J. Math. Tokushima Univ.
Vol. 53 (2019), 5 – 26

Axiomatic Method of Measure and Integration (III). Definition and Existence Theorem of the Lebesgue Measure

(Yoshifumi Ito "Theory of Lebesgue Integral", Chapter 5)

By

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(Received February 28, 2019)

Abstract

In this paper, we define the Lebesgue measure on $\boldsymbol{R}^d$, $(d \geq 1)$ by prescribing the complete system of axioms. Then we prove the uniqueness and existence theorem of the Lebesgue measure. This is a new result.

2000 Mathematics Subject Classification. Primary 28Axx.

Introduction

This paper is the part III of the series of papers on the axiomatic method of measure and integration on the Euclidean space.

As for the details, we refer to Ito [11]. Further, we refer to Ito [1] $\sim$ [10], [12] $\sim$ [14].

In this paper, we define the concept of the d-dimensional Lebesgue measure and its uniqueness and existence theorem. Here we assume $d \geq 1$.

The d-dimensional Jordan measure is a conditionally completely additive positive measure. On the other hand, the d-dimensional Lebesgue measure is a completely additive positive measure. This is the completion of the d-dimensional Jordan measure.

The Lebesgue measure is a completely additive positive measure defined on the completely additive family of all Lebesgue measurable sets and it is an invariant measure with respect to the group of congruent transformations.

In this paper, it is the new characterization that we define the Lebesgue measure by describing the complete system of axioms.

Further we prove the uniqueness and existence theorem of the Lebesgue measure by way of constructing the measure which satisfies the conditions of the system of axioms. Thus, the definition of the Lebesgue measure and its uniqueness and existence theorem are the new results. We call this the axiomatic method of the measure and integration.

Until now, we construct the Lebesgue measure as one of the set functions without defining the concept of the Lebesgue measure. Then there is one question that there is or not another measure than the well known Lebesgue measure. When we define the Lebesgue measure by giving the complete system of axioms, we can prove that there is the unique measure satisfying this system of axioms.

Thereby we know that there is no other Lebesgue measure than the measure constructed by Lebesgue himself. In this point, it is important that the theory of the Lebesgue measure is completed.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Definition of the intervals, the blocks of intervals and the Borel sets

In this section, we prepare the necessary facts for the Lebesgue measure. For that purpose, we study the families of the intervals and the blocks of intervals and the Borel sets in the d-dimensional Euclidean space $\boldsymbol{R}^d$. Here we assume $d \geq 1$.

At first, we study the intervals which are the fundamental subsets of the d-dimensional Euclidean space $\boldsymbol{R}^d$.

We say that an interval I in $\boldsymbol{R}^d$ is the direct product set of the d intervals $J_1,\ J_2,\ \cdots,\ J_d$ in $\boldsymbol{R}$. We denote this as follows:

$$I = \prod_{p=1}^{d} J_p.$$

Then we denote the interior of I by the symbol

$$I^\circ = (a_1,\ b_1) \times (a_2,\ b_2) \times \cdots \times (a_d,\ b_d).$$

Here we assume that $a_p,\ b_p,\ (1 \leq p \leq d)$ are certain real numbers or $-\infty$ or ∞. Further we assume that $a_p \leq b_p$ holds for $1 \leq p \leq d$ and $-\infty$ or ∞ is not a point in the interval $J_p,\ (1 \leq p \leq d)$. Further we assume that the interior of J_p is equal to

$$J_p^\circ = (a_p,\ b_p)$$

for $1 \leq p \leq d$. Then I° is an open interval.

Further we denote the closure $\overline{I}$ of I by the symbol

$$\overline{I} = [a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times [a_d,\ b_d].$$

Then $\overline{I}$ is a closed interval. The empty set ϕ is considered as an interval.

Now we denote the family of all intervals in $\boldsymbol{R}^d$ by $\mathcal{P}$.

Then we have the following theorem.

Theorem 1.1 *Assume that $\mathcal{P}$ is the family of all intervals in $\boldsymbol{R}^d$. Then we have the following* (1) *and* (2):

(1) *For $A,\ B \in \mathcal{P}$, we have $A \cap B \in \mathcal{P}$.*

(2) *For $A,\ B \in \mathcal{P}$, there exist a certain positive natural number n and the mutually disjoint n intervals $C_1,\ C_2,\ \cdots, C_n$ in $\mathcal{P}$, we have the equality*

$$A - B = C_1 + C_2 + \cdots + C_n.$$

Therefore, the family of all intervals in $\boldsymbol{R}^d$ is a **semi-ring** of sets.

Next, we study the blocks of intervals in $\boldsymbol{R}^d$. We say that a subset E in $\boldsymbol{R}^d$ is a **block of intervals** if there exist a finite number of mutually disjoint intervals $I_1,\ I_2,\ \cdots,\ I_n$ such that E is equal to the direct sum

$$E = \bigcup_{p=1}^{n} I_p = \sum_{p=1}^{n} I_p = I_1 + I_2 + \cdots + I_n.$$

We call this as the **division** of the block of intervals.

In general, there are infinitely many kinds of the divisions of a block of intervals E.

Now we denote by $\mathcal{R}$ the family of all blocks of intervals in $\boldsymbol{R}^d$.

Then we have the following theorem 1.2.

Theorem 1.2 *Let $\mathcal{R}$ be the family of all blocks of intervals in $\boldsymbol{R}^d$. Then we have the following* (1) $\sim$ (3):

(1) *$\phi \in \mathcal{R}$ holds.*

(2) *For* $A \in \mathcal{R}$, *we have* $A^c = \{x \in \boldsymbol{R}^d;\ x \notin A\} \in \mathcal{R}$.

(3) *For* $A,\ B \in \mathcal{R}$, *we have* $A \cup B \in \mathcal{R}$.

Therefore the family of sets $\mathcal{R}$ is a **ring** of sets.

Corollary 1.1 *Let* $\mathcal{R}$ *be the same as in Theorem* 1.2. *Then we have the following* (1) $\sim$ (3):

(1) *We have* $\boldsymbol{R}^d \in \mathcal{R}$.

(2) *For* $A,\ B \in \mathcal{R}$, *we have* $A - B \in \mathcal{R}$.
Here, the difference $A - B$ *of the sets* A *and* B *is defined by the relation*

$$A - B = A \cap B^c = \{x \in \boldsymbol{R}^d;\ x \in A,\ x \notin B\}.$$

(3) *For* $A_p \in \mathcal{R},\ (1 \leq p \leq n)$, *we have*

$$\bigcup_{p=1}^{n} A_p \in \mathcal{R},\ \bigcup_{p=1}^{n} A_p \in \mathcal{R}.$$

Therefore, the ring of sets $\mathcal{R}$ is an algebra of sets because the condition (1) of Corollary 1.1 is satisfied.

Definition 1.2 We define that the nonempty family $\boldsymbol{B}$ of sets of $\boldsymbol{R}^d$ is a σ**-ring** if the following conditions (i) and (ii) are satisfied:

(i) For $A,\ B \in \boldsymbol{B}$, we have $A - B \in \boldsymbol{B}$.

(ii) For $A_p \in \boldsymbol{B},\ (1 \leq p < \infty)$, we have $\displaystyle\bigcup_{p=1}^{\infty} A_p \in \boldsymbol{B}$.

Corollary 1.2 *For* $\boldsymbol{B}$ *be a* σ*-ring of subsets of* $\boldsymbol{R}^d$. *Then, for* $A_p \in \boldsymbol{B}$, $(1 \leq p < \infty)$, *the sets*

$$\bigcap_{p=1}^{\infty} A_p,\ \overline{\lim_{p\to\infty}}\ A_p \text{ and } \underline{\lim_{p\to\infty}}\ A_p$$

also belong to $\boldsymbol{B}$.

In Corollary 1.2, we define the superior limit $\overline{\lim}\ A_p$ and the inferior limit $\underline{\lim}\ A_p$ of a sequence of subsets $\{A_p\}$ by virtue of the relations

$$\overline{\lim}\ A_p = \overline{\lim_{p\to\infty}}\ A_p = \bigcap_{n=1}^{\infty} \bigcup_{p=n}^{\infty} A_p,$$

$$\underline{\lim} A_p = \underline{\lim}_{p\to\infty} A_p = \bigcup_{n=1}^{\infty} \bigcap_{p=n}^{\infty} A_p.$$

Now we denote by $\sigma(\mathcal{F})$ the smallest σ-ring which includes the family $\mathcal{F}$ of subsets of $\boldsymbol{R}^d$. Then we say that this $\sigma(\mathcal{F})$ is the **σ-ring** generated by the family $\mathcal{F}$ of subsets.

Then we denote the σ-ring generated by the family $\mathcal{O}$ of all open sets in $\boldsymbol{R}^d$ as $\boldsymbol{B} = \sigma(\mathcal{O})$. Then we say that $\boldsymbol{B} = \sigma(\mathcal{O})$ is the **Borel ring** and an element of $\boldsymbol{B}$ is a **Borel set** of $\boldsymbol{R}^d$.

Further we denote the σ-ring generated by the family $\mathcal{P}$ of sets as $\sigma(\mathcal{P})$. We also denote the σ-ring generated by the family $\mathcal{R}$ of sets as $\sigma(\mathcal{R})$.

Then we have the following Corollary.

Corollary 1.3 *We use the notation in the above. Then we have the following* (1) $\sim$ (4):

(1) *We have* $\mathcal{P} \subset \mathcal{R} \subset \boldsymbol{B}$.

(2) *We have* $\boldsymbol{B} = \sigma(\mathcal{P}) = \sigma(\mathcal{R})$.

(3) *We have* $\boldsymbol{R}^d \in \boldsymbol{B}$.

(4) *For* $A \in \boldsymbol{B}$, *we have* $A^c \in \boldsymbol{B}$.

Therefore $\boldsymbol{B}$ is a σ-algebra by virtue of the conditions (3) and (4) of Corollary 1.3. Then we say that $\boldsymbol{B}$ is the **Borel algebra**.

Remark 1.1 When we study a semi-ring, a ring, an algebra, a σ-ring, a σ-algebra and the Borel algebra as a families of subsets of $\boldsymbol{R}^d$, we can well understand their meaning and the reason why we study such a families of subsets by way of studying them at the point of view of the calculation of sets.

In general, we have the following proposition for the Borel sets.

Proposition 1.1 *An arbitrary element of the* Borel *algebra* $\boldsymbol{B}$ *is included in a union of a certain countable elements of* $\mathcal{P}$. *Further an arbitrary element of the* Borel *algebra* $\boldsymbol{B}$ *is included in a union of a certain countable elements of* $\mathcal{R}$.

2 Definition of the Lebesgue measure

In this section, we define the concept of the Lebesgue measure. By way of the inductive reasoning on the bases of the knowledge gained through the study of the Lebesgue measure until now, we define the Lebesgue measure and the Lebesgue measure space as follows.

Definition 2.1(Lebesgue measure) If the family $\mathcal{M}$ of sets on the d-dimensional Euclidean space $\boldsymbol{R}^d$ and the set function μ on $\mathcal{M}$ satisfy the following Axioms (I) $\sim$ (III), we define that the triplet $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is the d**-dimensional Lebesgue measure space**. Then we say that an element in $\mathcal{M}$ is a **Lebesgue measurable set** and μ is the d**-dimensional Lebesgue measure**.

(I) We have $\boldsymbol{B} \subset \mathcal{M}$.

(II) We have the following (i) $\sim$ (iv):

(i) For $A \in \mathcal{M}$, we have $0 \leq \mu(A) \leq \infty$.

(ii) If a countable elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ in $\mathcal{M}$ are mutually disjoint, then we have the condition

$$A = \bigcup_{p=1}^{\infty} A_p = \sum_{p=1}^{\infty} A_p \in \mathcal{M}$$

and we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

(iii) For $I_0 = [0,\ 1]^d$, we have $\mu(I_0) = 1$.

(iv) If $A,\ B \in \mathcal{M}$ are congruent, we have $\mu(A) = \mu(B)$.

(III) $A \in \mathcal{M}$ if and only if, for any bounded $E \in \boldsymbol{B}$, we have the equality

$$\mu^*(A \cap E) = \mu_*(A \cap E).$$

Then we have the equality

$$\mu(A) = \sup\{\mu^*(A \cap E);\ E \in \boldsymbol{B} \text{ is bounded}\}.$$

Here μ^* and μ_* denote the outer measure and the inner measure respectively which are defined by the measure μ on $\boldsymbol{B}$ obtained by the restriction of μ on $\mathcal{M}$. Namely, $\mu^*(A \cap E)$ and $\mu_*(A \cap E)$ are defined by the formulas

$$\mu^*(A \cap E) = \inf\{\mu(B);\ B \supset A \cap E,\ B \in \boldsymbol{B}\},$$

$$\mu_*(A \cap E) = \sup\{\mu(B);\ A \cap E \supset B,\ B \in \boldsymbol{B}\}$$

respectively.

For simplicity, we call the d-dimensional Lebesgue measure space and the d-dimensional Lebesgue measure as the **Lebesgue measure space** and the **Lebesgue measure** respectively.

Further, we call a Lebesgue measurable set as a measurable set.

The condition (ii) of the axiom (II) means that the Lebesgue measure is a completely additive measure.

In the condition (iv) of the axiom (II), we say that $A,\ B \in \mathcal{M}$ are congruent if we can put A on B by several operations of the rotations and the parallel translations of $\boldsymbol{R}^d$.

We remark that the condition (iv) of the axiom (II) may be replaced by the following condition (II), (iv)$'$:

(II), (iv)$'$ If $A + x$ is the parallel translation of a set $A \in \mathcal{M}$ for a vector $x \in R^d$, then we have the condition $A + x \in \mathcal{M}$ and we have the equality $\mu(A + x) = \mu(A)$.

In the following, we prove the existence theorem of the Lebesgue measure defined in Definition 2.1.

In order to do so, we have only to determine the family $\mathcal{M}$ of the Lebesgue measurable sets and the Lebesgue measure μ on $\boldsymbol{R}^d$ concretely.

At first, on the assumption of the existence of the Lebesgue measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ which satisfies the system of axioms in Definition 2.1, we have to determine the following (1) and (2):

(1) What kind of sets should be an element of $\mathcal{M}$.

(2) How is defined the value of $\mu(A)$ for any element A in $\mathcal{M}$.

Further, by virtue of the axiom (III) of Definition 2.1, we see that we have the definition of the completely additive measure which is equal to the Lebesgue measure μ on the Borel algebra $\boldsymbol{B}$ and its existence. We say that such a measure on $\boldsymbol{B}$ is the **Borel measure**.

3 Definition of the Borel measure

In this section, we define the Borel measure and prove its existence theorem. At first, we see that we should define the Borel measure as in the following Definition 3.1 by virtue of Definition 2.1.

Definition 3.1 (Borel measure) We define that a set function μ on the Borel algebra $\boldsymbol{B}$ of $\boldsymbol{R}^d$ is the d-dimensional Borel measure if we have the following axioms (i) $\sim$ (iv):

(i) For $A \in \boldsymbol{B}$, we have $0 \leq \mu(A) \leq \infty$.

(ii) If a countable elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ of $\boldsymbol{B}$ are mutually disjoint, then the direct sum

$$A = \bigcup_{p=1}^{\infty} A_p = \sum_{p=1}^{\infty} A_p$$

belongs to $\boldsymbol{B}$ and we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

(iii) For $I_0 = [0,\ 1]^d$, we have $\mu(I_0) = 1$.

(iv) If $A,\ B \in \boldsymbol{B}$ are congruent, we have $\mu(A) = \mu(B)$.

Then we say that the triplet $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ is the **d-dimensional Borel measure space**.

For simplicity, we say that the d-dimensional Borel measure space and the d-dimensional measure are the **Borel measure space** and the **Borel measure** respectively.

We remark that the condition (iv) of Definition 3.1 may be replaced by the following condition (iv)$'$:

(iv)$'$ If $A+x$ is the set of the translation of a set $A \in \boldsymbol{B}$ for a vector $x \in \boldsymbol{R}^d$, we have the condition $A + x \in \boldsymbol{B}$ and the equality

$$\mu(A + x) = \mu(A).$$

Corollary 3.1 *For the* Borel *measure space* $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$, *we have the following* (1) $\sim$ (3):

(1) *If* $A_1,\ \cdots,\ A_n \in \boldsymbol{B}$ *are mutually disjoint, we have the equality*

$$\mu(\sum_{p=1}^{n} A_p) = \sum_{p=1}^{n} \mu(A_p).$$

(Finite additivity).

(2) *If* $A,\ B \in \boldsymbol{B}$ *satisfy the relation* $A \supset B$, *then we have the inequality* $\mu(A) \geq \mu(B)$. *Especially, if* $\mu(B) < \infty$ *holds, we have the equality*

$$\mu(A \backslash B) = \mu(A) - \mu(B).$$

(3) *For* $A_p \in \boldsymbol{B},\ (p \geq 1)$, *we have the inequality*

$$\mu(\bigcup_{p=1}^{\infty} A_p) \leq \sum_{p=1}^{\infty} \mu(A_p).$$

(Complete sub-additivity).

Especially, for $A_1,\ A_2,\ \cdots,\ A_n \in \boldsymbol{B}$, *we have the inequality*

$$\mu(\bigcup_{p=1}^{n} A_p) \leq \sum_{p=1}^{n} \mu(A_p).$$

(Finite sub-additivity).

4 Existence theorem of the Borel measure

In this section, we prove the existence theorem of the Borel measure. For that purpose, we determine the set function μ on $\boldsymbol{B}$ so that it satisfies the system of axioms in Definition 3.1.

Since $\mathcal{R} \subset \boldsymbol{B}$ holds by virtue of the condition (2) of Corollary 1.3, we obtain the set function μ on $\mathcal{R}$ by restricting the Borel measure μ on $\boldsymbol{B}$ to $\mathcal{R}$.

Since this set function μ is completely additive on $\boldsymbol{B}$, it is a conditionally completely additive measure on $\mathcal{R}$. Because the uniqueness and existence theorem of the Jordan measure has been proved as such a measure on $\mathcal{R}$, the measure which is obtained by restricting the Borel measure to $\mathcal{R}$ coincides with the Jordan measure on $\mathcal{R}$.

Namely we have the following theorem.

Theorem 4.1 *The measure which is obtained by restricting the* Borel *measure* μ *on* $\boldsymbol{B}$ *to* $\mathcal{R}$ *coincides with the* Jordan *measure on* $\mathcal{R}$. *Namely the following axioms* (1) $\sim$ (4) *are satisfied*:

(1) *For* $A \in \mathcal{R}$, *we have* $0 \leq \mu(A) \leq \infty$.

(2) *If at most countable elements* $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ *in* $\mathcal{R}$ *are mutually disjoint and the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p \in \mathcal{R}$$

is satisfied, we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(3) *For* $I_0 = [0,\ 1]^d$, *we have* $\mu(I_0) = 1$.

(4) *If* $E + x$ *is the set of the translation of a set* $E \in \mathcal{R}$ *for a vector* $x \in \boldsymbol{R}^d$, *we have the condition* $E + x \in \mathcal{R}$ *and the equality* $\mu(E + x) = \mu(E)$.

The uniqueness and existence theorem of the Jordan measure has already been proved.

Then we have the following theorem.

Theorem 4.2 *For the* Jordan *measure* μ *on* $\mathcal{R}$, *we have the following* (1) $\sim$ (4):

(1) *If* $A_1,\ A_2,\ \cdots,\ A_n \in \mathcal{R}$ *are mutually disjoint, then we have the condition*

$$A = \bigcup_{p=1}^{n} A_p = \sum_{p=1}^{n} A_p \in \mathcal{R}$$

and we have the equality

$$\mu(A) = \sum_{p=1}^{n} \mu(A_p).$$

(2) *If we have* $A \supset B$ *for* $A,\ B \in \mathcal{R}$, *we have the inequality* $\mu(A) \geq \mu(B)$. *Especially, if* $\mu(A) < \infty$ *holds, we have the equality* $\mu(A - B) = \mu(A) - \mu(B)$. *Further we have the equality* $\mu(\phi) = 0$.

(3) *If we have the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p \in \mathcal{R}$$

for at most countable elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ *of* $\mathcal{R}$, *we have the inequality*

$$\mu(A) \leq \sum_{p=1}^{(\infty)} \mu(A_p).$$

(4) *If at most countable intervals* $I_1,\ I_2,\ \cdots,\ I_n,\ \cdots$ *are mutually disjoint and the direct sum*

$$I = \bigcup_{p=1}^{(\infty)} I_p = \sum_{p=1}^{(\infty)} I_p$$

is also an interval, we have the equality

$$\mu(I) = \sum_{p=1}^{(\infty)} \mu(I_p).$$

Therefore the Jordan measure on $\mathcal{R}$ is determined by the conditions in the following theorem.

Theorem 4.3 *The* Jordan *measure* μ *on* R *satisfies the following* (1) $\sim$ (4):

(1) *For a bounded closed interval* $[a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times \cdots [a_d,\ b_d]$ *or an interval* I *obtained by removing the part or the whole of its boundary, we have the equality*

$$\mu(I) = \prod_{p=1}^{d} (b_p - a_p).$$

Here we assume that $a_p,\ b_p,\ (1 \leq p \leq d)$ *are some real numbers such as* $a_p \leq b_p,\ (1 \leq p \leq d)$ *hold.*

(2) *For a unbounded interval* I, *we have either one of the following* (a) *and* (b):

 (a) *When* I *is not included in any hyperplane which is parallel to a certain coordinate axis, we have the equality* $\mu(I) = \infty$.

 (b) *When* I *is included in a certain hyperplane which is parallel to a certain coordinate axis, we have the equality* $\mu(I) = 0$.

(3) *If we decompose a block* A *of intervals* $I_1,\ I_2,\ \cdots,\ I_n$ *and denote it as*

$$A = I_1 + I_2 + \cdots + I_n,$$

we have the equality

$$\mu(A) = \mu(I_1) + \mu(I_2) + \cdots + \mu(I_n).$$

(4) *If at most countable intervals* $I_1,\ I_2,\ I_3,\ \cdots$ *are mutually disjoint and the direct sum*

$$I = \bigcup_{p=1}^{(\infty)} I_p = \sum_{p=1}^{(\infty)} I_p$$

is also an interval, we have the equality

$$\mu(I) = \sum_{p=1}^{(\infty)} \mu(I_p).$$

By virtue of Theorem 4.1 $\sim$ Theorem 4.3, the Borel measure μ must satisfy the conditions (1) $\sim$ (4) of Theorem 4.3 for an element in $\mathcal{R}$.

Further we have the following theorem.

Theorem 4.4 *For an arbitrary element $A \in \boldsymbol{B}$, we have the equality*

$$\mu(A) = \inf \sum_{p=1}^{\infty} \mu(E_p).$$

Here inf is taken over the all sequences $\{E_p\}$ of a countable elements of $\mathcal{R}$ such that its union includes A.

Conversely, the existence theorem of the Borel measure is given in the following theorem.

Theorem 4.5 *Assume that μ is the* Jordan *measure on $\mathcal{R}$. Then we put*

$$\widetilde{\mu}(A) = \inf \sum_{p=1}^{\infty} \mu(E_p)$$

for an arbitrary element $A \in \boldsymbol{B}$.

Here inf is taken over the all sequences $\{E_p\}$ of a countable elements of $\mathcal{R}$ such that its union includes A. Then $\widetilde{\mu}$ is the Borel *measure on $\boldsymbol{B}$.*

For simplicity, in the sequel, we denote $\widetilde{\mu}$ as μ.

In Theorem 4.5, the uniquness existence theorem of the Borel measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ is proved.

5 Existence theorem of the Lebesgue measure

In this section, we prove the existence theorem of the Lebesgue measure.

When the Borel measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ of section 4 is given, we prove the existence theorem of the Lebesgue measure by constructing the Lebesgue measure in the following way by using this Borel measure.

Definition 5.1 For an arbitrary subset A of $\boldsymbol{R}^d$, we define that

$$\mu^*(A) = \inf\{\mu(B);\ B \supset A,\ B \in \boldsymbol{B}\},$$

$$\mu_*(A) = \sup\{\mu(B);\ A \supset B,\ B \in \boldsymbol{B}\}$$

are the **Lebesgue outer measure** and the **Lebesgue inner measure** of A respectively. For simplicity, we call them the **outer measure** and the **inner measure** of A respectively.

Corollary 5.1 *For $A \in \boldsymbol{B}$, we have the following equality*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

Here the third side denote the Borel measure of the Borel set A.

By virtue of the definitions of the Lebesgue outer measure and the Lebesgue inner measure, we have the following three propositions immediately.

In the following, let $A,\ A_1,\ A_2$ be some subsets of $\boldsymbol{R}^d$.

Proposition 5.1 *We have the following* (1) *and* (2):

(1) $0 \leq \mu_*(A) \leq \mu^*(A) \leq \infty$. (2) $\mu^*(\phi) = \mu_*(\phi) = 0$.

Proposition 5.2 *If $A_1 \subset A_2$ holds, we have the following* (1) *and* (2):

(1) $\mu^*(A_1) \leq \mu^*(A_2)$. (2) $\mu_*(A_1) \leq \mu_*(A_2)$.

Proposition 5.3 *We have the following inequality*

$$\mu^*(A_1 \cup A_2) \leq \mu^*(A_1) + \mu^*(A_2).$$

Proposition 5.4 *If we put*

$$A = \bigcup_{p=1}^{\infty} A_p$$

for a countable subsets $A_1,\ A_2,\ A_3,\ \cdots$ of $\boldsymbol{R}^d$, we have the inequality

$$\mu^*(A) \leq \sum_{p=1}^{\infty} \mu^*(A_p).$$

Proposition 5.5 *If a countable subsets $A_1,\ A_2,\ A_3,\ \cdots$ of $\boldsymbol{R}^d$ are mutually disjoint and we put*

$$A = \sum_{p=1}^{\infty} A_p,$$

we have the inequality

$$\mu_*(A) \geq \sum_{p=1}^{\infty} \mu_*(A_p).$$

Proposition 5.6 *Assume that A is an arbitrary subset of $\boldsymbol{R}^d$. Assume that $E_1,\ E_2,\ \cdots$ are some sequence of bounded Borel sets of $\boldsymbol{R}^d$ such that we have the following conditions* (1) *and* (2):

$$(1) \quad E_1 \subset E_2 \subset \cdots, \quad (2) \quad \bigcup_{p=1}^{\infty} E_p = \boldsymbol{R}^d.$$

Then we have the equalities

$$\mu^*(A) = \lim_{p \to \infty} \mu^*(A \cap E_p),$$

$$\mu_*(A) = \lim_{p \to \infty} \mu_*(A \cap E_p).$$

Definition 5.2 We say that an arbitrary subset A of $\boldsymbol{R}^d$ is **Lebesgue measurable** if we have the equality

$$\mu^*(A \cap E) = \mu_*(A \cap E)$$

for an arbitrary bounded set $E \in \boldsymbol{B}$. Then we say that

$$\mu(A) = \sup\{\mu^*(A \cap E);\ E \text{ is an arbitrary bounded Borel set}\}$$

is the **d-dimensional Lebesgue measure** of A.

Then, for simplicity, we say that A is **measurable** and $\mu(A)$ is the **Lebesgue measure** of A.

Remark 5.1 The measurability of a subset A of $\boldsymbol{R}^d$ means that, for any bounded part of A, the outer measure $\mu^*(A \cap E)$ which is the approximation of the measures of bounded Borel sets from outer side and the inner measure $\mu_*(A \cap E)$ which is the approximation of the measures of bounded Borel sets from inner side are both identical.

Now we denote the family of all Lebesgue measurable sets of $\boldsymbol{R}^d$ as $\mathcal{M}$.

Corollary 5.2 *For $A \in \mathcal{M}$, we have the equality*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

Then, by using the d-dimensional Lebesgue measure μ defined in Definition 5.2, we have the measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$.

In order to prove that this measure space is really the d-dimensional Lebesgue measure space, we have only to prove that the axioms (I) $\sim$ (III) of Definition 2.1 are satisfied.

The axiom (III) is evidently satisfied by virtue of Definition 5.2.

Further, the axiom (I) is satisfied because we have the following Corollary.

Corollary 5.3 *We have $\boldsymbol{B} \subset \mathcal{M}$. Namely a* Borel *set A of $\boldsymbol{R}^d$ is a* Lebesgue *measurable set and the* Borel *measure $\mu(A)$ of A coincides with the* Lebesgue *measure $\mu(A)$ of A.*

Therefore the Lebesgue measure μ is the extension of the Borel measure.

Remark 5.2 Since we can determine whether the measure $\mu(A)$ of a subset A of $\boldsymbol{R}^d$ is the Lebesgue measure or the Borel measure according to the condition that A is a Lebesgue measurable set or a Borel measurable set respectively, it is not confused to use the same symbol μ for the Lebesgue measure and the Borel measure.

In the following, we prove that the axiom (II) in Definition 2.1 is satisfied. By virtue of the definition of μ, we have evidently the following two Corollaries.

Corollary 5.4 *For $A \in \mathcal{M}$, we have $0 \leq \mu(A) \leq \infty$.*

Corollary 5.5 *For $I_0 = [0,\ 1]^d$, we have $\mu(I_0) = 1$.*

The axiom (II), (ii) follows from the following theorem.

Theorem 5.1 *If a countable elements $A_1,\ A_2,\ \cdots,\ A_p,\ \cdots$ of $\mathcal{M}$ are mutually disjoint and we put*

$$A = \bigcup_{p=1}^{\infty} A_p = \sum_{p=1}^{\infty} A_p,$$

then we have the condition $A \in \mathcal{M}$ and we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

Theorem 5.2 *The* Lebesgue *measure does not depend on the choice of an orthogonal coordinate system.*

Theorem 5.3 *If two subsets of $\boldsymbol{R}^d$ are congruent, then, if one of them is* Lebesgue *measurable, the other is also* Lebesgue *measurable and the* Lebesgue *measures of these two subsets are equal.*

By the considerations in the above, we see that the measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ satisfies the system of axioms (I) $\sim$ (III) of Definition 2.1.

Thus $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is the d-dimensional Lebesgue measure space. By virtue of the process of its construction, we see that the measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is only one d-dimensional Lebesgue measure space.

Thus we have the following theorem.

Theorem 5.4(Existence theorem) *In the space $\boldsymbol{R}^d$, there exists only one d-dimensional* Lebesgue *measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$. Here $\mathcal{M}$ is the family* *of all* Lebesgue *measurable sets of* $\boldsymbol{R}^d$ *and* μ *is the* d*-dimensional* Lebesgue *measure.*

Next we prove that all Jordan measurable sets are Lebesgue measurable. Namely we have the following theorem.

Theorem 5.5 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{B},\ \mu)$ *is the* Jordan *measure space. Then we have* $\mathcal{B} \subset \mathcal{M}$*. Further, for* $A \in \mathcal{B}$*, the* Jordan *measure* $\mu(A)$ *of* A *is equal to the* Lebesgue *measure* $\mu(A)$*.*

6 Lebesgue measurable sets

In this section, we study the fundamental properties of the operations of sets in the family $\mathcal{M}$ of all d-dimensional Lebesgue measurable sets in $\boldsymbol{R}^d$.

At first, we study the condition that a subset A of $\boldsymbol{R}^d$ is Lebesgue measurable.

Proposition 6.1 *Assume that* A *is a subset of* $\boldsymbol{R}^d$*. Then, for an arbitrary bounded set* $E \in \boldsymbol{B}$*, we have the equality*

$$\mu_*(A \cap E) = \mu(E) - \mu^*(A^c \cap E).$$

Here $\mu(E)$ *denotes the* Borel *measure.*

Proposition 6.2 *Assume that* A *is an arbitrary subset of* $\boldsymbol{R}^d$*. Then* $A \in \mathcal{M}$ *holds if and only if, for an arbitrary bounded set* $E \in \boldsymbol{B}$*, we have the equality*

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu(E).$$

The condition of this proposition is the condition which Lebesgue used for the definition of the measurable set.

Proposition 6.3 *Assume that* A *is an arbitrary subset of* $\boldsymbol{R}^d$*. Then the following* (1) $\sim$ (3) *are equivalent*:

(1) $A \in \mathcal{M}$ *holds.*

(2) *For an arbitrary subset* B *of* $\boldsymbol{R}^d$*, we have the equality*

$$\mu^*(A \cap B) + \mu^*(A^c \cap B) = \mu^*(B).$$

(3) *Assume that* A_1 *and* A_2 *are two arbitrary subsets of* $\boldsymbol{R}^d$ *such that* $A_1 \subset A$ *and* $A_2 \subset A^c$ *holds. Then we have the equality*

$$\mu^*(A_1 + A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Proposition 6.4 *Assume that A is a bounded set of $\boldsymbol{R}^d$. Then, $A \in \mathcal{M}$ holds if and only if, for an arbitrary positive number $\varepsilon > 0$, there exist $B_1,\ B_2 \in \boldsymbol{B}$ such that we have $B_1 \subset A \subset B_2$ and $\mu(B_2 \backslash B_1) < \varepsilon$.*

Proposition 6.5 *Assume that A is an arbitrary subset of $\boldsymbol{R}^d$. Then, $A \in \mathcal{M}$ holds if and only if, for an arbitrary positive number $\varepsilon > 0$, there exist $B_1,\ B_2 \in \boldsymbol{B}$ such that we have $B_1 \subset A \subset B_2$ and $\mu(B_2 \backslash B_1) < \varepsilon$.*

Proposition 6.6 *Assume that $\mathcal{M}$ is the family of all d-dimensional* Lebesgue *measurable sets. Then we have the following* (1) *and* (2):

(1) *For $A \in \mathcal{M}$, we have $A^c \in \mathcal{M}$.*

(2) *For $A_1,\ A_2 \in \mathcal{M}$, we have $A_1 \bigcup A_2,\ A_1 \bigcup A_2,\ A_1 \backslash A_2 \in \mathcal{M}$.*

Theorem 6.1 *$\mathcal{M}$ is a σ-algebra. Namely, we have the following* (1) $\sim$ (3):

(1) *$\boldsymbol{R}^d \in \mathcal{M}$ holds.*

(2) *For $A, B \in \mathcal{M}$, we have $A \backslash B \in \mathcal{M}$.*

(3) *For $A_p \in \mathcal{M},\ (p \geq 1)$, we have*

$$\bigcup_{p=1}^{\infty} A_p \in \mathcal{M}.$$

Corollary 6.1 *We use the notation in the above. We have the following* (1) $\sim$ (4):

(1) *$\phi \in \mathcal{M}$ holds.*

(2) *For $A \in \mathcal{M}$, we have $A^c \in \mathcal{M}$.*

(3) *For $A_p \in \mathcal{M},\ (1 \leq p \leq n)$, we have*

$$\bigcup_{p=1}^{n} A_p \in \mathcal{M},\ \bigcap_{p=1}^{n} A_p \in \mathcal{M}.$$

(4) *For $A_p \in \mathcal{M},\ (1 \leq p < \infty)$, we have*

$$\bigcap_{p=1}^{\infty} A_p \in \mathcal{M}.$$

Here we remark that an open set and a closed set in $\boldsymbol{R}^d$ are the Borel sets. Further, by virtue of Corollary 5.3, a Borel set in $\boldsymbol{R}^d$ is Lebesgue measurable. Thus we have the following theorem.

Theorem 6.2 *Assume that $\mathcal{O}$ is the family of all open sets in $\boldsymbol{R}^d$, $\mathcal{C}$ is the family of all closed sets in $\boldsymbol{R}^d$ and $\boldsymbol{B}$ is the family of all* Borel *sets in $\boldsymbol{R}^d$. Then we have the inclusion relations*

$$\mathcal{O} \cup \mathcal{C} \subset \boldsymbol{B} \subset \mathcal{M}.$$

By virtue of Definition 2.1, Theorem 6.1 and Corollary 6.1, we have the following Corollary.

Corollary 6.2 *Assume that $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is the d-dimensional* Lebesgue *measure space. Then we have the following* (1) $\sim$ (3):

(1) *If $A_1,\ \cdots,\ A_n \in \mathcal{M}$ are mutually disjoint, we have the equality*

$$\mu(\sum_{p=1}^{n} A_p) = \sum_{p=1}^{n} \mu(A_p).$$

(Finite additivity).

(2) *For $A,\ B \in \mathcal{M}$ such that $A \supset B$ holds, we have the inequality $\mu(A) \geq \mu(B)$. Especially, if $\mu(B) < \infty$ holds, we have the equality $\mu(A \backslash B) = \mu(A) - \mu(B)$.*

(3) *For $A_p \in \mathcal{M},\ (1 \leq p < \infty)$, we have the inequality*

$$\mu(\bigcup_{p=1}^{\infty} A_p) \leq \sum_{p=1}^{\infty} \mu(A_p).$$

(Complete sub-additivity).

Especially, for $A_p \in \mathcal{M},\ (1 \leq p \leq n)$, we have the inequality

$$\mu(\bigcup_{p=1}^{n} A_p) \leq \sum_{p=1}^{n} \mu(A_p).$$

(Finite sub-additivity).

Example 6.1 Let A be the set of all rational points in $[0,\ 1]^d$. Since a set $\{a\}$ of one point a is an interval and A is a countable set, A is a Borel set. Therefore A is Lebesgue measurable and we have the equality

$$\mu^*(A) = \mu_*(A) = \mu(A) = \sum_{a \in A} \mu(\{a\}) = 0.$$

Nevertheless A is not Jordan measurable.

The empty set ϕ is a null set. Conversely, any null set is not necessarily the empty set.

Proposition 6.7 *A null set e is* Lebesgue *measurable and $\mu(e) = 0$ holds.*

All null sets have the following properties.

Proposition 6.8 *We have the following* (1) *and* (2):

(1) *A subset of a null set is a null set.*

(2) *A union*

$$e = \bigcup_{p=1}^{(\infty)} e_p$$

of at most countable null sets $e_1,\ e_2,\ \cdots$ is a null set.

Next, we study the fundamental properties of the Lebesgue measurable sets and the Lebesgue measure.

Especially, the relations of the limit sets and the measures are important. Since the Lebesgue measure is completely additive, it is characteristic to calculate very well the measure of the limit set.

Now, for a sequence of subsets $A_1,\ A_2,\ \cdots$ of $\boldsymbol{R}^d$, we define

$$\overline{\lim_{p\to\infty}}\ A_p = \bigcap_{n=1}^{\infty}(\bigcup_{p=n}^{\infty} A_p),$$

$$\underline{\lim_{p\to\infty}}\ A_p = \bigcup_{n=1}^{\infty}(\bigcap_{p=n}^{\infty} A_p)$$

and we call them a **superior limit** and an **inferior limit** of the sequence of sets $\{A_p\}$ respectively.

Especially, when we have the condition

$$\overline{\lim_{p\to\infty}}\ A_p = \underline{\lim_{p\to\infty}}\ A_p,$$

we define

$$\lim_{p\to\infty}\ A_p = \overline{\lim_{p\to\infty}}\ A_p = \underline{\lim_{p\to\infty}}\ A_p$$

and we call it a limit of the sequence of sets $\{A_p\}$.

Then, by the fact that the family $\mathcal{M}$ of all Lebesgue measurable sets is a σ-algebra, we have the following proposition.

Proposition 6.9 *We have the following* (1) *and* (2):

(1) *For* $A_p \in \mathcal{M},\ (1 \leq p < \infty)$, *we have*

$$\overline{\lim_{p\to\infty}}\ A_p,\ \underline{\lim_{p\to\infty}}\ A_p \in \mathcal{M}.$$

(2) *If there exists* $\lim_{p\to\infty} A_p$, *we have*

$$\lim_{p\to\infty} A_p \in \mathcal{M}.$$

Theorem 6.3 *For* $A_p \in \mathcal{M},\ (1 \leq p < \infty)$, *we have the following* (1) $\sim$ (4):

(1) *If either one of the conditions*

(i) $A_1 \subset A_2 \subset \cdots$,

(ii) $A_1 \supset A_2 \supset \cdots$, *and* $\mu(A_1) < \infty$

is satisfied, then we have the equality

$$\mu(\lim_{p\to\infty} A_p) = \lim_{p\to\infty} \mu(A_p).$$

(2) *We have the inequality*

$$\mu(\underline{\lim_{p\to\infty}}\ A_p) \leq \underline{\lim_{p\to\infty}}\ \mu(A_p).$$

(3) *If* $\mu(\bigcup_{p=1}^{\infty} A_p) < \infty$ *holds, we have the inequality*

$$\mu(\overline{\lim_{p\to\infty}}\ A_p) \geq \overline{\lim_{p\to\infty}}\ \mu(A_p).$$

(4) *If* $\mu(\bigcup_{p=1}^{\infty} A_p) < \infty$ *holds and there exists* $\lim_{p\to\infty} A_p$, *we have the equality*

$$\mu(\lim_{p\to\infty} A_p) = \lim_{p\to\infty} \mu(A_p).$$

Theorem 6.4 *Assume that* A *is an arbitrary bounded set of* $\boldsymbol{R}^d$ *and* A *is not necessarily measurable. Then, for an arbitrary positive number* $\varepsilon > 0$, *there exist an open set* G *and a closed set* F *such that we have the inequalities*

$$A \subset G,\ \mu(G) < \mu^*(A) + \varepsilon,$$

$$F \subset A,\ \mu(F) > \mu_*(A) - \varepsilon.$$

Theorem 6.5 *If A is a* Lebesgue *measurable set in $\boldsymbol{R}^d$, then, for any positive number $\varepsilon > 0$, there exist an open set G and a closed set F such that we have the relations*

$$F \subset A \subset G,\ \mu(G \backslash A) < \varepsilon,\ \mu(A \backslash F) < \varepsilon.$$

Especially, if $\mu(A) < \infty$ holds, we may have a bounded closed set F.

Corollary 6.3 *For a* Lebesgue *measurable set A, there exists a* Borel *set B such that we have the relations*

$$A \subset B,\ \mu(B \backslash A) = 0.$$

Corollary 6.4 *For a* Lebesgue *measurable set A, there exists a* Borel *set B such that we have the relations*

$$B \subset A,\ \mu(A \backslash B) = 0.$$

By virtue of these Corollaries, it is known that a Lebesgue measurable set is equal to a difference or a union of a Borel set and a null set.

Now, for two sets A and B, we define

$$A \Delta B = (A \backslash B) + (B \backslash A).$$

Then we define that a sequence of sets $\{A_p\}$ converges to A in measure if we have the condition

$$\mu^*(A_p \Delta A) \to 0,\ (p \to \infty).$$

Theorem 6.6 *a bounded set A in $\boldsymbol{R}^d$ is* Lebesgue *measurable if and only if there exists a sequence of* Borel *sets $\{A_p\}$ such that we have the condition*

$$\mu^*(A_p \Delta A) \to 0,\ (p \to \infty).$$

Then we have the equality

$$\lim_{p \to \infty} \mu(A_p) = \mu(A).$$

Theorem 6.7 *Assume that A is a bounded set in $\boldsymbol{R}^d$ and $\{A_p\}$ is a sequence of bounded* Lebesgue *measurable sets. Then, if we have the relation*

$$\mu^*(A_p \Delta A) \to 0,\ (p \to \infty),$$

A is Lebesgue *measurable.*

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