Le and Olaya-Castro Reply: In a recent Comment [1] on our Letter [2], Feller *et al.* [1] identified a mistake in our mathematical expression of "strong independence" for system-environment states that satisfy spectrum broadcast structure. We concede that we wrote a mathematical condition that is necessary but not sufficient. However, we used the original and correct qualitative definition for strong independence throughout the Letter and in our proofs, therefore the proofs and statements, aside from the aforementioned mathematical expression, remain correct.

The concept of strong independence was originally, qualitatively, introduced by Horodecki *et al.* [3] as the situation where "...the only correlation between the environments should be the common information about the system. In other words, conditioned by the information about the system, there should be no correlations between the environments." We adhered to this qualitative definition all throughout our Letter and our proofs of the connection between strong quantum Darwinism and spectrum broadcast structure [2] and, therefore, our proofs in [2] contain no error.

The mistake in our Letter is in the formal expression (9) in [2] for the written statement of strong independence. As the authors of the Comment [1] indicate, the correct formal expression for Eq. (9) should be given by the conditional multipartite mutual information, rather than the conditional pairwise mutual information, for when there are more than two environments. Therefore, Eq. (9) in [2] shall read

$$I(\mathcal{E}_1, \ldots, \mathcal{E}_F | \mathcal{S}) = 0,$$

which is consistent with the statement we wrote in our Letter right after Eq. (9): "Strong independence means that there are no correlations between the environments *conditioned* on the information about the system".

We also note that the purpose of our original Letter [2] was twofold: to introduce the concept of strong quantum Darwinism and to make the connection with spectrum broadcast structure [3]; i.e., strong quantum Darwinism with strong independence is equivalent to spectrum broadcast structure. Since strong quantum Darwinism is not at all conditioned on strong independence, our proposed strong quantum Darwinism framework remains completely untouched by the Comment [1], both conceptually and mathematically. Its importance in establishing the minimum conditions for the objectivity of a state therefore remains true.

In conclusion, we are grateful to the authors of the Comment [1] for catching the error in Eq. (9) in the original Letter [2] and replacing it with the correct mathematical expression.

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[3] R. Horodecki, J. K. Korbicz, and P. Horodecki, Phys. Rev. A **91**, 032122 (2015).