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Axiomatic Method of Measure and Integration (V). Definition and Existence Theorem of the RS-measure

(Yoshifumi Ito, “RS-integral and LS-integral”, Chapter 3)

By

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Abstract

In this paper, we define the RS-measure on $\boldsymbol{R}^d$, $(d \geq 1)$ by prescribing the complete system of axioms. Then we prove the existence theorem of the RS-measure and determine all the RS-measures. This is a new result.

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Introduction

This paper is the part V of the series of papers on the axiomatic method of measure and integration on the Euclidean space. As for the details, we refer to Ito [6], [14]. Further we refer to Ito [1]∼[5], [7]∼[13] and [15]∼ [21].

In this paper, we study the definition of the Riemann-Stieltjes measure on the d-dimensional Euclidean space and prove its existence theorem and study the fundamental properties of the d-dimensional Riemann-Stieltjes measurable sets. Here we assume $d \geq 1$.

For simplicity, we say that the d-dimensional Riemann-Stieltjes measure on the d-dimensional Euclidean space is the d-dimensional RS-measure. Further, for simplicity, we say that a d-dimensional Riemann-Stieltjes measurable set is a d-dimensional RS-measurable set.

In this paper, in the sequel, we happen to omit the adjective "d-dimensional".

A RS-measure is conditionally completely additive real-valued measure and all conditionally completely additive real-valued measure are the RS-measures.

Thereby the set of all conditionally completely additive real-valued measure on the Euclidean space is determined.

Namely this set is the set of all RS-measures on the Euclidean space.

The most fundamental property of the measure considered here is the fact that this measure is an additive set function defined on a certain family of sets.

We can characterize the whole of such measures by the complete system of axioms which is the conditions of defining the RS-measure and prove the existence theorem of such a RS-measure.

Thereby we have succeeded in characterizing the RS-measures on $\boldsymbol{R}^d$ on the whole.

We prove that a RS-measure is determined by the corresponding distribution function. Thereby we give the characterization of the RS-measure.

A Jordan measure is an additive set function defined on the ring of all Jordan measurable sets and is characterized by the conditions such as the positivity, the conditionally complete additivity and the invariance with respect to the congruence transformation. Therefore the value of the Jordan measure is determined by deciding the measure of the unit figure as a unit quantity.

The RS-measure considered in this paper is an additive set function on the ring of all RS-measurable sets and it is the measure characterized by only two conditions such as the real-valuedness and the conditionally complete additivity. Further the Jordan measure is the special example of the RS-measure.

Here I express my heartfelt gratitude to my wife Mutuko for help of typesetting this manuscript.

1 Intervals and blocks of intervals

In this section, we prepare the terminology which is necessary for giving the system of axioms of this d-dimensional RS-measure.

At first we study the intervals and the blocks of intervals as the fundamental subsets of the d-dimensional Euclidean space $\boldsymbol{R}^d$.

We define a subset E of $\boldsymbol{R}^d$ is an interval if E is a direct product set

$$E = \prod_{j=1}^{d} I_j.$$

Here $I_1,\ I_2,\ \cdots,\ I_d$ are the intervals of $\boldsymbol{R}$ and each one of them is a subset of $\boldsymbol{R}$ which is equal to either one of the intervals of $\boldsymbol{R}$ of the forms:

$$\begin{aligned} &(a,\ b) = \{x;\ a < x < b\},\ [a,\ b) = \{x;\ a \leq x < b\}, \\ &(a,\ b] = \{x;\ a < x \leq b\},\ [a,\ b] = \{x;\ a \leq x \leq b\}. \end{aligned} \tag{1.1}$$

Here a and b are equal to a real number or $-\infty$ or ∞. Then $-\infty$ and ∞ are not a point of the intervals $I_j,\ (1 \leq j \leq d)$.

Then we denote the interior of E as

$$E^\circ = \prod_{j=1}^{d} I_j^\circ. \tag{1.2}$$

Here I_j° denotes the interior of $I_j,\ (1 \leq j \leq d)$ and we consider that the empty set $\emptyset$ is the interval.

We define that a subset E of $\boldsymbol{R}^d$ is a **block of intervals** if E is equal to the direct sum

$$E = \bigcup_{p=1}^{n} E_p = \sum_{p=1}^{n} E_p = E_1 + E_2 + \cdots + E_n \tag{1.3}$$

of a certain family of mutually disjoint intervals $E_1,\ E_2,\ \cdots,\ E_n$. We say that the formula (1.3) is a division of E with the intervals.

In general, the choice of the division of a block of intervals has infinitely many varieties.

Here we denote the family of all blocks of intervals of $\boldsymbol{R}^d$ as $\mathcal{R}$. Then $\mathcal{R}$ is the smallest ring which includes all intervals of $\boldsymbol{R}^d$.

Namely $\mathcal{R}$ is a ring of sets generated by the family $\mathcal{P}$ of all intervals of $\boldsymbol{R}^d$.

Then we have the following theorem.

Theorem 1.1 *Assume that $\mathcal{R}$ is the family of all blocks of intervals of $\boldsymbol{R}^d$. Then we have the following conditions* (1)∼(3):

(1) $\emptyset \in \mathcal{R}$ *holds.*

(2) *For $A \in \mathcal{R}$, we have*

$$A^c = \{x \in \boldsymbol{R}^d;\ x \notin A\} \in \mathcal{R}.$$

(3) *For $A,\ B \in \mathcal{R}$, we have $A \cup B \in \mathcal{R}$.*

Corollary 1.1 *Assume that $\mathcal{R}$ is the same as in Theorem* 1.1. *Then we have the following* (1)∼(3):

(1) $\boldsymbol{R}^d \in \mathcal{R}$ *holds.*

(2) *For* $A,\ B \in \mathcal{R}$, *we have* $A - B \in \mathcal{R}$. *Here the difference* $A - B = A \backslash B$ *of the sets* A *and* B *is defined by the formula*

$$A \backslash B = A \cap B^c = \{x \in \boldsymbol{R}^d;\ x \in A,\ x \notin B\}.$$

(3) *For* $A_p \in \mathcal{R},\ (1 \leq p \leq n)$, *we have*

$$\bigcup_{p=1}^{n} A_p \in \mathcal{R},\ \bigcap_{p=1}^{n} A_p \in \mathcal{R}.$$

Therefore, since $\mathcal{R}$ satisfies the condition (1) of Corollary 1.1, the ring of sets $\mathcal{R}$ is an algebra of sets.

2 Function of locally bounded variation

In this section, we prepare the notations and the concepts necessary for the study of RS-measures.

For that purpose, we prepare the notation.

We consider a variable-wise left continuous function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on $\boldsymbol{R}^d$.

Now, for an interval in $\boldsymbol{R}^d$

$$E = \prod_{j=1}^{d} I_j,\ (I_j = [a_j,\ b_j),\ (1 \leq j \leq d)),$$

we put

$$\Delta_{I_j} f(x) = f(x_1,\ \cdots,\ x_{j-1},\ b_j,\ x_{j+1},\ x_{j+1},\ \cdots,\ x_d)$$
$$-f(x_1,\ \cdots,\ x_{j-1},\ a_j,\ x_{j+1},\ \cdots,\ x_d),$$

for $j,\ (1 \leq j \leq d)$ and

$$\Delta_E f(x) = \Delta_{I_1} \Delta_{I_2} \cdots \Delta_{I_d} f(x) = \Delta_{I_1}(\Delta_{I_2}(\cdots(\Delta_{I_d} f(x))).$$

Especially, for $I_j = \{a_j\},\ (1 \leq j \leq d)$,

$$\Delta_{I_j} f(x) = f(x_1,\ \cdots,\ x_{j-1},\ a_j + 0,\ x_{j+1},\ \cdots,\ x_d)$$
$$-f(x_1,\ \cdots,\ x_{j-1},\ a_j,\ x_{j+1},\ \cdots,\ x_d).$$

Here we happen to denote $I_j = \{a_j\} = [a_j,\ a_j + 0),\ (1 \leq j \leq d)$. Further, when either one of $a_j,\ b_j,\ (1 \leq j \leq d)$ is equal to ∞ or $-\infty$, we consider the limit such as $a_j \to -\infty$ or $b_j \to \infty$ in the symbol in the above.

When we consider a variable-wise right continuous function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on $\boldsymbol{R}^d$, we use the similar notation $\Delta_E f(x)$ for an interval in $\boldsymbol{R}^d$

$$E = \prod_{j=1}^{d} I_j.$$

Here $I_j = (a_j,\ b_j]$ or $I_j = \{a_j\} = (a_j - 0,\ b_j],\ (i = 1,\ 2,\ \cdots,\ d)$.

We say that a real-valued function $f(x) = f(x_1,\ x_2, \cdots,\ x_d)$ of the real variables on $\boldsymbol{R}^d$ is of locally bounded variation if, for an arbitrary natural number $n \geq 1$ and an arbitrary bounded interval E, the set of values

$$\sum_{p=1}^{n} |\Delta_{E^p} f(x)|$$

is bounded for an arbitrary family

$$\Big\{\ E^p = \prod_{j=1}^{d}(a_j^p,\ b_j^p);\ p = 1,\ 2,\ \cdots,\ n\ \Big\}$$

of mutually disjoint n bounded open intervals included in E.

Further a real-valued function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ of the real variables on $\boldsymbol{R}^d$ is **absolutely continuous** if, for an arbitrary positive number ε, there exists some positive number δ such that the condition

$$\sum_{p=1}^{n} \prod_{j=1}^{d}(b_j - a_j) < \delta$$

holds for an arbitrary natural number n and an arbitrary family of mutually disjoint n bounded open intervals

$$\Big\{\ E^p = \prod_{j=1}^{d}(a_j^p,\ b_j^p);\ p = 1,\ 2,\ \cdots,\ d\ \Big\},$$

the estimate

$$\Big|\sum_{p=1}^{n} \Delta_{E^p} f(x)\ \Big| < \varepsilon$$

holds.

3 Definition of the d-dimensional RS-measure

In this section, we define the concept of the d-dimensional RS-measure. Here we assume $d \geq 1$.

Since a d-dimensional RS-measure is a set function, it is defined by the following three conditions:

(i) Its domain is a family of sets $\mathcal{M}_0$.

(ii) Its range is a subset of $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{\pm\infty\}$.

(iii) We determine the rule of correspondence such that we define the value $\mu(A)$ in $\overline{\boldsymbol{R}}$ for an element A in $\mathcal{M}_0$.

We say that the system of the conditions of the definition of a d-dimensional RS-measure is the system of axioms of the d-dimensional RS-measure.

In the sequel, we define the concept of the RS-measure.

Then we define the concepts of the RS-measure space and the RS-measure in the following Definition 3.1.

Here we prepare the terminology which is used in the definition of the RS-measure.

In general, we consider a certain σ-finite completely additive measure space $(X,\ \mathcal{F},\ \mu)$. Here we may consider the case of a σ-finite conditionally completely additive measure space. We assume that the range of μ is a subset of $\overline{\boldsymbol{R}}$. Here we assume that the range of μ does not include ∞ and $-\infty$ at the same time.

Then, for $A \in \mathcal{F}$, we put

$$|\mu|(A) = \sup \sum_{j=1}^{n} |\mu(A_j)|.$$

Here sup is taken for all choices of the finite division of A such as

$$A = A_1 + A_2 + \cdots + A_n,\ (A_j \in \mathcal{F},\ (1 \leq j \leq n))$$

holds. Then we say that the set function $|\mu|$ is the **total variation** of μ. Further, for $A \in \mathcal{F}$, we put

$$\mu^+(A) = \sup_{E \subset A} \{\mu(E),\ 0\},$$
$$\mu^-(A) = -\inf_{E \subset A} \{\mu(E),\ 0\}.$$

Here sup and inf are considered for all $E \in \mathcal{F}$ such as $E \subset A$ holds.

Then we say that the set functions μ^+ and μ^- are the **positive variation** and the **negative variation** respectively. Then we have the following theorem for these variations.

Theorem 3.1 *We use the notation in the above. Then* $|\mu|$, μ^+ *and* μ^- *are the completely additive positive measures and we have the equalities*

$$\mu(A) = \mu^+(A) - \mu^-(A),\ |\mu|(A) = \mu^+(A) + \mu^-(A)$$

for $A \in \mathcal{F}$.

Definition 3.1(RS-measure) We define that the triplet $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ is a d**-dimensional RS-measure space** if the family of sets $\mathcal{M}_0$ on the d-dimensional Euclidean space $\boldsymbol{R}^d$ and the set function μ on $\mathcal{M}_0$ satisfy the following axioms (I)∼(IV).

Then we say that an element in $\mathcal{M}_0$ is a **RS-measurable set** and μ is a d**-dimensional RS-measure**.

Now we assume that ν is the total variation μ and μ^+, μ^- are the positive variation and the negative variation respectively.

(I) $\mathcal{R} \subset \mathcal{M}_0$ holds.

(II) We have the following (i)∼(ii):

(i) One of the following (a) or (b) holds:

(a) For $A \in \mathcal{M}_0$, we have $-\infty < \mu(A) \leq \infty$.

(b) For $A \in \mathcal{M}_0$, we have $-\infty \leq \mu(A) < \infty$.

(ii) If at most countable elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ of $\mathcal{M}_0$ are mutually disjoint and their direct sum

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p$$

belongs to $\mathcal{M}_0$, we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(III) $A \in \mathcal{M}_0$ holds if and only if, for an arbitrary bounded set $E \in \mathcal{R}$, we have the equality

$$\nu^*(A \cap E) = \nu_*(A \cap E).$$

Here ν^* and ν_* are the outer measure and the inner measure respectively which are defined by using the restricted measure ν on $\mathcal{R}$ of ν on $\mathcal{M}_0$.

Namely $\nu^*(A \cap E)$ and $\nu_*(A \cap E)$ are defined by the formulas

$$\nu^*(A \cap E) = \inf\{\nu(B);\ B \supset A \cap E,\ B \in \mathcal{R}\},$$

$$\nu_*(A \cap E) = \sup\{\nu(B);\ A \cap E \supset B,\ B \in \mathcal{R}\}.$$

(IV) For $A \in \mathcal{M}_0$, we have the equalities

$$\mu(A) = \mu^+(A) - \mu^-(A),\ \nu(A) = \mu^+(A) + \mu^-(A).$$

Especially we say that a RS-measure is the positive RS-measure if its range is included in $[0,\ \infty]$. Then $\nu,\ \mu^+$ and μ^- are all the positive RS-measures.

For simplicity, we say that a d-dimensional RS-measure space is the **RS-measure space** and a d-dimensional RS-measure is the **RS-measure**.

The symbol used in the axiom (III), (ii)

$$\bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p$$

denotes the finite or countable direct sum of the sets $A_p,\ (p \geq 1)$ and the symbol

$$\sum_{p=1}^{(\infty)} \mu(A_p)$$

denotes the finite or countable sum of $\mu(A_p),\ (p \geq 1)$.

Every RS-measure satisfies only one of the conditions (a) or (b) in the axiom (II), (i) of Definition 3.1. This means that the range of a RS-measure does not contain ∞ and $-\infty$ simultaneously. Therefore at least one of μ^+ or μ^- has the finite total measure. Then we say that the measure with the finite total measure is a **finite measure**. The condition (ii) of the axiom (II) means that a d-dimensional RS-measure is the conditionally completely additive measure. For simplicity, we happen to say that this d-dimensional RS-measure is completely additive. Then, since $\mathcal{M}_0$ is a ring, we can understand that a d-dimensional RS-measure on $\mathcal{M}_0$ is not a completely additive measure in the primitive sense and it is a conditionally completely additive measure.

Further, by virtue of the condition (ii) of the axiom (II), a conditionally completely additive measure is of course a finitely additive measure.

Corollary 3.1 *We use the notation in* Definition 3.1. *Then, for* $A \in \mathcal{M}_0$, *we have the equalities*

$$\nu(A) = \nu^*(A) = \sup\{\nu^*(A \cap E);\ E \in \mathcal{R} \text{ is bounded}\}$$
$$= \nu_*(A) = \sup\{\nu_*(A \cap E);\ E \in \mathcal{R} \text{ is bounded}\}.$$

4 Existence theorem of the d -dimensional RS-measure

In this section, we prove the existence theorem of the d-dimensional RS-measure defined in Definition 3.1.

For that purpose, we have only to determine the family $\mathcal{M}_0$ of RS-measurable sets in $\boldsymbol{R}^d$ and the RS-measure μ concretely.

At first, assuming that there exists a RS-measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ satisfying the system of axioms in Definition 3.1, we have to determine what kind of the set should be an element of $\mathcal{M}_0$ and how should we define the value of μ for an arbitrary element $A \in \mathcal{M}_0$.

Further, by virtue of the axiom (IV) of Definition 3.1, we see that we have to determine the two positive RS-measure μ^+ and μ^- on $\mathcal{M}_0$. Therefore we prove the existence theorem of the positive RS-measure satisfying the system of axioms of Definition 3.1 in the following.

Here we prove the following two Lemmas.

Lemma 4.1 *We use the notation in the above. Assume that a real-valued function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on $\boldsymbol{R}^d$ satisfies the following conditions* (i) *and* (ii):

(i) *$f(x)$ is a variable-wise left continuous function.*

(ii) *For an arbitrary interval*

$$E = \prod_{j=1}^{\alpha} I_j, \tag{4.1}$$

the condition

$$\Delta_E f(x) \geq 0$$

is satisfied. Here we denote

$$I_j = [x_j,\ y_j) \text{ or } I_j = \{x_j\} = [x_j,\ x_j + 0),$$

$$(x_j,\ y_j \in \boldsymbol{R},\ x_j < y_j,\ (1 \leq j \leq d)).$$

We assume that $\mathcal{P}$ is the family of all intervals in $\boldsymbol{R}^d$. Then there exists one and only one conditionally completely additive positive measure μ such that we have the following condition (1):

(1) *For the interval E in the formula* (4.1), *we have the formula*

$$\mu(E) = \Delta_E f(x).$$

Lemma 4.2 *We use the notation in* Lemma 4.1. *Let $\mathcal{R}$ be the ring of sets composed of all blocks of intervals in $\boldsymbol{R}^d$. Then $\mathcal{R}$ is a ring of sets generated by $\mathcal{P}$.*

Then there exists one and only one conditionally completely additive positive measure μ on $\mathcal{R}$ such that we have the following conditions (1) *and* (2):

(1) *For an interval E in the formula* (4.1), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

(2) *If $E \in \mathcal{R}$ has a division by using the finite number of mutually disjoint intervals $E_1,\ E_2,\ \cdots,\ E_n$ as follows*:

$$E = E_1 + E_2 + \cdots + E_n,$$

we have the equality

$$\mu(E) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_n).$$

Further the value of $\mu(E)$ is determined uniquely and independently with the choice of the division of E by using the intervals.

Then, for $E \in \mathcal{P}$, $\mu(E)$ coincides with the value of the interval functions defined in Lemma 4.1.

Theorem 4.1(Existence theorem of the RS-measure) *Assume that a function $f(x)$ is the same as in* Lemma 4.1. *Then there exists one and only one positive* RS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ *such that the following condition* (1) *is satisfied*:

(1) *For an interval E in the formula* (4.1), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Further the converse is also true.

We say that the function $f(x)$ considered in Theorem 4.1 is the **distribution function** of the positive RS-measure μ.

Corollary 4.1 *We use the notation in* Corollary 3.1 *and* Theorem 4.1. *Let A be a subset of $\boldsymbol{R}^d$. Then, for the outer measure $\mu^*(A)$ and the inner measure $\mu_*(A)$ of A, we have the following* (1) *and* (2):

(1) $\mu^*(A) = \sup\{\mu^*(A \cap E);\ E \in \mathcal{R}$ *is bounded* $\}$.

(2) $\mu_*(A) = \sup\{\mu_*(A \cap E);\ E \in \mathcal{R}$ *is bounded* $\}$.

Corollary 4.2 *We use the notation in* Corollary 4.1. *Then, if a subset A of $\boldsymbol{R}^d$ is a* RS-*measurable set, we have the equality*

$$\mu(A) = \mu^*(A) = \mu_*(A).$$

Theorem 4.2 *Assume that a real-valued function $f(x)$ is the same as in* Theorem 4.1.

Further we assume that μ is a positive RS-*measure on $\boldsymbol{R}^d$ and the function $f(x)$ in the above is the distribution function of μ. Then $f(x)$ is continuous if and only if, for each point $x_j \in \boldsymbol{R}$, we have $\mu(\{x_j\} \times E_{x'}) = 0,\ (1 \leq j \leq d)$. Here we put $x' = (x_1,\ \cdots,\ x_{j-1},\ x_{j+1},\ \cdots,\ x_d)$ and $E_{x'}$ is an interval in $\boldsymbol{R}^{d-1}_{x'}$.*

In the same way as Theorem 4.1, we can prove the following theorem.

Theorem 4.3(Existence theorem of the RS-measure) *Assume that a real-valued function $f(x)$ defined on $\boldsymbol{R}^d$ satisfies the following conditions* (i) *and* (ii):

(i) *$f(x)$ is a variable-wise right continuous function.*

(ii) *For an arbitrary interval*

$$E = \prod_{j=1}^{d} I_j, \tag{4.2}$$

the condition

$$\Delta_E f(x) \geq 0$$

is satisfied. Here we put

$$I_j = (x_j,\ y_j] \text{ or } I_j = \{x_j\} = (x_j - 0,\ x_j],$$

$$(x_j,\ y_j \in \boldsymbol{R} \text{ and } x_j < y_j,\ (1 \leq j \leq d)).$$

Then there exists one and only one positive RS-*measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0.\ \mu)$ such that we have the following condition* (1):

(1) *For an interval E in the formula* (4.2), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Further the inverse is also true.

We have the similar results as in Theorem 4.1 if $f(x)$ and μ satisfy the conditions of Theorem 4.3.

Theorem 4.4(Existence theorem of the RS-measure) *Assume that a real-valued function $f(x)$ defined on $\boldsymbol{R}^d$ satisfies the following conditions* (i) *and* (ii):

(i) *$f(x)$ is a variable-wise left continuous function of locally bounded variation.*

(ii) *The two functions $f^+(x)$ and $f^-(x)$ satisfy the same conditions as in* Theorem 4.1 *and the equality*

$$f(x) = f^+(x) - f^-(x)$$

holds.

Then there exists one and only one RS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ *such that we have the following condition* (1):

(1) *For an interval*

$$E = \prod_{j=1}^{d} I_j,$$

we have the equality

$$\mu(E) = \Delta_E f(x).$$

Here we put

$$I_j = [x_j,\ y_j) \text{ or } I_j = \{x_j\} = [x_j,\ x_j + 0),$$

$$(x_j,\ y_j \in \boldsymbol{R},\ x_j < y_j,\ (1 \leq j \leq d).$$

Further the inverse is also true,

We consider the two functions $f^+(x)$ and $f^-(x)$ as in Theorem 4.3. Then these functions satisfy the conditions of Theorem 4.1. Therefore we can define the positive RS-measures μ^+ and μ^- corresponding to $f^+(x)$ and $f^-(x)$ respectively. Then, if we put

$$\mu = \mu^+ - \mu^-,$$

we can define the RS-measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ and it is evident that it satisfies the conditions of Theorem 4.4.

The inverse can be proved by the similar way.

By the similar way to Theorem 4.4, we can characterize the RS-measure space by using a variable-wise right continuous function of locally bounded variation.

Theorem 4.5(Existence theorem of RS-measure) *Assume that a real-valued function $f(x)$ defined on $\boldsymbol{R}^d$ satisfics the following conditions* (i) *and* (ii):

(i) *$f(x)$ is a variable-wise right continuous function of locally bounded variation.*

(ii) *Two functions* $f^+(x)$ *and* $f^-(x)$ *satisfy the same conditions as in* Theorem 4.3 *and the equality*

$$f(x) = f^+(x) - f^-(x)$$

holds.
Then there exists one and only one RS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ *such that it satisfies the following condition* (1):

(1) *For an interval*

$$E = \prod_{j=1}^{d} I_j,$$

we have the equality

$$\mu(E) = \Delta_E f(x).$$

Here we put

$$I_j = (x_j,\ y_j] \text{ or } I_j = \{x_j\} = (x_j - 0,\ x_j],$$

$$(x_j,\ y_j \in \boldsymbol{R},\ x_j < y_j,\ (1 \leq j \leq d)).$$

Further the inverse is also true.

Remark 4.1 We see that the positive RS-measure on $\boldsymbol{R}^d$ corresponding to the function $f(x) = x_1 x_2 \cdots x_d$ is the d-dimensional Jordan measure. Further the positive RS-measure on $\boldsymbol{R}^d$ corresponding to the function which is a multiplication of $f(x) = x_1 x_2 \cdots x_d$ by an arbitrary positive constant is the d-dimensional Riemann-Haar measure.

Theorem 4.6 *If* μ *is a* RS-*measure on* $\boldsymbol{R}^d$, *there exist two positive* RS-*measures* μ^+ *and* μ^- *such that we have the unique expression*

$$\mu = \mu^+ - \mu^-.$$

We say that the result of Theorem 4.6 is the **Jordan decomposition**.

Theorem 4.7 *Assume that* μ *is a* RS-*measure on* $\boldsymbol{R}^d$ *and* λ *is the Jordan measure on* $\boldsymbol{R}^d$. *Further, assume that* $f(x)$ *is a continuous function of locally bounded variation which defines the* RS-*measure* μ. *Then* $f(x)$ *is absolutely continuous if and only if* μ *is absolutely continuous with respect to* λ.

If $f(x)$ is absolutely continuous, $f(x)$ is continuous. Thus, we may assume, as the precondition in Theorem 4.7, that $f(x)$ is a continuous function of locally bounded variation.

5 *d*-dimensional RS-measurable sets

In this section, we characterize the d-dimensional RS-measure defined in Theorem 4.1. And we prove that the family $\mathcal{M}_0$ of all RS-measurable sets is a ring of sets or an algebra of sets. Then we study their relation with the RS-measure μ.

We restrict the RS-measure μ in Definition 3.1 to the family $\mathcal{R}$ of all blocks of intervals. Thereby we obtain the concept of the RS-measure of the blocks of intervals in the following.

Definition 5.1 Let $\mathcal{R}$ be the ring of all blocks of intervals in $\boldsymbol{R}^d$. Then we say that a set function μ on $\mathcal{R}$ is a **RS-measure** of the blocks of intervals in $\boldsymbol{R}^d$ if we have the following conditions (i) $\sim$ (iii):

(i) We have either one of the following (a) or (b):

(a) For $A \in \mathcal{R}$, we have $-\infty < \mu(A) \leq \infty$.

(b) For $A \in \mathcal{R}$, we have $-\infty \leq \mu(A) < \infty$.

(ii) If at most countable elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ in $\mathcal{R}$ are mutually disjoint and the direct sum

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p$$

belongs to $\mathcal{R}$, we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(iii) For $A \in \mathcal{R}$, we have the following equalities

$$\mu(A) = \mu^+(A) - \mu^-(A),\ \nu(A) = \mu^+(A) + \mu^-(A).$$

Here ν is the total variation of μ and $\mu^+,\ \mu^-$ are the positive variation and the negative variation of μ respectively.

Then we say that the value $\mu(E)$ of μ at $E \in \mathcal{R}$ is the **RS-measure** of the block of intervals.

Corollary 5.1 *For the* RS-*measure* μ *of the blocks of intervals, we have the following* (1) $\sim$ (4):

(1) *If the elements* $A_1,\ A_2,\ \cdots,\ A_n$ *of* $\mathcal{R}$ *are mutually disjoint, we have the condition*

$$A = \bigcup_{p=1}^{n} A_p = \sum_{p=1}^{n} A_p \in \mathcal{R}$$

and we have the equality

$$\mu(A) = \sum_{p=1}^{n} \mu(A_p).$$

(2) *For* $A,\ B \in \mathcal{R}$ *with* $A \supset B$, *we have the inequalities* $\mu^{\pm}(A) \geq \mu^{\pm}(B),\ \nu(A) \geq \nu(B)$. *Especially, if* $\nu(B) < \infty$ *holds, we have the equality* $\mu(A \backslash B) = \mu(A) - \mu(B)$. *Especially we have the equality* $\mu(\phi) = 0$.

(3) *If at most countable elements* $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ *of* $\mathcal{R}$ *satisfy the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p \in \mathcal{R},$$

we have the inequality

$$\lambda(A) \leq \sum_{p=1}^{(\infty)} \lambda(A_p).$$

Here λ *denotes either one of the measures* $\nu,\ \mu^+$ *and* μ^-.

(4) *If at most countable intervals* $I_1,\ I_2,\ \cdots,\ I_n,\ \cdots$ *are mutually disjoint and their direct sum*

$$I = \bigcup_{p=1}^{(\infty)} I_p = \sum_{p=1}^{(\infty)} I_p$$

is also an interval, we have the equality

$$\mu(I) = \sum_{p=1}^{(\infty)} \mu(I_p).$$

Proposition 5.1 *Assume that* μ *is the* RS-*measure of the blocks of intervals in* $\mathcal{R}$.

Then, for the finite division

$$E = I_1 + I_2 + \cdots + I_n \tag{5.1}$$

of the block of intervals E *by using the mutually disjoint intervals* $I_1,\ I_2,\ \cdots,\ I_n$, *we have the equality*

$$\mu(E) = \mu(I_1) + \mu(I_2) + \cdots + \mu(I_n). \tag{5.2}$$

Here the value of the right hand side of the formula (5.2) *does not depends on the choice of the finite division of* E *by using the intervals.*

Conversely, we have the theorem concerning the existence of the RS-measure of the blocks of intervals. By virtue of Definition 5.1, we have only to prove the existence theorem of the positive RS-measure.

Theorem 5.1 *Assume that a real-valued function* $f(x) = f(x_1, x_2, \cdots, x_d)$ *is the same as in* Theorem 4.1.

Further we assume that, for an arbitrary interval

$$E = \prod_{j=1}^{d} I_j, \tag{5.3}$$

we have the condition

$$\Delta_E f(x) \geq 0.$$

Here we denote

$$I_j = [x_j, y_j) \text{ or } I_j = \{x_j\} = [x_j, x_j + 0),$$

$$(x_j, y_j \in \boldsymbol{R}, x_j < y_j, (1 \leq j \leq d)).$$

Then we define the set function μ *on* $\mathcal{R}$ *in the following*:

(i) *For an interval* E *in the formula* (5.3), *we put*

$$\mu(E) = \Delta_E f(x).$$

(ii) *If we have a finite division*

$$A = E_1 + E_2 + \cdots + E_n \tag{5.4}$$

of the block of intervals A *by using the intervals* $E_1, E_2, \cdots, E_n$, *we put*

$$\mu(A) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_n).$$

(iii) *If at most countable intervals* $E_1, E_2, \cdots, E_n, \cdots$ *are mutually disjoint and the direct sum*

$$E = \bigcup_{p=1}^{(\infty)} E_p = \sum_{p=1}^{(\infty)} E_p$$

is also an interval, we have the equality

$$\mu(E) = \sum_{p=1}^{\infty} \mu(E_p).$$

Then μ *is the positive* RS-*measure of the blocks of intervals.*

Next, we determine the d-dimensional positive RS-measure μ concretely and we prove the existence theorem of the d-dimensional positive RS-measure.

For that purpose, we make some preparation.

We have only to prove the existence of the d-dimensional positive RS-measure by virtue of Definition 3.1, (IV). Therefore we assume that the RS-measure μ on $\mathcal{R}$ is positive.

Definition 5.2 Assume that μ is the positive RS-measure on $\mathcal{R}$. We define that, for an arbitrary subset A in $\boldsymbol{R}^d$,

$$\mu^*(A) = \inf\{\mu(B);\ B \supset A,\ B \in \mathcal{R}\},$$

$$\mu_*(A) = \sup\{\mu(B);\ A \supset B,\ B \in \mathcal{R}\}.$$

are the **outer measure** and the **inner measure** of A respectively.

Corollary 5.2 *For $A \in \mathcal{R}$, we have the equalities*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

Here the third side is the positive RS-*measure of the blocks of intervals in the sense of* Theorem 5.1.

We have the following three propositions immediately from the definitions of the outer measure and the inner measure. In the following, assume that A, A_1 and A_2 are some subsets of $\boldsymbol{R}^d$.

Proposition 5.2 *We have the inequalities* $0 \le \mu_*(A) \le \mu^*(A) \le +\infty$. *Especially, we have the equalities* $\mu^*(\emptyset) = \mu_*(\emptyset) = 0$.

Proposition 5.3 *If $A_1 \subset A_2$ holds, we have the following*:

(1) $\mu^*(A_1) \le \mu^*(A_2)$. (2) $\mu_*(A_1) \le \mu_*(A_2)$.

Proposition 5.4 *We have the following inequality*

$$\mu^*(A_1 \cup A_2) \le \mu^*(A_1) + \mu^*(A_2).$$

Proposition 5.5 *For at most countable subsets $A_1,\ A_2,\ ,\cdots$ of $\boldsymbol{R}^d$, we put*

$$A = \bigcup_{p=1}^{(\infty)} A_p.$$

Then we have the inequality

$$\mu^*(A) \leq \sum_{p=1}^{(\infty)} \mu^*(A_p).$$

Proposition 5.6 *If a most countable sets* $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ *of* $\boldsymbol{R}^d$ *are mutually disjoint, we put*

$$A = \sum_{p=1}^{(\infty)} A_p.$$

Then we have the inequality

$$\mu_*(A) \geq \sum_{p=1}^{(\infty)} \mu_*(A_p).$$

Proposition 5.7 *Choose an arbitrary subset* A *of* $\boldsymbol{R}^d$*. Then, for an arbitrary bounded set* $E \in \mathcal{R}$*, we have the equality*

$$\mu_*(A \cap E) = \mu(E) - \mu^*(A^c \cap E).$$

Here μ *is the positive* RS-*measure of the blocks of intervals defined in* Theorem 5.1.

Proposition 5.8 *Choose an arbitrary subset* A *of* $\boldsymbol{R}^d$*. Assume that* $E_1,\ E_2,\ \cdots$ *are some bounded blocks of intervals of* $\boldsymbol{R}^d$ *such that we have the following conditions*:

$$E_1 \subset E_2 \subset \cdots,\ \bigcup_{n=1}^{\infty} E_n = \boldsymbol{R}^d.$$

Then we have the equalities

$$\mu^*(A) = \lim_{n\to\infty} \mu^*(A \cap E_n),$$
$$\mu_*(A) = \lim_{n\to\infty} \mu_*(A \cap E_n).$$

Definition 5.3 We use the notation in Definition 5.2. We define that an arbitrary subset A of $\boldsymbol{R}^d$ is **RS-measurable** if, for an arbitrary bounded set $E \in \mathcal{R}$, we have the equality $\mu^*(A \cap E) = \mu_*(A \cap E)$. Then we say that

$$\mu(A) = \sup\{\mu^*(A \cap E);\ E \text{ is a bounded block of intervals }\}$$

is the **positive RS-measure** of A.

Remark 5.1 In Definition 5.3, the RS-measurability of a subset A of $\boldsymbol{R}^d$ means that, for any bounded part $A\cap E$ of A, the outer measure $\mu^*(A\cap E)$ and the inner measure $\mu_*(A\cap E)$ coincide. Here the outer measure $\mu^*(A\cap E)$ is the approximation of $A\cap E$ by using the measures of the bounded blocks of intervals from the outer side and the inner measure $\mu_*(A\cap E)$ is the approximation of $A\cap E$ by using the measures of the bounded blocks of intervals from the inner side.

Corollary 5.3 *We use the notation in* Definition 5.3. *Then, for an arbitrary* RS-*measurable set A of $\boldsymbol{R}^d$, we have the equalities*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

In the following, we prove that the set function μ defined in Definition 5.3 satisfies the conditions of the positive RS-measure in Definition 5.1.

By virtue of Corollary 5.2, we see that the concept of the positive RS-measure of a measurable set coincides with the positive RS-measure of a block of intervals for the block of intervals.

Theorem 5.2 *Assume that A is an arbitrary subset of $\boldsymbol{R}^d$. Then A is measurable if and only if, for any $E\in\mathcal{R}$, we have the following equality*

$$\mu^*(A\cap E) + \mu^*(A^c\cap E) = \mu(E).$$

Theorem 5.3 *Assume that A is an arbitrary subset of $\boldsymbol{R}^d$. Then A is measurable if and only if, for an arbitrary subset B of $\boldsymbol{R}^d$, we have the following equality*

$$\mu^*(A\cap B) + \mu^*(A^c\cap B) = \mu^*(B).$$

Theorem 5.4 *Assume that A is an arbitrary subset of $\boldsymbol{R}^d$. Then A is measurable if and only if, for two arbitrary subsets A_1 and A_2 with $A_1\subset A$ and $A_2\subset A^c$, we have the following equality*

$$\mu^*(A_1 + A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Theorem 5.5 *We assume that μ is the positive* RS-*measure on $\mathcal{R}$. Assume that A is an arbitrary subset of $\boldsymbol{R}^d$. Then $A\in\mathcal{M}_0$ holds if and only if, for an arbitrary $\varepsilon>0$, there exist $A_1,\ A_2\in\mathcal{R}$ such that we have the following conditions* (1) *and* (2):

(1) $A_1 \subset A \subset A_2$ *holds.* (2) $\mu(A_2 \backslash A_1) < \varepsilon$ *holds.*

Theorem 5.6 *If* $\mathcal{M}_0$ *is the family of all* RS-*measurable subsets, then* $\mathcal{M}_0$ *satisfies the following* (1) $\sim$ (3):

(1) $\mathcal{R} \subset \mathcal{M}_0$ *holds. Especially* $\emptyset \in \mathcal{M}_0$ *holds.*

(2) *For* $A \in \mathcal{M}_0$, *we have* $A^c \in \mathcal{M}_0$.

(3) *For* $A,\ B \in \mathcal{M}_0$, *we have* $A \cup B \in \mathcal{M}_0$.

Corollary 5.4 *Let* $\mathcal{M}_0$ *be as same as in* Theorem 5.6. *Then we have the following* (1) $\sim$ (3):

(1) $\boldsymbol{R}^d \in \mathcal{M}_0$ *holds.*

(2) *For* $A,\ B \in \mathcal{M}_0$, *we have* $A - B \in \mathcal{M}_0$.

(3) *For* $A_p \in \mathcal{M}_0,\ (1 \leq p \leq n)$, *we have*

$$\bigcup_{p=1}^{n} A_p \in \mathcal{M}_0,\ \bigcap_{p=1}^{n} A_p \in \mathcal{M}_0.$$

Therefore $\mathcal{M}_0$ is the algebra of sets.

Theorem 5.7 *For* $A,\ B \in \mathcal{M}_0$ *with* $A \cap B = \emptyset$, *we have the equality*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Theorem 5.8 *If at most countable elements* $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ *of* $\mathcal{M}_0$ *are mutually disjoint and we have the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p \in \mathcal{M}_0,$$

we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

Theorem 5.9 *For the algebra* $\mathcal{M}_0$ *of all* RS-*measurable sets of* $\boldsymbol{R}^d$ *and the set function* μ *defined in Definition* 5.3, *the measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ *is the d-dimensional positive* RS-*measure space.*

Since the measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ in Theorem 5.9 satisfies the system of axioms of the d-dimensional positive RS-measure space in Definition 5.1, we prove the existence theorem of the d-dimensional positive RS-measure space.

At the same time, we prove the existence theorem of the general d-dimensional RS-measure space.

Namely we have the following theorem.

Theorem 5.10 *There exists a d-dimensional* RS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$.

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