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Inequalities for the Difference $A^{-1}g(A) - B^{-1}g(B)$ when g is Operator Monotone on $[0, \infty)$

By

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Abstract

In this paper we show that, if $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ with $g(0) = 0$, $A \geq 0$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then

$$\begin{aligned} A^{-1}g(A) - B^{-1}g(B) &\geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c} \right] 1_H \\ &\geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|} \right] 1_H > 0 \end{aligned}$$

and

$$\begin{aligned} &A^{-1}g(A) - B^{-1}g(B) \\ &\geq c \left(\frac{g(\|A\|)}{(d + \|A\|)\|A\|} - \frac{g(d + \|A\|) - g(\|A\|)}{d(d + \|A\|)} \right) 1_H \geq 0. \end{aligned}$$

Some applications for particular functions of interest are also given.
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1 Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $[0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1 *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{ts}{t+s}dm(s) \tag{1}$$

where $b\geq 0$ and a positive measure m on $[0,\infty)$ such that

$$\int_0^\infty \frac{s}{1+s}dm(s)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:[0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [5]. Also the function ln is operator monotone on the open interval $(0,\infty)$. Let $f(t)$ be a continuous function $(0,\infty)\to(0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t)=t/f(t)=:f^*(t)$ is also operator monotone, see for instance [3] or [7].

Let A and B be strictly positive operators on a Hilbert space H such that $B-A\geq m1_H>0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$\begin{aligned} f(B)-f(A) &\geq [f(\|A\|+m)-f(\|A\|)]1_H \\ &\geq [f(\|B\|)-f(\|B\|-m)]1_H>0. \end{aligned} \tag{2}$$

If $B>A>0$, then

$$\begin{aligned} f(B)-f(A) &\geq \left[f\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-f(\|A\|)\right]1_H \\ &\geq \left[f(\|B\|)-f\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)\right]1_H>0. \end{aligned} \tag{3}$$

The inequality between the first and third term in (3) was obtained earlier by H. Zuo and G. Duan in [9].

By taking $f(t)=t^r$, $r\in(0,1]$ in (3) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality

$$\begin{aligned} B^r-A^r &\geq \left[\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r-\|A\|^r\right]1_H \\ &\geq \left[\|B\|^r-\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)^r\right]1_H>0 \end{aligned} \tag{4}$$

provided $B>A>0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\ln B - \ln A \geq \left[\ln\left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|}\right) - \ln(\|A\|)\right] 1_H \tag{5}$$
$$\geq \left[\ln(\|B\|) - \ln\left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|}\right)\right] 1_H > 0.$$

Notice that the inequalities between the first and third terms in (4) and (5) were obtained earlier by M. S. Moslehian and H. Najafi in [8].

Motivated by the above results, in this paper we show that, if $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ with $g(0) = 0$, $A \geq 0$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then

$$A^{-1}g(A) - B^{-1}g(B) \geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c}\right] 1_H$$
$$\geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|}\right] 1_H > 0$$

and

$$A^{-1}g(A) - B^{-1}g(B)$$
$$\geq c\left(\frac{g(\|A\|)}{(d + \|A\|)\|A\|} - \frac{g(d + \|A\|) - g(\|A\|)}{d(d + \|A\|)}\right) 1_H \geq 0.$$

Some applications for particular functions of interest are also given.

2 Main Results

We start with the following lemma that is of interest in itself.

Lemma 2 *Assume that $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$. Then the function $f : (0, \infty) \to \mathbb{R}$,*

$$f(t) := \frac{g(0) - g(t)}{t} \tag{6}$$

is operator monotone on $(0, \infty)$. If $g(0) = 0$, then $f(t) = -g(t)t^{-1}$ is operator monotone on $(0, \infty)$.

Proof. Since g is operator monotone on $[0, \infty)$, then there exists $b \geq 0$ and w is a positive measure satisfying

$$\int_0^\infty \frac{\lambda}{1 + \lambda} dw(\lambda) < \infty$$

such that [1, p. 144-145]

$$g(t) = g(0) + bt + \int_0^\infty \frac{\lambda t}{t+\lambda} dw(\lambda). \tag{7}$$

We have for $t > 0$ that

$$h(t) := \frac{g(t) - g(0)}{t} - b = \int_0^\infty \frac{\lambda}{t+\lambda} dw(\lambda)$$

Therefore for all A, $B > 0$

$$h(B) - h(A) = \int_0^\infty \lambda \left[(B + \lambda 1_H)^{-1} - (A + \lambda 1_H)^{-1} \right] dw(\lambda). \tag{8}$$

Let T, $S > 0$. The function $g(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S) := \lim_{t \to 0} \left[\frac{g(T + tS) - g(T)}{t} \right] = T^{-1} S T^{-1} \tag{9}$$

for T, $S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0, 1]$ by

$$g_{C,D}(t) := g((1-t)C + tD), \ t \in [0, 1].$$

Then we have, by the properties of the Bochner integral, that

$$g(D) - g(C) = \int_0^1 \frac{d}{dt}(g_{C,D}(t))\, dt = \int_0^1 \nabla g_{(1-t)C+tD}(D - C)\, dt. \tag{10}$$

If we write this equality for the function $g(t) = -t^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} dt. \tag{11}$$

Now, if we replace in (11) $C = B + \lambda 1_H$ and $D = A + \lambda 1_H$ for $\lambda > 0$, then we get

$$\begin{aligned} &(B + \lambda 1_H)^{-1} - (A + \lambda 1_H)^{-1} \\ &= \int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (A - B) ((1-t)B + tA + \lambda 1_H)^{-1} dt. \end{aligned} \tag{12}$$

Therefore, by (8),

$$\begin{aligned} h(B) - h(A) &= \int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (A - B) \right. \qquad (13) \\ &\quad \left. \times ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda) \\ &= -\int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\quad \left. \times ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda). \end{aligned}$$

If $B \geq A > 0$, then

$$((1-t)B + tA + \lambda 1_H)^{-1} (B - A) ((1-t)B + tA + \lambda 1_H)^{-1} \geq 0$$

for all t, $\lambda > 0$, which implies that

$$\begin{aligned} &\int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\left. \times ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda) \geq 0, \end{aligned}$$

namely

$$\begin{aligned} f(B) - f(A) &= h(A) - h(B) \\ &= \int_0^\infty \lambda \left(\int_0^1 ((1-t)B + tA + \lambda 1_H)^{-1} (B - A) \right. \\ &\quad \left. \times ((1-t)B + tA + \lambda 1_H)^{-1} dt \right) dw(\lambda) \\ &\geq 0. \end{aligned}$$

Therefore, the function f is operator monotone on $(0, \infty)$. ∎

Theorem 3 *Assume that $g : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$. If $A > 0$ and there exists $c > 0$ such that $B - A \geq c1_H > 0$, then*

$$\begin{aligned} &A^{-1}g(A) - B^{-1}g(B) - g(0)\left(A^{-1} - B^{-1}\right) \qquad (14) \\ &\geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c} - g(0) \frac{c}{(\|A\| + c)\|A\|} \right] 1_H \\ &\geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|} - g(0) \frac{c}{(\|B\| - c)\|B\|} \right] 1_H > 0. \end{aligned}$$

If $g(0) = 0$, then

$$\begin{aligned} A^{-1}g(A) - B^{-1}g(B) &\geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\| + c)}{\|A\| + c} \right] 1_H \qquad (15) \\ &\geq \left[\frac{g(\|B\| - c)}{\|B\| - c} - \frac{g(\|B\|)}{\|B\|} \right] 1_H > 0. \end{aligned}$$

Proof. If we write the inequality (2) for $f(t) = \frac{g(0)-g(t)}{t}$, $t > 0$, which, by Lemma 2, is operator monotone, then we have

$$\begin{aligned} & B^{-1}[g(0)-g(B)] - A^{-1}[g(0)-g(A)] \\ & \geq \left[\frac{g(0)-g(\|A\|+c)}{\|A\|+c} - \frac{g(0)-g(\|A\|)}{\|A\|}\right] 1_H \\ & \geq \left[\frac{g(0)-g(\|B\|)}{\|B\|} - \frac{g(0)-g(\|B\|-c)}{\|B\|-c}\right] 1_H > 0. \end{aligned} \tag{16}$$

Observe that

$$\begin{aligned} & B^{-1}[g(0)-g(B)] - A^{-1}[g(0)-g(A)] \\ & = A^{-1}g(A) - B^{-1}g(B) - g(0)\left(A^{-1} - B^{-1}\right), \end{aligned}$$

$$\begin{aligned} & \frac{g(0)-g(\|A\|+c)}{\|A\|+c} - \frac{g(0)-g(\|A\|)}{\|A\|} \\ & = \frac{g(\|A\|)}{\|A\|} - \frac{g(\|A\|+c)}{\|A\|+c} - g(0)\frac{c}{(\|A\|+c)\|A\|} \end{aligned}$$

and

$$\begin{aligned} & \frac{g(0)-g(\|B\|)}{\|B\|} - \frac{g(0)-g(\|B\|-c)}{\|B\|-c} \\ & = \frac{g(\|B\|-c)}{\|B\|-c} - \frac{g(\|B\|)}{\|B\|} - g(0)\frac{c}{(\|B\|-c)\|B\|} \end{aligned}$$

and by (16) we get (14). ■

Its is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned} 0 & \leq \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\ & \leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} 1_H \leq T. \tag{17}$$

Corollary 4 *Assume that* $g : [0, \infty) \to \mathbb{R}$ *is operator monotone on* $[0, \infty)$. *If* $A > 0$ *and* $B - A > 0$, *then*

$$
\begin{aligned}
&A^{-1}g(A) - B^{-1}g(B) - g(0)\left(A^{-1} - B^{-1}\right) \qquad (18)\\
&\geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g\left(\|A\| + \left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\| + \left\|(B-A)^{-1}\right\|^{-1}}\right] 1_H\\
&- g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|A\| + \left\|(B-A)^{-1}\right\|^{-1}\right)\|A\|} 1_H\\
&\geq \left[\frac{g\left(\|B\| - \left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\| - \left\|(B-A)^{-1}\right\|^{-1}} - \frac{g(\|B\|)}{\|B\|}\right] 1_H\\
&- g(0)\frac{\left\|(B-A)^{-1}\right\|^{-1}}{\left(\|B\| - \left\|(B-A)^{-1}\right\|^{-1}\right)\|B\|} 1_H\\
&> 0.
\end{aligned}
$$

If $g(0) = 0$, *then*

$$
\begin{aligned}
&A^{-1}g(A) - B^{-1}g(B) \qquad (19)\\
&\geq \left[\frac{g(\|A\|)}{\|A\|} - \frac{g\left(\|A\| + \left\|(B-A)^{-1}\right\|^{-1}\right)}{\|A\| + \left\|(B-A)^{-1}\right\|^{-1}}\right] 1_H\\
&\geq \left[\frac{g\left(\|B\| - \left\|(B-A)^{-1}\right\|^{-1}\right)}{\|B\| - \left\|(B-A)^{-1}\right\|^{-1}} - \frac{g(\|B\|)}{\|B\|}\right] 1_H > 0.
\end{aligned}
$$

We have the following lower bound as well:

Theorem 5 *Assume that* $f : [0, \infty) \to \mathbb{R}$ *is operator monotone on* $[0, \infty)$. *Let* $A \geq 0$ *and assume that there exist positive numbers* $d > c > 0$ *such that*

$$d1_H \geq B - A \geq c1_H > 0. \qquad (20)$$

Then

$$f(B) - f(A) \geq c\frac{f(d + \|A\|) - f(\|A\|)}{d} 1_H \geq 0. \qquad (21)$$

Proof. Since the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$, then f can be written as in the equation (1) and for A, $B\geq 0$ we have the representation

$$\begin{aligned} & f(B)-f(A) \\ & = b(B-A)+\int_0^\infty s\left[B(B+s1_H)^{-1}-A(A+s1_H)^{-1}\right]dm(s). \end{aligned} \tag{22}$$

Observe that for $s>0$

$$\begin{aligned} & B(B+s1_H)^{-1}-A(A+s1_H)^{-1} \\ & =(B+s1_H-s1_H)(B+s1_H)^{-1}-(A+s1_H-s1_H)(A+s1_H)^{-1} \\ & =(B+s1_H)(B+s1_H)^{-1}-s1_H(B+s1_H)^{-1} \\ & -(A+s1_H)(A+s1_H)^{-1}+s1_H(A+s1_H)^{-1} \\ & =1_H-s1_H(B+s1_H)^{-1}-1_H+s1_H(A+s1_H)^{-1} \\ & =s\left[(A+s1_H)^{-1}-(B+s1_H)^{-1}\right]. \end{aligned}$$

Therefore, (22) becomes, see also [4]

$$f(B)-f(A)=b(B-A)+\int_0^\infty s^2\left[(A+s1_H)^{-1}-(B+s1_H)^{-1}\right]dm(s). \tag{23}$$

Now, if we replace in (11) $C=A+s1_H$ and $D=B+s1_H$ for $s>0$, then we get

$$\begin{aligned} & (A+s1_H)^{-1}-(B+s1_H)^{-1} \\ & =\int_0^1((1-t)A+tB+s1_H)^{-1}(B-A)((1-t)A+tB+s1_H)^{-1}dt. \end{aligned} \tag{24}$$

By the representation (23), we derive the following identity of interest

$$\begin{aligned} f(B)-f(A) & =b(B-A) \\ & +\int_0^\infty s^2\left[\int_0^1((1-t)A+tB+s1_H)^{-1}\right. \\ & \left.\times(B-A)((1-t)A+tB+s1_H)^{-1}dt\right]dm(s) \end{aligned} \tag{25}$$

for A, $B>0$.

From the representation (25) we get for $B=x1_H$, $A=0$ that

$$f(x)-f(0)-bx=\int_0^\infty s^2\left(\int_0^1(tx+s1_H)^{-1}x(tx+s1_H)^{-1}dt\right)dm(s),$$

which gives for $x>0$ that

$$\frac{f(x)-f(0)}{x}-b=\int_0^\infty s^2\left(\int_0^1(tx+s)^{-2}dt\right)dm(s). \tag{26}$$

Since $0 < c1_H \leq B - A$, hence

$$c\left(\left(1-t\right)A + tB + s1_H\right)^{-2}$$
$$\leq \left(\left(1-t\right)A + tB + s1_H\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A + tB + s1_H\right)^{-1}$$

for $t \in [0,1]$ and $s > 0$ and by (25) we get

$$c\int_0^\infty s^2\left(\int_0^1 \left(\left(1-t\right)A + tB + s1_H\right)^{-2}dt\right)dm\left(s\right) \tag{27}$$
$$\leq f\left(B\right) - f\left(A\right) - b\left(B-A\right).$$

Observe that for $t \in [0,1]$ and $s > 0$, we have

$$\begin{aligned}\left(1-t\right)A + tB + s1_H &= A + t\left(B-A\right) + s1_H \leq A + td1_H + s1_H \\ &= \left(1-t\right)A + t\left(d1_H + A\right) + s1_H.\end{aligned}$$

Since $A \leq \|A\| 1_H$ then

$$\left(1-t\right)A + t\left(d1_H + A\right) + s1_H \leq \left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)1_H,$$

which implies that

$$\left(1-t\right)A + tB + s1_H \leq \left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)1_H$$

for $t \in [0,1]$ and $s > 0$.

This implies that

$$\left(\left(1-t\right)A + tB + s1_H\right)^{-1} \geq \left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)^{-1}1_H$$

and

$$\left(\left(1-t\right)A + tB + s1_H\right)^{-2} \geq \left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)^{-2}1_H$$

for $t \in [0,1]$ and $s > 0$.

Therefore

$$\int_0^\infty s^2\left(\int_0^1 \left(\left(1-t\right)A + tB + s1_H\right)^{-2}dt\right)dm\left(s\right)$$
$$\geq \int_0^\infty s^2\left(\int_0^1 \left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)^{-2}dt\right)dm\left(s\right)1_H\ (\geq 0)$$
$$= \frac{1}{d}\int_0^\infty s^2\left(\int_0^1 \left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)^{-1}\left(d + \|A\| - \|A\|\right)\right.$$
$$\left.\times\left(\left(1-t\right)\|A\| + t\left(d + \|A\|\right) + s\right)^{-1}dt\right)dm\left(s\right)1_H$$
$$= \frac{1}{d}\left[\left(f\left(d + \|A\|\right) - f\left(\|A\|\right) - bd\right)\right]1_H \text{ (by identity (26))}$$
$$= \left(\frac{f\left(d + \|A\|\right) - f\left(\|A\|\right)}{d} - b\right)1_H \geq 0.$$

By (27) we get

$$\begin{aligned} & f(B)-f(A)-b(B-A) \\ & \geq c\int_0^\infty s^2\left(\int_0^1((1-t)A+tB+s1_H)^{-2}\,dt\right)dm(s) \\ & \geq c\left(\frac{f(d+\|A\|)-f(\|A\|)}{d}-b\right)1_H\geq 0. \end{aligned} \tag{28}$$

From (28) we derive

$$\begin{aligned} f(B)-f(A) & \geq b(B-A)+c\left(\frac{f(d+\|A\|)-f(\|A\|)}{d}-b\right)1_H \\ & = b[(B-A)-c]+c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H \\ & \geq c\frac{f(d+\|A\|)-f(\|A\|)}{d}1_H\geq 0 \end{aligned}$$

since $b[(B-A)-c]\geq 0$ and the inequality (21) is obtained. ■

Corollary 6 *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A\geq 0$ and $B-A>0$, then*

$$\begin{aligned} f(B)-f(A) & \geq \frac{f(\|B-A\|+\|A\|)-f(\|A\|)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H \\ & \geq \frac{f(\|B\|)-f(\|A\|)}{\left\|(B-A)^{-1}\right\|\|B-A\|}1_H\geq 0. \end{aligned} \tag{29}$$

The first inequality follows by (21) for $d=\|B-A\|$ and $c=\left\|(B-A)^{-1}\right\|^{-1}$. The second and third inequalities are obvious.

Theorem 7 *Assume that $g:[0,\infty)\to\mathbb{R}$ is operator monotone on $[0,\infty)$. If $A>0$ and there exist positive numbers $d>c>0$ such that the condition (20) is satisfied, then*

$$\begin{aligned} & g(0)\left(B^{-1}-A^{-1}\right)+A^{-1}g(A)-B^{-1}g(B) \\ & \geq c\left(\frac{g(\|A\|)-g(0)}{(d+\|A\|)\|A\|}-\frac{g(d+\|A\|)-g(\|A\|)}{d(d+\|A\|)}\right)1_H\geq 0. \end{aligned} \tag{30}$$

If $g(0)=0$, then

$$\begin{aligned} & A^{-1}g(A)-B^{-1}g(B) \\ & \geq c\left(\frac{g(\|A\|)}{(d+\|A\|)\|A\|}-\frac{g(d+\|A\|)-g(\|A\|)}{d(d+\|A\|)}\right)1_H\geq 0. \end{aligned} \tag{31}$$

Proof. Since g is operator monotone, then by Lemma 2 the function $f(t) := \frac{g(0)-g(t)}{t}$ is operator monotone on $(0,\infty)$ and by (21) we obtain

$$\frac{g(0)-g(B)}{B} - \frac{g(0)-g(A)}{A} \geq c\frac{\frac{g(0)-g(d+\|A\|)}{d+\|A\|} - \frac{g(0)-g(\|A\|)}{\|A\|}}{d} 1_H \geq 0. \tag{32}$$

Observe that

$$\begin{aligned} &\frac{g(0)-g(B)}{B} - \frac{g(0)-g(A)}{A} \\ &= g(0)\left(B^{-1} - A^{-1}\right) + A^{-1}g(A) - B^{-1}g(B) \end{aligned}$$

and

$$\begin{aligned} &\frac{g(0)-g(d+\|A\|)}{d+\|A\|} - \frac{g(0)-g(\|A\|)}{\|A\|} \\ &= \frac{[g(0)-g(d+\|A\|)]\|A\| - [g(0)-g(\|A\|)](d+\|A\|)}{(d+\|A\|)\|A\|} \\ &= \frac{g(0)\|A\| - g(d+\|A\|)\|A\| - g(0)d + g(\|A\|)d - g(0)\|A\| + g(\|A\|)\|A\|}{(d+\|A\|)\|A\|} \\ &= \frac{g(\|A\|)d - g(0)d + g(\|A\|)\|A\| - g(d+\|A\|)\|A\|}{(d+\|A\|)\|A\|} \\ &= d\frac{g(\|A\|)-g(0)}{(d+\|A\|)\|A\|} - \frac{g(d+\|A\|)-g(\|A\|)}{(d+\|A\|)}, \end{aligned}$$

which gives

$$\frac{\frac{g(0)-g(d+\|A\|)}{d+\|A\|} - \frac{g(0)-g(\|A\|)}{\|A\|}}{d} = \frac{g(\|A\|)-g(0)}{(d+\|A\|)\|A\|} - \frac{g(d+\|A\|)-g(\|A\|)}{d(d+\|A\|)}.$$

Then by (32) we get (30). ■

Corollary 8 *Assume that $g : [0,\infty) \to \mathbb{R}$ is operator monotone on $[0,\infty)$. If $A > 0$ and $B - A > 0$, then*

$$\begin{aligned} &g(0)\left(B^{-1} - A^{-1}\right) + A^{-1}g(A) - B^{-1}g(B) \\ &\geq \left\|(B-A)^{-1}\right\|^{-1} \\ &\times \left(\frac{g(\|A\|)-g(0)}{(\|B-A\|+\|A\|)\|A\|} - \frac{g(\|B-A\|+\|A\|)-g(\|A\|)}{\|B-A\|(\|B-A\|+\|A\|)}\right) 1_H \\ &\geq 0. \end{aligned} \tag{33}$$

If $g(0)=0,$ *then*

$$\begin{aligned}
&A^{-1}g(A)-B^{-1}g(B) \qquad (34)\\
&\geq \left\|(B-A)^{-1}\right\|^{-1}\\
&\times\left(\frac{g(\|A\|)}{(\|B-A\|+\|A\|)\|A\|}-\frac{g(\|B-A\|+\|A\|)-g(\|A\|)}{\|B-A\|(\|B-A\|+\|A\|)}\right)1_H\\
&\geq 0.
\end{aligned}$$

3 Some Examples

Consider the function $g(t)=t^r$, $r\in(0,1]$. This function is operator monotone and by (15) we have

$$\begin{aligned}
A^{r-1}-B^{r-1} &\geq \left[\|A\|^{r-1}-(\|A\|+c)^{r-1}\right]1_H \qquad (35)\\
&\geq \left[(\|B\|-c)^{r-1}-\|B\|^{r-1}\right]1_H>0
\end{aligned}$$

provided that $A>0$ and $B-A\geq c1_H>0$.

If $A>0$ and $B-A>0$, then

$$\begin{aligned}
A^{r-1}-B^{r-1} &\geq \left[\|A\|^{r-1}-\left(\|A\|+\left\|(B-A)^{-1}\right\|^{-1}\right)^{r-1}\right]1_H \qquad (36)\\
&\geq \left[\left(\|B\|-\left\|(B-A)^{-1}\right\|^{-1}\right)^{r-1}-\|B\|^{r-1}\right]1_H>0.
\end{aligned}$$

From (21) we obtain

$$B^r-A^r\geq c\frac{(d+\|A\|)^r-\|A\|^r}{d}1_H\geq 0 \qquad (37)$$

provided that there exist positive numbers $d>c>0$ such that condition (20) is satisfied.

If $A>0$ and $B-A>0$, then

$$\begin{aligned}
B^r-A^r &\geq \left\|(B-A)^{-1}\right\|^{-1}\frac{(\|B-A\|+\|A\|)^r-\|A\|^r}{\|B-A\|}1_H \qquad (38)\\
&\geq \left\|(B-A)^{-1}\right\|^{-1}\frac{\|B\|^r-\|A\|^r}{\|B-A\|}1_H\geq 0.
\end{aligned}$$

From (30) we have

$$\begin{aligned}
&A^{r-1}-B^{r-1} \qquad (39)\\
&\geq c\left(\frac{\|A\|^{r-1}}{d+\|A\|}-\frac{(d+\|A\|)^r-\|A\|^r}{d(d+\|A\|)}\right)1_H\geq 0,
\end{aligned}$$

provided that there exist positive numbers $d > c > 0$ such that condition (20) is satisfied.

If $A > 0$ and $B - A > 0$, then

$$\begin{aligned} A^{r-1} - B^{r-1} &\geq \left\| (B-A)^{-1} \right\|^{-1} \\ &\times \left(\frac{\|A\|^{r-1}}{\|B-A\| + \|A\|} - \frac{(\|B-A\| + \|A\|)^r - \|A\|^r}{\|B-A\| (\|B-A\| + \|A\|)} \right) 1_H \\ &\geq 0. \end{aligned} \tag{40}$$

Consider the function $g(t) = \ln(t+1)$, which is operator monotone on $[0, \infty)$ and $g(0) = 0$. By Lemma 2 we get that the function $f(t) = -t^{-1}\ln(t+1)$ is operator monotone on $(0, \infty)$.

From (15) we get

$$\begin{aligned} &A^{-1}\ln(A + 1_H) - B^{-1}\ln(B + 1_H) \\ &\geq \left(\frac{\ln(\|A\| + 1)}{\|A\|} - \frac{\ln(\|A\| + 1 + c)}{\|A\| + c} \right) 1_H \\ &\geq \left(\frac{\ln(\|B\| + 1 - c)}{\|B\| - c} - \frac{\ln(\|B\| + 1)}{\|B\|} \right) 1_H > 0 \end{aligned} \tag{41}$$

provided that $A > 0$ and $B - A \geq c1_H > 0$.

If $A > 0$ and $B - A > 0$, then

$$\begin{aligned} &A^{-1}\ln(A + 1_H) - B^{-1}\ln(B + 1_H) \\ &\geq \left(\frac{\ln(\|A\| + 1)}{\|A\|} - \frac{\ln\left(\|A\| + 1 + \left\| (B-A)^{-1} \right\|^{-1} \right)}{\|A\| + \left\| (B-A)^{-1} \right\|^{-1}} \right) 1_H \\ &\geq \left(\frac{\ln\left(\|B\| + 1 - \left\| (B-A)^{-1} \right\|^{-1} \right)}{\|B\| - \left\| (B-A)^{-1} \right\|^{-1}} - \frac{\ln(\|B\| + 1)}{\|B\|} \right) 1_H > 0. \end{aligned} \tag{42}$$

From (21) we derive

$$\ln(B + 1_H) - \ln(A + 1_H) \geq c\frac{\ln(d + \|A\| + 1) - \ln(\|A\| + 1)}{d} 1_H \geq 0 \tag{43}$$

provided that there exist positive numbers $d > c > 0$ such that the condition (20) is satisfied.

If $A > 0$ and $B - A > 0$, then

$$\begin{aligned} &\ln(B + 1_H) - \ln(A + 1_H) \\ &\geq \left\| (B-A)^{-1} \right\|^{-1} \frac{\ln(\|B-A\| + \|A\| + 1) - \ln(\|A\| + 1)}{\|B-A\|} 1_H \\ &\geq \left\| (B-A)^{-1} \right\|^{-1} \frac{\ln(\|B\| + 1) - \ln(\|A\| + 1)}{\|B-A\|} 1_H \geq 0. \end{aligned} \tag{44}$$

From (30) we have

$$
\begin{aligned}
& A^{-1}\ln(A+1) - B^{-1}\ln(B+1) \\
& \geq c\left(\frac{\ln(\|A\|+1)}{(d+\|A\|)\|A\|} - \frac{\ln(d+\|A\|+1)-\ln(\|A\|+1)}{d(d+\|A\|)}\right)1_H \geq 0
\end{aligned}
\tag{45}
$$

provided that there exist positive numbers $d > c > 0$ such that the condition (20) is satisfied.

Finally, from (34) we derive

$$
\begin{aligned}
& A^{-1}\ln(A+1_H) - B^{-1}\ln(B+1_H) \\
& \geq \left\|(B-A)^{-1}\right\|^{-1} \\
& \times\left(\frac{\ln(\|A\|+1)}{(\|B-A\|+\|A\|)\|A\|} - \frac{\ln(\|B-A\|+\|A\|+1)-\ln(\|A\|+1)}{\|B-A\|(\|B-A\|+\|A\|)}\right)1_H \\
& \geq 0,
\end{aligned}
\tag{46}
$$

provided $A > 0$ and $B - A > 0$.

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