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# Square, Rectangular and Triangular Nim Games

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**Abstract**

Let $p$ be an integer with $p \geq 2$. We shall investigate the following two piles Nim games. Let $S$ be the set of positive integers $\{1 \leq i \leq p-1\}$. Each player can remove the number of tokens $s_1 \in \{0\} \cup S$ from the first pile and $s_2 \in \{0\} \cup S$ from the second pile with $0 < s_1 + s_2$ at the same time. We shall identify $(m, n)$ to a position of this Nim game, where $m$ is the number of tokens in the first pile and $n$ is the number of tokens in the second pile. We shall show the Sprague-Grundy sequence (or simply G-sequences) $g_S(m, n)$ satisfy the periodic relation $g_S(m+p, n+p) = g_S(m, n)$ for any position $(m, n)$. We will call this two piles Nim *Square Nim*. In case $m$ and $n$ are sufficiently large, we will show that G-sequences $g_S(m, n)$ are also periodic for each row and column with the same period p. Finally we shall introduce several related games, such as *Rectangular Nim*, *Triangular Nim* and *Polytope Nim*.

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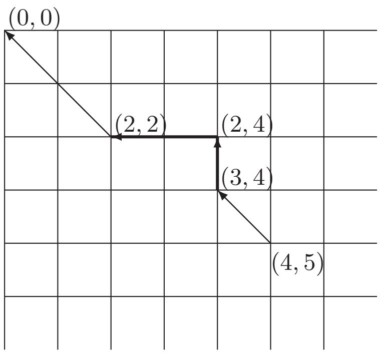
# 1 Introduction

Wythoff's Nim is a game played by two players on two piles of stones. Each player can remove the stones from one or both piles, where the number of stones removed from each pile must be the same. The player who removes the last stone(or stones) wins the game. It is well known that this game is equivalent to the following *Corner the Queen Game*.
A single chess queen is placed somewhere on a large grid of square. Each player can move the queen toward the upper left corner of the grid. Let $(m, n)$ be the first position of the queen(one can identify the two piles of $(m, n)$ stones in Wythoff's Nim). The winner is the player who moves the queen into the corner $(0, 0)$. The following is an example of sequence of movements of the queen in corner the queen game. The winner of the following example is the player $B$.

Example Player $A$ : Moves the queen from $(4, 5)$ to $(3, 4)$,
Player $B$ : Moves the queen from $(3, 4)$ to $(2, 4)$,
Player $A$ : Moves the queen from $(2, 4)$ to $(2, 2)$,
Player $B$ : Moves the queen from $(2, 2)$ to $(0, 0)$.




A position from which the player who made the last move, the previous player, can always win is called a *P-position*. Then the *grundy number* $g(m, n)$ is defined for any position $(m, n)$ of two pile Nim by using the notion of minimal excluded number. Thus $(m, n)$ is a P-position if and only if $g(m, n) = 0$.
Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the set of all the non-negative integers. Then the grundy number of original Nim of Bouton is $g(m, n) = m \oplus n$, where $\oplus$ is Nim-sum and $(m, n)$ is a P-position if and only if $m = n$. The P-positions of Wythoff's Nim are known to be closely related to the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$ as follows.

**Proposition 1.1 (Wythoff (1905))** *$(m, n)$ is a P-position $\iff$ $(m, n) = (m_s, m_s + s)$ or $(m_s + s, m_s)$, where $m_s$ is determined by $m_s = [s\phi]$ for any $s \in \mathbb{N}_0$.*

Thus the grundy number $g(m,n)$ of Wythoff's Nim have been settled for the special cases $g(m,n) = 0$. Although it is still an open problem to write down $g(m,n)$ explicitly in the closed form of $m$ and $n$.

**Table 1** Table of the grundy numbers $g_S(m,n)$ of Wythoff's Nim game for small $m$ and $n$.

| $m\backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\ldots$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\ldots$ |
| 1 | 1 | 2 | 0 | 4 | 5 | 3 | 7 | 8 | $\ldots$ |
| 2 | 2 | 0 | 1 | 5 | 3 | 4 | 8 | 6 | $\ldots$ |
| 3 | 3 | 4 | 5 | 6 | 2 | 0 | 1 | 9 | $\ldots$ |
| 4 | 4 | 5 | 3 | 2 | 7 | 6 | 9 | 0 | $\ldots$ |
| 5 | 5 | 3 | 4 | 0 | 6 | 8 | 10 | 1 | $\ldots$ |
| 6 | 6 | 7 | 8 | 1 | 9 | 10 | 3 | 4 | $\ldots$ |
| 7 | 7 | 8 | 6 | 9 | 0 | 1 | 4 | 5 | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\ldots$ |

Recently the following interesting properties has been found for each row $n$ and column $m$, which are called the additive periodicities.

**Proposition 1.2 (Dress, Flammenkamp, Pink (1999))** *The grundy number $g(m,n)$ of Wythoff's Nim has the additive periodicity for fixed $m$, that is, there exist $a_m \geq 0$ and $p_m > 0$ which satisfy*

$$n \geq a_m \Rightarrow g(m, n + p_m) = g(m,n) + p_m$$

**Table 2** Table of the additive periodicity of $g(m,n)$ ($p_m$ and $a_m$ are taken from [2])

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $p_m$ | 1 | 3 | 3 | 6 | 12 | 24 | 12 | 24 | 24 |
| $a_m$ | 0 | 0 | 0 | 8 | 9 | 27 | 37 | 92 | 102 |

| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|
| 24 | 24 | 48 | 48 | 96 | 96 | 96 | 24 |
| 127 | 224 | 227 | 347 | 382 | 613 | 693 | 771 |

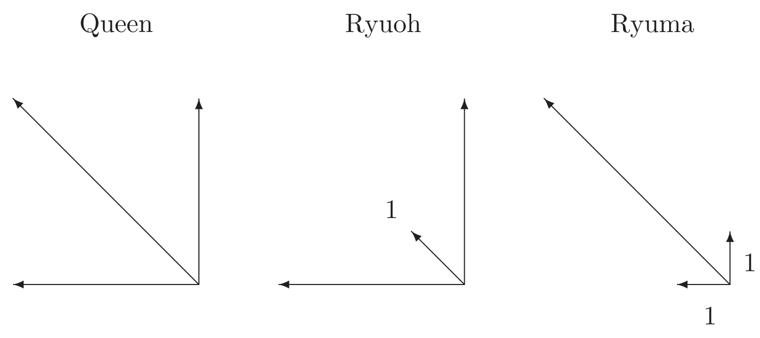
| 17 | 18 | 19 | 20 | $21 \ldots 43$ |
|---|---|---|---|---|
| 192 | 384 | 384 | 384 | 768 |
| 865 | 919 | 1032 | 1165 | $1252 \ldots 13849$ |

Here we will recall the notion of periodicity and additive periodicity for one pile nim. Fix a subset $S \subset \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Consider the restricted nim that one can remove only the stones $s \in S$ from the one pile and denote this restricted

grundy number of this game by $g_S(n)$.
If $|S| < \infty$, it is known the grundy number $g_S(m)$ is periodic, i.e., there exist non negative integers $a$ and $p > 0$ such that $g_S(n+p) = g_S(n)$ for any $n \geq a$. Here $p$ is called the period of $g_S(n)$.
Let $S = \mathbb{N}_0 - T$ with $|T| < \infty$. Then it is known that the grundy number $g_S(m)$ is additively periodic, i.e., there exist $a \geq 0$ and $p, q > 0$ such that $g_S(n+p) = g_S(n) + q$ for any $n \geq a$. Here $p$ and $q$ are called the period and the incremental of $g_S(n)$, respectively.

## 2 Variants of Wythoff's Nim

Recently several variants of Wythoff's original Nim have been investigated by many mathematicians. Here we shall introduce *Ryuoh Nim* and *Ryuma Nim* which have been investigated by Miyadera, Fukui, Nakaya and Tokuni in [6] (2016). The key idea of their paper is to replace the Queen(of Chess Game) of the corner the queen game(or equivallently Wythoff's Nim) to Ryuoh and Ryuma (of Japanese Shogi Game).
Here we shall recall the results of Ryuoh Nim. We note that originally Ryuoh can move $(m, n)$ to $(m-x, n)$ or $(m, n-y)$ or $(m-1, n-1)$ , where $0 \leq x \leq m$ and $0 \leq y \leq n$. We characterized the movements of queen, ryuoh and ryuma as follows.




Let $p$ be any fixed positive integer. In their generalized Ryuoh Nim, Ryuoh can move $(m, n)$ to $(m-a, n)$ or $(m, n-b)$, where $(0 < a \leq m)$ and $(0 < b \leq n)$, or $(m-x, n-y)$, where $0 < x + y \leq p - 1$. It should be noted that the original Ryuoh Nim is nothing but the special case $p = 3$ and the original Bouton's Nim can be regarded as the special case $p = 2$ of this generalized Ryuoh nim.

**Theorem 2.1 (Miyadera, Fukui, Nakaya and Tokuni (2016))** *The grundy number $g(m, n)$ of Ryuoh Nim satifies*

$$g(m,n) = ((m+n) \mod p) + p(\left\lfloor \frac{m}{p} \right\rfloor \oplus \left\lfloor \frac{n}{p} \right\rfloor)$$

Now we shall recall another variant of Wythoff's Nim. Let $S_i$ $(1 \leq i \leq 3)$ be the set of positive integers. Each player can remove the number of tokens $s_1 \in S_1$ from the first pile and $s_2 \in S_2$ from the second pile and remove the same number of tokens $s_3 \in S_3$ from both piles. We shall identify $(m, n)$ to a position of this nim, where $m$ is the number of tokens in the first pile and $n$ is the number of tokens in the second pile. In 2018 we will have shown the G-sequences $g_S(m, n)$ is periodic for fixed $m$ in case $|S_2| < \infty$. Let $S_2 = \{s_1, s_2, \ldots, s_{s(2)} \mid 0 < s_1 < s_2 < \cdots < s_{s(2)}\}$ be the set of positive integers. The player is restricted to remove the number of tokens $s \in S_2$ from the second pile.

**Proposition 2.2 (Katayama and Kubo (2018))** *Under the above notations, $g_S(m, n)$ has a period $p_m$ for any fixed $m$. There exist $a_m$ and $p_m$ which satisfy*

$$n \geq a_m \Longrightarrow g_S(m, n + p_m) = g_S(m, n), \quad \textit{for any } n \geq a_m.$$

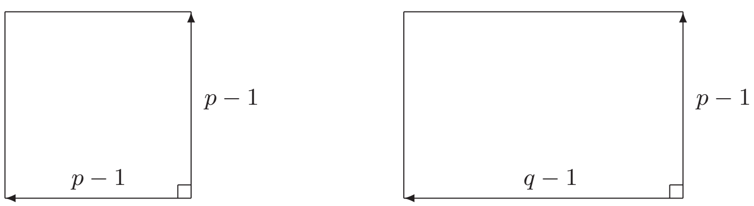
# 3 Square Nim

Let $p$ be an integer with $p \geq 2$. We shall study the following two pile Nim games. Let $S$ be the set of positive integers $\{1 \leq i \leq p - 1\}$. Each player can remove the number of tokens $s_1 \in \{0\} \cup S$ from the first pile and $s_2 \in \{0\} \cup S$ from the second pile with $0 < s_1 + s_2$ at the same time. Let $(m, n)$ denotes a position of this nim game, where $m$ is the number of tokens in the first pile and $n$ is the number of tokens in the second pile. We shall call this new nim game *Square Nim game*. We shall prove the G-sequence $g_S(m, n)$ satisfies $g_S(m + p, n + p) = g_S(m, n)$ for any position $(m, n)$.

We shall also verify that if $m$ and $n$ are sufficiently large, the G-sequences $g_S(m, n)$ are periodic for each row and column with the same period p.

The area of the movements of tokens of Square Nim and Rectangular Nim

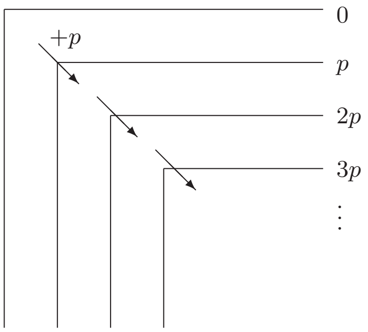
Square Nim $(p)$ Rectangular Nim $(p, q)$




Here we list a table of the Square Nim for the special case $S = \{1\}$.

| $m\backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 2 | 0 | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 1 | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 4 | 0 | 3 | 0 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 2 | 3 | 2 |
| 6 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 1 | 0 | 1 |
| 7 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 2 |
| 8 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 1 |
| 9 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |

Then we see $g_S(m+2, n+2) = g_S(m, n)$ for any $m$ and $n$ for this special case. Moreover the G-sequences $g_S(m, n)$ are periodic for each row $n$ and column $m$ with the period 2 for this case. We shall generalize this fact for general $S = \{1, 2, \ldots, p-1\}$. Then the G-sequence $g_S(m, n)$ satisfies $g_S(m+p, n+p) = g_S(m, n)$ for any $(m, n)$. Thus $g_S(m, n)$ diagonally repeats the following L-shaped hook structure with the period $p$.




Since $g_S(m, n) = g_S(n, m)$, one can assume $m \leq n$ without loss of generality. Hence we shall restrict ourselves to the case $m \leq n$ in the following and prove this fact for general $S = \{1, 2, \ldots, p-1\}$ as follows.
The first step 1.
Let $m = r$ with $0 \leq r < p$.
Then $g_S(r, n) = r + n$, for the case $r + n < rp$.
$g_S(r, n) = rp + ((r + n) \mod p)$, for the case $r + m \geq rp$.

The second step.
We shall show $g_S(m + p, n + p) = g_S(m, n)$ for any $(m, n)$ by induction.

To understand the first step, we shall give here a table for the grundy numbers for small $m$ in the case $S = \{1, \ldots, p-1\}$.

| $m \backslash n$ | $0$ | $\cdots$ | $p-1$ | $p$ | $\cdots$ | $2p-1$ | $\cdots$ | $(p-1)p$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $\cdots$ | $p-1$ | $0$ | $\cdots$ | $p-1$ | $\cdots$ | $0$ | $\cdots$ |
| $1$ | $1$ | $\cdots$ | $p$ | $p+1$ | $\cdots$ | $p$ | $\cdots$ | $p+1$ | $\cdots$ |
| $2$ | $2$ | $\cdots$ | $p+1$ | $p+2$ | $\cdots$ | $2p+1$ | $\cdots$ | $p+2$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\cdots$ |
| $p-1$ | $p-1$ | $\cdots$ | $2(p-1)$ | $2p-1$ | $\cdots$ | $3p-2$ | $\cdots$ | $p^2-1$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\cdots$ |

# 4 Proof of the Main Theorem

It is well known that the G-sequence $g_S(0,n) = n \mod p$ and hence is purely periodic with the period $p$. Next, by definition, we have $g_s(1,n) = mex\{g_S(0,n), g_S(0,n-i), g_S(1,n-i) |\ 1 \le i \le p-1 \text{ and } \ i \le n\}$.
Hence $g_S(1,n) = 1+n$ for the case $n < p-1$ and $g_S(1,n) = p + ((1+n) \mod p)$ for the case $n \ge p-1$. Similarly, using the definition of minimal excluded number, we obtain the following facts by induction on $m$.

(1) $g_S(r,0) = r$ and $g_S(r,n+1) = g_S(r,n)+1$ for the cases $r+n < rp$.

(2) $g_S(r,n) = rp + ((r+p) \mod p)$ for the cases $r+n \ge rp$.

Thus one can easily verify the following explicit lemma from these facts (1) and (2).

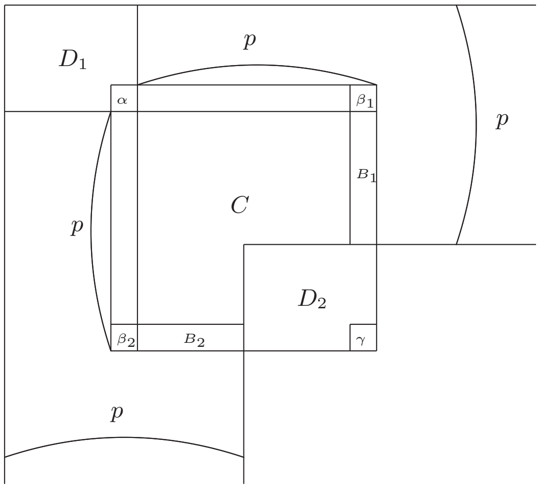
**Lemma 4.1** *Assume* $0 \le m < m+1 \le p-1$.
*If* $g_S(m,n)$ *is monotone increasing for the interval* $a \le n \le b$, *then* $g_S(m+1,n) = g_S(m,n)+1$ *and is also monotone increasing for the interval* $a \le n \le b$.

Now we shall show the second step, i.e.,

$$g_S(m+p, n+p) = g_S(m,n) \text{ for any } 0 \le n \le p-1.$$

Denote $g_S(m,n) = \alpha$, $g_S(m,n+p) = \beta_1$, $g_S(m+p,n) = \beta_2$ and $g_S(m+p,n+p) = \gamma$, respectively.
Put
$D_1 = \{g_S(m-x,n-y) \mid x \le m, y \le n \text{ and } 0 < x+y \le p-1\}$,
$D_2 = \{g_S(m+p-x,n+p-y) \mid x \le m, y \le n \text{ and } 0 < x+y \le p-1\}$,
$C = \{g_S(m+x,n+y) \mid 0 \le x,y \le p-1 \text{ and } 0 < x+y \le p-1\}$,
$B_1 = \{g_S(m+x,n+p) \mid 1 \le x \le p-1-m\}$,
$B_2 = \{g_S(m+p,n+y) \mid 0 \le y \le p-1\}$.
Then the set $E = \{g_S(m+p+x,n+p+y) \mid 0 \le x,y \le p-1 \text{ and } 0 < x+y \le p-1\}$ $= B_1 \cup B_2 \cup C \cup D_1$, and $g_S(m+p,n+p) = \text{mex } E$.

We shall give a figure of these sets as follows.




Let us start the induction from the case $D_1 = D_2 = \emptyset$.
Since $g_S(0,n) = g_S(n,0)$ is purely periodic, we see $\alpha = \beta_1 = \beta_2 = 0$. Moreover any $g_S(m,n) > 0$ for any $(m,n) \in E$, we obtain $\gamma = 0$ from the definition of the mex function.
Assume $D_1 = D_2$ inductively. Then $\{0, 1, \ldots, \alpha - 1\} \subset D_2 = D_1$ and $\alpha \notin D_2$. Then, by the definition of the minimal excluded number, it is obvious $\alpha \notin C$. We can show $\alpha \notin B_1$ as follows.
If $\beta_1 = \alpha$, i.e., $g_S(m, n + y)$ is periodic, $\sigma \neq \alpha$ for any $\sigma \in B_1$. If $\beta_1 > \alpha$, i.e., $g_S(m, n+y)$ is monotone increasing, we know any $\sigma \in B_1$ satisfies $\sigma > \beta_1 > \alpha$. Hence $\alpha \notin B_1$. Similarly one gets $\alpha \notin B_2$.
Thus we have verified $\alpha \notin E$ and $\{0, 1, \ldots, \alpha - 1\} \subset E$, and so we have shown that $\gamma = \alpha$, which completes the induction for the cases $0 \leq r \leq p - 1$.
For the case $r \geq p$, we may assume $g_S(m + p, n + p) = g_S(m, n)$ for any $m < r$ or $n < r$ from the above proof. Then, by the definition of the minimal excluded number, we have
$g_S(r + p, n + p)$
$= mex\{g_S(r+p-i, n+p), g_S(r+p, n+p-j), g_S(r-i+p, n-j+p) | 1 \leq i, j \leq p-1\}$
$= mex\{g_S(r - i, n), g_S(r, n - j), g_S(r - i, n - j) | 1 \leq i, j \leq p - 1\} = g_S(r, n)$.
Thus we have verified $g_S(m + p, n + p) = g_S(m, n)$ for any $(m, n)$ and satisfies the $L$-shaped hook structure.
Finally, we shall write down the grundy number $g_S(m, n)$ explicitly. Let $h_S(m, n)$ be the following function on $(m, n) \in \mathbb{N}_0^2$.

$$h_S(m,n) = \begin{cases} m+n-2\left[\frac{m}{p}\right]p \\ \qquad \text{for the case } n-m < (m-\left[\frac{m}{p}\right]p)(p-2) \\ (m-\left[\frac{m}{p}\right]p)p + (m+n \mod p) \\ \qquad \text{for the case } n-m \geq (m-\left[\frac{m}{p}\right]p)(p-2) \end{cases}$$

Then by induction, one can easily verify $g_S(m,n) = h_S(m.n)$.

**Theorem 4.2**

$$g_S(m,n) = \begin{cases} m+n-2\left[\frac{m}{p}\right]p \\ \qquad \textit{for } \ 0 \leq n-m < (m-\left[\frac{m}{p}\right]p)(p-2), \\ (m-\left[\frac{m}{p}\right]p)p + (m+n \pmod p) \\ \qquad \textit{for } \ n-m \geq (m-\left[\frac{m}{p}\right]p)(p-2). \end{cases}$$

*and* $g(m+p, n+p) = g(m,n)$ *for any* $m, n \in \mathbb{N}_0$

One can write down $g_S(m,n)$ using sign function in the following way.

**Remark 4.3** *Let* $m \leq n$ *and* $f(m,n) = m-n+(m-\left[\frac{m}{p}\right]p)(p-2)$. *Then*

$$g_S(m,n) = m+n-2\left[\frac{m}{p}\right]p + \frac{1+\text{sign}(f(m,n)+1/2)}{2}(f(m,n)+(m+n \pmod p)).$$

# 5 Rectangular, Triangular and Polytope Nim

## 5.1 Rectangular Nim

Now we shall slightly generalize Square Nim and call this new game *Rectangular Nim*. Let $p$ and $q$ be positive integers. We shall introduce the following two pile nim game. Let $S = (S_1, S_2)$ be the set of positive integers $S_1 = \{1 \leq i \leq p\}$ and $S_2 = \{1 \leq i \leq q\}$. Each player can remove the number of tokens $s_1 \in \{0\} \cup S_1$ from the first pile and $s_2 \in \{0\} \cup S_2$ from the second pile with $0 < s_1 + s_2$ at the same time. We shall identify $(m,n)$ to a position of this nim game as above, where $m$ is the number of tokens in the first pile and $n$ is the number of tokens in the second pile. We see the G-sequences $g_S(m,n)$ is periodic for each column $m$ with the period $p$ and for each row $n$ with the period $q$ for large $m,n$. More precisely, there exists some $a_m$ for any fixed $m$, such as $g_S(m,n+q) = g_S(m,n)$ for any $n \geq a_m$ Similarly, there exists some $b_n$ for any fixed $n$, such as $g_S(m,n+q) = g_S(m,n)$ for any $m \geq b_n$. Moreover one can show the following theorem using similar procedure as the case $p = q$.

**Theorem 5.1** *$g_S(m,n)$ satisfies that $g(m+p, n+q) = g(m,n)$ for any $m, n \in \mathbb{N}_0$*

The following table is an example of a Rectangular Nim with $S = (S_1 = \{1\}, S_2 = \{1,2\})$

| $m \backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 |
| 1 | 1 | 2 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 |
| 2 | 0 | 3 | 4 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 |
| 3 | 1 | 2 | 5 | 1 | 2 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 |
| 4 | 0 | 3 | 4 | 0 | 3 | 4 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 |
| 5 | 1 | 2 | 5 | 1 | 2 | 5 | 1 | 2 | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 |

## 5.2 Triangular Nim

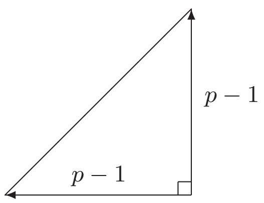
Now we shall refer to *Triangular Nim* game. Let $p$ be a positive integer greater than 2. In this game, one can move $(m,n)$ to $(m-x, n-y)$ with the restriction $0 < x+y \leq p-1$. Here we call this nim Triangular Nim.
Actually this Nim is the special case $p = q$ of the following Suetsugu and Fukui's variant of Wythoff's Nim and has the periodic grundy number $g_S(m,n) = (m+n) \mod p$. Suetsugu and Fukui's variant of Wythoff's Nim game is defined as follows. Let $p, q$ be a pair of positive integers such that $p \mid q$. From the two piles $(m,n)$, one can remove $0 \leq x \leq q-1$ tokens from each pile, or can remove $(x,y)$ tokens from two piles at the same time with $0 < x+y \leq p-1$. Then the grundy numbers $g_S(m,n)$ satisfy the following formula.

**Proposition 5.2 (Suetsugu and Fukui (2017))** *With the above notations,*

$$g_S(m,n) = ((m \mod q) + (n \mod q) \mod p) + p(\left[\frac{m \mod q}{p}\right] \oplus \left[\frac{n \mod q}{p}\right]).$$

The following figure is an image of the Triangular Nim.

Triangular Nim ($p$)



## 5.3 Polytopes Nim

Finally we shall generalize Triangular Nim to the following *Polytopes Nim*. Let $S$ be the set of positive integers $S \subset \mathbb{N}$. We shall call the following game *Polytopes Nim* with the restriction to $S$.
Let $(m_1, m_2, \ldots, m_n)$ be the set of $n$ piles. $\rho$ denotes the summation map

$$\rho((m_1, m_2, \ldots, m_n)) = m_1 + m_2 + \cdots + m_n.$$

Then $\rho$ is a surjective map from $\mathbb{N}_0^n$ to $\mathbb{N}_0$ and $\rho(\boldsymbol{u} + \boldsymbol{v}) = \rho(\boldsymbol{u}) + \rho(\boldsymbol{v})$ for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}_0$.
The player of this game can remove $\boldsymbol{w} = (x_1, x_2, \ldots, x_n)$ tokens from $\boldsymbol{v} = (m_1, m_2, \ldots, m_n)$ with the restriction such that the sum $\rho(\boldsymbol{w}) = x_1 + x_2 + \cdots + x_n \in S$.
It should be noted that Triangular Nim can be regarded as the special case of this Nim when the dimension $n = 2$ and $S = \{1, 2 \ldots, p-1\}$.
$N(v)$ denotes the set of next positions of $\boldsymbol{v} = (m_1, m_2, \ldots, m_n)$. Then $N(\boldsymbol{v}) = \{\boldsymbol{v} - \boldsymbol{w} \mid \rho(\boldsymbol{w}) \in S \text{ with } m_i \geq x_i \ \ (1 \leq i \leq n)\}$. Consider one pile Nim with the restriction to $S$ and denote the set of all the next positions of $m \in \mathbb{N}_0$ by $N(m)$. Then $N(m) = \{m - x \mid s \in S, \text{and } m - s \geq 0\}$ and the Grundy number is defined by $g_S(m) = \text{mex}(N(m))$. Using the summension map $\rho$, we shall show one can identify the set of tokens(or positions) $\{(m_1, m_2, \ldots, m_n) | m_1 + m_2 + \cdots + m_n = m\}$ in $\mathbb{N}_0^n$ of Polytopes Nim with restriction to $S$ to $m \in \mathbb{N}_0$ in one pile nim with the restriction to $S$. We shall show this fact by induction as follows.
Let $m = \rho((m_1, m_2, \ldots, m_n)) = m_1 + m_2 + \cdots + m_n$ as above. Put $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{N}_0$. Then by definition, $g_S(\mathbf{0})$ of the Polytote nim equals to $0 = g_S(0)$ of one pile nim. Hence $g_S(\mathbf{0}) = g_S(\rho(\mathbf{0}))$ holds for $0 = \rho(\mathbf{0})$. Assume $g_S(\boldsymbol{v}') = g_S(m')$ holds for any $\boldsymbol{v}'$ with $m' = \rho(\boldsymbol{v}') < m$. Put $m = \rho(\boldsymbol{v})$, where $\boldsymbol{v} = (m_1, m_2, \ldots, m_n)$ and put $x = \rho(\boldsymbol{w})$, where $\boldsymbol{w} = (x_1, x_2, \ldots, x_n)$, with $x \in S$. Then, by induction, the Grundy number of polytopes Nim $g_S(\boldsymbol{v})$ satisfies
$g_S(\boldsymbol{v}) = \text{mex } \{g_S(N(\boldsymbol{v}))\} = \text{mex } \{g_S(\boldsymbol{v}') \ |\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{w} \mid x = \rho(\boldsymbol{w}) \in S\}$.
From the assumption, we have $g_S(\boldsymbol{v} - \boldsymbol{w}) = g_S(\rho(\boldsymbol{v} - \boldsymbol{w})) = g_S(\rho(\boldsymbol{v}) - \rho(\boldsymbol{w})) = g_S(m - x)$. Hence we know
$\text{mex } \{g_S(\boldsymbol{v}') \ |\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{w} \mid x = \rho(\boldsymbol{w}) \in S\} = \text{mex } \{g_S(m - x) \mid x = \rho(\boldsymbol{w}) \in S\} = g_S(m)$.
Hence we have shown the following theorem.

**Theorem 5.3** *With the above notations, the Grundy number* $\boldsymbol{v} = (m_1, m_2, \ldots, m_n)$ *satisfies*

$$g_S(\boldsymbol{v}) = g_S(\rho(\boldsymbol{v})).$$

Let $p$ be a positive integer. Consider the special polytopes Nim with the restriction to the special set $S = \{1, 2, \ldots, p-1\}$. Then each player can remove $\boldsymbol{w} = (x_1, x_2, \ldots, x_n)$ from the piles of tokens $\boldsymbol{v} = (m_1, m_2, \ldots, m_n)$ with $x = x_1 + \cdots + x_n = x \in S$. Then we have the following slightly generalized version of the Triangular Nim.

**Corollary 5.4** *With the above notations,*

$$g_S((m_1, m_2, \ldots, m_n)) = (m_1 + m_2 + \cdots m_n) \pmod{p}.$$

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