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Decay Rates of Solutions for Non-Degenerate Kirchhoff Type Dissipative Wave Equations

By

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Abstract

Consider the Cauchy problem for the non-degenerate Kirchhoff type dissipative wave equations with the initial data belonging to $(H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times (H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$. Using the Fourier transform method in the $L^2 \cap L^1$ -frame, we can improve the decay rates of the energies given by the energy method of the L^2 -frame.

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1 Introduction

In this paper we study on decay estimates of the solution $u(t)$ for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + a(\|A^{1/2}u(t)\|^2) Au + u' = 0 & \text{in } \mathbb{R}^N \times [0, \infty), \\ u(x, 0) = u_0(x) \quad \text{and} \quad u'(x, 0) = u_1(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $u = u(x, t)$ is an unknown real value function, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N)$, ρ is positive constant, and $\|\cdot\|$ is the usual norm of $L^2(\mathbb{R}^N)$, that is,

$$\|f\| = \left(\int_{\mathbb{R}^N} |f(x)|^2 dx \right)^{\frac{1}{2}} \quad \text{for } f \in L^2(\mathbb{R}^N).$$

For the non-local nonlinear term $a(M) \in C^0([0, \infty)) \cap C^2((0, \infty))$, we assume that as follows :

Hyp.1 $K_1 \leq a(M) \leq K_2 + K_3 M^\gamma$ for $M \geq 0$

Hyp.2 $0 \leq a'(M)M \leq K_4 a(M)$ for $M > 0$

Hyp.3 $a'(M)M + |a''(M)|M^2 \leq K_5 M^\gamma$ for $M > 0$

with $\gamma > 0$ and $K_j > 0$ ($j = 1, 2, 3, 4, 5$).

For a typical example, we have

$$a(M) = 1 + M^\gamma \quad \text{with} \quad \gamma > 0.$$

Equation (1.1) describes small amplitude vibrations of an elastic string when the dimension is one (see Kirchhoff [5] for the original equation, and [1], [2], [8]).

In the previous papers [12] and [13], we have derived the following decay estimates of the solution $u(t)$ of (1.1) (see [3], [9], [11] for degenerate equations).

Theorem 1.1 *Suppose that the initial data (u_0, u_1) belong to $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^1)$ and $u_0 \neq 0$, and Hyp.1 and Hyp.2 are fulfilled. Then, there exists $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$ the problem (1.1) admits a unique global solution $u(t)$ in the class $C^0([0, \infty); H^2(\mathbb{R}^N)) \cap C^1([0, \infty); H^1(\mathbb{R}^N)) \cap C^2([0, \infty); L^2(\mathbb{R}^N))$ satisfying $\|A^{1/2}u(t)\|^2 \geq Ce^{-\alpha t}$ with some $\alpha > 0$.*

Moreover, if Hyp.3 is fulfilled, then the solution $u(t)$ satisfies

$$\|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1}, \quad (1.2)$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \leq C(1+t)^{-\theta}, \quad (1.3)$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \leq C(1+t)^{-\sigma} \quad \text{for } t \geq 0, \quad (1.4)$$

where

$$\theta = \min\{2, 1 + 2\gamma\},$$

$$\sigma = \min\{3, (1 + \gamma)(1 + 2\gamma)\}.$$

In order to derive (1.2)–(1.4) in the previous paper [12], we use the functions $E(t)$, $F(t)$, $L(t)$ defined by

$$E(t) = \rho\|u'(t)\|^2 + \int_0^{M(t)} a(\mu) d\mu, \quad M(t) = \|A^{1/2}u(t)\|^2, \quad (1.5)$$

$$F(t) = \rho\|A^{1/2}u'(t)\|^2 + a(M(t))\|Au(t)\|^2, \quad (1.6)$$

$$L(t) = \rho\|u''(t)\|^2 + a(M(t))\|A^{1/2}u'(t)\|^2 + \frac{a'(M(t))}{2}|M'(t)|^2, \quad (1.7)$$

respectively.

Moreover, we have derived that the functions $E(t)$, $F(t)$, $L(t)$ satisfy the

following inequalities :

$$\sup_{t \leq s \leq t+1} E(s)^2 \leq C \left(E(t) + \sup_{t \leq s \leq t+1} \|u(s)\|^2 \right) (E(t) - E(t+1)), \quad (1.8)$$

$$\begin{aligned} \sup_{t \leq s \leq t+1} F(s)^2 &\leq C \left(F(t) + \sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} M(s) \right) (F(t) - F(t+1)) \\ &+ C \left(\sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} M(s) \right) \sup_{t \leq s \leq t+1} f(s)^2, \end{aligned} \quad (1.9)$$

where

$$f(t)^2 = \min\{M(t)^{2\gamma+1}, M(t)^{2\gamma}\|Au(t)\|^2\}, \quad (1.10)$$

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(s)^2 &\leq C \left(L(t) + \sup_{t \leq s \leq t+1} h(s)^2 + \sup_{t \leq s \leq t+1} \|u'(s)\|^2 \right) (L(t) - L(t+1)) \\ &+ C \left(\sup_{t \leq s \leq t+1} h(s)^2 + \sup_{t \leq s \leq t+1} \|u'(s)\|^2 \right) \sup_{t \leq s \leq t+1} h(s)^2, \end{aligned} \quad (1.11)$$

where

$$h(t)^2 = \begin{cases} \|u'(t)\|^{2\gamma} \|A^{1/2}u'(t)\|^2 & \text{if } 0 < \gamma < \frac{1}{2} \\ M(t)^{\gamma-\frac{1}{2}} \|u'(t)\| \|A^{1/2}u'(t)\|^2 & \text{if } \gamma \geq \frac{1}{2}. \end{cases} \quad (1.12)$$

When the initial data (u_0, u_1) belong to $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$, we can derive the decay rate of L^2 -norm of the solution $u(t)$, and then, we will improve the decay rates of (1.2)–(1.4),

Our main result is as follows.

Theorem 1.2 *If the initial data (u_0, u_1) belong to $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ in addition to the assumption of Theorem 1.1, then the solution $u(t)$ of (1.1) satisfies*

$$\|u(t)\|^2 \leq C(1+t)^{-\eta}, \quad \|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1-\eta}, \quad (1.13)$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \leq C(1+t)^{-\omega}, \quad (1.14)$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \leq C(1+t)^{-\mu} \quad \text{for } t \geq 0, \quad (1.15)$$

where

$$\eta = \min\{N/2, 2\},$$

$$\omega = \min\{2 + \eta, (1 + 2\gamma)(1 + \eta)\},$$

$$\mu = \min\{3 + \eta, (1 + \gamma)(2 + \eta), (1 + \gamma)(1 + 2\gamma)(1 + \eta)\}.$$

The notations we use in the paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Integral Forms

We denote the Fourier transform of $g(x)$ by

$$\mathcal{F}(g(x))(\xi) \equiv \widehat{g}(\xi) \equiv (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} g(x) dx,$$

where $\xi \cdot x = \sum_{j=1}^N \xi_j x_j$.

Through the Fourier transform, we can rewrite (1.1) to the following equation :

$$\begin{cases} \rho \widehat{u}'' + \widehat{u}' + a(0)|\xi|^2 \widehat{u} = f(M(t)) \widehat{A}u & \text{in } \mathbb{R}_\xi^N \times [0, \infty), \\ \widehat{u}(\xi, 0) = \widehat{u}_0(\xi) \quad \text{and} \quad \widehat{u}'(\xi, 0) = \widehat{u}_1(\xi) & \text{in } \mathbb{R}_\xi^N, \end{cases} \quad (2.1)$$

where $f(M) = a(0) - a(M)$. Then, we obtain the integral form for (2.1) :

$$\widehat{u}(\xi, t) = \widehat{u}_L(\xi, t) + \widehat{u}_N(\xi, t) \quad (2.2)$$

where we set

$$\widehat{u}_L(\xi, t) = \frac{1}{2} (\phi_1(\xi, t) + \phi_2(\xi, t)) \widehat{u}_0(\xi) + \phi_2(\xi, t) \widehat{u}_1(\xi), \quad (2.3)$$

$$\widehat{u}_N(\xi, t) = \int_0^t \phi_2(\xi, t-s) f(M(s)) \widehat{A}u(\xi, s) ds, \quad (2.4)$$

and

$$\phi_1(\xi, t) = e^{\frac{-1+\lambda}{2\rho}t} + e^{\frac{-1-\lambda}{2\rho}t} = \begin{cases} 2e^{-\frac{t}{2\rho}} \cosh \frac{\lambda t}{2\rho} & \text{if } \xi \in X_1 \cup X_2, \\ 2e^{-\frac{t}{2\rho}} \cos \frac{\sigma t}{2\rho} & \text{if } \xi \in X_3 \cup X_4, \end{cases}$$

$$\phi_2(\xi, t) = \frac{1}{\lambda} \left(e^{\frac{-1+\lambda}{2\rho}t} - e^{\frac{-1-\lambda}{2\rho}t} \right) = \begin{cases} \frac{t}{\rho} e^{-\frac{t}{2\rho}} \frac{2\rho}{\lambda t} \sinh \frac{\lambda t}{2\rho} & \text{if } \xi \in X_1 \cup X_2, \\ \frac{t}{\rho} e^{-\frac{t}{2\rho}} \frac{2\rho}{\lambda t} \sin \frac{\sigma t}{2\rho} & \text{if } \xi \in X_3 \cup X_4, \end{cases}$$

with $\lambda = \sqrt{1 - 4\rho a(0)|\xi|^2}$ and $\sigma = \sqrt{4\rho a(0)|\xi|^2 - 1}$,

$$\begin{aligned} X_1 &= \{\xi \mid |\xi| < (2^3 \rho a(0))^{-\frac{1}{2}}\}, \\ X_2 &= \{\xi \mid (2^3 \rho a(0))^{-\frac{1}{2}} \leq |\xi| < (2^2 \rho a(0))^{-\frac{1}{2}}\}, \\ X_3 &= \{\xi \mid (2^2 \rho a(0))^{-\frac{1}{2}} \leq |\xi| < (2\rho a(0))^{-\frac{1}{2}}\}, \\ X_4 &= \{\xi \mid (2\rho a(0))^{-\frac{1}{2}} \leq |\xi|\}. \end{aligned}$$

We denote the cut-off functions $\chi_j(\cdot)$ for $j = 1, 2, 3, 4$ by

$$\chi_j(\xi) = \begin{cases} 1 & \text{if } \xi \in X_j, \\ 0 & \text{if } \xi \notin X_j. \end{cases}$$

Proposition 2.1 *The solution $u(t)$ of (1.1) satisfies*

$$\begin{aligned} \|u(t)\| &\leq C(1+t)^{-\frac{N}{4}} + C \int_0^t (1+t-s)^{-1} |f(M(s))| \|u(s)\| ds \\ &\quad + \int_0^t e^{-\frac{t-s}{2\rho}} |f(M(s))| \|Au(s)\| ds \quad \text{for } t \geq 0. \end{aligned} \quad (2.5)$$

Proof. (1) First, we estimate the linear part (2.3). From the Parseval identity, we observe

$$\begin{aligned} \|u_L(t)\| &\leq C \sum_{j=1}^4 \|\chi_j(\xi)(|\phi_1(\xi, t)| + |\phi_2(\xi, t)|)(|\widehat{u_0}(\xi)| + |\widehat{u_1}(\xi)|)\| \\ &\leq CI_1(t)(\|\widehat{u_0}(\xi)\|_{L^\infty} + \|\widehat{u_1}(\xi)\|_{L^\infty}) + C \sum_{j=2}^4 I_j(t)(\|\widehat{u_0}(\xi)\| + \|\widehat{u_1}(\xi)\|) \\ &\leq CI_1(t)(\|u_0\|_{L^1} + \|u_1\|_{L^1}) + C \sum_{j=2}^4 I_j(t)(\|u_0\| + \|u_1\|) \end{aligned}$$

where we set

$$I_j(t) = \begin{cases} \|\chi_1(\xi)(|\phi_1(\xi, t)| + |\phi_2(\xi, t)|)\| & \text{if } j = 1, \\ \sup_{\xi \in X_j} (|\phi_1(\xi, t)| + |\phi_2(\xi, t)|) & \text{if } j = 2, 3, 4. \end{cases}$$

(i) When $\xi \in X_1$, we observe that $1/\sqrt{2} < \lambda \leq 1$ and $-1 + \lambda \leq -2\rho a(0)|\xi|^2$ and

$$\sup_{\xi \in X_1} (|\phi_1(\xi, t)| + |\phi_2(\xi, t)|) \leq C e^{-a(0)|\xi|^2 t},$$

and hence,

$$\begin{aligned} I_1(t) &\leq C \|\chi_1(\xi) e^{-a(0)|\xi|^2 t}\| \leq C \left(\int_{X_1} e^{-2a(0)|\xi|^2 t} d\xi \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^\delta |\xi|^{N-1} e^{-2a(0)|\xi|^2 t} d|\xi| \right)^{\frac{1}{2}} \leq C(1+t)^{-\frac{N}{4}} \end{aligned}$$

with $\delta = (2^3 \rho a(0))^{-\frac{1}{2}} > 0$.

(ii) When $\xi \in X_2$, we observe that $0 < \lambda \leq 1/\sqrt{2}$ and

$$\begin{aligned} \sup_{\xi \in X_2} |\phi_1(\xi, t)| &\leq Ce^{-(1-\frac{1}{\sqrt{2}})\frac{t}{2\rho}}, \\ \sup_{\xi \in X_2} |\phi_2(\xi, t)| &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_2} \left| \frac{2\rho}{\lambda t} \sinh \frac{\lambda t}{2\rho} \right| \\ &= Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_2} \left| \frac{2\rho}{\lambda t} \int_0^1 \frac{d}{d\theta} \left(\sinh \frac{\lambda t}{2\rho} \theta \right) d\theta \right| \\ &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_2} \left| \cosh \frac{\lambda t}{2\rho} \right| \leq Cte^{-(1-\frac{1}{\sqrt{2}})\frac{t}{2\rho}}, \end{aligned}$$

and hence,

$$I_2(t) \leq Ce^{-\nu t}$$

with some $0 < \nu \leq (1 - 1/\sqrt{2})/(2\rho)$.

(iii) When $\xi \in X_3$, we observe that $0 \leq \sigma < \sqrt{2}$ and

$$\begin{aligned} \sup_{\xi \in X_3} |\phi_1(\xi, t)| &\leq Ce^{-\frac{t}{2\rho}}, \\ \sup_{\xi \in X_3} |\phi_2(\xi, t)| &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_3} \left| \frac{2\rho}{\sigma t} \sin \frac{\sigma t}{2\rho} \right| \\ &= Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_3} \left| \frac{2\rho}{\sigma t} \int_0^1 \frac{d}{d\theta} \left(\sin \frac{\sigma t}{2\rho} \theta \right) d\theta \right| \leq Cte^{-\frac{t}{2\rho}}, \end{aligned}$$

and hence,

$$I_3(t) \leq Ce^{-\nu t}$$

with some $0 < \nu \leq 1/(2\rho)$.

(iv) When $\xi \in X_4$, we observe that $\sigma \geq \sqrt{2}$ and

$$\sup_{\xi \in X_4} (|\phi_1(\xi, t)| + |\phi_2(\xi, t)|) \leq Ce^{-\frac{t}{2\rho}} \quad \text{and} \quad I_4(t) \leq Ce^{-\frac{t}{2\rho}}.$$

Therefore, we obtain

$$\|u_L(t)\| \leq C(1+t)^{-\frac{N}{4}} (\|u_0\|_{L^1} + \|u_1\|_{L^1}) + Ce^{-\nu t} (\|u_0\| + \|u_1\|) \quad (2.6)$$

with some $\nu > 0$.

(2) Next, we estimate the nonlinear part (2.4). From the Parseval identity, we observe

$$\|u_N(t)\| \leq \int_0^t \|\phi_2(\xi, t-s) \widehat{A}u(\xi, s)\| \|f(M(s))\| ds$$

and

$$\begin{aligned}
 \|\phi_2(\xi, t-s)\widehat{Au}(\xi, s)\| &\leq \sum_{j=1}^4 \|\chi_j(\xi)\phi_2(\xi, t-s)\widehat{Au}(\xi, s)\| \\
 &\leq \sum_{j=1}^3 \|\chi_j(\xi)|\xi|^2\phi_2(\xi, t-s)\widehat{u}(\xi, s)\| + \|\chi_4(\xi)\phi_2(\xi, t-s)\widehat{Au}(\xi, s)\| \\
 &\leq C \sum_{j=1}^3 J_j(t-s)\|u(s)\| + CJ_4(t-s)\|Au(s)\|
 \end{aligned}$$

where

$$J_j(t) = \begin{cases} \sup_{\xi \in X_1} |\xi|^2 |\phi_2(\xi, t)| & \text{if } j = 1, \\ \sup_{\xi \in X_j} |\phi_2(\xi, t)| & \text{if } j = 2, 3, 4. \end{cases}$$

(i) When $\xi \in X_1$, we observe $1/\sqrt{2} < \lambda \leq 1$ and $-1 + \lambda \leq -2\rho a(0)|\xi|^2$ and

$$\sup_{\xi \in X_1} |\phi_2(\xi, t)| \leq Ce^{-a(0)|\xi|^2 t},$$

and hence,

$$J_1(t) \leq C \sup_{\xi \in X_1} |\xi|^2 e^{-a(0)|\xi|^2 t} \leq C(1+t)^{-1}.$$

(ii) When $\xi \in X_2$, we observe that $0 < \lambda \leq 1/\sqrt{2}$ and

$$\sup_{\xi \in X_2} |\phi_2(\xi, t)| \leq Cte^{-(1-\frac{1}{\sqrt{2}})\frac{t}{2\rho}} \quad \text{and} \quad J_2(t) \leq Ce^{-\nu t}$$

with some $0 < \nu \leq (1 - 1/\sqrt{2})/(2\rho)$.

(iii) When $\xi \in X_3$, we observe $0 \leq \sigma < \sqrt{2}$ and

$$\sup_{\xi \in X_3} |\phi_2(\xi, t)| \leq Cte^{-\frac{t}{2\rho}} \quad \text{and} \quad J_3(t) \leq Ce^{-\nu t}$$

with some $0 < \nu \leq 1/(2\rho)$.

(iv) When $\xi \in X_4$, we observe that $\sigma \geq \sqrt{2}$ and

$$\sup_{\xi \in X_4} |\phi_2(\xi, t)| \leq Ce^{-\frac{t}{2\rho}} \quad \text{and} \quad J_4(t) \leq Ce^{-\frac{t}{2\rho}}.$$

Therefore, we obtain

$$\begin{aligned}
 \|u_N(t)\| &\leq C \int_0^t (1+t-s)^{-1} |f(M(s))| \|u(s)\| ds \\
 &\quad + C \int_0^t e^{-\frac{t-s}{2\rho}} |f(M(s))| \|Au(s)\| ds.
 \end{aligned} \tag{2.7}$$

Thus, from (2.6) and (2.7) we conclude (2.5). \square

3 Energy Decay

Proposition 3.1 *Suppose that the initial data (u_0, u_1) belong to $(H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times (H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$. Then, the solution $u(t)$ of (1.1) satisfies*

$$\|u(t)\|^2 \leq C(1+t)^{-\eta}, \quad (3.1)$$

$$E(t) = \rho \|u'(t)\|^2 + \int_0^{M(t)} a(\mu) d\mu \leq C(1+t)^{-1-\eta}, \quad (3.2)$$

$$M(t) = \|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1-\eta}, \quad (3.3)$$

where $\eta = \min\{N/2, 2\}$.

Proof. We observe from Hyp.2 that

$$\begin{aligned} F(M) &= a(0) - a(M) = - \int_0^1 \frac{d}{dt} a(\theta M) d\theta \\ &\leq \int_0^1 K_4 \frac{(\theta M)^\gamma}{\theta M} M d\theta \leq CM^\gamma, \end{aligned}$$

and from (2.5) that

$$\begin{aligned} \|u(t)\| &\leq C(1+t)^{-\frac{N}{4}} + C \int_0^t (1+t-s)^{-1} M(s)^\gamma \|u(s)\| ds \\ &\quad + C \int_0^t e^{-\frac{t-s}{2\rho}} M(s)^\gamma \|Au(s)\| ds, \end{aligned} \quad (3.4)$$

and from (1.2) and (1.3) that

$$\begin{aligned} \|u(t)\| &\leq C(1+t)^{-\frac{N}{4}} + C \int_0^t (1+t-s)^{-1} (1+s)^{-\gamma} \|u(s)\| ds \\ &\quad + C \int_0^t e^{-\frac{t-s}{2\rho}} (1+s)^{-(\frac{3}{2}+\gamma)} ds \\ &\leq C(1+t)^{-\frac{\eta}{2}} + C \int_0^t (1+t-s)^{-1} (1+s)^{-\gamma} \|u(s)\| ds \end{aligned}$$

with $\eta = \min\{N/2, 2\}$.

Setting

$$\mu(t) = \sup_{0 \leq s \leq t} (1+s)^{\frac{\eta}{2}} \|u(s)\|,$$

since $\|u(t)\|$ is bounded, we have that for $0 \leq \tau \leq t$ and any $\varepsilon > 0$,

$$\begin{aligned} &(1+\tau)^{\frac{\eta}{2}} \|u(\tau)\| \\ &\leq C + C(1+\tau)^{\frac{\eta}{2}} \left(\int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\tau} \right) (1+\tau-s)^{-1} (1+s)^{-\gamma-\frac{\eta}{2}(1-\varepsilon)} ds \mu(\tau)^{1-\varepsilon} \\ &\leq C + C\mu(t)^{1-\varepsilon}, \end{aligned}$$

and hence,

$$\mu(t) \leq C \quad \text{or} \quad \|u(t)\|^2 \leq C(1+t)^{-\eta}. \quad (3.5)$$

Using (1.8) together with (3.5) again, we have

$$\sup_{t \leq s \leq t+1} E(s)^2 \leq C (E(t) + (1+t)^{-\eta}) (E(t) - E(t+1)). \quad (3.6)$$

Applying Lemma 3.2 to (3.6), we obtain

$$E(t) \leq C(1+t)^{-1-\eta}$$

which implies (3.2) and (3.3). \square

We used the following Lemma for the energy estimate (see [4], [6], [7], [10] for the proof).

Lemma 3.2 *Let $\phi(t)$ be a non-negative function on $[0, \infty)$ and satisfy*

$$\sup_{t \leq s \leq t+1} \phi(s)^{1+\alpha} \leq (k_0 \phi(t)^\alpha + k_1(1+t)^{-\beta})(\phi(t) - \phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants $k_0, k_1, k_2 \geq 0$, $\alpha > 0$, $\beta > -1$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \leq C_0(1+t)^{-\theta}, \quad \theta = \min\left\{\frac{1+\beta}{\alpha}, \frac{\gamma}{1+\alpha}\right\}$$

for $t \geq 0$ with some constant C_0 depending on $\phi(0)$.

4 Improved Decay Estimates

Proposition 4.1 *Under the assumptions of Theorem 1.1 and Proposition 3.1, the solution $u(t)$ of (1.1) satisfies*

$$F(t) = \rho \|A^{1/2}u'(t)\|^2 + a(M(t)) \|Au(t)\|^2 \leq C(1+t)^{-\omega}, \quad (4.1)$$

$$\|u'(t)\|^2 + \|u''(t)\|^2 \leq C(1+t)^{-\omega}, \quad (4.2)$$

where $\omega = \min\{2 + \eta, (1 + 2\gamma)(1 + \eta)\}$.

Proof. From (1.6), (1.10), (3.3) we observe

$$f(t) \leq C(1+t)^{-(1+2\gamma)(1+\eta)} \quad \text{and} \quad f(t) \leq C(1+t)^{-2\gamma(1+\eta)} F(t). \quad (4.3)$$

Using (1.9) together with (3.3) and (4.3), we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} F(s)^2 &\leq C \left(F(t) + (1+t)^{-(1+\eta)} \right) (F(t) - F(t+1)) \\ &\quad + C(1+t)^{-(1+2\gamma)(1+\eta)} \sup_{t \leq s \leq t+1} F(s) \end{aligned}$$

and from the Young inequality, we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} F(s)^2 &\leq C \left(F(t) + (1+t)^{-(1+\eta)} \right) (F(t) - F(t+1)) \\ &\quad + C(1+t)^{-2(1+2\gamma)(1+\eta)}. \end{aligned} \quad (4.4)$$

Thus, applying Lemma 3.2 to (4.4), we have

$$F(t) \leq C(1+t)^{-\omega} \quad \text{with} \quad \omega = \min\{2 + \eta, (1 + 2\gamma)(1 + \eta)\}.$$

Multiplying (1.1) by $2u'(t)$ and integrating it over \mathbb{R}^N , we have

$$\rho \frac{d}{dt} \|u'(t)\|^2 + 2\|u'(t)\|^2 = -2a(M(t))(Au(t), u(t)),$$

and from the Young inequality, we have

$$\rho \frac{d}{dt} \|u'(t)\|^2 + \|u'(t)\|^2 \leq a(M(t))^2 \|Au(t)\|^2 \leq C(1+t)^{-\omega},$$

and hence, we derive the desired estimate (4.2). \square

Proposition 4.2 *Under the assumptions of Theorem 1.1 and Proposition 3.1, the solution $u(t)$ of (1.1) satisfies*

$$\begin{aligned} L(t) &= \rho \|u''(t)\|^2 + a(M(t)) \|Au'(t)\|^2 + \frac{a'(M(t))}{2} |M'(t)|^2 \\ &\leq C(1+t)^{-\mu}, \end{aligned} \quad (4.5)$$

where $\mu = \{3 + \eta, (1 + \gamma)(2 + \eta), (1 + \gamma)(1 + 2\gamma)(1 + \eta)\}$.

Proof. (i) When $0 < \gamma < \frac{1}{2}$, we observe from (1.12), (4.1), (4.2) that

$$h(t)^2 \leq C(1+t)^{-(1+\gamma)\omega} \quad \text{and} \quad h(t)^2 \leq C(1+t)^{-\gamma\omega} L(t). \quad (4.6)$$

Using (1.11) together with (4.2) and (4.6), we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(s)^2 &\leq C \left(L(t) + (1+t)^{-\omega} \right) (L(t) - L(t+1)) \\ &\quad + C(1+t)^{-(1+\gamma)\omega} \sup_{t \leq s \leq t+1} L(s) \end{aligned}$$

and from the Young inequality, we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(s)^2 &\leq C \left(L(t) + (1+t)^{-\omega} \right) (L(t) - L(t+1)) \\ &\quad + C(1+t)^{-2(1+\gamma)\omega} \end{aligned} \quad (4.7)$$

Thus, applying Lemma 3.2 to (4.7), we have

$$L(t) \leq C(1+t)^{-\mu} \quad \text{for } \mu = \min\{1+\omega, (1+\gamma)\omega\},$$

which implies (4.5) for $0 < \gamma < \frac{1}{2}$.

(ii) When $\gamma \geq \frac{1}{2}$, we observe from (1.12), (3.3), (4.1), (4.2) that

$$h(t)^2 \leq C(1+t)^{-\frac{3}{2}\omega} \quad \text{and} \quad h(t)^2 \leq C(1+t)^{-\frac{1}{2}\omega} L(t). \quad (4.8)$$

Using (1.11) together with (4.2) and (4.8), we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(s)^2 &\leq C(L(t) + (1+t)^{-\omega})(L(t) - L(t+1)) \\ &\quad + C(1+t)^{-\frac{3}{2}\omega} \sup_{t \leq s \leq t+1} L(s) \end{aligned}$$

and from the Young inequality, we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(s)^2 &\leq C(L(t) + (1+t)^{-\omega})(L(t) - L(t+1)) \\ &\quad + C(1+t)^{-3\omega}. \end{aligned} \quad (4.9)$$

Thus, applying Lemma 3.2 to (4.9), we have

$$L(t) \leq C(1+t)^{-\mu} \quad \text{with } \mu = \left\{1+\omega, \frac{3}{2}\omega\right\} = 1+\omega,$$

which implies (4.5) for $\gamma \geq \frac{1}{2}$. \square

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