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Joint Invariants of Linear Symplectic Actions

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Abstract: We review computations of joint invariants on a linear symplectic space, discuss variations for an extension of group and space and relate this to other equivalence problems and approaches, most importantly to differential invariants.

Keywords: polynomial and rational invariants; syzygy; free resolution; discretization

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1. Introduction

The classical invariant theory [1–3] investigates polynomial invariants of linear actions of a Lie group G on a vector space V, i.e., describes the algebra $(S\,V^*)^G$. For instance, the case of binary forms corresponds to $G = \mathrm{SL}(2,\mathbb{C})$ and $V = \mathbb{C}^2$; equivalently for $G = \mathrm{GL}(2,\mathbb{C})$ one studies instead the algebra of relative invariants. The covariants correspond to invariants in the tensor product $V \otimes W$ for another representation W. Changing to the Cartesian product $V \times W$ leads to joint invariants of G.

In this paper, we discuss joint invariants corresponding to the (diagonal) action of G on the iterated Cartesian product $V^{\times m}$ for increasing number of copies $m \in \mathbb{N}$. We will focus on the case $G = \mathrm{Sp}(2n,\mathbb{R})$, $V = \mathbb{R}^{2n}$ and discuss the conformal $G = \mathrm{CSp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \times \mathbb{R}_+$ and affine $G = \mathrm{ASp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \ltimes \mathbb{R}^{2n}$ versions later.

This corresponds to invariants of m-tuples of points in V, i.e., finite ordered subsets. By the Hilbert-Mumford [1] and Rosenlicht [4] theorems, the algebra of polynomial invariants (for the semi-simple G) or the field of rational invariants (in all other cases considered) can be interpreted as the space of functions on the quotient space $V^{\times m}/G$.

For $G = \mathrm{Sp}(2n,\mathbb{C})$ the algebra of invariants is known [5]. Generators and relations (syzygies) are described in the first and the second fundamental theorems, respectively. We review this in Theorem 1 (real version), and complement by explicit examples of free resolutions of the algebra. In addition, we describe the field of rational invariants.

We also discuss invariants with respect to the group $G = \mathrm{Sp}(2n,\mathbb{R}) \times S_m$, in which case considerably less is known. Another generalization we consider is the field of invariants for the conformal symplectic Lie group $G = \mathrm{CSp}(2n,\mathbb{R})$ on the contact space.

When approaching invariants of infinite sets, like curves or domains with smooth boundary, the theory of joint invariants is not directly applicable and the equivalence problem is solved via differential invariants [6]. In the case of a group G and a space V as above this problem was solved in [7]. We claim that the differential invariants from this reference can be obtained in a proper limit of joint invariants, i.e., via a certain discretization and quasiclassical limit, and demonstrate it explicitly in several cases.

In this paper, we focus on discussion of various interrelations of joint invariants. In particular, at the conclusion we note that joint invariants can be applied to the equivalence problem of binary forms. Since these have been studied also via differential invariants [2,8] a further link to the above symplectic discretization is possible.

The relation to binary forms mentioned above is based on the Sylvester theorem [9], which in turn can be extended to more general Waring decompositions, important in algebraic geometry [10]. Our computations should carry over to the general case. This note is partially based on the results of [11], generalized and elaborated in several respects.

2. Recollection: Invariants

We briefly recall the basics of invariant theory, referring to [3,12] for more details.

Let G be a Lie group acting on a manifold V. A point $x \in V$ is regular if a neighborhood of the orbit $G \cdot x$ is fibred by G-orbits. A point $x \in V$ is weakly regular, if its (not necessary G-invariant) neighborhood is foliated by the orbits of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. In general, the action can lack regular points, but a generic point is weakly regular. For algebraic actions a Zariski open set of points is regular.

2.1. Smooth Invariants

If G and V are only smooth (and non-compact), there is little one can do to guarantee regularity a priori. An alternative is to look for local invariants, i.e., functions $I = I(x)$ in a neighborhood $U \subset V$ such that $I(x) = I(g \cdot x)$ as long as $x \in U$ and $g \in G$ satisfy $g \cdot x \in U$.

The standard method to search for such I is by elimination of group parameters, namely by computing quasi-transversals [3] or using normalization and moving frame [2]. Another way is to solve the linear PDE system $L_\xi(I) = 0$ for $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$.

Given the space of invariants $\{I\}$ one can extend $U \subset V$ and address regularity. In our case the invariants are easy to compute and we do not rely on any of these methods; however instead we describe the algebra and the field of invariants depending on specification of the type of functions I.

2.2. Polynomial Invariants

If G is semi-simple and V is linear, then by the Hilbert-Mumford theorem generic orbits can be separated by polynomial invariants $I \in (S\,V^*)^G$, where $S\,V^* = \oplus_{k=0}^{\infty} S^k V^*$ is the algebra of homogeneous polynomials on V. With a choice of linear coordinates $\boldsymbol{x} = (x_1, ..., x_n)$ on V we identify $S\,V^* = \mathbb{R}[\boldsymbol{x}]$.

Moreover, by the Hilbert basis theorem, the algebra of polynomial invariants $\mathcal{A}_G = (S\,V^*)^G$ is Noetherian, i.e., finitely generated by some $\boldsymbol{a} = (a_1, \dots, a_s)$, $a_j = a_j(\boldsymbol{x}) \in \mathcal{A}_G$.

Denote by $\mathcal{R} = \mathbb{R}[\boldsymbol{a}]$ the free commutative $\mathbb{R}$-algebra generated by $\boldsymbol{a}$. It forms a free module F_0 over itself. $\mathcal{A}_G$ is also an $\mathcal{R}$-module with surjective $\mathcal{R}$-homomorphism $\phi_0 : F_0 \to \mathcal{A}_G$, $\phi_0(a_j) = a_j(\boldsymbol{x})$. The first syzygy module $S_1 = \mathrm{Ker}(\phi_0)$ fits the exact sequence

$$0 \to S_1 \to F_0 \to \mathcal{A}_G \to 0.$$

A *syzygy* is an element of S_1, i.e., a relation $r = r(\boldsymbol{a})$ between the generators of $\mathcal{A}_G$ of the form $\sum_{p=1}^{k} r_{i_p} a_{j_p} = 0$, $r_{i_p} \in \mathcal{R}$.

The module S_1 is Noetherian, i.e., finitely generated by some $\boldsymbol{b} = (b_1, \dots, b_t)$. Denote the free $\mathcal{R}$-module generated by $\boldsymbol{b}$ by $F_1 = \mathcal{R}[\boldsymbol{b}]$. The natural homomorphism $\phi_1 : F_1 \to S_1 \subset F_0$, $\phi_1(b_j) = b_j(\boldsymbol{a})$, defines the second syzygy module $S_2 = \mathrm{Ker}(\phi_1)$, and we can continue obtaining $S_2 \subset F_2 = \mathcal{R}[\boldsymbol{c}]$, etc. This yields the exact sequence of $\mathcal{R}$-modules:

$$\dots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} \mathcal{A}_G \to 0.$$

The Hilbert syzygy theorem states that q-th module of syzygies S_q is free for $q \geq s = \#\boldsymbol{a}$. In particular, the minimal free resolution exists and has length $\leq s$, see [13].

To emphasize the generating sets, we depict free resolutions as follows:

$$\mathbb{R}[\boldsymbol{x}] \supset \mathcal{A}_G \leftarrow \mathbb{R}[\boldsymbol{a}] \leftarrow \mathcal{R}[\boldsymbol{b}] \leftarrow \mathcal{R}[\boldsymbol{c}] \leftarrow \cdots \leftarrow 0.$$

2.3. Rational Invariants

If G is algebraic, in particular reductive, then by the Rosenlicht theorem [4] generic orbits can be separated by rational invariants $I \in \mathcal{F}_G$. Here $\mathbb{R}(\boldsymbol{x})$ is the field of rational functions on V and $\mathcal{F}_G = \mathbb{R}(\boldsymbol{x})^G$.

Let d be the transcendence degree of $\mathcal{F}_G$. This means that there exist $(a_1, \ldots, a_d) = \bar{\boldsymbol{a}}$, $a_j \in \mathcal{F}_G$, such that $\mathcal{F}_G$ is an algebraic extension of $\mathbb{R}(\bar{\boldsymbol{a}})$. Then either $\mathcal{F}_G = \mathbb{R}(\boldsymbol{a})$ for $\boldsymbol{a} = \bar{\boldsymbol{a}}$ or $\mathcal{F}_G$ is generated by a set $\boldsymbol{a} \supset \bar{\boldsymbol{a}}$, which by the primitive element theorem can be assumed of cardinality $s = \#\boldsymbol{a} = d+1$, i.e., $\boldsymbol{a} = (a_1, \ldots, a_d, a_{d+1})$. In the latter case there is one algebraic relation on $\boldsymbol{a}$. Please note that $d \leq n$ because $\mathbb{R}(\bar{\boldsymbol{a}}) \subset \mathbb{R}(\boldsymbol{x})$.

We adopt the following convention for depicting this:

$$\mathbb{R}(\boldsymbol{x}) \supset \mathcal{F}_G \overset{\text{alg}}{\supset} \mathbb{R}(\bar{\boldsymbol{a}}) \overset{d}{\supset} \mathbb{R}.$$

2.4. Our Setup

If the Lie group G acts effectively on V, then for some q it acts freely on $V^{\times q}$, and hence on all $V^{\times m}$ for $m \geq q$. The number of rational invariants separating a generic orbit in $V^{\times m}$ is equal to the codimension of the orbit.

It turns out that knowing all those invariants I on $V^{\times q}$ is enough to generate the invariants on $V^{\times m}$ for $m > q$. Indeed, let $\pi_{i_1,\ldots,i_q} : V^{\times m} \to V^{\times q}$ be the projection to the factors $(i_1, \ldots, i_q)$. Then the union of $\pi^*_{i_1,\ldots,i_q} I$ for I from the field $\mathcal{F}_G(V^{\times q})$ gives the generating set of the field $\mathcal{F}_G(V^{\times m})$, and similarly for the algebra of invariants.

Below we denote $\mathcal{A}^m_G = \mathcal{A}_G(V^{\times m})$ and $\mathcal{F}^m_G = \mathcal{F}_G(V^{\times m})$.

2.5. The Equivalence Problem

For a semi-simple Lie group G the field $\mathcal{F}_G$ is obtained from the ring $\mathcal{A}_G$ by localization (field of fractions): $\mathcal{F}_G = F(\mathcal{A}_G)$. Hence we discuss a solution to the equivalence problem through rational invariants.

Let $I_1, \ldots, I_s$ be a generating set of invariants of the action of G on $V^{\times q}$. If $s = d+1$, this set of generators is subject to an algebraic condition, which constrains the generators to an algebraic set $\Sigma \subset \mathbb{R}^s$. If $s = d$ then $\Sigma = \mathbb{R}^d$. This Σ is the signature space, cf. [14].

Now the q-tuple of points $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q)$ is mapped to $I_1(X), \ldots, I_s(X) \in \Sigma$. Denote this map by Ψ. Two generic configurations of points $X', X'' \in V^{\times q}$ are G-equivalent iff their signatures coincide $\Psi(X') = \Psi(X'')$.

3. Invariants on Symplectic Vector Spaces

Let $V = \mathbb{R}^{2n}(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be equipped with the standard symplectic form $\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$. The group $G = \mathrm{Sp}(2n, \mathbb{R})$ acts almost transitively on V, preserving the origin O. Thus, there are no continuous invariants of the action, $\mathcal{F}^1_G = \mathbb{R}$. The first invariant occurs already for two copies of V. Namely for a pair of points $A_i, A_j \in V$ the double symplectic area of the triangle OA_iA_j is

$$a_{ij} = \omega(OA_i, OA_j) = \boldsymbol{x}_i \boldsymbol{y}_j - \boldsymbol{x}_j \boldsymbol{y}_i = \sum_{k=1}^{n} x^k_i y^k_j - x^k_j y^k_i.$$

3.1. The Case $n = 1$

Consider at first the case of dimension 2, where $V = \mathbb{R}^2(x, y)$, $\omega = dx \wedge dy$. The invariant $a_{12} = x_1y_2 - x_2y_1$ on $V \times V$ generates pairwise invariants a_{ij} on $V^{\times m}$ for $m \geq 2$ induced through the pull-back of the projection $\pi_{i,j} : V^{\times m} \to V \times V$ to the corresponding factors. Below we describe minimal free resolutions of $\mathcal{A}^m_G$ for $m \geq 2$.

3.1.1. $V \times V$

Here the algebra is generated by one element, whence the resolution:

$$\mathbb{R}[x_1, x_2, y_1, y_2] \supset \mathcal{A}^2_G \leftarrow \mathbb{R}[a_{12}] \leftarrow 0$$

In other words, $\mathcal{A}^2_G \simeq \mathcal{R} := \mathbb{R}[a_{12}]$. Please note that $\mathcal{F}^2_G = \mathbb{R}(a_{12})$.

3.1.2. $V^{\times 3} = V \times V \times V$

Here the action is free on the level of $m = 3$ copies of V and we get $3 = \dim V^{\times 3} - \dim G$ independent invariants a_{12}, a_{13}, a_{23}. They generate the entire algebra, and we get the following minimal free resolution:

$$\mathbb{R}[x_1, x_2, x_3, y_1, y_2, y_3] \supset \mathcal{A}^3_G \leftarrow \mathbb{R}[a_{12}, a_{13}, a_{23}] \leftarrow 0$$

Once again, $\mathcal{A}^3_G \simeq \mathcal{R} := \mathbb{R}[a_{12}, a_{13}, a_{23}]$. Also $\mathcal{F}^3_G = \mathbb{R}(a_{12}, a_{13}, a_{23})$.

3.1.3. $V^{\times 4}$

Here $\dim V^{\times 4} = 8$, $\dim G = 3$ and we have 6 invariants $\boldsymbol{a} = \{a_{ij} : 1 \leq i < j \leq 4\}$. To obtain a relation, we try eliminating the variables $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, but this fails with the standard MAPLE command. Yet, using the transitivity of the G-action we fix A_1 at $(1, 0)$ and A_2 at $(0, a_{12})$, and then obtain the only relation

$$b_{1234} := a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$$

that we identify as the *Plücker relation*. Thus, the first syzygy is a module over $\mathcal{R} := \mathbb{R}[\boldsymbol{a}]$ with one generator, hence the minimal free resolution is:

$$\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}] \supset \mathcal{A}^4_G \leftarrow \mathbb{R}[a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}] \leftarrow \mathbb{R}[b_{1234}] \leftarrow 0.$$

For the field of rational invariants one of the generators is superfluous, for instance we can resolve the relation $b_{1234} = 0$ for $a_{34} = (a_{13}a_{24} - a_{14}a_{23})/a_{12}$, and get

$$\mathbb{R}(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \supset \mathcal{F}^4_G \simeq \mathbb{R}(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}) \overset{5}{\supset} \mathbb{R}$$

3.1.4. $V^{\times 5}$

The algebra of invariants $\mathcal{A}^5_G$ is generated by $\boldsymbol{a} = \{a_{ij} : 1 \leq i < j \leq 5\}$. This time the number of generators is 10, while codimension of the orbit is $10 - 3 = 7$. Using the same method we obtain that the first syzygy module is generated by the Plücker relations

$$b_{ijkl} := a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0.$$

We have 5 of those: $\boldsymbol{b} = \{b_{ijkl} : 1 \leq i < j < k < l \leq 5\}$. Thus, there should be relations among relations, or equivalently second syzygies. If $F_0 = \mathbb{R}[\boldsymbol{a}] =: \mathcal{R}$ and $F_1 = \mathcal{R}[\boldsymbol{b}]$ then this module is

$S_2 = \mathrm{Ker}(\phi_1 : F_1 \to S_1 \subset F_0)$. Using elimination of parameters, we find that S_2 is generated by $\boldsymbol{c} = \{c_i : 1 \le i \le 5\}$ with

$$c_i := \sum_{j=1}^{5} (-1)^j a_{ij} b_{1\ldots\check{j}\ldots 5}.$$

For instance, $c_1 = a_{12}b_{1345} - a_{13}b_{1245} + a_{14}b_{1235} - a_{15}b_{1234}$. Then we look for relations between the generators $\boldsymbol{c}$ of S_2, defining the third syzygy module S_3. It is generated by one element

$$\begin{aligned} d &:= (a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34})c_1 + (-a_{13}a_{45} + a_{14}a_{35} - a_{15}a_{34})c_2 \\ &\quad + (a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24})c_3 + (-a_{12}a_{35} + a_{13}a_{25} - a_{15}a_{23})c_4 \\ &\quad + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})c_5 = 0. \end{aligned}$$

Thus, the minimal free resolution of $\mathcal{A}_G^5$ is (note that here, as well as in our other examples, the length of the resolution is smaller than what the Hilbert theorem predicts):

$$\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}] \supset \mathcal{A}_G^5 \leftarrow \mathbb{R}[\boldsymbol{a}] \leftarrow \mathcal{R}[\boldsymbol{b}] \leftarrow \mathcal{R}[\boldsymbol{c}] \leftarrow \mathcal{R}[\boldsymbol{d}] \leftarrow 0.$$

As before, to generate the field of rational invariants, we express superfluous generators in terms of the others using the first syzygies. Specifically, we express a_{34}, a_{35}, a_{45} from the relations $b_{1234}, b_{1235}, b_{1245}$; the other 2 syzygies follow from the higher syzygies. Removing these generators, we obtain a set of 7 independent generators $\bar{\boldsymbol{a}} = \boldsymbol{a} \setminus \{a_{34}, a_{35}, a_{45}\}$ whence

$$\mathbb{R}(\boldsymbol{x}, \boldsymbol{y}) \supset \mathcal{F}_G^5 \simeq \mathbb{R}(\bar{\boldsymbol{a}}) \overset{7}{\supset} \mathbb{R}.$$

3.1.5. General $V^{\times m}$

The previous arguments generalize straightforwardly to conclude that $\mathcal{A}_G^m$ is generated by $\boldsymbol{a} = \{a_{ij} : 1 \le i < j \le m\}$. The first syzygy module is generated by the Plücker relations $\boldsymbol{b} = \{b_{ijkl} : 1 \le i < j < k < l \le m\}$. In other words we have:

$$\mathcal{A}_G^m = \langle \boldsymbol{a} \,|\, \boldsymbol{b} \rangle.$$

Similarly, the field of rational invariants is generated by $\boldsymbol{a}$, yet all of them except for a_{1j}, a_{2j} can be expressed (rationally) through the rest via the Plücker relations b_{12kl}. Denote $\bar{\boldsymbol{a}} := \{a_{12}, a_{13}, \ldots, a_{1m}, a_{23}, \ldots, a_{2m}\}$, $\#\bar{\boldsymbol{a}} = 2m - 3$. Then we get for $m \ge 2$:

$$\mathbb{R}(\boldsymbol{x}, \boldsymbol{p}) \supset \mathcal{F}_G^m \simeq \mathbb{R}(\bar{\boldsymbol{a}}) \overset{2m-3}{\supset} \mathbb{R}.$$

3.2. The General Case: Algebra of Polynomial Invariants

Minimal free resolutions can be computed in many examples for $n \ge 1$. However, in what follows we restrict our attention to describing generators/relations of $\mathcal{A}_G^m$.

Let us count the number of local smooth invariants. The action of G on V is almost transitive, so the stabilizer of a nonzero point A_1 has $\dim G_{A_1} = \binom{2n+1}{2} - 2n = \binom{2n}{2}$. For a generic A_2 there is only one invariant a_{12} (the orbit has codimension 1) and the stabilizer of A_2 in G_{A_1} has $\dim G_{A_1,A_2} = \binom{2n}{2} - (2n-1) = \binom{2n-1}{2}$. For a generic A_3 there are two more new invariants a_{13}, a_{23} (the orbit has codimension $2 + 1 = 3$) and the stabilizer of A_3 in G_{A_1,A_2} has $\dim G_{A_1,A_2,A_3} = \binom{2n-1}{2} - (2n-2) = \binom{2n-2}{2}$. By the same reason for $k \le 2n$ the stabilizer of a generic k-tuple of points $A_1, \ldots, A_k$ has $\dim G_{A_1,\ldots,A_k} = \binom{2n-k+1}{2}$. Finally, for $k = 2n$ the stabilizer of generic $A_1, \ldots, A_{2n}$ is trivial.

Thus, we get the expected number of invariants a_{ij}. For $m \leq 2n+1$ there are no relations between them, and the first comes at $m = 2n+2$. These can be obtained by successively studying cases of increasing n resulting in the *Pfaffian relation*:

$$b_{i_1 i_2 \ldots i_{2n+1} i_{2n+2}} := \mathrm{Pf}(a_{i_p i_q})_{1 \leq p,q \leq 2n+2} = 0.$$

Recall that the Pfaffian of a skew-symmetric operator S on V with respect to ω is $\mathrm{Pf}(S) = \mathrm{vol}_\omega(Se_1, \ldots, Se_{2n})$ for any symplectic basis e_i of V. The properties of the Pfaffian are: $\mathrm{Pf}(S)^2 = \det(S)$, $\mathrm{Pf}(TST^t) = \det(T)\,\mathrm{Pf}(S)$. For $n = 1$ we get

$$b_{1234} = \mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Similarly, for $n = 2$ we get

$$\begin{aligned} b_{123456} = & a_{12}a_{34}a_{56} - a_{12}a_{35}a_{46} + a_{12}a_{36}a_{45} - a_{13}a_{24}a_{56} + a_{13}a_{25}a_{46} - a_{13}a_{26}a_{45} + \\ & a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} - a_{15}a_{23}a_{46} + a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} + \\ & a_{16}a_{23}a_{45} - a_{16}a_{24}a_{35} + a_{16}a_{25}a_{34} = 0. \end{aligned}$$

Denote $\boldsymbol{b} = \{b_{i_1 i_2 \ldots i_{2n+1} i_{2n+2}} : 1 \leq i_1 < i_2 < \cdots < i_{2n+1} < i_{2n+2} \leq m\}$.

Theorem 1. *The algebra of G-invariants is generated by $\boldsymbol{a}$ with syzygies $\boldsymbol{b}$:*

$$\mathcal{A}_G^m = \langle \boldsymbol{a} \,|\, \boldsymbol{b} \rangle.$$

Proof. Let us first prove that the invariants a_{ij} generate the field $\mathcal{F}_G^m$ of rational invariants for $m = 2n$. We use the symplectic analog of Gram-Schmidt normalization: given points $A_1, \ldots, A_{2n}$ in general position, we normalize them using $G = \mathrm{Sp}(2n, \mathbb{R})$ as follows.

Let $e_1, \ldots, e_{2n}$ be a symplectic basis of V, i.e., $\omega(e_{2k-1}, e_{2k}) = 1$ and $\omega(e_i, e_j) = 0$ else. At first A_1 can be mapped to the vector e_1. The point A_2 can be mapped to the line $\mathbb{R}e_2$, and because of $\omega(OA_1, OA_2) = a_{12}$ it is mapped to the vector $a_{12}e_2$. Next in mapping A_3 we have two constraints $\omega(OA_1, OA_3) = a_{13}$, $\omega(OA_2, OA_3) = a_{23}$, and the point can be mapped to the space spanned by e_1, e_2, e_3 satisfying those constraints. Continuing like this, we arrive to the following matrix with columns OA_i:

$$\begin{pmatrix} 1 & 0 & -\frac{a_{23}}{a_{12}} & -\frac{a_{24}}{a_{12}} & \cdots & -\frac{a_{2,2n-1}}{a_{12}} & -\frac{a_{2,2n}}{a_{12}} \\ 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1,2n-1} & a_{1,2n} \\ 0 & 0 & 1 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \frac{b_{1234}}{a_{12}} & \vdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{2n-1,2n} \end{pmatrix}$$

where $b_{1234} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$ (this does not vanish in general if $n > 1$) and by $*$ we denote some rational expressions in a_{ij} that do not fit the table.

If $m < 2n$ then only the first m columns of this matrix have to be kept. If $m > 2n$ then the remaining points $A_{2n+1}, \ldots, A_m$ have all their coordinates invariant as the stabilizer of the first $2n$ points is trivial. Thus, the invariants are expressed rationally in a_{ij}.

To obtain polynomial invariants one clears the denominators in these rational expressions, and so $\mathcal{A}_G^m$ is generated by $\boldsymbol{a}$ as well.

Now the Pfaffian of the skew-symmetric matrix $(a_{ij})_{2k\times 2k}$ is the square root of the determinant of the Gram matrix of the vectors OA_i, $1 \le i \le k$, with respect to ω. If we take $k = n+1$ then the vectors are linearly dependent and therefore the Pfaffian vanishes. Thus, $\boldsymbol{b}$ are syzygies among the generators $\boldsymbol{a}$. That they form a complete set follows from the same normalization procedure as above. $\square$

Remark 1. *Theorem 1 is basically known: H. Weyl described the generators $\boldsymbol{a}$ as the first fundamental theorem; his second fundamental theorem gives not only the syzygy denoted above by $\boldsymbol{b}$, but also several different Pfaffians of larger sizes. Namely he lists in ([5], VI.1) the syzygies $b_{i_1\dots i_{2n+2k}} := \mathrm{Pf}(a_{i_p i_q})_{1\le p,q\le 2n+2k} = 0$, $1 \le k \le n$. Those however are abundant. For instance, in the simplest case $n = 2$*

$$b_{12345678} = a_{12}b_{345678} - a_{13}b_{245678} + a_{14}b_{235678} - a_{15}b_{234678} + a_{16}b_{234578} - a_{17}b_{234568} + a_{18}b_{234567}.$$

In general, the larger Pfaffians can be expressed via the smallest through the expansion by minors [15] (this fact was also noticed in [16]). Here is the corresponding Pfaffian identity (below we denote $S_{2n+1} = \{\sigma \in S_{2n+2} : \sigma(1) = 1\}$)

$$b_{i_1 i_2 \dots i_{2n+1} i_{2n+2}} = \frac{1}{n!} \sum_{\sigma \in S_{2n+1}} (-1)^{\mathrm{sgn}(\sigma)} a_{i_1 i_{\sigma(2)}} b_{i_{\sigma(3)} \dots i_{\sigma(2n+2)}}.$$

In ([3], §9.5) another set of syzygies was added: $q_{i_1\dots i_{4n+2}} = \det(a_{i_s, i_{t+2n+1}})_{s,t=1}^{2n+1} = 0$. These are also abundant, and should be excluded. For instance, for $n = 1$ we get

$$q_{123456} = a_{12}b_{3456} - a_{34}b_{1256} + a_{35}b_{1246} - a_{36}b_{1245}.$$

3.3. The General Case: Field of Rational Invariants

Since G is simple, the field of rational invariants is the field of fractions of the algebra of polynomial invariants: $\mathcal{F}_G^m = F(\mathcal{A}_G^m)$. To obtain its basis one can use the syzygies $b_{i_1\dots i_{2n+2}} = 0$ to express all invariants through $\bar{\boldsymbol{a}} = \{a_{ij} : 1 \le i \le 2n;\, i < j \le m\}$.

This can be done rationally (with $b_{1\dots 2n} \not\equiv 0$ in the denominator), for instance for $n = 2$ we can express a_{56} from the syzygy $b_{123456} = 0$ as follows:

$$\begin{aligned} a_{56} = (&a_{12}a_{35}a_{46} - a_{12}a_{36}a_{45} - a_{13}a_{25}a_{46} + a_{13}a_{26}a_{45} + a_{14}a_{25}a_{36} - a_{14}a_{26}a_{35} + a_{15}a_{23}a_{46} \\ &- a_{15}a_{24}a_{36} + a_{15}a_{26}a_{34} - a_{16}a_{23}a_{45} + a_{16}a_{24}a_{35} - a_{16}a_{25}a_{34}) / (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}). \end{aligned}$$

In general, we have $\#\bar{\boldsymbol{a}} = 2nm - n(2n+1)$ for $m \ge 2n$, in summary:

$$\boxed{\mathbb{R}(\boldsymbol{x},\boldsymbol{y}) \supset \mathcal{F}_G^m \simeq \mathbb{R}(\bar{\boldsymbol{a}}) \overset{d(m,n)}{\supset} \mathbb{R},}$$

where

$$d(m,n) = \begin{cases} 2nm - n(2n+1) & \text{for } m \ge 2n \\ \binom{m}{2} & \text{for } m \le 2n. \end{cases}$$

4. Variation on the Group and Space

Let us consider inclusion of symmetrization, scaling and translations to the transformation group G. We also discuss contactization of the action.

4.1. Symmetric Joint Invariants

Invariants of the extended group $\hat{G} = \mathrm{Sp}(2n, \mathbb{R}) \times S_m$ on $V^{\times m}$ are equivalent to G-invariants on configurations of unordered sets of points $V^{\times m}/S_m$ (which is an orbifold). Denote the algebra of polynomial $\hat{G}$-invariants on $V^{\times m}$ by $\mathcal{S}_G^m \subset \mathcal{A}_G^m$. The projection $\pi : \mathcal{A}_G^m \to \mathcal{S}_G^m$ is given by

$$\pi(f) = \frac{1}{m!} \sum_{\sigma \in S_m} \sigma \cdot f.$$

As a Noetherian algebra $\mathcal{S}_G^m$ is finitely generated, yet it is not easy to establish its generating set explicitly. All linear terms average to zero, $\pi(a_{ij}) = 0$, but there are several invariant quadratic terms in terms of the homogeneous decomposition $\mathcal{A}_G^m = \oplus_{k=0}^{\infty} \mathcal{A}_k^m$.

For example, for $n = 1$, $m = 4$ we have $\mathcal{A}_0^4 = \mathbb{R}$, $\mathcal{A}_1^4 = \mathbb{R}^6 = \langle a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \rangle$, $\mathcal{A}_2^4 = \mathbb{R}^{20}$ (21 monomials $a_{ij}a_{kl}$ modulo 1 Plücker relation), etc. Then $\pi(\mathcal{A}_0^4) = \mathbb{R}$, $\pi(\mathcal{A}_1^4) = 0$, and $\pi(\mathcal{A}_2^4) = \mathbb{R}^2$ has generators

$$\begin{aligned} 6\pi(a_{12}^2) &= a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2, \\ 12\pi(a_{12}a_{13}) &= a_{12}a_{13} + a_{12}a_{14} + a_{13}a_{14} - a_{12}a_{23} - a_{12}a_{24} + a_{23}a_{24} \\ &\quad + a_{13}a_{23} - a_{13}a_{34} - a_{23}a_{34} + a_{14}a_{24} + a_{14}a_{34} + a_{24}a_{34}. \end{aligned}$$

Theorem 2. *The field of symmetric rational invariants $\mathfrak{F}_G^m = \pi(\mathcal{F}_G^m)$ is the field of fractions $\mathfrak{F}_G^m = F(\mathcal{S}_G^m)$ and its transcendence degree is $d(m, n)$.*

Proof. This follows from general theorems ([17], §2.5) and discussion in Section 2. □

The last statement can be made more constructive: Let ℓ numerate indices (ij) of the basis $\bar{\boldsymbol{a}}$ of $\mathcal{F}_G^m$ as in Section 3.3, $1 \leq \ell \leq d = d(m, n)$. One can check that $q_k = \pi(\prod_{\ell \leq k} a_\ell^2)$ are algebraically independent. Thus, denoting $\boldsymbol{q} = (q_1, \ldots, q_d)$ we obtain the presentation

$$\boxed{\mathbb{R}(\boldsymbol{x}, \boldsymbol{y}) \supset \mathfrak{F}_G^m \overset{\text{alg}}{\supset} \mathbb{R}(\boldsymbol{q}) \overset{d(m,n)}{\supset} \mathbb{R}.}$$

Here is an algorithm to obtain generators of $\mathcal{S}_G^m$.

Proposition 1. *Fix an order on generators a_{ij} of $\mathcal{A}_G^m$, and induce the total lexicographic order on monomials $a^\sigma \in \mathcal{R} = \mathbb{R}[\boldsymbol{a}]$. Let Σ be the Gröbner basis of the $\mathcal{R}$-ideal generated by $\pi(a^\sigma)$. Then elements $\pi(a^\sigma)$, contributing to Σ, generate $\mathcal{S}_G^m = \pi(\mathcal{A}_G^m)$.*

Proof. Please note that the algorithm proceeds in total degree of a^σ until the Gröbner basis stabilizes. That the involved $\pi(a^\sigma)$ generate $\mathcal{S}_G^m$ as an algebra (initially they generate the ideal $\mathcal{R} \cdot \pi(\mathcal{A}_G^m) \subset \mathcal{A}_G^m$) follows from the same argument as in the proof of Hilbert's theorem on invariants [1]. (The above π is the Reynolds operator used there.) □

Let us illustrate how this works in the first nontrivial case $m = 3$, for any n.

In this case, the graded components of $\mathcal{S}_G^3 = \pi(\mathcal{A}_G^3)$ have the following dimensions: $\dim \mathcal{S}_0^3 = 1$, $\dim \mathcal{S}_1^3 = 0$, $\dim \mathcal{S}_2^3 = 2$, $\dim \mathcal{S}_3^3 = 1$, $\dim \mathcal{S}_4^3 = 4$, $\dim \mathcal{S}_5^3 = 2$, $\dim \mathcal{S}_6^3 = 7$, etc., encoded into the Poincaré series

$$P_{\mathcal{S}}^3(z) = 1 + 2z^2 + z^3 + 4z^4 + 2z^5 + 7z^6 + 4z^7 + 10z^8 + 7z^9 + \ldots = \frac{1 + z^4}{(1 - z^2)^2(1 - z^3)}.$$

For the monomial order $a_{12} > a_{13} > a_{23}$ the invariants

$$I_{2a} = 3\pi(a_{12}^2) = a_{12}^2 + a_{13}^2 + a_{23}^2, \quad I_{2b} = 3\pi(a_{12}a_{13}) = a_{12}a_{13} - a_{12}a_{23} + a_{13}a_{23},$$
$$I_3 = 6\pi(a_{12}^2 a_{13}) = a_{12}^2(a_{13} + a_{23}) - a_{23}^2(a_{12} + a_{13}) + a_{13}^2(a_{12} - a_{23}),$$
$$I_4 = 3\pi(a_{12}^2 a_{13}^2) = a_{12}^2 a_{13}^2 + a_{12}^2 a_{23}^2 + a_{13}^2 a_{23}^2$$

generate a Gröbner basis of the ideal $\mathcal{R} \cdot \pi(A_G^m)$ with the leading monomials of the corresponding Gröbner basis equal: $a_{12}^2, a_{12}a_{13}, a_{13}^3, a_{12}a_{23}^3, a_{13}^2 a_{23}^2, a_{13}a_{23}^3, a_{23}^4$.

The Gröbner basis also gives the following syzygy R_8:

$$(4I_{2a}^2 + 4I_{2a}I_{2b} + 3I_{2b}^2)I_{2b}^2 - (8I_{2a}^2 + 4I_{2a}I_{2b} + 14I_{2b}^2)I_4 + 4(I_{2a} - 2I_{2b})I_3^2 + 27I_4^2 = 0.$$

In other words, $\mathcal{S}_G^3 = \langle I_{2a}, I_{2b}, I_3, I_4 \mid R_8 \rangle$. We also derive a presentation of the field of rational invariants (2 : 1 means quadratic extension)

$$\mathbb{R}(\boldsymbol{x}, \boldsymbol{y}) \supset \mathfrak{F}_G^3 \overset{2:1}{\supset} \mathbb{R}(I_{2a}, I_{2b}, I_3) \overset{3}{\supset} \mathbb{R}.$$

4.2. Conformal and Affine Symplectic Groups

For the group $G_1 = \mathrm{CSp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \times \mathbb{R}_+$ the scaling makes the invariants a_{ij} relative, yet of the same weight, so their ratios $[a_{12} : a_{13} : \cdots : a_{m-1,m}]$ or simply the invariants $I_{ij} = \frac{a_{ij}}{a_{12}}$ are absolute invariants. These generate the field of invariants of transcendence degree $d(m, n) - 1$.

For the group $G_2 = \mathrm{ASp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \ltimes \mathbb{R}^{2n}$ the translations do not preserve the origin O and this makes a_{ij} non-invariant. However due to the formula $2\omega(A_1A_2A_3) = a_{12} + a_{23} - a_{13}$ (or more symmetrically: $a_{12} + a_{23} + a_{31}$), with the proper orientation of the triangle $A_1A_2A_3$, we easily recover the absolute invariants $a_{ij} + a_{jk} + a_{ki}$.

Alternatively, using the translational freedom, we can move the point A_1 to the origin O. Then its stabilizer in G_2 is $G = \mathrm{Sp}(2n, \mathbb{R})$ and we compute the invariants of $(m-1)$ tuples of points $A_2, \ldots, A_m$ as before. In particular they generate the field of invariants of transcendence degree $d(m-1, n)$.

4.3. Invariants in the Contact Space

Infinitesimal symmetries of the contact structure $\Pi = \mathrm{Ker}(\alpha)$, $\alpha = du - \boldsymbol{y}\, d\boldsymbol{x}$ in the contact space $M = \mathbb{R}^{2n+1}(\boldsymbol{x}, \boldsymbol{y}, u)$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$, are given by the contact vector field X_H with the generating function $H = H(\boldsymbol{x}, \boldsymbol{y}, u)$. Taking quadratic functions H with weights $w(\boldsymbol{x}) = 1$, $w(\boldsymbol{y}) = 1$, $w(u) = 2$ results in the conformally symplectic Lie algebra, which integrates to the conformally symplectic group $G_1 = \mathrm{CSp}(2n, \mathbb{R})$ (taking H of degree ≤ 2 results in the affine extension of it by the Heisenberg group).

Alternatively, one considers the natural lift of the linear action of $G = \mathrm{Sp}(2n, \mathbb{R})$ on $V = \mathbb{R}^{2n}$ to the contactization M and makes a central extension of it. We will discuss the invariants of this action. Please note that this action is no longer linear, so the invariants cannot be taken to be polynomial, but can be assumed rational.

4.3.1. The Case $n = 1$

In the 3-dimensional case the group $G_1 = \mathrm{GL}(2, \mathbb{R})$ acts on $M = \mathbb{R}^3(x, y, u)$ as follows:

$$G_1 \ni g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (x, y, u) \mapsto (\alpha x + \beta y, \gamma x + \delta y, f(x, y, u)),$$
$$\text{where } f(x, y, u) = (\alpha\delta - \beta\gamma)\left(u - \frac{xy}{2}\right) + \frac{(\alpha x + \beta y)(\gamma x + \delta y)}{2}.$$

This action is almost transitive (no invariants); however there are singular orbits and a relative invariant $R = xy - 2u$. Extending the action to multiple copies of M, i.e., considering the diagonal action of G_1 on $M^{\times m}$, results in m copies of this relative invariant, but also in the lifted invariants from various $V^{\times 2}$:

$$R_k = x_k y_k - 2u_k \ (1 \le k \le m), \quad R_{ij} = x_i y_j - x_j y_i \ (1 \le i < j \le m).$$

These are all relative invariants of the same weight, therefore their ratios are absolute invariants:

$$T_k = \frac{R_k}{R_m} \ (1 \le k < m), \quad T_{ij} = \frac{R_{ij}}{R_m} \ (1 \le i < j \le m).$$

Since u_k enter only R_k there are no relations involving those, and the relations on T_{ij} are the same as for a_{ij}, namely they are Plücker relations (since those are homogeneous, they are satisfied by both R_{ij} and T_{ij}). As previously, we can use them to eliminate all invariants except for $\bar{\boldsymbol{T}} = \{T_k, T_{1i}, T_{2i}\}$:

$$T_{kl} = \frac{T_{1k}T_{2l} - T_{1l}T_{2k}}{T_{12}}, \quad 3 \le k < l \le m.$$

The field of rational invariants for $m > 1$ is then described as follows:

$$\mathbb{R}(x, y, u) \supset \mathcal{F}^m_{G_1} \simeq \mathbb{R}(\bar{\boldsymbol{T}}) \overset{3m-4}{\supset} \mathbb{R}.$$

4.3.2. The General Case

In general, we also have no invariants on M and the following relative invariants on $M^{\times m}$

$$R_k = \boldsymbol{x}_k \boldsymbol{y}_k - 2u_k \ (1 \le k \le m), \quad R_{ij} = \boldsymbol{x}_i \boldsymbol{y}_j - \boldsymbol{x}_j \boldsymbol{y}_i \ (1 \le i < j \le m)$$

resulting in absolute invariants T_k, T_{ij} given by the same formulae. Again, using the Pfaffian relations we can rationally eliminate superfluous generators, and denote the resulting set by $\bar{\boldsymbol{T}} = \{T_k, T_{ij} : 1 \le k < m, i < j \le m, 1 \le i \le 2n\}$. This set is independent and contains $\bar{d}(m, n)$ elements, where

$$\bar{d}(m, n) = \begin{cases} (2n+1)m - n(2n+1) - 1 & \text{for } m \ge 2n \\ \binom{m}{2} + m - 1 = \binom{m+1}{2} - 1 & \text{for } m \le 2n. \end{cases}$$

This $\bar{d}(m, n)$ is thus the transcendence degree of the field of rational invariants:

$$\boxed{\mathbb{R}(\boldsymbol{x}, \boldsymbol{y}, u) \supset \mathcal{F}^m_{G_1} \simeq \mathbb{R}(\bar{\boldsymbol{T}}) \overset{\bar{d}(m,n)}{\supset} \mathbb{R}.}$$

5. From Joint to Differential Invariants

When we pass from finite to continuous objects the equivalence problem is solved through differential invariants. In [7] this was done for submanifolds and functions with respect to our groups G. After briefly recalling the results, we will demonstrate how to perform the discretization in several different cases.

5.1. Jets of Curves in Symplectic Vector Spaces

Locally a curve in $\mathbb{R}^{2n}$ is given as $\boldsymbol{u} = \boldsymbol{u}(t)$ for $t = x_1$ and $\boldsymbol{u} = (x_2, \ldots, x_n, y_1, \ldots, y_n)$ in the canonical coordinates $(x_1, x_2, \ldots, x_n, y_1, \ldots, y_n)$, $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. The corresponding jet-space $J^\infty(V, 1)$ has coordinates $t, \boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{u}_{tt}, \ldots$, and J^k is the truncation of it. For instance, $J^1(V, 1) = \mathbb{R}^{4n-1}(t, \boldsymbol{u}, \boldsymbol{u}_t)$. Please note that dim $J^k(V, 1) = 2n + k(2n - 1)$.

In the case of dimension $2n = 2$, the jet-space is $J^k(V,1) = \mathbb{R}^{k+2}(x, y, y_x, \dots, y_{x..x})$. Here $G = \mathrm{Sp}(2,\mathbb{R})$ has an open orbit in $J^1(V,1)$, and the first differential invariant is of order 2:

$$I_2 = \frac{y_{xx}}{(xy_x - y)^3}.$$

There is also an invariant derivation ($\mathcal{D}_x$ is the total derivative with respect to x)

$$\nabla = \frac{1}{xy_x - y}\mathcal{D}_x.$$

By differentiation we get new differential invariants $I_3 = \nabla I_2$, $I_4 = \nabla^2 I_2$, etc. The entire algebra of differential invariants is free:

$$\mathcal{A}_G = \langle I_2\,;\,\nabla\rangle.$$

In the general case we denote the canonical coordinates on $V = \mathbb{R}^{2n}$ by $(t, \boldsymbol{x}, y, \boldsymbol{z})$, where $\boldsymbol{x}$ and $\boldsymbol{z}$ and $(n-1)$-dimensional vectors. $G = \mathrm{Sp}(2n,\mathbb{R})$ acts on $J^\infty(V,1)$. The invariant derivation is equal to

$$\nabla = \frac{1}{(ty_t - y + \boldsymbol{x}\boldsymbol{z}_t - \boldsymbol{x}_t\boldsymbol{z})}\mathcal{D}_t.$$

and the first differential invariant of order 2 is

$$I_2 = \frac{\boldsymbol{x}_t\boldsymbol{z}_{tt} - \boldsymbol{x}_{tt}\boldsymbol{z}_t + y_{tt}}{(ty_t - y + \boldsymbol{x}\boldsymbol{z}_t - \boldsymbol{x}_t\boldsymbol{z})^3}.$$

There is one invariant I_3 of order 3 independent of $I_2, \nabla(I_2)$, one invariant I_4 of order 4 independent of $I_2, \nabla(I_2), I_3, \nabla^2(I_2), \nabla(I_3)$, and so on up to order $2n$. Then the algebra of differential invariants of G is freely generated ([7], §4) so:

$$\mathcal{A}_G = \langle I_2, I_3, \dots, I_{2n}\,;\,\nabla\rangle.$$

5.2. Symplectic Discretization

Consider first the case $n = 1$ with coordinates (x, y) on $V = \mathbb{R}^2$. Let $A_i = (x_i, y_i)$, $i = 0, 1, 2$, be three close points lying on the curve $y = y(x)$. We assume A_1 is in between A_0, A_2 and omit indices for its coordinates, i.e., $A_1 = (x, y)$.

Let $x_0 = x - \delta$ and $x_2 = x + \epsilon$. Denote also $y' = y'(x)$, $y'' = y''(x)$, etc. Then from the Taylor formula we have:

$$\begin{aligned} y_0 &= y - \delta y' + \tfrac{1}{2}\delta^2 y'' - \tfrac{1}{6}\delta^3 y''' + o(\delta^3),\\ y_2 &= y + \epsilon y' + \tfrac{1}{2}\epsilon^2 y'' + \tfrac{1}{6}\epsilon^3 y''' + o(\epsilon^3). \end{aligned}$$

Therefore, the symplectic invariants $a_{ij} = x_i y_j - x_j y_i$ are:

$$\begin{aligned} a_{12} &= \epsilon(xy' - y) + \tfrac{1}{2}\epsilon^2 xy'' + \tfrac{1}{6}\epsilon^3 xy''' + o(\epsilon^3),\\ a_{01} &= \delta(xy' - y) - \tfrac{1}{2}\delta^2 xy'' + \tfrac{1}{6}\delta^3 xy''' + o(\delta^3),\\ a_{02} &= (\epsilon + \delta)(xy' - y) + \tfrac{1}{2}(\epsilon^2 - \delta^2)xy''\\ &\quad + \tfrac{1}{6}(\epsilon^3 + \delta^3)xy''' - \tfrac{1}{2}(\epsilon + \delta)\epsilon\delta y'' + o((|\delta| + |\epsilon|)^3). \end{aligned}$$

This implies:

$$\frac{a_{01} - a_{02} + a_{12}}{a_{01}a_{02}a_{12}} = \frac{1}{2}\frac{y''}{(xy' - y)^3} + o(|\delta| + |\epsilon|).$$

Thus, we can extract the invariant exploiting no distance (like $\epsilon = \delta$) but only the topology ($\epsilon, \delta \to 0$) and the symplectic area. This works in any dimension n, and using the coordinates from the previous subsection we get

$$\lim_{A_0,A_2\to A_1} \frac{\mathrm{Area}_\omega(A_0A_1A_2)}{\mathrm{Area}_\omega(OA_0A_1)\,\mathrm{Area}_\omega(OA_0A_2)\,\mathrm{Area}_\omega(OA_1A_2)} = \frac{2(\boldsymbol{x}_t\boldsymbol{z}_{tt} - \boldsymbol{x}_{tt}\boldsymbol{z}_t + y_{tt})}{(ty_t - y + \boldsymbol{x}\boldsymbol{z}_t - \boldsymbol{x}_t\boldsymbol{z})^3} = 2I_2.$$

Similarly, we obtain the invariant derivation (it uses only two points and hence is of the first order)

$$\lim_{A_0\to A_1} \frac{\overrightarrow{A_0A_1}}{\mathrm{Area}_\omega(OA_0A_1)} = \frac{2\mathcal{D}_t}{(ty_t - y + \boldsymbol{x}\boldsymbol{z}_t - \boldsymbol{x}_t\boldsymbol{z})} = 2\nabla.$$

The other generators $I_3, I_4, \dots$ (important for $n > 1$) can be obtained by a higher order discretization, but the formulae become more involved.

5.3. Contact Discretization

Now we use joint invariants to obtain differential invariants of curves in contact 3-space $W = \mathbb{R}^3(x, y, u)$ with respect to the group $G = \mathrm{GL}(2, \mathbb{R})$, acting as in §4.3. The curves will be given as $y = y(x), u = u(x)$ and their jet-space is $J^k(W, 1) = \mathbb{R}^{2k+3}(x, y, u, y_x, u_x, \dots, y_{x..x}, u_{x..x})$. The differential invariants are generated in the Lie–Tresse sense ([7], §8.1) as

$$\mathcal{A}_G = \langle I_1, I_2\,;\,\nabla\rangle.$$

where

$$I_1 = \frac{u_x - y}{xy_x - y}, \quad I_2 = \frac{(xy - 2u)^2}{(xy_x - y)^3}\, y_{xx}, \quad \nabla = \frac{xy - 2u}{xy_x - y}\,\mathcal{D}_x.$$

Instead of exploiting the absolute rational invariants T_i, T_{ij} we will work with the relative polynomial invariants R_i, R_{ij} from Section 4.3. To get absolute invariants we will then have to pass to weight zero combinations.

Consider three close points $\hat{A}_i = (x_i, y_i, u_i)$, $i = 0, 1, 2$, lying on the curve. We again omit indices for the middle point, so $x_0 = x - \delta$, $x_1 = x$ and $x_2 = x + \epsilon$. Using the Taylor decomposition as in the preceding subsection, we obtain

$$\begin{aligned} R_1 &= xy - 2u, & R_0 - R_1 &= \delta(2u' - y - xy') + o(\delta),\\ R_{01} &= \delta(xy' - y) + o(\delta), & R_{02} &= (\epsilon + \delta)(xy' - y) + o(|\epsilon| + |\delta|),\\ R_{12} &= \epsilon(xy' - y) + o(\epsilon), & R_{01} + R_{12} - R_{02} &= \tfrac{1}{2}\epsilon\delta(\epsilon + \delta)y'' + o((|\epsilon| + |\delta|)^3) \end{aligned}$$

as well as

$$\overrightarrow{A_0A_1} = \delta(\partial_x + y'\partial_y + u'\partial_y) + o(\delta).$$

Passing to jet-notations, we obtain the limit formulae for basic differential invariants:

$$\begin{aligned} I_1 &= \lim_{A_0\to A_1} \frac{R_0 - R_1}{2R_{01}} + \frac{1}{2} = \lim_{A_0\to A_1} \frac{T_0 - 1 + T_{01}}{2T_{01}},\\ \frac{1}{2}I_2 &= \lim_{A_0,A_2\to A_1} \frac{R_1^2(R_{01} + R_{12} - R_{12})}{R_{01}R_{02}R_{12}} = \lim_{A_0,A_2\to A_1} \frac{T_{01} + T_{12} - T_{12}}{T_{01}T_{02}T_{12}},\\ \nabla &= \lim_{A_0\to A_1} \frac{R_1}{R_{01}}\overrightarrow{A_0A_1} = \lim_{A_0\to A_1} \frac{\overrightarrow{A_0A_1}}{T_{01}}. \end{aligned}$$

These formulae straightforwardly generalize to invariants of jets of curves in contact manifolds of dimension $2n + 1$, $n > 1$, in which case there are also other generators obtained by higher order discretizations.

5.4. Functions and Other Examples

Let us discuss invariants of jets of functions on the symplectic plane. The action of $G = \mathrm{Sp}(2, \mathbb{R})$ on $J^0V = V \times \mathbb{R}(u) \simeq \mathbb{R}^3(x, y, u)$, with $I_0 = u$ invariant, prolongs to $J^\infty(V) = \mathbb{R}^\infty(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots)$. Please note that functions can be identified as surfaces in J^0V through their graphs.

For any finite set of points $\hat{A}_k = (x_k, y_k, u_k)$ the values u_k are invariant, and the other invariants a_{ij} are obtained from the projections $A_k = (x_k, y_k)$. In this way we get the basic first order invariant (as before we omit indices $x_1 = x$, $y_1 = y$, $u_1 = y$ for the reference point A_1 in the right-hand side)

$$I_1 = \lim_{A_0, A_2 \to A_1} \frac{a_{01}(u_1 - u_2) + a_{12}(u_1 - u_0)}{a_{01} - a_{02} + a_{12}} = xu_x + yu_y$$

as well as two invariant derivations

$$\nabla_1 = \overrightarrow{OA_1} = x\mathcal{D}_x + y\mathcal{D}_y, \quad \nabla_2 = \lim_{A_0 \to A_1} \frac{I_1}{a_{01}} \overrightarrow{A_0A_1} - \frac{u_1 - u_0}{a_{01}} \overrightarrow{OA_1} = u_x\mathcal{D}_y - u_y\mathcal{D}_x.$$

To obtain the second order invariant $I_{2c} = u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}$ let A_0 belong to the line through A_1 in the direction ∇_2 (this constraint reduces the second order formula to depend on only two points), i.e., $A_0 = (x + \epsilon u_y, y - \epsilon u_x)$, $A_1 = (x, y)$. Then $u_0 - u_1 = \frac{\epsilon^2}{2} I_{2c} + o(\epsilon^2)$, $a_{01} = \epsilon I_1$ and letting $\epsilon \to 0$ we obtain

$$\lim_{\substack{A_0 \to A_1 \\ A_0A_1 \parallel \nabla_2}} \frac{u_0 - u_1}{a_{01}^2} = \frac{I_{2c}}{2I_1^2}.$$

In the same way we get $I_{2a} = x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy}$ and $I_{2b} = xu_y u_{xx} - yu_x u_{yy} + (yu_y - xu_x)u_{xy}$. These however are not required as the algebra of differential invariants is generated as follows ([7], §3.1) for some differential syzygies $\mathcal{R}_i$:

$$\mathcal{A}_G = \langle I_0, I_{2c} \,;\, \nabla_1, \nabla_2 \,|\, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \rangle.$$

Similarly, one can consider surfaces in the contact 3-space (with the same coordinates x, y, u but different lift of $\mathrm{Sp}(2, \mathbb{R})$ extended to $\mathrm{GL}(2, \mathbb{R})$) and higher-dimensional cases. The idea of discretization of differential invariants applies to other problems treated in [7].

6. Relation to Binary and Higher Order Forms

According to the Sylvester theorem [9] a general binary form $p \in \mathbb{C}[x, y]$ of odd degree $2m - 1$ with complex coefficients can be written as

$$p(x, y) = \sum_{i=1}^{m} (\alpha_i x + \beta_i y)^{2m-1}.$$

This decomposition is determined up to permutation of linear factors and independent multiplication of each of them by a $(2m - 1)$-th root of unity.

In other words, we have the branched cover of order $k_m = (2m - 1)^m m!$

$$\times^m(\mathbb{C}^2) \to S^{2m-1}\mathbb{C}^2$$

and the deck group of this cover is $S_m \ltimes \mathbb{Z}_{2m-1}^{\times m}$.

Please note that in the real case, due to uniqueness of the odd root of unity, the corresponding cover over an open subset of the base

$$\times^m(\mathbb{R}^2) \to S^{2m-1}\mathbb{R}^2$$

has the deck group S_m.

With this approach the invariants of real binary forms are precisely the joint symmetric invariants studied in this paper, and for complex forms one must additionally quotient by $\mathbb{Z}_{2m-1}^{\times m}$, which is equivalent to passing from a_{ij} to a_{ij}^{2m-1} and other invariant combinations (example for $m = 4$: $a_{12}^3 a_{13}^2 a_{14}^2 a_{23}^2 a_{24}^2 a_{34}^3$) and subsequently averaging by the map π.

Other approaches to classification of binary forms, most importantly through differential invariants [2,8], can be related to this via symplectic discretization.

Remark 2. *Please note that the standard "root cover" $\mathbb{C}^{2m} \to S^{2m-1}\mathbb{C}^2$:*

$$(a_0, a_1, \dots, a_{2m-1}) \mapsto (p_0, p_1, \dots, p_{2m-1}), \quad \sum_{i=0}^{2m-1} p_i x^i y^{2m-i-1} = a_0 \prod_{i=1}^{2m-1} (x - a_i y)$$

has order $(2m-1)! < k_m$. Polynomial $\mathrm{SL}(2,\mathbb{C})$-invariants of binary forms with this approach correspond to functions on the orbifold $\mathbb{C}^{2m}/S_{2m}$.

The above idea extends further to ternary and higher valence forms (see [18] for the differential invariants approach and [19] for an approach using joint differential invariants) with the Waring decompositions [10] as the cover, but here the group G is no longer symplectic. We expect all the ideas of the present paper to generalize to the linear and affine actions of other reductive groups G.

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References

1. Hilbert, D. *Theory of Algebraic Invariants*; Cambridge University Press: Cambridge, UK, 1993.
2. Olver, P.J. Classical invariant theory. In *London Mathematical Society Student Texts*; Cambridge University Press: Cambridge, UK, 1999; Volume 44.
3. Popov, V.L.; Vinberg, E.B. Invariant theory. In *Algebraic Geometry IV, Encyclopaedia of Mathematical Sciences*; Springer: Berlin, Germany, 1994; Volume 55.
4. Rosenlicht, M. Some basic theorems on algebraic groups. *Am. J. Math.* **1956**, *78*, 401–443. [CrossRef]
5. Weyl, H. *Classical Groups*; Princeton University Press: Princeton, NJ, USA, 1946.
6. Kruglikov, B.; Lychagin, V. Global Lie-Tresse theorem. *Selecta Math.* **2016**, *22*, 1357–1411. [CrossRef]
7. Jensen, J.O.; Kruglikov, B. Differential Invariants of Linear Symplectic Actions. *Symmetry* **2020**, *12*, 2023. [CrossRef]
8. Bibikov, P.; Lychagin, V. $\mathrm{GL}_2(\mathbb{C})$-orbits of Binary Rational Forms. *Lobachevskii J. Math.* **2011**, *32*, 95–102. [CrossRef]
9. Sylvester, J.J. *An Essay on Canonical Forms*; George Bell: 1851; *On a Remarkable Discovery in the Theory of Canonical Forms and of Hyperdeterminants*; Philosophical Magazine: 1851; *Mathematical Papers*; Chelsea: New York, NY, USA, 1973; pp. 34, 203–216, 41, 265–283.
10. Alexander, J.; Hirschowitz, A. Polynomial interpolation in several variables. *J. Algebr. Geom.* **1995**, *4*, 201–222
11. Andreassen, F. Joint Invariants of of Symplectic and Contact Lie Algebra Actions. Master's Thesis in Mathematics, UiT the Arctic University of Norway, Tromsø, Norway, June 2020. Available online: https://hdl.handle.net/10037/19003 (accessed on 20 August 2020).
12. Mumford, D.; Fogarty, J.; Kirwan, F. *Geometric Invariant Theory*; Springer: Berlin/Heidelberg, Germany, 1994.
13. Eisenbud, D. The geometry of syzygies: A second course in algebraic geometry and commutative algebra. In *Graduate Texts in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2006; Volume 229.

14. Olver, P.J. Joint invariant signatures. *Found. Comput. Math.* **2001**, *1*, 3–68. [CrossRef]
15. Ishikawa, M.; Wakayama, M. Minor summation formula of Pfaffians, Survey and a new identity. *Adv. Stud. Pure Math.* **2000**, *28*, 133–142.
16. Vust, T. Sur la théorie des invariants des groupes classiques. *Ann. Inst. Fourier* **1976**, *26*, 1–31. [CrossRef]
17. Springer, T.A. Invariant Theory. In *Lecture Notes in Math*; Springer: Berlin/Heidelberg, Germany, 1977; Volume 585.
18. Bibikov, P.; Lychagin, V. Classification of linear actions of algebraic groups on spaces of homogeneous forms. *Dokl. Math.* **2012**, *85*, 109–112. [CrossRef]
19. Gün Polat, G.; Olver, P.J. Joint differential invariants of binary and ternary forms. *Port. Math.* **2019**, *76*, 169–204. [CrossRef]