

Supplemental Material to  
Equilibrium Uniqueness in Entry Games with Private Information

By

José-Antonio Espín-Sánchez, Álvaro Parra, and Yuzhou Wang

April 2018

Revised May 2021

COWLES FOUNDATION DISCUSSION PAPER NO. 2126RS



COWLES FOUNDATION FOR RESEARCH IN ECONOMICS  
YALE UNIVERSITY  
Box 208281  
New Haven, Connecticut 06520-8281

<http://cowles.yale.edu/>

# Online Appendix

## Equilibrium Uniqueness in Entry Games with Private Information

by José-Antonio Espín-Sánchez and Álvaro Parra  
Supplemental Material –Not for Publication

### C Equilibrium Exists and is in Cutoff Strategies

An entry strategy for firm  $i$  is a mapping from the firm's type  $v_i$  to a probability of entering in the market  $\tau_i : [a, b] \rightarrow [0, 1]$ . We assume that the strategy of firm  $i$  is an integrable function with respect to its own type  $v_i$ . We study the Bayesian Equilibria of the entry game. Denote by  $\tau = (\tau_1, \tau_2, \dots, \tau_n)$  the vector of entry strategies. Given a strategy profile  $\tau$ , the expected profit of firm  $i$  *after* drawing the type  $v_i$  but *before* entry decisions are realized is

$$\Pi_i(v_i, \tau) = \tau_i(v_i) \left[ \sum_{e \in E_i} \left\{ \int_{[a, b]^{n-1}} \pi_i(v_e) \Pr[e | \tau_{-i}, v_{-i}] \phi(v_{-i}) d^{n-1} v_{-i} \right\} \right] \quad (\text{C.1})$$

where  $\Pr[e | \tau_{-i}, v_{-i}]$  is the probability of observing market structure  $e$ , given the vector of strategies  $\tau_{-i}$  and the realizations of types  $v_{-i}$ . The integral is over each of the  $n - 1$  dimensions of firm  $i$ 's competitors types,  $v_{-i}$ . Conditional on  $i$ 's entry, which occurs with probability  $\tau_i(v_i)$ , the expected profit of firm  $i$  consists on the expected sum of profit that firm  $i$  would get under each feasible market structure, which is induced by the vector of strategies  $\tau$  and the realization of types  $v_{-i}$ , integrated over all possible realizations of the competitors' types,  $\phi(v_{-i})$ .

**Definition** (Cutoff Strategy). A strategy  $\tau_i(v_i)$  is called *cutoff* if there exists a threshold  $x > 0$  such that

$$\tau_i(v_i) = \begin{cases} 1 & \text{if } v_i \geq x \\ 0 & \text{if } v_i < x \end{cases} .$$

A cutoff strategy specifies whether a firm enters a market with certainty depending on whether its type is above or below some given threshold. In any best response, there exists a type,  $v_i$ , that makes a firm indifferent to enter the market. We break this indifference by assuming that firms enter. For a cutoff strategy, this means that a firm enters when its type is greater or equal to its cutoff. Given a vector  $\tau_{-i}$ , a best response is given by the strategy  $\hat{\tau}_i$  that maximizes (C.1) at every value of  $v_i$ .

A Bayesian Nash equilibrium is defined by a vector of strategies  $\tau$  in which every firm best respond to each other. The next proposition establishes the existence of an equilibrium and that, without loss of generality, we can restrict our analysis to cutoff strategies.

**Lemma C.1.** *For any game  $(\pi_i, F_i)_{i=1}^n$  satisfying assumptions A1-A4, there exists an equilibrium. For any vector  $\tau_{-i}$ , firm  $i$ 's best response is a cutoff strategy. Therefore, every equilibrium of the game is in cutoff strategies.*

**Proof of Lemma C.1.**

*Best responses are cutoff strategies:* Fix any firm  $i$  and vector of strategies  $\tau$ . By

assumptions A4 and A2, we know that in equilibrium no firm will enter if they draw  $v_j < \underline{v}_j$ . For relevance, impose that  $\tau$  satisfies the restriction  $\tau_j(v_j) = 0$  in that range. Because firm  $i$ 's profit is linear in  $\tau_i$ , firm  $i$ 's best response is to participate with probability one whenever there is a positive payoff of doing so. Suppose firm  $i$  enters the market with certainty ( $\tau_i(v_i) = 1$ ). The restriction above implies that there is positive probability that firm  $i$  is the sole entrant to the market and, consequently, by A1, profits are strictly increasing in  $v_i$ . By A4,  $\Pi_i(\underline{v}_i, \tau) < 0$ , and  $\Pi_i(\bar{v}_i, \tau) > 0$ . Thus,  $\Pi_i(v_i, \tau)$  single crosses zero and  $i$ 's best response to  $\tau_{-i}$  is the cutoff strategy defined by the value  $x_i$  that satisfies  $\Pi_i(x_i, \tau_i = 1, \tau_{-i}) = 0$ .

*Existence:* We check the conditions of Brouwer's fixed-point theorem. Because  $F_i$  is atomless and has full support and  $\pi_i(v_e)$  being continuous and differentiable in  $v_i$ , firm  $i$ 's best response lies in the compact and convex set  $[\underline{v}_i, \bar{v}_i]$ . Thus the  $n$ -dimensional function of best responses is a continuous mapping from  $\times_{i=1}^n [\underline{v}_i, \bar{v}_i]$  to itself and the conditions for the theorem are met. ■

Existence follows from Brouwer's fixed-point theorem. The restriction to cutoff strategies is quite intuitive: regardless of which strategy competitors are playing, assumption A1 implies that firm  $i$ 's expected profit is strictly increasing in its type. Because  $i$ 's expected profit is linear in its entry probability (see eq. (C.1)),  $i$  either prefers to enter with certainty, when it is profitable to do so, or to stay out. The next Lemma characterizes all cutoff equilibria.

**Lemma C.2.** *The vector  $\mathbf{x}$  of cutoff strategies constitutes an equilibrium if and only if  $\Pi_i(\mathbf{x}) = 0$  for every firm  $i$ .*

**Proof of Lemma C.2.** By the previous proof a cutoff strategy is defined as the value  $x_i$  satisfying  $\Pi_i(x_i, \tau_i = 1, \tau_{-i}) = 0$ . Because in a cutoff equilibrium  $\Pr[e|\tau, v_i]$  is either zero or one. Integrating (C.1) over payoff-irrelevant firms delivers (1). ■

Lemma C.2 characterizes every equilibrium of the entry game. Firm  $i$ 's best response to  $\mathbf{x}_{-i}$  is defined by a cutoff  $x_i$  equal to the value of  $v_i$  that satisfies  $\Pi_i(v_i, \mathbf{x}_{-i}) = 0$ . A profile of equilibrium cutoffs  $\mathbf{x}$  is, thus, constructed by the collection of functions  $\Pi_i(\mathbf{x})$  evaluated at a point in which every firm  $i$  is indifferent between entering the market when drawing type  $x_i$ .

## D Uniqueness in the Linear Model

In this section we derive the condition for uniqueness used in Examples 4, 5 and 7. Consider the following linear model

$$\pi_i(v_e) = \eta_i - \delta_i \sum_{k=1}^{n_e-1} r_i^{k-1} + v_i.$$

In this context, for a given vector of cutoff strategies  $\mathbf{x}$ , equation (1) is given by

$$\Pi_i(v_i, \mathbf{x}_{-i}) = \eta_i + v_i - \delta_i \mathbb{I}_{n_e > 1} \sum_{e \in E_i} \left\{ \left( \prod_{j \in O_i(e)} F_j(x_j) \right) \left( \prod_{\ell \in I_i(e)} (1 - F_\ell(x_\ell)) \right) r_i^{n_e-2} \right\}$$

and  $\Pi'(\mathbf{x}) = 1$ . Similarly, noticing that  $\pi(v_i, v_{e \setminus i}) - \pi(v_i, v_j, v_{e \setminus i}) = r^{n_e - 1} \delta$  we obtain

$$\Delta_{i,j}(\mathbf{x}) = \delta F_j(x_j) \prod_{\ell \neq i,j} (r + F_\ell(x_\ell)(1 - r)).$$

Then,

$$\frac{\Delta_{i,j}(\mathbf{x})}{\Pi'_i(\mathbf{x})} = \delta F_j(x_j) \prod_{\ell \neq i,j} (r + F_\ell(x_\ell)(1 - r)).$$

Noticing that  $F_\ell(x_\ell)$  for  $\ell \neq i$  increases in  $x_\ell$  we can replace  $x_\ell = \bar{v}_\ell$  in the previous expression, which leads to

$$\frac{\Delta_{i,j}(\mathbf{x})}{\Pi'_i(\mathbf{x})} \leq \delta F_j(\bar{v}_j) \prod_{\ell \neq i,j} (r + F_\ell(\bar{v}_\ell)(1 - r)).$$

When firms are symmetric, the previous expression can be used to derive equation (5). When  $r = 1$ , the expression simplifies to:

$$\frac{\Delta_{i,j}(\mathbf{x})}{\Pi'_i(\mathbf{x})} \leq \delta F_j(\bar{v}_j).$$

which can be used to construct conditions (8) and (10).

## E Uniqueness in a Selective-Entry Auction

Here we describe the model and derive the sufficient condition to determine whether the selective-entry model of Roberts and Sweeting (2013, 2016) has a unique equilibrium when there are one potential entrant from each group.

There are two bidders, a logger and a miller, which for simplicity we call  $i \in \{1, 2\}$ . Each bidder observes a signal  $v_i = \theta_i \varepsilon_i$  where  $\varepsilon_i \sim LN(0, \sigma_\varepsilon^2)$  and  $\theta_i \sim LN(\mu_i, \sigma_\theta^2)$ . Consequently,  $v_i \sim LN(\mu_i, \sigma_\theta^2 + \sigma_\varepsilon^2)$ . We call the CDF of this distribution  $F_i(v_i)$ . Conditional on  $v_i$  the posterior of  $\theta_i | v_i \sim LN(\alpha \mu_i + (1 - \alpha) \ln(v_i), \alpha \sigma_\theta^2)$ , where

$$\alpha = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\theta^2}.$$

We denote the PDF of this distribution  $h_i(\theta_i | v_i)$ . For completeness

$$h_i(\theta_i | v_i) = \frac{1}{\theta_i \sqrt{2\alpha\sigma_\theta^2\pi}} \exp\left(-\frac{(\ln \theta_i - (\alpha\mu_i + (1 - \alpha) \ln(v_i)))^2}{2\alpha\sigma_\theta^2}\right)$$

$$f_i(v_i) = \frac{1}{v_i \sqrt{2(\sigma_\theta^2 + \sigma_\varepsilon^2)\pi}} \exp\left(-\frac{(\ln v_i - \mu_i)^2}{2(\sigma_\theta^2 + \sigma_\varepsilon^2)}\right)$$

**Table 1:** Roberts and Sweeting (2013, 2016) estimates.

$\mu_1$	$\mu_2$	$\sigma_\varepsilon^2$	$\sigma_\theta^2$	$\alpha$	$K$	$r$
3.9607	3.5824	0.8578	0.3321	0.689	2.0543	27.77

Note: From Tables 3 and 4 from the cited papers.

The sufficient condition for uniqueness hold if, for every  $x_k \in [\underline{v}_k, \bar{v}_k]$ ,

$$\frac{f_i(x_i)}{F_i(x_i)} \frac{\Delta_{i,j}(x_i, x_j)}{\Pi'_i(x_i, x_j)} < 1.$$

In the current scenario, these expressions become

$$\begin{aligned} \Delta_{i,j}(x_i, x_j) &= F_j(x_j) (\pi_i(x_i) - \pi_i(x_i, x_j)) \\ \Pi'_i(x_i, x_j) &= F_j(x_j) \pi'_i(x_i) + \int_{x_j}^{\infty} \pi'_i(x_i, v_j) dF_j(v_j) \end{aligned}$$

where

$$\begin{aligned} \pi_i(x_i) &= \int_r^{\infty} (\theta_i - r) h_i(\theta_i|x_i) d\theta_i - K \\ \pi_i(x_i, x_j) &= \int_r^{\infty} \left( \int_0^{\theta_i} (\theta_i - \max\{r, \theta_j\}) h_j(\theta_j|x_j) d\theta_j \right) h_i(\theta_i|x_i) d\theta_i - K \\ \pi'_i(x_i) &= \int_r^{\infty} (\theta_i - r) \frac{\partial h_i(\theta_i|x_i)}{\partial x_i} d\theta_i \\ \pi'_i(x_i, v_j) &= \int_r^{\infty} \left( \int_0^{\theta_i} (\theta_i - \max\{r, \theta_j\}) h_j(\theta_j|v_j) d\theta_j \right) \frac{\partial h_i(\theta_i|x_i)}{\partial x_i} d\theta_i \end{aligned}$$

and

$$\frac{\partial h_i(\theta_i|x_i)}{\partial x_i} = h_i(\theta_i|x_i) \frac{(1 - \alpha) (\ln \theta_i - (\alpha \mu_i + (1 - \alpha) \ln x_i))}{x_i \alpha \sigma_\theta^2}$$

Using the estimates provided in Table 1, we can now compute all the necessary elements to verify sufficient condition (7).

**Cutoffs lower bound:** The lower bound for a firm's feasible cutoff,  $\underline{v}_i$ , is given by the unique solution to:

$$\int_r^{\infty} (\theta_i - r) h_i(\theta_i|\underline{v}_i) d\theta_i = K.$$

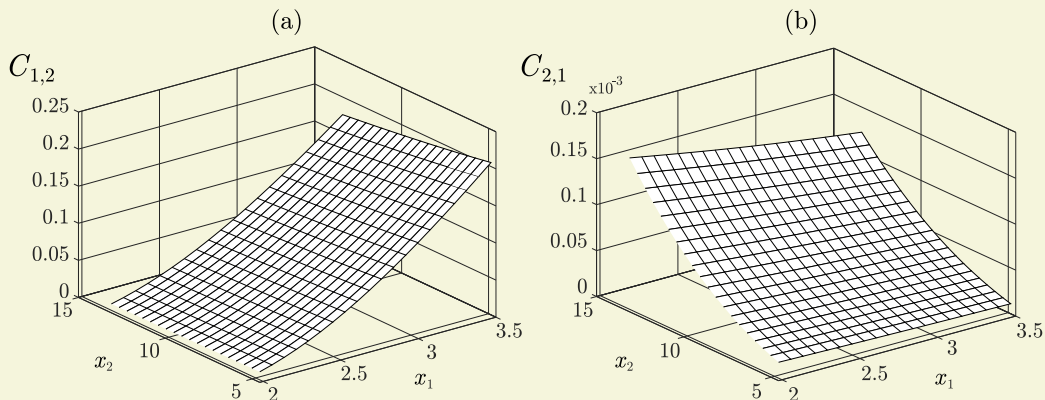
Computing, we obtain  $\underline{v}_1 = 1.978$  and  $\underline{v}_2 = 4.573$ .

**Upper bound:** The upper bound for a firm's feasible cutoff,  $\bar{v}_i$ , is given by the unique solution to:

$$\int_0^{\infty} \pi(\bar{v}_i, v_j) dF_j(v_j) = 0$$

Computing, we obtain  $\bar{v}_1 = 3.468$  and  $\bar{v}_2 = 13.109$ .

**Sufficient condition:** The left-hand side of sufficient condition is plotted for the relevant range of cutoff. As it can be observed in the figure below, the conditions are always less than one.



**Strength and herculean equilibrium:** For completeness, we also present the strength of each firm. Firm  $i$ 's strength is given by the unique solution to  $\sigma_i(s_i) = 0$ , where

$$\sigma_i(s_i) = F_j(s_i) \pi_i(s_i) + \int_{s_i}^{\infty} \pi_i(s_i, v_j) dF_j(v_j) - K$$

The strength of each firm is given by  $s_1 = 3.465$  and  $s_2 = 12.146$ . Finally, the herculean equilibrium is given by the unique solution to the system  $\Pi_1(x_1, x_2) = 0$  and  $\Pi_2(x_1, x_2) = 0$ , where

$$\Pi_i(x_i, x_j) = F_j(x_j) \pi_i(x_i) + \int_{x_j}^{\infty} \pi_i(x_i, v_j) dF_j(v_j)$$

We find that  $x_1 = 3.314 \in (\underline{v}_1, s_1)$  and  $x_2 = 12.999 \in (s_2, \bar{v}_2)$ .