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ON COLORING ORIENTED GRAPHS OF LARGE GIRTH

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We prove that for every oriented graph D and every choice of positive integers k and ℓ, there exists an oriented graph D^* along with a surjective homomorphism $\psi : D^* \to D$ such that: (i) girth$(D^*) \geq \ell$; (ii) for every oriented graph C with at most k vertices, there exists a homomorphism from D^* to C if and only if there exists a homomorphism from D to C; and (iii) for every D-pointed oriented graph C with at most k vertices and for every homomorphism $\varphi : D^* \to C$ there exists a unique homomorphism $f : D \to C$ such that $\varphi = f \circ \psi$. Finding the chromatic number of an oriented graph D is equivalent to finding the smallest integer k such that there is a homomorphism from D to a tournament on k vertices, so our main theorem provides results about girth and chromatic number of oriented graphs. While we prove our main theorem probabilistically (i.e. nonconstructively), we conclude with a construction of an oriented graph with any given girth $\ell \geq 3$ and chromatic number $k \geq 5$.

Keywords: oriented graph, chromatic number, girth

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Chapter 1 | Introduction

Graph theory is a comparatively new branch of mathematics, having begun only in the early 18th century with Leonhard Euler's solution [6] to the Seven Bridges of Königsberg problem, and the first textbook [9] on the subject not being published until the 20th century. The simplicity of the notion of a graph and of some elementary theorems provide graph theory with a certain elegance. Our natural intuition here can lead us to many true results, but also to false assumptions about graphs. One of the most fascinating examples of the false assumptions is that a graph with only large cycles will require relatively few colors for a proper vertex coloring. In 1959, Paul Erdős famously proved [5] that we can indeed have graphs of arbitrarily large girth and arbitrarily large chromatic number, contradicting untrained intuition at that time.

In Erdős' groundbreaking paper, he was able to demonstrate probabilistically the existence of such counterintuitive graphs. We discuss the history of this and related topics with a debt of gratitude to Mark Kayll and Esmaeil Parsa for their brief history of the topic in [8]. We begin by noting that [5] presents a probabilistic proof and does not suggest a way to construct such a graph. This work then leads naturally in a few directions: (1) to generalize Erdős' result; (2) to consider analogues of his results for other types of graphs, specifically of interest to us, directed graphs; and (3) to find a way to construct these graphs whose existence is guaranteed by Erdős.

Both refinements and generalizations of [5] have followed in the intervening six-plus decades. In 1976, Bollobás and Sauer [3] refined Erdős' result by showing that for any positive integer n there are graphs of arbitrarily large girth that are *uniquely* n-colorable. In 1996, Zhu [17], working with graph homomorphisms as a generalization of coloring, was able to carry forward the work of [3] by showing that for any 'core' H, there are uniquely H-colorable graphs of arbitrarily large girth. We note that complete graphs are cores, so Zhu's work provides a generalization of [5]. Zhu's result in [17] was further generalized by Nešetřil and Zhu [12] in 2004 to the notion of 'pointed' graphs. We end up following a similar trajectory in our work here.

We now shift our attention to digraphs. In 2004, Bokal, Fijavž, Juvan, Kayll, and Mohar [2] studied the circular chromatic number of digraphs and showed that the coloring theory of digraphs is similar to that of undirected graphs. In the world of undirected graphs, the circular chromatic number, $\chi_C(G)$, can be considered as a refinement of the chromatic number, $\chi(G)$, because $\chi(G) - 1 < \chi_C(G) \leq \chi(G)$ [16]. When we consider circular coloring of a digraph D, we are looking for a map $\rho : V(D) \to S_p$ where S_p is a circle with circumference p and the images of all adjacent vertices in $V(D)$ are at least one (directed) unit apart. Then the circular chromatic number of D is the infimum of real numbers

p such that D admits a circular p-coloring. We do note that a problem arises because the infimum may not be achieved.

By using acyclic homomorphisms, [2] resolves both the potential problem of an infimum not being achieved and, perhaps more importantly, the issue of finding a natural way to create a digraph analogue of the chromatic number of an undirected graph. As a result, they define the chromatic number $\chi(D)$ of a digraph D to be the minimum integer k such that $V(D)$ can be partitioned into k acyclic subsets. In addition to this yielding the relations $\chi(D) - 1 < \chi_C(D) \leq \chi(D)$—making this a good analogue of chromatic number of an undirected graph—another nice consequence of using acyclic homomorphisms to define circular chromatic number is that there is no longer the possibility of having a circular chromatic number that is not a minimum. Although acyclic homomorphisms do have the nice properties just discussed, they also introduce complications. For example, the authors of [8] had to use a lot of care to demonstrate that certain mappings fail to be acyclic homomorphisms. The fact that we use oriented coloring in our work here means we have no need to rely on acyclic homomorphisms to resolve the sorts of problems that they address. We hope the reader will appreciate the way this has simplified our work in comparison with that of [8].

Following [2], a subset of the authors and their doctoral students in [7] completed work in the realm of digraphs analogous to that of Zhu for graphs in [17]. Then [8] generalized the results of [2, 7] just as Nešetřil and Zhu in [12] generalized [5, 17]. One of our successes in the present work is a similar sequence of generalizations for oriented graphs.

We delay definitions for a little longer (until Section 2) and proceed to give our main result and a few of its consequences:

Theorem 1. *For every oriented graph D and every choice of positive integers k and ℓ, there exists an oriented graph D^* along with a surjective homomorphism $\psi : D^* \to D$ such that:*

(i) *girth*$(D^*) \geq \ell$*;*

(ii) *For every oriented graph C with at most k vertices, there exists a homomorphism from D^* to C if and only if there exists a homomorphism from D to C; and*

(iii) *For every D-pointed oriented graph C with at most k vertices and for every homomorphism $\varphi : D^* \to C$ there exists a unique homomorphism $f : D \to C$ such that $\varphi = f \circ \psi$.*

We hope that at this point an attentive reader familiar with [8] will be concerned that our Theorem 1 is an immediate consequence of [8, Theorem 1]. After all, their theorem is proven for digraphs in general, and oriented graphs are a specific type of digraph. Furthermore, oriented colorings are homomorphisms from oriented graphs to oriented graphs, so in particular they are acyclic homomorphisms. We can see this because preimages under a homomorphism must be independent sets, and are hence acyclic. The important difference is that in our Theorem 1, we are able to get an *oriented* D^* and an *oriented coloring* ψ, whereas in [8] we are only guaranteed a *digraph* D^* and an *acyclic* *homomorphism* ψ. Although D^* of [8] will in fact be an oriented graph when D is an oriented graph, one can readily check that the acyclic homomorphism ψ of [8] in general will not be an oriented coloring, so their results do not guarantee us the desired results in the world of oriented graphs. The importance of this distinction becomes clear as we discuss two consequences of Theorem 1, which we now state.

Theorem 2. *If D and C are oriented graphs such that D is not C-colorable, then for every positive integer ℓ, there exists an oriented graph D^* of girth at least ℓ that is D-colorable but not C-colorable.*

Theorem 3. *For every oriented core D and every positive integer ℓ, there is an oriented graph D^* of girth at least ℓ that is uniquely D-colorable.*

To see that Theorem 1 implies Theorem 2, if we have D and C as in Theorem 2 with a given integer ℓ and take k to be the order of C, then (i) of Theorem 1 gives us a D^* of required girth such that we have $\psi : D^* \to D$, so D^* is D-colorable. But as D is not C-colorable, condition (ii) of Theorem 1 implies that D^* is not C-colorable.

To see that Theorem 1 implies Theorem 3 follows a similar argument as in [8]. We note that cores D are D-pointed. So if we are given a positive integer ℓ and a core D, we can take $k = |V(D)|$. Then Theorem 1 gives us D^* of girth at least ℓ and a D-coloring $\psi : D^* \to D$. We can set $C = D$ in part (iii) of Theorem 1, which gives us that for every D-coloring $\varphi : D^* \to D$ there is a (unique) homomorphism $f : D \to D$ such that $\varphi = f \circ \psi$. Because D is a core,

f is an automorphism, so φ and ψ differ by this automorphism and D^* is indeed uniquely D-colorable.

Chapter 2 | Terminology and Notation

We assume basic familiarity with graphs and digraphs and refer the reader to [4] for any missing concepts not addressed here. For our work we consider oriented graphs and oriented colorings going forward unless otherwise indicated. An *oriented graph* D is a digraph in which for every pair of vertices u, v, at most one of uv and vu is an element of $A(D)$, the *arc set* of D. Our oriented graphs will always be finite, simple, that is, loopless without multiple arcs, and opposite arcs are precluded by the definition of oriented graphs. It can be easier to think about an oriented graph as one obtained by assigning directions to each edge of some (undirected) graph G. Recall that a *tournament* D on n vertices is an oriented graph obtained by assigning a direction to each edge of the complete graph K_n. When we discuss *cycles* of oriented graphs, we mean directed cycles, and the *girth* of an oriented graph D is the length of a shortest directed cycle in D. Finally, for oriented graphs D and C, an *oriented graph homomorphism* is a map $f : V(D) \to V(C)$ such that for $xy \in A(D)$ we have $f(x)f(y) \in A(C)$.

We are now ready to define an 'oriented coloring' of an oriented graph D. An *oriented k-coloring*, then, is a map $c : V(D) \to \{1, ..., k\}$ such that:

1. $c(x) \neq c(y)$ for every arc $xy \in A(D)$, and

2. $c(u) \neq c(y)$ for every two arcs $uv \in A(D)$ and $xy \in A(D)$ with $c(v) = c(x)$.

This is by now a standard definition; see, e.g., [15].

This definition of an oriented coloring is equivalent to that of a homomorphism to a tournament on k vertices. First, it is clear that a homomorphism to a tournament satisfies condition (1) of being an oriented coloring because it is a homomorphism, and condition (2) is satisfied because tournaments have no opposite arcs. On the other hand, given such a map c, we can construct an oriented graph C^* with $V(C^*) = \{1, ..., k\}$ and $A(C^*) = \{xy \colon x, y \in V(C^*) \text{ and } xy = c(a)c(b) \text{ for some } ab \in A(D)\}$. Then it is clear that C^* is a subgraph of a tournament C' on k vertices by property (2) of c. Furthermore, C^* was constructed so that c is a homomorphism to C^* and thus a homomorphism to C', so c is a homomorphism to a tournament on k vertices. In our work ahead we always consider oriented colorings to be homomorphisms to oriented graphs.

For terminology more directly related to our theorem statements, we say that a homomorphism of oriented graphs of D to C is a *C-coloring of D*, and we say that D is *C-colorable*. We say that D is *uniquely C-colorable* if there is a homomorphism of D onto C, and for any two C-colorings ψ and φ of D, these homomorphisms 'differ by an automorphism'. That is, there is some $f \in \mathrm{Aut}(C)$ such that $\psi = f \circ \varphi$. For an oriented graph D, we say that D is a *core* if every homomorphism $f : V(D) \to V(D)$ is an automorphism. Finally, we say that for oriented graphs C and D, the digraph C is *D-pointed* if there do not exist two distinct C-colorings of D that agree on all but one vertex of D.

Chapter 3 | Setup for the Proof of Theorem 1

For a given oriented graph D, we begin the 'construction' of the digraph D^*, and we do so by first constructing a digraph D_0 again inspired by [8]. We define

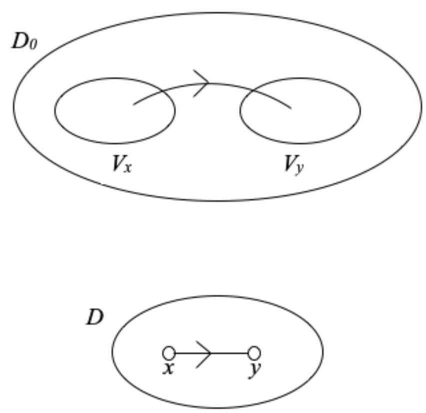

Figure 1: The view in D_0 of elements resulting from one arc xy in D.

$V(D_0) = V_1 \cup V_2 \cup \cdots \cup V_a$ where $V(D) = \{1, 2, ..., a\}$, and each $|V_i| = n$ for some fixed n large enough to satisfy necessary probabilistic inequalities. Then we define the arc set $A(D_0) = \{xy : x \in V_i, y \in V_j \text{ and } ij \in A(D)\}$. We can view each V_i simply as the preimage of a vertex $i \in V(D)$ under the natural homomorphism $\psi : D_0 \to D$, mapping each V_i to i, for $i \in \{1, ..., a\}$. See Fig. 1.

Now we use D_0 to 'construct' an oriented graph D^* probabilistically. First we fix an ε with $0 < \varepsilon < 1/(4\ell)$ where ℓ is chosen as in the statement of Theorem 1. Then our random oriented graph model $\mathcal{D}(n,p)$ consists of spanning subgraphs of D_0 where arcs are chosen randomly and independently with probability $p = n^{\varepsilon - 1}$ with n sufficiently large. We now introduce two lemmas from [8].

Lemma 1. *(i) The expected number of cycles of length less than ℓ in a digraph $\hat{D} \in \mathcal{D}(n,p)$ is bounded above by $n^{\varepsilon\ell} n^{-\varepsilon/2}$;*

(ii) The expected number of pairs of cycles of length less than ℓ in a digraph $\hat{D} \in \mathcal{D}(n,p)$ which intersect in at least one vertex is bounded above by $n^{-1/2}$.

This is Lemma 5 of [8], except that our oriented graph model $\mathcal{D}(n,p)$ differs. In particular, our D_0 has fewer arcs than the analogue in [8], so the lemma remains true in our case. This along with the First Moment Method [1] shows that asymptotically almost all oriented graphs in $\mathcal{D}(n,p)$ have at most $n^{\varepsilon\ell}$ cycles of length less than ℓ which are pairwise vertex-disjoint, see, e.g., [8].

We introduce some definitions from [8] (which itself adopted these from [12]), first calling a set $\mathcal{A} \subseteq V(D_0)$ *large* if there are distinct $i, j \in \{1, ..., a\}$ with $ij \in A(D)$ such that $|\mathcal{A} \cap V_i| \geq n/k$ and $|\mathcal{A} \cap V_j| \geq n/k$, and calling $ij \in A(D)$ in this case a *good arc* for $\mathcal{A}$. Then given a large $\mathcal{A}$, we denote by $|\hat{D}/\mathcal{A}|$ the minimum number of arcs of a random $\hat{D}$ which lie in the set $\{xy : x \in \mathcal{A} \cap V_i, y \in \mathcal{A} \cap V_j\}$ with ij a good arc. Then we have:

Lemma 2. *If $\hat{D} \in \mathcal{D}(n,p)$ and $\mathcal{A}$ is large, then $P(|\hat{D}/\mathcal{A}| \geq n) = 1 - o(1)$.*

Again the space $\mathcal{D}(n,p)$ in [8] differs from ours, but the proof still follows through unchanged because the arcs counted in $|\hat{D}/\mathcal{A}|$ in [8] are all present in the current model.

Finally, we need the following version of Chernoff's bounds on the tail distributions of binomial random variables; see, e.g., [8]:

Theorem 4. *If X is a binomial random variable and $0 < \delta < 3/2$, then*

$$P(|X - E(X)| \geq \delta E(X)) \leq 2e^{-\delta^2 E(X)/3}.$$

Now we can move on to the proof of our main theorem.

Chapter 4 | Proof of Theorem 1

Lemma 1 and its consequences mean that asymptotically almost all $D' \in \mathcal{D}(n,p)$ have at most $n^{\varepsilon\ell}$ pairwise-disjoint cycles of length less than ℓ. Similarly, Lemma 2 guarantees that asymptotically almost all $D' \in \mathcal{D}(n,p)$ have the property that all good arcs of D for large sets $\mathcal{A}$ induce at least n arcs of D'. In the proof of (iii) we introduce a third property that almost all $D' \in \mathcal{D}(n,p)$ possess. Therefore, there exists some $D' \in \mathcal{D}(n,p)$ enjoying the two stated properties and a property to be named later, and we select such a D'. Now we pick one arc from each of the at most $n^{\varepsilon\ell}$ cycles of length less than ℓ in D', giving an independent arc set M, and define $D^* = D' - M = (V(D_0), A(D') \setminus M)$. It is clear then that D^* has girth at least ℓ, and that $\psi : D^* \to D$ defined by $\psi(x) = i$ if and only if $x \in V_i$ gives a surjective homomorphism, yielding (i) from Theorem 1. Note that since $\varepsilon < 1/(4\ell)$, the deleted arc set satisfies $|M| \leq n^{\varepsilon\ell} < n^{1/4}$.

Now we work toward (ii) from Theorem 1. Let us fix an oriented graph C of order at most k, and assume that there is a homomorphism $\varphi : D^* \to C$. Then for every $i \in V(D)$ there is a vertex $x \in V(C)$ such that $|V_i \cap \varphi^{-1}(x)| \geq n/k$. This is because there are at most k vertices in C, so there are at most k such intersections ranging over all $x \in V(C)$, and they must cover the n vertices of V_i. Therefore, the Pigeonhole Principle implies that at least one such intersection has cardinality at least n/k. Then let us define $f : V(D) \to V(C)$ by $f(i) = x$ for some $x \in V(C)$ such that $|V_i \cap \varphi^{-1}(x)| \geq n/k$. We must show this f is a homomorphism.

Let $ij \in A(D)$ and consider all possible $a, b \in V(D^*)$ where $a \in V_i \cap \varphi^{-1}(f(i))$ and $b \in V_j \cap \varphi^{-1}(f(j))$. If there is one such arc $ab \in A(D^*)$, this will guarantee the existence of an arc $f(i)f(j) \in A(C)$ by the existence of φ. Recall that f satisfies $|V_i \cap \varphi^{-1}(f(i))| \geq n/k$ and $|V_j \cap \varphi^{-1}(f(j))| \geq n/k$. Then $\mathcal{A} = \big(V_i \cap \varphi^{-1}(f(i))\big) \cup \big(V_j \cap \varphi^{-1}(f(j))\big)$ is large as defined for Lemma 2, so by our choice of D' relying on that lemma, D' has at least n arcs with endpoints in $\mathcal{A}$. Then since we have removed at most $n^{1/4}$ arcs from D' to construct D^*, there exists at least one such arc $ab \in A(D^*)$, and in fact many such arcs. So we have $\varphi(a)\varphi(b) \in A(C)$, and we have that $f(i) = \varphi(a)$ and $f(j) = \varphi(b)$ with $f(i) \neq f(j)$ because φ is a homomorphism. So $f(i)f(j) \in A(C)$, and f maps arcs to arcs and is thus a homomorphism.

Conversely, if we assume that there is a homomorphism $f : V(D) \to V(C)$, then we get the homomorphism $\varphi : V(D^*) \to V(C)$ by $\varphi = f \circ \psi$, completing our proof of (ii).

Now we look at (iii), letting C be a D-pointed oriented graph of order at most k, and $\varphi : V(D^*) \to V(C)$ be a homomorphism. We will use $f : V(D) \to V(C)$ as in the proof of (ii). As in [8], the D-pointedness of C forces for every $i \in V(D)$ the existence of a unique $x_i \in V(C)$ such that $|\varphi^{-1}(x_i) \cap V_i| \geq n/k$. If some x_i

were not unique and x_i' also satisfies $|\varphi^{-1}(x_i') \cap V_i| \geq n/k$, then we could define f' by

$$f'(j) = \begin{cases} f(j) & \text{for } j \neq i \\ x_i' & \text{for } j = i \end{cases}$$

giving another homomorphism differing at one vertex of D and contradicting the D-pointedness of C. This establishes the uniqueness of a homomorphism f chosen in this way. If we assume that $\varphi \neq f \circ \psi$, there must be some vertex $z \in V(D^*)$ such that $\varphi(z) \neq (f \circ \psi)(z)$. So if $z \in V_j$, then $(f \circ \psi)(z) = f(j) \neq \varphi(z)$. So $V_j \setminus \left(\varphi^{-1}(f(j)) \cap V_j\right) \neq \emptyset$ (as it contains z), which leads to a contradiction as we proceed to show.

Now let $i_0 \in \{1, ..., a\}$ be such that $t := |\varphi^{-1}(f(i_0)) \cap V_{i_0}|$ is minimum, so we have $n/k \leq t < n$. Because $t < n$, the set $\varphi^{-1}(f(i_0)) \cap V_{i_0}$ is a proper subset of V_{i_0}, and so we can choose $x \in V(C)$ with $x \neq f(i_0)$ such that $|\varphi^{-1}(x) \cap V_{i_0}| > 0$. Then we define $f' : V(D) \to V(C)$ by:

$$f'(i) = \begin{cases} f(i) & \text{for } i \neq i_0 \\ x & \text{for } i = i_0. \end{cases}$$

Because f and f' differ only at i_0 and C is D-pointed, f' is not a homomorphism. Thus, it fails to send arcs to arcs. So it must be for some $v \in V(D)$ not equal to i_0, either $vi_0 \in A(D)$ and $f(v)x \notin A(C)$ or $i_0v \in A(D)$ and $xf(v) \notin A(C)$. Without loss of generality, we may assume that $i_0v \in A(D)$ and $xf(v) \notin A(C)$. By the definition of f we have that $|\varphi^{-1}(f(v)) \cap V_v| \geq n/k$. Then because $n^{\varepsilon\ell} \leq n^{1/4} = o(n/k)$ we can choose $\mathcal{A} \subseteq \varphi^{-1}(f(v)) \cap V_v$ with

$|\mathcal{A}| \geq n/2k$ so that $\mathcal{A}$ is not incident with any arc of M that was removed to construct D^* from D'. We can also choose $\mathcal{B} = \varphi^{-1}(x) \cap V_{i_0}$, which we observed above is nonempty. If we let the random variable Y count the arcs from $\mathcal{B}$ to $\mathcal{A}$ in D^* we can bound the expected value $E(Y)$ by:

$$E(Y) \geq \frac{n}{2k}p = \frac{n \cdot n^{\varepsilon-1}}{2k} = \frac{n^{\varepsilon}}{2k}.$$

Then by Theorem 4 with $\delta = 1$ we have

$$P(Y = 0) \leq P(|Y - E(Y)| \geq E(Y)) \leq 2e^{-E(Y)/3} \leq 2e^{-n^{\varepsilon}/6k}.$$

It is clear this is asymptotically zero. So for sufficiently large n, for almost all $D' \in \mathcal{D}(n,p)$, the corresponding D^* will have $ba \in A(D^*)$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This is the third property for which we selected D' at the beginning of the proof, so we in fact have $ba \in A(D^*)$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then because φ is a homomorphism, we have $\varphi(b)\varphi(a) \in A(C)$. But $\varphi(a) = f(v)$ and $\varphi(b) = x$, so $xf(v) \in A(C)$ which is a contradiction. Thus our assumption of the existence of a vertex $z \in V(D^*)$ such that $\varphi(z) \neq (f \circ \psi)(z)$ is incorrect, and we must have $\varphi = f \circ \psi$. Finally, we note that the surjectivity of ψ implies that such a homomorphism f is unique. $\square$

Chapter 5 | Constructions

Our last natural direction of exploration from Erdős' original theorem is that of actually constructing those graphs which we have probabilistically proven exist. These constructions are generally challenging and delicate. The common approach is to proceed by induction, constructing a (di)graph of chromatic number $n+1$ with girth ℓ using copies of a (di)graph of chromatic number n with girth ℓ. The first such construction was completed by Lovász [11] in 1968 using hypergraphs intermediately. It was not until 1989 that Křìž in [10] was able to create a purely graph-theoretic construction of highly chromatic graphs without short cycles. Similarly, Severino in [13] demonstrated constructions of highly chromatic digraphs without short cycles and in [14] constructed uniquely n-colorable digraphs with arbitrarily large girth.

Ideally, we would like to construct the digraph D^* with all the properties described in Theorem 1. We shall content ourselves with a construction of an oriented graph of a given girth and chromatic number and leave the construction of such a D^* for a future author.

Theorem 5. *For integers $k \geq 5$ and $\ell \geq 3$, there exists an oriented graph D*

with chromatic number k and girth ℓ.

Remark: Some instances of (k,ℓ) with $k = 3$ or $k = 4$ are also feasible. In particular, $k = 3$ is feasible for $\ell \equiv 0 \pmod 3$, and $k = 4$ is feasible for all $\ell \geq 3$ with $\ell \neq 5$. However, we state Theorem 5 as such because when $\ell = 5$, of necessity our basis starts at $k = 5$. Readers may find it illustrative to convince themselves that the directed 5-cycle admits no homomorphism to a tournament on four vertices, while a directed cycle of any other order admits such a homomorphism.

Proof. We follow the common approach to which we alluded above and proceed by induction on k, so let us fix integers k and ℓ. Then we begin by considering $\overrightarrow{C_\ell}$, an oriented cycle of length ℓ (and girth ℓ). We define $V(\overrightarrow{C_\ell}) = \{v_0, v_1, ..., v_{\ell-1}\}$, and there is a homomorphism $c : V(\overrightarrow{C_\ell}) \to V(T_5)$ where $V(T_5) = \{t_0, t_1, t_2, t_3, t_4\}$ and $\{t_0t_1, t_1t_2, t_2t_3, t_3t_4, t_2t_0, t_3t_0, t_4t_0\} \subset A(T_5)$. Then if $\ell \equiv 0 \pmod 3$, we have $c : V(\overrightarrow{C_\ell}) \to V(T_5)$ defined by

$$c(v_r) = t_{r \bmod 3}.$$

If $\ell \equiv 1 \pmod 3$, then c is defined by

$$c(v_r) = \begin{cases} t_{r \bmod 3} & \text{for } r < \ell - 1 \\ t_3 & \text{for } r = \ell - 1. \end{cases}$$

And finally, if $\ell \equiv 2 \pmod 3$, then c is defined by

$$c(v_r) = \begin{cases} t_{r \text{ mod } 3} & \text{for } r < \ell - 2 \\ t_3 & \text{for } r = \ell - 2 \\ t_4 & \text{for } r = \ell - 1. \end{cases}$$

We note that our base cases have given us an oriented graph of girth ℓ with chromatic number $k \leq 5$. The verification of our induction below will then guarantee the existence of an oriented graph of girth ℓ with any given chromatic number $k \geq 5$.

Having established our base cases, we now proceed with the induction. So assume we have an oriented graph D_k of girth ℓ, chromatic number k, and order m, and then define $V(D_k) = \{v_0, v_1, ..., v_{m-1}\}$. Because D_k has chromatic number k, there exists a tournament T_k with $V(T_k) = \{t_0, t_1, ..., t_{k-1}\}$ and a homomorphism $\varphi_k : V(D_k) \to V(T_k)$. Now we construct D_{k+1} and the corresponding T_{k+1}. Define the vertex set $V(D_{k+1}) = V(D_k) \cup \{v_m\}$, and define the arc set $A(D_{k+1}) = A(D_k) \cup \{(v_i v_m : i \in \{0, 1, ..., m-1\}\}$. Then we construct T_{k+1} in exactly the same fashion; i.e., $V(T_{k+1}) = V(T_k) \cup \{t_k\}$ and $A(T_{k+1}) = A(T_k) \cup \{(t_i t_k : i \in \{0, 1, ..., k-1\}\}$.

We now examine the girth and chromatic number of D_{k+1}. First of all, it is immediately clear that we have created no new oriented cycles in this construction, so D_{k+1} also has girth ℓ. It is equally clear that we have a homomorphism

$\varphi_{k+1} : V(D_{k+1}) \to V(T_{k+1})$ defined by

$$\varphi_{k+1}(v) = \begin{cases} \varphi_k(v) & \text{for } v \neq v_m \\ \\ t_k & \text{for } v = v_m. \end{cases}$$

Therefore, $\chi(D_{k+1}) \leq k + 1$.

To complete the proof, it remains to show that D_{k+1} admits no homomorphism to a tournament on k vertices. Assume to the contrary that for some order-k tournament T'_k the digraph D_{k+1} admits a homomorphism $\psi : V(D_{k+1}) \to V(T'_k)$. Let's say that $\psi(v_m) = x \in V(T'_k)$. Then because every vertex $v \in V(D_{k+1}) \setminus \{v_m\}$ forms an arc vv_m, we know that $\psi(v) \neq x$ for every $v \neq v_m$. If we let Λ be the subgraph of D_{k+1} induced by the vertex set $\{v_0, ..., v_{m-1}\}$, then Λ is isomorphic to D_k. Similarly, if we let Γ be the subgraph of T'_k induced by $V(T'_k) \setminus \{x\}$, then Γ is a tournament on $k-1$ vertices. But then $\psi|_{V(\Lambda)}$ gives a homomorphism from Λ to Γ, a tournament on $k-1$ vertices, contradicting the fact that D_k has chromatic number k. Therefore, D_{k+1} indeed has chromatic number $k+1$. $\square$

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