{\overset{\pi_2^{FT}}{\rightleftarrows}} & \mathcal{H}' \\
\Vert & & \downarrow & & \downarrow T \\
\mathcal{N} & \xrightarrow[\pi_{1*}^{F}]{} & \mathrm{Gl}(F) & \underset{\pi_{2*}^{F}}{\overset{\pi_2^{F}}{\rightleftarrows}} & \mathcal{H}
\end{array}$$

The functor $\mathrm{Ext}(T,\mathcal{N})$ acts on morphisms via the universal property of the 2-pullback. Let Ψ be a morphism of extensions corresponding to the natural transformation $\psi\colon F_2 \to F_1$ and consider the following diagram, noting that $T\pi_2^{F_1 T} = \pi_2^{F_2}\Psi Q$ and hence we have a 2-cone. Also recall that Ψ can always be to chosen to correspond to pulling back along ψ.



We shall now show that we may take $\mathrm{Ext}(T,\mathcal{N})(\Psi)$ to be $(\Gamma_{\mathcal{H}',\mathcal{N}} \circ \mathrm{Hom}(T,\mathcal{N}) \circ \Gamma^{-1}_{\mathcal{H},\mathcal{N}})(\Psi) = \Gamma_{\mathcal{H}',\mathcal{N}}(\psi T)$ — that is, the latter functor is given by the universal property of the 2-pullback. We must also supply two 2-morphisms corresponding to the left and top faces of the above cube. Both may be taken to be the identity. Note in fact that now each face of the cube commutes strictly.

We only need to check that $\pi_2^{F_2T}\Gamma_{\mathcal{H}',\mathcal{N}}(\psi T) = \pi_2^{F_1T}$ and that $Q'\Gamma_{\mathcal{H}',\mathcal{N}}(\psi T) = \Psi Q$. The former is immediate and the latter follows because $\Psi Q(N, H, \ell) = \Psi(N, T(H), \ell)$ is given by the following pullback.

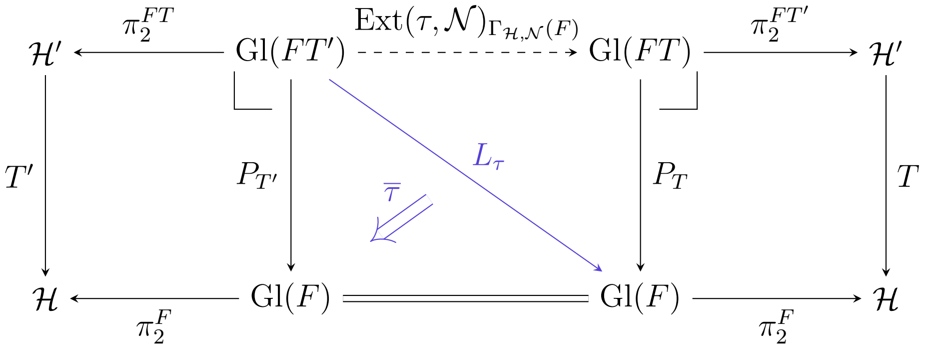


This is readily seen to be the same pullback which determines $\Gamma_{\mathcal{H}',\mathcal{N}}(\psi T)(N, H, \ell)$.

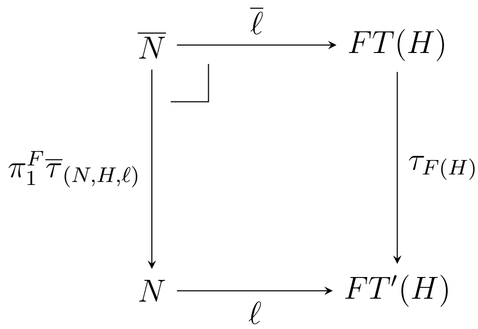
Finally, we discuss how $\mathrm{Ext}(-,\mathcal{N})$ acts on 2-morphisms. We follow the construction outlined above for the codomain 2-fibration. Let $\tau\colon T \to T'$ be a natural transformation. Then we describe the natural transformation $\mathrm{Ext}(\tau,\mathcal{N})\colon \mathrm{Ext}(T',\mathcal{N}) \to \mathrm{Ext}(T,\mathcal{N})$ componentwise.

Without loss of generality we may describe each component at extensions of the form

$\Gamma_{\mathcal{H},\mathcal{N}}(F)$. Consider the following diagram.



As discussed in the case of the codomain fibration, we may define a functor L_τ as follows. We have that $P_{T'}(N,H,\ell) = (N, T'(H), \ell)$ lies above the codomain of $\tau_H \colon T(H) \to T'(H)$ with respect to the fibration π_2^F (see Proposition 6.2.6) and so by the universal property of the fibration we get a map $\overline{\tau}_{(N,H,\ell)} \colon (\overline{N}, T(H), \overline{\ell}) \to (N, T'(H), \ell)$ as given by the pair of vertical morphisms in the following pullback diagram.



These morphisms form a natural transformation $\overline{\tau} \colon L_\tau \to P_{T'}$, where L_τ is the functor which sends (N,H,ℓ) to $(\overline{N}, T(H), \overline{\ell})$. As above, this functor factors through P_T to give a functor $\mathrm{Ext}(\tau,\mathcal{N})_{\Gamma_{\mathcal{H},\mathcal{N}}(F)} \colon \mathrm{Gl}(FT') \to \mathrm{Gl}(FT)$, which sends (N,H,ℓ) to $(\overline{N}, H, \overline{\ell})$. It remains to show that this gives a morphism of split extensions, but it is clear from the above pullback diagram that this functor is the morphism of split extensions corresponding to τF (which itself is equal to $\mathrm{Hom}(\tau,\mathcal{N})_F$).

The morphisms $\mathrm{Ext}(\tau,\mathcal{N})_{\Gamma_{\mathcal{H},\mathcal{N}}(F)}$ define the desired natural transformation $\mathrm{Ext}(\tau,\mathcal{N})$. (Naturality follows from the interchange law or from the general theory of the codomain 2-fibration.)

The 2-functor $\mathrm{Ext}(-,\mathcal{N})$ composes strictly, and for the unitors note that $\mathrm{Ext}(\mathrm{Id}_\mathcal{H},\mathcal{N})$ is equal to $\Gamma_{\mathcal{H},\mathcal{N}}\Gamma_{\mathcal{H},\mathcal{N}}^{-1} \colon \mathrm{Ext}(\mathcal{H},\mathcal{N}) \to \mathrm{Ext}(\mathcal{H},\mathcal{N})$ so that we make take as our unitors the unit of the adjunction $\Gamma_{\mathcal{H},\mathcal{N}}$. The necessary 2-functor axioms can then easily be seen to hold.

Remark 6.5.2. The above argument also proves that the 2-functor sending adjoint extensions with fixed kernel $\mathcal{N}$ to their cokernels is a 2-fibration. △

Theorem 6.5.3. *Let $\mathcal{N}$ be a topos. The 2-functors* $\mathrm{Ext}(-, \mathcal{N})$ *and* $\mathrm{Hom}_{\mathrm{op}}(-, \mathcal{N})$ *are 2-naturally equivalent via* $\Gamma^{\mathcal{N}} \colon \mathrm{Hom}_{\mathrm{op}}(-, \mathcal{N}) \to \mathrm{Ext}(-, \mathcal{N})$ *defined as follows:*

i) for each $\mathcal{H} \in \mathrm{Top}_{\mathrm{lex}}$ we have the equivalence $\Gamma^{\mathcal{N}}_{\mathcal{H}} = \Gamma_{\mathcal{H},\mathcal{N}} \colon \mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}} \to \mathrm{Ext}(\mathcal{H}, \mathcal{N})$,

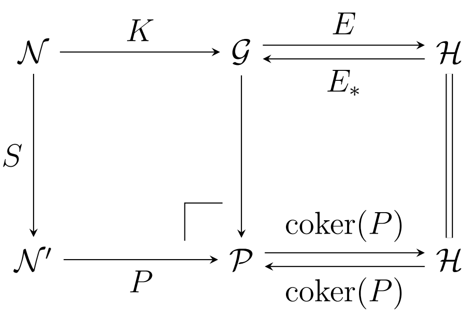
ii) for each $T \colon \mathcal{H}' \to \mathcal{H}$ we have the identity $\Gamma_{\mathcal{H}',\mathcal{N}} \mathrm{Hom}(T, \mathcal{N})^{\mathrm{op}} = \mathrm{Ext}(T, \mathcal{N}) \Gamma_{\mathcal{H},\mathcal{N}}$.

Proof. The equality in point (ii) is clear by inspection of the definition of $\mathrm{Ext}(T, \mathcal{N})$. The proof of the coherence conditions is easy. In particular, the first coherence condition is satisfied because each morphism of the diagram is the identity. Similarly, for the second condition again all morphisms are the identity (though marginally more work is required to show that the unitor at H whiskered with $\Gamma_{\mathcal{H},\mathcal{N}}$ is in fact the identity). □

We turn our attention to the functor $\mathrm{Ext}(\mathcal{H}, -) \colon \mathrm{Top}^{\mathrm{co}}_{\mathrm{lex}} \to \mathrm{Cat}$. We could not find an elegant description of this in terms of a 2-fibration. However, we believe a reasonable definition can be given by dualising our arguments for $\mathrm{Ext}(-, \mathcal{N})$.

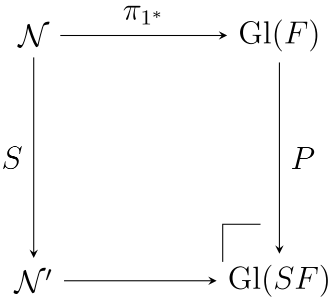
Naturally, for an object $\mathcal{N}$ we have that $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ is just the category of adjoint split extensions.

For 1-morphisms, consider $S \colon \mathcal{N} \to \mathcal{N}'$. We would have $\mathrm{Ext}(\mathcal{H}, S)$ act on objects by sending $\mathcal{N} \overset{K}{\rightarrowtail} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ to the extension resulting from a pushout of K along S and on morphisms via the universal property of the pushout.



To see that this is well-defined we prove the following result dual to Proposition 6.5.1.

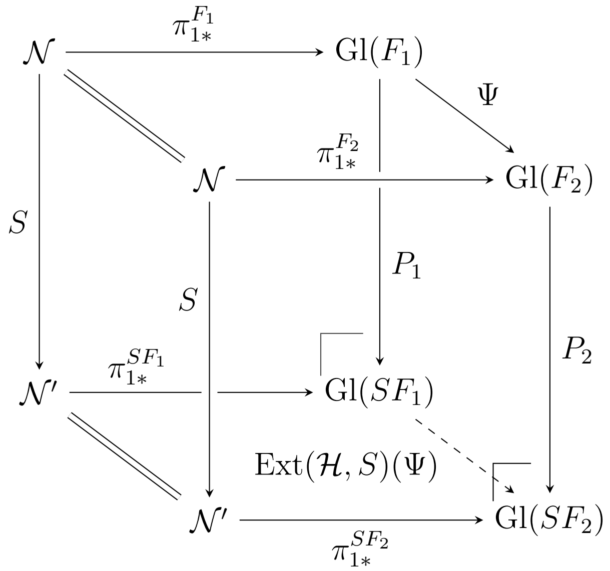
Proposition 6.5.4. *Let $F \colon \mathcal{H} \to \mathcal{N}$ and $S \colon \mathcal{N} \to \mathcal{N}'$ be finite-limit-preserving functors. Then $\mathrm{Gl}(SF)$ is given by the following 2-pushout.*



Proof. By Proposition 6.4.7, we know that $\mathrm{Gl}(F)$ is a cocomma object in the 2-category $\mathrm{Cat}_{\mathrm{lex}}$. Thus it is a comma object in $\mathrm{Cat}_{\mathrm{lex}}^{\mathrm{op}}$. Applying Proposition 6.5.1 and then reversing the arrows gives the desired result. It is not hard to see that $P(N, H, \ell) = (S(N), H, S(\ell))$ and $P(f, g) = (S(f), g)$. □

Thus, fixing particular pushouts we can describe $\mathrm{Ext}(\mathcal{H}, S)$ concretely as sending an extension $\mathcal{N} \xrightarrow{K} \mathcal{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathcal{H}$ to $\mathcal{N} \xrightarrow{\pi_{1*}} \mathrm{Gl}(SK^*E_*) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} \mathcal{H}$.

As mentioned above, $\mathrm{Ext}(\mathcal{H}, S)$ should act on morphisms by the universal property of the pushout. Let $\Psi = (\Psi, \alpha_1, \beta_1, \gamma_1)$ be a morphism of extensions and consider the 2-cocone given by $P_2\Psi\pi_{1*}^{F_1} \xrightarrow{P_2\alpha_1^{-1}} P_2\pi_{1*}^{F_2}\mathrm{Id}_{\mathcal{N}} = \pi_{1*}^{SF_2}\mathrm{Id}_{\mathcal{N}'}S$ as in the following pasting diagram (where we omit the 2-morphisms do avoid clutter).



Here the front, back and left faces commute strictly and the top face has associated invertible 2-morphism α_1^{-1}.

Let ψ be the natural transformation associated to Ψ. We will show that the map given by the universal property is $\Gamma_{\mathcal{H},\mathcal{N}'}(S\psi) = (\Gamma_{\mathcal{H},\mathcal{N}'}(S\psi), \alpha_2, \beta_2, \gamma_2)$. We define the associated 2-morphisms for the universal property of the 2-pushout as follows.

For the bottom face we use $\alpha_2^{-1} \colon \Gamma_{\mathcal{H},\mathcal{N}'}(S\psi)\pi_{1*}^{SF_1} \to \pi_{1*}^{SF_2}$.

As for the right-hand face, we note that $P_2\Psi(N, H, \ell)$ is given by the following diagram.

$$
\begin{array}{ccc}
S(\overline{N}) & \xrightarrow{S(\overline{\ell})} & SF_2(H) \\
{\scriptstyle S(\overline{\psi_H})}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle S\psi_H} \\
S(N) & \xrightarrow[S(\ell)]{} & SF_1(H)
\end{array}
$$

On the other hand, we have that $\Gamma_{\mathcal{H},\mathcal{N}'}(S\psi)P_1(N, H, \ell)$ is given by a similar diagram as follows.

$$
\begin{array}{ccc}
\overline{S(N)} & \xrightarrow{\overline{S(\ell)}} & SF_2(H) \\
{\scriptstyle \overline{S(\psi_H)}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle S\psi_H} \\
S(N) & \xrightarrow[S(\ell)]{} & SF_1(H)
\end{array}
$$

Since S preserves pullbacks, there is a natural family of isomorphisms $\mu_{(N,H,\ell)} = (\widehat{\mu}_{(N,H,\ell)}, \mathrm{id}_H)$ from $(S(\overline{N}), H, S(\overline{\ell}))$ to $(\overline{S(N)}, H, \overline{S(\ell)})$, which in particular satisfies that $\overline{S(\psi_H)}\widehat{\mu}_{(N,H,\ell)} = S(\overline{\psi_H})$. We take μ to be the 2-morphism associated to the right-hand face.

Now let us discuss the 2-cocone 2-morphism $P_2\alpha_1^{-1}$ in more detail. The component at N is given by $(S\widehat{\alpha}_{1,N}^{-1}, \mathrm{id})$ for $\widehat{\alpha}$ as defined Section 6.4.2. Notice that $S(\widehat{\alpha}_{1,N}^{-1})$ is given by the following diagram.

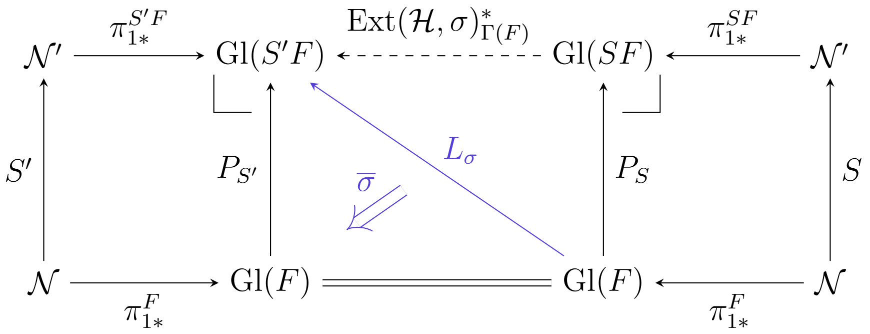
$$
\begin{array}{ccc}
S(\overline{N}) & \xrightarrow{S(!)} & SF_2(1) \\
{\scriptstyle S(\widehat{\alpha}_{1,N}^{-1})}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle S\psi_1} \\
S(N) & \xrightarrow[S(!)]{} & SF_1(1)
\end{array}
$$

We see that $S(\widehat{\alpha}_1^{-1}) = S(\overline{\psi_1})$. Similarly, we have $\overline{S(\psi_1)} = \widehat{\alpha}_2^{-1}S$. Thus, the equation

$\overline{S(\psi_H)}\mu_N = S(\overline{\psi_H})$ from above reduces in this case to $\widehat{\alpha}_{2,N}^{-1} S \circ \widehat{\mu}_{(N,1,!)} = S\widehat{\alpha}_{1,N}^{-1}$. Consequently, we have $\alpha_2^{-1} S \circ \mu \pi_{1*}^{F_1} = P_2 \alpha_1^{-1}$, and hence $\Gamma_{\mathcal{H},\mathcal{N}}(S\psi)$ equipped with α_2^{-1} and (μ, id) is indeed the desired map from the universal property of the pushout.

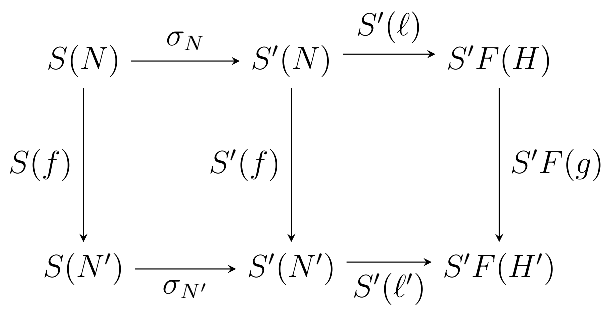
In summary, we have $\mathrm{Ext}(\mathcal{H}, S)(\Psi) = \Gamma_{\mathcal{H},\mathcal{N}'}(S\psi)$, which completes the description of $\mathrm{Ext}(\mathcal{H}, S)$.

Finally, let $\sigma\colon S \to S'$ be a natural transformation. We shall describe the natural transformation $\mathrm{Ext}(\mathcal{H}, \sigma)\colon \mathrm{Ext}(\mathcal{H}, S') \to \mathrm{Ext}(\mathcal{H}, S)$ componentwise. We define $\mathrm{Ext}(\mathcal{H}, \sigma)_{\Gamma_{\mathcal{H},\mathcal{N}}(F)}$ through its left adjoint as follows.



Notice the resemblance of the above diagram to the one that arose when defining $\mathrm{Ext}(\tau, \mathcal{N})_{\Gamma_{\mathcal{H},\mathcal{N}}(F)}$. It has the same basic structure, but all 1-morphisms are pointing in the opposite direction.

First we define L_σ and $\overline{\sigma}$. Note that $P_{S'}(N, H, \ell) = (S'(N), H, S'(\ell))$ lies above the codomain of σ_N, with respect to the fibration $\pi_1^{S'F}$ (see Proposition 6.2.7). Thus by the universal property, we may lift σ_N to a map $\overline{\sigma_N}\colon (S(N), H, S'(\ell)\sigma_N) \to (S'(N), H, S'(\ell))$. We may define a functor $L_\sigma\colon \mathrm{Gl}(F) \to \mathrm{Gl}(S'F)$ which sends objects (N, H, ℓ) to $(S(N), H, S'(\ell)\sigma_N)$ and which sends morphisms $(f, g)\colon (N, H, \ell) \to (N', H', \ell')$ to $(S(f), g)$. The pair $(S(f), g)$ can be seen to be a morphism in $\mathrm{Gl}(S'F)$ by considering the following diagram.



The left-hand square commutes by naturality of σ and the right-hand square commutes since (f, g) is morphism in $\mathrm{Gl}(F)$. Now the $\overline{\sigma_N}$ arrange into a natural transformation $\overline{\sigma}\colon L_\sigma \to P_{S'}$.

If things are to behave dually, we should have L_σ factor through P_S. By the naturality of σ we have that the following diagram commutes.

$$\begin{array}{ccc} S(N) & \xrightarrow{\sigma_N} & S'(N) \\ {\scriptstyle S(\ell)}\downarrow & & \downarrow{\scriptstyle S'(\ell)} \\ S(F(H)) & \xrightarrow[\sigma_{F(H)}]{} & S'(F(H)) \end{array}$$

Thus, observe that $L_\sigma(N, H, \ell) = (S(N), H, S'(\ell)\sigma_N) = (S(N), H, \sigma_{F(H)}S(\ell))$. This perspective allows us to factor L_σ as $L_\sigma = \Gamma_{\mathcal{H},\mathcal{N}'}(\sigma F)^* \circ P_S$. (Recall that the left adjoint to the functor $\Gamma_{\mathcal{H},\mathcal{N}'}(\sigma F)$ sends an object (N, H, ℓ) to $(N, H, \sigma_{F(H)}\ell)$.)

We might now hope to take $\mathrm{Ext}(\mathcal{H}, \sigma)_{F(H)}$ to be this resulting factor $\Gamma_{\mathcal{H},\mathcal{N}'}(\sigma F)^*$. However, this map goes in the 'wrong' direction. We can remedy this by taking the right adjoint and setting $\mathrm{Ext}(\mathcal{H}, \sigma)_{F(H)} = \Gamma_{\mathcal{H},\mathcal{N}'}(\sigma F)$.

In order to specify $\mathrm{Ext}(\mathcal{H}, -)$ completely, it only remains to discuss the compositors and unitors. As before we have that $\mathrm{Ext}(\mathcal{H}, -)$ composes strictly and we take the unitors to be the unit of the adjunction $\Gamma^*_{\mathcal{H},\mathcal{N}} \dashv \Gamma_{\mathcal{H},\mathcal{N}}$ in Theorem 6.4.12.

We can express the relationship between $\mathrm{Ext}(\mathcal{H}, -)$ and $\mathrm{Hom}(\mathcal{H}, -)$ as follows.

Theorem 6.5.5. *Let $\mathcal{H}$ be a topos.* $\mathrm{Ext}(\mathcal{H}, -)$ *and* $\mathrm{Hom}_{\mathrm{op}}(\mathcal{H}, -)$ *are 2-naturally equivalent via the mapping* $\Gamma^{\mathcal{H}} \colon \mathrm{Hom}_{\mathrm{op}}(\mathcal{H}, -) \to \mathrm{Ext}(\mathcal{H}, -)$ *defined as follows.*

i) for each $\mathcal{N} \in \mathrm{Top}_{\mathrm{lex}}$ *we have the equivalence* $\Gamma^{\mathcal{H}}_{\mathcal{N}} = \Gamma_{\mathcal{H},\mathcal{N}} \colon \mathrm{Hom}(\mathcal{H}, \mathcal{N})^{\mathrm{op}} \to \mathrm{Ext}(\mathcal{H}, \mathcal{N})$,

ii) for each $S \colon \mathcal{N} \to \mathcal{N}'$ *we have the identity* $\Gamma_{\mathcal{H},\mathcal{N}}\mathrm{Hom}(\mathcal{H}, S)^{\mathrm{op}} = \mathrm{Ext}(\mathcal{H}, S)\Gamma_{\mathcal{H},\mathcal{N}'}$.

Proof. Just as before, the equality in point (ii) is clear by inspection of the definition of $\mathrm{Ext}(\mathcal{H}, S)$ and the necessary coherence conditions hold, because each involved morphism is an identity. □

A bifunctor theorem for 2-functors was discussed in (Faul, Manuell, and Siqueira [17]) and gives the precise conditions that allow two families of 2-functors $M_B \colon \mathcal{C} \to \mathcal{D}$ and $L_C \colon \mathcal{B} \to \mathcal{D}$ to be collated into a bifunctor $P \colon \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ for which $P(B, -)$ is isomorphic to M_B and $P(-, C)$ is isomorphic to L_C. These conditions are that $L_C(B) = M_B(C)$ and that for each $f \colon B_1 \to B_2$ in $\mathcal{B}$ and $g \colon C_1 \to C_2$ in $\mathcal{C}$ there exists an invertible 2-morphism $\chi_{f,g} \colon L_{C_2}(g)M_{B_1}(f) \to M_{B_2}(f)L_{C_1}(g)$ satisfying certain coherence conditions reminiscent of those for distributive laws for monads. Such families together with χ are called a *distributive law of pseudofunctors*.

For the families $\mathrm{Ext}(-, \mathcal{N})$ and $\mathrm{Ext}(\mathcal{H}, -)$ the first condition is immediate. Moreover, it is not hard to see that if $T \colon \mathcal{H}' \to \mathcal{H}$ and $S \colon \mathcal{N} \to \mathcal{N}'$ that $\mathrm{Ext}(\mathcal{H}, S)\mathrm{Ext}(T, \mathcal{N}) = \mathrm{Ext}(T, \mathcal{N}')\mathrm{Ext}(\mathcal{H}', S)$ so that we might choose $\chi_{T,S}$ to be the identity. The coherence conditions are then immediate.

Thus, we may apply the results of (Faul, Manuell, and Siqueira [17]) to arrive at the 2-functor $(\mathrm{Ext}, \omega, \kappa) \colon \mathrm{Top}_{\mathrm{lex}}^{\mathrm{op}} \times \mathrm{Top}_{\mathrm{lex}} \to \mathrm{Cat}$ defined below.

Definition 6.5.6. Let $(\mathrm{Ext}, \omega, \kappa) \colon \mathrm{Top}_{\mathrm{lex}}^{\mathrm{co\,op}} \times \mathrm{Top}_{\mathrm{lex}}^{\mathrm{co}} \to \mathrm{Cat}$ be the 2-functor defined as follows.

i) $\mathrm{Ext}(\mathcal{H}, \mathcal{N})$ is the category of extensions of $\mathcal{H}$ by $\mathcal{N}$,

ii) $\mathrm{Ext}(T, S) = \mathrm{Ext}(T, \mathcal{N}')\mathrm{Ext}(\mathcal{H}', S)$ for functors $S \colon \mathcal{N} \to \mathcal{N}'$ and $T \colon \mathcal{H}' \to \mathcal{H}$,

iii) $\mathrm{Ext}(\tau, \sigma) = \mathrm{Ext}(\tau, \mathcal{N}') * \mathrm{Ext}(\mathcal{H}', \sigma)$ for 2-morphisms $\sigma \colon S \to S'$ and $\tau \colon T \to T'$,

iv) ω is the identity,

v) $\kappa_{\mathcal{H},\mathcal{N}}$ is given by the unit Φ of the adjunction $\Gamma^*_{\mathcal{H},\mathcal{N}} \dashv \Gamma_{\mathcal{H},\mathcal{N}}$ as defined in Theorems 6.3.15 and 6.4.12. △

The 2-bifunctor $\mathrm{Hom}_{\mathrm{op}}$ can be recovered as the collation of the functors obtained by fixing one of its components, $\mathrm{Hom}_{\mathrm{op}}(\mathcal{H}, -)$ and $\mathrm{Hom}_{\mathrm{op}}(-, \mathcal{N})$. It is shown in (Faul, Manuell, and Siqueira [17]) that 'morphisms between distributive laws' can also be collated to give 2-natural transformations between the corresponding bifunctors. The 2-natural equivalences in Theorems 6.5.3 and 6.5.5 can be collected into a 2-natural equivalence $\Gamma \colon \mathrm{Ext} \to \mathrm{Hom}_{\mathrm{op}}$ provided that $\Gamma^{\mathcal{N}}_{\mathcal{H}} = \Gamma^{\mathcal{H}}_{\mathcal{N}}$ and the Yang–Baxter equation holds. These conditions are immediate in our setting and so we obtain the following theorem.

Theorem 6.5.7. *The 2-functors* Ext *and* $\mathrm{Hom}_{\mathrm{op}}$ *are 2-naturally equivalent via* $\Gamma \colon \mathrm{Ext} \to \mathrm{Hom}_{\mathrm{op}}$ *in which* $\Gamma_{\mathcal{H},\mathcal{N}} = \Gamma^{\mathcal{H}}_{\mathcal{N}}$ *and* $\Gamma_{T,S}$ *is the identity for all functors* $S \colon \mathcal{N} \to \mathcal{N}'$ *and* $T \colon \mathcal{H}' \to \mathcal{H}$.

Bibliography

[1] M. Artin, A. Grothendieck, and J.-L. Verdier. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos (SGA4-1)*, volume 259 of *Lecture Notes in Mathematics*. Springer, Berlin, 1972.

[2] B. Billhardt. On a wreath product embedding and idempotent pure congruences on inverse semigroups. *Semigroup Forum*, pages 45–54, 1992.

[3] F. Borceux and D. Bourn. *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*, volume 566 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dordrecht, 2004.

[4] F. Borceux, G. Janelidze, and G. M. Kelly. Internal object actions. *Comment. Math. Univ. Carolin.*, 46(2):235–255, 2005.

[5] F. Borceux, G. Janelidze, and G. M. Kelly. On the representability of actions in a semi-abelian category. *Theory Appl. Categ.*, 14(11):244–286, 2005.

[6] D. Bourn. Partial Mal'tsevness and partial protomodularity. arXiv:1507.02886v1, 2015.

[7] D. Bourn and G. Janelidze. Protomodularity, descent, and semidirect products. *Theory Appl. Categ.*, 4(2):37–46, 1998.

[8] D. Bourn, N. Martins-Ferreira, A. Montoli, and M. Sobral. Monoids and pointed S-protomodular categories. *Homology, Homotopy Appl.*, 18:151–172, 2015.

[9] G. Bruns and H. Lakser. Injective hulls of semilattices. *Canad. Math. Bull.*, 13 (1):115–118, 1970.

[10] M. Buckley. Fibred 2-categories and bicategories. *J. Pure Appl. Algebra*, 218(6): 1034–1074, 2014.

[11] D. DeWolf. *Restriction Category Perspectives of Partial Computation and Geometry*. PhD thesis, Dalhousie University, 2017.

[12] P. F. Faul. Baer sums for a natural class of monoid extensions. *Semigroup Forum (in press), arXiv:2005.09063*, 2020.

[13] P. F. Faul. A characterization of weakly Schreier extensions of monoids. *J. Pure Appl. Algebra*, 2020.

[14] P. F. Faul. λ-semidirect products of inverse monoids are weakly Schreier extensions. *Semigroup Forum, doi:10.1007/s00233-020-10131-4*, 2020.

[15] P. F. Faul and G. Manuell. Artin glueings of frames as semidirect products. *J. Pure Appl. Algebra*, 224(8):106334, 2020.

[16] P. F. Faul and G. Manuell. Quotients of monoid extensions and their interplay with Baer sums. arXiv:2006.10537, 2020.

[17] P. F. Faul, G. Manuell, and J. Siqueira. 2-dimensional bifunctor theorems and distributive laws. arXiv:2010.07926, 2020.

[18] P. F. Faul, G. Manuell, and J. Siqueira. Artin glueings of toposes as adjoint split extensions. arXiv:2012.04963, 2020.

[19] I. Fleischer. Monoid extension theory. *J. Pure Appl. Algebra*, 21(2):151–159, 1981.

[20] R. Fulp and J. Stepp. Structure of the semigroup of semigroup extensions. *Trans. Amer. Math. Soc.*, 158(1):63–73, 1971.

[21] R. Fulp and J. Stepp. Semigroup extensions. *J. Reine Angew. Math.*, 248:28–41, 1971.

[22] P. A. Grillet. Left coset extensions. In *Semigroup Forum*, volume 7, pages 200–263. Springer, 1974.

[23] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume 1. Oxford University Press, Oxford, 2002.

[24] S. Kasangian, G. Metere, and E. M. Vitale. Split extensions, semidirect product and holomorph of categorical groups. *Homology Homotopy Appl.*, 8(1):145–167, 2006.

[25] P. Köhler. Quasi-decompositions of semigroups. *Houston J. Math*, 5:525–542, 1979.

[26] S. Lack. A 2-categories companion. In *Towards Higher Categories*, pages 105–191. Springer, New York, 2010.

[27] M. V. Lawson. *Inverse Semigroups: The Theory of Partial Symmetries.* World Scientific, 1998.

[28] J. Leech. Extending groups by monoids. *J. Algebra*, 74(1):1–19, 1982.

[29] S. Mac Lane. *Homology.* Springer Science & Business Media, Berlin, 2012.

[30] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer Science & Business Media, 2012.

[31] D. Marsden. Category theory using string diagrams. arXiv:1401.7220, 2014.

[32] N. Martins-Ferreira. Semi-biproducts in monoids. arXiv:2002.05985, 2020.

[33] N. Martins-Ferreira, A. Montoli, and M. Sobral. Semidirect products and crossed modules in monoids with operations. *J. Pure Appl. Algebra*, 217(2):334–347, 2013.

[34] N. Martins-Ferreira, A. Montoli, and M. Sobral. Baer sums of special Schreier extensions of monoids. *Semigroup Forum*, 93(2):403–415, 2016.

[35] D. McAlister and T. Blyth. Split orthodox semigroups. *J. Algebra*, 51(2): 491–525, 1978.

[36] D. B. McAlister. Groups, semilattices and inverse semigroups. II. *Trans. Amer. Math. Soc.*, 196:351–370, 1974.

[37] S. Niefield. The glueing construction and double categories. *J. Pure Appl. Algebra*, 216(8–9):1827–1836, 2012.

[38] A. Patchkoria. Extensions of semimodules by monoids and their cohomological characterization. *Bull. Georgian Acad. Sci*, 86:21–24, 1977.

[39] A. Patchkoria. *On Schreier extensions of semimodules.* PhD thesis, Tbilisi State University, 1979.

[40] A. Patchkoria. Crossed semimodules and Schreier internal categories in the category of monoids. *Georgian Math. J.*, 5(6):575–581, 1998.

[41] J. Picado and A. Pultr. *Frames and Locales: Topology Without Points.* Springer Science & Business Media, Basel, 2011.

[42] L. Rédei. Die Verallgemeinerung der Schreierschen Erweiterungstheorie. *Acta Sci. Math. (Szeged)*, 14:252–273, 1952.

[43] R. Rosebrugh and R. J. Wood. Cofibrations in the bicategory of topoi. *J. Pure Appl. Algebra*, 32(1):71–94, 1984.

[44] N. X. Tuyen. Non-abelian extensions of monoids. *Bull. Acad. Sci. Georgian SSR*, 84:37–39, 1976.

[45] G. Wraith. Artin glueing. *J. Pure Appl. Algebra*, 4(3):345–348, 1974.