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Gauge-Invariant Uniqueness and Reductions of Ordered Groups

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GAUGE-INVARIANT UNIQUENESS AND

REDUCTIONS OF ORDERED GROUPS

by

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GAUGE-INVARIANT UNIQUENESS AND

REDUCTIONS OF ORDERED GROUPS

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A reduction φ of an ordered group (G, P) to another ordered group is an order homomorphism which maps each interval $[1, p]$ bijectively onto $[1, \varphi(p)]$. We show that if (G, P) is weakly quasi-lattice ordered and reduces to an amenable ordered group, then there is a gauge-invariant uniqueness theorem for P-graph algebras. We also consider the class of ordered groups which reduce to an amenable ordered group, and show this class contains all amenable ordered groups and is closed under direct products, free products, and hereditary subgroups.

DEDICATION

Dedicated to my incredible math family at UNL and around the world.

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Chapter 1

Introduction

Cuntz-Krieger algebras and their generalizations (Exel-Laca algebras, graph algebras, higher-rank graph algebras, etc) are all, broadly speaking, the universal algebras generated by partial isometries whose range and source projections satisfy certain combinatorial relations. But defining these algebras by their universality comes at a cost, since it becomes difficult to check if any particular collection of partial isometries (a "representation") is universal. Mathematicians responded with uniqueness theorems, conditions on the representation which guarantee that it is universal. A classic example is the *gauge-invariant uniqueness theorem* for graph algebras, which states that so long as the canonical generators of the algebra are nonzero and there is a gauge action (meaning an action α of the circle satisfying $\alpha_z(T_\gamma) = z^{\mathrm{length}(\gamma)}T_\gamma$ for any $z \in \mathbb{T}$ and path γ in the graph), then any other representation of the graph is a quotient of this representation. This construction and uniqueness theorem has been generalized to higher rank graphs by Kumjian and Pask [15], where paths are given lengths in $\mathbb{N}^k$ instead of $\mathbb{N}$, and an action of $\mathbb{T}^k = \widehat{\mathbb{Z}^k}$ replaces the gauge action of $\mathbb{T} = \widehat{\mathbb{Z}}$.

One might then ask that paths be allowed to have "lengths" in any positive cone P of a group G. The first attempt at such a generalization was by Brownlowe, Sims, and Vittadello [3], who studied P-graphs when (G, P) is quasi-lattice ordered, but surprisingly the authors construct a P-graph algebra which is not universal but *co-*

universal, meaning it is a quotient of any sufficiently large representation.

In Section 4 of this paper, we build on their work to show that if the grading group (G, P) is weakly quasi-lattice ordered and has a certain kind of quotient map called a reduction such that the quotient is amenable, this algebra is both universal and co-universal. This allows us to generalize the gauge-invariant uniqueness theorems for graphs and k-graphs (such as [18, Theorem 2.2] or [15, Theorem 3.4]) to make a gauge-invariant uniqueness theorem for P-graphs:

Theorem 4.4.2. *Let (G, P) be a weakly quasi-lattice ordered group which reduces to an amenable ordered group, and let Λ be a finitely-aligned P-graph. Then there is exactly one representation (up to canonical isomorphism) of Λ which is Λ-faithful, tight, and has a gauge coaction of G. This representation is universal for tight representations and co-universal for representations which are Λ-faithful and have a gauge coaction by G.*

This diagram provides a visual summary of the result:
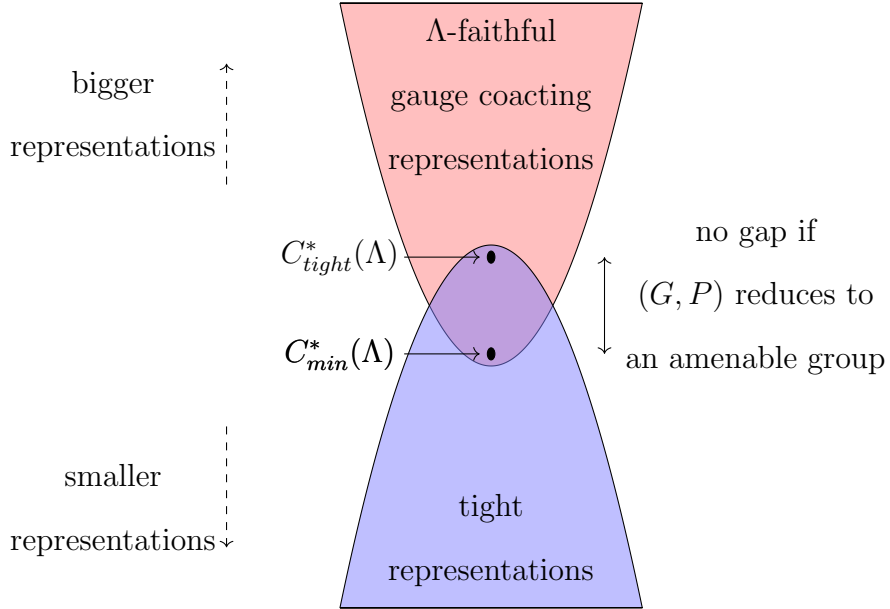
We believe that the algebra generated by this simultaneously universal and co-universal representation deserves the title of *the* Cuntz-Krieger algebra of the P-graph.

The notion of a reduction of ordered groups is introduced in Section 3, culminating in this theorem showing that several natural operations on ordered groups preserve the existence of a reduction onto an amenable group:

Theorem 3.3.1. *The class of ordered groups which have (strong) reductions onto amenable groups contains all amenable ordered groups and is closed under hereditary subgroups, finite direct products, and finite free products.*

In Theorem 4.4.2, being reducible to an amenable group plays the role of amenability in guaranteeing a unique representation, but by Theorem 3.3.1, this condition is, in at least one sense, more robust than amenability. In particular, *being reducible to* an amenable group is preserved under free products, whereas *being* amenable is almost always destroyed by a free product (for instance if G and H contain copies of $\mathbb{Z}$, then $G * H$ is not amenable).

The notion of being reducible to an amenable group is sufficiently robust that, as corollary to Theorem 3.3.1, the important example $(\mathbb{Z}^2 * \mathbb{Z}, \mathbb{N}^2 * \mathbb{N})$ from the literature ([21, 3]) reduces to an amenable group. In Proposition 4.6.2, we use this fact in combination with the results of [3] to show that every Kirchberg algebra in the UCT class is stably isomorphic to the simultaneously universal and co-universal algebra of a $(\mathbb{N}^2 * \mathbb{N})$-graph.

Besides the results listed in Theorem 3.3.1, a great deal is not yet known about how reductions interact with other group constructions. The limits of our understanding are discussed in Section 3.3.

We hope that this new approach will provide a useful tool for the analysis of P-graphs and their algebras, and possibly have implications for product systems and Fell bundles.

Chapter 2

Background

2.1 Ordered Groups

Ordered groups have been studied in both a context of group theory and operator theory, and we largely follow the conventions of [3] and [10, Chapter 32]. The reader should note that other authors, such as [4], use the term "ordered group" to refer to a group with a total order, but we follow the convention that the group has a partial order.

Definition 2.1.1. *A* positive cone *in a group G is a submonoid P such that $P \cap P^{-1} = \{1\}$.*

A left-invariant partial order *on a group G is a partial order $\leq$ such that for all $a, b, c \in G$, $a \leq b$ if and only if $ca \leq cb$.*

Every left-invariant partial order on a group arises naturally from a positive cone P by saying that $a \leq b$ if and only if $a^{-1}b \in P$. We will say that an ordered group *(G, P) is a group along with a positive cone, with the implication that the group takes on a left-invariant partial order in this way.*

A group is totally ordered *if its partial order is a total order. This is equivalent to the condition that $P \cup P^{-1} = G$.*

Although our theory of reductions does not require any additional structure on our ordered groups, the relators for a graph algebra will ask for an additional property on our orderings from [10, Chapter 32]:

Definition 2.1.2. *We say (G, P) is <u>weakly quasi-lattice ordered</u> (WQLO) if whenever $x, y \in P$ have a common upper bound in P, they have a supremum (least common upper bound), denoted $x \vee y$. Note that any upper bound of a positive element is itself positive by transitivity.*

While many mathematicians (including[3, 20, 22]) worked in the context of quasi-lattice ordered groups, we have found that the slightly more general condition of weak quasi-lattice order suffices. The condition of weak quasi-lattice order is also friendlier than quasi-lattice order since it is entirely a property of the submonoid P, and therefore one can "forget" the ambient group. The same is not true for quasi-lattice order. For a more thorough introduction to quasi-lattice order, weak quasi-lattice order, and their connections, the reader is directed to [10, Chapter 32].

Notation 2.1.3. *In this work, we will make use of the following notation:*

- *$\mathbb{N} = \{0, 1, 2, 3...\}$ includes 0.*

- *Given a group G and subset $S \subseteq G$, $\langle S \rangle$ will denote the group generated by the elements of S, and S^* will denote the monoid generated by the elements of S.*

- *The identity element in a multiplicative group G will be denoted by 1_G, or 1 if the group is clear from context. Less commonly we will denote the identity element by e.*

Example 2.1.4. *Let $G = \mathbb{Z}^k$, and let $P = \mathbb{N}^k$. Then (G, P) is WQLO. Every pair of elements $p = (p_1, ..., p_k)$ and $q = (q_1, ..., q_k)$ have a supremum $p \vee q =$*

$(\max(p_1, q_1), ..., \max(p_k, q_k))$. *(In fact, this ordering is a "lattice order", meaning any two elements have both a supremum and an infimum.)*

Let $G = F_k = \langle a_1, ..., a_k \rangle$ be the free group on k generators, and let $P = \{a_1, ..., a_k\}^*$ be the free monoid generated by the generators of G. Then (G, P) is WQLO, since any $p, q \in P$ have a common upper bound if and only if $p \leq q$ or $q \leq p$, in which case $p \vee q = \max\{p, q\}$.

Notation 2.1.5. *Let (G, P) be an ordered group. We use the following standard notation for intervals: for $a, b \in G$, write*

$$[a, b] := \{g \in G | a \leq g \leq b\}.$$

It is immediate that such an interval is nonempty if and only if $a \leq b$. We will most often be interested in intervals of the form $[1, p]$ for some $p \in P$.

Definition 2.1.6. *Let (G, P) and (H, Q) be ordered groups. We say that $\varphi : G \to H$ is an <u>order homomorphism</u> if it is a group homomorphism with $\varphi(P) \subseteq Q$. We will write $\varphi : (G, P) \to (H, Q)$ to denote an order homomorphism.*

Order homomorphisms preserve both the group structure and the order structure, as the following lemma shows:

Lemma 2.1.7. *Let (G, P) and (H, Q) be ordered groups, and $\varphi : G \to H$ a group homomorphism. Then φ is an order homomorphism if and only if $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ for all $x, y \in G$.*

In particular, if φ is an order homomorphism and $p \in P$, then $\varphi([1, p]) \subseteq [1, \varphi(p)]$.

Proof. If φ is an order homomorphism and $x \leq y$, then $x^{-1}y \in P$, so $\varphi(x^{-1}y) = \varphi(x)^{-1}\varphi(y) \in Q$. Thus $\varphi(x) \leq \varphi(y)$.

Conversely, if φ is not an order homomorphism, then there is $p \in P$ such that $\varphi(p) \notin Q$. Then $1_G \leq p$, and $1_H = \varphi(1_G) \not\leq \varphi(p)$, as desired.

For the "in particular", if $s \in [1, p]$, then $1 \leq s \leq p$, so $1 = \varphi(1) \leq \varphi(s) \leq \varphi(p)$, so $\varphi(s) \in [1, \varphi(p)]$ as desired. $\qquad\square$

2.2 C^*-algebras

2.2.1 The Factors Through Theorem

The following result is an elementary notion from algebra, which we state here for completeness since we will make frequent use of it:

Lemma 2.2.1 (Factors Through Theorem)**.** *Let $A, B,$ and C be sets, let $f : A \to B$ and $g : A \to C$ be functions, and suppose that f is surjective.*

$$A \xrightarrow{\;f\;} B$$
$$\downarrow{\scriptstyle g} \quad \diagdown{\scriptstyle \exists h}$$
$$C$$

1. *Suppose that whenever $f(a) = f(a')$, then $g(a) = g(a')$. Then there is a "bonding map" $h : B \to C$ such that $h \circ f = g$. In particular, h is (well-)defined by $h(f(a)) = g(a)$ for all $f(a) \in B$.*

2. *If $A, B,$ and C are groups, f and g are group homomorphisms, and $\ker f \subseteq \ker g$, then hypothesis of (1) that $f(a) = f(a')$ implies $g(a) = g(a')$ is satisfied. Therefore, the conclusion of (1) is true.*

3. *If A, B, C have a binary operation $\cdot$ (respectively a unary operation *) which is preserved by f and g, then $\cdot$ (respectively *) is also preserved by h.*

Proof. For (1), for all $b \in B$, since f is surjective, there is some $a \in A$ such that $f(a) = b$. Now, define $h(b) = g(a)$, which is well-defined since if there were some

other $a' \in A$ with $f(a') = b$, then $g(a) = g(a')$ by hypothesis. That is, we have defined h by $h(f(a)) = g(a)$, so $h \circ f = g$.

For (2), if $f(a) = f(a')$, then $a^{-1}a' \in \ker f \subseteq \ker g$, so $g(a) = g(a')$, as desired.

For (3), let $b_1, b_2 \in B$. Since f is surjective, there is some $a_1, a_2 \in A$ such that $f(a_1) = b_1, f(a_2) = b_2$. Then

$$
\begin{aligned}
h(b_1 \cdot b_2) &= h(f(a_1) \cdot f(a_2)) = h(f(a_1 \cdot a_2)) = g(a_1 \cdot a_2) \\
&= g(a_1) \cdot g(a_2) = h(f(a_1)) \cdot h(f(a_2)) = h(b_1) \cdot h(b_2)
\end{aligned}
$$

and

$$
h(b_1^*) = h(f(a_1)^*) = h(f(a_1^*)) = g(a_1^*) = g(a_1)^* = h(f(a_1))^* = h(b_1)^*
$$

as desired.

$\square$

2.2.2 Universal Algebras

The following construction of a "universal algebra" is a slight simplification of the one outlined by Blackadar in [2]. We state it here for completeness.

Lemma 2.2.2 (Construction of a Universal Algebra, [2]). *Let $\mathcal{G} = \{x_\alpha\}_{\alpha \in I}$ denote a set of formal symbols, $\mathcal{G}^* = \{x_\alpha^*\}_{\alpha \in I}$ another set of formal symbols over the same indexing set, and $\mathcal{R}$ a set of relators of the form $p(x_{\alpha_1}, ..., x_{\alpha_n}, x_{\alpha_1}^*, ..., x_{\alpha_n}^*) = 0$ where p is some polynomial in $2n$ non-commuting variables and complex coefficients.*

We define a <u>representation</u> of $(\mathcal{G}, \mathcal{R})$ to be a collection of elements $\{y_\alpha\}_{\alpha \in I}$ in a C^*-algebra that satisfy $p(y_{\alpha_1}, ..., y_{\alpha_n}, y_{\alpha_1}^*, ..., y_{\alpha_n}^*) = 0$ for each relator in $\mathcal{R}$.

Suppose that the relators on $(\mathcal{G}, \mathcal{R})$ imply that in any representation, the images of the generators are partial isometries. Then there is a representation $\{z_\alpha\}_{\alpha \in I}$ such that for any other representation $\{y_\alpha\}_{\alpha \in I}$ there is a surjective $*$-homomorphism $\pi :$ $C^*(\{z_\alpha\}_{\alpha \in I}) \to C^*(\{y_\alpha\}_{\alpha \in I})$ satisfying $z_\alpha \mapsto y_\alpha$.

The existence of this surjective $*$-homomorphism is called the <u>universal property</u> of $C^*(\{z_\alpha\}_{\alpha \in I})$. We call this representation the <u>universal representation</u> and say that $C^*(\{z_\alpha\}_{\alpha \in I})$ is the <u>universal algebra</u> for $(\mathcal{G}, \mathcal{R})$.

Proof. Let $\mathcal{F}(\mathcal{G})$ denote the free $*$-algebra over $\mathbb{C}$ generated by $\mathbb{C}$ and $\mathcal{G} \cup \mathcal{G}^*$ where the $*$-map is given by $(x_\alpha)^* = x_\alpha^*$ and $(x_\alpha^*)^* = x_\alpha$ and extended to the entire algebra via conjugate-linearity and $(ab)^* = b^* a^*$. Note that any representation $\varphi : x_\alpha \mapsto y_\alpha$ of $(\mathcal{G}, \mathcal{R})$ extends uniquely to a $*$-algebra homomorphism $\bar{\varphi} : \mathcal{F}(\mathcal{G}) \to C^*(\{y_\alpha\}_{\alpha \in I})$.

For $X \in \mathcal{F}(\mathcal{G})$, let $\|\|X\|\| = \sup\{\|\bar{\varphi}(X)\| : \varphi$ a rep. of $(\mathcal{G}, \mathcal{R})\}$. Note that since $(\mathcal{G}, \mathcal{R})$ can be represented by the zero map, this supremum is bounded below by 0, and since each generator must be represented by a partial isometry, the image of each generator has norm 1, and thus the supremum is bounded above by the sum of the absolute values of its $\mathbb{C}$-coefficients. Thus $\|\|X\|\|$ is a well-defined, non-negative finite number.

Let $J = \{X \in \mathcal{F}(\mathcal{G}) : \|\|X\|\| = 0\}$, and note that $J = \bigcap_{\varphi \text{ a rep.}} \ker \bar{\varphi}$ where the intersection is taken over all representations. Thus J is the intersection of $*$-ideals, so it is a $*$-ideal. Now let $\psi : \mathcal{F}(\mathcal{G}) \to \mathcal{F}(\mathcal{G})/J$ denote the quotient map, and for all α, let $z_\alpha = \psi(x_\alpha)$. For any relator $p(x_{\alpha_1}, ..., x_{\alpha_n}, x_{\alpha_1}^*, ..., x_{\alpha_n}^*) = 0 \in \mathcal{R}$, since this relator is satisfied for every representation, then $\|\|p(x_{\alpha_1}, ..., x_{\alpha_n}, x_{\alpha_1}^*, ..., x_{\alpha_n}^*)\|\| = 0$, and thus $p(x_{\alpha_1}, ..., x_{\alpha_n}, x_{\alpha_1}^*, ..., x_{\alpha_n}^*) \in J$, so $p(z_{\alpha_1}, ..., z_{\alpha_n}, z_{\alpha_1}^*, ..., z_{\alpha_n}^*) = 0$. That is, the $\{z_\alpha\}_{\alpha \in I}$ satisfy the relators.

Now note that $\|\|\cdot\|\|$ is a C^*-seminorm, so the norm $\|X + J\| := \|\|X\|\|$ is a well-

defined C^*-norm on $\mathcal{F}(\mathcal{G})/J$, and thus the completion of $\mathcal{F}(\mathcal{G})/J$ with respect to this norm is a C^*-algebra. Thus $\psi : x_\alpha \to z_\alpha$ is a representation of $(\mathcal{G}, \mathcal{R})$.

Now, given any other presentation $\varphi : x_\alpha \mapsto y_\alpha$, note that $\ker \bar\varphi = \{X \in \mathcal{F}(\mathcal{G}) : \|\bar\varphi(X)\| = 0\} \subseteq \{X \in \mathcal{F}(\mathcal{G}) : \||X|\| = 0\} = \ker \bar\psi$, so by Lemma 2.2.1(2), there is a function on the $*$-algebras generated by y and z given by $y_\alpha \mapsto z_\alpha$, and by Lemma 2.2.1(3) applied to $+, \times,$ and *, this function is a $*$-homomorphism, so it extends to a $*$-homomorphism $\pi : C^*(\{z_\alpha\}_{\alpha \in I}) \to C^*(\{y_\alpha\}_{\alpha \in I})$ as desired. $\square$

2.2.3 Conditional Expectations

We will also make frequent use of conditional expectations, whose definition we include here. For a more thorough treatment of conditional expectations, the reader is directed to [23, Chapter III.3], where they are called projections of norm one.

Definition 2.2.3. *Let C be a C^*-algebra, and $D \subseteq C$ a C^*-subalgebra. We say a conditional expectation or projection of norm one is a linear map $\Phi : C \to D$ such that:*

- *Φ is contractive, meaning $\|\Phi(c)\| \leq \|c\|$ for all $c \in C$ (and is thus continuous).*

- *Φ is idempotent, meaning $\Phi \circ \Phi = \Phi$.*

- *$\Phi(d) = d$ for all $d \in D$.*

If additionally $c > 0$ implies $\Phi(c) > 0$, we say Φ is faithful.

Example 2.2.4. *In a matrix algebra M_n, there is a faithful conditional expectation Φ given by "restricting to the diagonal", meaning $\Phi([a_{ij}]) = [b_{ij}]$ where*

$$b_{ij} = \begin{cases} a_{ij} & if\ i = j \\ 0 & if\ i \neq j \end{cases}.$$

The following result is [24, Theorem 1] and is known as Tomiyama's Theorem. It says that a conditional expectation has some additional properties "for free". Some authors will define conditional expectations as requiring these properties, and some will define them to only require the shorter list of conditions (as we have done here).

Theorem 2.2.5 (Tomiyama's Theorem)**.** *If* $\Phi : C \to D$ *is a conditional expectation, then* Φ *is positive, meaning that* $x \geq 0$ *implies* $\Phi(x) \geq 0$. *Furthermore,* Φ *is a* D-*bimodule map, meaning* $\Phi(d_1 c d_2) = d_1 \Phi(c) d_2$ *for all* $d_1, d_2 \in D$ *and* $c \in C$.

2.2.4 Tensor Products

Here we will give a very brief introduction to the minimal tensor product. A more thorough introduction can be found in [23, Chapter 4]. Throughout this paper, as in much of the coaction literature, we will use unadorned $\otimes$ for the minimal tensor product of C^*-algebras.

Definition 2.2.6. *Given two vector spaces* V *and* W, *we let* $V \odot W$ *denote the algebraic tensor product of* V *and* W, *meaning the* $\mathbb{C}$-*vector space spanned by formal symbols* $\{v \otimes w : v \in V, w \in W\}$ *with the relations of linearity over either coordinate.*

Let A *and* B *be* C^*-*algebras. Then one can give* $A \odot B$ *a* $*$-*algebra structure by*

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb') \text{ and } (a \otimes b)^* = a^* \otimes b^*.$$

If $\pi : A \to \mathcal{B}(H)$ *and* $\rho : B \to \mathcal{B}(\mathcal{K})$ *are representations of* A *and* B *on Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively, then there exists a* $*$-*algebra representation* $\pi \otimes \rho :$ $A \odot B \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ *satisfying* $[(\pi \otimes \rho)(a \otimes b)](h \otimes k) = \pi(a)h \otimes \rho(b)k$ *for all* $a \in$ $A, b \in B, h \in \mathcal{H}, k \in \mathcal{K}$. *The* minimal norm *on* $A \odot B$ *is defined by*

$$\left\| \sum_{i=1}^{n} a_i \otimes b_i \right\|_{min} = \sup \left\{ \left\| \sum_{i=1}^{n} \pi(a_i) \otimes \rho(b_i) \right\| \right\}$$

where the supremum is taken over all representations π and ρ of A and B, respectively (and this is indeed a norm which is minimal in an appropriate sense). Note that $\|a \otimes b\|_{min} = \|a\| \cdot \|b\|$ (that is, $\|\cdot\|_{min}$ is a "C^*-cross norm"). The <u>minimal tensor product</u> $A \otimes_{min} B$ is the C^*-algebra created by completing $A \odot B$ with respect to this norm. Since all of our tensor products will be minimal, we will henceforth write $\|\cdot\|$ for $\|\cdot\|_{min}$ and $A \otimes B$ for $A \otimes_{min} B$.

The following result will be used occasionally, and it is a minor variant of [6, Lemma A.1]:

Lemma 2.2.7. *Let $\varphi : A \to C$ and $\psi : B \to D$ be $*$-homomorphisms of C^*-algebras. Then there is a homomorphism $\varphi \otimes \psi : A \otimes B \to C \otimes D$ satisfying $(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \otimes \psi(b)$. If φ and ψ are nondegenerate (respectively, faithful), then so is $\varphi \otimes \psi$.*

Proof. Consider C and D as subalgebras (respectively) of their multiplier algebras $M(C)$ and $M(D)$, and therefore consider φ and ψ as mapping into $M(C)$ and $M(D)$, respectively. By [6, Lemma A.1], there is a $*$-homomorphism $\varphi \otimes \psi : A \otimes B \to M(C \otimes D)$ satisfying $(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \otimes \psi(b)$, and if φ and ψ are nondegenerate (respectively, faithful), then so is $\varphi \otimes \psi$.

It then suffices to show that $\varphi \otimes \psi$ maps into $C \otimes D$ properly, instead of mapping into $M(C \otimes D)$. But for all $a \in A, b \in B$, we have $\varphi(a) \in C$ and $\varphi(b) \in D$, so $(\varphi \otimes \psi)(a \otimes b) \in C \otimes D$ for all $a \in A, b \in B$. Taking closed spans, we get that for any $x \in A \otimes B$, we have $(\varphi \otimes \psi)(x) \in C \times D$, as desired. $\square$

2.3 Connections between Groups and C^*-Algebras

In this section we will remind the reader of some of the many connections between groups and C^*-algebras, including group C^*-algebras, amenability, Fell bundles, gradings, coactions, and actions. Our context is relatively simple since we are limiting our attention to discrete groups.

2.3.1 Group C^*-algebras and Amenable Groups

The following discussion of group C^*-algebras and amenability is adapted from [5, Chapter VII], to which the reader is directed for a more thorough treatment.

Definition 2.3.1. *Let G be a discrete group. Then a <u>representation</u> of G in a C^*-algebra is a collection of unitary operators $\{u_g\}_{g \in G}$ such that $u_g u_h = u_{gh}$ and $u_g^* = u_{g^{-1}}$ for all $g, h \in G$.*

Since these relators are all polynomials, then by Lemma 2.2.2 there is a representation of G which is universal for all representations. We denote it by $\{U_g\}_{g \in G}$, and call $C^*(G) := C^*(\{U_g\}_{g \in G})$ the <u>full group C^*-algebra</u>.

A particularly nice representation of a group G, called the <u>left-regular representation of a group</u>, comes from the natural action of G multiplying on itself. In particular, let $\{e_g\}_{g \in G}$ denote the standard orthonormal basis for $\ell^2(G)$, and define an operator L_g by $L_g e_{g'} = e_{gg'}$ for $g, g' \in G$. Then $\{L_g\}_{g \in G}$ is a representation of G in $\mathcal{B}(\ell^2(G))$. We call $C_r^*(G) := C^*(\{L_g\}_{g \in G})$ the <u>reduced group C^*-algebra</u>.

The left-regular representation is more commonly denoted by λ instead of L, but our choice of notation will help avoid ambiguity with paths $\lambda \in \Lambda$ later in the text.

Note that $C^*(G)$ and $C_r^*(G)$ are the closed spans of $\{U_g\}_{g \in G}$ and $\{L_g\}_{g \in G}$, respectively.

Famously, there are many equivalent definitions of amenability, but for our purposes, this one is the most convenient:

Definition 2.3.2. *By the universal property of $C^*(G)$, there is a surjective $*$-homomorphism $\pi_L^U : C^*(G) \to C_r^*(G)$ given by $U_g \mapsto L_g$. We say that G is* <u>amenable</u> *if and only if π_L^U is injective (and hence an isomorphism).*

Readers more familiar with another definition of amenability may wish to read [5, Theorem VII.2.5] which proves the equivalence of this definition with a more common one.

The following remark summarizes some of the well-known results about amenability of groups.

Remark 2.3.3. *By [5, Proposition VII.2.3], for discrete groups amenability is preserved under:*

1. *Subgroups*

2. *Quotients*

3. *Direct Limits*

4. *Extensions (meaning that if $1 \to N \to G \to H \to 1$ is a short exact sequence of groups, with N and H amenable, then G is amenable).*

5. *Finite direct products (which is immediate from being closed under extensions).*

Furthermore, by [5, Proposition VII.2.2], every abelian group is amenable. Every finite group is amenable since $C^*(G)$ and $C_r^*(G)$ have the same finite dimension $|G|$, so by the rank-nullity theorem the surjective map π_L^U is injective.

Notably, the free group on two generators is not amenable by [5, Example VII.2.4]. By the subgroup property, any group containing a free group on two (or more) generators is also not amenable.

2.3.2 Fell Bundles, Topologically Graded C^*-algebras, and Coactions

In this section, we will give a short introduction to Fell bundles, topologically graded C^*-algebras, and coactions, and show some of the ways these closely-related structures overlap. For a more detailed treatment, the reader is directed to [10, 9] for Fell bundles, [10] for topological gradings, and [6, Appendix A] for coactions.

Throughout this section, all of our groups will be discrete, which simplifies some definitions. The following is from [10, Definition 16.1]:

Definition 2.3.4. *Let G be a discrete group. Let $\mathcal{B} = \{B_g\}_{g \in G}$ be a collection of Banach spaces, and write $\mathscr{B}$ for the disjoint union of the $\{B_g\}_{g \in G}$, called the <u>total space</u>. Suppose $\mathscr{B}$ has a binary operation $\cdot$ called multiplication, and an involution $*$ which satisfy the following properties for all $g, h \in G$ and $b, c \in \mathscr{B}$:*

a. $B_g B_h \subseteq B_{gh}$,

b. *Multiplication is bilinear from $B_g \times B_h$ to B_{gh},*

c. *Multiplication on $\mathscr{B}$ is associative,*

d. $\|bc\| \leq \|b\| \cdot \|c\|$,

e. $(B_g)^* \subseteq B_{g^{-1}}$,

f. *Involution is conjugate-linear from B_g to $B_{g^{-1}}$,*

g. $(bc)^* = c^* b^*$,

h. $b^{**} = b$,

i. $\|b^*\| = \|b\|$,

j. $\|b^*b\| = \|b\|^2$,

k. $b^*b \geq 0$ *in* B_1.

Then we say that $\mathcal{B}$ is a <u>Fell bundle</u> over G. We call each B_g a <u>fiber</u>.

The following is from [10, Definitions 16.2 and 19.2]:

Definition 2.3.5. *Let A be a C^*-algebra, and let G be a (discrete) group. We say that a <u>(C^*-) grading</u> for A is a collection $\{A_g\}_{g \in G}$ of linearly independent closed subspaces such that $\bigoplus_{g \in G} A_g$ is dense in A, $A_g A_h \subseteq A_{gh}$, and $A_g^* \subseteq A_{g^{-1}}$ for $g, h \in G$. Each A_g is called a <u>graded subspace</u> or <u>graded component</u>.*

If there is also a conditional expectation $\Phi : A \to A_e$ satisfying

$$\Phi(a) = \begin{cases} a & \text{if } a \in A_e \\ 0 & \text{if } a \in A_g \text{ for } g \neq e \end{cases},$$

then we say that $\{A_g\}_{g \in G}$ is a <u>topological grading</u>.

Remark 2.3.6. *It is straightforward to verify that if A is a topologically graded C^*-algebra, then its graded components form a Fell bundle. Most of the Fell bundles we will use arise in this way.*

Remark 2.3.7. *Given a Fell bundle $\mathcal{B}$, one can construct a "reduced" and "full" cross sectional algebra representing this bundle, respectively denoted $C_r^*(\mathcal{B})$ and $C^*(\mathcal{B})$ (see Definition 2.3 of [9] and the comment following it).*

By [9, Theorem 3.3], if B is a topologically graded C^*-algebra, and $\mathcal{B}$ denotes its associated Fell bundle, then there is a $*$-homomorphism $L : B \to C_r^*(\mathcal{B})$ called

the left-regular representation of the Fell bundle. Combining this result with their Proposition 3.7, L is an isomorphism if and only if the conditional expectation from the topological grading is faithful.

In a Fell bundle $\mathcal{B}$, the full cross sectional algebra $C^*(\mathcal{B})$ has a topological grading, and the bundle is called amenable if its left-regular representation $L : C^*(\mathcal{B}) \to C_r^*(\mathcal{B})$ is injective (and hence an isomorphism). The reader may notice the parallel with our definition of amenability for a group (Definition 2.3.2).

The following two results are respectively [9, Theorem 4.7] and [9, Proposition 4.2] :

Lemma 2.3.8. *Let G be a discrete amenable group. Then every Fell bundle over G is amenable.*

Lemma 2.3.9. *If $\mathcal{B}$ is an amenable Fell bundle, then all topologically graded C^*-algebras whose associated Fell bundles coincide with $\mathcal{B}$ are isomorphic to each other.*

Finally, we will give a brief introduction to coactions. The following is both the simplest example of a coaction, and a necessary component of its definition.

Example 2.3.10. *If G is a discrete group, then the operators $\{U_g \otimes U_g\}_{g \in G} \subset C^*(G) \otimes C^*(G)$ are a representation of G. Therefore, by the universal property of $C^*(G)$, there is a $*$-homomorphism $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ given by $U_g \mapsto U_g \otimes U_g$.*

Note that letting $\mathrm{id}_G := \mathrm{id}_{C^*(G)}$, then $(\delta_G \otimes \mathrm{id}_G) \circ \delta_G = (\mathrm{id}_G \otimes \delta_G) \circ \delta_G$, which can be easily verified by checking that both send $U_g \mapsto U_g \otimes U_g \otimes U_g \in C^*(G) \otimes C^*(G) \otimes C^*(G)$.

The following definition of coactions for discrete groups is taken from [7, Section 2]:

Definition 2.3.11. *Let G be a discrete group. A <u>coaction</u> of G on a C^*-algebra A is an injective, nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$ satisfying the <u>coaction identity</u> that $(\delta \otimes \mathrm{id}_G) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta$ as maps from A into $A \otimes C^*(G) \otimes C^*(G)$, summarized in this diagram:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \delta\ \ } & A \otimes C^*(G) \\
\Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta \otimes \mathrm{id}_G} \\
A \otimes C^*(G) & \xrightarrow{\ \mathrm{id}_A \otimes \delta_G\ } & A \otimes C^*(G) \otimes C^*(G)
\end{array}
$$

In this context, nondegeneracy means that $\overline{\mathrm{span}}\,[\delta(A)(A \otimes C^*(G))] = A \otimes C^*(G)$. We call the triple (A, G, δ) a <u>cosystem</u>.

If G is discrete, a cosystem has a nice topological grading as described in [20, Proposition A.3]:

Lemma 2.3.12. *Let (A, G, δ) be a discrete cosystem, and for $g \in G$ let*

$$
A_g = \{a \in A \mid \delta(a) = a \otimes U_g\}.
$$

Then the collection $\{A_g\}_{g \in G}$ is a topological grading of A. We will write $\Phi_A : A \to A_e$ for the conditional expectation.

Some coactions are particularly nice, having a condition called normality. There are many definitions of normality, but we will find a definition depending on this conditional expectation to be the most convenient:

Definition 2.3.13. *We say a discrete coaction (A, G, δ) is <u>normal</u> if the conditional expectation Φ_A is faithful.*

A more common definition of normality of a coaction (such as [6, Definition A.50] or the comments preceding Definition 1.1 of [17]) is that the coaction is normal if and

only if the map $j_A := (\text{id}_A \otimes \pi_L^U) \circ \delta$ is injective. Our definition is proved equivalent to the more common one in [17, Lemma 1.4].

For a coaction over a discrete amenable group, this property is automatic:

Lemma 2.3.14. *Let (A, G, δ) be a cosystem. If G is amenable and discrete, then δ is normal.*

Proof. For a cosystem (A, G, δ) over an amenable group, by Lemma 2.3.12 there is a topological grading $\{A_g\}_{g \in G}$ of A. By Remark 2.3.6, these graded components form a Fell bundle $\mathcal{B}$, and this Fell bundle is amenable by Lemma 2.3.8.

Since $\mathcal{B}$ is amenable, by Lemma 2.3.9, $A \cong C_r^*(\mathcal{B})$. But for any Fell bundle, $C_r^*(\mathcal{B})$ has a conditional expectation by [9, Proposition 2.9], which is faithful by [9, Proposition 2.12]. □

Finally, we will show that coactions, topological gradings, and Fell bundles are equivalent over amenable groups.

Lemma 2.3.15. *Let G be an amenable group, and let A be a C^*-algebra. Then the following are equivalent:*

1. *A has a coaction by G.*

2. *A has a topological grading $\{A_g\}_{g \in G}$.*

3. *There is a Fell bundle $\mathcal{B} = \{B_g\}_{g \in G}$ whose fibers are linearly independent closed subspaces of A such that $A \cong C^*(\mathcal{B})$.*

And under these conditions, the structures coincide in the sense that the fibers $\{B_g\}_{g \in G}$ of the Fell bundle are equal to the graded components $\{A_g\}_{g \in G}$, and the graded components arise from the coaction via $A_g = \{a \in A : \delta(a) = a \otimes U_g\}$.

Proof. By Lemma 2.3.12, (1) implies (2).

To show that (2) implies (3), suppose that A has a topological grading $\{A_g\}_{g \in G}$. By Remark 2.3.6, these fibers form a Fell bundle. To see that the fibers are linearly independent, fix a finite sum $\sum_{i=1}^{n} a_{g_i}$ where each $a_{g_i} \in A_{g_i}$ for some distinct $g_i \in G$, and suppose this sum equals 0. Then by [10, Corollary 19.6], there are contractive linear maps $F_g : A \to A_g$ satisfying $F_{g_j}(\sum_{i=1}^{n} a_{g_i}) = a_{g_j}$. But since $0 = \sum_{i=1}^{n} a_{g_i}$, then for each g_j, we have $0 = F_{g_j}(\sum_{i=1}^{n} a_{g_i}) = a_{g_j}$, so each summand is 0, so the graded components are linearly independent. Finally, we must show that $A \cong C^*(\mathcal{B})$. By Lemma 2.3.8, $\mathcal{B}$ is amenable, and by [10, Proposition 19.3], $C^*(\mathcal{B})$ is a topologically graded C^*-algebra. Since this amenable bundle coincides with the bundle in A, then by Lemma 2.3.9, $A \cong C^*(\mathcal{B})$, as desired.

To see that (3) implies (1), suppose that $\mathcal{B} = \{B_g\}_{g \in G}$ is a Fell bundle of linearly independent closed subspaces of A such that $A \cong C^*(\mathcal{B})$. Since G is amenable, $C^*(G) \cong C_r^*(G)$, and by Lemma 2.3.8, $\mathcal{B}$ is an amenable Fell bundle, so $C^*(\mathcal{B}) \cong C_r^*(\mathcal{B})$. Combining these simplifications with [10, Proposition 18.7], there is an injective $*$-homomorphism $\delta : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*(G)$ satisfying $\delta(b_g) = b_g \otimes U_g$ for all $g \in G$ and $b_g \in B_g$. It is routine to verify that this is a coaction, and that the graded parts of this coaction coincide with the original Fell bundle $\mathcal{B}$.

$\square$

2.3.3 Coactions as Actions of the Dual Group

In this section we will show that if G is discrete and abelian, then a coaction by G is equivalent to an action by its dual group $\widehat{G}$. For a non-abelian group, this correspondence breaks down because there is no dual group.

First, we will remind the reader of actions by groups and dual groups. The

following definition is well-known, and can be found for instance in the remarks preceding Proposition 2.1 in [18]:

Definition 2.3.16. *Given a locally compact group G, an <u>action</u> of G on a C^*-algebra A is a strongly continuous homomorphism $\alpha : G \to \mathrm{Aut}(A)$, the group of automorphisms of A. (Here, strong continuity means that for all $a \in A$, the map $g \mapsto \alpha_g(a)$ is continuous as a function from G to A.) The trio (A, G, α) is called a <u>(C^*-dynamical) system</u>.*

We will now define the dual group. A more detailed treatment of dual groups can be found in [12, Chapter 1.7].

Definition 2.3.17. *Given a abelian locally compact group G, we say that a <u>character</u> of G is a continuous homomorphism $\chi : G \to \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle in the complex numbers. The set of characters $\widehat{G}$ forms an abelian locally compact group called the <u>dual group</u> under the operation $(\chi_1 * \chi_2)(g) := \chi_1(g)\chi_2(g)$ and the topology of uniform convergence on compacta.*

Some examples of dual groups are that $\widehat{\mathbb{Z}^k} \cong \mathbb{T}^k$, $\widehat{\mathbb{R}^k} \cong \mathbb{R}^k$, and that if G is finite and abelian then $\widehat{G} \cong G$.

The following two theorems are well-known results about the duals of abelian locally compact groups. They can be found in [12, Theorem 1.85] and [12, Theorem 1.88] respectively.

Theorem 2.3.18 (Pontryagin duality theorem)**.** *An abelian locally compact group is naturally isomorphic to its double dual by the map $g \mapsto [\chi \mapsto \chi(g)]$. That is, $G \cong \widehat{\widehat{G}}$.*

Theorem 2.3.19. *If G is a locally compact abelian group, then G is discrete if and only if $\widehat{G}$ is compact.*

We are now ready to prove the equivalence between coactions and actions by a dual group.

Lemma 2.3.20. *Let G be a discrete abelian group and let A be a C^*-algebra. Then for every C^*-dynamical system $(A, \widehat{G}, \alpha)$, there is a unique cosystem (A, G, δ) which satisfies*

$$A_g = \{a \in A : \alpha_\chi(a) = \chi(g) \cdot a \text{ for all } \chi \in \widehat{G}\}$$

for all $g \in G$. All cosystems by discrete abelian groups arise in this way.

Proof. Since G is abelian, it is amenable. Thus by Lemma 2.3.15, a cosystem is equivalent to a topological grading whose graded components are the $\{A_g\}_{g \in G}$. By [19, Theorem 3], such a topological grading is equivalent to the desired group action. $\square$

2.4   Graphs and P-graphs

Definition 2.4.1. *Let (G, P) be an ordered group, and consider P as a category with one object where the morphisms are the elements of P under their multiplication structure. A <u>P-graph</u> is a countable category Λ along with a functor $d : \Lambda \to P$ with the <u>unique factorization property</u>: if $\lambda \in \Lambda$ and $p_1, p_2 \in P$ with $p_1 p_2 = d(\lambda)$, then there exists unique $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda = \lambda_1 \lambda_2$ and $d(\lambda_1) = p_1$ (and hence necessarily $d(\lambda_2) = p_2$).*

We refer to the morphisms of Λ as <u>paths</u>, and identity morphisms in Λ as <u>vertices</u>. We let Λ^0 denote the set of vertices in Λ, and for $p \in P$, we write $\Lambda^p = \{\mu \in \Lambda : d(\mu) = p\}$. As is common in the literature, we identify the objects in Λ with the identity morphism at those objects, so we will refer to an object v with its identity morphism id_v interchangeably. Given a path $\lambda \in \Lambda$, let $r(\lambda)$ denote its range vertex

and $s(\lambda)$ its source vertex. We write the composition of paths "working backwards" so that given $\alpha, \beta \in \Lambda$, the product $\alpha\beta$ is defined if and only if $s(\alpha) = r(\beta)$, in which case $s(\alpha\beta) = s(\beta)$ and $r(\alpha\beta) = r(\alpha)$. This can be summarized in the following diagram:

$$
\begin{array}{c}
r(\alpha) \\
\alpha \Big\uparrow \quad \overset{\alpha\beta}{\nwarrow\text{-}\text{-}\text{-}\text{-}} \\
s(\alpha) = r(\beta) \xleftarrow{\;\;\beta\;\;} s(\beta)
\end{array}
$$

Remark 2.4.2. *The existence of the degree functor and the unique factorization property gives several nice properties to the graph.*

First, $\Lambda^0 = \{\lambda \in \Lambda : d(\lambda) = 1_G\}$, since for a vertex v, we have $v^2 = v$, so $d(v)^2 = d(v)$, so $d(v) = 1_G$, and conversely if $d(v) = 1_G$, then $r(v)v = v = vs(v)$ are two factorizations with the same degrees, so $r(v) = v = s(v)$ and hence v is a vertex.

Second, having $\alpha\beta = s(\beta)$ implies $\alpha = \beta = s(\beta)$, since then $d(\alpha)d(\beta) = d(s(\beta)) = 1$, so $d(\alpha), d(\beta) \in P \cap P^{-1} = \{1\}$.

Finally, the category has both left- and right-cancellation. For left cancellation, if $\alpha\beta = \alpha\gamma$, then this provides two factorizations, so by the uniqueness of the factorizations, we have $\beta = \gamma$, and a nearly identical argument shows right-cancellation.

These properties together imply that a P-graph is a <u>category of paths</u> in the sense of [22]. Our representation of a P-graph will be a special case of their representation of a category of paths, and the reader is invited to compare our Definitions 2.5.2 and 2.5.5c to Theorems 6.3 and 8.2 of [22], respectively.

Definition 2.4.3. *Let (G, P) be an ordered group, let Λ be a P-graph. For $\alpha \in \Lambda$, we write $\alpha\Lambda = \{\alpha\mu : \mu \in \Lambda\}$.*

We can give Λ a partial order $\leq$ by saying $\alpha \leq \beta$ if there exists some $\alpha_1 \in \Lambda$ such that $\alpha\alpha_1 = \beta$. That is, $\alpha \leq \beta$ if and only if $\beta \in \alpha\Lambda$.

We say that α and β have a _common extension_ if there is a $\mu \in \Lambda$ such that $\alpha \leq \mu$ and $\beta \leq \mu$.

Given $\alpha, \beta \in \Lambda$, we say that $\mu \in \Lambda$ is a _minimal common extension_ of α and β if μ is a common extension of α and β, and for all other common extensions ν, $\nu \leq \mu$ implies $\nu = \mu$. We let $MCE(\alpha, \beta)$ denote the set of minimal common extensions of α and β.

For general ordered groups, the ordering on Λ may be sufficiently poorly behaved that there are no _minimal_ common extensions, even if there are common extensions. However, the humble hypothesis of weak quasi-lattice ordering on (G, P) prevents this catastrophe, and therefore will recur as a hypothesis in almost all results relating to P-graphs. The following lemma is the main "entry-point" for the hypothesis of weak quasi-lattice ordering into P-graphs:

Lemma 2.4.4. _Let (G, P) be an ordered group, let Λ be a P-graph, and $\alpha, \beta \in \Lambda$. If (G, P) is weakly quasi-lattice ordered (WQLO), then:_

1. _Let $MDCE(\alpha, \beta) = \{\mu \in \alpha\Lambda \cap \beta\Lambda : d(\mu) = d(\alpha) \vee d(\beta)\}$ denote the set of minimal degree common extensions. For every common extension λ of α and β, there is a $\mu \in MDCE(\alpha, \beta)$ with $\mu \leq \lambda$._

2. $MCE(\alpha, \beta) = MDCE(\alpha, \beta)$.

3. _For every common extension λ of α and β, there is a $\mu \in MCE(\alpha, \beta)$ with $\mu \leq \lambda$._

4. $\alpha\Lambda \cap \beta\Lambda = \displaystyle\bigsqcup_{\mu \in MCE(\alpha,\beta)} \mu\Lambda$, _where $\bigsqcup$ denotes a disjoint union._

Proof. (1) Suppose that λ is a common extension of α and β, so $\lambda = \alpha\alpha_1 = \beta\beta_1$.

Note that $d(\lambda) \geq d(\alpha)$ and $d(\lambda) \geq d(\beta)$, so $d(\lambda) \geq d(\alpha) \vee d(\beta)$. Thus by factorization, we may write $\lambda = \mu\mu_1$ where $d(\mu) = d(\alpha) \vee d(\beta)$. Now, $d(\mu) \geq d(\alpha)$, so μ may be additionally factored as $\mu = \alpha'\alpha_2$, where $d(\alpha') = d(\alpha)$. Then we have $\lambda = \alpha\alpha_1 = \mu\mu_1 = \alpha'\alpha_2\mu_1$, meaning we have two factorizations of λ whose initial segments are of equal length $d(\alpha') = d(\alpha)$. Then by uniqueness of factorizations we have $\alpha' = \alpha$, so μ is an extension of α. Similarly, μ will be an extension of β, so μ is a common extension of α and β of length $d(\alpha) \vee d(\beta)$, so $\mu \in MDCE(\alpha, \beta)$ as desired.

(2) If $\nu \in MCE(\alpha, \beta)$, then by (1) there is some $\mu \in MDCE(\alpha, \beta)$ with $\mu \leq \nu$. Then by minimality of ν, we have $\mu = \nu$, so $\nu \in MDCE(\alpha, \beta)$ and thus $MCE(\alpha, \beta) \subseteq MDCE(\alpha, \beta)$.

Conversely, if $\mu \in MDCE(\alpha, \beta)$, suppose there were some $\nu \in \Lambda$ with $\alpha \leq \nu, \beta \leq \nu, \nu \leq \mu$. Then by (1), there would be some $\mu' \in MDCE(\alpha, \beta)$ with $\mu' \leq \nu \leq \mu$. Then $\mu' \leq \mu$, so we may write $\mu s(\mu) = \mu = \mu'\mu_1$ for some $\mu_1 \in \Lambda$, and since $d(\mu) = d(\alpha) \vee d(\beta) = d(\mu')$, then by the uniqueness of the factorization we have $\mu = \mu'$. Then $\mu = \nu$, so μ is "minimal" in the appropriate sense such that $\mu \in MCE(\alpha, \beta)$. Thus $MDCE(\alpha, \beta) = MCE(\alpha, \beta)$.

(3) Immediately follows from (1) and (2).

(4) It is immediate that $\bigcup_{\mu \in MCE(\alpha, \beta)} \mu\Lambda \subseteq \alpha\Lambda \cap \beta\Lambda$, and by (3) we have $\alpha\Lambda \cap \beta\Lambda \subseteq \bigcup_{\mu \in MCE(\alpha, \beta)} \mu\Lambda$, so it suffices to show the union is disjoint. To this end, if $\mu, \nu \in MCE(\alpha, \beta)$ and $\lambda \in \mu\Lambda \cap \nu\Lambda$, then we would have $\lambda = \mu\mu_1 = \nu\nu_1$, but this presents two factorizations with equal degrees $d(\mu) = d(\alpha) \vee d(\beta) = d(\nu)$, so we must have $\mu = \nu$. Thus the $\{\mu\Lambda : \mu \in MCE(\alpha, \beta)\}$ are pairwise disjoint, as desired. $\square$

Remark 2.4.5. *It should be noted that much of the literature (e.g. [3, Definition 2.3]) defines $MCE(\alpha, \beta)$ as we have defined $MDCE(\alpha, \beta)$, which the previous lemma shows are equivalent. We have chosen to emphasize the definition arising purely from*

the category structure because later we will be considering a category as a P-graph and as a Q-graph, and this definition clarifies that the category structure does not depend on the choice of the grading (as long as one exists).

2.5   Representations of P-graphs and P-graph Algebras

Two additional notions will be needed in our study of P-graph algebras.

Definition 2.5.1. *Let (G, P) be a WQLO group, and let Λ be a P-graph. We say that Λ is <u>finitely-aligned</u> if for all $\mu, \nu \in \Lambda$, $MCE(\mu, \nu)$ is finite.*

We say that a set $A \subseteq \Lambda$ is <u>exhaustive</u> for a set $B \subseteq \Lambda$ if for all $\beta \in B$ there is an $\alpha \in A$ such that α and β have a common extension.

We are at last ready to define our main object of study, the representations of P-graphs in a C^*-algebra. The following definition is due to [3], with the slight modification to apply to WQLO groups:

Definition 2.5.2. *Let (G, P) be a WQLO group and Λ a finitely-aligned P-graph. A <u>representation</u> of Λ in a C^*-algebra B is a function $t : \Lambda \to B, \lambda \mapsto t_\lambda$ such that:*

(T1) $t_v = t_v^*$ and $t_v t_w = \delta_{v,w} t_v$ for all $v, w \in \Lambda^0$ (here δ denotes the Kronecker delta).

That is, $\{t_v : v \in \Lambda^0\}$ is a collection of pairwise orthogonal projections.

(T2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$.

(T3) $t_\mu^* t_\mu = t_{s(\mu)}$ for all $\mu \in \Lambda$.

(T4) $t_\mu t_\mu^* t_\nu t_\nu^* = \displaystyle\sum_{\lambda \in MCE(\mu,\nu)} t_\lambda t_\lambda^*$ for all $\mu, \nu \in \Lambda$.

We denote by $C^*(t)$ the C^*-algebra generated by the $\{t_\lambda\}_{\lambda \in \Lambda}$, and $C^*(t)$ is called a <u>P-graph C^*-algebra</u> or just P-graph algebra.

We require that our graphs be finitely-aligned so that the expression in the (T4) relator is a finite sum. The fact that an infinite sum of projections does not (usually) converge in the norm topology is a perennial problem in the field of generalizing Cuntz-Krieger algebras, and our solution is to limit our attention to the finitely-aligned case where the issue does not arise.

We will often want to check the properties of $C^*(t)$ on a dense spanning set, so the following lemma is useful. This result is widely known (see for instance [3, Remark 5.2]).

Lemma 2.5.3. *Let (G, P) be a WQLO group, let Λ a finitely aligned P-graph, and let t be a representation of a P-graph. Then $C^*(t) = \overline{\operatorname{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in \Lambda\}$.*

Proof. Fix some $\alpha, \beta, \mu, \nu \in \Lambda$. If $s(\alpha) \neq s(\beta)$ or $s(\mu) \neq s(\nu)$, then by the T1 relator we have $t_\alpha t_\beta^* = 0$ or $t_\mu t_\nu^* = 0$ respectively, and in either case $(t_\alpha t_\beta^*)(t_\mu t_\nu^*) = 0$. If instead $s(\alpha) = s(\beta)$ and $s(\mu) = s(\nu)$, then

$$
\begin{aligned}
(t_\alpha t_\beta^*)(t_\mu t_\nu^*) &= (t_\alpha t_\beta^*)(t_\beta t_\beta^*)(t_\mu t_\mu^*)(t_\mu t_\nu^*) \text{ by T3 and T2} \\
&= (t_\alpha t_\beta^*) \left(\sum_{\lambda \in MCE(\beta,\mu)} t_\lambda t_\lambda^* \right) (t_\mu t_\nu^*) \text{ by T4} \\
&= \sum_{\lambda \in MCE(\beta,\mu)} t_{\alpha(\beta^{-1}\lambda)} t_{\nu(\mu^{-1}\lambda)} \text{ by T2 and T3}
\end{aligned}
$$

where $\beta^{-1}\lambda$ denotes the unique path for which $\beta(\beta^{-1}\lambda) = \lambda$, and similarly $\mu^{-1}\lambda$ denotes the unique path for which $\mu(\mu^{-1}\lambda) = \lambda$.

This calculation shows that $\operatorname{span}\{t_\alpha t_\beta^* : \alpha, \beta \in \Lambda\}$ is closed under multiplication, so it is a $*$-subalgebra of $C^*(t)$, and it contains each t_α since $t_\alpha = t_\alpha t_{s(\alpha)}^*$. Thus

$\overline{\operatorname{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in \Lambda\}$ is a C^*-subalgebra of $C^*(t)$ containing all of the generators, so $C^*(t) = \overline{\operatorname{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in \Lambda\}$.

$\square$

Example 2.5.4. *As we have already seen with groups and Fell bundles, a simple way of representing an object often comes from its own action on itself, and such an example arises for representations of P-graphs as well. To define the* <u>*left-regular representation*</u> *of a P-graph Λ, let $\mathcal{H} = \ell^2(\Lambda)$, with the typical orthonormal basis $\{e_\alpha\}_{\alpha \in \Lambda}$. (Kribs and Power call this the* <u>*Fock space*</u> *of Λ in their papers [13, 14].) For $\mu \in \Lambda$ define a "forward shift by μ" operator L_μ by*

$$L_\mu e_\alpha = \begin{cases} e_{\mu\alpha} & \text{if } s(\mu) = r(\alpha) \\ 0 & \text{if } s(\mu) \neq r(\alpha) \end{cases}$$

and extending this linearly. A short computation on the inner product shows that L_μ^* is the "backwards shift by μ" operator, which is given by

$$L_\mu^* e_\beta = \begin{cases} e_\alpha & \text{if there exists } \alpha \text{ with } \mu\alpha = \beta \\ 0 & \text{if no such } \alpha \text{ exists} \end{cases}$$

(noting that if such an α does exist, it is unique by left cancellation).

We will now show that $\{L_\mu\}_{\mu \in \Lambda}$ is a representation of Λ. The T1 and T2 relations are immediate.

For the T3 relation, observe that

$$L_\mu^* L_\mu e_\alpha = L_\mu^* \left(\begin{cases} e_{\mu\alpha} & \text{if } s(\mu) = r(\alpha) \\ 0 & \text{if } s(\mu) \neq r(\alpha) \end{cases} \right) = \begin{cases} e_\alpha & \text{if } s(\mu) = r(\alpha) \\ 0 & \text{if } s(\mu) \neq r(\alpha) \end{cases} = L_{s(\mu)} e_\alpha$$

as desired.

For the T4 relation, first note that

$$L_\mu L_\mu^* e_\alpha = \begin{cases} e_\alpha & \text{if } \alpha \in \mu\Lambda \\ 0 & \text{if } \alpha \notin \mu\Lambda \end{cases}.$$

Thus

$$L_\mu L_\mu^* L_\nu L_\nu^* e_\alpha = \begin{cases} e_\alpha & \text{if } \alpha \in \mu\Lambda \cap \nu\Lambda \\ 0 & \text{if } \alpha \notin \mu\Lambda \cap \nu\Lambda \end{cases}.$$

But since $\mu\Lambda \cap \nu\Lambda = \bigsqcup_{\lambda \in MCE(\mu,\nu)} \lambda\Lambda$ by Lemma 2.4.4(4), then if $\alpha \in \mu\Lambda \cap \nu\Lambda$ there exists a unique $\lambda_0 \in MCE(\mu,\nu)$ with $\alpha \in \lambda\Lambda$, so $L_{\lambda_0} L_{\lambda_0}^* e_\alpha = e_\alpha$, and $L_\lambda L_\lambda^* e_\alpha = 0$ for $\lambda \in MCE(\mu,\nu) \setminus \lambda_0$. Thus

$$\sum_{\lambda \in MCE(\mu,\nu)} L_\lambda L_\lambda^* e_\alpha = e_\alpha$$

when $\alpha \in \mu\Lambda \cap \nu\Lambda$, so $L_\mu L_\mu^* L_\nu L_\nu^* = \sum_{\lambda \in MCE(\mu,\nu)} L_\lambda L_\lambda^*$ as desired.

Thus $\{L_\mu\}_{\mu \in \Lambda}$ is a representation, as desired.

There are several additional properties that we will sometimes ask of our representation. In order, we will ask for our generators to be nonzero, that our representation preserve its knowledge of the P-graded structure through a certain coaction called a gauge coaction, and that our representation be "tight" in the sense of [8]. More precisely:

Definition 2.5.5. *Let (G, P) be a WQLO group and Λ a finitely aligned P-graph. We say that a representation t of Λ...*

a. is Λ-_faithful_ if each t_λ is nonzero.

b. has a _gauge coaction_ if there is a G-coaction δ on $C^*(t)$ such that

$$\delta(t_\lambda) = t_\lambda \otimes U_{d(\lambda)}$$

for all $\lambda \in \Lambda$.

c. is _tight_ if whenever $\mu \in \Lambda$ and $E \subset \mu\Lambda$ is finite and exhaustive for $\mu\Lambda$, we have

$$\prod_{\alpha \in E}(t_\mu t_\mu^* - t_\alpha t_\alpha^*) = 0.$$

(Recall that E is exhaustive for $\mu\Lambda$ if for all $\nu \in \mu\Lambda$, there is an $\alpha \in E$ such that α and ν have a common extension.) This terminology is motivated by [8], which defines a tight representation of a semilattice. We show in Appendix 1 that this notion of tight is equivalent to the one in [8].

d. _canonically covers_ another representation s if there is a (necessarily surjective) $*$-homomorphism from $C^*(t)$ to $C^*(s)$ given by $t_\lambda \mapsto s_\lambda$. In such a case, we will write $\pi_s^t : C^*(t) \to C^*(s)$ to denote the $*$-homomorphism, which we call the _canonical covering_. The notation π_s^t should hopefully be suggestive of t "covering" s.

e. is _canonically isomorphic_ to another representation s if there is an isomorphism between $C^*(s)$ and $C^*(t)$ given by $t_\lambda \mapsto s_\lambda$ (i.e. if s and t canonically cover each other).

It should be noted that the conditions (T1)-(T4) and tightness are all polynomial relations of the form suitable for Lemma 2.2.2 (finite-alignment being necessary in

the case of (T4)). However, Λ-faithfulness and the existence of a gauge coaction are not polynomial relators in an obvious way.

Remark 2.5.6. *The left-regular representation L from Example 2.5.4 is Λ-faithful.*

However, this representation is as far from tight as possible: given $\mu \in \Lambda$ and $E \subset \mu\Lambda$ which is finite and exhaustive for $\mu\Lambda$, $\prod_{\alpha \in E}(L_\mu L_\mu^* - L_\alpha L_\alpha^*) = 0$ if and only if $\mu \in E$ (i.e. when one of the terms of the product is 0). To see this, note that if $\alpha \in \mu\Lambda \setminus \mu$, then $L_\alpha L_\alpha^* e_\mu = 0$, so $(L_\mu L_\mu^* - L_\alpha L_\alpha^*)e_\mu = e_\mu$, and thus taking the product over $\alpha \in E \subset \mu\Lambda \setminus \mu$, we get that $\prod_{\alpha \in E}(L_\mu L_\mu^* - L_\alpha L_\alpha^*)e_\mu = e_\mu \neq 0$.

We will show in Corollary 4.5.6 that L has a gauge coaction if (G, P) has a strong reduction to an amenable group.

Lemma 2.5.7. *Let (G, P) be a WQLO group, and let Λ be a finitely aligned P-graph. One may apply Lemma 2.2.2 to the relators (T1)-(T4), so there is a representation of Λ which is universal.*

Proof. Observe that (T1) and (T3) together imply that the generators are partial isometries. Also, all of the relators are polynomials in the generators (in the case of (T4), we must note that the expression is finite since the graph is finitely aligned). Thus there is a universal representation of Λ. $\square$

Definition 2.5.8. *We call the universal representation of a P-graph Λ the <u>Toeplitz(-Cuntz-Krieger) representation</u> of Λ. We will use $\mathcal{T}$ to denote this representation, and write $\mathcal{T}C^*(\Lambda)$ or $C^*(\mathcal{T})$ for the algebra generated by it which we call the <u>Toeplitz(-Cuntz-Krieger) algebra</u> of the graph.*

Lemma 2.5.9. *Let (G, P) be a WQLO group, and let Λ be a finitely aligned P-graph.*

1. *Let $Rep(\Lambda)$ denote the set of all representations of Λ up to canonical isomorphism. Then $Rep(\Lambda)$ can be given a partial ordering $\geq_{rep}$ by saying that $s \geq_{rep} t$*

if and only if s canonically covers t (and $t \cong s$ if and only if they are canonically isomorphic).

2. *There is a bijection Ψ from $Rep(\Lambda)$ to ideals in $\mathcal{T}C^*(\Lambda)$ given by $t \mapsto \ker \pi_t^{\mathcal{T}}$, which is order-reversing in the sense that $t \leq_{rep} s \iff \Psi(t) \supseteq \Psi(s)$.*

Proof. For (1), we must check the partial order properties of reflexivity, transitivity, and antisymmetry.

Certainly each representation canonically covers itself by the identity map, so $\leq_{rep}$ is reflexive.

If $t \leq_{rep} s$ and $s \leq_{rep} r$, then there are canonical coverings $\pi_s^r : C^*(r) \to C^*(s)$ and $\pi_t^s : C^*(s) \to C^*(t)$. Then the composition $\pi_t^r := \pi_t^s \circ \pi_s^r$ is a canonical covering of t by r, so $t \leq_{rep} r$, giving transitivity.

If $t \leq_{rep} s$ and $s \leq_{rep} t$, then there are canonical coverings $\pi_t^s : C^*(s) \to C^*(t)$ and $\pi_s^t : C^*(t) \to C^*(s)$. Composing these maps in either direction shows that they fix each generator, and therefore are the identity maps. Therefore π_t^s and π_s^t are inverses and hence canonical isomorphisms.

Thus $\leq_{rep}$ is indeed a partial order, as desired.

For (3), it is certainly true that Ψ is a well-defined function from $Rep(\Lambda)$ to $\{$ideals in $\mathcal{T}C^*(\Lambda)\}$.

We will first check that if $s, t \in Rep(\Lambda)$, then $t \leq_{rep} s \iff \Psi(t) \supseteq \Psi(s)$. If $t \leq_{rep} s$, then there is a canonical covering $\pi_t^s : C^*(s) \to C^*(t)$, and since $\pi_t^s \circ \pi_s^{\mathcal{T}} = \pi_t^{\mathcal{T}}$, then $\Psi(t) = \ker \pi_t^{\mathcal{T}} \supseteq \pi_s^{\mathcal{T}} = \Psi(s)$. Conversely, if $\Psi(t) \supseteq \Psi(s)$, then by Lemma 2.2.1 applied to $\pi_t^{\mathcal{T}}$ and $\pi_s^{\mathcal{T}}$, there is a $*$-homomorphism $\pi_t^s : C^*(s) \to C^*(t)$ with $\pi_t^s \circ \pi_s^{\mathcal{T}} = \pi_t^{\mathcal{T}}$ so π_t^s is a canonical covering and $t \leq_{rep} s$. This completes the proof that Ψ is order-reversing.

Now to show that Ψ is a bijection, if $\Psi(s) = \Psi(t)$, then by the above $s \leq_{rep} t$ and $t \leq_{rep} s$, so $s \cong t$ and hence Ψ is injective. For surjectivity, if $J \triangleleft \mathcal{T}C^*(\Lambda)$ is an ideal, represent Λ by $\lambda \mapsto \mathcal{T}_\lambda + J$. Then this representation has kernel J, so Ψ is surjective, as desired. $\qquad\square$

As a consequence of this, representations have the order structure of the family of ideals of an algebra, and in particular have a lattice order (where meet and join are represented by intersection and addition of ideals).

Remark 2.5.10. Λ-*faithfulness is a "largeness" condition in that if $s \leq t$ and s is Λ-faithful, then so is t.*

Tightness is a "smallness" condition in that if $s \leq t$ and t is tight, then so is s.

2.5.1 Gauge coactions and Coactionization

It may not be obvious what role "has a gauge coaction" plays: is it saying a representation is "large" or "small"?

The following proposition is a tangled web of results, but it shows that the existence of a gauge coaction is properly considered a "largeness" condition: any representation can be "lifted" (1) to have a gauge coaction (4), the lifted version covers the original (2) and is the smallest gauge coacting representation to do so (6), and they are isomorphic if and only if the original had a gauge coaction (5). We also simplify the notion of a gauge coaction by showing that any homomorphism satisfying $t_\lambda \mapsto t_\lambda \otimes U_{d(\lambda)}$ is automatically a gauge coaction (3), and show that a representation is Λ-faithful or tight if and only if its coactionization is (7).

Proposition 2.5.11. *Let (G, P) be a WQLO group, and let Λ be a finitely aligned P-graph. Let t be a representation of Λ.*

1. *There is a representation t' of Λ given by $t'_\lambda = t_\lambda \otimes U_{d(\lambda)}$.*

2. *There is a canonical covering $t' \mapsto t$.*

3. *If there is a $*$-homomorphism $\delta : C^*(t) \to C^*(t) \otimes C^*(G)$ satisfying $\delta(t_\lambda) = t'_\lambda$, then δ is a coaction, and hence a gauge coaction.*

4. *t' has a gauge coaction.*

5. *t is canonically isomorphic to t' if and only if t has a gauge coaction.*

6. *If $s \leq t$, then $s' \leq t'$, and in particular for any representation s, s' is the smallest gauge coacting representation that covers s.*

7. *t' is Λ-faithful (respectively, tight) if and only if t is.*

Due to these properties, we believe that t' deserves to be called the <u>coactionization</u> of t.

Proof. The proof of (1) is a routine confirmation of the T1-T4 relators, which we include here. We have:

T1: $\quad t'_v t'_w \quad = \quad (t_v \otimes U_e)(t_w \otimes U_e) = t_v t_w \otimes U_e = \delta_{v,w} t_v \otimes U_e = \delta_{v,w} t'_v$

T2: $\quad t'_\mu t'_\nu \quad = \quad (t_\mu \otimes U_{d(\mu)})(t_\nu \otimes U_{d(\nu)}) = (t_\mu t_\nu) \otimes U_{d(\mu)} U_{d(\nu)}$

$\qquad\qquad = \quad t_{\mu\nu} \otimes U_{d(\mu\nu)} = t'_{\mu\nu}$

T3: $\quad t'_\mu{}^* t'_\mu \quad = \quad (t_\mu \otimes U_{d(\mu)})^*(t_\mu \otimes U_{d(\mu)}) = t_\mu^* t_\mu \otimes U_{d(\mu)}^* U_{d(\mu)}$

$\qquad\qquad = \quad t_{s(\mu)} \otimes U_e = t'_{s(\mu)}$

T4: $\quad t'_\nu t'_\nu{}^* t'_\mu t'_\mu{}^* \quad = \quad (t_\nu \otimes U_{d(\nu)})(t_\nu \otimes U_{d(\nu)})^*(t_\mu \otimes U_{d(\mu)})(t_\mu \otimes U_{d(\mu)})^*$

$\qquad\qquad = \quad (t_\nu t_\nu^* t_\mu t_\mu^*) \otimes U_e = \sum_{\lambda \in MCE(\mu,\nu)} t_\lambda t_\lambda^* \otimes U_e$

$\qquad\qquad = \quad \sum_{\lambda \in MCE(\mu,\nu)} t'_\lambda t'_\lambda{}^*$

all as desired.

For (2), let V denote the trivial representation of G (given by $V_g = 1 \in \mathbb{C}$ for all $g \in G$). Then by the universal property of $C^*(G)$, there is a covering map $\pi_V^U : C^*(G) \to \mathbb{C}$ given by $\pi_V^U(U_g) = V_g = 1$.

Now by Lemma 2.2.7, there is a map $\mathrm{id}_{C^*(t)} \otimes \pi_V^U : C^*(t) \otimes C^*(G) \to C^*(t) \otimes \mathbb{C}$ satisfying

$$
\begin{aligned}
(\mathrm{id}_{C^*(t)} \otimes \pi_V^U)(t'_\lambda) &= (\mathrm{id}_{C^*(t)} \otimes \pi_V^U)(t_\lambda \otimes U_{d(\lambda)}) \\
&= t_\lambda \otimes 1.
\end{aligned}
$$

By identifying $C^*(t)$ with $C^*(t) \otimes \mathbb{C}$ in the natural way, this gives the desired canonical covering.

For (3), we must check that δ is injective, nondegenerate, and satisfies the coaction identity of Definition 2.3.11. To this end, note that if $\pi_t^{t'}$ is the canonical covering from the previous part, then $\pi_t^{t'} \circ \delta = \mathrm{id}_{C^*(t)}$, so δ must be injective. To check nondegeneracy, we must check that

$$
\overline{\mathrm{span}}\,[\delta(C^*(t))(C^*(t) \otimes C^*(G))] = C^*(t) \otimes C^*(G).
$$

To this end, note that for all $g \in G$ and $\mu, \nu \in \Lambda$,

$$
\delta(t_{s(\mu)})(t_\mu t_\nu^* \otimes U_g) = (t_{s(\mu)} \otimes U_1)(t_\mu t_\nu^* \otimes U_g) = t_\mu t_\nu^* \otimes U_g.
$$

That is, $\{t_\mu t_\nu^* \otimes U_g : \mu, \nu \in \Lambda, g \in G\} \subseteq \delta(C^*(t))(C^*(t) \otimes C^*(G))$. But since $C^*(t) = \overline{\mathrm{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$ and $C^*(G) = \overline{\mathrm{span}}\{U_g\}_{g \in G}$, then these simple tensors have dense span $C^*(t) \otimes C^*(G)$. Thus

$$C^*(t) \otimes C^*(G) = \overline{\operatorname{span}}\{t_\mu t_\nu^* \otimes U_g\}_{\mu,\nu \in \Lambda, g \in G} = \overline{\operatorname{span}}\left[\delta(C^*(t))(C^*(t) \otimes C^*(G))\right]$$

as desired.

Finally, it suffices to check the coaction identity on each generator t_μ. To this end,

$$
\begin{aligned}
((\delta \otimes \operatorname{id}_G) \circ \delta)(t_\mu) &= (\delta \otimes \operatorname{id}_G)(t_\mu \otimes U_{d(\mu)}) \\
&= t_\mu \otimes U_{d(\mu)} \otimes U_{d(\mu)} \\
&= t_\mu \otimes \delta_G(U_{d(\mu)}) \\
&= ((\operatorname{id}_{C^*(t)} \otimes \delta_G)(t_\mu \otimes U_{d(\mu)}) \\
&= ((\operatorname{id}_{C^*(t)} \otimes \delta_G) \circ \delta)(t_\mu)
\end{aligned}
$$

as desired.

For (4), recall that there is a coaction $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ given by $U_g \mapsto U_g \otimes U_g$. By Lemma 2.2.7 we may define $\bar{\delta} : C^*(t) \otimes C^*(G) \to C^*(t) \otimes C^*(G) \otimes C^*(G)$ by $\bar{\delta} = \operatorname{id}_{C^*(t)} \otimes \delta_G$. Let $\delta = \bar{\delta} \mid_{C^*(t')}$. Then for $\lambda \in \Lambda$,

$$\delta(t') = \delta(t_\lambda \otimes U_{d(\lambda)}) = t_\lambda \otimes \delta_G(U_{d(\lambda)}) = t_\lambda \otimes U_{d(\lambda)} \otimes U_{d(\lambda)} = t'_\lambda \otimes U_{d(\lambda)}.$$

Thus by (3), δ is a gauge coaction of $C^*(t')$.

For (5), if t is canonically isomorphic to t', then there is a homomorphism between their C^*-algebras given by $t_\lambda \mapsto t'_\lambda = t_\lambda \otimes U_{d(\lambda)}$. By (3) this is precisely a gauge

coaction.

For (6), if $s \le t$, recall that π_s^t denotes the canonical covering. Then by Lemma 2.2.7 there is a $*$-homomorphism $\pi_{s'}^{t'} : C^*(t') \to C^*(s')$ given by $\pi_{s'}^{t'} = \pi_s^t \otimes \mathrm{id}_{C^*(G)}$, and it is immediate that this is a canonical covering. For the "in particular", if $s \le t$ and t has a gauge coaction, then $t \cong t'$ by (5), so $s' \le t' \cong t$, and thus $s' \le t$, as desired.

For (7), observe that for a fixed $\lambda \in \Lambda$,

$$\left\| t'_\lambda \right\|^2 = \left\| t_\lambda \otimes U_{d(\lambda)} \right\|^2 = \left\| t_\lambda^* t_\lambda \otimes 1 \right\| = \left\| t_\lambda^* t_\lambda \right\| = \left\| t_\lambda \right\|^2$$

so any t'_λ is 0 if and only if t_λ is 0. Thus t is Λ-faithful if and only if t' is Λ-faithful.

For tightness, fix a $\mu \in \Lambda$ and finite $E \subset \mu\Lambda$ which is exhaustive for $\mu\Lambda$. Then

$$\left\| \prod_{\alpha \in E} t'_\mu {t'_\mu}^* - t'_\alpha {t'_\alpha}^* \right\| = \left\| \left(\prod_{\alpha \in E} t_\mu t_\mu^* - t_\alpha t_\alpha^* \right) \otimes 1 \right\|$$

$$= \left\| \prod_{\alpha \in E} t_\mu t_\mu^* - t_\alpha t_\alpha^* \right\|$$

and in particular $\prod_{\alpha \in E} t'_\mu {t'_\mu}^* - t'_\alpha {t'_\alpha}^* = 0$ if and only if $\prod_{\alpha \in E} t_\mu t_\mu^* - t_\alpha t_\alpha^* = 0$, as desired.

□

Finally, note that property (3) says that the notion of coaction is dramatically simplified when considering just gauge coactions. Indeed, the necessary and sufficient condition to having a gauge coaction is that the map $t_\lambda \mapsto t_\lambda \otimes U_{d(\lambda)}$ extends to a $*$-homomorphism on $C^*(t)$.

2.6 The Classical Gauge-Invariant Uniqueness Theorem

In this section we will introduce the gauge-invariant uniqueness theorem for directed graphs (what we would call $\mathbb{N}$-graphs), and explain how we hope to generalize it.

The following definitions and theorems are adapted from Chapters 1 and 2 of [18].

Definition 2.6.1. *A* <u>directed graph</u> *is a tuple* $E = (E^0, E^1, r, s)$ *where* E^0 *and* E^1 *are countable sets, and* $r, s : E^1 \to E^0$ *are functions. We think of* E^0 *as the set of vertices in our directed graph,* E^1 *as our set of edges, and* r *and* s *as the range and source maps for* E^1*. We say a directed graph is* <u>row-finite</u> *if for each* $v \in V$*,* $|r^{-1}(v)| < \infty$*.*

A <u>Cuntz-Krieger E-family</u> *of a row-finite directed graph* (E^0, E^1, r, s) *is a collection of operators* $\{P_v\}_{v \in E^0} \cup \{S_e\}_{e \in E^1}$ *in a* C^**-algebra such that* $\{P_v\}_{v \in E^0}$ *are a family of pairwise orthogonal projections, and*

(CK1) $P_{s(e)} = S_e^* S_e$ *for all* $e \in E^1$ *and*

(CK2) $P_v = \displaystyle\sum_{\substack{e \in E^1 \\ r(e)=v}} S_e S_e^*$ *whenever* $r^{-1}(v)$ *is nonempty.*

The reader may wish to compare this definition to the definition of a representation of a P-graph (Definition 2.5.2). Having done so, there are a few differences that we will wish to reconcile.

Remark 2.6.2. *Given a directed graph* (E^0, E^1, r, s)*, for* $n > 1$ *we let* $E^n = \{(e_1, e_2, ..., e_n) : s(e_i) = r(e_{i+1}) \text{ for } 1 \le i < n\}$*. Then* $E^* = \bigcup_{n \ge 0} E^n$ *forms an* $\mathbb{N}$*-graph in a natural way, where the operation is concatenation of tuples, and* $d(e_1, ..., e_n) = n$*. This* $\mathbb{N}$*-graph is always finitely aligned, since for any* $\mathbb{N}$*-graph,*

$$MCE(\alpha, \beta) = \begin{cases} \{\alpha\} & \text{if } \beta \leq \alpha \\ \{\beta\} & \text{if } \alpha \leq \beta \\ \emptyset & \text{otherwise} \end{cases}$$

so $|MCE(\alpha, \beta)| \leq 1 < \infty$.

Any Cuntz-Krieger E-family extends to a representation of E^* by $S_{(e_1, e_2, \ldots, e_n)} = S_{e_1} S_{e_2} \ldots S_{e_n}$. In proving this, the (T1) and (T2) relators are immediate, the (T3) relator follows rapidly from (CK1), and the (T4) relator follows from the expression above for $MCE(\alpha, \beta)$.

One should note that the (CK2) condition is analogous to tightness, since for any vertex v, $r^{-1}(v)$ is finite and exhaustive for $v\Lambda$, and since $\{S_e S_e^*\}_{e \in r^{-1}(v)}$ are pairwise orthogonal projections, then

$$\prod_{e \in r^{-1}(v)} (P_v P_v^* - S_e S_e^*) = 0 \text{ if and only if } P_v = \sum_{\substack{e \in E^1 \\ r(e)=v}} S_e S_e^*.$$

In other words, the (CK2) condition is saying that the representation is tight in the case that $\mu \in \Lambda^0$ and $E \subset \Lambda^1$. Since every path in an $\mathbb{N}$-graph can be uniquely factored into paths of length 1, tightness in all cases is equivalent to this.

Due to this last point, readers should be aware that in the literature representations of directed graphs (and higher rank graphs) are what we would call tight representations.

In the notation of [18], $C^*(E)$ denotes the universal tight representation of E, and if $\{T_v\}_{v \in E^0} \cup \{Q_e\}_{e \in E^1}$ is a Cuntz-Krieger E-family, then $\pi_{T,Q} : C^*(E) \to C^*(T, Q)$ denotes the canonical covering.

The foundational result we wish to generalize is this, as stated in [18, Theorem 2.2]:

Theorem 2.6.3 (Gauge Invariant Uniqueness Theorem for Graphs). *Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph, and suppose that $\{T_v\}_{v \in E^0} \cup \{Q_e\}_{e \in E^1}$ is a Cuntz-Krieger E-family in a C^*-algebra B with each $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to \mathrm{Aut}\, B$ such that $\beta_z(T_e) = zT_e$ for every $e \in E^1$ and $\beta_z(Q_v) = Q_v$ for every $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

Remark 2.6.4. *The reader may recognize many of the hypotheses of Theorem 2.6.3 from our earlier list of terminology for representations (Definition 2.5.5). The statement $Q_v \neq 0$ for $v \in E^0$ implies that $T_\mu \neq 0$ for all $\mu \in E^*$, so it is equivalent to our Λ-faithfulness. The continuous action β is called a <u>gauge action</u> and is equivalent by Lemma 2.3.20 to a gauge coaction. Also recall that tightness is a built-in hypothesis in the representations considered in [18].*

Therefore, we may restate Theorem 2.6.3 like so: for any $\mathbb{N}$-graph, there is exactly one Λ-faithful, tight, gauge coacting representation up to canonical isomorphism.

What we mean by a gauge invariant uniqueness theorem for P-graphs is a similar statement: for any P-graph, there is exactly one Λ-faithful, tight, gauge coacting representation up to canonical isomorphism.

We have already seen that in generalizing from $\mathbb{N}$-graphs to P graphs, we needed an additional hypothesis (finite alignment). We will see in Lemma 4.2.2 that such a gauge invariant uniqueness theorem need not be true in general, so another hypothesis is needed. We spend Chapter 3 developing this hypothesis on (G, P), and then in Chapter 4 prove a gauge invariant uniqueness theorem for P-graphs that satisfy this new hypothesis.

Chapter 3

Reductions of Ordered Groups

In this section we will develop the notion of a reduction of an ordered group. Roughly speaking, a reduction occurs when a positive cone P in a group G does not carry "enough" information about the group, and one is able to replace (G, P) with some (H, Q) while preserving the essential properties that will be required for a P-graph (namely, the structure of intervals $[1, p]$). A special case of a reduction (called a "strong reduction") is when one can embed the positive cone P in another group H such that this embedding extends to an order homomorphism from (G, P) to (H, P).

3.1 Definition and Basic Properties

Definition 3.1.1. *An order homomorphism $\varphi : (G, P) \to (H, Q)$ is a <u>reduction</u> if for all $p \in P$, φ is a bijection between the interval $[1, p] := \{x \in P : 1 \le x \le p\}$ and the interval $[1, \varphi(p)] := \{y \in Q : 1 \le y \le \varphi(p)\}$.*

We say that φ is a <u>strong reduction</u> if $\varphi \mid_P : P \to Q$ is a bijection between P and Q.

We say (G, P) has a (strong) reduction to an amenable group or (strongly) reduces to an amenable group if there is an amenable group H, a positive cone $Q \subset H$, and a (strong) reduction $\varphi : (G, P) \to (H, Q)$.

Intuitively, a reduction is a map that preserves the order structure of (G, P) "locally", meaning on every interval. For strong reductions, the reader should think of them as arising from embedding the positive cone P in two distinct groups (G, P) and (H, P) such that there is a homomorphism from one to the other taking P to itself bijectively.

Example 3.1.2. *Let $G = \langle a, b \rangle$, the free group on two generators. If $P = \{a, b\}^*$, then define $\varphi : (G, P) \to (\mathbb{Z}, \mathbb{N})$ by $\varphi(a) = \varphi(b) = 1$, which is an order homomorphism.*

Then φ is a reduction since, for example, it maps the interval $[e, abba] = \{e, a, ab, abb, abba\}$ bijectively onto the interval $[\varphi(e), \varphi(abba)] = [0, 4] = \{0, 1, 2, 3, 4\}$. Since $\mathbb{Z}$ is amenable, then this shows that (G, P) reduces to an amenable group.

However, φ is not injective even when restricted to P, as $\varphi(a) = 1 = \varphi(b)$, so φ is not a strong reduction.

It may not be immediately obvious from the definition that a strong reduction is a reduction, so we will prove this as a short lemma:

Lemma 3.1.3. *Let $\varphi : (G, P) \to (H, Q)$ be a strong reduction. Then it is a reduction.*

Proof. We must show that if φ is bijective as a map from P to Q, then it is bijective from $[1, p]$ to $[1, \varphi(p)]$ for each $p \in P$. Fixing a $p \in P$, since $[1, p] \subset P$, then φ is injective on $[1, p]$, so it suffices to show it is surjective onto $[1, \varphi(p)]$.

To this end, fix some $q_1 \in [1, \varphi(p)]$. Since $q_1 \leq \varphi(p)$, there is a $q_2 \in Q$ such that $q_1 q_2 = \varphi(p)$. Now since $q_1, q_2 \in Q$ and φ is a bijection from P to Q, there is a $p_1, p_2 \in P$ such that $\varphi(p_1) = q_1$ and $\varphi(p_2) = q_2$. Then $\varphi(p) = q_1 q_2 = \varphi(p_1)\varphi(p_2) = \varphi(p_1 p_2)$. Since $p_1 p_2 \in P$ and φ is injective on P, then $p_1 p_2 = p$, so $p_1 \in [1, p]$, and in particular there is a $p_1 \in [1, p]$ such that $\varphi(p_1) = q_1$. That is, φ is a bijection from $[1, p]$ onto $[1, \varphi(p)]$, as desired. $\square$

We'll now provide several alternative characterizations of being a reduction and a strong reduction.

Proposition 3.1.4. *Let $\varphi : (G, P) \to (H, Q)$ be an order homomorphism. Then the following are equivalent:*

1. *φ is a reduction in the sense of Definition 3.1.1.*

2. *$d_Q^P := \varphi \mid_P: P \to Q$ is a functor that makes P into a Q-graph.*

3. *For all $q \in Q, p \in P$ such that $q \leq \varphi(p)$, there exist unique $p_1, p_2 \in P$ such that $p_1 p_2 = p$ and $\varphi(p_1) = q$.*

4. *For each nonempty interval $[x, y] \subset G$, φ is an order isomorphism between $[x, y]$ and $[\varphi(x), \varphi(y)]$.*

5. *The following two statements together:*

    a) *$S \cap \ker \varphi = \{1\}$ where $S = \bigcup_{p \in P} [1, p][1, p]^{-1}$.*

    b) *For all $p \in P$, $\varphi([1, p]) = [1, \varphi(p)]$.*

Proof. Fix $\varphi : (G, P) \to (H, Q)$ an order homomorphism. We will prove that $(2) \Rightarrow (3) \Rightarrow (1) \Rightarrow (4) \Rightarrow (2)$ and $(1) \iff (5)$.

$(2 \Rightarrow 3)$ Fix $q \in Q$, $p \in P$ with $q \leq \varphi(p) = d_Q^P(p)$. Then since P is a Q-graph, we can uniquely factorize p as $p = p_1 p_2$ where $q = d_Q^P(p_1) = \varphi(p_1)$, as desired.

$(3 \Rightarrow 1)$ Fix $p \in P$. By Lemma 2.1.7, $\varphi([1, p]) \subset [1, \varphi(p)]$. It then suffices to show that if $q \in [1, \varphi(p)]$, then there exists a unique $s \in [1, p]$ with $\varphi(s) = q$. To this end, if $q \in [1, \varphi(p)]$, then by (3), there exists $p_1, p_2 \in P$ such that $p_1 p_2 = p$ and with $\varphi(p_1) = q$. Then $p_1 \in [1, p]$, so p_1 is such an s. If there were some $p_1' \in [1, p]$ with $\varphi(p_1) = \varphi(p_1')$, then by the former, there exists $p_2' \in P$ with $p_1' p_2' = p$, so by the

uniqueness condition of (2), we have $p_1' = p_1$ and $p_2' = p_2$, and thus our p_1 is unique, as desired.

$(1 \Rightarrow 4)$ Fix $x, y \in G$, and suppose the interval $[x, y] = \{z \in G : x \leq z \leq y\}$ is nonempty. Since this interval is nonempty, then $x \leq y$, so $1 \leq x^{-1}y$, and we will write $p = x^{-1}y \in P$. By left-invariance, $x \leq z \leq y$ if and only if $1 \leq x^{-1}z \leq x^{-1}y$, so $[x, y] = x[1, x^{-1}y] = x[1, p]$. By left-invariance, $x[1, p]$ is order-isomorphic to $[1, p]$, and $\varphi(x)[1, \varphi(p)]$ is order-isomorphic to $[1, \varphi(p)]$, so it suffices to show that $[1, p]$ is order-isomorphic to $[1, \varphi(p)]$. By (1), φ is a bijection between $[1, p]$ and $[1, \varphi(p)]$, so it suffices to show that $s \leq t \iff \varphi(s) \leq \varphi(t)$ for $s, t \in [1, p]$. The $\Rightarrow$ direction is immediate from Lemma 2.1.7. For the other direction, suppose $s, t \in [1, p]$ and $\varphi(s) \leq \varphi(t)$. Then $\varphi(s) \in [1, \varphi(t)]$, and by (3), φ maps $[1, t]$ surjectively onto $[1, \varphi(t)]$, so there exists some $s' \in [1, t]$ such that $\varphi(s') = \varphi(s)$. But $s' \in [1, t] \subseteq [1, p]$, and φ is injective on $[1, p]$, so $s' = s$. Thus $s \in [1, t]$, so $s \leq t$.

$(4 \Rightarrow 2)$ Certainly $d_Q^P := \varphi \mid_P$ is a functor from P to Q. It then suffices to check unique factorization. To this end, suppose $p \in P$ and $q_1, q_2 \in Q$ with $d_Q^P(p) = q_1 q_2$. Then $q_1 \in [1, d_Q^P(p)] = [1, \varphi(p)]$, so by (4) there is a unique $p_1 \in [1, p]$ with $q_1 = \varphi(p_1) = d_Q^P(p_1)$. Since $p_1 \leq p$, then $p_2 := p_1^{-1}p \in P$. Thus p is uniquely factorized as $p = p_1 p_2$ with $d_Q^P(p_1) = q_1$.

$(1 \Rightarrow 5a)$ Suppose that $s \in S \cap \ker \varphi$. Then $s = qr^{-1}$ where $q, r \in [1, p]$ for some $p \in P$. Then $\varphi(r) = \varphi(s)\varphi(r) = \varphi(sr) = \varphi(q)$. By (1), $\varphi \mid_{[1,p]}$ is injective, so $q = r$ and thus $s = qr^{-1} = 1$, as desired. .

$(1 \Rightarrow 5b)$ is immediate from the fact that φ is a bijection between the intervals $[1, p]$ and $[1, \varphi(p)]$.

$(5 \Rightarrow 1)$ Fix $p \in P$. By Lemma 2.1.7, $\varphi([1, p]) \subseteq [1, \varphi(p)]$. It then suffices to show that if $q \in [1, \varphi(p)]$, then there exists a unique $s \in [1, p]$ with $\varphi(s) = q$. By (5b), there exists at least one $s \in [1, p]$ with $\varphi(s) = q$. Assume for the sake of contradiction

that there were distinct $s_1, s_2 \in [1,p]$ with $\varphi(s_1) = q = \varphi(s_2)$. Then $s_1 s_2^{-1} \in \ker \varphi$. But since $s_1 s_2^{-1} \in [1,p][1,p]^{-1} \subseteq S$, then by (5a) we have $s_1 s_2^{-1} = 1$, so $s_1 = s_2$, a contradiction of distinctness. Thus φ is bijective as a map from $[1,p]$ to $[1, \varphi(p)]$.

$\square$

Lemma 3.1.5. *Let (G, P), (H, Q) be ordered groups, and $\varphi : G \to H$ a group homomorphism. The following are equivalent:*

1. *φ is a strong reduction in the sense of Definition 3.1.1.*

2. *$\varphi \mid_P$ is an order isomorphism from P to Q.*

Proof. $(2 \Rightarrow 1)$ is immediate.

$(1 \Rightarrow 2)$ Since $\varphi \mid_P$ is already a bijection, we must show that for $p_1, p_2 \in P$, $p_1 \le p_2 \iff \varphi(p_1) \le \varphi(p_2)$. The $\Rightarrow$ direction is immediate from Lemma 2.1.7. For the other direction, suppose that $\varphi(p_1) \le \varphi(p_2)$. Then there is a $q \in Q$ such that $\varphi(p_1)q = \varphi(p_2)$. Since φ is a surjection from P to Q, there exists a $p \in P$ such that $\varphi(p) = q$, and hence $\varphi(p_1 p) = \varphi(p_2)$. Since φ is injective on P, then $p_1 p = p_2$, and thus $p_1 \le p_2$, as desired. $\square$

While the theory of reductions of ordered groups can be entirely severed from the amenability of the groups involved, it is perhaps unsurprising that a reduction to an amenable group will allow us to do more interesting analysis of the C^*-algebras of P-graphs that will appear in later sections.

The following example shows that the existence of a reduction to an amenable group (or even a non-injective reduction!) depends on not just the group G, but also the positive cone P.

Example 3.1.6. Let $G = \langle a, b \rangle$, the free group on two generators. If $P = \{a, b\}^*$, then we've seen that (G, P) is an ordered group which reduces to an amenable group via the map $\varphi : (G, P) \to (\mathbb{Z}, \mathbb{N})$ given by $\varphi(a) = \varphi(b) = 1$.

Suppose that (G, R) is a total ordering on a group (meaning that $R \cup R^{-1} = G$ in addition to $R \cap R^{-1} = \{1\}$). Then for any reduction φ, by Proposition 3.1.4(5), since $S = \bigcup_{r \in R} [1, r][1, r]^{-1} = G$, then $\ker \varphi = \ker \varphi \cap G = \{1\}$. That is, every reduction φ is injective.

In particular, taking $G = F_2$ and R a total ordering on G (such as the ordering arising from the Magnus expansion given in [4, Section 3.2]), (F_2, R) cannot reduce to an amenable group since the range group of a reduction will always contain a copy of F_2.

Example 3.1.7. For any group (G, P), id_G is a reduction, so if G is amenable, id_G is a reduction to an amenable group.

Example 3.1.8. Let $G = \langle a, b \rangle$, the free group on two generators and $P = \{a, b\}^*$. One can give a strong reduction of (G, P) onto the amenable group $H = BS(1, 2) = \langle c, t | tc = c^2 t \rangle$. Let $Q = \{t, ct\}^*$, which is a positive cone in $BS(1, 2)$, and let $\varphi : (G, P) \to (H, Q)$ by $\varphi(a) = ct, \varphi(b) = t$. Certainly φ is an order homomorphism. Then one may verify that φ is bijective from P to Q in this way: if $\varphi(p_1) = \varphi(p_2)$, then write this word in H as $c^i t^j$. It must be that j is the length of p_1 and the length of p_2, so p_1 and p_2 have equal length. Then, i will be the number that comes from substituting 1 for a and 0 for b in the original word and interpreting the result as a binary number. Two binary numbers with the same number of digits are equal if and only if their digits are equal, so it must be that $p_1 = p_2$. Thus φ is injective on P, and hence bijective on P, meaning φ is a strong reduction.

The following result shows that reductions preserve weakly quasi-lattice order.

Lemma 3.1.9. *Let $\varphi : (G, P) \to (H, Q)$ be a reduction. Then:*

1. *If (H, Q) is WQLO, then so is (G, P).*

2. *If $\varphi(P) = Q$ and (G, P) is WQLO, then so is (H, Q).*

Proof. For (1), assume for the sake of contradiction that (H, Q) is WQLO but (G, P) is not. Then there are $x, y \in P$ with a common upper bound but no least common upper bound. That is, there are upper bounds $b_1 > b_2 > b_3 > \ldots$ such that $x, y \leq b_i$ for all $i \in \mathbb{N}$.

For all $i \in \mathbb{N}$, since $1 \leq x \leq b_i$, then $1 \leq b_i$. In particular, $b_i \in [1, b_1]$ for all $i \in \mathbb{N}$. Also note that $x, y \in [1, b_1]$.

Now, since φ is a reduction, φ is an order isomorphism from $[1, b_1]$ to $[1, \varphi(b_1)]$, so in Q, $\varphi(b_1) > \varphi(b_2) > \ldots$ is a strictly decreasing sequence of common upper bounds of $\varphi(x)$ and $\varphi(y)$. This contradicts the fact that (H, Q) is WQLO.

For (2), assume for the sake of contradiction that $\varphi(P) = Q$ and (G, P) is WQLO but that (H, Q) is not WQLO. Then there are $z, w \in Q$ with a common upper bound but no least common upper bound. That is, there are upper bounds $c_1 > c_2 > c_3 > \ldots$ such that $z, w \leq c_i$ for all $i \in \mathbb{N}$.

For all $i \in \mathbb{N}$, since $1 \leq z \leq c_i$, then $1 \leq c_i$. In particular, $c_i \in [1, c_1]$ for all $i \in \mathbb{N}$. Also note that $z, w \in [1, c_1]$.

Now, since $\varphi(P) = Q$, there is some $p \in P$ such that $\varphi(p) = c_1$. Since φ is a reduction, φ is an order isomorphism from $[1, p]$ to $[1, \varphi(p)] = [1, c_1]$. Let x, y, and b_i denote the unique elements in $[1, p]$ such that $\varphi(x) = z, \varphi(y) = w$, and $\varphi(b_i) = c_i$ for all $i \in \mathbb{N}$. Then in P, $b_1 > b_2 > b_3 \ldots$ is a strictly decreasing sequence of common upper bounds of x and y. This contradicts the fact that (G, P) is WQLO.

$\square$

Remark 3.1.10. *There is a notion of amenability for semigroups, and (G, P) may strongly reduce to an amenable group even if neither G nor P is amenable as a group or semigroup (respectively). For example, in (F_2, P_2), the free group on 2 generators with positive cone P_2 which is the free monoid on 2 generators, we will show that (F_2, P_2) strongly reduces to $(\mathbb{Z} \wr \mathbb{Z}, \mathbb{N} \wr \mathbb{N})$, which is amenable, but F_2 is famously not amenable, and P_2 is also not amenable as a semigroup.*

The fact that P_2 is not amenable but can be embedded in an amenable group has been known for more than half a century. In [11], the author shows an embedding of P_2 into $\mathbb{Z} \wr \mathbb{Z}^2$.

3.2 Constructions

In this section, we will show that the notion of "(strongly) reduces to an (amenable) group" behaves well with the usual group theory constructions such as composition, hereditary subgroups, direct products, and free products.

3.2.1 Composition

Lemma 3.2.1. *A composition of (strong) reductions is a (strong) reduction. That is, if $\varphi : (G, P) \to (H, Q)$ and $\psi : (H, Q) \to (F, R)$ are two (strong) reductions, then $\psi \circ \varphi$ is a (strong) reduction.*

Proof. Suppose $\varphi : (G, P) \to (H, Q)$ and $\psi : (H, Q) \to (F, R)$ are two reductions. Then we will show that $\psi \circ \varphi$ is a reduction.

Certainly, $\psi \circ \varphi$ is an order homomorphism, so by Lemma 3.1.4 (3), we must show that for all $p \in P$, $\psi \circ \varphi$ bijectively maps $[1, p]$ onto $[1, \psi(\varphi(p))]$.

To this end, we know that $\varphi : [1, p] \to [1, \varphi(p)]$ is bijective, and $\psi : [1, \varphi(p)] \to [1, \psi(\varphi(p))]$ are bijections since φ and ψ are reductions, so therefore their composition

is a bijection, as desired.

For the composition of strong reductions, if $\varphi : (G, P) \to (H, Q)$ and $\psi : (H, Q) \to (F, R)$ are two strong reductions, then φ maps P bijectively onto Q and ψ maps Q bijectively onto R, so $\psi \circ \varphi$ maps P bijectively onto R, as desired.

$\square$

3.2.2 Hereditary Subgroups

In the context of ordered groups, the "correct" notion of a subgroup is often a hereditary subgroups in the sense of [3, Corollary 5.6]. In this section we will remind the reader of the definition and some basic results about the concept.

Definition 3.2.2. *Given an ordered group (G, P), if $Q \subseteq P$ is a subsemigroup, we say that Q is $\underline{hereditary}$ in P if $p_1, p_2 \in P$, $p_1 p_2 \in Q$ implies that $p_1, p_2 \in Q$.*

If (G, P) is an ordered group, we will say that (H, Q) is a $\underline{hereditary\ subgroup}$ if H is a subgroup of G, Q is a subsemigroup of $P \cap H$, and Q is hereditary in P.

Lemma 3.2.3. *Let (H, P) be an ordered group, and $Q \subseteq P$ a hereditary subsemigroup. For the sake of clarity, let $\leq_P$ and $\leq_Q$ denote the orderings on G arising from P and Q, respectively. Then for all $q \in Q$ and $p \in P$, $1 \leq_P p \leq_P q$ if and only if $1 \leq_Q p \leq_Q q$.*

In particular, for $q \in Q$, the intervals $[1, q]_{\leq_P}$ and $[1, q]_{\leq_Q}$ are equal as sets.

Proof. Fix some $q \in Q, p \in P$.

Suppose $1 \leq_P p \leq_P q$. Then $p \in P$ and there exists an $s \in P$ such that $ps = q$. Since $q \in Q$ and Q is hereditary, then this implies that $p, s \in Q$, so $1 \leq_Q p \leq_Q q$, as desired.

Suppose $1 \leq_Q p \leq_Q q$. Then $p \in Q$ and there exists an $s \in Q$ such that $ps = q$. Since $Q \subseteq P$, then $1 \leq_P p \leq_P q$, as desired.

The "in particular" part is immediate.

□

Lemma 3.2.4. *Let* $(G, P), (H, Q)$ *be ordered groups, and* $\varphi : G \to H$ *a homomorphism. Let* $H' = \varphi(G)$, $Q' = \varphi(P)$, *and let* φ' *denote the codomain restriction of* φ *to the codomain* H'. *Then the following are equivalent:*

1. φ *is a reduction.*

2. (H', Q') *is a hereditary subgroup of* (H, Q) *and* φ' *is a surjective reduction of* (G, P) *onto* (H', Q').

Proof. $(1 \Rightarrow 2)$ If φ is a reduction, we will first show that $Q' = \varphi(P)$ is a hereditary subsemigroup of Q. If $q = \varphi(p) \in \varphi(P)$, and $rr' = q$ for some $r, r' \in Q$, then $r \in [1, \varphi(p)]$, so by Proposition 3.1.4(3), there exists a unique $s \in [1, p]$ such that $\varphi(s) = r$. Then $\varphi(s)r' = \varphi(p)$, so $r' = \varphi(s^{-1}p)$. Since $s \leq p$, then there exists $s' \in P$ such that $ss' = p$, so $r' = \varphi(s')$. Thus $r, r' \in \varphi(P)$, so $\varphi(P)$ is a hereditary subsemigroup of Q.

Since additionally $H' < H$ and $Q' < Q \cap \varphi(G)$, then (H', Q') is a hereditary subgroup of (H, Q).

To show that φ' is a surjective reduction, it is immediately surjective since its codomain was restricted to the image. Now fix a $p \in P$. Since φ is a reduction, then by Proposition 3.1.4 (3), φ maps $[1, p]$ bijectively onto $[1, \varphi(p)]$. By Lemma 3.2.3, we have $[1, \varphi(p)] = [1, \varphi'(p)]$ as sets, so φ' maps $[1, p]$ bijectively onto $[1, \varphi'(p)]$. Then by Proposition 3.1.4 (3), φ' is a reduction.

$(2 \Rightarrow 1)$ Fix a $p \in P$. Since φ' is a reduction, then by Proposition 3.1.4 (3), φ' maps $[1, p]$ bijectively onto $[1, \varphi'(p)]$. By Lemma 3.2.3, we have $[1, \varphi'(p)]_{\leq Q'} = [1, \varphi(p)]_{\leq Q}$

as sets, so φ maps $[1, p]$ bijectively onto $[1, \varphi(p)]$. Then by Proposition 3.1.4 (3), φ is a reduction.

$\square$

Lemma 3.2.5. *Let (G, P) be an ordered group and (H, Q) a hereditary subgroup of (G, P). Then the inclusion map $i : (H, Q) \to (G, P)$ is a reduction.*

Proof. Observe that the inclusion map, restricted to its image, is the identity map, and thus a surjective reduction. Then the inclusion map is a reduction by Lemma 3.2.4.

$\square$

We can now conclude that "reduces to an amenable group" is closed under taking hereditary subgroups.

Corollary 3.2.6. *Let (G, P) be an ordered group and (H, Q) a hereditary subgroup of (G, P). If (G, P) reduces to an amenable group, then (H, Q) reduces to an amenable group.*

Proof. Let $i : H \to G$ denote the inclusion map, which by the previous lemma is a reduction.

If $\varphi : (G, P) \to (F, R)$ is a reduction to an amenable group, then $\varphi \circ i : (H, Q) \to (F, R)$ is a composition of reductions, and hence a reduction by Lemma 3.2.1. It has amenable range, so it is a reduction to an amenable group. $\square$

3.2.3 Direct Products

Now, we'll show that reductions respect direct products, which implies that "reduces to an amenable group" is preserved under direct products.

First, let us check that direct products preserve WQLO:

Lemma 3.2.7. *Let (G, P) and (H, Q) be ordered groups. Let $P \times Q$ denote the submonoid of $G \times H$ generated by (the canonical copies of) P and Q. Then $(G \times H, P \times Q)$ is an ordered group, and if (G, P) and (H, Q) are WQLO, then so is $(G \times H, P \times Q)$.*

Proof. By definition, $P \times Q$ is a submonoid of $G \times H$, and it is immediate that $P \times Q \cap (P \times Q)^{-1} = (P \cap P^{-1}) \times (Q \cap Q^{-1}) = \{1_G\} \times \{1_H\}$, so $(G \times H, P \times Q)$ is an ordered group.

If (G, P) and (H, Q) are WQLO, then one may quickly confirm that given $(p_1, q_1), (p_2, q_2) \in P \times Q$, their least upper bound is $(p_1 \vee p_2, q_1 \vee q_2)$.

$\square$

Lemma 3.2.8. *For $i = 1, 2$, let (G_i, P_i) and (H_i, Q_i) be ordered groups, $\varphi_i : G_i \to H_i$ a homomorphism. Let $G = G_1 \times G_2$, and similarly define P, H, Q. Then there is a homomorphism $\varphi : G \to H$ given by $\varphi(g_1, g_2) = (\varphi_1(g_1), \varphi_2(g_2))$, and φ is an order homomorphism (respectively, reduction or strong reduction) if φ_1 and φ_2 both are.*

Proof. The existence of such a homomorphism is an elementary fact of group theory.

If φ_1, φ_2 are both order homomorphisms, then $\varphi_1(P_1) \subseteq Q_1$ and $\varphi_2(P_2) \subseteq Q_2$, so $\varphi(P_1 \times P_2) \subseteq \varphi_1(P_1) \times \varphi_2(P_2) \subseteq Q_1 \times Q_2$, as desired.

If φ_1, φ_2 are both reductions, then we will use Proposition 3.1.4 (3) as our notion of a reduction, so it suffices to show that for all $p_1 \in P_1, p_2 \in P_2$, $[(1, 1), (p_1, p_2)]$ is mapped bijectively onto $[(1, 1), (\varphi_1(p_1), \varphi_2(p_2))]$. But in $(G_1 \times G_2, P_1 \times P_2)$, we have that $[(1, 1), (x_1, x_2)] = [1, x_1] \times [1, x_2]$, so we know that $\varphi = \varphi_1 \times \varphi_2$ bijectively carries $[(1, 1), (p_1, p_2)] = [1, p_1] \times [1, p_2]$ onto $[(1, 1), (\varphi_1(p_1), \varphi_2(p_2))] = [1, \varphi_1(p_1)] \times [1, \varphi_2(p_2)]$, as desired.

If φ_1, φ_2 are both strong reductions, then φ_1 and φ_2 map P_1 and P_2 bijectively onto Q_1 and Q_2 respectively, so for all $q = (q_1, q_2) \in Q$ there is a unique $p_1 \in P_1, p_2 \in P_2$

such that $\varphi_1(p_1) = q_1, \varphi_2(p_2) = q_2$, and hence $p = (p_1, p_2)$ is the unique element of P with $\varphi(p) = q$, so φ is a bijection from P to Q, as desired.

$\square$

Corollary 3.2.9. *If (G_1, P_1), (G_2, P_2) are ordered groups which reduce to amenable ordered groups, then $(G_1 \times G_2, P_1 \times P_2)$ reduces to an amenable group.*

Proof. For $i = 1, 2$, let $\varphi_i : (G_i, P_i) \to (H_i, Q_i)$ denote the reduction of (G_i, P_i) to an amenable ordered group.

By the previous Lemma, we know that $(G_1 \times G_2, P_1 \times P_2)$ reduces to $(H_1 \times H_2, Q_1 \times Q_2)$. Since H_1 and H_2 are amenable, then $H_1 \times H_2$ is amenable, so this is a reduction to an amenable group. $\square$

3.2.4 Free Products

In this section we will show that the class of groups which have reductions to amenable groups is closed under (finite) free products. That is, we will show that if (G, P) and (H, Q) reduce to amenable groups, then $(G * H, P * Q)$ also reduces to an amenable group.

This will consist of two steps, analogous to Lemma 3.2.8 and Corollary 3.2.9 from the direct product case. A free product analogue of Lemma 3.2.8 is straightforward. However, in the free product case, an analogue of Corollary 3.2.9 is more difficult, since the free product of two amenable groups is almost never amenable. Therefore, our second step is longer.

Given ordered groups (G_1, P_1) and (G_2, P_2), we denote by $P_1 * P_2$ the submonoid of $G_1 * G_2$ generated by the naturally embedded copies of P_1 and P_2. First, we will prove a lemma characterizing the order on $(G_1 * G_2, P_1 * P_2)$:

Lemma 3.2.10. *Let $(G_1, P_1), (G_2, P_2)$ be ordered groups, and let $(G, P) = (G_1 * G_2, P_1 * P_2)$.*

1. *For $p \in P$, we may write $p = p_1 p_2 ... p_{2n}$ where $p_i \in P_1$ if i is odd, and $p_i \in P_2$ if i even, and $p_i \neq 1$ for $1 < i < 2n$.*

2. *This representation of p is unique. That is, there is exactly one such choice of n and elements p_i such that $p_i \in P_1$ if i is odd, and $p_i \in P_2$ if i even, and $p_i \neq 1$ for $1 < i < 2n$.*

3. *Given such a representation $p = p_1 p_2 ... p_{2n}$, we have $[1, p] = \bigcup_{i=1}^{2n} X_i$, where $X_i = p_1 ... p_{i-1}[1, p_i]$.*

4. *Given such a decomposition $[1, p] = \bigcup_{i=1}^{2n} X_i$, if $a \in X_i$ and $b \in X_j$ for $i < j$, then $a \leq b$.*

Proof. For (1), by definition every element of P can be written as a product of elements of P_1 and P_2. If two consecutive elements are from the same P_i, they can be combined, and if any intermediate term is a 1, it can be removed and the now-consecutive terms combined. Finally, if necessary a term of 1 can be put at the beginning or end of the expression to make the alternating product begin with a term from P_1 and end with a term from P_2.

For (2), this representation is unique because of the Normal Form Theorem for Free Products ([16, Chapter IV, Theorem 1.2]).

For (3), fix a $p = p_1 p_2 ... p_{2n}$. It is immediate that each X_i is a subset of $[1, p]$, so it suffices to show containment in the other direction. To this end, suppose $q \in [1, p]$, so $q \in P$ and $p = qr$ for some $r \in P$. Then writing $q = q_1 ... q_{2m}$ and $r = r_1 ... r_{2k}$, we have that $p = p_1 p_2 ... p_{2n} = q_1 ... q_{2m} r_1 ... r_{2k}$. We now have a few cases depending on whether or not q_{2m} and r_1 are the identity.

- $q_{2m} = 1, r_1 = 1$: Then $p = p_1 p_2 ... p_{2n} = q_1 ... q_{2m-1} r_2 ... r_{2k}$ is the unique alternating presentation with nonidentity terms, so $p_1 = q_1, p_2 = q_2$, etc. In particular, $q = q_1 ... q_{2m-1} = p_1 ... p_{2m-1} \in X_{2m} = p_1 ... p_{2m-1}[1, p_{2m}]$.

- $q_{2m} \neq 1, r_1 \neq 1$: Then $p = p_1 p_2 ... p_{2n} = q_1 ... q_{2m} r_1 ... r_{2k}$ is the unique alternating presentation with nonidentity terms, so $p_1 = q_1, p_2 = q_2$, etc. In particular, $q = q_1 ... q_{2m} = p_1 ... p_{2m} \in X_{2m} = p_1 ... p_{2m-1}[1, p_{2m}]$.

- $q_{2m} \neq 1, r_1 = 1$: Then $p = p_1 p_2 ... p_{2n} = q_1 ... (q_{2m} r_2) r_3 ... r_{2k}$ is the unique alternating presentation with nonidentity terms, so $p_1 = q_1, p_2 = q_2$, etc, ending with $p_{2m} = q_{2m} r_2$. Thus $q_{2m} \in [1, p_{2m}]$, so $q = q_1 ... q_{2m} = p_1 p_2 ... p_{2m-1} q_{2m} \in X_{2m} = p_1 ... p_{2m-1}[1, p_{2m}]$.

- $q_{2m} = 1, r_1 \neq 1$: Then $p = p_1 p_2 ... p_{2n} = q_1 ... q_{2m-2}(q_{2m-1} r_1) r_2 ... r_{2k}$ is the unique alternating presentation with nonidentity terms, so $p_1 = q_1, p_2 = q_2$, etc, ending with $p_{2m-1} = q_{2m-1} r_1$. Thus $q_{2m-1} \in [1, p_{2m-1}]$, so $q = q_1 ... q_{2m-1} = p_1 p_2 ... p_{2m-2} q_{2m-1} \in X_{2m-1} = p_1 ... p_{2m-2}[1, p_{2m-1}]$.

In all four cases, q is in some X_i, so $[1, p] = \bigcup_{i=1}^{2n} X_i$, as desired.

For (4), note that for each $y \in X_i$, we have $p_1 p_2 ... p_{i-1} \leq y \leq p_1 p_2 ... p_i$, so $a \leq p_1 p_2 ... p_i \leq p_1 p_2 ... p_{j-1} \leq b$, as desired. $\square$

We can now confirm that weak quasi-lattice order is preserved under free products.

Lemma 3.2.11. *If (G_1, P_1) and (G_2, P_2) are ordered groups (respectively, WQLO groups) then so is $(G_1 * G_2, P_1 * P_2)$.*

Proof. If (G_1, P_1) and (G_2, P_2) are ordered groups, then by definition $P_1 * P_2$ is a submonoid of $G * H$. We must now check that $(P_1 * P_2) \cap (P_1 * P_2)^{-1} = \{1\}$, or equivalently that if $a, b \in P_1 * P_2$ with $ab = 1$, then $a = b = 1$. But if $ab = 1$,

then by writing $a = p_1 p_2 ... p_{2n}$ and $b = q_1 q_2 ... q_{2m}$ as in the previous theorem, then $1 = ab = p_1 p_2 ... p_{2n} q_1 q_2 ... q_{2m}$. By the Normal Form Theorem for Free Products ([16, Chapter IV, Theorem 1.2], a reduced sequence in a free product can equal 1 if and only if its length is 1, so all the p_is and q_is in the products must cancel with each other. But since each term of the product is positive, they can only cancel if each term is equal to 1. Thus $a = b = 1$, as desired, so $(G_1 * G_2, P_1 * P_2)$ is an ordered group.

If (G_1, P_1) and (G_2, P_2) are WQLO, we will check that $(G_1 * G_2, P_1 * P_2)$ is WQLO. To this end, fix some $y_1, y_2 \in P_1 * P_2$ and suppose y_1, y_2 have some common upper bound $y_3 \in P_1 * P_2$. By part (1) of the previous lemma, we can write $y_3 = p_1 p_2 p_3 ... p_{2n-1} p_{2n}$, where each $p_i \in P_1$ for i odd and $p_i \in P_2$ for i even. By part (3) of that lemma, $[1, y_3] = \bigcup_{i=1}^{2n} X_i$ where $X_i = p_1 ... p_{i-1} [1, p_i]$. Since $y_1, y_2 \in [1, y_3]$, then $y_1 \in X_i, y_2 \in X_j$ for some $i, j \le 2n$. Without loss of generality, suppose $i \le j$.

If $i < j$, then by part (4) of the previous Lemma, we have $y_1 \le y_2$, so $y_1 \vee y_2 = y_2$.

If $i = j$, then $y_1, y_2 \in X_i = p_1 ... p_{i-1} [1, p_i]$, so there are r_1, r_2 such that $y_1 = p_1 ... p_{i-1} r_1$ and $y_2 = p_1 ... p_{i-1} r_2$ where $r_1, r_2 \in P_1$ if i is odd, and $r_1, r_2 \in P_2$ if i is even. In either case, $p_1 ... p_{i-1} (r_1 \vee r_2)$ will be the supremum of y_1 and y_2.

$\square$

We will now show that if two groups have a reduction, their free product reduces to the free product of the reductions.

Lemma 3.2.12. *For $i = 1, 2$, let $\varphi_i : (G_i, P_i) \to (H_i, Q_i)$ be a reduction. Then let*

$$G = G_1 * G_2, P = P_1 * P_2, H = H_1 * H_2, \text{ and } Q = Q_1 * Q_2,$$

and let $\varphi : G \to H$ be the homomorphism satisfying $\varphi \mid_{G_i} = \varphi_i$ for $i = 1, 2$. Then

$\varphi : (G, P) \to (H, Q)$ *is a reduction.*

Proof. Certainly, φ is a homomorphism. By Proposition 3.1.4 (3), it suffices to show that for all $p \in P$, the interval $[1, p]$ is mapped bijectively onto $[1, \varphi(p)]$.

Fix some $p \in P$, and then by Lemma 3.2.10, we may write $p = p_1 p_2 ... p_{2n}$ where p_i is in P_1 if i is odd, and in P_2 if i is even. Then $[1, p] = \bigcup_{i=1}^{2n} X_i$, where $X_i = p_1 ... p_{i-1}[1, p_i]$.

Now, for $1 \le i \le 2n$, let $q_i = \varphi_1(p_i)$ if i is odd, and $q_i = \varphi_2(p_i)$ if i is even. Thus $\varphi(p) = q_1 ... q_{2n}$. Note that for $1 < i < 2n$, $q_i \ne 1$ since $p_i \ne 1$ and by Proposition 3.1.4 (4), both φ_1 and φ_2 send strictly positive elements to strictly positive elements.

Thus $\varphi(p) = q_1 ... q_{2n}$ is the unique way to write $\varphi(p) \in Q$ as an alternating product of non-unit elements, so by Lemma 3.2.10, $[1, \varphi(p)] = \bigcup_{i=1}^{2n} Y_i$ where $Y_i = q_1 ... q_{i-1}[1, q_i]$. But since φ_1 and φ_2 are reductions, we know that each X_i is carried bijectively to its Y_i, so it suffices to explain why bijectivity is preserved when taking the union of the X_i and Y_is. It is immediate that the map from $\bigcup X_i$ to $\bigcup Y_i$ is surjective.

For injectivity, first note that for $1 \le i \le j \le n$,

$$Y_i \cap Y_j = \begin{cases} \emptyset & j > i + 1 \\ \{q_1 ... q_i\} & j = i + 1 \\ Y_i & i = j \end{cases}$$

and similarly,

$$X_i \cap X_j = \begin{cases} \emptyset & j > i + 1 \\ \{p_1 ... p_i\} & j = i + 1 \\ X_i & i = j \end{cases}$$

Then, if $\varphi(s_1) \in Y_i$ and $\varphi(s_2) \in Y_j$ for $s_1, s_2 \in [1, p]$ such that $\varphi(s_1) = \varphi(s_2)$, either $i = j$ so $s_1, s_2 \in X_i$ and thus by injectivity on X_i we have $s_1 = s_2$, or $j = i+1$, so $\varphi(s_1) = \varphi(s_2) = q_1...q_i$, so $s_1 = s_2 = p_1...p_i$. In either case, $s_1 = s_2$, so we have injectivity of φ. Thus φ is bijective on $[1, p]$ as desired, so it is a reduction.

$\square$

This completes the first step, showing that if two groups have reductions to amenable groups, their free product reduces to a free product of amenable groups. Now comes the harder step: showing that a free product of amenable groups reduces to an amenable group!

The key is a construction from group theory called a wreath product. Recall the following definition of a wreath product:

Definition 3.2.13. *Let G and H be groups. Let*

$$G^H = \{f : H \to G \text{ a function} | \operatorname{supp}(f) \text{ is finite}\}$$

where $\operatorname{supp}(f) = \{h \in H : f(h) \neq 1\}$, *and give G^H a group structure by pointwise multiplication. Note that G^H is isomorphic to a direct sum of $|H|$ copies of G, hence the notation.*

Give G^H an action $\alpha : H \to Aut(G^H)$ by translation: $[\alpha_h(f)](h') = f(h^{-1}h')$.

Then we define the (restricted) wreath product to be the semidirect product $G \wr H :=$ $G^H \rtimes_\alpha H$, so there is a (split) short exact sequence:

$$1 \to G^H \to G \wr H \to H \to 1.$$

The following remark establishes the basic properties of wreath products, and are all routine to verify.

Remark 3.2.14. *For each $g \in G$ and $h \in H$, let $g\delta_h$ denote the function in G^H*

given by $g\delta_h(h') = \begin{cases} g & h = h' \\ 1 & h \neq h' \end{cases}$. *Then there is a natural embedding of G into $G \wr H$*

by $g \mapsto (g\delta_{1_H}, 1_H)$ and there is a natural embedding of H into $G \wr H$ by $h \mapsto (1_{G^H}, h)$.

In a slight abuse of notation, we will write g or $g\delta_1$ for the embedded element of G in

$G \wr H$, and write h for the embedded element of H in $G \wr H$.

Now note that

$$
\begin{aligned}
[\alpha_{h_2}(g\delta_{h_1})](h_3) &= g\delta_{h_1}(h_2^{-1}h_3) \\
&= \begin{cases} g & h_1 = h_2^{-1}h_3 \\ 1 & h_1 \neq h_2^{-1}h_3 \end{cases} \\
&= \begin{cases} g & h_2 h_1 = h_3 \\ 1 & h_2 h_1 \neq h_3 \end{cases} \\
&= g\delta_{h_2 h_1}(h_3)
\end{aligned}
$$

so unbinding the h_3, we have that $\alpha_{h_2}(g\delta_{h_1}) = g\delta_{h_2 h_1}$. Now,

$$
\begin{aligned}
h_2(g\delta_{h_1})h_2^{-1} &= (1_G, h_2)(g\delta_{h_1}, 1_H)(1_G, h_2^{-1}) \\
&= (1_G, h_2)(g\delta_{h_1}, h_2^{-1}) \\
&= (1_G, h_2)(g\delta_{h_1}, h_2^{-1}) \\
&= (\alpha_{h_2}(g\delta_{h_1}), 1_H) \\
&= g\delta_{h_2 h_1}
\end{aligned}
$$

Since G^H is generated by the $g\delta_h$, then together (the embedded copies of) G and H generate $G \wr H$.

Given P, Q positive cones in G, H, we let $P \wr Q$ denote the monoid generated by the naturally embedded copies of P and Q within $G \wr H$.

With these natural embeddings of G and H into $G \wr H$, by the universal property of free products of groups, we get a homomorphism $\varphi : G * H \to G \wr H$ given by sending (the naturally embedded copies of) G and H to (the naturally embedded copies of) G and H. We will call this the natural homomorphism of $G * H$ into $G \wr H$. Since $G \wr H$ is generated by G and H, it is immediate that the natural homomorphism is surjective. For the same reason, $P * Q$ will be mapped surjectively onto $P \wr Q$. We will show later that $\varphi : (G * H, P * Q) \to (G \wr H, P \wr Q)$ is a strong reduction.

Finally, note that every element of G^H can be written as a (possibly empty) product of a finite number of $g_i \delta_{h_i}$ where each g_i is nonunital and each h_i is distinct, and this presentation is unique up to a permutation of the $g_i \delta_{h_i}$. Note that the element $g_i \delta_{h_i}$ commutes with the element $g_j \delta_{h_j}$ as long as $h_i \neq h_j$. When $f \in G^H$ is written as $f = g_1 \delta_{h_1} ... g_n \delta_{h_n}$ where the h_i are distinct, then

$$f(h) = \begin{cases} g_1 & \text{if } h = h_1 \\ g_2 & \text{if } h = h_2 \\ \vdots & \\ g_n & \text{if } h = h_n \\ 1 & \text{otherwise} \end{cases}.$$

Finally, we will remind the reader of this fact about amenability of wreath products:

Lemma 3.2.15. *Let G and H be amenable groups. Then $G \wr H$ is amenable.*

Proof. Recall that the class of amenable groups are closed under finite direct sums, direct limits, and extensions.

From our definition of the wreath product, there is a short exact sequence

$$1 \to G^H \to G \wr H \to H \to 1.$$

Recall also that G^H is isomorphic to the direct sum of $|H|$ copies of G. Thus G^H is a direct limit of $\{G^n\}_{n \in \mathbb{N}}$. Since amenable groups are closed under finite direct sum and G is amenable by hypothesis, then each G^n is amenable, and since amenable groups are closed under direct limit, then G^H is amenable.

Also, H is amenable by hypothesis, so $G \wr H$ is an extension of an amenable group by an amenable group. Thus $G \wr H$ is amenable.

$\square$

Now, we will have two results that establish why $\varphi : (G_1 * G_2, P_1 * P_2) \to (G_1 \wr G_2, P_1 \wr P_2)$ is a reduction.

Lemma 3.2.16. *Let (G_1, P_1) and (G_2, P_2) be ordered groups, and let $p = p_1 p_2 ... p_{2n} \in P_1 \wr P_2$ be an element such that $p_i \in P_1$ for i odd and $p_i \in P_2$ for i even and $p_i \neq 1$ for $1 < i < n$. Then, as an element of $P_1 \wr P_2 \subset G_1^{G_2} \rtimes G_2$, $p = (f, p_{\leq 2n})$ where $p_{\leq 2n} = p_2 p_4 ... p_{2n}$ and*

$$f(h) = \begin{cases} p_1 & \textit{if } h = 1_{G_2} \\[6pt] p_3 & \textit{if } h = p_2 \\[6pt] p_5 & \textit{if } h = p_2 p_4 \\[6pt] \vdots & \\[6pt] p_{2n-1} & \textit{if } h = p_2...p_{2n-2} \\[6pt] 1 & \textit{otherwise} \end{cases}$$

Proof. By elementary algebra, we may rewrite the product $p = p_1 p_2 p_3 ... p_{2n}$ as:

$$p = p_1 p_2 p_3 ... p_{2n} = p_1 (p_2 p_3 p_2^{-1})...(p_2 p_4 ... p_{2n-2} p_{2n-1} p_{2n-2}^{-1} ... p_4^{-1} p_2^{-1})(p_2 p_4 ... p_{2n})$$

With the shorthand that $p_{\leq 0} = 1_H$ and $p_{\leq 2i} = p_2 p_4 ... p_{2i}$ for $1 \leq i \leq n$, we then have that

$$p = \left[(p_{\leq 0} p_1 p_{\leq 0}^{-1})(p_{\leq 2} p_3 p_{\leq 2}^{-1})(p_{\leq 4} p_5 p_{\leq 4}^{-1})...(p_{\leq 2n-2} p_{2n-1} p_{\leq 2n-2}^{-1}) \right] p_{\leq 2n}.$$

Let $f = (p_{\leq 0} p_1 p_{\leq 0}^{-1})(p_{\leq 2} p_3 p_{\leq 2}^{-1})(p_{\leq 4} p_5 p_{\leq 4}^{-1})...(p_{\leq 2n-2} p_{2n-1} p_{\leq 2n-2}^{-1})$ be the bracketed term, so then p is represented in $G_1^{G_2} \rtimes G_2$ by $(f, p_{\leq 2n})$. It then suffices to show that f assumes the values as claimed.

Recall that for i odd, p_i is embedded as $p_i \delta_{1_H}$, and for i even, p_i is embedded as an element of G_2. By the calculation in the above remark, we then have that $p_{\leq 2i} p_{2i+1} p_{\leq 2i}^{-1} = p_{\leq 2i}(p_{2i+1} \delta_{1_H}) p_{\leq 2i}^{-1} = p_{2i+1} \delta_{p_{\leq 2i}}$, so $f = (p_1 \delta_{p_{\leq 0}})(p_3 \delta_{p_{\leq 2}})...(p_{2n-1} \delta_{p_{\leq 2n-2}})$.

Note that $p_{\leq 2i+2} = p_{\leq 2i} p_{2i+2}$, and since $p_{2i+2} > 1$ for $0 \leq i < n-1$, we have that $p_{\leq 2i} < p_{\leq 2i+2}$ for $0 \leq i < n-1$, and in particular the $p_{\leq 2i}$ are distinct (except

possibly that $p_{\leq 2n-2} = p_{\leq 2n}$).

Since the $\{p_{\leq 0}, ..., p_{\leq 2n-2}\}$ are distinct, then by the above remark, we know that

$$f = (p_1 \delta_{p_{\leq 0}})(p_3 \delta_{p_{\leq 2}})...(p_{2n-1} \delta_{p_{\leq 2n-2}})$$

is the unique representation of f as a product of $g_i \delta_{h_i}$ (except possibly that the $p_1 \delta_{p_{\leq 0}}$ term is equal to 1), so

$$f(h) = \begin{cases} p_1 & \text{if } h = p_{\leq 0} \\ p_3 & \text{if } h = p_{\leq 2} \\ \vdots & \\ p_{2n-1} & \text{if } h = p_{\leq 2n-2} \\ 1 & \text{otherwise} \end{cases}$$

as desired.

$\square$

Proposition 3.2.17. *Let (G_1, P_1), (G_2, P_2) be two ordered groups. Let φ denote the natural homomorphism from $G_1 * G_2$ onto $G_1 \wr G_2$. If $p, q \in P := P_1 * P_2$ and $\varphi(p) = \varphi(q)$, then $p = q$.*

In particular, φ is a strong reduction of $G_1 * G_2$ onto $G_1 \wr G_2$.

Proof. Since $p, q \in P$, then by Lemma 3.2.10 we may write $p = p_1 p_2 ... p_{2n}$ where p_i is in P_1 if i is odd, and in P_2 if i is even, and all $p_i \neq 1$ for $1 < i < n$. Similarly, we may write $q = q_1 q_2 ... q_{2m}$ where q_i is in P_1 if i is odd, and in P_2 if i is even, and all $p_i \neq 1$ for $1 < i < m$.

Now, by the previous Lemma, we may write $\varphi(p) = (f_p, p_{\leq 2n})$ and $\varphi(q) = (f_q, q_{\leq 2m})$ where $p_{\leq 2i} = p_2 p_4 ... p_{2i}$, $q_{\leq 2i} = q_2 q_4 ... q_{2i}$,

$$
f_p(h) = \begin{cases} p_1 & \text{if } h = 1_{G_2} \\[1ex] p_3 & \text{if } h = p_2 \\[1ex] p_5 & \text{if } h = p_2 p_4 \\[1ex] \vdots \\[1ex] p_{2n-1} & \text{if } h = p_2 ... p_{2n-2} \\[1ex] 1 & \text{otherwise} \end{cases} \quad \text{and } f_q(h) = \begin{cases} q_1 & \text{if } h = 1_{G_2} \\[1ex] q_3 & \text{if } h = q_2 \\[1ex] q_5 & \text{if } h = q_2 q_4 \\[1ex] \vdots \\[1ex] q_{2m-1} & \text{if } h = q_2 ... q_{2m-2} \\[1ex] 1 & \text{otherwise} \end{cases}
$$

Since $\varphi(p) = \varphi(q)$, then $f_p = f_q$, so $p_1 = q_1$. Furthermore, $\{p_2, p_2 p_4, ..., p_2...p_{2n-2}\} = \operatorname{supp}(f_p) \setminus \{1\} = \operatorname{supp}(f_q) \setminus \{1\} = \{q_2, q_2 q_4, ..., q_2...q_{2m-2}\}$, and these sets are strictly increasing sequences, so it must be that $n = m$ and $p_2 = q_2$, $p_2 p_4 = q_2 q_4$, and so on. By cancelling common terms, we get that $p_{2i} = q_{2i}$ for $1 \leq i < n$. Returning to our expressions for f_p and f_q, we can now compare their values at $p_2 = q_2, p_2 p_4 = q_2 q_4$, etc to get that $p_3 = q_3, p_5 = q_5$, and so on. Finally, recall that $(f_p, p_{\leq 2n}) = \varphi(p) = \varphi(q) = (f_q, q_{\leq 2n})$ so $p_{\leq 2n} = q_{\leq 2n}$ and by cancelling the common factor of $p_2...p_{2n-2} = q_2...q_{2n-2}$, we have that $p_{2n} = q_{2n}$.

Thus for all $1 \leq i \leq 2n$, we have that $p_i = q_i$, so $p = q$, as desired.

For the "in particular", by Corollary 3.1.5 it suffices to show that φ is a bijection of $P_1 * P_2$ onto $P_1 \wr P_2$. It is immediate that φ is surjective, and by the previous part of the proposition it must be injective. Thus φ is a bijection of $P_1 * P_2$ onto $P_1 \wr P_2$, so φ is a strong reduction as desired.

$\square$

We have done the hard work already, so we may now reap our rewards:

Proposition 3.2.18. *Let $(G, P), (H, Q)$ be ordered groups, and suppose that G and H are amenable. Then the natural homomorphism $\varphi : (G * H, P * Q) \to (G \wr H, P \wr Q)$ is a strong reduction to an amenable group.*

Proof. By Propositon 3.2.17, φ is a strong reduction. By Lemma 3.2.15, $G \wr H$ is amenable, so φ is a strong reduction to an amenable group.

$\square$

And combining this with a previous result gives:

Corollary 3.2.19. *The class of ordered groups which reduce to an amenable group is closed under finite free products.*

Proof. It suffices to check two-term free products. Suppose that (G_1, P_1) and (G_2, P_2) are ordered groups with reductions φ_1, φ_2 to amenable groups (H_1, Q_1) and (H_2, Q_2) respectively. Let $G = G_1 * G_2$, $P = P_1 * P_2$, $H = H_1 * H_2$, and $Q = Q_1 * Q_2$.

Now by Lemma 3.2.12 there is a reduction φ of (G, P) onto (H, Q). By Proposition 3.2.18, since (H, Q) is the free product of two amenable groups, then there is a strong reduction $\varphi' : (H_1 * H_2, Q_1 * Q_2) \to (H_1 \wr H_2, Q_1 \wr Q_2)$ where $H_1 \wr H_2$ is amenable. Then $\varphi'\varphi : (G, P) \to (F, R)$ is a composition of reductions, hence a reduction by Lemma 3.2.1, and since its range is amenable, (G, P) reduces to an amenable group. $\square$

3.3   Summary

To summarize our results on reductions to amenable groups:

Theorem 3.3.1. *The class of ordered groups which have (strong) reductions onto amenable groups contains all amenable ordered groups and is closed under hereditary subgroups, finite direct products, and finite free products.*

Proof. If (G, P) is an ordered group with G amenable, then $\text{id}_G : (G, P) \to (G, P)$ is a reduction to an amenable group, so every amenable ordered group contains a reduction to an amenable group.

We've seen that this class is closed under hereditary subgroups in Corollary 3.2.6, direct products in Lemma 3.2.9, and free products in Corollary 3.2.19.

$\square$

Corollary 3.3.2. *Let $G = \mathbb{Z}^2 * \mathbb{Z}$ and $P = \mathbb{N}^2 * \mathbb{N}$. Then (G, P) strongly reduces to an amenable group.*

Proof. By Proposition 3.2.17, $(\mathbb{Z}^2 * \mathbb{Z}, \mathbb{N}^2 * \mathbb{N})$ strongly reduces to $(\mathbb{Z}^2 \wr \mathbb{Z}, \mathbb{N}^2 \wr \mathbb{N})$, which is amenable by Lemma 3.2.15. $\square$

Although the theory we have developed gives a rich class of examples of groups which reduce to amenable groups, there are still many open questions worthy of exploration:

Question 3.3.3. *Are there any ordered groups that reduce to an amenable group, but do not strongly reduce to an amenable group?*

Question 3.3.4. *Is the class of ordered groups which reduce to amenable groups closed under other group theoretic constructions (direct limits, HNN extensions, amalgamated free products, graph products, etc)?*

For the latter question, we would conjecture that the answer is yes, and that this should follow from this class being closed under quotients by "positive relators":

Conjecture 3.3.5. *Let (G, P) be an ordered group. Let $p_1, p_2 \in P$ be incomparable (meaning $p_1 \nleq p_2$ and $p_1 \ngeq p_2$), and let N denote the normal subgroup of G generated by $p_1 p_2^{-1}$. Then:*

a. If (G, P) *reduces to an amenable group, then* $(G/N, P/N)$ *reduces to an amenable group.*

b. If (G, P) *strongly reduces to an amenable group, then* $(G/N, P/N)$ *strongly reduces to an amenable group.*

Of these two sub-conjectures, (b) seems more plausible: one could imagine "carrying the quotient forward" into the reduction. However, attempting that in the case of a non-strong reduction of (a) seems more difficult. For instance, we've seen that (F_2, P_2) reduces to $(\mathbb{Z}, \mathbb{N})$, and taking $p_1 = ab, p_2 = ba$ results in the relator $ab = ba$, so $F_2/N \cong \mathbb{Z}^2$ and $P_2/N \cong \mathbb{N}^2$, but $(\mathbb{Z}^2, \mathbb{N}^2)$ does not reduce to $(\mathbb{Z}, \mathbb{N})$ or any quotient thereof. While $(\mathbb{Z}^2, \mathbb{N}^2)$ reduces to a different amenable group (itself), it is not clear in general how one would avoid this type of issue.

Chapter 4

Gauge-Invariant Uniqueness for P-graphs

In this section, our goal is to prove a gauge-invariant uniqueness theorem for P-graphs: there is exactly one representation of a P-graph which is Λ-faithful, tight, and has a gauge coaction. However, we will show in Lemma 4.2.2 that the gauge-invariant uniqueness theorem can fail in general, so we require an additional hypothesis. That additional hypothesis is that (G, P) can reduce to an amenable ordered group, and we will prove a gauge-invariant uniqueness theorem for such P-graphs in Theorem 4.4.2.

4.1 A Co-Universal Algebra

In this section, we will prove a weaker form of a gauge invariant uniqueness theorem which says that there is exactly one representation of a P-graph which is Λ-faithful, tight, and has a *normal* gauge coaction. This result is a slight generalization of [3, Theorem 5.3] to the context of *weakly* quasi-lattice ordered groups.

To prove this, we define and study the balanced algebra of a representation, culminating in Theorem 4.1.15, which classifies the balanced algebras of a graph.

To give a brief summary of the argument:

1. Use the structure of "infinite paths" in Λ to construct a tight, Λ-faithful repre-

sentation called the ultrafilter representation f.

2. For a representation t, we define the balanced algebra $\mathcal{B}(t) = \overline{\text{span}}\{t_\mu t_\nu^* : d(\mu) = d(\nu)\}$ (Lemma 4.1.7). If there is a gauge coaction on t, there is a conditional expectation $\Phi_t : C^*(t) \to \mathcal{B}(t)$ which is given by

$$\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise} \end{cases},$$

and this conditional expectation is faithful if and only if the gauge coaction is normal (Lemma 4.1.8). We define a balanced covering to be a map $\psi_s^t : \mathcal{B}(t) \to \mathcal{B}(s)$ given by $t_\mu t_\nu^* \mapsto s_\mu s_\nu^*$.

3. Recalling that $\mathcal{T}$ denotes the Toeplitz representation of Λ from Definition 2.5.8, in Theorem 4.1.15 we show that $\ker \psi_t^{\mathcal{T}}$ is generated by the "bolts" and range projections it contains. Using this, we show in Lemma 4.1.16 that any tight representation is balanced covered by any Λ-faithful representation.

4. In Lemma 4.1.21 we show that in the presence of appropriate coactions, we can lift a balanced covering to a covering of the entire algebras.

5. Finally, we show in Theorem 4.1.22 that there is a unique tight, Λ-faithful representation with a normal gauge coaction, and that this representation is co-universal for Λ-faithful coacting representations in the sense of [3, Theorem 5.3].

4.1.1 The Ultrafilter Representation

In this subsection, we construct the ultrafilter representation f of a P-graph Λ, and demonstrate that it is Λ-faithful and tight. This construction follows the one in [3,

Section 3] and is included here for completeness.

Definition 4.1.1. *Let (G, P) be a WQLO group, and Λ a finitely aligned P-graph. Recall that Λ has a partial order given by $\alpha \leq \beta$ if $\beta \in \alpha\Lambda$.*

A <u>filter</u> of Λ is a nonempty subset $U \subseteq \Lambda$ such that:

(F1) If $\mu \in U$ and $\lambda \leq \mu$, then $\lambda \in U$

(F2) If $\mu, \nu \in U$, there exists a $\lambda \in U$ such that $\mu \leq \lambda$ and $\nu \leq \lambda$.

Given a filter U, it is nonempty so it contains some μ. By F1, $r(\mu) \in U$, and by F2 since no distinct vertices have a common extension, then $r(\mu)$ is the only vertex in U. Therefore, for any $\mu, \nu \in U$, $r(\mu) = r(\nu)$, so we write $r(U)$ for this unique vertex.

A <u>ultrafilter</u> of Λ is a maximal filter. We denote the set of filters by $\widehat{\Lambda}$ and the set of ultrafilters by $\widehat{\Lambda}_\infty$.

Lemma 4.1.2. *Every filter is contained in an ultrafilter.*

Proof. The proof is a standard Zorn's Lemma argument: fix a filter U, and let $\widehat{\Lambda}_U = \{V \in \widehat{\Lambda} : U \subseteq V\}$. We will show that $\widehat{\Lambda}_U$ contains a maximal element by Zorn's lemma, and that this maximal element of $\widehat{\Lambda}_U$ is also maximal in $\widehat{\Lambda}$.

To apply Zorn's Lemma, we must check that $\widehat{\Lambda}_U = \{V \in \widehat{\Lambda} : U \subseteq V\}$ is nonempty and that chains in $\widehat{\Lambda}_U$ have an upper bound in $\widehat{\Lambda}_U$. The former is immediate since $U \in \widehat{\Lambda}_U$. If $\mathcal{C}$ is a chain in $\widehat{\Lambda}_U$, then let $W = \bigcup_{V \in \mathcal{C}} V$, which we will show is a filter. To check F1, if $\mu \in W$ and $\nu \leq \mu$, then $\mu \in V$ for some $V \in C$, so $\nu \in V$ and thus $\nu \in W$. To check F2, if $\mu, \nu \in W$, then $\mu \in V_1$, $\nu \in V_2$ for some $V_1, V_2 \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$. Without loss of generality assuming the former, we have that $\mu, \nu \in V_2$, so there is some $\lambda \in V_2 \subseteq W$ with $\mu, \nu \leq \lambda$ as desired.

Since W also contains U, then $W \in \widehat{\Lambda}_U$, so by Zorn's Lemma $\widehat{\Lambda}_U$ contains a maximal element U_∞.

Finally, we will show that U_∞ is an ultrafilter, meaning that we will check that U_∞ is maximal in $\widehat{\Lambda}$. If it were not, there would be some $V \supsetneq U_\infty \supseteq U$, but then we'd have $V \in \widehat{\Lambda}_U$, so U_∞ was not maximal in $\widehat{\Lambda}_U$, a contradiction. Thus U_∞ is indeed maximal in $\widehat{\Lambda}$, so $U_\infty \in \widehat{\Lambda}_\infty$ as desired.

$\square$

Lemma 4.1.3. *Let U be an ultrafilter, and suppose $\mu \in U$, $E \subset \mu\Lambda$, and that E is exhaustive for $\mu\Lambda$. Then $E \cap U$ is nonempty.*

Proof. We split into two cases: U is finite and U is infinite. If U is finite, it has a greatest element ω by F2. Since $\mu \leq \omega$, then $\omega \in \mu\Lambda$, and since E is exhaustive for $\mu\Lambda$ there is some $\alpha \in E$ with $MCE(\alpha, \omega) \neq \emptyset$. That is, we can find some $\beta \in MCE(\alpha, \omega)$, so define $U' = \{\nu : \nu \leq \beta\}$. Then U' is a filter containing U, and since U was an ultrafilter then $U' = U$. Thus $\beta \in U$ and by F1, $\alpha \leq \beta \in U$, as desired.

Suppose instead that U is infinite. Since Λ is countable, so is U, so let give U an enumeration $U = \{\nu_1, \nu_2, ...\}$ and without loss of generality suppose $\mu = \nu_1$. By the F2 property, for each $n \in \mathbb{N}$, we may choose an η_n such that $\eta_{n-1} \leq \eta_n$ if $n > 1$ and $\nu_i \leq \eta_n$ for all $1 \leq i \leq n$. Then in particular $\{\eta_n\}_{n \in \mathbb{N}}$ is a rising sequence of elements of U such that $\eta_n \in \mu\Lambda$ for all n, and by the F1 property $\lambda \in U$ if and only if $\lambda \leq \eta_n$ for some n. Finally, we may assume that the sequence $\{\eta_n\}_{n \in \mathbb{N}}$ is strictly increasing by passing to a subsequence.

Now, for each η_n, since $\eta_n \in \mu\Lambda$ and E is exhaustive is $\mu\Lambda$, there is an $\alpha_n \in E$ such that η_n has a common extension with α_n. But E is finite, so one $\alpha \in E$ must appear infinitely many times in the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$. Thus there are infinitely many

n with $MCE(\eta_n, \alpha)$ nonempty. Our goal now is to show that $\alpha \in U$ by constructing a potentially bigger ultrafilter that contains α, but since U is already an ultrafilter it must be that α is already in U.

To this end, for $i \leq j$, if $\lambda \in MCE(\eta_j, \alpha)$, we can factorize $\lambda = \zeta\zeta'$ where $d(\zeta) = d(\eta_i) \vee d(\alpha)$, which implies $\zeta \in MCE(\eta_i, \alpha)$, and therefore $\lambda \in \zeta\Lambda$ for some $\zeta \in MCE(\eta_i, \alpha)$. Thus

$$MCE(\eta_j, \alpha) = \bigcup_{\zeta \in MCE(\eta_i, \alpha)} [MCE(\eta_j, \alpha) \cap \zeta\Lambda].$$

Considering i as fixed but taking the union over all $j \geq i$, we have that

$$\begin{aligned}
\bigcup_{j \geq i} MCE(\eta_j, \alpha) &= \bigcup_{j \geq i} \left(\bigcup_{\zeta \in MCE(\eta_i, \alpha)} [MCE(\eta_j, \alpha) \cap \zeta\Lambda] \right) \\
&= \bigcup_{\zeta \in MCE(\eta_i, \alpha)} \left(\bigcup_{j \geq i} MCE(\eta_j, \alpha) \cap \zeta\Lambda \right)
\end{aligned}$$

Since infinitely many $MCE(\eta_j, \alpha)$ are nonempty and the sequence $\{\eta_n\}_{n \in \mathbb{N}}$ is strictly increasing, the lefthand side is infinite, and thus the righthand side must have one of the terms $\bigcup_{j \geq i} MCE(\eta_j, \alpha) \cap \zeta\Lambda$ being infinite. In particular, when $i = 1$, there is some ζ_1 such that $MCE(\eta_j, \alpha) \cap \zeta_1\Lambda$ is infinite. Then we may repeat a version of this argument: we have

$$\bigcup_{j \geq 2} MCE(\eta_j, \alpha) \cap \zeta_1\Lambda = \bigcup_{\zeta \in MCE(\eta_2, \alpha)} \left(\bigcup_{j \geq 2} MCE(\eta_j, \alpha) \cap \zeta\Lambda \cap \zeta_1\Lambda \right)$$

so one of the terms $\bigcup_{j \geq 2} MCE(\eta_j, \alpha) \cap \zeta\Lambda \cap \zeta_1$ must be infinite, so we take ζ_2 to be the element of $MCE(\eta_2, \alpha)$ giving that infinite term. Next, since $\zeta_1 \leq \zeta_2$ then $\zeta_2\Lambda \cap \zeta_1 = \zeta_2\Lambda$, and we may repeat this process inductively.

In this way we create a sequence $\{\zeta_n\}_{n \in \mathbb{N}}$ with $\zeta_n \in MCE(\eta_n, \alpha)$ and $\zeta_{n-1} \leq \zeta_n$ for all n. Finally, let $U_\infty = \{\lambda : \lambda \leq \zeta_n \text{ for some } n \in \mathbb{N}\}$. Then it is immediate to verify that U_∞ is a filter. Furthermore, if $\lambda \in U$, then $\lambda \leq \eta_n$ for some n, so $\lambda \leq \eta_n \leq \zeta_n$, so $\lambda \in U_\infty$. That is, $U \subseteq U_\infty$, but U was an ultrafilter, so it must be that $U = U_\infty$. Since $\alpha \leq \zeta_1$, then $\alpha \in U_\infty = U$, as desired.

In either case, we have shown that there is some $\alpha \in E \cap U$. $\qquad\square$

The following result is [3, Lemma 3.4]:

Lemma 4.1.4. *Let (G, P) be a WQLO group, Λ a finitely-aligned P-graph, $\lambda \in \Lambda$ and let U and V be filters. If $r(U) = s(\lambda)$ and if $\lambda \in V$, we define*

$$\lambda \cdot U := \bigcup_{\mu \in U} \{\alpha \leq \lambda\mu\} \ and$$

$$\lambda^* \cdot V := \{\mu \in \Lambda : \lambda\mu \in V\}$$

Then $\lambda \cdot U$ and $\lambda^* \cdot V$ are filters and if U and V are ultrafilters, then so are $\lambda \cdot U$ and $\lambda^* \cdot V$. Finally, $\lambda^* \cdot (\lambda \cdot U) = U$ and $\lambda \cdot (\lambda^* \cdot V) = V$.

We are now ready to define our Λ-faithful tight representation by letting Λ act by translation on its set of ultrafilters:

Lemma 4.1.5. *Let (G, P) be a WQLO group, and Λ a finitely-aligned P-graph. For $\lambda \in \Lambda$, define an operator $f_\lambda \in \mathcal{B}(\ell^2(\widehat{\Lambda}_\infty))$ by*

$$f_\lambda e_U = \begin{cases} e_{\lambda \cdot U} & \text{if } r(U) = s(\lambda) \\ 0 & \text{otherwise} \end{cases}.$$

Then f is a representation of Λ which is Λ-faithful and tight. We call f the
ultrafilter representation.

Proof. Defining the operators f_λ as given above, a typical inner product argument
shows that $f_\lambda^* e_V = \begin{cases} e_{\lambda^* \cdot V} & \text{if } \lambda \in V \\ 0 & \text{otherwise} \end{cases}$.

To check the T1 operator, it is immediate that if $v \in \Lambda^0$, then f_v is a projection,
namely a projection onto the subspace spanned by the set of ultrafilters containing
v. To show that the $\{f_v\}_{v \in \Lambda^0}$ are orthogonal, first note that this family commutes,
and fix distinct $v, w \in \Lambda^0$. For any ultrafilter U, $r(U)$ is the unique vertex contained
in U, so either $v \notin U$ or $w \notin U$. In the former case, $f_w f_v e_U = 0$ and in the latter case
$f_v f_w e_U = 0$, so in either case $f_v f_w = f_w f_v = 0$.

The T2 operator is immediate.

The T3 operator is immediate from the previous lemma.

For the T4 operator, observe that $f_\lambda f_\lambda^*$ is projection onto the ultrafilters contain-
ing λ. Therefore, $f_\mu f_\mu^* f_\nu f_\nu^*$ is projection onto the ultrafilters containing μ and ν. By
the F2 and F1 properties, such an ultrafilter would contain some $\lambda \in MCE(\mu, \nu)$. In
fact, it would contain exactly one such term, since if it contained two distinct $\lambda_1, \lambda_2 \in
MCE(\mu, \nu)$, by the F2 property it would contain a common extension of λ_1 and λ_2,
but no such extension exists by the uniqueness of factorizations. Thus the set of ultra-
filters containing μ and ν is precisely the disjoint union of the ultrafilters containing
a $\lambda \in MCE(\mu, \nu)$. As operators, this is to say that $f_\mu f_\mu^* f_\nu f_\nu^* = \sum_{\lambda \in MCE(\mu,\nu)} f_\lambda f_\lambda^*$.

Thus f is indeed a representation.

To show that f is Λ-faithful, fix some $\lambda \in \Lambda$. Since $\{s(\lambda)\}$ is a filter, then by
Lemma 4.1.2, there is an ultrafilter U containing $s(\lambda)$, in which case $r(U) = s(\lambda)$.
Then, $f_\lambda \cdot e_U = e_{\lambda \cdot U} \neq 0$, so $f_\lambda \neq 0$.

To show that f is tight, fix some $\mu \in \Lambda$ and $E \subset \mu\Lambda$ which is finite and exhaustive for $\mu\Lambda$. Let $B = \prod_{\alpha \in E}(f_\mu f_\mu^* - f_\alpha f_\alpha^*)$ be the corresponding bolt (see Definition 4.1.10). We'll now show that for any ultrafilter U, $Be_U = 0$, and this implies that $B = 0$ as desired.

First, if U does not contain μ, then it also does not contain any of the $\alpha \in E$, so $f_\mu f_\mu^* e_U = 0 = f_\alpha f_\alpha^* e_U$, and thus $Be_U = 0$.

If instead U does contain μ, by the previous lemma, then U contains one of the $\alpha \in E$.

Then, since $\mu, \alpha \in U$, $f_\mu f_\mu^* e_U = e_U = f_\alpha f_\alpha^* U$, so $Be_U = \prod_{\alpha \in E}(f_\mu f_\mu^* - f_\alpha f_\alpha^*)e_U = 0$.

We have now shown that $Be_U = 0$ for all ultrafilters U, so $B = 0$ as an operator in $\mathcal{B}(\ell^2(\widehat{\Lambda}_\infty))$. Thus f is tight.

$\square$

Since the ultrafilter representation is Λ-faithful and tight, it is natural to ask if it also has a gauge coaction. The answer is often no, as we will show in the next example.

Example 4.1.6. *Let (G, P) be a WQLO group, and suppose that P is directed, meaning that any two elements of P have a common upper bound. Examples of directed positive cones include $\mathbb{N}^k$ and any total order. Then letting $\Lambda = P$, Λ itself is a filter, so Λ is the unique ultrafilter. Thus $\widehat{\Lambda}_\infty = \{\Lambda\}$, so the ultrafilter representation is given by $f_\mu = 1$ for all $\mu \in \Lambda$. In particular, the ultrafilter representation fails to have a gauge coaction if there are distinct $\mu, \nu \in \Lambda = P$, since such a coaction δ cannot send $f_\mu = f_\nu$ to the distinct elements $f_\mu \otimes U_{d(\mu)}$ and $f_\mu \otimes U_{d(\nu)}$.*

4.1.2 Balanced Algebras and Balanced Coverings

We will now discuss a particularly nice AF subalgebra of a P-graph algebra. Our approach in this section mimics [3, Section 4], although we use slightly different notation.

Lemma 4.1.7. *Let (G, P) be a WQLO group, Λ a finitely-aligned P-graph, and t a representation of Λ. Then $\mathcal{B}(t) = \overline{\text{span}}\{t_\mu t_\nu^* : d(\mu) = d(\nu)\}$ is a closed $*$-subalgebra of $C^*(t)$. We call $\mathcal{B}(t)$ the* balanced (sub)algebra.

Proof. It is immediate that $\mathcal{B}(t)$ is a closed $*$-invariant subspace, so it suffices to check that it is closed under multiplication. To this end, suppose $t_\mu t_\nu^*, t_\alpha t_\beta^*$ satisfy $d(\mu) = d(\nu)$ and $d(\alpha) = d(\beta)$. Then,

$$
\begin{aligned}
(t_\mu t_\nu^*)(t_\alpha t_\beta^*) &= (t_\mu t_\nu^*)(t_\nu t_\nu^* t_\alpha t_\alpha^*)(t_\alpha t_\beta^*) \\
&= (t_\mu t_\nu^*)\Big(\sum_{\lambda \in MCE(\nu,\alpha)} t_\lambda t_\lambda^* \Big)(t_\alpha t_\beta^*) \\
&= \sum_{\lambda \in MCE(\nu,\alpha)} t_{\mu(\nu^{-1}\lambda)} t_{\beta(\alpha^{-1}\lambda)}^*
\end{aligned}
$$

where $\nu^{-1}\lambda$ denotes the unique path such that $\nu(\nu^{-1}\lambda) = \lambda$ and similarly $\alpha^{-1}\lambda$ denotes the unique path such that $\alpha(\alpha^{-1}\lambda) = \lambda$. Then note that

$$
d(\mu(\nu^{-1}\lambda)) = d(\mu)d(\nu)^{-1}d(\lambda) = d(\lambda) = d(\alpha)d(\beta)^{-1}d(\lambda) = d(\alpha(\beta^{-1}\lambda))
$$

so the product is in $\mathcal{B}(t)$ as desired. $\square$

In the following lemma, we rephrase Lemma 2.3.12 for the context of a gauge coaction on a P-graph C^*-algebra:

Lemma 4.1.8. *Let (G, P) be a WQLO group, Λ a finitely-aligned P-graph, t a representation of Λ and suppose t has a gauge coaction δ. Then there is a conditional expectation $\Phi_t : C^*(t) \to \mathcal{B}(t)$ such that*

$$
\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise} \end{cases}.
$$

Proof. Let $A = C^*(t)$. For any $\mu, \nu \in \Lambda$, observe that $\delta(t_\mu t_\nu^*) = t_\mu t_\nu^* \otimes U_{d(\mu)d(\nu)^{-1}}$, so $t_\mu t_\nu^* \in A_{d(\mu)d(\nu)^{-1}}$. Then $t_\mu t_\nu^* \in A_e$ if and only if $d(\mu) = d(\nu)$, so $\mathcal{B}(t) = A_e$.

Let $\Phi_t : C^*(t) \to \mathcal{B}(t)$ be the conditional expectation arising from Lemma 2.3.12. If $d(\mu) = d(\nu)$, then $t_\mu t_\nu^* \in A_e$, and Φ_t fixes its range space, so $\Phi_t(t_\mu t_\nu^*) = t_\mu t_\nu^*$. If $d(\mu) \neq d(\nu)$, then $t_\mu t_\nu^* \in A_g$ for $g = d(\mu)d(\nu)^{-1} \neq e$, and by Lemma 2.3.12, Φ_t vanishes on all A_g for $g \neq e$, so $\Phi_t(t_\mu t_\nu^*) = 0$, as desired.

$\square$

Recall from Definition 2.3.13 that Φ_t is a faithful conditional expectation if and only if δ is a normal coaction.

Definition 4.1.9. *Let (G, P) be a WQLO group, Λ a P-graph, and s, t two representations of Λ. Let us say that a <u>balanced covering</u> is a (necessarily surjective) $*$-homomorphism $\psi_s^t : \mathcal{B}(t) \to \mathcal{B}(s)$ given by*

$$
\psi_s^t(t_\mu t_\nu^*) = s_\mu s_\nu^*
$$

for all $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$.

If such a $*$-homomorphism exists, we will write $t \geq_{bal} s$. It is immediate that $\geq_{bal}$ is reflexive and transitive, but it may not be a partial ordering since it may not be antisymmetric (that is, there may be two representations whose balanced algebras are isomorphic, but which are not canonically isomorphic as representations). If $t \geq_{bal} s$ and $s \geq_{bal} t$, we will write $t \cong_{bal} s$, and say that their balanced algebras are canonically isomorphic.

When there may be ambiguity between a balanced covering a canonical covering, we will write $\geq_{rep}$ to clarify that we mean a canonical covering of the full P-graph C^*-algebra. Note that a canonical covering gives rise to a balanced covering by restricting the canonical covering to the balanced algebra. That is, $t \geq_{rep} s$ implies $t \geq_{bal} s$. The converse can fail, but we prove a partial converse in Lemma 4.1.21.

4.1.3   The Kernel of $\psi_t^{\mathcal{T}}$

This section is devoted to proving the following fact which is our Theorem 4.1.15: letting $\mathcal{T}$ denote the Toeplitz representation from Definition 2.5.8, then for any representation t, $\ker \psi_t^{\mathcal{T}}$ is generated (as an ideal) by the "bolts" (see Definition 4.1.10) and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains. This is our analogue of [3, Theorem 4.9], although slightly generalized to allow the $\ker \psi_t^{\mathcal{T}}$ to contain $\mathcal{T}_\mu \mathcal{T}_\mu^*$.

This result is highly involved in the sense that the proof is many times longer than the statement. The key to the argument is to show that the property "$\ker \psi_t^{\mathcal{T}} \cap A$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains" is preserved under direct limits, and that this property is true for a dense collection of subalgebras $A \subseteq \mathcal{B}(\mathcal{T})$. This will take several lemmas to establish.

Definition 4.1.10. *Let (G, P) be a WQLO group, let Λ be a finitely aligned P-graph and let t be a representation of Λ. We will say that a <u>bolt</u> in $C^*(t)$ is an element of*

the form

$$\prod_{\alpha \in E} (t_\mu t_\mu^* - t_\alpha t_\alpha^*)$$

where $\mu \in \Lambda$ and $E \subset \mu\Lambda$ is finite and exhaustive for $\mu\Lambda$. We say it is a <u>proper bolt</u> if $E \subset s(\mu)\Lambda \setminus s(\mu)$.

The term "bolt" comes from a tortured metaphor: a representation is tight if all of its bolts are fastened down (i.e. equal to 0).

Lemma 4.1.11. *Let (G, P) be a WQLO group, let Λ be a finitely aligned P-graph, and let t a representation of Λ.*

Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing sequence of subalgebras of $\mathcal{B}(\mathcal{T})$, and let $A = \overline{\bigcup_{n \in \mathbb{N}} A_n}$. Suppose that for each $n \in \mathbb{N}$, $\ker \psi_t^\mathcal{T} \cap A_n$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains. Then $\ker \psi_t^\mathcal{T} \cap A$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains.

Proof. Let $J = \ker \psi_t^\mathcal{T} \cap A$ which is an ideal in A, and let I denote the ideal generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ contained in J. Certainly $I \subseteq J$, and it suffices to show $I = J$.

By [5, Lemma III.4.1],

$$J = \overline{\bigcup_{n \in \mathbb{N}} (J \cap A_n)}$$

But note that $J \cap A_n = \ker \psi_t^\mathcal{T} \cap A \cap A_n = \ker \psi_t^\mathcal{T} \cap A_n$, so $J \cap A_n$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains. Such a bolt or $\mathcal{T}_\mu \mathcal{T}_\mu^*$ is certainly in $J \supseteq J \cap A_n$, and thus $J \cap A_n \subseteq I$, and taking the union and closure, we have

$$I \supseteq \overline{\bigcup_{n \in \mathbb{N}} (J \cap A_n)} = J$$

so $I = J$ as desired.

$\square$

We will now build towards our dense family of subalgebras.

Definition 4.1.12. *Let (G, P) be a WQLO group, let Λ be a finitely aligned P-graph, and let $S \subseteq \Lambda$. We will say that S is <u>MCE closed</u> if for all $\mu, \nu \in S$, $MCE(\mu, \nu) \subseteq S$.*

We will say that S is <u>substitution closed</u> if for all $\mu, \nu \in S$ with $d(\mu) = d(\nu)$ and $s(\mu) = s(\nu)$, and for all $\alpha \in \Lambda$, we have that $\mu\alpha \in S$ implies $\nu\alpha \in S$.

For $S \subseteq \Lambda$, we will write $D(S) = \{d(\mu) : \mu \in S\}$.

As we will see in the next lemma, the condition that S is MCE closed and substitution closed is the correct condition in order to make $A_S = \overline{\operatorname{span}}\{\mathcal{T}_\mu \mathcal{T}_\nu^* : \mu, \nu \in S, d(\mu) = d(\nu)\}$ a subalgebra of the balanced algebra.

Lemma 4.1.13. *Let (G, P) be a WQLO group and let Λ be a finitely-aligned P-graph. For each $S \subseteq \Lambda$, let $A_S = \overline{\operatorname{span}}\{\mathcal{T}_\mu \mathcal{T}_\nu^* : \mu, \nu \in S, d(\mu) = d(\nu)\}$. Then:*

1. *If S is MCE closed and substitution closed, then A_S is a closed *-subalgebra of $\mathcal{B}(\mathcal{T})$.*

2. *If S is MCE closed and substitution closed, and $D(S) = \{d(\mu) : \mu \in S\}$ contains a minimal element m, then $S' = \{\mu \in S : d(\mu) \neq m\}$ is MCE closed and substitution closed.*

3. *Let $S, D(S), m$, and S' be as above. For any representation t of Λ, if $\mu, \nu \in \Lambda^m \cap S = \{\lambda \in \Lambda \cap S : d(\lambda) = m\}$ with $s(\mu) = s(\nu)$, then $t_\mu t_\nu^* \in \psi_t^{\mathcal{T}}(A_{S'})$ implies that either $t_\mu = 0$ or there is a finite set $E \subset \mu\Lambda \cap S'$ which is exhaustive for $\mu\Lambda$ such that the bolt $B = \prod_{\alpha \in E}(t_\mu t_\mu^* - t_\alpha t_\alpha^*)$ is equal to 0.*

Proof. (1): It is immediate that A_S is a closed $*$-invariant subspace of $\mathcal{B}(\mathcal{T})$. It then suffices to check that it is closed under multiplication. If $\mathcal{T}_\mu \mathcal{T}_\nu^*, \mathcal{T}_\alpha \mathcal{T}_\beta^* \in A_S$, then

$$\begin{aligned}
(\mathcal{T}_\mu \mathcal{T}_\nu^*)(\mathcal{T}_\alpha \mathcal{T}_\beta^*) &= (\mathcal{T}_\mu \mathcal{T}_\nu^*) \sum_{\lambda \in MCE(\nu,\alpha)} \mathcal{T}_\lambda \mathcal{T}_\lambda^* (\mathcal{T}_\alpha \mathcal{T}_\beta^*) \\
&= \sum_{\lambda \in MCE(\nu,\alpha)} \mathcal{T}_{\mu(\nu^{-1}\lambda)} \mathcal{T}_{\alpha(\beta^{-1}\lambda)}^*.
\end{aligned}$$

Note that since S is MCE closed, then each $\lambda \in MCE(\nu,\alpha)$ is in S, and since S is substitution closed, then $\mu(\nu^{-1}\lambda), \alpha(\beta^{-1}\lambda) \in S$. Thus $(\mathcal{T}_\mu \mathcal{T}_\nu^*)(\mathcal{T}_\alpha \mathcal{T}_\beta^*) \in A_S$, as desired.

(2): Given $\mu, \nu \in S'$, $MCE(\mu,\nu) \subseteq S$ since S is MCE closed, and for $\lambda \in MCE(\mu,\nu)$, we have $d(\lambda) \geq d(\mu) \neq m$, so $d(\lambda) \neq m$, so $\lambda \in S'$. Similarly, given $\mu, \mu\alpha, \nu \in S'$ with $d(\mu) = d(\nu)$, we have that $\nu\alpha \in S$ since S is substitution closed, but $d(\nu\alpha) = d(\mu\alpha) \neq m$, so $\nu\alpha \in S'$. Thus S' is MCE closed, and substitution closed.

(3): Suppose that $t_\mu t_\nu^* \in \psi_t^{\mathcal{T}}(A_{S'}) = \overline{\operatorname{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in S', d(\alpha) = d(\beta)\}$. Then there is an element of $\operatorname{span}\{t_\alpha t_\beta^* : \alpha, \beta \in S', d(\alpha) = d(\beta)\}$ within a distance of 1 of $t_\mu t_\nu^*$, which is to say there are $c_i \in \mathbb{C}$ and $\alpha_i, \beta_i \in S'$ with $d(\alpha_i) = d(\beta_i)$ such that writing $x = t_\mu t_\nu^* - \sum_{i=1}^n c_i t_{\alpha_i} t_{\beta_i}^*$, then $\|x\| < 1$.

Now, let $y = t_\mu t_\mu^* x$, we have $\|y\| \leq \|t_\mu\|^2 \cdot \|x\| < 1$, and

$$\begin{aligned}
y &= t_\mu t_\mu^* \left(t_\mu t_\nu^* - \sum_{i=1}^n c_i t_{\alpha_i} t_{\beta_i}^* \right) \\
&= t_\mu t_\nu^* - \sum_{i=1}^n c_i t_\mu t_\mu^* t_{\alpha_i} t_{\beta_i}^* \\
&= t_\mu t_\nu^* - \sum_{i=1}^n \sum_{\lambda \in MCE(\alpha_i,\mu)} c_i t_\lambda t_{\beta(\alpha^{-1}\lambda)} \\
&= t_\mu t_\nu^* - \sum_{j=1}^N c_j t_{\lambda_j} t_{\eta_j}
\end{aligned}$$

where the last line is simply relabelling the previous line. Note that the sum is still indeed finite since Λ is finitely aligned. Also, since S is substitution closed and MCE closed, then $\lambda_j, \eta_j \in S$ for all j. Since $d(\lambda_j) \geq d(\alpha_i) \neq m$ and m was minimal, then $d(\lambda_j) \neq m$ and thus $\lambda_j \in S'$. Also, λ_j is a common extension of μ and some α_i, so $\lambda_j \in \mu\Lambda$.

Now let $E = \{\lambda_1, ..., \lambda_N\}$, which is finite and satisfies $E \subseteq \mu\Lambda \cap S'$ by our previous remarks. Suppose for the sake of contradiction that E were not exhaustive for $\mu\Lambda$. Then there would be some $\gamma \in \mu\Lambda$ such that $MCE(\gamma, \lambda_i) = \emptyset$ for all $\alpha_i \in E$. Then, $\left\|(t_\gamma t_\gamma^*)y\right\| \leq \|t_\gamma\|^2 \|y\| < 1$, and

$$
\begin{aligned}
(t_\gamma t_\gamma^*)y &= (t_\gamma t_\gamma^*)(t_\mu t_\nu^* - \sum_{j=1}^{N} c_j t_{\lambda_j} t_{\eta_j}) \\
&= t_\gamma t_\gamma^* t_\mu t_\nu^*
\end{aligned}
$$

since each $MCE(\lambda_i, \gamma) = \emptyset$ for $1 \leq i \leq N$. Next, since $\gamma \in \mu\Lambda$, then $\gamma = \mu\gamma'$ for some $\gamma' \in \Lambda$, so we may simplify this expression to

$$
(t_\gamma t_\gamma^*)(t_\mu t_\nu^*) = t_{\mu\gamma'} t_{\gamma'}^* t_\mu^* t_\mu t_\nu^* = t_{\mu\gamma'} t_{\nu\gamma'}^*
$$

which is a partial isometry. Since partial isometries have norm either 0 or 1, and we've seen it has norm less than 1, we have that $0 = t_{\mu\gamma'} t_{\nu\gamma'}^*$, and since $s(\mu\gamma') = s(\mu) = s(\nu) = s(\nu\gamma')$, we have $0 = t_{s(\mu\mu_1)} = t_{s(\mu)} = t_\mu^* t_\mu = t_\mu$, showing that either E is exhaustive or $t_\mu = 0$, as desired.

Finally, we wish to show that $B = 0$, where B is the bolt $B = \prod_{\lambda \in E}(t_\mu t_\mu^* - t_\lambda t_\lambda^*)$. To this end, we will make a similar argument that By is a partial isometry of norm less than 1, so it must be 0. The latter is immediate: $\|By\| \leq \|B\| \cdot \|y\| < 1$. Now

let us simplify the expression By:

$$
\begin{aligned}
By &= B(t_\mu t_\nu^* - \sum_{j=1}^{N} c_j t_{\lambda_j} t_{\eta_j}) \\
&= Bt_\mu t_\nu^* - \sum_{j=1}^{N} c_j \left(\prod_{k=1}^{N} (t_\mu t_\mu^* - t_{\lambda_k} t_{\lambda_k}^*) \right) t_{\lambda_j} t_{\eta_j}) \\
&= Bt_\mu t_\nu^* - \sum_{j=1}^{N} c_j \left(\prod_{k \neq j}^{N} (t_\mu t_\mu^* - t_{\lambda_k} t_{\lambda_k}^*) \right) \left((t_\mu t_\mu^* - t_{\lambda_j} t_{\lambda_j}^*) t_{\lambda_j} \right) t_{\eta_j}) \\
&= Bt_\mu t_\nu^* - \sum_{j=1}^{N} 0 \\
&= Bt_\mu t_\nu^*
\end{aligned}
$$

where the main simplification occurs since $(t_\mu t_\mu^* - t_{\lambda_j} t_{\lambda_j}^*) t_{\lambda_j} = t_{\lambda_j} - t_{\lambda_j} = 0$.

Thus $By = Bt_\mu t_\nu^*$, so it is again a partial isometry of norm strictly less than 1, and thus $By = 0$. Finally, since $Bt_\mu t_\nu^* = 0$, then

$$
\begin{aligned}
0 &= (Bt_\mu t_\nu^*)(Bt_\mu t_\nu^*)^* \\
&= Bt_\mu t_\nu^* t_\nu t_\mu^* B \\
&= Bt_\mu t_\mu^* B
\end{aligned}
$$

and recalling that B is a subprojection of $t_\mu t_\mu^*$, we have that $0 = B$, completing the proof of (3).

$\square$

The next lemma shows that a sufficient collection of A_S have the property that

$A_S \cap \ker \psi_t^{\mathcal{T}}$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$s it contains.

Lemma 4.1.14. *Let (G, P) be a WQLO group and let Λ be a finitely-aligned P-graph. For each $S \subseteq \Lambda$ which is MCE closed and substitution closed, let $A_S = \overline{\text{span}}\{\mathcal{T}_\mu \mathcal{T}_\nu^* : \mu, \nu \in S, d(\mu) = d(\nu)\}$. Let t be a representation of Λ, and let $\psi_t^{\mathcal{T}} : \mathcal{B}(\mathcal{T}) \to \mathcal{B}(t)$ be the balanced covering. If $D(S) = \{d(\mu) : \mu \in S\}$ is finite, then $\ker \psi_t^{\mathcal{T}} \cap A_S$ is generated (as an ideal) by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$s it contains.*

Proof. Given a MCE-closed and substitution closed subset $S \subseteq \Lambda$, let J_S denote $\ker \psi_t^{\mathcal{T}} \cap A_S$, which is an ideal in A_S. Let I_S denote the ideal in A_S generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$s contained in J_S. Then certainly $I_S \subseteq J_S$, and our claim is equivalent to proving that $J_S / I_S = 0$.

We proceed by induction on $|D(S)|$. In the base case of $|D(S)| = 0$, the claim is trivial since $A_S = J_S = I_S = 0$.

In the inductive case, suppose that $|D(S)| > 0$, and that the claim is true for all D' with $|D'| < |D(S)|$. Since $D(S)$ is finite, it has some minimal element m. First consider the case where $S^m = \{\mu \in S : d(\mu) = m\}$ is finite. Note that S^m is itself substitution closed and MCE-closed and that $A_S = A_{S'} + A_{S^m}$.

Fix some $x + I_S \in J_S / I_S$, in which case $x \in J_S$. Since $x \in J_S \subseteq A_S = A_{S'} + A_{S^m}$, we may write $x = x_m + x'$, where $x_m \in A_{S^m}$ and $x' \in A_{S'}$. Additionally, since A_{S^m} is finite dimensional, then we may write $x_m = \sum_{i=1}^{n} c_i \mathcal{T}_{\mu_i} \mathcal{T}_{\nu_i}^*$ where $d(\mu_i) = d(\nu_i) = m$ for all i. There are potentially many such representations of $x + I_S$ as $x_m + x' + I_S$. We will assume without loss of generality we have chosen the decomposition with the fewest terms in the sum $x_m = \sum_{i=1}^{n} c_i \mathcal{T}_{\mu_i} \mathcal{T}_{\nu_i}^*$, and our goal is to show that in fact there are zero terms in the sum and thus $x_m = 0$.

To this end, suppose for the sake of contradiction that $x_m = \sum_{i=1}^{n} c_i \mathcal{T}_{\mu_i} \mathcal{T}_{\nu_i}^*$ has a nonzero summand, and without loss of generality that it is the first summand. That

is, suppose $c_1 \mathcal{T}_{\mu_1} \mathcal{T}_{\nu_1}^* \neq 0$.

Now,

$$
\begin{aligned}
(\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^*) x (\mathcal{T}_{\nu_1} \mathcal{T}_{\nu_1}^*) &= (\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^*) \left(\sum_{i=1}^{n} c_i \mathcal{T}_{\mu_i} \mathcal{T}_{\nu_i}^* + x' \right) (\mathcal{T}_{\nu_1} \mathcal{T}_{\nu_1}^*) \\
&= c_1 \mathcal{T}_{\mu_1} \mathcal{T}_{\nu_1}^* + (\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^*) x' (\mathcal{T}_{\nu_1} \mathcal{T}_{\nu_1}^*).
\end{aligned}
$$

Since $x \in J_S \subseteq \ker \psi_t^{\mathcal{T}}$, by applying $\ker \psi_t^{\mathcal{T}}$ to both sides, we have that

$$
0 = c_i t_{\mu_1} t_{\nu_1}^* + \psi_t^{\mathcal{T}}((\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^*) x' (\mathcal{T}_{\nu_1} \mathcal{T}_{\nu_1}^*))
$$

so solving for $t_{\mu_1} t_{\nu_1}^*$, we have

$$
t_{\mu_1} t_{\nu_1}^* = \psi_t^{\mathcal{T}}(\frac{-1}{c_1}(\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^*) x' (\mathcal{T}_{\nu_1} \mathcal{T}_{\nu_1}^*))
$$

but $x' \in A_{S'}$, and $A_{S^m} A_{S'} \subseteq A_{S'}$, so the righthand side is an element of $\psi^{\mathcal{T}_t}(A_{S'})$. Now by part (3) of Lemma 4.1.13, we have either $t_{\mu_1} = 0$ or there is a finite exhaustive set $E \subset \mu_1 \Lambda$ with $d(\alpha) \in S \setminus \{m\}$ for all $\alpha \in E$, and such that the bolt $B = \prod_{\alpha \in E}(t_{\mu_1} t_{\mu_1}^* - t_\alpha t_\alpha^*)$ is equal to 0.

In the former case, $\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^* \in I_S$, so taking $y = c_1 (\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^*) \mathcal{T}_{\mu_1} \mathcal{T}_{\nu_1}^* = c_1 \mathcal{T}_{\mu_1} \mathcal{T}_{\nu_1}^*$, we have $y \in I_S$, so $x + I_S = (x_m - y) + x' + I_S$ is a decomposition with one fewer terms in the sum $x_m - y$, a contradiction. In the latter case, $\prod_{\alpha \in E}(\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^* - \mathcal{T}_\alpha \mathcal{T}_\alpha^*)$ is a bolt in $\ker \pi_t^{\mathcal{T}}$, so $B_1 = \prod_{\alpha \in E}(\mathcal{T}_{\mu_1} \mathcal{T}_{\mu_1}^* - \mathcal{T}_\alpha \mathcal{T}_\alpha^*) \in I_S$. Let $y = c_1 B_1 \mathcal{T}_{\mu_1} \mathcal{T}_{\nu_1}^*$, and then we may write $x + I_S = x_m - y + x' + I_S$ which will have zeroed out the $c_1 \mathcal{T}_{\mu_1} \mathcal{T}_{\nu_1}^*$ summand in the x_m term (while possibly adding more terms to the x' part, which is acceptable), again a contradiction.

In either case, we found a contradiction with x_m being the representative with the

fewest summands. Thus x_m had no terms in its summation, and thus $x_m = 0$. That is to say, $x = x_m + x' = x'$, where $x' \in A_{S'}$, and therefore $x \in A_{S'} \cap \ker_t^{\mathcal{T}} = J_{S'}$. By the inductive hypothesis, since $D' = \{d(\mu) : \mu \in S'\} = D(S) \setminus \{m\}$ has fewer terms than $D(S)$, we have that $J_{S'} = I_{S'}$ so $x + I_S \in I_{S'} + I_S = I_S$, so $J_S \subseteq I_S$, as desired.

Now in the case that S^m is not finite, we let F_n be an increasing sequence of finite subsets of S^m with $S^m = \bigcup F_n$, in which case $S' \cup F_n$ is MCE closed and substitution-closed, so the previous case applies to $A_{S' \cup F_n}$. But $A_S = \overline{\bigcup_{n=1}^{\infty} A_{S' \cup F_n}}$, and so the general case follows from Lemma 4.1.11. $\qquad\square$

We can now add a second dash of Lemma 4.1.11 to get our desired result:

Theorem 4.1.15. *Let (G, P) be a WQLO group and let Λ be a finitely-aligned P-graph. Let t be a representation of Λ, and $\psi_t^{\mathcal{T}} : \mathcal{B}(\mathcal{T}) \to \mathcal{B}(t)$ be the balanced covering. Then $\ker \psi_t^{\mathcal{T}}$ is generated (as an ideal) by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$s it contains.*

Proof. As above, for any $S \subseteq \Lambda$, let $D(S) = \{d(\mu) : \mu \in \Lambda\}$. Note that taking a substitution preserves the degree of a path, and taking minimal common extensions of paths can take joins of degrees (that is, replace paths of length p and q with paths of length $p \vee q$). Therefore, if $F \subseteq \Lambda$ and S is the set of all paths in Λ obtained by taking substitutions and MCEs of F, then $D(S) \subseteq \{p_1 \vee p_2 \vee ... \vee p_n : p_i \in D(F)\}$. In particular, if F is finite, then $|D(S)| \leq 2^{|F|} < \infty$, so any finite $F \subseteq \Lambda$ is contained in a set $S \subseteq \Lambda$ which is MCE closed and substitution closed and for which $D(S)$ is finite.

Since Λ is countable, we can enumerate the elements as $\lambda_1, \lambda_2, ...$ For each $n \in \mathbb{N}$, let $F_n = \{\lambda_1, ..., \lambda_n\}$, and let S_n denote the set of all paths in Λ obtained by taking substitutions and MCEs of F_n.

Then S_n is substitution closed and MCE closed and by the above argument, $D(S_n) < \infty$. Therefore, by Lemma 4.1.14, $\ker \psi_t^{\mathcal{T}} \cap A_{S_n}$ is generated by the bolts and

$\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains.

But recall that $\mathcal{B}(\mathcal{T}) = \overline{\mathrm{span}}\{\mathcal{T}_\mu \mathcal{T}_\nu^* : d(\mu) = d(\nu)\}$, and for any $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$, since we have an enumeration, there exists some $m, n \in \mathbb{N}$ with $\mu = \lambda_m, \nu = \lambda_n$, so $\mu, \nu \in F_{\max(m,n)} \subset S_{\max(m,n)}$, so $\mathcal{T}_\mu \mathcal{T}_\nu^* \in A_{S_{\max(m,n)}}$. Thus $\mathcal{B}(\mathcal{T}) = \overline{\bigcup_{n \in \mathbb{N}} A_{S_n}}$.

Then by Lemma 4.1.11, $\ker \psi_t^{\mathcal{T}} \cap \mathcal{B}(\mathcal{T}) = \ker \psi_t^{\mathcal{T}}$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ it contains. $\qquad\square$

The above result classifies ideals in the balanced algebra $\mathcal{B}(\mathcal{T})$, and when combined with the Factors Through Theorem (Lemma 2.2.1) shows that two balanced algebras are isomorphic if and only if they have the same set of bolts and range projections which are set equal to 0. We make use of this fact in the following lemma:

Lemma 4.1.16. *Let (G, P) be a WQLO group, Λ a P-graph, and let s and t be two representations of (G, P). Then*

1. *If t is tight and s is Λ-faithful, then $t \leq_{bal} s$.*

2. *If t and s are both tight and Λ-faithful, then $t \cong_{bal} s$.*

Proof. For the first claim, let $\mathcal{T}$ denote the universal representation of Λ. Then, there are balanced coverings $\psi_s^{\mathcal{T}} : \mathcal{B}(\mathcal{T}) \to \mathcal{B}(s)$ and $\psi_t^{\mathcal{T}} : \mathcal{B}(\mathcal{T}) \to \mathcal{B}(t)$. We wish to use the Factors Through Theorem (Lemma 2.2.1) to conclude there is a balanced covering $\psi_t^s : \mathcal{B}(s) \to \mathcal{B}(t)$ given by $s_\mu s_\nu^* \to t_\mu t_\nu^*$, which we can do if and only if $\ker \psi_s^{\mathcal{T}} \subseteq \ker \psi_t^{\mathcal{T}}$.

By Theorem 4.1.15, $\ker \psi_s^{\mathcal{T}}$ and $\ker \psi_t^{\mathcal{T}}$ are generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$ they contain. Since s is Λ-faithful, its kernel contains no $\mathcal{T}_\mu \mathcal{T}_\mu^*$ (and possibly some bolts), and since t is tight, its kernel contains every bolt (and possibly some $\mathcal{T}_\mu \mathcal{T}_\mu^*$). Thus $\ker \psi_s^{\mathcal{T}} \subseteq \ker \psi_t^{\mathcal{T}}$, so $t \leq_{bal} s$ as desired.

For the second claim, by applying (1) twice, we have $t \leq_{bal} s$ and $s \leq_{bal} t$, so $t \cong_{bal} s$. $\qquad\square$

4.1.4 Normalizations of Coactions

We will now touch briefly on the normalization of a coaction. For a more thorough introduction, the reader is directed to [6, Appendix A.7]

Remark 4.1.17. *Given a discrete coaction* (A, G, δ), *define* $j_A : A \to A \otimes C_r^*(G)$ *by* $j_A = (\mathrm{id}_A \otimes \pi_L^U) \circ \delta$, *which is a $*$-homomorphism.*

Let $A^n = j_A(A) \cong A/\ker j_A$. *Then by Lemma A.55 and Definition A.56 of [6] there is a coaction* (A^n, G, δ^n) *which is normal, and such that* j_A *is* δ-δ^n *equivariant, meaning that* $\delta^n \circ j_A = (j_A \otimes \mathrm{id}_G) \circ \delta$. *We call* (A^n, G, δ^n) *the* <u>*normalization*</u> *of* (A, G, δ).

Note that [6] uses a definition of normality which is equivalent to ours, but not identical. Readers may wish to read [17, Lemma 1.4] for a proof of the equivalence of the two definitions.

The following lemma is analogous to Lemma 2.5.11 but refers to a "normalization" process for representations instead of a "coactionization" process. Note that while coactionization made the representation larger with respect to $\leq_{rep}$, normalization will make the representation smaller.

Lemma 4.1.18. *Let* (G, P) *be a WQLO group, and* Λ *a P-graph. Let t be a representation of* Λ *with a gauge coaction* δ. *Then,*

1. *Let* $j_A : C^*(t) \to C^*(t)^n$ *denote the quotient map taking the cosystem* $(C^*(t), G, \delta)$ *to its normalization* $(C^*(t)^n, G, \delta^n)$, *and for* $\lambda \in \Lambda$ *let* $\tilde{t}_\lambda := j_A(t_\lambda)$. *Then* $\tilde{t}$ *is a representation of* Λ.

2. There is a canonical covering $t \mapsto \tilde{t}$.

3. The coaction δ^n on $C^*(\tilde{t})$ is normal and is a gauge coaction.

4. t is canonically isomorphic to $\tilde{t}$ if and only if δ is a normal coaction.

5. There is a balanced covering $\psi_{\tilde{t}}^t : \mathcal{B}(t) \to \mathcal{B}(\tilde{t})$, which is a balanced isomorphism.

6. $\tilde{t}$ is Λ-faithful (respectively, tight) if and only if t is.

Definition 4.1.19. *We call the representation $\tilde{t}$ the given in the previous lemma the* normalization *of t.*

Proof of 4.1.18. (1) The fact that $\tilde{t}_\lambda$ is a representation is immediate since it is the image of a representation under a homomorphism.

(2) j_A is a canonical covering.

(3) δ^n is normal by construction. To show it is a gauge coaction, recall from Definition A.56 and Lemma A.55 of [6] that the canonical covering j_A is δ-δ^n equivariant, meaning that $\delta^n \circ j_A = (j_A \otimes \mathrm{id}_G) \circ \delta$. Then, for any $\mu \in \Lambda$,

$$
\begin{aligned}
\delta^n(\tilde{t}_\mu) &= \delta^n(j_A(t_\mu)) \\
&= (j_A \otimes \mathrm{id}_G) \circ \delta(t_\mu) \\
&= (j_A \otimes \mathrm{id}_G)(t_\mu \otimes U_{d(\mu)}) \\
&= \tilde{t}_\mu \otimes U_{d(\mu)}
\end{aligned}
$$

as desired.

(4) By [17, Lemma 1.4], a coaction is normal (in the sense of its conditional expectation Φ being faithful) if and only if j_A is injective, which is equivalent to j_A being a canonical isomorphism.

(5) The balanced covering $\psi_{\tilde{t}}^t$ exists because it is a restriction of the canonical covering j_A from part (2). Since it is a balanced covering, it is surjective, so it suffices to show $\psi_{\tilde{t}}^t$ is injective. To this end, recall that $\psi_{\tilde{t}}^t = j_A \mid_{\mathcal{B}(t)}$ and that $j_A := (\mathrm{id}_A \otimes \pi_L^U) \circ \delta$. Now for $x \in \mathcal{B}(t)$ we have $\delta(x) = x \otimes U_e$, so $j_A(x) = ((\mathrm{id}_A \otimes \pi_L^U) \circ \delta)(x) = x \otimes L_e = x \otimes 1$. Then since $\|\cdot\|$ is a C^*-cross norm, $\|j_A(x)\| = \|x\| \cdot \|1\| = \|x\|$, so $x = 0$ if and only if $\psi_{\tilde{t}}^t(x) = j_A(x) = 0$, as desired.

(6) Having proven (5), this is immediate: since tightness is a relation among elements of the balanced algebra, t is tight if and only if $\tilde{t}$ is tight. Similarly, t is Λ-faithful if and only if every $t_\lambda \neq 0$ if and only if every $t_\lambda^* t_\lambda \neq 0$, which is a relation among elements of the balanced algebra, so t is Λ-faithful if and only if $\tilde{t}$ is.

$\square$

Corollary 4.1.20. *Let (G, P) be a WQLO group, and let Λ be a finitely aligned P-graph. For any Λ-faithful, tight representation t of Λ, the normalization of the coactionization of t is a Λ-faithful, tight representation with a normal gauge coaction.*

Proof. Recall that t' denotes the coactionization of t as in Lemma 2.5.11 and let $\widetilde{t'}$ denoting the normalization of t' as in Lemma 4.1.18. By those two lemmas, $\widetilde{t'}$ has a normal gauge coaction, and since t is Λ-faithful and tight, so is $\widetilde{t'}$. $\square$

4.1.5 The Co-Universal Algebra

The following result shows that a balanced covering extends to a covering of the entire algebras in the context of a normal coaction:

Lemma 4.1.21. *Let (G, P) be a WQLO group, let Λ be a finitely-aligned P-graph, and let s and t be two representations of Λ. If $s \leq_{bal} t$, s and t have gauge coactions, and the gauge coaction on s is normal, then $s \leq_{rep} t$.*

Proof. Our argument will make use of several functions, organized according to the following diagram (which will commute):

$$
\begin{array}{ccccc}
C^*(\mathcal{T}) & \xrightarrow{\pi_t^{\mathcal{T}}} & C^*(t) & \xrightarrow{\Phi_t} & \mathcal{B}(t) \\
\downarrow{\scriptstyle \pi_s^{\mathcal{T}}} & \overset{\exists \pi_s^t}{\nwarrow} & & & \downarrow{\scriptstyle \psi_s^t} \\
C^*(s) & & \xrightarrow{\quad\quad \Phi_s \quad\quad} & & \mathcal{B}(s)
\end{array}
$$

Here $\mathcal{T}$ denotes the universal representation of Λ, and $\pi_t^{\mathcal{T}}$ and $\pi_s^{\mathcal{T}}$ are the canonical coverings of $C^*(t)$ and $C^*(s)$ respectively. We wish to show the existence of the canonical covering π_s^t. The algebras $\mathcal{B}(t)$ and $\mathcal{B}(s)$ are the balanced subalgebras as in Lemma 4.1.7 and the maps $\Phi_t : C^*(t) \to \mathcal{B}(t)$ and $\Phi_s : C^*(s) \to \mathcal{B}(s)$ are the conditional expectations arising from Lemma 4.1.8. From that lemma, these conditional expectations satisfy

$$
\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise} \end{cases} \quad \text{and similarly } \Phi_s(s_\mu s_\nu^*) = \begin{cases} s_\mu s_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise} \end{cases}.
$$

Also note that by Definition 2.3.13, Φ_s is faithful because the gauge coaction on s is normal. Finally, ψ_s^t is the balanced covering that exists because $t \geq_{bal} s$.

We will now verify that this diagram commutes. That is, we will show that

$$
\psi_s^t \circ \Phi_t \circ \pi_t^{\mathcal{T}} = \Phi_s \circ \pi_s^{\mathcal{T}}.
$$

Since $C^*(\mathcal{T}) = \overline{\text{span}}\{\mathcal{T}_\mu \mathcal{T}_\nu : \mu, \nu \in \Lambda\}$ and all the maps are linear and continuous, it suffices to check that the two functions agree on each $\mathcal{T}_\mu \mathcal{T}_\nu^*$. To this end, we fix some $\mu, \nu \in \Lambda$, and we have

$$
\begin{aligned}
\psi_s^t \circ \Phi_t \circ \pi_t^{\mathcal{T}}(\mathcal{T}_\mu \mathcal{T}_\nu^*) &= \psi_s^t \circ \Phi_t(t_\mu t_\nu^*) \\
&= \psi_s^t \left(\begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise} \end{cases} \right) \\
&= \begin{cases} s_\mu s_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{otherwise} \end{cases} \\
&= \Phi_s(s_\mu s_\nu^*) \\
&= \Phi_s \circ \pi_s^{\mathcal{T}}(\mathcal{T}_\mu \mathcal{T}_\nu^*)
\end{aligned}
$$

as desired.

We will now show that $\ker \pi_t^{\mathcal{T}} \subseteq \ker \pi_s^{\mathcal{T}}$. To this end, suppose that $x \in \ker \pi_t^{\mathcal{T}}$. Then $x^*x \in \ker \pi_t^{\mathcal{T}}$, so

$$
\begin{aligned}
0 &= \psi_s^t \circ \Phi_t \circ \pi_t^{\mathcal{T}}(x^*x) \\
&= \Phi_s \circ \pi_s^{\mathcal{T}}(x^*x)
\end{aligned}
$$

and since $\pi_s^{\mathcal{T}}(x^*x) \geq 0$ and Φ_s is faithful, then we must have that $\pi_s^{\mathcal{T}}(x^*x) = 0$, so $\pi_s^{\mathcal{T}}(x) = 0$, and thus $x \in \ker \pi_s^{\mathcal{T}}$ as desired.

Thus by Lemma 2.2.1 (the Factors Through Theorem), there exists a map π_s^t satisfying $\pi_s^{\mathcal{T}} = \pi_s^t \circ \pi_t^{\mathcal{T}}$, from which it is immediate that π_s^t is a canonical covering.

Thus $s \leq_{rep} t$.

$\square$

Now we may prove the main theorem of this section, which is a slight generalization of [3, Theorem 5.3] from the context of quasi-lattice ordered groups to weakly quasi-lattice ordered groups. We have also rephrased the result:

Theorem 4.1.22. *Let (G, P) be a WQLO group, and let Λ be a finitely aligned P-graph. If s is a Λ-faithful, tight representation with a normal gauge coaction, and t is a Λ-faithful gauge-coacting representation, then $s \leq t$.*

In particular, there is a unique representation S of Λ which is Λ-faithful, tight, and has a normal gauge coaction, and $S \leq t$ for any Λ-faithful gauge-coacting representation t.

We write $C^*_{min}(\Lambda)$ for $C^*(S)$, and call $C^*_{min}(\Lambda)$ the <u>co-universal</u> algebra of the graph.

Proof. Suppose that s was a Λ-faithful, tight representation with a normal gauge coaction, and that t was any Λ-faithful gauge-coacting representation. Since s is tight and t is Λ-faithful, by Lemma 4.1.16, $s \leq_{bal} t$. Then since s has a normal gauge coaction and t has a gauge coaction, by Lemma 4.1.21, $s \leq_{rep} t$.

To show the existence of such an S, recall that f denotes the ultrafilter representation from Lemma 4.1.5, which is Λ-faithful and tight by that lemma. By Corollary 4.1.20, $S := \widetilde{f'}$ is a Λ-faithful, tight representation with a normal gauge coaction.

If t were another Λ-faithful, tight representation that had a normal gauge coaction, then by the first part of the claim $t \leq_{rep} S$ and $S \leq_{rep} t$, so $t \cong_{rep} S$. That is, S is unique up to canonical isomorphism.

$\square$

4.2 The Tight Algebra

The following result is a relatively straightforward combination of the properties of $C^*_{min}(\Lambda)$ and the construction of a universal algebra (Lemma 2.2.2).

Proposition 4.2.1. *Let (G, P) be a WQLO group, and Λ a finitely-aligned P-graph.*

1. *There is a tight representation T of Λ which is universal for tight representations. We will denote the algebra generated by this representation as $C^*_{tight}(\Lambda)$.*

2. *There is a canonical covering $\pi^T_S : C^*_{tight}(\Lambda) \to C^*_{min}(\Lambda)$.*

3. *T is Λ-faithful.*

4. *T has a gauge coaction δ.*

5. *The gauge coaction δ on $C^*_{tight}(\Lambda)$ is normal if and only if π^T_S is an isomorphism, in which case $C^*_{tight}(\Lambda) \cong C^*_{min}(\Lambda)$.*

6. *If δ is normal, there is a gauge-invariant uniqueness theorem of this form: if t is another Λ-faithful, gauge coacting, tight representation, then $C^*_{min}(\Lambda)$ is canonically isomorphic to $C^*(t)$.*

Proof. For (1), the relators (T1)-(T4) and the tightness condition are all polynomial relations in the generators, and $(T1)$ and $(T3)$ together imply that all the generators are partial isometries, so by Lemma 2.2.2, there is a universal C^*-algebra for such tight representations. Let us denote this representation by T and the algebra it generates by $C^*_{tight}(\Lambda)$.

For (2), by Theorem 4.1.22, the representation S which generates $C^*_{min}(\Lambda)$ is a tight representation, so by the universality of T, there is a canonical covering $\pi^T_S :$ $C^*_{tight}(\Lambda) = C^*(T) \to C^*(S) = C^*_{min}(\Lambda)$.

For (3), by Theorem 4.1.22(1), S is Λ-faithful, so each S_λ is nonzero, and since $\pi_S^T(T_\lambda) = S_\lambda$, then each T_λ is nonzero as well.

For (4), let T' denote the coactionization of T given by Proposition 2.5.11. By part (7) of that proposition, since T is tight, then T' is also tight, and therefore by the universality of T, there is a canonical covering $T_\lambda \mapsto T'_\lambda = T_\lambda \otimes U_{d(\lambda)}$, which is the desired gauge coaction.

For (5), if $S \cong T$, then the gauge coaction on T is normal since the gauge coaction on S is normal. Conversely, if the gauge coaction on T is normal, then by the uniqueness part of Theorem 4.1.22, the canonical covering π_S^T is an isomorphism.

For (6), suppose that δ is normal and t is a Λ-faithful gauge coacting, tight representation. Then by the universality of $C_{tight}^*(\Lambda) = C^*(T)$, $T \geq_{rep} t$ (here we are using the partial order notation of Lemma 2.5.9). Similarly, by the co-universality of $C_{min}^*(\Lambda) = C^*(S)$, $t \geq_{rep} S$. Finally, by (5) since δ is normal, then $T \cong S$. Putting these together, we have that $S \cong T \geq_{rep} t \geq_{rep} S$, so $t \cong S$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

This is a pleasant result if δ is normal, but there are examples arising from non-amenable groups where δ is not normal:

Lemma 4.2.2. *Let (G, P) be a group with a total ordering (meaning $G = P \cup P^{-1}$ in addition to $P \cap P^{-1} = \{1\}$), and let $\Lambda = P$, thought of as a small category with one object and with degree functor id_P. Then any tight representation t of $\Lambda = P$ extends uniquely to a unitary representation $\bar{t}$ of G by $\bar{t}_g = \begin{cases} t_g & \text{if } g \in P \\ t_{g^{-1}}^* & \text{if } g \in P^{-1} \end{cases}.$*

Therefore, $C_{tight}^*(\Lambda) \cong C^*(G)$, and in particular the gauge coaction on $C_{tight}^*(\Lambda)$ is normal if and only if G is amenable.

Proof. Let (G, P) be a group with a total ordering, and let $\Lambda = P$, thought of as a small category with the natural degree functor. Let t be a tight representation of P.

We will first show that t_p is a unitary for each $p \in P$. For all $p, q \in P$, $\max(p, q) \geq p, q$, so p and q have a common extension. That is, every $p \in P$ is exhaustive for P, so by tightness, we have that $1 = t_e t_e^* = t_p t_p^*$ for all $p \in P$. Since also $t_p t_p^* = t_{s(p)} = t_e = 1$ by the (T3) relator, this says that the t_ps are unitaries.

Now, extend t to $\bar{t}$ on all of $G = P \cup P^{-1}$ by $\bar{t}_g = \begin{cases} t_g & \text{if } g \in P \\ t_{g^{-1}}^* & \text{if } g \in P^{-1} \end{cases}$, or equivalently $\bar{t}_p = t_p$ and $\bar{t}_{p^{-1}} = t_p^*$ for $p \in P$. We wish to show that $\bar{t}$ is a representation of all of G. To this end, it suffices to check that it is a group homomorphism, and since it extends a representation of P, it suffices to check that multiplication is correct for one element in P and one element in P^{-1}.

To this end, suppose $p, q \in P$. We have several cases depending on which products are positive. If $1 \leq p^{-1}q$, then

$$\bar{t}_{p^{-1}}\bar{t}_q = t_p^* t_q = t_p^* t_p t_{p^{-1}q} = t_{p^{-1}q} = \bar{t}_{p^{-1}q}$$

and if instead $1 \geq p^{-1}q$, then $1 \leq q^{-1}p$, so

$$\bar{t}_{p^{-1}}\bar{t}_q = t_p^* t_q = (t_q t_{q^{-1}p})^* t_q = t_{q^{-1}p}^* t_q^* t_q = t_{q^{-1}p}^* = \bar{t}_{p^{-1}q}$$

so in either case $\bar{t}_{p^{-1}}\bar{t}_q = \bar{t}_{p^{-1}q}$.

For products the other way, if $pq^{-1} \geq 1$, then

$$\bar{t}_p \bar{t}_{q^{-1}} = t_p t_q^* = t_{pq^{-1}} t_q t_q^* = t_{pq^{-1}} = \bar{t}_{pq^{-1}}$$

while if $pq^{-1} \leq 1$, then $qp^{-1} \geq 1$, so

$$\bar{t}_p \bar{t}_{q^{-1}} = t_p t_q^* = t_p(t_{qp^{-1}} t_p)^* = t_p t_p^* t_{qp^{-1}}^* = t_{qp^{-1}}^* = \bar{t}_{pq^{-1}}$$

so in either case $\bar{t}_p \bar{t}_{q^{-1}} = \bar{t}_{pq^{-1}}$. Thus $\bar{t}$ is indeed a representation of G.

To show that $\bar{t}$ is the unique extension of t to a unitary representation of G, observe that if $\tilde{t}$ were another such representation, then for all $p \in P$, $1 = \tilde{t}_p \tilde{t}_{p^{-1}} = t_p \tilde{t}_{p^{-1}}$, so since t_p is a unitary we get that $t_p^* = \tilde{t}_{p^{-1}}$, so $\tilde{t} = \bar{t}$.

Thus the algebra generated by the universal tight representation T is the algebra generated by the universal group representation. That is, $C_{tight}^*(\Lambda) = C^*(G)$, and the gauge coaction on $C_{tight}^*(\Lambda)$ is the standard coaction δ_G on $C^*(G)$ given in Example 2.3.10.

Now for normality, by Lemma 2.3.14, if G is amenable, then all of its coactions are amenable, and in particular δ_G is amenable. Conversely, if the gauge coaction on $C_{tight}^*(\Lambda) \cong C^*(G)$ is normal, then considering $\{\mathbb{C}U_g\}_{g \in \mathbb{C}}$ as a topological grading over G with conditional faithful expectation, by [9, Proposition 3.7], $C^*(G)$ is naturally isomorphic to $C_r^*(\{\mathbb{C}U_g\}_{g \in \mathbb{C}}) = C_r^*(G)$, so the full and reduced C^*-algebras are isomorphic and thus G is amenable.

$\square$

Corollary 4.2.3. *If $(G, P) = (F_2, R)$ where R a total ordering on G such as the Magnus expansion given in Section 3.2 of [4], then $C_{tight}^*(\Lambda) \not\cong C_{min}^*(\Lambda)$. Since both $C_{tight}^*(\Lambda)$ and $C_{min}^*(\Lambda)$ are Λ-faithful, tight, gauge coacting representations, there is no gauge-invariant uniqueness theorem for this R-graph.*

Proof. By the previous lemma, $C_{tight}^*(\Lambda) \cong C^*(G)$, and by Proposition 4.2.1 $C_{tight}^*(\Lambda)$ is a tight, Λ-faithful representation of Λ with a gauge coaction. Since $G = F_2$ is not amenable, the gauge coaction is not normal. But by part (5) of 4.2.1, since the gauge coaction is not normal, then π_S^T is not an isomorphism, so $C_{min}^*(\Lambda)$ is another tight,

Λ-faithful representation of Λ with a gauge coaction, but which is not canonically isomorphic to $C^*_{tight}(\Lambda)$.

$\square$

The reader may recognize the ordered group (F_2, R) from Example 3.1.6 earlier in this manuscript, where we used it as an example of an ordering that had no reduction to an amenable group. This recognition may ignite a spark of hope: perhaps the existence of a reduction to an amenable group prevents the obstruction seen in the previous lemma and guarantees that δ is normal. We shall see in the next section that this is indeed the case.

4.3   Reductions and Representations

In this section, we develop the notion that if (G, P) reduces to (H, Q), then P-graphs are Q-graphs and they have the same representations.

Remark 4.3.1. *In this section, we will be considering a category Λ as a graph with respect to more than one semigroup (ex: as a P-graph and as a Q-graph). In cases where there may be confusion, we will write $C^*_{min}(\Lambda, P)$ and $C^*_{tight}(\Lambda, P)$ to indicate the semigroup P, and so on for other semigroups.*

First we will show that if Λ is a P-graph, and (G, P) has reduction (H, Q), then Λ is also a Q-graph in a natural way.

Lemma 4.3.2. *Let (G, P) be an ordered group, and let (H, Q) be a reduction of (G, P) with degree functor $d^P_Q : P \to Q$. Then a P-graph Λ with degree functor d^Λ_P is also a Q-graph with degree functor $d = d^P_Q \circ d^\Lambda_P$.*

Proof. Certainly, d maps Λ into Q, so, it suffices to check that Λ has unique factorization with respect to d.

Let $\lambda \in \Lambda$, and $q \in Q$ satisfy $q \leq d(\lambda)$. We must show that there is a unique $\alpha, \beta \in \Lambda$ such that $\lambda = \alpha\beta$ and $d(\alpha) = q$.

For existence, since $d_P^\Lambda(\lambda) \in P$ is a path of length $d_Q^P(d_P^\Lambda(\lambda)) = d(\lambda) \geq q$, then by unique factorization of d_Q^P, there exist unique $p, p_1 \in P$ such that $pp_1 = d_P^\Lambda(\lambda)$ and $d_Q^P(p) = q$. Then $p \leq d_P^\Lambda(\lambda)$, so by unique factorization of d_P^Λ, there exist unique $\alpha, \beta \in \Lambda$ such that $\alpha\beta = pp_1$ and $d_P^\Lambda(\alpha) = p$. Then $d(\alpha) = d_Q^P(d_P^\Lambda(\alpha)) = d_Q^P(p) = q$, so the desired α, β exist.

For uniqueness, if $\alpha', \beta' \in \Lambda$ were two other paths with $\lambda = \alpha'\beta'$ and $d(\alpha') = q$, then $d_P^\Lambda(\lambda) = d_P^\Lambda(\alpha')d_P^\Lambda(\beta')$ would be a factorization of $d_P^\Lambda(\lambda)$ where the first term has degree $d_Q^P(d_P^\Lambda(\alpha')) = d(\alpha') = q$, so by uniqueness of the factorization of d_Q^P, we have that $d_P^\Lambda(\alpha') = p$. Then by the uniqueness of the factorization d_P^Λ, we have $\alpha' = \alpha$, as desired.

$\square$

Next, we will show that whether Λ is regarded as a P-graph or a Q-graph, it has the same representations.

Lemma 4.3.3. *Let (G, P) and (H, Q) be two WQLO groups, and Λ a small category with two functors $d_P^\Lambda : \Lambda \to P$ and $d_Q^\Lambda : \Lambda \to Q$ such that Λ is a P-graph with respect to d_P^Λ and a Q-graph with respect to d_Q^Λ. Then,*

1. *Λ is finitely-aligned as a P-graph if and only if it is finitely-aligned as a Q-graph.*

2. *The (T1)-(T4) relators are the same for Λ independently of being a P-graph or a Q-graph.*

3. *A representation is tight independently of Λ being a P-graph or a Q-graph.*

Proof. For (1), recall that $MCE(\mu, \nu)$ depends only on the category structure of Λ by definition, so it is finite regardless of the degree functor given to Λ.

Similarly for (2) and (3), the T1-T4 and tightness relations as written only depend on the category structure of Λ.

$\square$

Corollary 4.3.4. *Let (G, P) and (H, Q) be two WQLO groups, and Λ a small category with two functors $d_P^\Lambda : \Lambda \to P$ and $d_Q^\Lambda : \Lambda \to Q$ such that Λ is a finitely-aligned P-graph with respect to d_P^Λ and a finitely-aligned Q-graph with respect to d_Q^Λ. Then $C_{tight}^*(\Lambda, P)$ is canonically isomorphic to $C_{tight}^*(\Lambda, Q)$.*

Proof. Let a and b be the universal tight representations of Λ as a P-graph and Q-graph, respectively, so that $C_{tight}^*(\Lambda, P) = C^*(a)$ and $C_{tight}^*(\Lambda, Q) = C^*(b)$.

By the previous Lemma, both the a_λs and the b_λs satisfy the equivalent T1-T4 and tightness relators, so by their respective universal properties, there are canonical covers taking $a_\lambda \mapsto b_\lambda$ and $b_\lambda \mapsto a_\lambda$, which are inverses, and hence canonical isomorphisms. $\square$

Lemma 4.3.5. *Let $\varphi : (G, P) \to (H, Q)$ be a reduction of WQLO groups. Let Λ be a finitely-aligned P-graph, and t a representation of Λ. If t has a gauge coaction by G, then it has a gauge coaction by H (when Λ inherits the structure of a Q-graph as described in 4.3.2).*

In particular, if δ is the gauge coaction by G, then $\epsilon := (\mathrm{id}_{C^*(t)} \otimes \bar{\varphi}) \circ \delta$ is the gauge coaction by H, where $\bar{\varphi} : C^*(G) \to C^*(H)$ is given by $U_g \mapsto V_{\varphi(g)}$.

Proof. First let us fix some notation. Let $\{U_g\}_{g \in G}$ and $\{V_h\}_{h \in H}$ denote the universal unitary representations of G and H respectively, so that $C^*(G)$ and $C^*(H)$ are the closed span of the U_g and V_h, respectively. Let δ denote the gauge coaction on $C^*(t)$, d_P^Λ the degree function on Λ as a P-graph, and recall that the degree functor on Λ as a Q-graph was $d_Q^\Lambda = \varphi \circ d_P^\Lambda$.

Then we have that $g \mapsto V_{\varphi(g)}$ is a unitary representation of G, so by the universal property of $C^*(G)$, there is a $*$-homomorphism $\bar\varphi : C^*(G) \to C^*(H)$ given by $U_g \mapsto V_{\varphi(g)}$. We will prove that there is a gauge coaction of H on $C^*(t)$ given by $\epsilon :=$ $(\mathrm{id}_{C^*(t)} \otimes \bar\varphi) \circ \delta$. First note that by Lemma 2.2.7, there is such a function ϵ, it is a $*$-homomorphism, and it maps $C^*(t) \to C^*(t) \otimes C^*(H)$.

But now observe that

$$
\begin{aligned}
\epsilon(t_\lambda) &= (\mathrm{id}_A \otimes \bar\varphi) \circ \delta(t_\lambda) \\
&= (\mathrm{id}_A \otimes \bar\varphi)(t_\lambda \otimes U_{d_P^\Lambda(\lambda)}) \\
&= t_\lambda \otimes V_{\varphi(d_P^\Lambda(\lambda))} \\
&= t_\lambda \otimes V_{d_Q^\Lambda(\lambda)}
\end{aligned}
$$

so by Proposition 2.5.11(3), ϵ is a gauge coaction by H, as desired. $\qquad\square$

4.4 Gauge-Invariant Uniqueness for Ordered Groups which Reduce to Amenable Groups

The following is the mathematical core of our gauge-invariant uniqueness theorem:

Proposition 4.4.1. *Let (G, P) be a WQLO group and suppose there is a reduction to an amenable group $\varphi : (G, P) \to (H, Q)$. Then for any finitely-aligned P-graph Λ, the following are canonically isomorphic:*

 a. $C^*_{tight}(\Lambda, P)$.

 b. $C^*_{tight}(\Lambda, Q)$.

 c. $C^*_{min}(\Lambda, P)$.

d. $C^*_{min}(\Lambda, Q)$.

e. $C^*(t)$, where t is any representation of Λ as a P-graph which is tight, Λ-faithful, and which has a gauge coaction by G.

Proof. For the sake of abbreviation, let us denote the representations generating $C^*_{tight}(\Lambda, P)$, $C^*_{tight}(\Lambda, Q)$, $C^*_{min}(\Lambda, P)$, and $C^*_{min}(\Lambda, Q)$ as A, B, C, and D, respectively. Our argument is organized by the following diagram:

$$
\begin{array}{ccc}
C^*(A) = C^*_{tight}(\Lambda, P) & \xrightarrow{\pi_B^A} & C^*(B) = C^*_{tight}(\Lambda, Q) \\
\downarrow{\scriptstyle \pi_C^A} & & \downarrow{\scriptstyle \pi_D^B} \\
C^*(C) = C^*_{min}(\Lambda, P) & \xrightarrow{\pi_D^C} & C^*(D) = C^*_{min}(\Lambda, Q)
\end{array}
$$

All arrows here denote canonical coverings.

The map π_B^A exists and is an isomorphism by Corollary 4.3.4.

The maps π_C^A and π_D^B are the canonical coverings of Proposition 4.2.1(2). Since H is amenable, then by Lemma 2.3.14 the gauge coaction by H on $C^*_{tight}(\Lambda, Q)$ is normal, and so π_D^B is a canonical isomorphism by Proposition 4.2.1 (5).

Combining these two, we have that $A \cong B \cong D$.

The map π_D^C arises from the work established in the previous lemmas: by Lemma 4.3.5 $C^*(C)$ carries a gauge coaction by H. Since also C is Λ-faithful by 4.1.22, then by the minimality of $C^*_{min}(\Lambda, Q)$, there is a canonical covering $\pi_D^C : C^*_{min}(\Lambda, P) \to C^*_{min}(\Lambda, Q)$.

Now, with the order notation of Lemma 2.5.9, we have that $A \geq_{rep} C \geq_{rep} D \cong B \cong A$, so $A \cong B \cong C \cong D$. That is, (a.)-(d.) are canonically isomorphic.

For (e.), we have that $A \geq_{rep} t \geq_{rep} C \cong A$ by the respective universality and co-universality of $C^*(A)$ and $C^*(C)$, so $A \cong t \cong C$. $\qquad \square$

One nice outcome is that this result is independent of the amenable group (H, Q) and the reduction φ, and simply requires that one exist. In particular, we have a

gauge-invariant uniqueness theorem for WQLO groups which have a reduction to an amenable group:

Theorem 4.4.2. *Let (G, P) be a weakly quasi-lattice ordered group which reduces to an amenable ordered group, and let Λ be a finitely-aligned P-graph. Then there is exactly one representation (up to canonical isomorphism) of Λ which is Λ-faithful, tight, and has a gauge coaction of G. This representation is universal for tight representations and co-universal for representations which are Λ-faithful and have a gauge coaction by G.*

Proof. Let T be the universal tight representation as in Proposition 4.2.1 (1), so $C^*(T) \cong C^*_{tight}(\Lambda)$. By that theorem, T is Λ-faithful, tight, and has a gauge coaction. It is also universal for tight representations.

By Proposition 4.4.1, $C^*(T) \cong C^*_{min}(\Lambda)$, so T is co-universal for representations which are Λ-faithful and have gauge coaction.

To see that T is the unique representation (up to canonical isomorphism) which is Λ-faithful, tight, and has a gauge coaction, if there were such another representation t, it would be covered by T (since t is tight, and T is universal for tight representations) and it would cover T (since t is Λ-faithful and has a gauge coaction and T is co-universal for such representations). Thus $t \cong T$.

$\square$

Remark 4.4.3. *In the case that (G, P) reduces to an amenable group, we believe that the C^*-algebra generated by this unique Λ-faithful, tight, gauge coacting representation deserves the title of* the C^*-algebra of the graph. *We will write $C^*(\Lambda)$ for this algebra.*

4.5 Additional Results in the case of a Strong Reduction

In this section we will consider what further results can be provided about a P-graph algebra under the assumption that (G, P) strongly reduces to an amenable group.

Lemma 4.5.1. *Let (G, P) be a WQLO group, and suppose (G, P) has a strong reduction to an amenable group. Then every gauge coaction on a finitely-aligned P-graph is normal.*

Proof. Let $\varphi : (G, P) \to (H, Q)$ be the strong reduction where H is amenable. Fix some finitely-aligned P-graph Λ and a representation s of Λ which has a gauge coaction by G.

Then by Lemma 4.3.5, s has a gauge coaction by H. For sake of clarity, let Φ_s and Ω_s denote the conditional expectations on $C^*(s)$ as in Lemma 2.3.12 which respectively arise from its G-coaction and H-coaction.

By Lemma 2.3.14, since H is amenable, then Ω_s is faithful. Our desired result is that Φ_s is faithful.

To this end, it suffices to show that $\Omega_s = \Phi_s$ as maps on $C^*(s)$, and since $C^*(s) = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in \Lambda\}$ and both functions are continuous and linear, it suffices to check that $\Omega_s = \Phi_s$ on a single term $s_\alpha s_\beta^*$.

Recall that by Lemma 2.3.12, we have

$$\Phi_s(s_\alpha s_\beta^*) = \begin{cases} s_\alpha s_\beta^* & \text{if } d(\alpha) = d(\beta) \\ 0 & \text{if } d(\alpha) \neq d(\beta) \end{cases}$$

and

$$\Omega_s(s_\alpha s_\beta^*) = \begin{cases} s_\alpha s_\beta^* & \text{if } \varphi(d(\alpha)) = \varphi(d(\beta)) \\ \\ 0 & \text{if } \varphi(d(\alpha)) \neq \varphi(d(\beta)) \end{cases}.$$

But φ is a strong reduction, so for the positive elements $d(\alpha), d(\beta) \in P$, we have $d(\alpha) = d(\beta)$ if and only if $\varphi(d(\alpha)) = \varphi(d(\beta))$.

Thus $\Omega_s = \Phi_s$ on each spanning element, so they are equal as functions, and thus Φ_s is faithful, so s is normal.

$\square$

Lemma 4.5.2. *Let (G, P) be a WQLO group with a strong reduction onto an amenable group, let Λ be a P-graph, and let s be a representations of (G, P). If s is Λ-faithful, has a gauge coaction, and every proper bolt in s is nonzero, then $s \cong \mathcal{T}$.*

Proof. Since $\mathcal{T}$ covers s, it suffices to show that $\ker \pi_s^{\mathcal{T}} = 0$.

To this end, consider the diagram

$$
\begin{array}{ccc}
C^*(\mathcal{T}) & \xrightarrow{\Phi_{\mathcal{T}}} & \mathcal{B}(\mathcal{T}) \\
\downarrow{\pi_s^{\mathcal{T}}} & & \downarrow{\psi_s^{\mathcal{T}}} \\
C^*(s) & \xrightarrow{\Phi_s} & \mathcal{B}(s)
\end{array}
$$

which commutes by a routine computation on the spanning elements $\mathcal{T}_\mu \mathcal{T}_\nu^*$.

Fix some $x \in \ker \pi_s^{\mathcal{T}} = 0$. Then $\pi_s^{\mathcal{T}}(x^*x) = 0$, so

$$0 = \Phi_s(\pi_s^{\mathcal{T}}(x^*x)) = \psi_s^{\mathcal{T}}(\Phi_{\mathcal{T}}(x^*x)).$$

Now by Theorem 4.1.15, since $\ker \psi_s^{\mathcal{T}}$ is generated by the bolts and $\mathcal{T}_\mu \mathcal{T}_\mu^*$s it contains, and by the hypotheses on s this kernel contains no proper bolts or $\mathcal{T}_\mu \mathcal{T}_\mu^*$, then we have $\ker \psi_s^{\mathcal{T}} = 0$, and thus $\Phi_{\mathcal{T}}(x^*x) = 0$.

By Lemma 4.5.1, $\Phi_{\mathcal{T}}$ is faithful, so $x^*x = 0$ and thus $x = 0$. Thus $\ker \pi_s^{\mathcal{T}} = \{0\}$, as desired.

$\square$

Corollary 4.5.3. *Let Λ be a finitely-aligned $\mathbb{N}^2 * \mathbb{N}$-graph. Then any gauge coaction by $G = \mathbb{Z}^2 * \mathbb{Z}$ is automatically normal.*

Proof. Recall that $(\mathbb{Z}^2 * \mathbb{Z}, \mathbb{N}^2 * \mathbb{N})$ strongly reduces to $(\mathbb{Z}^2 \wr \mathbb{Z}, \mathbb{N}^2 \wr \mathbb{N})$, which is amenable. Then by Lemma 4.5.1, the gauge coaction is normal.

$\square$

Lemma 4.5.4. *Let (G, P) be a WQLO group and suppose that (G, P) has a strong reduction to an amenable group. Let Λ be a finitely-aligned P-graph, and let t be a representation of Λ. Then the following are equivalent:*

1. *t has a gauge coaction by G*

2. *There is a bounded linear map $\Phi_t : C^*(t) \to \mathcal{B}(t)$ satisfying*

$$\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{if } d(\mu) \neq d(\nu) \end{cases}$$

Proof. $(1 \Rightarrow 2)$ If t has a gauge coaction, then such a map Φ_t exists by Lemma 4.1.8.

$(2 \Rightarrow 1)$ If such a Φ_t exists, then by Theorem 19.1 and Definition 19.2 of [10], writing $B_g = \overline{\text{span}}\{t_\mu t_\nu^* : d(\mu)d(\nu)^{-1} = g\}$, we have that $\mathcal{B} = \{B_g\}_{g \in G}$ are linearly independent and form a topological grading of $C^*(t)$.

In particular, $\mathcal{B}$ is a Fell bundle, and we can form the cross-sectional algebra $C^*(\mathcal{B})$ as in Remark 2.3.7. Each graded component B_g is naturally embedded in $C^*(\mathcal{B})$, so for each $\lambda \in \Lambda$, let s_λ denote the operator arising from embedding $t_\lambda \in B_{d(\lambda)}$ into $C^*(\mathcal{B})$.

But one may immediately check the T1-T4 relators to check that $\{s_\lambda\}_{\lambda \in \Lambda}$ is a representation of Λ in $C^*(\mathcal{B})$. Note that since the $\{s_\lambda\}_{\lambda \in \Lambda}$ generate each fiber B_g, then they generate all of $C^*(\mathcal{B})$, so $C^*(\mathcal{B}) = C^*(s)$.

Our goal is now to show that s has a gauge coaction and that $s \cong t$. For the former, by [17, Proposition 3.3], there is a coaction δ on $C^*(\mathcal{B})$ satisfying

$$\delta(b) = b \otimes U_g \text{ for all } b \in B_g, g \in G.$$

In particular, since $s_\lambda \in B_{d(\lambda)}$, then $\delta(s_\lambda) = s_\lambda \otimes U_{d(\lambda)}$, so δ is a gauge coaction on s.

Now we must show that $s \cong t$. By [10, Theorem 19.5], since $C^*(t)$ is a topologically graded C^*-algebra, with grading $\mathcal{B}$ there is a commutative diagram of surjective $*$-homomorphisms

$$C^*(\mathcal{B}) = C^*(s) \xrightarrow{\quad L \quad} C^*_r(\mathcal{B})$$

$$\pi^s_t \searrow \qquad \nearrow \psi$$

$$C^*(t)$$

where L denotes the regular representation from [10, Definition 17.6] (in that definition, this map is called Λ, but we have changed its name to avoid confusion with the P-graph Λ).

Recall that $C^*(\mathcal{B})$ is topologically graded with conditional expectation Φ_s. Since G strongly reduces to an amenable group, then by Lemma 4.5.1, the gauge action s is normal and thus the conditional expectation Φ_s is faithful. But by [10, Proposition 19.6], the kernel of the regular representation L is given by

$$\ker(L) = \{x \in C^*(\mathcal{B}) : \Phi_s(x^*x) = 0\}$$

and since Φ_s is faithful, then $\ker L = \{0\}$. Since L is injective and $L = \psi \circ \pi^t_s$, then π^t_s is injective, so $s \cong t$, so t has a gauge coaction as desired.

$\square$

The reader should note that if such a bounded linear map exists, then there is a gauge coaction, so that bounded linear map is the conditional expectation of Lemma 2.3.12.

The next result shows that if you represent a P-graph on a Hilbert space in such a way that the Hilbert space also has a "P-grading" which interacts with the grading on Λ in a natural way, then this representation has a gauge coaction.

Lemma 4.5.5. *Let (G, P) be a WQLO group and suppose that (G, P) has a strong reduction to an amenable group. Let Λ be a finitely-aligned P graph. Let t be a representation of Λ in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and suppose that $\mathcal{H} = \bigoplus_{p \in P} \mathcal{H}_p$ such that $t_\mu \mathcal{H}_p \subseteq \mathcal{H}_{d(\mu)p}$ for all $\mu \in \Lambda$, $p \in P$. Then t has a gauge coaction.*

Proof. By Lemma 4.5.4, it suffices to show that there is a bounded linear map $\Phi_t : C^*(t) \to \mathcal{B}(t)$ satisfying

$$
\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{if } d(\mu) \neq d(\nu) \end{cases}.
$$

For each $p \in P$, let $Q_p \in \mathcal{B}(\mathcal{H})$ denote the projection onto $\mathcal{H}_p$. Then define

$$
\Phi_t(x) = \sum_{\substack{p \in P \\ WOT \text{ limit}}} Q_p x Q_p
$$

which is bounded and converges in the weak operator topology because the $\{Q_p\}_{p \in P}$ are pairwise orthogonal projections. We then wish to show that

$$
\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{if } d(\mu) \neq d(\nu) \end{cases}
$$

for all $\mu, \nu \in \Lambda$.

To this end, fix some $q \in P$, $h_q \in \mathcal{H}_q$, and $x \in \mathcal{H}$. Then we have that

$$
\begin{aligned}
\langle \Phi_t(t_\mu t_\nu^*)h_q, x \rangle &= \underset{\substack{p \in P \\ WOT \text{ limit}}}{\sum} \langle Q_p t_\mu t_\nu^* Q_p h_q, x \rangle \\
&= \langle Q_q t_\mu t_\nu^* Q_q h_q, x \rangle \\
&= \langle Q_q t_\mu t_\nu^* h_q, x \rangle
\end{aligned}
$$

since $Q_p h_q = 0$ if $p \neq q$. Since our choice of $x \in \mathcal{H}$ was arbitrary, we may unbind the x term, so we have shown that $\Phi_t(t_\mu t_\nu^*)h_q = Q_q t_\mu t_\nu^* h_q$.

But note that $t_\mu t_\nu^* h_q \in \mathcal{H}_{d(\mu)d(\nu)^{-1}q}$ and $q = d(\mu)d(\nu)^{-1}q$ if and only if $d(\mu) = d(\nu)$, so

$$
\Phi_t(t_\mu t_\nu^*)h_q = Q_q t_\mu t_\nu^* h_q = \begin{cases} t_\mu t_\nu^* h_q & \text{if } d(\mu) = d(\nu) \\ 0 & \text{if } d(\mu) \neq d(\nu) \end{cases}.
$$

Since this is true for each $q \in P$ and $h_q \in \mathcal{H}_q$, and the $\mathcal{H}_q$ span all of $\mathcal{H}$, we may unbind the h_q to get that

$$
\Phi_t(t_\mu t_\nu^*) = \begin{cases} t_\mu t_\nu^* & \text{if } d(\mu) = d(\nu) \\ 0 & \text{if } d(\mu) \neq d(\nu) \end{cases}
$$

as desired.

$\square$

Corollary 4.5.6. *Let (G, P) be a WQLO group and suppose that (G, P) has a strong reduction to an amenable group. Let Λ be a finitely-aligned P-graph. Then $L \cong \mathcal{T}$, where L denotes the regular representation of Λ and $\mathcal{T}$ denotes the Toeplitz representation of Λ.*

Proof. To show that L is universal, by Lemma 4.5.2 it suffices to show that L it is Λ-faithful, has a gauge coaction, and that every proper bolt is nonzero. By Remark 2.5.6, L is Λ-faithful and its proper bolts are nonzero.

To show that L has a gauge coaction, we will appeal to Lemma 4.5.5. Recall that L is a representation in $\mathcal{B}(\ell^2(\Lambda))$ by

$$
L_\mu e_\alpha =
\begin{cases}
e_{\mu\alpha} & \text{if } s(\mu) = r(\alpha) \\
0 & \text{if } s(\mu) \neq r(\alpha)
\end{cases}.
$$

Therefore, writing $\Lambda^p = \{\mu \in \Lambda : d(\mu) = p\}$, we can see that $\ell^2(\Lambda) = \bigoplus_{p \in P} \ell^2(\Lambda^p)$, and a simple calculation shows that $L_\mu \ell^2(\Lambda^p) \subseteq \ell^2(\Lambda^{d(\mu)p})$. Therefore, by Lemma 4.5.5, L has a gauge coaction.

Thus by by Lemma 4.5.2, $L \cong \mathcal{T}$.

$\square$

The following result shows that if Λ has no "infinite paths", then the ultrafilter representation is the universal tight representation.

Lemma 4.5.7. *Let (G, P) be a WQLO group and suppose that (G, P) has a strong reduction to an amenable group. Let Λ be a finitely-aligned P-graph. Recall that Λ has a partial order given by $\alpha \leq \beta$ if $\beta \in \alpha\Lambda$. Suppose that with respect to this partial order, every chain in Λ has an upper bound in Λ. Then the ultrafilter representation f of Λ is the universal tight representation of Λ.*

Proof. By our gauge-invariant uniqueness theorem for P-graphs, it suffices to show that f is Λ-faithful, tight, and has a gauge coaction. The former two properties are shown in Lemma 4.1.5. It then suffices to use Lemma 4.5.5 to show that f has a gauge coaction.

To this end, we wish to show that the ultrafilters in f correspond to maximal paths in Λ. More formally, for each path $\alpha \in \Lambda$, let $U_\alpha = \{\mu \in \Lambda : \mu \leq \alpha\}$. It is straightforward to check that U_α is a filter, and that it is an ultrafilter if and only if α is maximal. We wish to show that all ultrafilters in Λ are of the form U_α.

Fix some ultrafilter U. If U is finite, it must contain a maximal element α, in which case $U = U_\alpha$. If instead U is infinite, since $U \subseteq \Lambda$ is countable, we may enumerate $U = \{\alpha_n\}_{n\in\mathbb{N}}$. Inductively define a sequence $\{\beta_n\}_{n\in\mathbb{N}} \subseteq U$ by $\beta_1 = \alpha_1$, and for $n > 1$, let β_n be a common upper bound of β_{n-1} and α_n which is guaranteed to exist by the $F2$ property of filters. Note that $U = \bigcup_{n=1}^{\infty} U_{\beta_n}$.

Then $\{\beta_n\}_{n\in\mathbb{N}}$ is a chain in Λ, so by our hypothesis, it has an upper bound $\beta \in \Lambda$. Then $U_\beta \supseteq U_{\beta_n}$ for all n, and since $U = \bigcup_{n=1}^{\infty} U_{\beta_n}$, then $U_\beta \supseteq U$, and since U is an ultrafilter, $U = U_\beta$, as desired.

Therefore, we can partition the set of ultrafilters by the degree of their unique maximal element. That is, writing $\widehat{\Lambda}_\infty^p = \{U_\alpha \in \widehat{\Lambda}_\infty : d(\alpha) = p\}$, we have $\widehat{\Lambda}_\infty = \bigsqcup_{p\in P} \widehat{\Lambda}_\infty^p$, and therefore

$$\ell^2(\widehat{\Lambda}_\infty) = \bigoplus_{p\in P} \ell^2(\widehat{\Lambda}_\infty^p).$$

Finally, a simple calculation shows that $f_\mu \ell^2(\widehat{\Lambda}_\infty^p) \subseteq \ell^2(\widehat{\Lambda}_\infty^{d(\mu)p})$, so by Lemma 4.5.5, f has a gauge coaction, as desired.

$\square$

Example 4.5.8. *Let* $G = BS(1,2) = \langle c,t | tc = c^2t \rangle$, *and let* $P = \{c,t\}^*$. *Note that* $BS(1,2)$ *is amenable, and therefore has a strong reduction to an amenable group (itself). Define a P-graph Λ as in this diagram:*

$$v_1 \xrightarrow{\ c_1\ } v_2$$

$$\downarrow t_1 \qquad\qquad \downarrow t_2$$

$$v_3 \xrightarrow{\ c_2\ } v_4 \xrightarrow{\ c_3\ } v_5$$

so $\Lambda = \{v_1, v_2, v_3, v_4, v_5, c_1, c_2, c_3, t_1, t_2, c_2t_1, c_3c_2, c_3c_2t_1 = c_1t_2\}$, *where the range and source maps are as indicated in the diagram, the operation is concatenation of paths, and the degrees are given by* $d(v_i) = 1$, $d(c_i) = c$, $d(t_i) = t$, *and so on.*

Since Λ *is finite, all chains have an upper bound, so by Lemma 4.5.7, the ultrafilter representation is the universal tight representation and the set of ultrafilters is in bijection with the maximal paths in* Λ. *Thus* $\widehat{\Lambda}_\infty = \{v_1, c_1, t_1, c_2t_1, c_3c_2t_1 = t_2c_1\}$, *so* $C^*(f)$ *is represented on a 5-dimension vector space and* $C^*(f) \subseteq M_5$, *the space of 5-by-5 matrices. A direct calculation with respect to the basis* $\{e_{v_1}, e_{c_1}, e_{t_1}, e_{c_2t_1}, e_{c_3c_2t_1} = e_{t_2c_1}\}$ *shows that*

$$
\begin{aligned}
f_{v_1} &= E_{11} \\
f_{c_1} &= E_{21} \\
f_{t_1} &= E_{31} \\
f_{c_2t_1} &= E_{41} \\
f_{t_2c_1} &= E_{51}
\end{aligned}
$$

where E_{ij} *denotes the matrix consisting of all 0s except for a single 1 in the ith row and jth column (a matrix unit). Since these matrix units generate* M_5, *then* $C^*(f) = C^*(\Lambda) = M_5$.

It is notable that the results in this section apply only to groups with strong reduc-

tions, instead of any reduction. The only "bottleneck" for this is Lemma 4.5.1, which says that gauge coactions are normal when (G, P) strongly reduces to an amenable group. We conjecture that a (non-strong) reduction to an amenable group would suffice:

Conjecture 4.5.9. *Let (G, P) be a WQLO group, let Λ be a finitely aligned P-graph, and suppose (G, P) reduces to an amenable group. Then every gauge coaction on a representation of Λ is normal.*

Therefore, the other results in this section only require the hypothesis that (G, P) reduces to an amenable group.

4.6 Applications to Kirchberg Algebras in the UCT Class

Definition 4.6.1. *A C^*-algebra A is called a <u>Kirchberg algebra</u> if it is purely infinite, simple, nuclear, and separable.*

A C^*-algebra is said to be in the <u>UCT class</u> if it satisfies the hypotheses of the universal coefficient theorem. (For a more thorough treatment, the reader is directed to the introduction of [1]).

Proposition 4.6.2. *Let A be a Kirchberg algebra in the UCT class. Then there is a finitely aligned $\mathbb{N}^2 * \mathbb{N}$-graph such that A is stably isomorphic to $C^*(\Lambda)$.*

Proof. By [3, Corollary 6.3], there is a finitely aligned $\mathbb{N}^2 * \mathbb{N}$-graph Λ such that A is stably isomorphic to $C^*_{min}(\Lambda)$. But since $\mathbb{N}^2 * \mathbb{N}$ strongly reduces to an amenable group by Corollary 3.3.2, then $C^*_{min}(\Lambda) = C^*(\Lambda)$ by the gauge-invariant uniqueness theorem for P-graphs (Theorem 4.4.2). Thus A is stably isomorphic to $C^*(\Lambda)$. $\quad\square$

Chapter 5

Appendix 1: Tight Representations

In this section, our goal is to justify our use of the term "tight" in the context of representations of P-graphs by showing that a representation is tight in the sense of Definition 2.5.5(c) if and only if it corresponds to a tight representation of a semilattice in the sense of Exel's definition in [8].

In this section, we fix (G, P) a WQLO group, Λ a finitely-aligned P-graph, and $t : \Lambda \to C^*(t)$ a representation. We will often write $q_\lambda = t_\lambda t_\lambda^*$.

It can be easy to be confused about what symbols denote paths, sets of paths, and sets of sets of paths. To avoid confusion, we will try to stick to the convention that greek letters like α denote paths in Λ, lower case letters (and $\mu\Lambda$ where $\mu \in \Lambda$) denote subsets of Λ, and capital letters denote collections of subsets.

5.1 Background

Recall the following definition of the tightness from 2.5.5(c), slightly modified to keep with our notation for this section:

Definition 5.1.1. *A representation t of a graph Λ is* <u>*tight*</u> *if for every $\mu \in \Lambda$ and finite $z \subseteq \mu\Lambda$ which is exhaustive for $\mu\Lambda$, we have that*

$$\prod_{\alpha \in z}(q_\mu - q_\alpha) = 0$$

where $q_\lambda = t_\lambda t_\lambda^*$.

This condition was referred to a tightness before, but in this section will be referred to as H-tightness to avoid confusion.

The following terminology is from [8, Section 2]:

Definition 5.1.2. *In [8]'s context, a partially ordered set contains a smallest element, denoted 0. A semilattice is a partially ordered set where for all $x, y \in X$, the set $\{z \in X : z \le x, y\}$ has a maximum element, denoted $x \wedge y$. That is, $x \wedge y$ is the infimum of x and y, and every two elements have an infimum (although it may be 0).*

Given, $x, y \in X$, we say x and y are disjoint and write $x \perp y$ if there is no nonzero $z \in X$ with $z \le x, y$. Otherwise we say that x and y intersect, and write $x \Cap y$.

Definition 5.1.3. *Let E be a semilattice. Given finite $X, Y \subseteq E$, let $E^{X,Y}$ denote the subset of E given by*

$$E^{X,Y} = \{z \in E : z \le x \ \forall x \in X, z \perp y \ \forall y \in Y\}.$$

Given any subset $F \subseteq E$, we shall say that a subset $Z \subseteq F$ is a cover for F if, for every nonzero $x \in F$ there exists $z \in Z$ such x and z have a common extension.

Definition 5.1.4. *Let $\sigma : E \to \mathcal{B}$ be a representation of a semilattice E in a Boolean algebra $\mathcal{B}$. We shall say that σ is tight if for every finite subsets $X, Y \subseteq E$ and for every finite cover Z for $E^{X,Y}$, one has that*

$$\bigvee_{z \in Z} \sigma(z) \ge \bigwedge_{x \in X} \sigma(x) \wedge \bigwedge_{y \in Y} \neg \sigma(y).$$

Since the reverse inequality is always true, tightness is equivalent to equality in this expression.

To avoid confusion with H-tightness, we will refer to this notion of tightness as E-tightness.

5.2 Building our Semilattice

We'll now define our semilattice E and develop its basic properties.

Remark 5.2.1. *Given Λ, let $E(\Lambda) = \{\bigcup_{i=1}^n \alpha_i \Lambda : \alpha_i \in \Lambda\}$ denote the collection of finite unions of $\alpha\Lambda s$, ordered under containment. When Λ is clear from context, we will write E for $E(\Lambda)$. The empty set serves as the zero element for E, and two elements a, b of E have a least lower bound $a \cap b$, which is also in E by finite alignment of Λ. Thus E is a semilattice.*

Recall that Λ is given a partial ordering $\leq$ by $\alpha \leq \beta$ if and only if $\beta = \alpha\alpha_1$. Note that $\alpha \leq \beta$ if and only if $\beta\Lambda \subseteq \alpha\Lambda$.

For $a \in E$, we let $m(a) = \{\alpha \in a : \alpha \text{ minimal in } a\}$ denote the set of maximal elements of a (with respect to the ordering on Λ). For any $a \in E$, we may write $a = \bigcup_{i=1}^n \alpha_i \Lambda$ for some $\alpha_i \in \Lambda$, so it follows that $m(a) \subseteq \{\alpha_i\}_{i=1}^n$, so $m(a)$ is finite. Note also that $m(a)$ is empty if and only if a is empty. Finally, note that $a = \bigcup_{\alpha \in m(a)} \alpha\Lambda$. So in this way, m provides a unique representation of each element of E, and provides a bijection between E and the set of finite, pairwise incomparable subsets of Λ.

If t be a representation of Λ, let $q_\alpha = t_\alpha t_\alpha^*$ for all $\alpha \in \Lambda$, and let $\mathcal{B}_t \subseteq C^*(t)$ be the Boolean algebra generated by 1 and $\{q_\alpha : \alpha \in \Lambda\}$, where $q_\alpha \wedge q_\beta = q_\alpha q_\beta$.

Finally, given such a representation t, define $\sigma_t : E \to \mathcal{B}_t$ by

$$\sigma_t : a = \bigcup_{\alpha \in m(a)} \alpha\Lambda \mapsto \bigvee_{\alpha \in m(a)} t_\alpha t_\alpha^* = \bigvee_{\alpha \in m(a)} q_\alpha.$$

When the representation t is clear from context, we will write σ and $\mathcal{B}$ for σ_t and $\mathcal{B}_t$, respectively.

5.3 Technical Lemmas

Lemma 5.3.1 (Intersection Criterion)**.** *If $a, b \in E$, then $a \Cap b$ if and only if there are $\alpha \in m(a), \beta \in m(b)$ such that $\alpha \Cap \beta$ (in the sense of having a common extension in Λ).*

Proof. Observe that $a \cap b = \bigcup_{\alpha \in m(a), \beta \in m(b)} \alpha\Lambda \cap \beta\Lambda$, so the lefthand side is nonempty if and only if one of the terms on the righthand side are nonempty. But $\alpha\Lambda \cap \beta\Lambda$ is nonempty if and only if α has a common extension with β. $\qquad\square$

Lemma 5.3.2. *Let Z be a finite cover of $E^{X,Y}$. Then there is another finite cover Z' of $E^{X,Y}$ such that every element of Z' is of the form $\alpha\Lambda$, and $\bigvee_{z \in Z} z = \bigvee_{z' \in Z'} z'$.*

Proof. Let $B = m(\bigcup_{z \in Z} z)$ and let $Z' = \{\beta\Lambda : \beta \in B\}$. Then by construction B and hence Z' are finite sets, and by our initial remark on m, we know that $\bigcup_{z \in Z} z = \bigvee_{z \in Z} z$ can be reconstituted from its minimal elements in the sense that $\bigvee_{z \in Z} z = \bigvee_{z' \in Z'} z'$.

It then suffices to show that Z' is a cover of $E^{X,Y}$. To first show that it is contained in $E^{X,Y}$, we know that if $\beta\Lambda \in Z'$, then $\beta \in B$, so β must be minimal in some $z_0 \in Z$. Then for all $x \in X$ and $y \in Y$, $\beta\Lambda \le z_0 \le x$, and $\beta\Lambda \wedge y \le z_0 \wedge y = 0$, so $\beta\Lambda \le x$ and $\beta\Lambda \perp y$ for all $x \in X, y \in Y$. Thus $Z' \subseteq E^{X,Y}$.

To show it covers, fix some nonzero $w \in E^{X,Y}$. Then since Z covers $E^{X,Y}$, there is a $z \in Z$ such that $z \Cap w$. Then by the intersection criterion, there is a $\mu \in m(z), \nu \in$

$m(w)$ such that $\mu \cap\!\!\!\!\!\cap \nu$. Since $\mu \in m(z)$, then $\mu\Lambda \le \bigcup_{\beta \in B} \beta\Lambda$, so there is some $\beta \in B$ with $\beta \le \mu$. Then $\beta \cap\!\!\!\!\!\cap \nu$, so $\beta\Lambda \cap\!\!\!\!\!\cap w$. Thus Z' is a finite cover.

$\square$

Lemma 5.3.3. *Let $X, Y \subseteq E$ be finite sets. Then there is a finite set $Y' \subseteq E$ such that $E^{X,Y} = E^{X,Y'}$ and $Y' \subseteq E^{X,\emptyset}$.*

Proof. By the remark in [8] following Definition 2.5, we may assume without loss of generality, that $X = \{x\}$, a singleton. Let $Y' = \{y \wedge x : x \in X\}$ which is finite. Then by construction, $y' = y \wedge x \le x$ for all $y' \in Y'$, so $Y' \subseteq E^{X,\emptyset}$. It then suffices to show that $E^{X,Y} = E^{X,Y'}$.

If $w \in E^{X,Y}$, then $w \le x$ and for all $y \in Y$, $w \perp y$, so $w \wedge y = 0$, and thus $w \wedge y' = w \wedge y \wedge x = 0$ for all $y' \in Y'$. Thus $w \in E^{X,Y'}$.

Conversely, if $w \in E^{X,Y'}$, then $w \le x$ and for all $y' \in Y$, $w \perp y'$, so $w \wedge y \wedge x = 0$. But $w \le x$, so $w \wedge y \wedge x = (w \wedge x) \wedge y = w \wedge y = 0$, so $w \in E^{X,Y}$.

Thus $E^{X,Y} = E^{X,Y'}$ as desired. $\square$

Lemma 5.3.4. *Let $\mu \in \Lambda$ and $B \subseteq \mu\Lambda$ be finite. Then $q_\mu = \bigvee_{\beta \in B} q_\beta$ if and only if $0 = \prod_{\beta \in B} (q_\mu - q_\beta)$.*

Proof. The result follows from a chain of equivalences:

$$
\begin{aligned}
q_\mu &= \bigvee_{\beta \in B} q_\beta \\
0 &= q_\mu - \bigvee_{\beta \in B} q_\beta \text{ by subtraction} \\
0 &= \bigwedge_{\beta \in B} (q_\mu - q_\beta) \text{ by De Morgan's laws} \\
0 &= \prod_{\beta \in B} (q_\mu - q_\beta) \text{ by definition of } \bigwedge.
\end{aligned}
$$

□

5.4 The Proof of Equivalence

Here we prove the following proposition. The main interest is the equivalence of the first and last criteria, and the middle statements are there to ease the proof.

Proposition 5.4.1. *If t is a representation of Λ, and $\sigma = \sigma_t$ is the associated representation of the semilattice $E = E(\Lambda)$, then the following are equivalent:*

1. *σ is a tight representation in the E-sense.*

2. *σ is a tight representation in the E-sense when $Y = \emptyset$.*

3. *σ is a tight representation in the E-sense when $Y = \emptyset$ and $X = \{\mu\Lambda\}$.*

4. *t is a tight representation in the H-sense.*

Proof. Certainly $(1) \Rightarrow (2) \Rightarrow (3)$. We will prove four non-obvious directions: $(3) \Rightarrow (4), (4) \Rightarrow (3), (3) \Rightarrow (2)$ and $(2) \Rightarrow (1)$.

$(3) \Rightarrow (4)$ Fix some $\mu \in \Lambda$. We wish to show that if $B \subseteq \mu\Lambda$ is exhaustive, then $Z_B = \{\beta\Lambda : \beta \in B\}$ is a finite cover of $E^{\{\mu\Lambda\},\emptyset}$. In particular, given such an exhaustive B, if $w \in E^{\{\mu\Lambda\},\emptyset}$ is nonzero, then taking $\alpha \in m(w)$, $\alpha \in w \leq \mu\Lambda$, so $\alpha \in \mu\Lambda$, so since B is exhaustive there is a $\beta \in B$ such that $\alpha \Cap \beta$, and by the intersection criterion, $\beta\Lambda \Cap w$. Thus Z_B covers.

Then by (3), we have that $\bigvee_{z \in Z_B} \sigma(z) = \sigma(\mu\Lambda)$ and substituting $\sigma(z) = \sigma(\beta\Lambda) = q_\beta$ and $\sigma(\mu\Lambda) = q_\mu$, we have

$$\bigvee_{\beta \in B} q_\beta = q_\mu$$

which is equivalent to H-tightness by Lemma 5.3.4.

Given such an exhaustive B, if $w \in E^{\{\mu\Lambda\},\emptyset}$ is nonzero, then taking $\alpha \in m(w)$, $\alpha \in w \leq \mu\Lambda$, so $\alpha \in \mu\Lambda$, so since B is exhaustive there is a $\beta \in B$ such that $\alpha \pitchfork \beta$, and by the intersection criterion, $\beta\Lambda \pitchfork w$. Thus Z_B covers.

(4) $\Rightarrow$ (3) Fix some $\mu \in \Lambda$. We wish to show that if $Z \subseteq E^{\{\mu\Lambda\},\emptyset}$ is a finite cover, then $B_Z = \bigcup_{z \in Z} m(z)$ is a finite exhaustive subset of $\mu\Lambda$. In particular, given $\alpha \in \mu\Lambda$, we have that $\alpha\Lambda \in E^{\{\mu\Lambda\},\emptyset}$, so there is $z \in Z$ such that $z \pitchfork \alpha\Lambda$. Then by the intersection property, there is a $\beta \in m(z)$ such that $\beta \pitchfork \alpha$. Thus B_Z is exhaustive for $\mu\Lambda$, as desired.

Then, by (4), and Lemma 5.3.4, we have that

$$q_\mu = \bigvee_{\beta \in B_Z} q_\beta.$$

Substituting $\sigma(\mu\Lambda) = q_\mu$ and $\sigma(\beta\Lambda) = q_\beta$, then regrouping the terms on the right, we have that

$$
\begin{aligned}
\sigma(\mu\Lambda) &= \bigvee_{\beta \in B_Z} \sigma(\beta\Lambda) \\
&= \bigvee_{z \in Z} \bigvee_{\beta \in m(z)} \sigma(\beta\Lambda) \\
&= \bigvee_{z \in Z} \sigma\Big(\bigvee_{\beta \in m(z)} \beta\Lambda \Big) \\
&= \bigvee_{z \in Z} \sigma(z)
\end{aligned}
$$

as desired.

(3 $\Rightarrow$ 2) Let $X \subseteq E$ and let Z be a finite cover of $E^{X,\emptyset}$. By the remark in

[8] following Definition 2.5, we may assume without loss of generality that X is a singleton, since otherwise we may replace X with $\{x_{min}\}$ where $x_{min} = \bigwedge_{x \in X} x$.

Let Z be a finite cover of $E^{X,\emptyset}$. By Lemma 5.3.2, we may assume without loss of generality that $Z = \{\beta_1 \Lambda, ..., \beta_n \Lambda\}$ for some $\beta_1, ..., \beta_n \in \Lambda$. Let $B = \{\beta_1, ... \beta_n\}$ and $A = m(x)$.

Now, for each $\alpha \in A$ and $\beta \in B$, let $Z_{\alpha,\beta} = \{\gamma \Lambda : \gamma \in MCE(\alpha, \beta)\}$. Note that $Z_{\alpha,\beta}$ is finite and $\gamma \Lambda \in Z_{\alpha,\beta}$ is in $\alpha \Lambda \cap \beta \Lambda$.

We wish to show that for fixed $\beta \in B$, $\bigcup_{\alpha \in A} Z_{\alpha,\beta}$ is a cover for $E^{\{\beta \Lambda\},\emptyset}$ and for fixed $\alpha \in A$, that $\bigcup_{\beta \in B} Z_{\alpha,\beta}$ is a cover for $E^{\{\alpha \Lambda\},\emptyset}$. Each is a finite union of finite sets, hence finite, and is contained in the appropriate space. It then suffices to show that they are covering.

For the former set, fix some nonzero $w \in E^{\{\beta \Lambda\},\emptyset}$. Since w is nonzero, there is some $\mu \in m(w)$. Then $\mu \Lambda \leq w \leq \beta \Lambda \leq x$, so $\mu \Lambda \wedge x = \mu \Lambda < \infty$, so $\mu \Lambda \Cap x$, and thus there is $\alpha \in m(x)$ such that $\mu \Cap \alpha$. That is, α and μ have a common extension ν. Since β is a prefix of μ, then ν is a common extension of α and β as well. Thus some prefix of ν is a $\gamma \in MCE(\alpha, \beta)$, so ν is a common extension of both γ and μ. That is, $\gamma \Lambda \Cap w$, and $\gamma \Lambda \in Z_{\alpha,\beta}$. Thus $\bigcup_{\alpha \in A} Z_{\alpha,\beta}$ is a cover for $E^{\{\beta \Lambda\},\emptyset}$, as desired.

For the latter set, fix some nonzero $w \in E^{\{\alpha \Lambda\},\emptyset}$. Then we know that $w \leq \alpha \Lambda \leq x$, so $w \in E^{X,\emptyset}$, and because Z is a cover, there is some $\beta \Lambda \in Z$ with $w \Cap \beta \Lambda$. Then by the intersection criterion, there is a $\mu \in m(w)$ such that $\mu \Cap \beta$. That is, μ and β have a common extension ν. However, $\alpha \leq \mu$ as $w \in E^{\{\alpha \Lambda\},\emptyset}$, so ν is a common extension of β and α, and thus has a prefix which is some $\gamma \in MCE(\beta, \alpha)$. Thus ν is a common extension of μ and γ, so $w \Cap \gamma \Lambda$, and $\gamma \Lambda \in Z_{\alpha,\beta}$. Thus $\bigcup_{\beta \in B} Z_{\alpha,\beta}$ is a cover for $E^{\{\alpha \Lambda\},\emptyset}$, as desired.

Now, using (3), the fact that these two sets cover gives us that

$$
\begin{aligned}
\sigma(\beta\Lambda) &= \bigvee_{\gamma\in\bigcup_{\alpha\in A} Z_{\alpha,\beta}} \sigma(\gamma\Lambda) \\
&= \bigvee_{\alpha\in A} \bigvee_{\gamma\in MCE(\alpha,\beta)} \sigma(\gamma\Lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma(\alpha\Lambda) &= \bigvee_{\gamma\in\bigcup_{\beta\in B} Z_{\alpha,\beta}} \sigma(\gamma\Lambda) \\
&= \bigvee_{\beta\in B} \bigvee_{\gamma\in MCE(\alpha,\beta)} \sigma(\gamma\Lambda)
\end{aligned}
$$

Finally, using the fact that $x = \bigcup_{\alpha\in A} \alpha\Lambda$, we have that

$$
\begin{aligned}
\sigma(x) &= \bigvee_{\alpha\in A} \sigma(\alpha\Lambda) \\
&= \bigvee_{\alpha\in A} \left(\bigvee_{\beta\in B} \bigvee_{\gamma\in MCE(\alpha,\beta)} \sigma(\gamma\Lambda) \right) \\
&= \bigvee_{\beta\in B} \left(\bigvee_{\alpha\in A} \bigvee_{\gamma\in MCE(\alpha,\beta)} \sigma(\gamma\Lambda) \right) \\
&= \bigvee_{\beta\in B} \sigma(\beta\Lambda) \\
&= \bigvee_{z\in Z} \sigma(z)
\end{aligned}
$$

which is the E-tightness condition for $E^{X,\emptyset}$.

$(2 \Rightarrow 1)$ Let $X, Y \subseteq E$ be finite, and let Z be a finite cover of $E^{X,Y}$. By Lemma 5.3.3, we may assume that Y is a subset of $E^{X,\emptyset}$. Let $Z' = Z \cup Y$. We claim that Z'

is a finite cover of $E^{X,\emptyset}$. In particular, given $w \in E^{X,\emptyset}$, if $w \perp y$ for all $y \in Y$, then $w \in E^{X,Y}$, so $w \pitchfork z$ for some $z \in Z \subseteq Z'$. If instead $w \not\perp y$ for some y, then $w \pitchfork y$ for some $y \in Y \subseteq Z'$.

Then by (2), we know that

$$\bigvee_{z' \in Z'} \sigma(z') = \bigwedge_{x \in X} \sigma(x)$$

and meeting with $\bigwedge_{y \in Y} \neg\sigma(y)$, we have that

$$\left(\bigvee_{z' \in Z'} \sigma(z') \right) \wedge \bigwedge_{y \in Y} \neg\sigma(y) = \bigwedge_{x \in X} \sigma(x) \wedge \bigwedge_{y \in Y} \neg\sigma(y)$$

Simplifying the lefthand side, we have that

$$
\begin{aligned}
\left(\bigvee_{z' \in Z'} \sigma(z') \right) \wedge \bigwedge_{y \in Y} \neg\sigma(y) &= \left(\bigvee_{z \in Z} \sigma(z) \vee \bigvee_{y' \in Y} \sigma(y') \right) \wedge \bigwedge_{y \in Y} \neg\sigma(y) \\
&= \left(\bigvee_{z \in Z} \sigma(z) \wedge \bigwedge_{y \in Y} \neg\sigma(y) \right) \vee \left(\bigvee_{y' \in Y} \sigma(y') \wedge \bigwedge_{y \in Y} \neg\sigma(y) \right) \\
&= \left(\bigvee_{z \in Z} \sigma(z) \wedge \bigwedge_{y \in Y} \neg\sigma(y) \right) \\
&= \bigvee_{z \in Z} \left(\sigma(z) \wedge \bigwedge_{y \in Y} \neg\sigma(y) \right)
\end{aligned}
$$

Now, for a fixed $z \in Z, y \in Y$, and for any $\alpha \in m(z)$, $\beta \in m(y)$, we have that $\alpha \perp \beta$, so $q_\alpha q_\beta = 0$. Then $\sigma(\alpha\Lambda) \wedge \neg\sigma(\beta\Lambda) = q_\alpha(1 - q_\beta) = q_\alpha$, so $\sigma(z) \vee \neg\sigma(y) = \sigma(z)$. Thus

$$\bigvee_{z \in Z} \left(\sigma(z) \wedge \bigwedge_{y \in Y} \neg\sigma(y) \right) = \bigvee_{z \in Z} \sigma(z)$$

so by transitivity,

$$\bigvee_{z \in Z} \sigma(z) = \bigwedge_{x \in X} \sigma(x) \wedge \bigwedge_{y \in Y} \neg\sigma(y)$$

as desired. This completes the proof that (2) implies (1).

$\square$

Bibliography

[1] Selçuk Barlak and Xin Li. Cartan subalgebras and the UCT problem, II. *Math. Ann.*, 378(1-2):255–287, 2020.

[2] Bruce Blackadar. Shape theory for C^*-algebras. *Math. Scand.*, 56(2):249–275, 1985.

[3] Nathan Brownlowe, Aidan Sims, and Sean T. Vittadello. Co-universal C^*-algebras associated to generalised graphs. *Israel J. Math.*, 193(1):399–440, 2013.

[4] Adam Clay and Dale Rolfsen. *Ordered groups and topology*, volume 176 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.

[5] Kenneth R. Davidson. C^*-*algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[6] Siegfried Echterhoff, S. Kaliszewski, John Quigg, and Iain Raeburn. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.*, 180(850):viii+169, 2006.

[7] Siegfried Echterhoff and John Quigg. Induced coactions of discrete groups on C^*-algebras. *Canad. J. Math.*, 51(4):745–770, 1999.

[8]   R. Exel. Tight representations of semilattices and inverse semigroups. *Semigroup Forum*, 79(1):159–182, 2009.

[9]   Ruy Exel. Amenability for Fell bundles. *J. Reine Angew. Math.*, 492:41–73, 1997.

[10] Ruy Exel. *Partial dynamical systems, Fell bundles and applications*, volume 224 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[11] M. Hochster. Subsemigroups of amenable groups. *Proc. Amer. Math. Soc.*, 21:363–364, 1969.

[12] Eberhard Kaniuth and Keith F. Taylor. *Induced representations of locally compact groups*, volume 197 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013.

[13] David W. Kribs and Stephen C. Power. Free semigroupoid algebras. *J. Ramanujan Math. Soc.*, 19(2):117–159, 2004.

[14] David W. Kribs and Stephen C. Power. The analytic algebras of higher rank graphs. *Math. Proc. R. Ir. Acad.*, 106A(2):199–218, 2006.

[15] Alex Kumjian and David Pask. Higher rank graph C^*-algebras. *New York J. Math.*, 6:1–20, 2000.

[16] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[17] John C. Quigg. Discrete C^*-coactions and C^*-algebraic bundles. *J. Austral. Math. Soc. Ser. A*, 60(2):204–221, 1996.

[18] Iain Raeburn. *Graph algebras*, volume 103 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.

[19] Iain Raeburn. On graded C^*-algebras. *Bull. Aust. Math. Soc.*, 97(1):127–132, 2018.

[20] Camila F. Sehnem. On C^*-algebras associated to product systems. *J. Funct. Anal.*, 277(2):558–593, 2019.

[21] Jack Spielberg. Graph-based models for Kirchberg algebras. *J. Operator Theory*, 57(2):347–374, 2007.

[22] Jack Spielberg. Groupoids and C^*-algebras for categories of paths. *Trans. Amer. Math. Soc.*, 366(11):5771–5819, 2014.

[23] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[24] Jun Tomiyama. On the projection of norm one in W^*-algebras. *Proc. Japan Acad.*, 33:608–612, 1957.

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