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Upper Decay Estimates for Non-Degenerate Kirchhoff Type Dissipative Wave Equations

By

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Abstract

We study on the Cauchy problem for non-degenerate Kirchhoff type dissipative wave equations $\rho u'' + a\left(\|A^{1/2}u(t)\|^2\right)Au + u' = 0$ and $(u(0), u'(0)) = (u_0, u_1)$, where $u_0 \neq 0$ and the nonlocal nonlinear term $a(M) = 1 + M^\gamma$ with $\gamma > 0$. Under the suitably smallness condition, we derive the upper decay estimates of the solution $u(t)$ for the case of $0 < \gamma < 1$ in addition to $\gamma \geq 1$.

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1 Introduction

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

In this paper we investigate on the upper decay estimates of the solution $u(t)$ for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + a\left(\|A^{1/2}u(t)\|^2\right)Au + u' = 0\,, \quad t \geq 0 \\ (u(0), u'(0)) = (u_0, u_1) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})\,, \end{cases} \tag{1.1}$$

where $u = u(t)$ is an unknown real value function, ρ is a positive constant, $' = d/dt$, A is a linear operator on H with dense domain $\mathcal{D}(A)$.

We assume that the operator A is self-adjoint and nonnegative such that $(Av, v) \geq 0$ for $v \in \mathcal{D}(A)$. The α-th power of A with dense domain $\mathcal{D}(A^\alpha)$ is denoted by A^α for $\alpha > 0$, and the graph-norm of A^α is denoted by $\|v\|_\alpha = \left(\|v\|^2 + \|A^\alpha v\|^2\right)^{\frac{1}{2}}$ for $v \in \mathcal{D}(A^\alpha)$. We use that $\|A^{1/2}v\|^2 = (Av, v)$ for $v \in \mathcal{D}(A^{1/2})$.

For the non-local nonlinear term $a(M) \in C^0([0,\infty)) \cap C^2((0,\infty))$, we assume that as follows :

Hyp.1 $K_1 \le a(M) \le K_2 + K_3 M^\gamma \quad$ for $M \ge 0$

Hyp.2 $0 \le a'(M)M \le K_4 a(M) \quad$ for $M > 0$

Hyp.3 $a'(M)M + |a''(M)|M^2 \le K_5 M^\gamma \quad$ for $M > 0$

with $\gamma > 0$ and $K_j > 0$ $(j = 1, 2, 3, 4, 5)$.

From Hyp.1, we see that

$$K_1 M \le \int_0^M a(\mu)\, d\mu \le \left(K_2 + \frac{K_3}{\gamma+1} M^\gamma\right) M\,. \tag{1.2}$$

For typical examples, we have that

$$a(M) = 1 + M^\gamma \quad \text{with} \quad \gamma > 0\,.$$

When the dimension is one, (1.1) describes small amplitude vibrations of an elastic string (see [3], [6]).

We denote the energy $E(t)$ for (1.1) by

$$E(t) = \rho\|u'(t)\|^2 + \int_0^{M(t)} a(\mu)\, d\mu \quad \text{with} \quad M(t) = \|A^{1/2}u(t)\|^2\,. \tag{1.3}$$

By fundamental calculation, we have the energy identity

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0 \tag{1.4}$$

and

$$E(t) + 2\int_0^t \|u'(s)\|^2\, ds = E(0) \tag{1.5}$$

with

$$E(0) = \rho\|u_1\|^2 + 2\int_0^{\|A^{1/2}u_0\|^2} a(\mu)\, d\mu\,.$$

Moreover, we introduce the quantities $G(0)$ and $B(0)$ on the initial data (u_0, u_1) :

$$G(0) = \frac{\|Au_0\|^2}{\|A^{1/2}u_0\|^2} + \rho\frac{\|A^{1/2}u_0\|^2\|A^{1/2}u_1\|^2 - |(A^{1/2}u_0, A^{1/2}u_1)|}{a(\|A^{1/2}u_0\|^2)\|A^{1/2}u_0\|^4}$$

and

$$B(0) = \max\{\frac{\|u_1\|^2}{\|A^{1/2}u_0\|^2}\,,\ \frac{1+K_4}{K_4}(K_2 + K_3(K_1^{-1}E(0))^\gamma)^2 G(0)\}\,.$$

In the previous paper [12], we have proved the following the global existence theorem (see [1], [2], [9], [13] for local solutions).

Theorem 1.1 *Suppose that Hyp.1 and Hyp.2 are fulfilled. If the initial data* (u_0, u_1) *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ *and* $u_0 \neq 0$*, and moreover, the coefficient* ρ *and the initial data* (u_0, u_1) *satisfy*

$$2\rho G(0)^{\frac{1}{2}} B(0)^{\frac{1}{2}} < \frac{1}{K_4 + 1},$$

then the problem (1.1) *admits a unique global solution* $u(t)$ *in the class*

$$C^0([0,\infty);\mathcal{D}(A)) \cap C^1([0,\infty);\mathcal{D}(A^{1/2})) \cap C^2([0,\infty);H)$$

and the solution $u(t)$ *satisfies*

$$\|u(t)\|^2 \leq C(\|u_0\|^2 + E(0)), \tag{1.6}$$

$$K_1 M(t) \leq E(t) \leq E(0), \tag{1.7}$$

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{K_4 + 1}, \tag{1.8}$$

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(0), \quad \frac{\|u'(t)\|^2}{M(t)} \leq B(0), \tag{1.9}$$

and $M(t) \geq Ce^{-\alpha t}$ *with some* $\alpha > 0$ *for* $t \geq 0$.

We do not need the assumption that $\gamma \geq 1$ in our argument (see [4] for $\gamma \geq 1$ that is, $a(\cdot) \in C^1([0,\infty))$, and $a'(M) \geq K_0 > 0$ for $\gamma > 0$ (see [11] for $a(M) = (1+M)^\gamma$ with $\gamma > 0$).

The purpose of this paper to derive upper decay estimates of the solution $u(t)$ of (1.1) for the case of $0 < \gamma < 1$ in addition to $\gamma \geq 1$, under Hyp.1, Hyp.2, Hyp.3.

Our main result is as follows.

Theorem 1.2 *Suppose that the assumption of Theorem 1.1 and Hyp.3 are fulfilled. Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$\|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1},$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \leq \begin{cases} C(1+t)^{-(1+2\gamma)} & \text{if } 0 < \gamma < \frac{1}{2}, \\ C(1+t)^{-2} & \text{if } \gamma \geq \frac{1}{2}, \end{cases}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \leq \begin{cases} C(1+t)^{-(1+\gamma)(1+2\gamma)} & \text{if } 0 < \gamma < \frac{1}{2}, \\ C(1+t)^{-3} & \text{if } \gamma \geq \frac{1}{2} \end{cases}$$

for $t \geq 0$.

The proof of Theorem 1.2 will be given by Propositions 2.2–2.5 in the next section.

The notations we use in the paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Decay Rates

The following generalized Nakao type inequality is useful to derive decay estimates of energies (see [5], [7], [8], [10] for the proof).

Lemma 2.1 *Let $\phi(t)$ be a non-negative function on $[0,\infty)$ and satisfy*

$$\sup_{t\le s\le t+1} \phi(s)^{1+\alpha} \le (k_0\phi(t)^\alpha + k_1(1+t)^{-\beta})(\phi(t)-\phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants $k_0, k_1, k_2 \ge 0$, $\alpha > 0$, $\beta > -1$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \le C_0(1+t)^{-\theta}\,, \qquad \theta = \min\{\frac{1+\beta}{\alpha}\,,\,\frac{\gamma}{1+\alpha}\}$$

for $t \ge 0$ with some constant C_0 depending on $\phi(0)$.

Using Lemma 2.1, we obtain the following energy decay for the energy $E(t)$.

Proposition 2.2 *Under the assumption of Theorem 1.1, the energy $E(t)$ satisfies*

$$E(t) = \rho\|u'(t)\|^2 + \int_0^{M(t)} a(\mu)\,d\mu \le C(1+t)^{-1}\,, \tag{2.1}$$

and the solution $u(t)$ satisfies

$$\|A^{1/2}u(t)\|^2 + \|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-1} \tag{2.2}$$

for $t \ge 0$.

Proof. Integrating (1.4) over $[t,t+1]$, we have

$$2\int_t^{t+1} \|u'(s)\|^2\,ds = E(t) - E(t+1) \quad (\equiv 2D(t)^2)\,. \tag{2.3}$$

Then there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u'(t_j)\|^2 \le 4D(t)^2 \quad \text{for} \quad j = 1,2\,. \tag{2.4}$$

On the other hand, taking the inner product of (1.1) with $u(t)$, we have

$$a(M(t))M(t) = \rho\left(\|u'(t)\|^2 - \frac{d}{dt}(u'(t),u(t))\right) - (u'(t),u(t))\,. \tag{2.5}$$

Integrating (2.5) over $[t_1, t_2]$, we have that

$$\int_{t_1}^{t_2} a(M(s))M(s)\, ds$$
$$\leq \rho \int_t^{t+1} \|u'(s)\|^2\, ds + \rho \sum_{j=1}^{2} \|u'(t_j)\|\|u(t_j)\| + \int_t^{t+1} \|u'(s)\|\|u(s)\|\, ds$$

and from (2.3), (2.4), and Hyp.1 that

$$K_1 \int_{t_1}^{t_2} M(s)\, ds \leq \rho D(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s) \quad \text{with} \quad g(t)^2 = \|u(t)\|^2, \quad (2.6)$$

and from (1.2), (1.3), (1.7), (2.3), (2.6) that

$$\int_{t_1}^{t_2} E(s)\, ds \leq \rho \int_t^{t+1} \|u'(s)\|^2 ds + \int_{t_1}^{t_2} \left(K_2 + \frac{K_3}{\gamma+1} M(s)^\gamma\right) M(s)\, ds$$
$$\leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s)\,. \quad (2.7)$$

Integrating (2.3) over $[t, t_2]$, we have (2.3) and (2.7) that

$$E(t) = E(t_2) + 2\int_t^{t_2} \|u'(s)\|^2\, ds$$
$$\leq 2\int_{t_1}^{t_2} E(s)\, ds + \int_t^{t+1} \|u'(s)\|^2\, ds$$
$$\leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s)\,.$$

Since it holds that $2D(t)^2 = E(t) - E(t+1) \leq E(t)$ by (2.3), we observe

$$E(t)^2 \leq C\left(D(t)^2 + \sup_{t \leq s \leq t+1} g(s)^2\right) D(t)^2$$
$$\leq C\left(E(t) + \sup_{t \leq s \leq t+1} g(s)^2\right) (E(t) - E(t+1))\,. \quad (2.8)$$

Thus, using $E(t) \leq E(0)$ and $g(t) = \|u(t)\|^2 \leq C$ by (1.6) and (1.7), we have

$$E(t)^2 \leq C(E(t) - E(t+1))\,, \quad (2.9)$$

and hence, applying Lemma 2.1 to (2.9), we obtain (2.1).

Moreover, we obtain that $M(t) \leq K_1^{-1}E(t) \leq C(1+t)^{-1}$ by (1.7), $\|Au(t)\|^2 + \|u'(t)\|^2 \leq CM(t) \leq C(1+t)^{-1}$ by (2.4), and furthermore, $\|u''(t)\|^2 \leq C(1+t)^{-1}$ by (1.1), that is, the desired estimate (2.2) holds true. □

Proposition 2.3 *Under the assumption of Theorem 1.2, it holds that*

$$F(t) \equiv \rho\|A^{1/2}u'(t)\|^2 + a(M(t))\|Au(t)\|^2 \le C(1+t)^{-\omega} \quad \textit{for} \quad t \ge 0 \tag{2.10}$$

with $\omega = \min\{2\,,\,1+2\gamma\}$.

Proof. Taking the inner product of (1.1) with $2Au'(t)$, we have that

$$\begin{aligned}\frac{d}{dt}F(t) + 2\|A^{1/2}u'(t)\|^2 &= a'(M(t))M'(t)\|Au(t)\|^2 \\ &\le CM(t)^{\gamma+\frac{1}{2}}\frac{\|Au(t)\|^2}{M(t)}\|A^{1/2}u'(t)\|\end{aligned} \tag{2.11}$$

and from the Young inequality that

$$\frac{d}{dt}F(t) + \|A^{1/2}u'(t)\|^2 \le Cf(t)^2 \quad \text{with} \quad f(t)^2 = M(t)^{2\gamma+1}\frac{\|Au(t)\|^4}{M(t)^2}\,. \tag{2.12}$$

Integrating (2.12) over $[t, t+1]$, we have

$$\int_t^{t+1}\|A^{1/2}u'(s)\|^2 ds = F(t) - F(t+1) + C\sup_{t\le s\le t+1} f(s)^2 \quad (\equiv D(t)^2)\,. \tag{2.13}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|A^{1/2}u'(t_j)\|^2 \le 4D(t)^2 \quad \text{for} \quad j = 1, 2\,. \tag{2.14}$$

On the other hand, taking the inner product of (1.1) with $Au(t)$, we have

$$a(M(t))\|Au(t)\|^2 = \rho\left(\|A^{1/2}u'(t)\|^2 - \frac{d}{dt}(A^{1/2}u', A^{1/2}u)\right) - (A^{1/2}u', A^{1/2}u)$$

and hence

$$F(t) = 2\rho\|A^{1/2}u'(t)\|^2 - \rho\frac{d}{dt}(A^{1/2}u', A^{1/2}u) - (A^{1/2}u', A^{1/2}u)\,. \tag{2.15}$$

Integrating (2.15) over $[t_1, t_2]$, we have from (2.13) and (2.14) that

$$\begin{aligned}&\int_{t_1}^{t_2} F(s)\,ds \\ &\le 2\rho\int_t^{t+1}\|A^{1/2}u'(s)\|^2\,ds + \rho\sum_{j=1}^{2}\|A^{1/2}u'(t_j)\|\|A^{1/2}u(t_j)\| \\ &\quad + \int_t^{t+1}\|A^{1/2}u'(s)\|\|A^{1/2}u(s)\|\,ds \\ &\le CD(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) \quad \text{with} \quad g(t)^2 = M(t)\,.\end{aligned} \tag{2.16}$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$F(t_*) \le 2 \int_{t_1}^{t_2} F(s)\, ds\,. \tag{2.17}$$

For $\tau \in [t, t+1]$, integrating (2.11) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (2.12) and (2.17) that

$$\begin{aligned} F(\tau) &= F(t_*) + \int_{\tau}^{t_*} \left(2\|A^{1/2}u'(s)\|^2 - a'(M(s))M'(s)\|Au(s)\|^2\right) ds \\ &\le 2\int_{t_1}^{t_2} F(s)\, ds + C\int_t^{t+1} \|A^{1/2}u'(s)\|^2 ds + C\int_t^{t+1} f(s)^2 ds \\ &\le CD(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) + C \sup_{t\le s\le t+1} f(s)^2\,. \end{aligned}$$

Since it holds that

$$D(t)^2 = F(t) - F(t+1) + C \sup_{t\le s\le t+1} f(s)^2 \le F(t) + \sup_{t\le s\le t+1} f(s)^2$$

by (2.13), we observe

$$\begin{aligned} &\sup_{t\le s\le t+1} F(s)^2 \\ &\quad\le C\left(D(t)^2 + \sup_{t\le s\le t+1} g(s)^2\right) D(t)^2 + C \sup_{t\le s\le t+1} f(s)^4 \\ &\quad\le C\left(F(t) + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(F(t) - F(t+1)) \\ &\qquad + CF(t) \sup_{t\le s\le t+1} f(s)^2 + C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2 \end{aligned}$$

and hence

$$\begin{aligned} &\sup_{t\le s\le t+1} F(s)^2 \\ &\quad\le C\left(F(t) + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(F(t) - F(t+1)) \\ &\qquad + C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2\,. \end{aligned} \tag{2.18}$$

Since it holds that

$$f(t)^2 = \begin{cases} M(t)^{2\gamma+1}\dfrac{\|Au(t)\|^4}{M(t)^2} \le CM(t)^{2\gamma+1} \le C(1+t)^{-(1+2\gamma)} \\ M(t)^{2\gamma}\dfrac{\|Au(t)\|^2}{M(t)}\|Au(t)\|^2 \le CM(t)^{2\gamma}\|Au(t)\|^2 \le C(1+t)^{-2\gamma}F(t) \end{cases}$$

and $g(t)^2 = M(t) \le C(1+t)^{-1}$, we have

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t) + (1+t)^{-1}\right)(F(t) - F(t+1)) \\ + C(1+t)^{-(1+2\gamma)} \sup_{t\le s\le t+1} F(s)$$

and hence

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t) + (1+t)^{-1}\right)(F(t) - F(t+1)) \\ + C(1+t)^{-2(1+2\gamma)}\,. \tag{2.19}$$

Thus, applying Lemma 2.1 to (2.19), we obtain

$$F(t) \le C(1+t)^{-\omega} \quad \text{with} \quad \omega = \min\{2\,,\,1+2\gamma\}$$

which implies the desired estimate (2.10). □

Proposition 2.4 *Under the assumption of Theorem 1.2, it holds that*

$$\|u'(t)\| \le C(1+t)^{-\omega} \quad \textit{for} \quad t \ge 0 \tag{2.20}$$

with $\omega = \min\{2\,,\,1+2\gamma\}$.

Proof. Taking the inner product of (1.1) with $2u'(t)$, we have

$$\rho\frac{d}{dt}\|u'(t)\|^2 + 2\|u'(t)\|^2 = -2a(M(t))(Au(t), u'(t))\,,$$

and by the Young inequality we observe

$$\rho\frac{d}{dt}\|u'(t)\|^2 + \|u'(t)\|^2 \le a(M(t))^2\|Au(t)\|^2\,.$$

Thus, from (1.7) and (2.10) we drive the desired estimate (2.20). □

Proposition 2.5 *Under the assumption of Theorem 1.2, it holds that*

$$L(t) \equiv \rho\|u''(t)\|^2 + a(M(t))\|A^{1/2}u'(t)\|^2 + \frac{a'(M(t))}{2}|M'(t)|^2 \\ \le C(1+t)^{-\sigma} \quad \textit{for} \quad t \ge 0 \tag{2.21}$$

with $\sigma = \min\{3\,,\,(1+\gamma)(1+2\gamma)\}$.

Proof. Taking the inner product of (1.1) differentiated with respect to t with $2u''(t)$, we have

$$\frac{d}{dt}L(t) + 2\|u''(t)\|^2 \\ = 3a'(M(t))M'(t)\|A^{1/2}u'(t)\|^2 + \frac{a''(M(t))}{2}(M'(t))^3 \tag{2.22}$$

$$\le Cf(t)^2 \quad \text{with} \quad f(t)^2 = M(t)^\gamma\frac{|M'(t)|}{M(t)}\|A^{1/2}u'(t)\|^2\,. \tag{2.23}$$

Integrating (2.23) over $[t, t+1]$, we have

$$2\int_t^{t+1} \|u''(s)\|^2 ds \le L(t) - L(t+1) + C \sup_{t\le s\le t+1} f(s)^2 \quad (\equiv 2D(t)^2)\,. \tag{2.24}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u''(t_j)\|^2 \le 4D(t)^2 \quad \text{for} \quad j = 1, 2\,. \tag{2.25}$$

On the other hand, taking the inner product of (1.1) differentiated with respect to t with $u'(t)$, we have

$$\begin{aligned} &a(M(t))\|A^{1/2}u'(t)\|^2 + \frac{a'(M(t))}{2}|M'(t)|^2 \\ &\quad = \rho\left(\|u''(t)\|^2 - \frac{d}{dt}(u''(t), u'(t))\right) - (u''(t), u'(t)) \end{aligned}$$

and hence

$$L(t) = 2\rho\|u''(t)\|^2 - \rho\frac{d}{dt}(u''(t), u'(t)) - (u''(t), u'(t))\,. \tag{2.26}$$

Integrating (2.26) over $[t_1, t_2]$, we observe from (2.24) and (2.25) that

$$\begin{aligned} &\int_{t_1}^{t_2} L(s)\,ds \\ &\quad \le 2\rho\int_t^{t+1} \|u''(s)\|^2 ds + \rho\sum_{j=1}^{2} \|u''(t_j)\|\|u'(t_j)\| + \int_t^{t+1} \|u''(s)\|\|u'(s)\|\,ds \\ &\quad \le CD(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) \qquad \text{with} \quad g(t)^2 = \|u'(t)\|^2\,. \end{aligned} \tag{2.27}$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$L(t_*) \le 2\int_{t_1}^{t_2} L(s)\,ds\,. \tag{2.28}$$

For $\tau \in [t, t+1]$, integrating (2.22) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (2.23) and (2.28) that

$$\begin{aligned} &L(\tau) = L(t_*) \\ &+ \int_\tau^{t_*} \left(2\rho\|u''(s)\|^2 - 3a'(M(t))M'(s)\|A^{1/2}u'(s)\|^2 + \frac{a(M(s))}{2}(M'(s))^3\right) ds \\ &\le 2\int_{t_1}^{t_2} L(s)\,ds + C\int_t^{t+1} \|u''(s)\|^2 ds + C\int_t^{t+1} f(s)^2 ds \\ &\le CD(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) + C \sup_{t\le s\le t+1} f(s)^2\,. \end{aligned}$$

Since it holds that

$$D(t)^2 = L(t) - L(t+1) + C \sup_{t\le s\le t+1} f(s)^2 \le L(t) + \sup_{t\le s\le t+1} f(s)^2$$

by (2.24), we observe

$$\begin{aligned}
&\sup_{t\le s\le t+1} L(s)^2 \\
&\quad \le C\left(D(t)^2 + \sup_{t\le s\le t+1} g(s)^2\right) D(t)^2 + C \sup_{t\le s\le t+1} f(s)^4 \\
&\quad \le C\left(L(t) + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(L(t) - L(t+1)) \\
&\qquad + CL(t) \sup_{t\le s\le t+1} f(s)^2 + C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{t\le s\le t+1} L(s)^2 \\
&\quad \le C\left(L(t) + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(L(t) - L(t+1)) \\
&\qquad + C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2 . \qquad (2.29)
\end{aligned}$$

(i) When $0 < \gamma < \frac{1}{2}$, we put $\omega = 1 + 2\gamma$. Since it holds that

$$\begin{aligned}
f(t)^2 &\le 2 \frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}} \frac{\|u'(t)\|^{1-2\gamma}}{M(t)^{\frac{1}{2}}} \|u'(t)\|^{2\gamma} \|A^{1/2}u'(t)\|^2 \\
&\le C\|u'(t)\|^{2\gamma}\|A^{1/2}u'(t)\|^2 \le \begin{cases} C(1+t)^{-(1+\gamma)\omega} \\ C(1+t)^{-\gamma\omega}L(t) \end{cases}
\end{aligned}$$

and $g(t)^2 = \|u'(t)\|^2 \le C(1+t)^{-\omega}$, we have

$$\begin{aligned}
\sup_{t\le s\le t+1} L(t)^2 &\le C\left(L(t) + (1+t)^{-\omega}\right)(L(t) - L(t+1)) \\
&\quad + C(1+t)^{-(1+\gamma)\omega} \sup_{t\le s\le t+1} L(s)
\end{aligned}$$

and hence

$$\begin{aligned}
\sup_{t\le s\le t+1} L(t)^2 &\le C\left(L(t) + (1+t)^{-\omega}\right)(L(t) - L(t+1)) \\
&\quad + C(1+t)^{-2(1+\gamma)\omega} . \qquad (2.30)
\end{aligned}$$

Thus, applying Lemma 2.1 to (2.30), we obtain

$$L(t) \le C(1+t)^{-\sigma} \quad \text{with} \quad \sigma = \{\omega + 1\,,\, (1+\gamma)\omega\} = (1+\gamma)(1+2\gamma)$$

which implies the desired estimate (2.21) for $0 < \gamma < \frac{1}{2}$.

(ii) When $\gamma \ge \frac{1}{2}$, we put $\omega = 2$. Since it holds that

$$\begin{aligned} f(t)^2 &\le 2M(t)^{\gamma-\frac{1}{2}} \frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}} \|u'(t)\| \|A^{1/2}u'(t)\| \\ &\le CM(t)^{\gamma-\frac{1}{2}} \|u'(t)\| \|A^{1/2}u'(t)\| \le \begin{cases} C(1+t)^{-(\gamma+\frac{3\omega-1}{2})} \\ C(1+t)^{-(\gamma+\frac{\omega-1}{2})} L(t) \end{cases} \end{aligned}$$

and $g(t)^2 = \|u'(t)\|^2 \le C(1+t)^{-\omega}$, we have

$$\begin{aligned} \sup_{t \le s \le t+1} L(t)^2 &\le C\left(L(t) + (1+t)^{-\omega}\right)(L(t) - L(t+1)) \\ &\quad + C(1+t)^{-(\gamma+\frac{3\gamma-1}{2})} \sup_{t \le s \le t+1} L(s) \end{aligned}$$

and hence

$$\begin{aligned} \sup_{t \le s \le t+1} L(t)^2 &\le C\left(L(t) + (1+t)^{-\omega}\right)(L(t) - L(t+1)) \\ &\quad + C(1+t)^{-2(\gamma+\frac{3\gamma-1}{2})}\,. \end{aligned} \tag{2.31}$$

Thus, applying Lemma 2.1 to (2.31), we obtain

$$L(t) \le C(1+t)^{-\sigma} \quad \text{with} \quad \sigma = \{\omega + 1\,,\, \gamma + \frac{3\gamma - 1}{2}\} = 3$$

which implies the desired estimate (2.21) for $\gamma \ge \frac{1}{2}$. □

Proof of Theorem 1.2. Gathering Propositions 2.2–2.5, we conclude Theorem 1.2. □

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