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The Plus-Minus Davenport Constant of Finite Abelian Groups

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The Plus-Minus Davenport Constant of Finite Abelian Groups

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

The Plus-Minus Davenport Constant of Finite Abelian Groups

Let G be a finite abelian group, written additively. The Davenport constant, $D(G)$, is the smallest positive number s such that any subset of the group G, with cardinality at least s, contains a non-trivial zero-subsum. We focus on a variation of the Davenport constant where we allow addition and subtraction in the non-trivial zero-subsum. This constant is called the plus-minus Davenport constant, $D_{\pm}(G)$. In the early 2000's, Marchan, Ordaz, and Schmid proved that if the cardinality of G is less than or equal to 100, then the $D_{\pm}(G) = \lfloor \log_2 \ n \rfloor + 1$, the basic upper bound, with few exceptions. The value of $D_{\pm}(G)$ is primarily known when the rank of G at most two and the cardinality of G is less than or equal to 100. In most cases, when $D_{\pm}(G)$ is known, $D_{\pm}(G) = \lfloor \log_2 \ |G| \rfloor + 1$, with the exceptions of when G is a 3-group or a 5-group. We have studied a class of groups where the cardinality of G is a product of two prime powers. We look more closely to when the primes are 2 and 3, since the plus-minus Davenport constant of a 2-group attains the basic upper bound and while the plus-minus Davenport constant of a 3-group does not. To help us compute $D_{\pm}(G)$, we define the even plus-minus Davenport constant, $De_{\pm}(G)$, that guarantees a pm zero-subsum of even length.

Let C_n be a cyclic group of order n. Then $D(C_n) = n$ and $D_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 1$. We have shown that $De_{\pm}(C_n)$ depends on whether n is even or odd. When n is even and not a power of 2, then $De_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 2$. When $n = 2^k$, then $De_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 1$. The case when n is odd, $De_{\pm}(C_n)$ varies depending on how close n is to a power of 2. We have also shown that a subset containing the Jacobsthal numbers provides a subset of C_n that does not contain an even pm zero-subsum for certain values of n.

When G is a finite abelian group, we provide bounds for $De_{\pm}(G)$. If $D_{\pm}(G)$ is known, then we given an improvement to the lower bound of $De_{\pm}(G)$. Additional improvements are shown when G is a direct sum an elementary abelian p-groups where p is prime. Then we compute the values of $De_{\pm}(C_3^r)$ when $2 \leq r \leq 9$ and provide an optimal lower bound for larger r. For the group $C_2 \oplus C_3^r$, $D_{\pm}(C_2 \oplus C_3^r) = De_{\pm}(C_3^r)$. When $r < 10$, $D_{\pm}(C_2 \oplus C_3^r)$ does not attain the basic upper bound. We conjecture that as r increases, $D_{\pm}(C_2 \oplus C_3^r)$ will not attain the basic upper bound. Now, let $G = C_2^q \oplus C_3^r$. We compute the values of $D_{\pm}(G)$ for general q and small r. In this case, we show that if $D_{\pm}(G)$ attains the basic upper bound then so does $De_{\pm}(G)$. We then look at the case when the cardinality of G is a product of two prime powers and show improvements on the lower bound by using the fractional part of $\log_2 p$ of each prime. Furthermore, we compute the values of $D_{\pm}(G)$ when $100 < |G| \leq 200$, with some exceptions.

KEYWORDS: zero-subsum problem, weighted Davenport constant, additive number theory

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April 22, 2021

The Plus-Minus Davenport Constant of Finite Abelian Groups

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Oh, the places you will go. $\sim$ Dr. Suess

Dedicated to Elizabeth Lavin and Jaime Perez.

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Chapter 1 Introduction

Let G be a finite abelian group. Note that all definitions and results below appear later in the dissertation and are unnumbered here.

Definition. The *Davenport constant*, $D(G)$, is the smallest positive number s such that for any set $\{g_1, g_2, \ldots, g_s\}$ of s elements in G, allowing repetition, there exists a non-trivial solution to

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_s g_s = 0,$$

where $\alpha_i \in \{0, 1\}$.

This constant was originally defined by Rogers [28]. It was then named after H. Davenport after his lecture at the Midwest Conference on Group Theory and Number Theory in 1966, 17. A survey of this zero-subsum problem was given by Caro [7, 8]. Gao and Geroldinger [14] provided a survey on variations of the Davenport constant and extended questions on zero-subsum problems. This dissertation focuses on a variation of the Davenport constant called the plus-minus Davenport constant, $D_\pm(G)$.

Definition. The *plus-minus Davenport constant*, $D_\pm(G)$, is defined similarly to $D(G)$ but instead requires that there exists a non-trivial solution to

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_s g_s = 0,$$

where $\alpha_i \in \{\pm 1, 0\}$.

The following results provide bounds for $D_\pm(G)$.

Lemma. *Let G be a finite abelian group. Then*

$$D_\pm(G) \leq \lfloor \log_2 \ |G| \rfloor + 1.$$

As a basic example, for cyclic groups C_n of order n,

$$D_\pm(C_n) = \lfloor \log_2 n \rfloor + 1.$$

Given the unique factorization provided by the Fundamental Theorem of Finite Abelian Groups 2.1.1, Adhikari, Grynkiewicz, and Sun 5, Thm. 1.3 provided the following bounds for $D_\pm(G)$.

Theorem. *[5] Let G be a finite abelian group with $G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with invariant factor decomposition. Then*

$$\sum_{i=1}^{r} \lfloor \log_2 \ n_i \rfloor + 1 \leq D_\pm(G) \leq \left\lfloor \sum_{i=1}^{r} \log_2 n_i \right\rfloor + 1$$

Marchan, Ordaz, and Schmid 20 provided general bounds for $D_{\pm}(G)$ that do not depend on the decomposition of G as direct sum of cyclic groups. To optimize the lower bound, they defined $D^*_{\pm}(G)$.

Definition. [20] Let G be a finite abelian group. Define

$$D^*_{\pm}(G) = max\left\{ \sum_{i=1}^{t} \lfloor \log_2\ m_i \rfloor + 1 : G \cong \oplus_{i=1}^{t} C_{m_1},\ \text{with } t,\ m_i \in \mathbb{N} \right\}.$$

For a finite abelian group G, the bounds below are referred to as our basic upper and lower bounds,

$$D^*_{\pm}(G) \leq D_{\pm}(G) \leq \lfloor \log_2\ |G| \rfloor + 1.$$

Let $S \subset G$ where $|S| = \lfloor \log_2\ |G| \rfloor$. If there exists a subset S where S does not contain a plus-minus zero-subsum, then $D_{\pm}(G)$ attains the basic upper bound. With the help of subgroups of G, we have an improvement to the lower bound.

Lemma. *[16] Let G be a finite abelian group and H be a subgroup of G. Then*

$$D_{\pm}(G) \geq D_{\pm}(G/H) + D_{\pm}(H) - 1.$$

Let $x \in \mathbb{Q}$ then $x = \lfloor \log_2\ x \rfloor + \{x\}$, where $\{x\}$ is the fractional part of x. An application of this Lemma is provided in the following result.

Lemma. *[20] Let H_1, H_2 be finite abelian groups such that*

$$D_{\pm}(H_i) = \lfloor \log_2\ |H_i| \rfloor + 1.$$

and such that $\{\log_2 |H_1|\} + \{\log_2 |H_2|\} < 1$. Then for every finite abelian group G containing H_1 such that $G/H_1 \cong H_2$,

$$D_{\pm}(G) = \lfloor \log_2\ |G| \rfloor + 1.$$

In particular, if G has a subgroup H such that $D_{\pm}(H) = \lfloor \log_2\ |H| \rfloor + 1$ and G/H is a 2-group, then $D_{\pm}(G) = \lfloor \log_2\ |G| \rfloor + 1$.

Marchan, Ordaz, and Schmid primarily provided results for when the rank of G is two and when $|G| \leq 100$.

Theorem. *[20] Let G be a finite abelian group with $|G| \leq 100$. Then, $D_{\pm}(G) = \lfloor \log_2\ |G| \rfloor + 1$, except when G is isomorphic to $C_3^2, C_3^3, C_3^2 \oplus C_9$, where the values are 3,4, 5, respectively, and $C_5 \oplus C_{15}$ where the value is either 6 or 7.*

In most cases, they show when a group attains the basic upper bound. As we begin to understand when $D_{\pm}(G)$ attains the basic upper bound, we were interested in investigating the following question.

Question. Let G be a finite abelian group. Then when does $D_{\pm}(G)$ fall between the basic lower and upper bounds?

It has been shown for p-groups, when $p \in \{3, 5\}$, that $D_{\pm}(C_p^n)$ does not attain the basic upper bound as n increases. Before providing these results, we first state a classical result that allows us to compute the plus-minus Davenport constant of p-groups. Let $\mathbb{F}_q$ be the finite field with q elements.

Theorem. *[Chevalley-Warning Theorem 29, pg 5] Let $f_1, \ldots, f_r$ be homogeneous polynomials of degree $d_1, \ldots, d_r$ respectively in n variables over $\mathbb{F}_q$. If $n > d_1 + \cdots + d_r$ then $\{f_1, \ldots, f_r\}$ has a common nontrivial zero over $\mathbb{F}_q$.*

Note that C_p^n is an n-dimensional vector space over $\mathbb{F}_p$ when p is prime. Thus, $C_p^n \cong \mathbb{F}_p^n$ as vector spaces. The Chevalley-Warning Theorem quickly leads to the value of $D_{\pm}(C_p^n)$ when $p \in \{3, 5\}$.

Theorem. *[32]*

$$D_{\pm}(C_3^s) = s + 1.$$

Mead and Narkiewicz [23], actually uses the Chevalley-Warning Theorem to show the following result.

Theorem. *[23]*

$$D_{\pm}(C_5^s) = 2s + 1.$$

Interestingly, $D_{\pm}(C_2 \oplus C_3^r)$ is unknown for all $n \in \mathbb{N}$ since $D_{\pm}(C_3^r)$ does not attain the basic upper bound. To help us compute $D_{\pm}(C_2 \oplus C_3^r)$ we look for when a subset of C_3^r has a plus-minus zero-subsum of even length.

Definition. Let $\{g_1, g_2, \ldots, g_k\} \subset G$ where $g_i \neq 0$. An *even PM zero-subsum* is an expression

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_k g_k = 0,$$

for some $\alpha_i \in \{0, \pm 1\}$ where an even number of $\alpha_i \neq 0$.

Given a set of elements S of $G \setminus \{0\}$, we want to know whether S contains a PM zero-subsum of even length.

Definition. Let G be a finite abelian group and $S \subset G$, where $S = \{g_1, \ldots, g_k\}$ and each g_i is non-zero. Define $De_{\pm}(G)$ to be the smallest possible k such that any such subset S of cardinality k contains an even PM zero-subsum.

The bounds for $De_{\pm}(G)$ are provided by the plus-minus Davenport constant of G and $C_2 \oplus G$.

Proposition. *Let G be a finite abelian group. Then*

$$D_{\pm}(G) \leq De_{\pm}(G) \leq D_{\pm}(C_2 \oplus G).$$

For cyclic groups C_n the bounds for $De_{\pm}(C_n)$ are shown in Corollary 2.4.7,

$$\lfloor \log_2\ n \rfloor + 1 \leq De_{\pm}(C_n) \leq \lfloor \log_2\ n \rfloor + 2.$$

We show that $D_{\pm}(C_n) = \lfloor \log_2\ n \rfloor + 1$ is the basic upper bound in Proposition 2.3.2. We have found that $De_{\pm}(C_n)$ is not always the upper bound. In Section 3.1, we introduce the Jacobsthal numbers which help us compute $De_{\pm}(C_n)$. Proposition 3.1.5 shows when

$$De_{\pm}(C_n) = \lfloor \log_2\ n \rfloor + 2,$$

for certain values of n. More generally, we have the following results for when n is even.

Proposition. *For $k \geq 2$ and $2^k < 2\ell < 2^{k+1}$, then*

$$De_{\pm}(C_{2\ell}) = \lfloor \log_2\ 2\ell \rfloor + 2 = k + 2.$$

Proposition. *Let $k \geq 1$. Then*

$$De_{\pm}(C_{2^k}) = \lfloor \log_2\ 2^k \rfloor + 1 = k + 1.$$

Furthermore, in Section 3.2, we compute values $De_{\pm}(C_n)$ when n is odd and

$$2^{k-1} < n < \frac{2}{3}(2^k - (-1)^k).$$

Within these values of n, we find cases where $De_{\pm}(C_n)$ attains the lower bound.

In Section 3.2, similar to the improvements for the lower bound of $D_{\pm}(G)$, we are also able to improve $De_{\pm}(G^r)$ for when G is a finite abelian group.

Proposition.

$$De_{\pm}(G^r) \geq De_{\pm}(G) + (r-1)(D_{\pm}(G) - 1).$$

For groups C_n^r, then

$$De_{\pm}(C_n^r) \geq r\lfloor \log_2\ n \rfloor + 1.$$

Also, when $De_{\pm}(C_n)$ attains the upper bound we can further improve this lower bound. In the case where $n = 2$, we can use linear algebra to compute the value of $De_{\pm}(C_2^r)$.

Theorem. *For $r \geq 2$,*

$$De_{\pm}(C_2^r) = r + 2.$$

When p is an odd prime, computing $De_{\pm}(C_p^r)$ is not as straightforward as in the 2-group case.

Lemma. *For $r \geq 2$,*

$$De_{\pm}(C_3^r) \geq r + 3.$$

In Section 3.4, we show $De_{\pm}(C_3^r)$ attains the lower bound above for when $2 \leq r \leq 9$. More generally, we provide an improved lower bound in Section 3.5,

Theorem. *Let $S \subset C_3^n$ such that $|S| = n+q$. If*

$$n \geq \sum_{m=0}^{q-2} 2^{q-m-1} \binom{q}{m}$$

then

$$De_{\pm}(C_3^r) \geq r+q+1.$$

To see a specific example of this lower bound, we provide a proof $De_{\pm}(C_3^r) \geq r+4$ for $r \geq 10$ in Proposition 3.4.3 and Corollary 3.4.4. While we have a better understanding of $De_{\pm}(C_3^r)$, computing $De_{\pm}(C_5^r)$ seems to be a different challenge. In Section 3.2, we discuss the difference between these two elementary p-groups.

The original motivation for computing $De_{\pm}(C_3^r)$ was to help us compute $D_{\pm}(C_2 \oplus C_3^r)$. In Section 3.6, we show the connection between $De_{\pm}(C_3^n)$ and $D_{\pm}(C_2 \oplus C_3^n)$.

Theorem. *Let $n \geq 1$.*

$$De_{\pm}(C_3^n) = D_{\pm}(C_2 \oplus C_3^n).$$

The following corollary directly follows from the results of $De_{\pm}(C_3^r)$.

Corollary.

$$\begin{aligned} D_{\pm}(C_2 \oplus C_3^r) &= r+3 && \textit{for } r < 10 \\ D_{\pm}(C_2 \oplus C_3^r) &\geq r+4 && \textit{for } r \geq 10 \end{aligned}$$

Notice that, for $2 \leq r \leq 9$, $D_{\pm}(C_2 \oplus C_3^r)$ does not obtain the basic upper bound, i.e.

$$D_{\pm}(C_2 \oplus C_3^r) = r+3 < \lfloor \log_2 \ 3^r \rfloor + 2.$$

We conjecture that this pattern will continue as r increases. If this conjecture holds, then $C_2 \oplus C_3^r$ is the first known finite abelian group that is not a p-group such that $D_{\pm}(C_2 \oplus C_3^r)$ consistantly does not attain the basic upper bound.

Next, we consider the group $C_2^q \oplus C_3^r$. For some values of q and r, we show that $D_{\pm}(C_2^q \oplus C_3^r)$ attains the basic upper bound.

Proposition. *Let $q \geq 1$ and $r \leq 3$. Then,*

$$D_{\pm}(C_2^q \oplus C_3^r) = \lfloor \log_2 \ 2^q 3^r \rfloor + 1 = \lfloor r \log_2 3 \rfloor + q + 1.$$

As r increases, the value of $D_{\pm}(C_2^q \oplus C_3^r)$ varies depending on the values of q and r. We show that if there exists a value q_0 where $D_{\pm}(C_2^{q_0} \oplus C_3^r)$ attains the basic upper bound then for all $q \geq q_0$ it also attains the basic upper bound.

Lemma. *Let $q_0 \geq 1$. If*

$$D_{\pm}(C_2^{q_0} \oplus C_3^r) = \lfloor \log_2 \ 2^{q_0} \cdot 3^r \rfloor + 1 = \lfloor \log_2 \ 3^r \rfloor + q_0 + 1,$$

and $q > q_0$, then

$$D_{\pm}(C_2^q \oplus C_3^r) = \lfloor \log_2 \ 2^q 3^r \rfloor + 1 = \lfloor \log_2 \ 3^r \rfloor + q + 1.$$

The following groups obtain the basic upper bound,

$$\begin{aligned} D_{\pm}(C_2^q \oplus C_3^r) \text{ for when } & q \geq 2 \text{ and } r \in \{4,5\} \\ & q \geq 3 \text{ and } r = 6 \\ & q \geq 4 \text{ and } r \in \{8,10\}. \end{aligned}$$

The following results are shown in Lemma 4.1.3 and Corollary 4.1.4. Also, we show some conditions of when $De_{\pm}(C_2^q \oplus C_3^r) = D_{\pm}(C_2^q \oplus C_3^r)$.

Proposition. *Let $q \geq q_0$. If $D_{\pm}(C_2^{q_0} \oplus C_3^r) = \lfloor \log_2 \ 2^{q_0} 3^r \rfloor + 1$ and*

$$De_{\pm}(C_2^{q_0} \oplus C_3^r) = D_{\pm}(C_2^{q_0+1} \oplus C_3^r),$$

then

$$De_{\pm}(C_2^q \oplus C_3^r) = D_{\pm}(C_2^{q+1} \oplus C_3^r).$$

We also investigated when groups have the form $C_p^m \oplus C_q^n$ where p, q are primes. Since p and q are prime, then we are able to compute $D_{\pm}^*(C_p^m \oplus C_q^n)$.

Lemma. *Let $k = \min\{p, q\}$. Then*

$$D_{\pm}^*(C_p^m \oplus C_q^n) = k\lfloor \log_2 \ pq \rfloor + (m-k)\lfloor \log_2 \ p \rfloor + (n-k)\lfloor \log_2 \ q \rfloor + 1.$$

Corollary. *If $\{\log_2 \ p\} + \{\log_2 \ q\} < 1$, then*

$$D_{\pm}^*(G) = m\lfloor \log_2 \ p \rfloor + n\lfloor \log_2 \ q \rfloor + 1.$$

When $\{\log_2 \ p\} + \{\log_2 \ q\} < 1$, we can improve the lower bound.

Lemma. *Let $G = C_p^m \oplus C_q^n$ and let $H = C_p^r \oplus C_q^s$ be a subgroup of G. So, $r \leq m$ and $s \leq n$. Assume $\{\log_2 \ p\} + \{\log_2 \ q\} < 1$. Suppose $D_{\pm}(H) - D_{\pm}^*(H) = \ell$. Then*

$$D_{\pm}(G) \geq D_{\pm}^*(G) + \ell.$$

In the case when $\{\log_2 \ p\} + \{\log_2 \ q\} > 1$, we do not have an improvement on the lower bound. For when $m = n$, then $C_p^m \oplus C_q^m \cong C_{pq}^m$. After some computations, we found that there exists a maximal N such that for every $m \leq N$,

$$D_{\pm}(C_{pq}^m) = \lfloor \log_2 \ (pq)^m \rfloor + 1.$$

In Section 4.2, Table 4.2 provides values of N for different primes p, q.

Next, we look more closely at groups $C_p^m \oplus C_q^n$ for relatively small primes p, q. First, we focus on when $p = 3$ and $q = 5$, thus, the group $C_3^m \oplus C_5^n$. Since $\{\log_2 \ 3\} + \{\log_2 \ 5\} < 1$, and since for small values of m, n, $D_{\pm}(C_3^m \oplus C_5^n)$ attains the basic upper bound, we were able to further improve the lower bound.

Lemma. *Let $k = \min\left\{\left\lfloor \frac{m}{2} \right\rfloor, n\right\}$. Then*

$$D_{\pm}(G) \geq m + 2n + k + 1.$$

This result allowed us to show that $D_{\pm}(C_3^m \oplus C_5^n)$ attains the basic upper bound for when $m = n = 2$, and $m = 4$ and $n = 2$. We suspect that the values of $D_{\pm}(C_3^m \oplus C_5^n)$ will oscillate between attaining the basic upper bound and just below the basic upper bound.

For C_7^m, $D_{\pm}(C_7^m)$ attains the basic upper bound for when $m \in \{1, 2\}$. All other values of m are unknown. In this case, we are able to improve the lower bound.

Lemma. *Let $G = C_7^r$ and $k = \lfloor \frac{r}{2} \rfloor$ for $r \geq 4$, then*

$$D_{\pm}(G) \geq \begin{cases} 5k+1 & \textit{when } r \textit{ is even} \\ 5k+3 & \textit{when } r \textit{ is odd} \end{cases}$$

Since $\{\log_2 \ 7\} > 4/5$, then $\{\log_2 \ p\} + \{\log_2 \ 7\} > 1$ for both $p \in \{3, 5\}$. For $C_3^m \oplus C_7^n$, we compute the values of $D_{\pm}(C_3^m \oplus C_7^n)$ for when

$$(m, n) \in \{(1, 2), (2, 2), (4, 3)\}$$

by understanding when a subgroup of $C_3^n \oplus C_7^n$ attains the basic upper bound. Notice that in this case, the value of N equals 2. For $C_5^m \oplus C_7^n$, we find that the value of value of N equals 7.

To finish, we look at all possible groups when $100 < |G| \leq 200$.

Theorem. *Let G be a finite abelian group. If $100 < |G| \leq 200$, then*

$$D_{\pm}(G) = \lfloor \log_2 \ G \rfloor + 1,$$

with the possible exception of the following cases,

$$G \in \left\{ C_5^3, C_2 \oplus C_3^4, C_3^3 \oplus C_5, C_3^3 \oplus C_7 \right\},$$

where

$$\begin{aligned} D_{\pm}(C_5^3) &= 6 < \lfloor \log_2 \ 5^3 \rfloor + 1 = 7 \\ D_{\pm}(C_2 \oplus C_3^4) &= 7 < \{\log_2 \ 3^4\} + 2 = 8 \\ 7 \leq D_{\pm}(C_3^3 \oplus C_5) &\leq 8 \\ 7 \leq D_{\pm}(C_3^3 \oplus C_7) &\leq 8 \end{aligned}$$

This follows the pattern shown by Marchan, Ordaz, and Schmid, [20], when $|G| \leq 100$ where only a small portion of the groups in this class do not obtain the upper bound.

Chapter 2 Preliminary Results for the Davenport Constant

2.1 Group Theory

Let G be a finite abelian group written additively. Then G is isomorphic to a direct product of cyclic groups,

$$G \cong C_{m_1} \oplus C_{m_2} \oplus \cdots \oplus C_{m_s},$$

where C_{m_i} is a cyclic group of order m_i. An element $g \in G$, can be written as an s-tuple, so

$$g = (g_1, g_2, \ldots, g_s)$$

where $g_i \in C_{m_i}$. The *order* of g, denoted by $ord(g)$, is the smallest positive $n = ord(g) \in \mathbb{N}$ such that $ng = 1_G$. For any finite abelian group, G, there is a unique *invariant factor decomposition* of G, as given in Theorem 2.1.1.

Theorem 2.1.1. *[11] [Fundamental Theorem of Finitely Generated Abelian Groups] Let G be a finitely generated abelian group. Then*

1. $$G \cong \mathbb{Z}^q \oplus C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}, T$$

   for some integers $q, n_1, n_2, \ldots, n_r$ satisfying the following conditions:

   (a) $q \geq 0$, $r \geq 0$, and $n_j \geq 2$ for all j, and

   (b) $n_{i+1} \mid n_i$ for $1 \leq i \leq r-1$

2. *the expression in (1) is unique: if $G \cong \mathbb{Z}^t \oplus C_{m_1} \oplus C_{m_2} \oplus \cdots C_{m_u}$, where t and $m_1, m_2, \ldots, m_u$ satisfy (a) and (b) (i.e., $t \geq 0, u \geq 0, m_j \geq 2$ and $m_{i+1} \mid m_i$ for $1 \leq i \leq u-1$), then $t = q$, $u = r$ and $m_i = n_i$ for all i.*

When G is finite, the *rank* of G, denoted by $rk(G)$, is the number of invariant factors of G. The Fundamental Theorem of Finite Abelian Groups provides that $rk(G)$ is uniquely determined.

Lemma 2.1.2. *Suppose $G \cong \oplus_{i=1}^{t} C_{m_i}$. Then $rk(G) \leq t$.*

Proof. Let p be a prime that divides n_r. Then p divides n_i for $1 \leq i \leq r$ because $n_{i+1} \mid n_i$. That means that G contains a subgroup isomorphis to C_p^r. In particular, G contains exactly p^r elements of order 1 and p.

Now consider the other direct sum decomposition of G. Since an arbitrary cyclic group contains either 1 or p elements of order 1 and p, it follows that G contains at most p^t elements of order 1 and p. This implies that $r \leq t$. □

Example 2.1.3. Let G be the following finite abelian group written in terms of prime factors,

$$G = C_2 \oplus C_3^3 \oplus C_7 \oplus C_{13} \oplus C_{23}.$$

We can use the known algorithm to find the invariant factor decomposition of G,

$$G \cong C_{2\cdot 3\cdot 7\cdot 13\cdot 23} \oplus C_3 \oplus C_3,$$

to show that $rk(G) = 3$. Any other possible decomposition of G will be a direct sum of three or more cyclic groups. For example,

$$\begin{aligned} G &\cong C_{2\cdot 3\cdot 7} \oplus C_{3\cdot 13} \oplus C_{3\cdot 23} \\ &\cong C_2 \oplus C_{3\cdot 7} \oplus C_{3\cdot 13} \oplus C_{3\cdot 23} \end{aligned}$$

2.2 The Davenport Constant

This constant was originally defined by Rogers [28]. It was then named after H. Davenport after his lecture at the Midwest Conference on Group Theory and Number Theory in 1966, 17.

Definition 2.2.1. The *Davenport constant*, $D(G)$, is the smallest positive number s such that for any set $\{g_1, g_2, \ldots, g_s\}$ of s elements in G, allowing repetition, there exists a non-trivial solution to

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_s g_s = 0,$$

where $\alpha_i \in \{0, 1\}$.

For some initial work in this area see [26, 27, 31, 25, 13]. For more current work see [9, 12, 10, 18]. A survey of this zero-subsum problem was given by [7, 8]. Gao and Geroldinger [14] provided a survey on variations of the Davenport constant and extended questions on zero-subsum problems. The Davenport constant has applications in non-unique factorization theory, ramsey theory type questions and finding solutions to homogeneous polynomials over n variables. Then [3] introduced the weighted Davenport constant. The weighted Davenport constant is defined similary to $D(G)$ but instead it allows different coefficients.

Definition 2.2.2. Let $A \subseteq \mathbb{Z} \setminus \{0\}$. The *weighted Davenport constant*, $D_A(G)$, is the smallest positive number s such that for any set $\{g_1, g_2, \ldots, g_s\}$ of s elements in G, allowing repetition, there exists a non-trivial solution to

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_s g_s = 0,$$

where $\alpha_i \in A \cup \{0\}$. We denote this sum as an A-zero subsum.

For current work on the weighted Davenport constant and similar weighted Davenport variants see [1, 15, 21, 24, 2, 19, 22]. We are interested in the case when $A = \{\pm 1\}$, so $D_\pm(G)$ known as the *plus-minus Davenport Constant.*

Definition 2.2.3. The *plus-minus Davenport constant*, $D_{\pm}(G)$, is defined similarly to $D(G)$ but instead requires that there exists a non-trivial solution to

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_s g_s = 0,$$

where $\alpha_i \in \{\pm 1, 0\}$.

Notice that if our subset $S = \{g_1, \ldots, g_s\} \subset G$ contains repeating element, so $g_i = g_j$, then since we have subtraction, S contains a PM zero-subsum.

2.3 General Bounds for $D_{\pm}(G)$

Bounds for $D_{\pm}(G)$ were provided in [16, 20, 5]. We begin with an upper bound of $D_{\pm}(G)$ and give the development of other upper and lower bounds.

Lemma 2.3.1. *Let G be a finite abelian group. Then*

$$D_{\pm}(G) \leq \lfloor \log_2 \ |G| \rfloor + 1.$$

Proof. Let $|G| = m$ and let S be a subset of G such that $|S| = \lfloor \log_2 \ m \rfloor + 1 = t$, where $S = \{c_1, \ldots, c_t\}$. It follows that

$$\begin{aligned} t = \lfloor \log_2 m \rfloor + 1 &> \log_2 m \\ 2^t &> m. \end{aligned}$$

Now, we construct each sum of the form

$$\sum_{k=1}^{t} a_k c_k, \quad a_i \in \{0, 1\}.$$

Notice we have 2^t distinct expressions. Since $2^t > m$, then by the pigeon hole principle, two of the sums are equivalent $\mod m$ in G. Now we pick two expressions that have the same sum,

$$\begin{aligned} \sum a_k c_k &= \sum b_k c_k \\ \sum (a_k - b_k) c_k &= 0 \end{aligned}$$

Since our expressions are distinct, then for some k, $a_k \neq b_k$. Note that $(a_k - b_k) \in \{0, \pm 1\}$. Thus we have a non-trivial plus-minus zero-subsum. Since S was arbitrary, it follows that

$$D_{\pm}(G) \leq t = \lfloor \log_2 \ m \rfloor + 1 = \lfloor \log_2 \ |G| \rfloor + 1.$$

$\square$

The inequality in Lemma 2.3.1 is called the *basic upper bound.* For cyclic groups, we have equality. The result below was initially shown by [4].

Proposition 2.3.2.
$$D_{\pm}(C_n) = \lfloor \log_2 n \rfloor + 1.$$

Proof. By Lemma 2.3.1, we have that $D_{\pm}(C_m) \leq \lfloor \log_2\ m \rfloor + 1$. This leaves us to show that $\lfloor \log_2\ m \rfloor + 1 \leq D_{\pm}(C_m)$. One can show this by finding a subset S where $|S| = \lfloor \log_2\ m \rfloor$ that has no non-trivial plus-minus zero-subsum. Let $j = \lfloor \log_2 m \rfloor$. Then the following statements are equivalent,

$$\begin{aligned} j &= \lfloor \log_2 m \rfloor \\ j &\leq \log_2 m \\ 2^j &< m + 1 \\ 2^j - 1 &< m. \end{aligned}$$

Consider $S = \{1, 2, 4, \dots, 2^{j-1}\}$. Suppose

$$\sum_{k=0}^{j-1} \alpha_k 2^k \equiv 0 \mod m, \tag{2.1}$$

where $\alpha_k \in \{\pm 1, 0\}$. By our choice of j,

$$1 + 2 + \cdots + 2^{j-1} = 2^j - 1 < m$$

by geometric series. Thus, our summand 2.1 must equal zero.

Let ℓ be the smallest subscript such that $\alpha_\ell \neq 0$. This provides

$$\begin{aligned} \sum_{i=0}^{j-1} \alpha_i 2^i = \sum_{i=\ell}^{j-1} \alpha_i 2^i &= 0 \\ \alpha_\ell 2^\ell &= -\sum_{i=\ell+1}^{j-1} \alpha_i 2^i. \end{aligned}$$

Notice that $2^{\ell+1} \nmid \alpha_\ell 2^\ell$, but $2^{\ell+1}$ divides the right side, a contradiction.

Thus $D_{\pm}(C_m) > j = \lfloor \log_2 m \rfloor$. Therefore $D_{\pm}(C_m) \geq \lfloor \log_2 m \rfloor + 1$. Hence, we have equality, so
$$D_{\pm}(C_m) = \lfloor \log_2\ m \rfloor + 1.$$

□

Given a subgroup H of a finite abelian group G, Grynkiewicz, Marchan, and Ordaz [16, Lemma 3.1], provided a lower bound for the weighted Davenport Constant, $D_A(G)$.

Lemma 2.3.3. *[16] Let G be a finite abelian group and H be a subgroup of G. Let $A \subset \mathbb{Z} \setminus \{0\}$ be a non-empty set. Then*

$$D_A(G) \geq D_A(G/H) + D_A(H) - 1.$$

We provide a proof for this result when $A = \{\pm 1\}$, so $D_A(G) = D_\pm(G)$.

Proof. Let $D_A(H) = s$ and $D_A(G/H) = t$. There exists $\{x_1, \ldots, x_{s-1}\} \subset H$ with no A-zero subsum. Similarly, there exists $\{Hy_1, \ldots, Hy_{t-1}\} \subset G/H$ with no A-zero subsum. Let $W = \{x_1, \ldots, x_{s-1}, y_1, \ldots, y_{t-1}\} \subset G$. Then $|W| = s + t - 2$. We will show W contains no non-trivial A-zero subsum. Consider

$$a_1x_1 + \cdots + a_{s-1}x_{s-1} + b_1y_1 + \cdots b_{t-1}y_{t-1} = 0,$$

where $a_i, b_j \in A \cup \{0\}$. Since $a_1x_1 + \cdots + a_{s-1}x_{s-1} \in H$, this equation implies that $b_1y_1 + \cdots b_{t-1}y_{t-1} \in H$. The assumption on G/H implies that each $b_i = 0$. Then the assumption on H implies that each $a_i = 0$. This implies W contains no non-trivial A-zero subsum. Hence $D_A(G) \geq D_A(G) + D_A(G/H) - 1$. $\square$

Given the unique factorization provided by the Fundamental Theorem of Finite Abelian Groups 2.1.1, Adhikari, Grynkiewicz, and Sun [5, Thm 1.3], provided the following bounds for $D_\pm(G)$.

Theorem 2.3.4. *[5] Let G be a finite abelian group with $G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with invariant factor decomposition. Then*

$$\sum_{i=1}^{r} \lfloor \log_2 \ n_i \rfloor + 1 \leq D_\pm(G) \leq \left\lfloor \sum_{i=1}^{r} \log_2 n_i \right\rfloor + 1$$

For the upper bound,

$$\left\lfloor \sum_{i=1}^{r} \log_2 n_i \right\rfloor + 1 = \lfloor \log_2 \ |G| \rfloor + 1$$

as shown in Lemma 2.3.1 as the basic upper bound. The result below is a small strengthening of Theorem 2.3.4, which shows that we can consider any direct sum decomposition of G.

Theorem 2.3.5. *[20] Let G be a finite abelian group and let $m_1, \ldots, m_t$ be positive integers such that G is isomorphic to $\oplus_{i=1}^{t} C_{m_i}$. Then*

$$\sum_{i=1}^{t} \lfloor \log_2 \ m_i \rfloor + 1 \leq D_\pm(G) \leq \left\lfloor \sum_{i=1}^{t} \log_2 \ m_i \right\rfloor + 1.$$

Proof. Let $G \cong \oplus_{i=1}^{t} C_{m_i}$ for fixed $t \geq rk(G)$. By Lemma 2.3.1

$$D_\pm(G) \leq \lfloor \log_2 \ |G| \rfloor + 1 = \lfloor \log_2 \ m_1 m_2 \cdots m_t \rfloor + 1 = \left\lfloor \sum_{i=1}^{t} \log_2 \ m_i \right\rfloor + 1$$

we obtain the upper bound. Next, we need to show the lower bound of $D_\pm(G)$.

We want to show

$$\sum_{i=1}^{t} \lfloor \log_2 \ m_i \rfloor + 1 \leq D_\pm(G).$$

We will show this by induction on t. Our base case is when $t = 1$ which is shown in Proposition 2.3.2. Now assume the equality holds for $k \leq t$ and assume that $G \cong \oplus_{i=1}^{t+1} C_{m_i}$.

Let $H \cong C_{m_{t+1}}$. Then $G/H \cong C_{m_1} \oplus \cdots \oplus C_{m_t}$. By Lemma 2.3.3,

$$\begin{aligned} D_{\pm}(G) &\geq D_{\pm}(G/H) + D_{\pm}(H) - 1 \\ &\geq \left(\sum_{i=1}^{t} \lfloor \log_2 \ m_i \rfloor + 1 \right) + \left(\lfloor \log_2 \ m_{t+1} \rfloor + 1 \right) - 1 \\ &= \sum_{i=1}^{t+1} \lfloor \log_2 \ m_i \rfloor + 1. \end{aligned}$$

The second inequality holds by our induction hypothesis and Proposition 2.3.2. Hence, by proof by induction, we have our desired lower bound. □

To help optimize the lower bound, Marchan, Ordaz, and Schmid, define the following constant.

Definition 2.3.6. [20] Let G be a finite abelian group. Define

$$D_{\pm}^{*}(G) = max \left\{ \sum_{i=1}^{t} \lfloor \log_2 \ m_i \rfloor + 1 : G \cong \oplus_{i=1}^{t} C_{m_1}, \text{ with } t, \ m_i \in \mathbb{N} \right\}.$$

To help illustrate $D_{\pm}^{*}(G)$, we provide an example using the same group G as in Example 2.1.3.

Example 2.3.7. Let

$$G = C_2 \oplus C_3^3 \oplus C_7 \oplus C_{13} \oplus C_{23}.$$

We provide the unique factorization of G on the left and another decompostion on the right,

$$C_{2 \cdot 3 \cdot 7 \cdot 13 \cdot 23} \oplus C_3 \oplus C_3 \cong C_{3 \cdot 7} \oplus C_{3 \cdot 13} \oplus C_{6 \cdot 23},$$

and

$$\begin{aligned} \lfloor \log_2 \ 2 \cdot 3 \cdot 7 \cdot 13 \cdot 23 \rfloor + \lfloor \log_2 \ 3 \rfloor + \lfloor \log_2 \ 3 \rfloor = 13 + 1 + 1 = 15 \\ \lfloor \log_2 \ 3 \cdot 7 \rfloor + \lfloor \log_2 \ 3 \cdot 13 \rfloor + \lfloor \log_2 \ 6 \cdot 23 \rfloor = 4 + 5 + 7 = 16. \end{aligned}$$

So, with a different decomposition, we are able to improve the lower bound of $D_{\pm}(G)$.

Finding the best decompostion of G is algorithmically difficult as the number of prime divisors increase. This new constant does provide new bounds for $D_{\pm}(G)$.

Corollary 2.3.8. *[20] Let G be a finite abelian group and $rk(G) = r$ such that $G \cong \oplus_{i=1}^{r} C_{n_i}$ where $n_{i+1} \mid n_i$. Then*

$$D_{\pm}^{*}(G) \leq D_{\pm}(G) \leq D_{\pm}^{*}(G) + r - 1.$$

Proof. The first inequality is immediate by Theorem 2.3.5 and the definition 2.3.6 of $D^*_{\pm}(G)$. Now for the second inequality, we first show the following

$$\left\lfloor \sum_{i=1}^{t} x_i \right\rfloor \leq \sum_{i=1}^{t} \lfloor x_i \rfloor + t - 1.$$

First, we consider x to be a real number. Then, $x < \lfloor x \rfloor + 1$. Now we consider a sum of a set of real numbers. So

$$\sum_{i=1}^{t} x_i < \sum_{i=1}^{t} \lfloor x_i \rfloor + t,$$
$$\left\lfloor \sum_{i=1}^{t} x_i \right\rfloor \leq \sum_{i=1}^{t} \lfloor x_i \rfloor + t - 1.$$

This gives the inequality stated above.

Now, to show that $D_{\pm}(G) \leq D^*_{\pm}(G) + r - 1$. From Lemma 2.3.1

$$D_{\pm}(G) \leq \lfloor \log_2 \ |G| \rfloor + 1 = \lfloor \log_2 \ m_1 m_2 \cdots m_t \rfloor + 1 = \left\lfloor \sum_{i=1}^{t} \log_2 \ m_i \right\rfloor + 1$$

We can write G as a direct sum of r cyclic groups,

$$G \cong \oplus_{i=1}^{r} C_{n_i},$$

such that $n_i \mid n_{i+1}$. Then,

$$\begin{aligned}
\left\lfloor \sum_{i=1}^{r} \log_2 n_i \right\rfloor + 1 &\leq \sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor + (r-1) + 1 && \text{from the inequality above,} \\
&\leq \max \left\{ \sum_{i=1}^{t} \lfloor \log_2 \ m_i \rfloor + 1 : G \cong \oplus_{i=1}^{t} C_{m_1}, \text{ with } t, \ m_i \in \mathbb{N} \right\} + r - 1 \\
&= D^*_{\pm}(G) + r - 1.
\end{aligned}$$

□

Within this proof, we see that

$$\lfloor \log_2 \ |G| \rfloor + 1 \leq D^*_{\pm}(G) + r - 1.$$

For some finite abelian groups, G, this inequality can be strict.

Example 2.3.9. Let $G = C_{2 \cdot 3 \cdot 7 \cdot 13 \cdot 23} \oplus C_3 \oplus C_3$. By the previous example, we found that $D^*_{\pm}(G) = 16$. Then we find that

$$\lfloor \log_2 \ |G| \rfloor + 1 = 17 < D^*_{\pm}(G) + r - 1 = 16 + 3 - 1 = 18.$$

Unless stated otherwise, the basic bounds that we use for $D_\pm(G)$ are

$$D^*_\pm(G) \leq D_\pm(G) \leq \lfloor \log_2 \ |G| \rfloor + 1.$$

Let $x \in \mathbb{Q}$, then we write $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the floor function and $\{x\}$ is the fractional part of x. Let H and K be subgroups of G. By understanding the sum of $\lfloor \log_2 \ |H| \rfloor$ and $\lfloor \log_2 \ |K| \rfloor$, we can classify when G attains the basic upper bound when $G = H \oplus K$.

Lemma 2.3.10. *[20] Let H_1, H_2 be finite abelian groups such that*

$$D_\pm(H_i) = \lfloor \log_2 \ |H_i| \rfloor + 1.$$

and such that $\{\log_2 |H_1|\} + \{\log_2 |H_2|\} < 1$. Then for every finite abelian group G containing H_1 such that $G/H_1 \cong H_2$,

$$D_\pm(G) = \lfloor \log_2 \ |G| \rfloor + 1.$$

In particular, if G has a subgroup H such that $D_\pm(H) = \lfloor \log_2 \ |H| \rfloor + 1$ and G/H is a 2-group, then $D_\pm(G) = \lfloor \log_2 \ |G| \rfloor + 1$.

Proof. Let G be a finite abelian group containing H_1 such that $G/H_1 \cong H_2$. So, by Lemma 2.3.3 and the assumption, we get that

$$D_\pm(G) \geq D_\pm(H_1) + D_\pm(H_2) - 1 = \lfloor \log_2 |H_1| \rfloor + \lfloor \log_2 |H_2| \rfloor + 1$$

Now, by assumption on the fractional parts

$$\lfloor \log_2 |H_1| \rfloor + \lfloor \log_2 |H_2| \rfloor = \lfloor \log_2(|H_1| \cdot |H_2|) \rfloor = \lfloor \log_2 |G| \rfloor,$$

this lower bound matches the general upper $\lfloor \log_2 |G| \rfloor + 1$, and implies the claimed equality. The additional statement follows directly as $\{\log_2 |G/H|\} = 0$ and

$$D_\pm(G/H) = \lfloor \log_2 |G/H| \rfloor + 1.$$

□

Let $\mathbb{F}_q$ be the finite field with q elements.

Theorem 2.3.11. *[Chevalley-Warning Theorem 29, pg 5] Let $f_1, \ldots, f_r$ be homogeneous polynomials of degree $d_1, \ldots, d_r$ respectively in n variables over $\mathbb{F}_q$. If $n > d_1 + \cdots + d_r$ then $\{f_1, \ldots, f_r\}$ has a common nontrivial zero over $\mathbb{F}_q$.*

Note that C_p^n is an n-dimensional vector space over $\mathbb{F}_p$. Thus, $C_p^n \cong \mathbb{F}_p^n$ as vector spaces. We can use the Chevalley-Warning Theorem to show some results for $D_\pm(C_p^n)$ when $p \in \{3, 5\}$.

Theorem 2.3.12. *[32]*

$$D_\pm(C_3^s) = s + 1.$$

Theorem 2.3.13. *[23]*

$$D_{\pm}(C_5^s) = 2s + 1.$$

Proof. Let $S = \{g_1, \dots, g_k\} \subset C_5^s$ for $k \in \mathbb{N}$. We want to find the smallest possible value of k such that

$$\sum_{i=1}^{k} \alpha_i g_i = 0, \qquad \text{for } \alpha_i \in \{\pm 1, 0\} \tag{2.2}$$

has a non-trivial solution for every possible S when $|S| = k$. For $g \in S$, g can be represented as an s-tuple,

$$g_1 = \begin{bmatrix} g_{11} \\ \vdots \\ g_{s1} \end{bmatrix}, \dots, g_k = \begin{bmatrix} g_{1k} \\ \vdots \\ g_{sk} \end{bmatrix}.$$

This allows us to take the equation 2.2 and write it as the matrix problem,

$$\begin{bmatrix} g_{11} & \dots & g_{1k} \\ \vdots & \vdots & \vdots \\ g_{s1} & \dots & g_{sk} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \bar{0}.$$

Note, for every $x \in \mathbb{F}_5$, $x^2 \in \{\pm 1, 0\}$. Hence, for

$$\begin{bmatrix} g_{11} & \dots & g_{1k} \\ \vdots & \vdots & \vdots \\ g_{s1} & \dots & g_{sk} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \bar{0},$$

the ith row of the matrix corresponds to the quadratic form

$$g_{i1}x_1^2 + \dots + g_{ik}x_k^2$$

where $x_i^2 = \alpha_i$.

Let $d = 2$, using the Chevalley-Warning Theorem, 2.3.11 , we know there exists a solution of this system of quadratic forms when $k = 2s + 1$. This provides an upper bound for $D_{\pm}(C_5^s) \leq 2s + 1$. Consider the following $S \times 2S$ matrix

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix}$$

where the ith row contains zeros except a 1 for $j = 2i - 1$ and 2 for $j = 2i$. Then M does not contain a plus-minus zero-subsum.

Therefore, we have equality. Hence $D_{\pm}(C_5^s) = 2s + 1$. $\square$

This is the largest prime where we know $D_{\pm}(C_p^n)$ for all $n \geq 1$. In general, the Chevalley-Warning upper bound is greater than our basic upper bound,

$$D_{\pm}(G) \leq \lfloor \log_2 \ |G| \rfloor + 1.$$

Later, in Section 4.2 , we consider the complexity of computing $D_{\pm}(C_7^n)$.

In general $D_{\pm}(C_m^2)$ is currently not known for all values of m and the current bounds are

$$2\lfloor \log_2 \ m \rfloor + 1 \leq D_{\pm}(C_m^2) \leq \lfloor \log_2 \ m^2 \rfloor + 1.$$

By Proposition 2.3.2 and Lemma 2.3.10, if $\{\log_2 \ m\} < 1/2$ then,

$$D_{\pm}(C_m^2) = \lfloor \log_2 \ m^2 \rfloor + 1.$$

In 1982, Mead and Narkiewicz, they computed $D_{\pm}(C_m^2)$, for other values m, using slightly different terminology.

Theorem 2.3.14. *[23] If*

$$3 \cdot 2^{a-1} < m < 2^{a+1}$$

for $a \geq 2$, then

$$D_{\pm}(C_m^2) = \lfloor \log_2 m^2 \rfloor + 1.$$

Proof. First assume

$$3 \cdot 2^{a-1} \leq m < 2^{a+1}$$

for $a \geq 2$. Notice that

$$\begin{aligned}
3 \cdot 2^{a-1} &\leq m < 2^{a+1} \\
\log_2(3 \cdot 2^{a-1}) &\leq \log_2 m < a+1 \\
\log_2(3) + a - 1 &\leq \log_2 m < a+1 \\
a + \{\log_2 \ 3\} &\leq \log_2 m < a+1 \\
a + \{\log_2 \ 3\} &\leq \lfloor \log_2 \ m \rfloor + \{\log_2 \ m\} < a+1 \\
a &\leq \lfloor \log_2 \ m \rfloor < a+1 \\
a &= \lfloor \log_2 \ m \rfloor.
\end{aligned}$$

Then notice that this provides

$$\begin{aligned}
a + \{\log_2 \ 3\} &\leq \lfloor \log_2 \ m \rfloor + \{\log_2 \ m\} \\
\{\log_2 \ 3\} &\leq \{\log_2 \ m\}.
\end{aligned}$$

Since

$$3 \cdot 2^{a-1} < m < 2^{a+1},$$

then

$$\{\log_2 \ 3\} < \{\log_2 \ m\}.$$

Consider the following tuples:

$$\begin{aligned} &u_1 = (1,0), u_2 = (0,1), v_1 = (1,2), v_2 = (1,-2) && \\ &w_i = (3\cdot 2^i, -3\cdot 2^i), && \text{for } 0 \le i \le a-3 \\ &z_j = (3\cdot 2^j, 3\cdot 2^j), && \text{for } 0 \le j \le a-2. \end{aligned}$$

Let $S_0 = \{u_1, u_2, v_1, v_2, w_i, z_j\}$ where $0 \le i, j \le a-3$. We first show that S_0 does not contain a plus-minus zero-subsum. Let $1 \le k \le 2$ and recall $0 \le i \le a-3$. Assume

$$\sum_{k=1}^{2} b_k u_k + \sum_{k=1}^{2} c_k v_k + \sum_{i=0}^{a-3} d_i w_i + \sum_{i=0}^{a-3} e_i z_i = 0 \tag{2.3}$$

where $b_k, c_k, d_i, e_i \in \{0, \pm 1\}$. Since

$$3\sum_{i=0}^{a-3} 2^{i+1} = 3(2^{a-1}-2) < m-6,$$

it follows that the equation (2.3) must be equal to zero. Thus, considering the first and second cordinates separately, we have

$$b_1 + c_1 + c_2 + 3\sum_{i=0}^{a-3} 2^i(d_i + e_i) = 0, \tag{2.4}$$

$$b_2 + 2c_1 - 2c_2 + 3\sum_{i=0}^{a-3} 2^i(-d_i + e_i) = 0. \tag{2.5}$$

First assume that each $d_i, e_i = 0$ and not all b_k, c_k, for $k \in \{1,2\}$, are zero. Then

$$\begin{aligned} b_1 + c_1 + c_2 &= 0 \\ b_2 + 2c_1 - 2c_2 &= 0. \end{aligned}$$

Since b_k, c_k are not all zeros, these equations do not hold for any choices of b_k, c_k. Thus, a contradiction to the assumption. Hence, if not all of b_k, c_k, d_i, e_i are zero, then some d_i or e_i are not zero. Assume some $e_i \neq 0$ and t be the largest i for which $e_i \neq 0$. Adding the equations (2.4) and (2.5), we find

$$b_1 + b_2 + 3c_1 - c_2 + 3\sum_{i=0}^{a-3} 2^{i+1}(e_i) = 0$$

$$b_1 + b_2 + 3c_1 - c_2 + 3\cdot 2\sum_{i=0}^{t} 2^{i}(e_i) = 0$$

$$\pm\left(b_1 + b_2 + 3c_1 - c_2 + 3\cdot 2\sum_{i=0}^{t-1} 2^{i}(e_i)\right) = 3\cdot 2^{t+1}$$

This implies that

$$6 \mid b_1 + b_2 + 3c_1 - c_2.$$

Since $b_k, c_k \in \{0, \pm 1\}$ and not all zero then

$$6 = b_1 + b_2 + 3c_1 - c_2.$$

Thus, it follows that $b_1 = b_2 = c_1 = -c_2 \neq 0$.

Next, we subtract equation (2.5) from (2.4), suppose that some $d_i \neq 0$. With s representing the largest i such that $d_i \neq 0$, we have

$$b_1 - b_2 - c_1 + 3c_2 + 3\sum_{i=0}^{a-3} 2^{i+1}(d_i) = 0$$

$$b_1 - b_2 - c_1 + 3c_2 + 3 \cdot 2\sum_{i=0}^{s} 2^{i}(d_i) = 0$$

$$\pm\left(b_1 - b_2 - c_1 + 3c_2 + 3 \cdot 2\sum_{i=0}^{s-1} 2^{i}(d_i)\right) = 3 \cdot 2^{s+1}$$

This implies that

$$6 \mid b_1 - b_2 - c_1 + 3c_2.$$

Since $b_k, c_k \in \{0, \pm 1\}$ and not all zero then

$$6 = b_1 - b_2 - c_1 + 3c_2.$$

We previously assumes that $b_1 = b_2 = c_1 = -c_2$. These equalities provide a contradiction to our equation. Hence, we conclude that S_0 does not contain a non-trivial plus-minus zero-subsum.

Let $S_1 = \{u_1, u_2, v_1, v_2, w_i, z_j\}$ where $0 \leq i \leq a-3$ and $0 \leq j \leq a-2$. Assume

$$z_{a-2} + \sum_{k=1}^{2} b_k u_k + \sum_{k=1}^{2} c_k v_k + \sum_{i=0}^{a-3} d_i w_i + \sum_{i=0}^{a-3} e_i z_i = 0 \tag{2.6}$$

where $b_k, c_k, d_i, e_i \in \{0, \pm 1\}$. Once again, we look at the first and second cordinate separately. Let

$$A = 3 \cdot 2^{a-2} + b_1 + c_1 + c_2 + 3\sum_{i} 2^{i}(d_i + e_i) \tag{2.7}$$

$$B = 3 \cdot 2^{a-2} + b_2 + 2c_1 - 2c_2 + 3\sum_{i} 2^{i}(-d_i + e_i) \tag{2.8}$$

To get bounds for A, B, we first let b_k, c_k, d_i, e_i be non-zero so that each entry takes the value of 1. Then notice

$$\begin{aligned}
A \leq 3 \cdot 2^{a-2} + 3 + 3 \cdot 2 \sum_{i=0}^{a-3} 2^i &= 3 \cdot 2^{a-2} + 3 + 3 \cdot 2(2^{a-2} - 1) \\
&= 3 \cdot 2^{a-2} + 3 \cdot 2^{a-1} - 3 \\
&< 2m \\
B \leq 3 \cdot 2^{a-2} + 5 + 3 \cdot 2 \sum_{i=0}^{a-3} 2^i &= 3 \cdot 2^{a-2} + 5 + 3 \cdot 2(2^{a-2} - 1) \\
&= 3 \cdot 2^{a-2} + 3 \cdot 2^{a-1} - 1 \\
&< 2m
\end{aligned}$$

since $3 \cdot 2^{a-1} \leq m$. Next, we let each b_k, c_k, d_i, e_i be non-zero so that each entry takes the value of -1. Similarly,

$$\begin{aligned}
A \geq 3 \cdot 2^{a-2} - 3 - 3 \cdot 2 \sum_{i=0}^{a-3} 2^i &= 3 \cdot 2^{a-2} - 3 - 3 \cdot 2^{a-1} + 6 \\
&> 3 \cdot 2^{a-2} - m + 3 \\
&> -m \\
B \geq 3 \cdot 2^{a-2} - 5 - 3 \cdot 2 \sum_{i=0}^{a-3} 2^i &= 3 \cdot 2^{a-2} - 5 - 3 \cdot 2^{a-1} + 6 \\
&> 3 \cdot 2^{a-2} - m + 1 \\
&> -m
\end{aligned}$$

Thus $-m < A, B < 2m$. Equation 2.6 can hold if and only if A and B are in the set $\{0, m\}$.

Assume $A = B = m$. Then,

$$\begin{aligned}
|A + B| &= \left| 2(3 \cdot 2^{a-2}) + b_1 + b_2 + 3c_1 - c_2 + 3 \sum_{i=0}^{a-3} 2^{i+1}(e_i) \right| \\
&\leq 3 \cdot 2^{a-1} + |b_1| + |b_2| + 3|c_1| + |c_2| + 3 \cdot 2 \sum_{i=1}^{a-3} 2^i |e_i| \\
&\leq 3 \cdot 2^{a-1} + 6 + 3 \cdot 2(2^{a-2} - 1) \\
&= 3 \cdot 2^a < 2m = A + B.
\end{aligned}$$

This contradiction shows that we can not have that $A = B = m$.

Assume one of A and B is m and the other zero. Then

$$\begin{aligned}|A-B| &= \left|b_1 - b_2 - c_1 + 3c_2 + 3\sum_{i=0}^{a-3} 2^{i+1}(d_i)\right| \\ &\le |b_1| + |b_2| + |c_1| + 3|c_2| + 3\cdot 2\sum_{i=0}^{a-3} 2^i |d_i| \\ &\le 6 + 3\cdot 2(2^{a-2}-1) \\ &= 3\cdot 2^{a-2} < m = |A-B|.\end{aligned}$$

This is a contradiction, therefore $A = B = 0$. Therefore

$$A + B = 3\cdot 2^{a-1} + b_1 + b_2 + 3c_1 - c_2 + 3\cdot 2\sum_{i=1}^{a-3} 2^i e_i = 0,$$

but

$$\begin{aligned}3\cdot 2^{a-1} &= \left|b_1 + b_2 + 3c_1 - c_2 + 3\cdot 2\sum_{i=0}^{a-3} 2^i e_i\right| \\ &\le 6 + 3\cdot 2(2^{a-2}-1) = 3\cdot 2^{a-1},\end{aligned}$$

this implies that $b_1 = b_2 = c_1 = -c_2 \neq 0$. So, $b_1 + b_2 + c_1$ is divisible by three. However, if these conditions are satisfied, then

$$A - B = b_1 - b_2 - c_1 + 3c_2 + 3\cdot 2\sum_{i=0}^{a-3} 2^i d_i \neq 0,$$

since $b_1 - b_2 - c_1$ is not divisible by three. From this final contradiction, we conclude that S_1 does not contain a plus-minus zero-subsum. Therefore, $D_{\pm}(C_m^2) > 2a + 1 = 2\lfloor \log_2\ m \rfloor + 1$. Thus

$$2\lfloor \log_2\ m \rfloor + 2 \le D_{\pm}(C_m^2) \le \lfloor \log_2\ m^2 \rfloor + 1.$$

Since $2\{\log_2\ m\} > 1$, then the lower bound is equal to the upper bound. Hence

$$D_{\pm}(C_m^2) = \lfloor \log_2\ m^2 \rfloor + 1.$$

$\square$

Marchan, Ordaz, and Schmid [20]continue with computing $D_{\pm}(G)$ when $rk(G)$ is relatively small. They were able to improve the lower bound for groups of the form $C_{m_1} \oplus C_{m_2}$ using the result below. Note that a finite abelian group, G, has the form $C_{m_1} \oplus C_{m_2}$ if and only if $rk(G) \le 2$.

Proposition 2.3.15. *[20] Let m_1, m_2 be integers with $m_1 \ge 4$ and $m_2 \ge 3$. Then*

$$D_{\pm}(C_{m_1} \oplus C_{m_2}) \ge \lfloor \log_2\ (m_1/3) \rfloor + \lfloor \log_2\ (m_2/3) \rfloor + 4.$$

Proof. Let e_1, e_2 be generating elements of $C_{m_1} \oplus C_{m_2}$ such that the orders of e_1 and e_2 are m_1 and m_2, respectively. We construct a sequence of length $\lfloor \log_2(m_1/3) \rfloor + \lfloor \log_2(m_2/3) \rfloor + 3$ that has no plus-minus zero-subsum.

Let $k = \lfloor \log_2(m_1/3) \rfloor$ and let $\ell = \lfloor \log_2(m_2/3) \rfloor$. Let $d = m - 2^k$ where m is the integer closest to $m_1/6$ (in case this is a half integer we round up).

We consider the sequences T_1, T_2, and T_3 formed by the elements

- $2^i e_1$ for $i \in [0, k-1]$
- $2^j 3e_2$ for $j \in [0, \ell - 1]$, and
- $de_1 + e_2$, $(d + 2^k)e_1 + e_2$, and $(d + 2^{k+1})e_1 + e_2$.

The length of the sequence $T_1 \cup T_2 \cup T_3$ is $k + \ell + 3$ and we will show that it has no plus-minus zero-subsum.

The sequence T_1, by the argument in the proof of Proposition 2.3.2 and $2^k - 1 < m_1$, has no plus-minus zero-subsum. For similar reasons, T_2 has no such plus-minus zero-subsum either. More precisely, the sets of plus-minus subsums are

$$A_1 = \{-(2^k - 1)e_1, \dots, -e_1, e_1, \dots, (2^k - 1)e_1\},$$

and

$$A_2 = \{-3(2^\ell - 1)e_2, \dots, -3e_2, 3e_2, \dots, 3(2^\ell - 1)e_2\},$$

respectively. If we add two elements from T_3, then in the e_2-coordinate we get $2e_2$ which is clearly not an element of A_2. Hence, we cannot attain a plus-minus zero-subsum with the sum of two elements from T_3. If we consider all three elements from T_3 in our sum where only two have the same sign and the last one has the opposite sign, then the e_2-coordinate will be $\pm e_2 \notin A_2$. Thus, a plus-minus zero-subsum of $T_1 \cup T_2 \cup T_3$ has to contain elements from T_3 and more precisely, either two with opposite signs or all three with the same sign.

First consider the former. We take the difference between any two elements in T_3. In this case, the e_1-coordinate from T_3 of the difference of these two elements, call it a, is $\pm 2^k e_1$ or $\pm 2^{k+1} e_1$. One can see that $\pm a \notin A_1$, since $2^k - 1 < 2^k < 2^{k+1}$. Also, since for every $x \in A_1$, $|x| \leq 2^k - 1$. Thus $x + a \neq 0$ for any $x \in A_1$ and each possible a. By our choice of k,

$$\begin{aligned} k = \lfloor \log_2 (m_1/3) \rfloor &\leq \log_2 (m_1/3) \\ 2^k &\leq m_1/3 \\ 3 \cdot 2^k &\leq m_1. \end{aligned}$$

Then $(2^k - 1) + 2^{k+1} = 3 \cdot 2^k - 1 < m_1$, so $x + a \not\equiv 0 \mod m_1$. Since all elements in T_2 have e_1-coordinate equal to 0, it is thus impossible to have a plus-minus zero subsum of this form.

Second assume the latter, and without loss of generality, we take the sum of the three elements from T_3. We can see the e_2-coordinate is contained A_2. We need to focus on the e_1-coordinate. Let b be the sum of these elements in the e_1-coordinate. So

$$\begin{aligned} b =& (3d + 2^k + 2^{k+1})e_1 \\ =& (3(m - 2^k) + 3 \cdot 2^k)e_1 \\ =& (3m)e_1. \end{aligned}$$

By our choice of m, we can write b as one of

$$\frac{m_1 + \epsilon}{2} e_1 \text{ with } -2 \leq \epsilon \leq 3;$$

which one precisely depends on m_1. For each of these we will show that $-b \notin A_1$. First assume that $m_1 \geq 9$. Then

$$2^k \leq \frac{m_1}{3} \leq \frac{m_1 - \epsilon}{2} \leq \frac{2m_1}{3} \leq m_1 - 2^k.$$

The first inequality, we have from the assumption on k. Then next inequality holds for all $3\epsilon \leq m_1$ by solving for 3ϵ in the inequality. Since we assumed $m_1 \geq 9$, this is true. For the third equality, similarly this inequality holds for all $-3\epsilon \leq m_1$. Since we assumed $m_1 \geq 9$, this is true. Finally, the last inequality holds from the fact that $2^k \leq \frac{m_1}{3}$. We will show that $2^k \leq b \leq m_1 - 2^k$, and this will imply that $-b_1 \notin A_1$.

Next, we need to show that $2^k \leq b \leq m_1 - 2^k$. Notice that

$$\frac{m_1 - 2}{2} \leq \frac{m_1 + \epsilon}{2} = b$$

since $-2 \leq \epsilon \leq 3$. Then notice that

$$2^k \leq 3 \cdot 2^{k-1} - 1 \leq \frac{m_1 - 2}{2}.$$

The first inequality holds since $k \geq 1$ because $m_1 \geq 9$. The second inequality holds because $3 \cdot 2^k \leq m_1$. Thus, we get that $2^k \leq b$. Now for the upper bound of b, the following statements are equivalent,

$$\begin{aligned} 2^k &\leq \frac{m_1 - \epsilon}{2} \\ \epsilon &\leq m_1 - 2^{k+1} \\ m_1 + \epsilon &\leq 2m_1 - 2^{k+1} \\ b = \frac{m_1 + \epsilon}{2} &\leq m_1 - 2^k. \end{aligned}$$

Therefore since $b \in [2^k, m_1 - 2^k]$ then $-b \notin A_1$.

Below we consider the cases where $m_1 < 9$. We need to show that $\pm b \notin A_1$ for each case.

- For $m_1 \in \{4,5\}$, $k = \lfloor \log_2\ m_1/3 \rfloor = 0$. So, $A_1 = \{\emptyset\}$. By our choice of m_1, this provides that $m = 1$ and so $d = 0$. Then we obtain T_3 using d and k,

$$T_3 = \{e_2, e_1 + e_2, 2e_1 + e_2\}.$$

Given that $m = 1$, the e_1-coordinate of the sum of the three elements in T_3 is $b = 3e_1$. Since $3 \not\equiv 0 \mod m_1$ for each m_1, we are not able to acquire a plus-minus zero-subsum by using elements of T_3.

- For $m_1 \in \{6,7,8\}$, $k = \lfloor \log_2\ m_1/3 \rfloor = 1$. So $A_1 = \{\pm e_1\}$. By our choice m_1, we obtain the values $m = 1$ so $d = m - 2^k = -1$. Then we obtain T_3 using d and k,

$$T_3 = \{-e_1, e_1 + e_2, 3e_1 + e_2\}.$$

Given that $m = 1$, the e_1-coordinate of the sum of the three elements in T_3 is $b = 3e_1$. Since $3 \not\equiv \pm 1 \mod m_1$, we are not able to acquire a plus-minus zero-subsum by using elements of T_3.

Thus, no plus-minus zero-subsum is possible. □

Corollary 2.3.16. *[20] Let $G \cong C_{m_1} \oplus C_{m_2}$, where $m_1 \geq 4$ and $m_2 \geq 3$. Then*

$$D_\pm(C_{m_1} \oplus C_{m_2}) \geq \lfloor \log_2\ m_1 \rfloor + \lfloor \log_2\ m_2 \rfloor + 1 + \delta$$

where

$$\delta = \begin{cases} 1 & \textit{if } \{\log_2\ m_i\} \geq \{\log_2 3\} \textit{ for both } i \in \{1,2\}. \\ 0 & \textit{if } \{\log_2\ m_i\} \geq \{\log_2 3\} \textit{ for exactly one } i \in \{1,2\}. \\ -1 & \textit{if } \{\log_2\ m_i\} \geq \{\log_2 3\} \textit{ for no } i \in \{1,2\}. \end{cases}$$

Proof. By Proposition 2.3.15,

$$\lfloor \log_2\ (m_1/3) \rfloor + \lfloor \log_2\ (m_2/3) \rfloor + 4$$

and we want to show this is equalent to

$$\lfloor \log_2\ m_1 \rfloor + \lfloor \log_2\ m_2 \rfloor + 1 + \delta$$

where

$$\delta = \begin{cases} 1 & \text{if } \{\log_2\ m_i\} \geq \{\log_2 3\} \text{ for both } i \in \{1,2\}. \\ 0 & \text{if } \{\log_2\ m_i\} \geq \{\log_2 3\} \text{ for exactly one } i \in \{1,2\}. \\ -1 & \text{if } \{\log_2\ m_i\} \geq \{\log_2 3\} \text{ for no } i \in \{1,2\}. \end{cases}$$

We show equality by considering each case of δ independently. First, consider $\{\log_2\ m_i\} \geq \{\log_2 3\}$ for both i.

$$\begin{aligned} &\lfloor \log_2(m_1/3) \rfloor + \lfloor \log_2(m_2/3) \rfloor + 4 \\ =&\lfloor \log_2\ m_1 \rfloor - 1 + \lfloor \log_2\ m_2 \rfloor - 1 + 4 \\ =&\lfloor \log_2\ m_1 \rfloor + \lfloor \log_2\ m_2 \rfloor + 2. \end{aligned}$$

This provides the condition when $\delta = 1$.

Now, consider $\{\log_2\ m_i\} \geq \{\log_2 3\}$ for one i. Without loss of generality, we pick m_1.

$$\begin{aligned}
&\lfloor \log_2(m_1/3) \rfloor + \lfloor \log_2(m_2/3) \rfloor + 4 \\
=&\lfloor \log_2\ m_1 \rfloor - 1 + \lfloor \log_2\ m_2 \rfloor - 2 + 4 \\
=&\lfloor \log_2\ m_1 \rfloor + \lfloor \log_2\ m_2 \rfloor + 1.
\end{aligned}$$

This provides the condition when $\delta = 0$.

Finally, assume $\{\log_2\ m_i\} \geq \{\log_2 3\}$ for neither i.

$$\begin{aligned}
&\lfloor \log_2(m_1/3) \rfloor + \lfloor \log_2(m_2/3) \rfloor + 4 \\
=&\lfloor \log_2\ m_1 \rfloor - 2 + \lfloor \log_2\ m_2 \rfloor - 2 + 4 \\
=&\lfloor \log_2\ m_1 \rfloor + \lfloor \log_2\ m_2 \rfloor.
\end{aligned}$$

This provides the condition when $\delta = -1$. Thus, we have shown equality for all cases of δ. $\square$

Here we see that the construction given in the proof of Proposition 2.3.15 provides an improvement to the basic lower bound when $\delta = 1$. Otherwise, this lower bound is less than or equal to $D^*_\pm(G)$. The next proposition show that there is a relation between Theorems 2.3.15 and 2.3.14.

Proposition 2.3.17. *Let $m \geq 4$. Proposition 2.3.15 implies Theorem 2.3.14.*

Proof. Since $m \geq 4$, by Proposition 2.3.15, we have

$$D_\pm(C_m^2) \geq 2\lfloor \log_2\ m/3 \rfloor + 4.$$

We want to show that if

$$m \geq 4 \qquad\qquad \text{and} \qquad\qquad 3 \cdot 2^{a-1} \leq m < 2^{a+1},$$

then

$$2\lfloor \log_2\ m/3 \rfloor + 4 = \lfloor \log_2\ m^2 \rfloor + 1.$$

Since $3 \cdot 2^{a-1} \leq m < 2^{a+1}$, the proof of Theorem 2.3.14 shows that $\{\log_2\ m\} \geq \{\log_2\ 3\}$. This gives

$$\begin{aligned}
2\lfloor \log_2\ m/3 \rfloor + 4 &= 2\Bigl\lfloor \log_2 m - \log_2 3 \Bigr\rfloor + 4 \\
&= 2\Bigl\lfloor \lfloor \log_2\ m \rfloor - \lfloor \log_2\ 3 \rfloor + \{\log_2\ m\} - \{\log_2\ 3\} \Bigr\rfloor + 4 \\
&= 2\Bigl\lfloor \lfloor \log_2\ m \rfloor - \lfloor \log_2\ 3 \rfloor \Bigr\rfloor + 4 \\
&= 2\bigl\{ \lfloor \log_2\ m \rfloor - 1 \bigr\} + 4 \\
&= 2\lfloor \log_2\ m \rfloor + 2
\end{aligned}$$

Since $\{\log_2\ m\} \geq \{\log_2\ 3\}$, then $2\{\log_2\ m\} > 1$. Hence

$$\begin{aligned}\lfloor \log_2\ m^2 \rfloor &= \lfloor 2\log_2 m \rfloor \\ &= \left\lfloor 2\Big(\lfloor \log_2\ m \rfloor + \{\log_2\ m\}\Big) \right\rfloor \\ &= 2\lfloor \log_2\ m \rfloor + 1\end{aligned}$$

Thus,

$$\lfloor \log_2\ m^2 \rfloor + 1 = 2\lfloor \log_2\ m \rfloor + 2 = 2\lfloor \log_2\ m/3 \rfloor + 4.$$

Therefore,

$$\lfloor \log_2\ m^2 \rfloor + 1 = 2\lfloor \log_2\ m/3 \rfloor + 4 \leq D_{\pm}(C_m^2) \leq \lfloor \log_2\ m^2 \rfloor + 1.$$

Hence, Proposition 2.3.15 implies Theorem 2.3.14. □

Even with the improvements of Theorem 2.3.15, we still do not have an answer to the following question.

Question 2.3.18. Let $G = C_n^2$. Let $1/2 < \{\log_2\ n\} < \{\log_2\ 3\}$. What is the values of $D_{\pm}(C_n^2)$?

When $n \leq 100$, the values $D_{\pm}(C_n^2)$ are unknown when $n \in \{23, 41, 42, 91, 92, 93, 94, 95\}$.

Theorem 2.3.19. *[20] Let $n \geq 2$ be an integer with either $\{\log_2\ (3n)\} < 1 - \{\log_2 3\}$ or $\{log_2\ (3n)\} \geq \{\log_2 3\}$. Then*

$$D_{\pm}(C_3 \oplus C_{3n}) = \lfloor \log_2(9n) \rfloor + 1.$$

Proof. By Theorem 2.3.5, $\lfloor \log_2(9n) \rfloor + 1$ is the basic upper bound; this is true for any n. If $\{\log_2(3n)\} < 1 - \{\log_2 3\}$, then $\lfloor \log_2(9n) \rfloor = \lfloor \log_2(3) \rfloor + \lfloor \log_2(3n) \rfloor$ and the claim follows by invoking the lower bound from Theorem 2.3.5.

Assume $\{\log_2(3n)\} \geq \{\log_2 3\}$. Since $\{\log_2 3\} > 1/2$, we can apply Corollary 2.3.16 to obtain

$$D_{\pm}(C_3 \oplus C_{3n}) \geq \lfloor \log_2(3) \rfloor + \lfloor \log_2(3n) \rfloor + 2.$$

Since $\{\log_2\ 3\} > 1/2$, then $\lfloor \log_2(9n) \rfloor = \lfloor \log_2(3) \rfloor + \lfloor \log_2(3n) \rfloor + 1$. Therefore, we find that

$$\lfloor \log_2(9n) \rfloor + 1 = \lfloor \log_2(3) \rfloor + \lfloor \log_2(3n) \rfloor + 2.$$

Hence, the lower bound is equal to the upper bound. □

We can show this result more generally.

Proposition 2.3.20. *Let $n \geq 2$ and $j, k \geq 1$ be an integer. Assume that either $\{\log_2\ (p^j n)\} < 1 - \{\log_2 p^k\}$ or $\{\log_2\ (p^j n)\} \geq \{\log_2 p^k\} \geq \{\log_2 3\}$ for prime $p \geq 3$. Then*

$$D_{\pm}(C_{p^k} \oplus C_{p^j n}) = \lfloor \log_2(p^{k+j} n) \rfloor + 1.$$

Proof. By Theorem 2.3.5, $\lfloor \log_2(p^{k+j}n) \rfloor + 1$ is the basic upper bound; this is true for any n. If $\{\log_2(p^k n)\} < 1 - \{\log_2 p^k\}$, then $\lfloor \log_2(p^{k+j}n) \rfloor = \lfloor \log_2(p^k) \rfloor + \lfloor \log_2(p^j n) \rfloor$ and the claim follows by invoking the lower bound from Theorem 2.3.5.

Since $\{\log_2(p^j n)\} \geq \{\log_2(p^k)\} \geq \{\log_2 3\}$, we can apply Corollary 2.3.16 to obtain

$$\begin{aligned} D_\pm(C_{p^j} \oplus C_{p^k n}) &\geq \lfloor \log_2(p^k) \rfloor + \lfloor \log_2(p^j n) \rfloor + 2, \\ &= \lfloor \log_2(p^{k+j}n) \rfloor + 1, \end{aligned}$$

since the assumption on the fractional parts implies

$$\lfloor \log_2(p^{k+j}n) \rfloor = \lfloor \log_2(p^k) \rfloor + \lfloor \log_2(p^j n) \rfloor + 1.$$

Hence, the lower bound equals the basic upper bound. □

Theorem 2.3.21. *[20] Let G be a finite abelian group with $|G| \leq 100$. Then, $D_\pm(G) = \lfloor \log_2\ |G| \rfloor + 1$, except when G is isomorphic to $C_3^2, C_3^3, C_3^2 \oplus C_9$, where the values are 3,4,6, respectively, and $C_5 \oplus C_{15}$ where the value is either 6 or 7.*

Proof. Before we begin, we recall some results shown previously in this section.

- When G be cyclic, by Theorem 2.3.2, $D_\pm(G)$ equals the basic upper bound. From here on, we assume that G is not cyclic.
- When $|G| = 2^\alpha$, one can verify that the lower and upper bound are equal in the bounds given by Theorem 2.3.4, thus attaining the basic upper bound.
- If $G \cong C_3^n$, by Theorem 2.3.12, we know that $D_\pm(C_3^n) = n + 1$. Notice that when $n \geq 2$, $D_\pm(G)$ does not attain the basic upper bound.
- If $G \cong C_5^n$, by Theorem 2.3.13, we know that $D_\pm(C_5^n) = 2n + 1$. Notice that when $n \geq 2$, $D_\pm(G)$ does not attain the basic upper bound.

Assume that $G \cong C_n^2$. Then $n \leq 10$. Outside of the cases shown above, this leaves us to consider when $n \in \{6, 7, 9, 10\}$. When $n \in \{6, 7\}$, the lower bound given by Proposition 2.3.15 equals the basic upper bound.

Let $H_i = C_n$, for $i \in \{1, 2\}$. We know that $D_\pm(H_i) = \lfloor \log_2\ n \rfloor + 1$, for any n. If $n \in \{9, 10\}$, we have that $2\{\log_2\ n\} < 1$. Thus, by Lemma 2.3.10, $D_\pm(G)$ equals the basic upper bound.

Next, consider the case when $2 \big| |G|$, so $|G| = 2^\alpha n$ where n is odd. If $\alpha = 1$ and G is not cyclic, one can verify that G is one of the following cases,

1. $|G| = 2p^2$ for $3 \leq p \leq 7$ and p is prime,
2. $|G| = 2 \cdot 3^3$,
3. $|G| = 2 \cdot 3^2 \cdot 5$.

Case (1) and Case (3) In both cases, $G \cong C_p \oplus C_{pn}$. Then, we can apply Proposition 2.3.20 to show that $D_\pm(G)$ equals the basic upper bound.

Case (2) Since G is not cyclic, this leaves us to consider the following two subcases,

1. $G \cong C_2 \oplus C_3^3$,

2. $G \cong C_2 \oplus C_3 \oplus C_9$.

In case (2.1), $G \cong C_6 \oplus C_3 \oplus C_3$. Let $H_1 \cong C_3$ and $H_2 \cong C_3 \oplus C_6$. By Proposition 2.3.2 and Theorem 2.3.19, we have that $D_{\pm}(H_i) = \lfloor \log_2 |H_i| \rfloor + 1$, for $i \in \{1,2\}$. Then, since $\{\log_2 |H_1|\} + \{\log_2 |H_2|\} < 1$, by Lemma 2.3.10, $D_{\pm}(G)$ equals the basic upper bound. In case (2.2), $G \cong C_9 \oplus C_6$. Then by Proposition 2.3.20, $D_{\pm}(G)$ equals the basic upper bound.

Next, we look at the cases where $\alpha = 2$, so

1. $|G| = 2^2p^2$, or

2. $|G| = 2^2pq$,

where $p, q \geq 3$ are primes and $p \neq q$. In case (1), since $|G| \leq 100$, then $p \leq 5$. First assume that $G \cong C_2 \oplus C_{2p^2}$. Then by Lemma 2.3.10, $D_{\pm}(G)$ equals the basic upper bound. Now, assume $G \cong C_{2^{\alpha_1}} \oplus C_{2^{\alpha_2}p} \oplus C_p$, where $\alpha_1 + \alpha_2 = \alpha$. Let $H_1 = C_{2^{\alpha_1}}$ and $H_2 = C_{2^{\alpha_2}p} \oplus C_p$. We have previously shown that $D_{\pm}(H_2)$ equals the basic upper bound. When $\alpha_1 \neq 0$, since H_1 is a 2-group, by Lemma 2.3.10, $D_{\pm}(G)$ equals the basic upper bound. In the case where $\alpha_1 = 0$, we apply Proposition 2.3.20 to show that $D_{\pm}(G)$ equals the upper bound. In case (2), $G \cong C_2 \oplus C_{2pq}$. By Lemma 2.3.10, $D_{\pm}(G)$ equals the upper bound.

Now, we need to consider the case wheren $\alpha = 3$, so $|G| = 2^3n$ where $n \leq 12$ and odd. One can verify that these are the only two cases to consider.

1. n is a prime, $n \leq 11$,

2. $n = 3^2$.

In case (1), $G \cong H \oplus C_n$ where $|H| = 8$. Since H is a 2-group, we can apply Lemma 2.3.10 to show that $D_{\pm}(G)$ equals the basic upper bound.

Case (2) has additional subcases,

1. $C_3 \oplus C_3$ is a subgroup of G, and

2. C_9 is a subgroup of G.

In case (2.1), let $H_2 = C_3 \oplus C_{2^\beta 3}$, where $\beta \leq \alpha$, and $H_1 = G/H_2$. Notice that H_1 is a 2-group and we have previously in Theorem 2.3.19 shown that $D_{\pm}(H_2)$ equals the basic upper bound. Then, by Lemma 2.3.10, $D_{\pm}(G)$ equals the basic upper bound. For case (2.2), $G \cong H \oplus C_{2^\beta 9}$ where $\beta < \alpha$ and H is a 2- group. Then, by Lemma 2.3.10, $D_{\pm}(G)$ equals the basic upper bound.

Furthermore, when $\alpha \geq 4$, then $G \cong H \oplus C_p$ where H is a 2-group and p is prime. Then by Lemma 2.3.10, we conclude that $D_{\pm}(G)$ equals the basic upper bound. $\square$

We can show that $D_{\pm}(C_5 \oplus C_{15}) = 6$, a proof is provided in the Appendix 4.3. In Section 4.3, we compute for $D_{\pm}(G)$ for $100 < |G| \leq 200$.

2.4 Preliminaries and Notation for $G = C_n^r$

Given a finite abelian group G, we want to compute $D_\pm(G)$. Denote plus-minus zero-subsums as PM zero-subsums. We provide a construction of how one can approach this problem using matrices for specific types of finite abelian groups. Let $G \cong C_n^r$ where r is the rank of G, and C_n is the cyclic group of order n. For $g \in G$, we may express g as an r-tuple,

$$g = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r \end{bmatrix}$$

where $a_i \in C_n$. Let $\{e_1, \ldots, e_r\}$ be the standard basis of C_n^r. Thus,

$$e_i^T = [0, \ldots, 0, 1, 0, \ldots, 0]$$

where the ith entry is 1 and all other entries are zero. For $g \in G$, then

$$g = \sum_{j=1}^{r} a_j e_j,$$

where $a_j \in C_n$. The *support* of g, denoted by $sp(g)$, is the set of those e_i's where $a_i \neq 0$ in the summation above.

Let $S = \{s_1, \ldots, s_k\} \subseteq G$, with repetition allowed. We construct an $r \times k$ matrix M using the elements of S. So

$$M = [s_1, s_2, \ldots, s_k]$$

where $k = |S|$. Let M' be the matrix obtained by performing elementary row operations on M corresponding to a new set S'. M' has the same row space of M. The null space of M is the orthogonal complement of the row space of M. Since the row spaces of M and M' are the same, the null space of M is the null space of M'. Thus, S has a PM zero-subsum if and only if S' has a PM zero-subsum.

We shall now assume that n is a prime number. Define the rank of M, $rk(M)$, to be the number of $(\mathbb{Z}/n\mathbb{Z})$-linearly independent rows of M. We shall always assume that $rk(M) = r \leq k$. We will always row reduce M so that the left $r \times r$ submatrix of M is the identity matrix,

$$M = [e_1, \ldots, e_r, m_{r+1}, \ldots, m_k].$$

For simplicity, we relabel the columns of M,

$$M = [e_1, \ldots, e_r, m_{r+1}, \ldots, m_k] = [I_r \mid t_1, \ldots, t_{k-r}] = [I_r \mid T],$$

where t_i are the columns of T. We label the rows of T as u_i, $1 \leq i \leq r$,

$$T = [t_1, \ldots, t_{k-r}] = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \in C_n^{r \times (k-r)}.$$

Now, let x be a column vector where $x^T = [x_1, \ldots, x_{k-r}]$ such that $x_i \in \{0, \pm 1\}$. Notice that $Tx \in C_n^r$. We are interested in when $|sp(Tx)|$ is even or odd for a given x. For fixed x, one can find $|sp(Tx)|$ by understanding when x annihilates a row u_i of T. Define

$$Ann_T(x) = \{u_\ell \mid u_\ell \cdot x = 0,\ 1 \leq \ell \leq r\}$$

to be the annihilating set of x. Then,

$$|sp(Tx)| = r - |Ann_T(x)|.$$

Lemma 2.4.1. *Assume $n = 3$, $r \geq 2$, and $k - r \geq 2$. Let u_i be a row of T where $|sp(u_i)| \geq 2$, for some $1 \leq i \leq r$. Then there exists an x with $x^T = (x_1, \ldots, x_{k-r})$ and $x_i \in \{\pm 1\}$, such that $u_i \cdot x = 0$.*

Proof. Assume u_i contains $\ell \leq k - r$ non-zero entries and $u_i = [u_{i1}, \ldots, u_{i,k-r}]$. First consider the case where $2 \mid \ell$. Since $D_\pm(C_3) = 2$, we can pair up non-zero entires of u_i and find a PM zero-sum for each pair. If $u_{ij} = 0$, then the choice for x_j is arbitrary.

Next, assume $2 \nmid \ell$. Then there exists an $m \in \mathbb{N}$ such that $\ell = 2m + 1 = 2(m-1) + 3$. We pair up the first $m - 1$ non-zero entires of u_i and find a PM zero-subsum for each pair. This leaves us with three non-zero entires of u_i. Without loss of generality, we can assume that these non-zero entires of u_i are u_{i1}, u_{i2} and u_{i3}. Our goal is find $x_1, x_2, x_3 \in \{\pm 1\}$, such that

$$u_{i1}x_1 + u_{i2}x_2 + u_{i3}x_3 \equiv 0 \mod 3.$$

Let $x_j = u_{ij}$. Then this provides a PM zero-subsum for the first three elements of u_i. Thus, we have constructed an x such that

$$u_i \cdot x = 0.$$

□

Lemma 2.4.2. *Let $x^T = (x_1, \ldots, x_{k-r})$. Then*

$$Ann_T(x) = Ann_T(-x).$$

Proof. $u_1 \cdot x = 0$ if and only if $-u_i \cdot x = 0$. Thus, we have our result. □

We also are interested in when our PM zero-subsum has even length.

Definition 2.4.3. Let $\{g_1, g_2, \ldots, g_k\} \subset G$ where $g_i \neq 0$. An *even PM zero-subsum* is an expression

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_k g_k = 0,$$

for some $\alpha_i \in \{0, \pm 1\}$ where an even number of $\alpha_i \neq 0$.

Given a set of elements S of $G \setminus \{0\}$, we want to know whether S contains a even PM zero-subsum.

Definition 2.4.4. Let G be a finite abelian group and $S \subset G$, where $S = \{g_1, \ldots, g_k\}$ and each g_i is non-zero. Define $De_{\pm}(G)$ to be the smallest possible k such that any such subset S of cardinality k contains an even PM zero-subsum.

The bounds for $De_{\pm}(G)$ can be expressed by the plus-minus Davenport constants of G and $C_2 \oplus G$.

Proposition 2.4.5. *Let G be a finite abelian group. Then*

$$D_{\pm}(G) \leq De_{\pm}(G) \leq D_{\pm}(C_2 \oplus G).$$

Proof. Notice that for the first inequality, if there is an even PM zero-subsum, then there is a PM zero-subsum. Next, we need to show the upper bound. It is equivalent to show that

$$De_{\pm}(G) - 1 < D_{\pm}(C_2 \oplus G).$$

Let $De_{\pm}(G) = n$. By the definition of $De_{\pm}(G)$, there exists a set

$$M = \{m_1, \ldots, m_{n-1}\} \subset G$$

that does not contain an even PM zero-subsum.

Now let $S \subseteq C_2 \oplus G$ such that each $s_i = (1, m_i) \in S$, $1 \leq i \leq n-1$. To obtain a PM zero-subsum, we need to use an even number of $s_i \in S$. By our assumptions, M does not contain an even PM zero-subsum. Therefore, S does not contain a PM zero-subsum. Thus

$$De_{\pm}(G) - 1 < D_{\pm}(C_2 \oplus G),$$

and we get the desired result. □

Corollary 2.4.6.

$$De_{\pm}(G) \leq \lfloor \log_2 |G| \rfloor + 2.$$

Proof. By Proposition 2.4.5, we know

$$De_{\pm}(G) \leq D_{\pm}(C_2 \oplus G).$$

Then by Lemma 2.3.1

$$\begin{aligned} D_{\pm}(C_2 \oplus G) &\leq \lfloor \log_2 2|G| \rfloor + 1 \\ &= \lfloor \log_2 |G| \rfloor + 2. \end{aligned}$$

□

Corollary 2.4.7. *Let $n \geq 2$.*

$$\lfloor \log_2 n \rfloor + 1 \leq De_{\pm}(C_n) \leq \lfloor \log_2 n \rfloor + 2.$$

Proof. By Proposition 2.3.2, $D_{\pm}(C_n) = \lfloor \log_2 n \rfloor + 1$. By Lemma 2.3.10, we find that $D_{\pm}(C_2 \oplus C_n) = \lfloor \log_2 n \rfloor + 2$. By Proposition 2.4.5, the result now follows. □

Remark 2.4.8. Let H be a subgroup of G and let $rk(H) = k$. Then

$$De_{\pm}(H) \leq De_{\pm}(G).$$

We compute specific values of $De_{\pm}(C_n)$ in Section 3.1. More general bounds are provided for $De_{\pm}(C_p^n)$, where p is prime in Section 3.2 and computed values of $De_{\pm}(C_3^n)$ can be found in Section 3.4.

Chapter 3 Even Plus-Minus Davenport Constant

3.1 Even Length PM zero-subsums in C_n

In this section, we compute the value of $De_{\pm}(C_n)$. After some computation, we recognized that a subset S that contains some Jacobsthal numbers, does not contain an even PM zero-subsum. The coming results build up to Proposition 3.1.5 where we are able to show for some n,

$$De_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 2,$$

attain the upper bound. We begin by introducing the Jacobsthal numbers that arise from the *Lucus sequence*.

Definition 3.1.1. [6] Let s, t be integers. The *Lucus sequence of the first kind* is defined by $U_0 = 0, U_1 = 1$ and for $n \geq 2$, $U_n = sU_{n-1} + tU_{n-2}$.

Notice that when $s = t = 1$, we have the Fibonacci numbers When $s = 1$ and $t = 2$, these are known as the Jacobsthal numbers. Let J_n be the n^{th} Jacabosthal number. So, the Jacobsthal numbers are defined by the following recurrance,

$$J_{n+1} = J_n + 2J_{n-1}.$$

By Sloane and Inc. [30],

J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}
0	1	1	3	5	11	21	43	85	171	341

This well studied sequence has some additional recurrances,

$$\begin{aligned} J_{n+1} &= J_n + 2J_{n-1} \\ &= 2J_n + (-1)^n \\ &= 2^n - J_n. \end{aligned}$$

Using the original recurrance and its generating function, we get

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{3.1}$$

We are interested in this sequence when $n \geq 3$.

Lemma 3.1.2. *Let $3 \leq n \leq k$ and J_n be the n^{th} Jacobsthal number. Let $\alpha \in \{0, 1\}$. Then*

$$3 + \sum_{n=3}^{k} J_n = 2J_k + \alpha,$$

where $\alpha = 0$ when k is odd and $\alpha = 1$ when k is even.

Proof. Notice that when $k = 3$, k is odd and

$$3 + J_3 = 3 + 3 = 6 = 2 \cdot 3 = 2J_3 + 0.$$

When $k = 4$, k is even and

$$3 + J_3 + J_4 = 3 + 3 + 5 = 11 = 10 + 1 = 2J_4 + 1.$$

Let $\alpha \in \{0, 1\}$. Assume that $k \geq 4$ and that the result holds when $3 \leq j \leq k$. Next, we show this holds for when $j = k + 1$. Let $\alpha_j \in \{0, 1\}$ be the value corresponding to J_j.

$$\begin{aligned}
3 + \sum_{n=3}^{k+1} J_n &= \left(3 + \sum_{n=3}^{k} J_n\right) + J_{k+1} \\
&= 2J_k + \alpha_k + J_{k+1} \\
&= \frac{2}{3}\left(2^k - (-1)^k\right) + \alpha_k + J_{k+1} && \text{By identity 3.1} \\
&= \frac{1}{3}\left(2^{k+1} - 2(-1)^k\right) + \alpha_k + J_{k+1} \\
&= \frac{1}{3}\left(2^{k+1} - (-1)^{k+1} - 3(-1)^k\right) + \alpha_k + J_{k+1} \\
&= \frac{1}{3}\left(2^{k+1} - (-1)^{k+1}\right) - (-1)^k + \alpha_k + J_{k+1} \\
&= 2J_{k+1} - (-1)^k + \alpha_k.
\end{aligned}$$

We are left to show that

$$\alpha_{k+1} = -(-1)^k + \alpha_k = \begin{cases} 0 & \text{if } k+1 \text{ is odd} \\ 1 & \text{if } k+1 \text{ is even.} \end{cases}$$

We first assume that k is even. Then by the assumption, we know that $\alpha_k = 1$. So

$$-(-1)^k + \alpha_k = -1 + 1 = 0.$$

Since k is even, $k + 1$ is odd. Thus, we have the corresponding α_{k+1}. Now, let k be odd. Then $\alpha_k = 0$. So,

$$-(-1)^k + \alpha_k = 1 + 0 = 1.$$

Since k is odd, $k + 1$ is even. Thus, we have the corresponding α_{k+1}. Therefore, by induction, we have the desired result. □

Lemma 3.1.3. *Let $k \geq 3$ be odd and $A_o = \{n \in \mathbb{N} \mid 2J_k \leq n \leq 2^k - 1\}$. Let $S = \{1, 2, J_3, \ldots, J_k\} \subset C_n$ where $n \in A_o$. Then S does not contain an even PM zero-subsum.*

Proof. One can verify when $k = 3$, $S = \{1, 2, 3\}$ does not contain an even PM zero-subsum for $n = 6, 7$. We now assume when $k \geq 3$ and odd, $S = \{1, 2, J_3, \ldots, J_k\}$ does not contain an even PM zero-subsum for each values of n where $2J_k \leq n \leq 2^k - 1$. Next, it is sufficient to show that $T = \{1, 2, J_3, \ldots, J_k, J_{k+1}, J_{k+2}\}$ also does not contain an even PM zero-subsum for values of n where $2J_{k+1} \leq n \leq 2^{k+2} - 1$. Let $s_i \in S$ and $t_i \in T$, be the i^{th} elements of S and T respectively. Notice that for $1 \leq i \leq k$, $s_i = t_i$.

By Lemma 3.1.2,

$$\sum_{i=1}^{k+2} t_i = 2t_{k+2} = 2J_{k+2}.$$

Since we assumed that $2J_{k+2} \leq n$, this is only congruent to zero when $n = 2J_{k+2}$, but this sequence has odd length, since $k+2$ is odd. Let $\beta_i \in \{0, \pm 1\}$. Therefore, if there exists an even PM zero-subsum, then it must have the following form

$$\sum_{i=1}^{k+2} \beta_i t_i = 0,$$

where only an even number of β_i are non-zero.

By our assumption, we assumed that S does not contain an even PM zero-subsum. Thus if T contains an even PM zero-subsum, without loss of generality, then

$$\sum_{i=1}^{k} \beta_i t_i = t_{k+1} \qquad \text{or} \qquad \sum_{i=1}^{k+1} \beta_i t_i = t_{k+2},$$

where there are an odd number of β_i that are non-zero. From the recursions above and since k is odd, we know

$$J_{k+1} = 2J_k - 1.$$

Let each $\beta_i = 1$. Then,

$$\sum_{i=1}^{k} \beta_i t_i = 1 + 2 + \sum_{i=3}^{k} J_i = 2J_k.$$

By the recursion above,

$$\sum_{i=2}^{k} t_i = 2J_k - 1 = J_{k+1} = t_{k+1}.$$

There are an even number of non-zero β_i. Thus we do not obtain our even PM zero-subsum. This also tells us that for any possible set of β_i,

$$\sum_{i=1}^{k} \beta_i t_i < t_{k+2} = J_{k+2}.$$

Then, by the given recursions,

$$\sum_{i=1}^{k} t_i + t_{k+1} = 2J_k + J_{k+1} = J_{k+2}.$$

Hence

$$\sum_{i=1}^{k+1} t_i = t_{k+2}$$

when each $\beta_i = 1$. This leaves us with an even number of non-zero β_i. Thus, we are not able to obtain our even PM zero-subsum.

Therefore, when $k \geq 3$ and odd and $S = \{1, 2, J_3, \dots, J_k\}$, S does not contain an even PM zero-subsum. $\square$

When consider the case where k is even, we need to exclude an element to be able to use similar methods.

Lemma 3.1.4. *Let $k \geq 3$ be even and $A_e = \{n \in \mathbb{N} \mid 2J_k \leq n \leq 2^k - 1; n \neq 2J_k + 1\}$. Let $S = \{1, 2, J_3, \dots, J_k\} \subset C_n$ where $n \in A_e$. Then S does not contain an even PM zero-subsum.*

Proof. One can verify when $k = 4$, $S = \{1, 2, 3, 5\}$ does not contain an even PM zero-subsum for $n \in \{10, 12, 13, 14, 15\}$. We now assume when $k \geq 3$ and even, $S = \{1, 2, J_3, \dots, J_k\}$ does not contain an even PM zero-subsum for each values of $n \in A = \{n \mid 2J_k \leq n \leq 2^k - 1; n \neq 2J_k + 1\}$. Next, it is sufficient to show that $T = \{1, 2, J_3, \dots, J_k, J_{k+1}, J_{k+2}\}$ also does not contain an even PM zero-subsum for values of

$$n \in \bar{A} = \{n \mid 2J_{k+2} \leq n \leq 2^{k+2} - 1; n \neq 2J_{k+2} + 1\}.$$

Let $s_i \in S$ and $t_i \in T$, be the i^{th} elements of S and T respectively. Notice that for $1 \leq i \leq k$, $s_i = t_i$.

By Lemma 3.1.2,

$$\sum_{i=1}^{k+2} t_i = 2J_{k+2} + 1.$$

Since $2J_{k+2} + 1 \notin \bar{A}$, this does not provide an even PM zero-subsum for any $n \in A$. Let $\beta_i \in \{0, \pm 1\}$. Therefore, if there exists an even PM zero-subsum, then it must have the following form

$$\sum_{i=1}^{k+2} \beta_i t_i = 0,$$

where only an even number of β_i are non-zero.

By our assumption, we assumed that S does not contain an even PM zero-subsum. Thus if T contains an even PM zero-subsum, without loss of generality, then

$$\sum_{i=1}^{k} \beta_i t_i = t_{k+1} \qquad \text{or} \qquad \sum_{i=1}^{k+1} \beta_i t_i = t_{k+2},$$

where there are an odd number of β_i that are non-zero. By Lemma 3.1.2, since k is even,

$$\sum_{i=1}^{k} t_i = 2J_k + 1 = J_{k+1} = t_{k+1}.$$

Since each $\beta_i = 1$, there are an even number of non-zero β_i. Thus we do not obtain our even PM zero-subsum. This also tells us that for any possible set of β_i,

$$\sum_{i=1}^{k} \beta_i t_i < t_{k+2} = J_{k+2}.$$

From the given recursions, we have the following identity

$$J_{k+2} = 2J_{k+1} - 1.$$

Thus, by Lemma 3.1.2 and recursion above,

$$\begin{aligned}\sum_{i=1}^{k} t_i + t_{k+1} &= 1 + 2 + \sum_{i=3}^{k+1} J_i = 2J_{k+1} \\ &= J_{k+2} + 1.\end{aligned}$$

Hence

$$\sum_{i=2}^{k+1} t_i = t_{k+2}$$

when each $\beta_i = 1$. This leaves us with an even number of non-zero β_i. Thus, we are not able to obtain our even PM zero-subsum.

Therefore, when $k \geq 3$ and even and $S = \{1, 2, J_3, \ldots, J_k\}$, S does not contain an even PM zero-subsum. □

Proposition 3.1.5. *Let $k \geq 3$ and*

$$\begin{aligned} A_o &= \{n \in \mathbb{N} \mid 2J_k \leq n \leq 2^k - 1\} && \textit{when } k \textit{ is odd} \\ A_e &= \{n \in \mathbb{N} \mid 2J_k \leq n \leq 2^k - 1; n \neq 2J_k + 1\} && \textit{when } k \textit{ is even.}\end{aligned}$$

If n is contained in A_o or A_e for some k value, then

$$De_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 2.$$

Proof. Notice that

$$2^{k-1} < 2J_k = \begin{cases} \frac{2}{3}\left(2^k + 1\right) & \text{when k is odd,} \\ \frac{2}{3}\left(2^k - 1\right) & \text{when k is even.}\end{cases}$$

So,

$$\begin{gathered} 2^{k-1} < n < 2^k \\ k - 1 < \log_2 n < k, \end{gathered}$$

which implies that $k = \lfloor \log_2 \ n \rfloor + 1$. By Corollary 2.4.7,

$$\lfloor \log_2 \ n \rfloor + 1 \leq De_{\pm}(C_n) \leq \lfloor \log_2 \ n \rfloor + 2.$$

Let $S = \{1, 2, J_3, \ldots, J_k\}$. Notice, that $|S| = \lfloor \log_2 \ n \rfloor + 1$ is the lower bound of $De_{\pm}(C_n)$. By Lemma 3.1.3 and 3.1.4, S does not contain an even PM zero-subsum. Therefore, $De_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 2$. □

Proposition 3.1.6. *For $k \geq 2$ and $2^k < 2\ell < 2^{k+1}$,*

$$De_{\pm}(C_{2\ell}) = k + 2.$$

Proof. First, let $k = 2$. So $4 < 2\ell < 8$ provides that $2\ell = 6$. By Corollary 2.4.7, we know

$$3 \leq De_{\pm}(C_6) \leq 4.$$

Now, let $S = \{1, 2, 3\}$. Notice that every PM zero-subsum of S has length three. Thus, S does not contain an even PM zero-subsum. Therefore,

$$De_{\pm}(C_6) = 4.$$

Now, assume that $k \geq 3$. Since $2^{k-2} + 2 \leq 2^{k-1} < \ell$, then $2 < \ell - 2^{k-2}$. Let

$$S = \{1, 2, \ell - 2^{k-2}, \cdots, \ell - 8, \ell - 4, \ell - 2, \ell\}.$$

Notice that S contains $k - 2$ elements of the form $\ell - 2^m$, for $1 \leq m \leq k - 2$. Since $1 < 2 < \ell - 2^{k-2}$ and $\ell \in S$, then $|S| = k - 2 + 3 = k + 1$. We will show that S does not contain an even PM zero-subsum. By geometric series

$$1 + 2 + \sum_{m=1}^{k-2} 2^m = 1 + 2 + 2^{k-1} - 2 = 2^{k-1} + 1. \tag{3.2}$$

Since $2^{k-1} < \ell < 2^k$, then $2^{k-1} + 1 \leq \ell < 2^k$.

Suppose

$$\begin{aligned}\sum_{i=1}^{k+1} a_i s_i &= a_1 + 2a_2 + \sum_{i=3}^{k} a_i(\ell - 2^{k-i+1}) + a_{k+1}\ell \\ &= a_1 + 2a_2 - \sum_{i=3}^{k} a_i 2^{k-i+1} + \ell \sum_{i=3}^{k+1} a_i \\ &\equiv 0 \mod 2\ell,\end{aligned}$$

where $a_i \in \{0, \pm 1\}$. Let

$$\begin{aligned}N &= a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot (-2^{k-2}) + \cdots + a_{k-1} \cdot (-4) + a_k \cdot (-2) + a_{k+1} \cdot 0 \\ &= a_1 + a_2 - \sum_{i=3}^{k} a_i 2^{k-i+1}.\end{aligned}$$

Since we assumed $\sum_{i=1}^{k+1} a_i s_i \equiv 0 \mod 2\ell$, then $N \equiv 0 \mod \ell$. Notice that $N \leq 2^{k-1} + 1$, given by Equation 3.2.

Case (1): Suppose $2^{k-1} + 2 \leq \ell$. Then, $N = 0$. This implies that $a_1 = 0$ and $a_2 \neq 0$, since a_i are distinct powers of two for $i \geq 3$. Recall,

$$\sum_{i=1}^{k+1} a_i s_i = \left[\sum_{i=3}^{k+1} a_i\right] \ell + N \equiv 0 \mod 2\ell.$$

It follows that

$$\sum_{i=3}^{k+1} a_i \equiv 0 \mod 2.$$

Thus, there are an even number $a_i \neq 0$ for $i \geq 3$. Since $a_1 = 0$ and $a_2 \neq 0$, there are an odd number of $a_i \neq 0$ in our zero-subsum of S. Therefore, S does not contain an even PM zero-subsum for Case (1).

Case (2): Suppose $2^{k-1} + 1 = \ell$. If $N = 0$, the argument in Case (1) holds. Now, assume that $N \neq 0$. Since $N \equiv 0 \mod \ell$, it follows that $N = \pm\ell$. Recall that $N \leq 2^{k-1} + 1$, so it follows that $a_1, \ldots, a_k$ are all non-zero. If $a_{k+1} = 0$, then

$$\begin{aligned}
0 &\equiv \sum_{i=1}^{k+1} a_i s_i \equiv a_1 + 2a_2 - \sum_{i=3}^{k} a_i 2^{k-i+1} + \ell \sum_{i=3}^{k} a_i \\
&\equiv \left[\sum_{i=3}^{k} a_i\right] \ell + N \equiv (k-2)\ell + \ell \equiv (k-1)\ell \mod 2\ell.
\end{aligned}$$

Therefore, $k - 1$ is even, so k is odd. Since the sum length is k, we do not obtain an even PM zero-subsum.

Suppose $a_{k+1} \neq 0$. Then

$$0 \equiv \sum_{i=1}^{k+1} a_i s_i \equiv (k-1)\ell + \ell \equiv k\ell \mod 2\ell.$$

Thus, k is even, but the sum length is $k + 1$ which is odd. Hence S does not contain an even PM zero-subsum. Therefore,

$$De_{\pm}(C_{2\ell}) = k + 2.$$

□

Proposition 3.1.7. *Let $k \geq 1$.*

$$De_{\pm}(C_{2^k}) = k + 1.$$

Proof. Let $S = [s_1, \ldots, s_{k+1}] \subset C_{2^k}$ where $s_i \neq 0$. We will prove the proposition by using induction on k. When $k = 1$, if $|S| = 2$, then it is clear that S contains an even PM zero-subsum, since each element of S is non-zero. Now, assume for $n = 2^\ell$ where $\ell < k$,

$$De_{\pm}(C_{2^\ell}) = \ell + 1.$$

We need to show this holds when $\ell + 1 = k$.

First assume that every s_i is even. Then let $S/2 = \{s_i/2 \mid s_i \in S\}$. Notice that $S/2 \subset C_{2^{k-1}}$. By the induction hypothesis, $De_{\pm}(C_{2^{k-1}}) = k$, thus $S/2$ contains an even PM zero-subsum. So there exists a set of $\gamma_i \in \{0, \pm1\}$ such that

$$\sum_{i=1}^{k+1} \gamma_i(s_i/2) \equiv 0 \mod 2^{k-1}.$$

This tells us that

$$\sum_{i=1}^{k+1} \gamma_i s_i \equiv 0 \mod 2^k.$$

Hence, S contains an even PM zero-subsum. Therefore, we can assume that there exists an $s_i \in S$ such that s_i is odd. Without loss of generality, let $s_{k+1} = 1$.

Consider all possible sums contained in the set R, defined by

$$R = \left\{ \sum_{j=1}^{k+1} \alpha_j s_j \mod 2^k \;\middle|\; s_j \in S; \alpha_j \in \{0,1\} \right\}.$$

Denote $r \in R$ as a residue of C_{2^k}. Since $|R| = 2^{k+1}$, we are left with two cases,

1. There exists at least one residue with at least three distinct sums.

2. Every residue has two distinct sums.

Let the length of a summation denote the number of non-zero α_j. In case (1), there exists a pair of summations whose length have the same parity. Let $r_0, r_1 \in R$ be such a pair. Then it is known $r_0 - r_1 = 0$ where $r_0 - r_1$ has an even number of non-zero entries. Thus, we have obtained an even PM zero-subsum.

Now, we consider case (2). Assume that every residue has two distinct sums. Let

$$\widetilde{R} = \left\{ \sum_{j=1}^{k} \alpha_j s_j \mod 2^k \;\middle|\; s_j \in S; \alpha_j \in \{0,1\} \right\}.$$

Notice that $\widetilde{R} \subset R$. Since we are removing the last odd term, s_{k+1}, we lose the fact that every residue of C_{2^k} has exactly two distinct summations. Thus, we have that the residues of $\widetilde{R}$ have at most two distinct summations. Let R_i be the set of residues that have i distinct summations. So $|R_0 \bigcup R_1 \bigcup R_2| = 2^k$.

Let $r \in R_2$. Then there exist two distinct summations

$$\sum_{i=1}^{k} \alpha_i s_i \equiv \sum_{i=1}^{k} \beta_i s_i \equiv r,$$

where $\alpha_i, \beta_i \in \{0,1\}$ and some $\alpha_i \neq \beta_i$. Next, let

$$R_i + s_{k+1} = \{r + s_{k+1} \mid r \in R_i\}, \text{for } i \in \{0,1,2\}.$$

Suppose $r + s_{k+1} \equiv r_0 \in R_1 \cup R_2$. Then we have at least three distinct summations,

$$\sum_{i=1}^{k} \alpha_i s_i + s_{k+1} \equiv \sum_{i=1}^{k} \beta_i s_i + s_{k+1} \equiv r + s_{k+1} \equiv r_0 \equiv \sum_{i=1}^{k} \gamma_i s_i,$$

where $\alpha_i, \beta_i, \gamma_i \in \{0,1\}$. Thus, this contracticts that r_0 has only two distinct summations in R. Therefore, $r_0 \in R_0$ and hence $R_2 + s_{k+1} \subseteq R_0$.

Now, let $r \in R_0$. Then no summation in $\widetilde{R}$ is equivalent to r. By our assumption that every residue of C_{2^k} has exactly two distinct summations in R, then

$$r \equiv \sum_{i=1}^{k} \alpha_i s_i + s_{k+1} \equiv \sum_{i=1}^{k} \beta_i s_i + s_{k+1}.$$

Thus, $r - s_{k+1} \in R_2$ and hence $R_0 - s_{k+1} \subseteq R_2$. By adding s_{k+1} to both sides, then $R_0 \subseteq R_2 + s_{k+1}$. Therefore, by double containment, $R_0 = R_2 + s_{k+1}$ and thus, $|R_0| = |R_2|$.

This leaves us to consider if R_1 is empty or non-empty. Assume that R_1 is non-empty. Let $r \in R_1$. By our assumption, there exists the following two distinct summations in R

$$r \equiv \sum_{i=1}^{k} \alpha_i s_i \equiv \sum_{i=1}^{k} \beta_i s_i + s_{k+1}.$$

Then, notice that

$$\sum_{i=1}^{k} \beta_i s_i \in R_1 \cup R_2.$$

Observe that this summation is not contained in R_0. Suppose that

$$\sum_{i=1}^{k} \beta_i s_i \in R_2.$$

So there exists another summation where

$$\sum_{i=1}^{k} \beta_i s_i \equiv \sum_{i=1}^{k} \gamma_i s_i.$$

Notice that then this provides three distinct summations for $r \in R_0$,

$$r \equiv \sum_{i=1}^{k} \alpha_i s_i \equiv \sum_{i=1}^{k} \beta_i s_i + s_{k+1} \equiv \sum_{i=1}^{k} \gamma_i s_i + s_{k+1}.$$

This contradicts our assumation that every residue has only two distinct summations. Therefore,

$$\sum_{i=1}^{k} \beta_i s_i \in R_1,$$

and thus, $r - s_{k+1} \in R_1$. Since $r \in R_1$ is arbitrary, this provides that $R_1 - s_{k+1} \subseteq R_1$. Since the cardinality is the same, $R_1 - s_{k+1} = R_1$. Therefore, $R_1 = R_1 + s_{k+1}$. Now, since s_{k+1} is odd and thus a unit $\bmod 2^k$, if follows that $R_1 = C_{2^k}$. Therefore, R_0 and R_2 are empty.

Assume $R_1 = C_{2^k}$. Let $R_1 = E \cup O$ where E contains the residues where the summation has even length and O contains the residues where the summations have odd length. Since $R_1 = C_{2^k}$, then $|E| = |O|$ and both sets are disjoint. Let $r \in E$.

If $r + s_{k+1} \in O$, then there exists an $r' \in O$ such that $r' = r + s_{k+1}$. Hence, we have two distinct summations with the same parity. Similar to case (1), we obtain an even PM zero-subsum. Hence, we can assume that if $r \in E$, then $r + s_{k+1} \in E$. Therefore, $E + s_{k+1} \subseteq E$. Since they have the same cardinality, $E + s_{k+1} = E$. Again, since s_{k+1} is odd and a unit $\bmod 2^k$, then $E = C_{2^k}$ which is a contradiction. Therefore, $R_1 = \emptyset$.

Since $R_1 = \emptyset$, then $|R_0 \cup R_2| = 2^k$. Then, since R_0 and R_2 are disjoint and have the same cardinality, each $|R_i| = 2^{k-1}$, for $i \in \{0, 2\}$. Recall that $R_0 = R_2 + s_{k+1}$ nd thus, it follows that $R_2 = R_0 + s_{k+1}$. Now, let $r \in R_0$. Then there exists a set of three summations such that

$$r + s_{k+1} \equiv r_1 \equiv r_2,$$

a contradiction to our assumption that every residues has two distinct sums.

Therefore, if $|S| = k+1$, we have shown that S contains an even PM zero-subsum. Thus,

$$De_{\pm}(C_{2^k}) = k + 1.$$

$\square$

When $k \in \{3, 4\}$ Proposition 3.1.5 and 3.1.7 exclude the values of $De_{\pm}(C_n)$ for $n \in \{5, 9, 11\}$. Also note that $De_{\pm}(C_3)$ is excluded in both of these results.

Lemma 3.1.8.

$$De_{\pm}(C_3) = 2 \qquad \textit{and} \qquad De_{\pm}(C_5) = 3.$$

Proof. First we compuet $De_{\pm}(C_3)$. Notice that if $S \subset C_3$ such that $|S| = 2$ and every element of S is non-zero than we have a unique even PM zero-subsum using both elements of S. Thus, $De_{\pm}(C_3) = 2$.

Now, let $S \subset C_5$ such that $|S| = 3$ where every element of S is non-zero. Notice that if we look at the non-zero elements of $C_5^{\times} = \{\pm 1, \pm 2\}$, then without loss of generality, $S \subset \{1, 2\}$. Since $|S| = 3$ then S contains a repeated element of C_5. Thus S has an even PM zero-subsum of length two. $\square$

Lemma 3.1.9. *Let* $n \in \{9, 11\}$

$$De_{\pm}(C_n) = 4.$$

Proof. With some computation, we have confirmed that $De_{\pm}(C_9) = 4$. We have suppressed this calculation.

If $g \in C_{11}$ is non-zero, then $g \in \pm\{1, 2, 3, 4, 5\} = [5]$. Let $S \subset C_{11} \setminus \{0\}$ such that $|S| = 5$. Since we are looking for an even PM zero-subsum, then without loss of generality, we can assume $S \subset [5]$. Notice that if two elements of S are equal, we have our even PM zero-subsum. This leaves us to consider the case where every element of S is distinct.

Now, let $S, T \subset [5]$ such that $S \neq T$. Notice that $|S \cap T| = 3$ for any choice of S and T. First assume $\{1, 2, 3\} \subset S$, so $S = \{1, 2, 3, s\}$ where $s \in \{4, 5\}$. In either choice s, we can use all four elements of S to find our even PM zero-subsum,

$$1 + 2 + 3 + 5 \equiv 0 \mod 11$$
$$(1 - 2) + (4 - 3) = 0.$$

Let $s_1, s_2 \in \{1, 2, 3\}$. Next, let $\{4, 5\} \subset T$, so $T = \{s_1, s_2, 4, 5\}$. If $|s_1 - s_2| = 1$, we obtain our even PM zero-subsumsince

$$|s_1 - s_2| = 5 - 4 = 1.$$

This leaves us to consider the case when $T = \{1, 3, 4, 5\}$. We see that

$$5 + 4 + 3 - 1 \equiv 0 \mod 11.$$

Since we have considered all possible subsets of size four of [5] and found an even PM zero-subsum for each subset, then we have shown that

$$De_{\pm}(C_{11}) = 4.$$

□

In Appendix C, we ask the following question using combinatorial methods. Let $n \in \mathbb{N}$ where $n \geq 5$. Let $4 \leq k < n$. Denote $[n] = \{1, \ldots, n\}$. Let $S = \{s_1, \ldots, s_k\} \subset [n]$ where $|S| = k$ and $s_1 < s_2 < \cdots < s_k$. Define

$$S - S \setminus \{0\} = \{s_i - s_j \mid s_i, s_j \in S, i < j\}$$

as our non-zero difference set of S. Let $\{a, b, c, d\} \subset [k]$ where each one is distinct.

Question (C.1)**.** Given $S \subset [n]$ where $|S| = k$, what subsets S of size k does the following property hold

$$|s_a - s_b| \neq |s_c - s_d| \tag{3.3}$$

for any choice of $\{a, b, c, d\}$?

The result below uses these combintorial methods when $S \subset \{1, 2, \ldots, 8\}$.

Lemma 3.1.10. *Let $S = \{s_1, s_2, s_3, s_4, s_5\} \subset \{1, \ldots, 8\} = [8]$ where each element of S is distinct. Let $\{i, j, k, l\} \subset \{1, 2, 3, 4, 5\} = [5]$ where i, j, k, l are distinct. If*

$$|s_i - s_j| \neq |s_k - s_l|$$

for any $\{i, j, k, l\}$, then

$$S \in \{\{1, 2, 3, 5, 8\}, \{1, 4, 6, 7, 8\}\}.$$

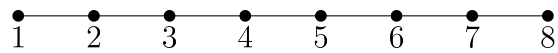
Proof. First notice that if

$$S \in \{\{1, 2, 3, 5, 8\}, \{1, 4, 6, 7, 8\}\},$$

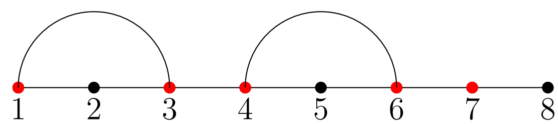
then

$$|s_i - s_j| \neq |s_k - s_l| \tag{3.4}$$

holds for any $\{i, j, k, l\}$. Now, we need to show that for all other possibe subsets S, 3.4 does not hold. To help illustrate our choice of S, one can think about about a number line from one to eight.



Then we pick our subset $S \subset [8]$ where $|S| = 5$. For example, let $S = \{1, 3, 4, 6, 7\}$. These points are highlighted in red in the figure below. If there exist two distinct arcs with distinct end points that have the same radius, then 3.4 does not hold, as shown below. We say S contains a symmetric arc pair if this occurs.



Let S' be the reflection set of S if

$$S' = \{n + 1 - s_i \mid s_i \in S\}.$$

Then one can see that if S has a symmetric arc pair then so will S' since it is the reflection set of S.

It is clear that if S contains four consecutive numbers then 3.4 does not hold. We first consider the case when there exists an $s \in S$ such that $\{s, s+1, s+2\} \subset S$. So, let $S = \{s, s+1, s+2, v, w\}$. Let $\alpha_i \in \{0, 1, 2\}$ for $0 \leq i \leq 2$. Since $|s+\alpha_i-(s+\alpha_j)| \in \{1, 2\}$, if $|v - w| \in \{1, 2\}$ then 3.4 does not hold. So, $|v - w| \geq 3$. Since $S \subset [8]$, in the case where s is maximal or minimal, S has the form

$$S = \{s, s+1, s+2, v, v+3\} \in \{\{1, 2, 3, 5, 8\}, \{1, 4, 6, 7, 8\}\}.$$

This leaves us to consider the cases where $2 \leq s \leq 5$. By symmetry of our line graph, we only need to show the cases when $s \in \{2, 3\}$.

Let $s = 2$, so $S = \{2, 3, 4, v, w\}$. Since $v, w \notin \{s - 1, s + 3\}$, this leaves, $v, w \in \{6, 7, 8\}$. For any choice of v, w, $|v - w| \in \{1, 2\}$. Thus, 3.4 does not hold. Next, let $s = 3$, so $S = \{3, 4, 5, v, w\}$. Since $|v-w| \notin \{1, 2\}$, then $1 \in S$. So, $S = \{1, 3, 4, 5, w\}$. This leaves $w \in \{7, 8\}$. In either case, one can verify that 3.4 does not hold.

Next, we assume that S does not contain three consecutive numbers. Then we split out number line in half. Since eight is even, we have four points on each side. Since it is symmetric, then without loss of generality, we pick three numbers from the left set, $\{1, 2, 3, 4\}$ and two numbers from the right set, $\{5, 6, 7, 8\}$. Since S does not contain three consective numbers, from the left side, we either have $\{1, 2, 4\} \subset S$ or $\{1, 3, 4, \} \subset S$. First, consider when $\{1, 2, 4\} \subset S$. Let $S = \{1, 2, 4, v, w\}$. If $|v - w| \in \{1, 2\}$ then 3.4 does not hold. Then $|v - w| = 3$, so $S = \{1, 2, 4, 5, 8\}$. Notice that $2 - 1 + 4 - 5 = 0$. Hence 3.4 does not hold. Next, consider when $\{1, 3, 4\} \subset S$. Let $S = \{1, 3, 4, v, w\}$. Since S does not contain three consecutive numbers then $5 \notin \{v, w\}$. Then for any other choice of v, w, $|v - w| \in \{1, 2\}$. Notice that $\{1, 4, 6, 7, 8\}$ is the reflection set of S. Hence 3.4 does not hold. $\square$

Let $S = \{s_1, s_2, s_3, s_4, s_5\} \subset \{1, 2, \ldots, 8\}$ as stated in Lemma 3.1.10 such that there exists a subset $\{i, j, k, l\} \subset \{1, 2, 3, 4, 5\}$ where

$$|s_i - s_j| = |s_k - s_l|.$$

Without loss of generality, assume that

$$s_i - s_j = s_k - s_l,$$

then it is clear that this is an even PM zero-subsum. Thus, when trying to find an S that does not contain an even PM zero-subsum, by Lemma 3.1.10, we do not need to consider the following subsets of C_n, for $n \geq 8$. We apply this lemma in the result below.

Lemma 3.1.11.

$$De_{\pm}(C_{17}) = 5.$$

Proof. By Corollary 2.4.7,

$$5 \leq De_{\pm}(C_{17}) \leq 6.$$

Let $S = \{s_1, \dots, s_5\} \subset C_{17}$ where $s_i \neq 0$ and each s_i is distinct. We choose to write each $s_i \in \pm\{1, 2, \dots, 8\}$. Since we are looking for plus-minus subsums, we can reduce to the case where each $s_i \in \{1, \dots, 8\}$. Lemma 3.1.10 tells us that if

$$S \notin \{\{1,2,3,5,8\}, \{1,4,6,7,8\}\},$$

then S contains an even PM zero-subsum. This leaves us only to consider the sets when

$$S \in \{\{1,2,3,5,8\}, \{1,4,6,7,8\}\}.$$

Notice that

$$\begin{aligned} 1+2+5-8 &= 0, \\ -1+4+6+8 &\equiv 0 \mod 17, \end{aligned}$$

are even PM zero-subsums. Since we have shown that every possible subset $S \subset \{1, \dots, 8\}$ contains an even PM zero-subsumfor when $|S| = 5$,

$$De_{\pm}(C_{17}) = 5.$$

$\square$

Lemma 3.1.12.

$$De_{\pm}(C_{21}) = 6.$$

Proof. Let $S = \{1, 2, 5, 7, 9\}$. Since each element of S is distinct and each $s_i \leq \left\lfloor \frac{n}{2} \right\rfloor$, then there is no even PM zero-subsumof length two. Since $2 \nmid 21$, if S contains an even PM zero-subsum, then we can exclude the subset of size four that contains only odd numbers. Thus, 2 must be contained in our even PM zero-subsum. Let $s_2, s_3, s_4 \in S$ and assume

$$2\alpha_1 + \alpha_2 s_2 + \alpha_3 s_3 + \alpha_4 s_4 \equiv 0 \mod 21$$

for some $\alpha_i \in \{\pm 1\}$. So,

$$\alpha_2 s_2 + \alpha_3 s_3 + \alpha_4 s_4 \in \{2, 19\}.$$

Consider when $\{5,7,9\} = \{s_2, s_3, s_4\}$. When each $\alpha_i = 1$, then

$$\alpha_2 5 + \alpha_3 7 + \alpha_4 9 \equiv 0 \mod 21.$$

While this provides a PM zero-subsum, it is not of even length. Therefore there exists some $\alpha_i = -1$. Without loss of generality, we only need to consider the case when exactly one $\alpha_i = -1$. For any choices of $\alpha_i = -1$, then

$$2 < \alpha_2 5 + \alpha_3 7 + \alpha_4 9 < 19.$$

Thus, $1 \in \{s_2, s_3, s_4\}$ and

$$2\alpha_1 + \alpha_2 + \alpha_3 s_3 + \alpha_4 s_4 \equiv 0 \mod 21.$$

So,

$$\alpha_3 s_3 + \alpha_4 s_4 \in \{1, 3, 18, 20\}.$$

Since $s_3, s_4 \in \{5,7,9\}$, then $|s_3 - s_4| \in \{2,4\}$ and $|s_3 + s_4| < 18$. Therefore S does not contain an even PM zero-subsum. Hence $De_{\pm}(C_{21}) = 6$. □

We believe that $De_{\pm}(C_{19})$ obtains the lower bound. Let $S = \{2,3,4,8,11\} \subset C_{23}$. One can verify that S does not contain an even PM zero-subsum, and hence $D_{\pm}(C_{23}) = \lfloor \log_2 \ 23 \rfloor + 2$.

Conjecture 3.1.13. *Let $2^k < n < 2^{k+1}$ be odd and $k \geq 3$. Then*

$$De_{\pm}(C_n) = \begin{cases} k+1 & \textit{if } \{\log_2 \ n\} < \frac{1}{2^{k-2}} \\ k+2 & \textit{if } \{\log_2 \ n\} \geq \frac{1}{2^{k-2}}. \end{cases}$$

Consider $n \in \{39, 41, 43\}$. Each odd n is not an odd considered in Proposition 3.1.5 and $\{\log_2 \ n\} \geq 1/4$. Notice that

$$6 \leq De_{\pm}(C_n) \leq 7.$$

Below are subsets $S \subset C_n$ for each n that do no contain an even PM zero-subsum. Thus, providing $De_{\pm}(C_n) = 7$.

$$S = \begin{cases} \{3,4,5,10,13,16\} & \text{for when } n = 39 \\ \{3,4,5,11,14,17\} & \text{for when } n = 41 \\ \{3,4,5,11,14,17\} & \text{for when } n = 43. \end{cases}$$

For $n \in \{33, 35, 37\}$, $\{\log_2 \ n\} < 1/4$ and we believe that $De_{\pm}(C_n)$ obtains the lower bound. In hope of proving this conjecture, we first would like to show the following result.

Conjecture 3.1.14.

$$De_{\pm}(C_{2^k+1}) = k + 1.$$

The results below provide a pathway to how we hope to prove Conjecture 3.1.14 in the near future.

Proposition 3.1.15. *If* $\{\log_2\ n\} \geq \{\log_2\ 3\}$*, then there exists an* $S = \{s_1, \ldots, s_k\} \subset C_n$*, where* $k = \lfloor \log_2\ n \rfloor + 1$*, such that* S *contains a unique PM zero-subsum.*

Proof. By Proposition 2.3.2 then $k = \lfloor \log_2\ n \rfloor + 1 = D_{\pm}(C_n)$. Hence, for any $S = \{s_1, \ldots, s_k\} \subset C_n$, S contains a PM zero-subsum. Now, let $\ell = \lfloor \log_2\ n \rfloor - 1$ and $S = \{1, 2, 4, \ldots, 2^\ell, a\}$ for some $a \in C_n$. Notice that by geometric series, $\sum_{i=0}^{\ell-1} 2^i < 2^\ell$. So, by our choice of ℓ, we know that $T = \{1, 2, \ldots, 2^\ell\} \subset S$ does not contain a PM zero-subsum. By geometric series, the largest possible sum from elements of T is $2^{\ell+1} - 1$. If $a > 2^{\ell+1} - 1$ and $a + 2^\ell = n$, then S contains a unique PM zero-subsum. Notice that $a + 2^\ell \equiv 0 \mod n$. Since T does not contain a PM zero-subsum, if there exists another PM zero-subsum it must contain a. Since T does not contain a PM zero-subsum equivalent to 2^ℓ then $a + 2^\ell \equiv 0 \mod n$ is unique.

Let $a = n - 2^\ell$. Since $\{\log_2\ n\} \geq \{\log_2\ 3\}$, then the following statements are equivalent

$$\begin{aligned} \{\log_2\ 3\} &\leq \{\log_2\ n\} \\ \{\log_2\ 3\} + \lfloor \log_2\ n \rfloor = \log_2 3 + \ell &\leq \log_2 n \\ 3 \cdot 2^\ell &\leq n \\ 2^{\ell+1} - 1 &< n - 2^\ell = a. \end{aligned}$$

Therefore, S contains a unique PM zero-subsum. $\square$

Data suggest that if $\{\log_2\ 3\} \leq \{\log_2\ n\} < \{\log_2\ 7\}$, then each set S that has a unique PM zero-subsum, the PM zero-subsum must have length two.

Conjecture 3.1.16. *If* $\{\log_2\ n\} \geq \{\log_2\ 7\}$*, then there exists an* $S = \{s_1, \ldots, s_k\} \subset C_n$*, where* $k = \lfloor \log_2\ n \rfloor + 1$*, such that* S *contains a unique PM zero-subsum of length greater than two.*

Conjecture 3.1.17. *If* $\{\log_2\ n\} < \{\log_2\ 3\}$*, then every* $S = \{s_1, \ldots, s_k\}$*, where* $k = \lfloor \log_2\ n \rfloor + 1$*, contains at least two PM zero-subsums.*

Lemma 3.1.18. *Let* $k \geq 5$ *and let* $S = \{s_1, \ldots, s_{k+1}\} \subset \{1, 2, 3, \ldots, 2^{k-1}\} \subset C_{2^k+1}$ *where each* s_i *is even. Then* S *contains an even PM zero-subsum that is zero.*

Proof. One can quickly verify that no such subset S where every s_i is even exists for when $k < 5$. Consider all possible sums contained in the set R, defined by

$$R = \left\{ \sum_{j=1}^{k+1} \alpha_j s_j \mod 2^k + 1 \;\middle|\; s_j \in S; \alpha_j \in \{0, 1\} \right\}.$$

The set R are the residues of C_{2^k+1} and $|R| = 2^{k+1}$. Since each s_i is even, then for every $r \in R$, r is even. So, there are only $2^{k-1} + 1$ distinct residues in R. By pigeon hole principle there exists a residue with at least three distinct sums. Therefore, a

2^k+1	k	$\ell = \|S\|$	$T \subset S$
17	4	5	$\{1,2,4,8\}, \{3,5,6,7\}$
33	5	6	$\{1,2,4,8,16\}, \{5,7,10,13,14\}$
65	6	7	$\{1,2,4,8,16,32\}$, $\{3,6,12,17,24,31\}$, $\{7,9,14,18,28,29\}$, $\{11,19,21,22,23,27\}$

Table 3.1: Subsets $T \subset S \subset C_{2^k+1}$ similar to those shown in Conjecture 3.1.20.

pair of summations have the same parity. Let $r_0, r_1 \in R$ be such a pair. Hence, $r_0 - r_1 = 0$ where $r_0 - r_1$ has an even number of non-zero entries. Thus, S contains an even PM zero-subsum that is zero. $\square$

Lemma 3.1.19. *Let $k \geq 5$ and let $S = \{s_1, \ldots, s_{k+1}\} \subset \{1, 2, 3, \ldots, 2^{k-1}\} \subset C_{2^k+1}$ where each s_i is odd. Then S contains an even PM zero-subsum that is zero.*

Proof. One can quickly verify that no such subset S where every s_i is odd exists for when $k < 5$. Consider all possible sums contained in the set R, defined by

$$R = \left\{ \sum_{j=1}^{k+1} \alpha_j s_j \mod 2^k + 1 \;\middle|\; s_j \in S; \alpha_j \in \{0,1\} \right\}.$$

The set R are the residues of C_{2^k+1} and $|R| = 2^{k+1}$. Let $\alpha = (\alpha_1, \ldots, \alpha_{k+1})$ and we say $|sp(\alpha)|$ is the number of non-zero α_j. Since each s_i is odd, then if $|sp(\alpha)|$ is even then $r = \sum_j \alpha_j s_j$ is also even. There are a total of 2^k distinct $r \in R$ that are even. By pigeon hole principle there are exists a residue $r \in R$ that has at least two summations since every s_i is odd then this pair of summations have the same parity. Let $r_0, r_1 \in R$ be such a pair. Thus $r_0 - r_1 = 0$. So, S contains an even PM zero-subsum. $\square$

Conjecture 3.1.20. *Let*

$$R = \left\{ \sum_{j=1}^{k+1} \alpha_j s_j \mod 2^k + 1 \;\middle|\; \begin{array}{c} s_j \in S = \{s_1, \ldots, s_\ell\} \subset \{1, 2, \ldots, m\}; \\ \ell = k+1,\ m = 2^{k-1},\ \alpha_j \in \{0,1\} \end{array} \right\}.$$

Denote $r \bmod 2^k + 1 \in R$ as a residue of C_{2^k+1}. If $T = \{1, 2, 2^2, \ldots, 2^{k-1}\} \subset S$, then every residue has at most two distinct equations.

Note that most other subsets S contain a residue with at least three distinct equations. If a residue has at least three such equations then there exists two such equations where the number of non-zero α_i have the same parity. Thus, we have an even PM zero-subsum. After some computation, other such subsets of S exists. These subsets are shown in Table 3.1.

3.2 General Bounds for Even Length PM zero-subsums G

By Proposition 2.4.5,

$$D_{\pm}(G) \leq De_{\pm}(G) \leq D_{\pm}(C_2 \oplus G).$$

In this section we improve the lower bound for $De_{\pm}(G)$ using similar methods used for $D_{\pm}(G)$. We first show a result that provides an example of when it is clear that we have an even PM zero-subsum. Then in directly improve the lower bounds for groups G, G^r, and C_p^r for when p is a prime.

Lemma 3.2.1. *Let G be an arbitrary finite abelian group where $rk(G) = r$. Let $q > r$ and let M be an $r \times q$ matrix of the form*

$$M = [e_1, \dots, e_r \mid t_1, t_2, \dots, t_{q-r}] = [I_r \mid T],$$

where the last m rows of T are zero rows. Let N be an $(r-m) \times (q-m)$ submatrix of M of the form

$$N = [e_1, \dots, e_{r-m} \mid \tilde{t_1}, \tilde{t_2}, \dots, \tilde{t_\ell}],$$

that removes the last m rows of M and indicated columns of I_r. Then M has an even PM zero-subsum if and only if N has an even PM zero-subsum.

Proof. Let $\ell = q - r$. Let $\alpha_i, \beta_j \in \{\pm 1, 0\}$ for $1 \leq i \leq r$ and $1 \leq j \leq \ell$. M contains an even PM zero-subsum if and only if there exists a set of α_i and β_j such that

$$\bar{0} = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_r e_r + \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_\ell t_\ell,$$

where an even number of α_i and β_j are non-zero. Since the last m rows of T are zero rows, such a sum would exist if and only if $\alpha_i = 0$ for $r - m + 1 \leq i \leq r$. Thus

$$\bar{0} = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{r-m} e_{r-m} + \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_\ell t_\ell,$$

if and only if

$$\bar{0} = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{r-m} e_{r-m} + \beta_1 \tilde{t_1} + \beta_2 \tilde{t_2} + \cdots + \beta_\ell \tilde{t_\ell}.$$

This proves the lemma. $\square$

Let G be a general finite abelian group and $S = \{s_1, \dots, s_\ell\} \subset G$. Let M be the corresponding matrix generated from S, so $M = [s_1, \dots, s_\ell]$. With some finite abelian groups G, we loose our ability to row reduce down to an identity matrix on the left side of M as saw in Lemma 3.2.1. The result below provides an improvement to the lower bound similar to the result for $D_{\pm}(G)$ given by Lemma 2.3.3.

Lemma 3.2.2. *Let H be a subgroup of G. Then*

$$De_{\pm}(G) \geq De_{\pm}(H) + D_{\pm}(G/H) - 1.$$

Proof. Let H be a subgroup of G. Suppose that $De_{\pm}(H) = h+1$. Let $S \subset H$ such that $|S| = h$ and S does not contain an even PM zero-subsum. Then let M be the corresponding matrix of S with h columns. Let $D_{\pm}(G/H) = k+1$ and $T \subset G$ such that the images of $T \in G/H$ does not contain a PM zero-subsum. Then let N be the corresponding matrix of T with k columns. Next, we construct the following block matrix,

$$P = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right],$$

where 0 are the corresponding zero matrices. Notice that P has $h+k$ columns and does not contain an even PM zero-subsum. Therefore

$$\begin{aligned} De_{\pm}(G) &\geq h+k+1 \\ &= De_{\pm}(H) - 1 + D_{\pm}(G/H) - 1 + 1 \\ &= De_{\pm}(H) + D_{\pm}(G/H) - 1. \end{aligned}$$

□

Proposition 3.2.3. *Let G be an arbitrary finite abelian group. Then*

$$De_{\pm}(G^r) \geq De_{\pm}(G) + (r-1)(D_{\pm}(G) - 1).$$

Proof. First let $r = 2$ and notice that G is a subgroup of G^2. Then by Lemma 3.2.2,

$$De_{\pm}(G^2) \geq De_{\pm}(G) + D_{\pm}(G) - 1.$$

Now assume that this lower bound holds for $k \leq r-1$. Then, we want to show the following lower bound holds for when $k = r$,

$$De_{\pm}(G^k) \geq De_{\pm}(G) + (k-1)(D_{\pm}(G) - 1).$$

Let $H = G^{r-1}$, so then $G^r/H \cong G$. Therefore by Lemma 3.2.2 and our induction hypothesis

$$\begin{aligned} De_{\pm}(G^r) &\geq De_{\pm}(G^{r-1}) + D_{\pm}(G) - 1 \\ &\geq (De_{\pm}(G) + (r-2)(D_{\pm}(G) - 1)) + D_{\pm}(G) - 1 \\ &= De_{\pm}(G) + (r-1)(D_{\pm}(G) - 1). \end{aligned}$$

□

Corollary 3.2.4. *Let $n \geq 3$, $r \geq 2$, and let $k = \lfloor \log_2 n \rfloor$. Then*

$$De_{\pm}(C_n^r) \geq rk + 1.$$

Proof. By Proposition 2.3.2, $D_{\pm}(C_n) = \lfloor \log_2 n \rfloor + 1 = k+1$. By Proposition 3.2.3,

$$De_{\pm}(C_n^r) \geq De_{\pm}(C_n^r) + (r-1)k.$$

Then by Corollary 2.4.7,

$$De_{\pm}(C_n^r) \geq k + 1 + (r-1)k = rk + 1.$$

□

Corollary 3.2.5. *Let $k = \lfloor \log_2\ n \rfloor$ and $r \geq 2$. If*

$$De_{\pm}(C_n) = k + 2,$$

then

$$De_{\pm}(C_n^r) \geq rk + 2.$$

Proof. By Proposition 2.3.2, $D_{\pm}(C_n) = \lfloor \log_2\ n \rfloor + 1 = k + 1$. Then, by assuming that $De_{\pm}(C_n^r) = r + 2$, and by Proposition 3.2.3,

$$\begin{aligned} De_{\pm}(C_n^r) &\geq De_{\pm}(C_n) + (r-1)(D_{\pm}(C_n) - 1) \\ &= k + 2 + (r-1)k \\ &= rk + 2. \end{aligned}$$

□

In the case where $p = 2$, we can use linear algebra to compute the value of $De_{\pm}(C_2^r)$.

Theorem 3.2.6. *For $r \geq 2$,*

$$De_{\pm}(C_2^r) = r + 2.$$

Proof. Since $r \geq 2$, there exist an $m \in C_2^r$ such that $|sp(m)|$ is even. Then consider

$$M_0 = [e_1, e_2, \ldots, e_r \mid m].$$

Since the first r columns are linearly independent and $|sp(m)|$ is even, M_0 does not contain an even PM zero-subsum. Thus,

$$De_{\pm}(C_2^r) > r + 1$$

which implies

$$De_{\pm}(C_2^r) \geq r + 2.$$

Let S be a subset of C_2^r with non-zero elements such that $|S| = r+2$ and let M be the corresponding matrix. Notice that M is an $r \times (r+2)$ matrix. Let x be a vector of dimension $r+2$ with every entries in $\{0, \pm 1\}$. Then Mx provides a homogeneous system of r linear equations in $r+2$ variables. Since $r < r+2$, we are guaranteed a non-trivial zero solution. We are looking for a solution where x has an even number of non-zero entries.

Let N be the $(r+1) \times (r+2)$ matrix where the first r rows are the rows of M and the last row of N is a row of all ones. Thus, there exists a non-zero y such that Ny is zero. Since the last row is all ones, we know that y contains an even number of non-zero entries. This provides that y is a non-trivial zero solution of even length. Hence $De_{\pm}(C_2^r) = r + 2$. □

In the case where $r = 1$, it is easy to check that $De_{\pm}(C_2) = 2$. We now assume that $p \geq 3$ is an odd prime.

Lemma 3.2.7. *For $r \geq 2$,*

$$De_{\pm}(C_3^r) \geq r + 3.$$

Proof. To show that

$$r + 3 \leq De_{\pm}(C_3^r)$$

it is equivalent to show that

$$r + 2 < De_{\pm}(C_3^r).$$

To show this strict inequality, we must find an $r \times (r+2)$ matrix with entries in C_3 that does not contain an even PM zero-subsum. We begin by showing that the following matrix does not have an even PM zero-subsum

$$A = [e_1, e_2, a, b] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

Notice that a, b are linearly independent and every PM zero-subsum has length three.

Consider the following block matrix

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & I_{r-2} \end{array}\right],$$

where 0 is the corresponding zero matrix. Since the last $r-2$ columns of M are linearly indpendent from the first four columns and since A does not contain an even PM zero-subsum, M also does not contain an even PM zero-subsum. □

For other primes, p, such an improvement does not always hold.

Proposition 3.2.8.

$$De_{\pm}(C_5^2) = 5.$$

Proof. By Lemma 3.2.4 and since $\lfloor \log_2 5 \rfloor = 2$,

$$De_{\pm}(C_5^2) \geq 5.$$

Let $S = \{s_1, s_2, s_3\} \subset C_5$ where each s_i is non-zero, for $1 \leq i \leq 3$. By Lemma 3.1.8, we know that $De_{\pm}(C_5) = 3$. This is shown by the fact that there exists an i, j such that $s_i = \pm s_j$. Thus, for every subset S where $|S| = 3$ we get our even PM zero-subsum.

Now, let $S = \{s_1, \ldots, s_5\} \subset C_5^2$ where each s_i is non-zero. Since 5 is prime, we let M be the corresponding row reduced matrix of S where

$$M = [I_2 \mid T] = [e_1, e_2 \mid t_1, t_2, t_3].$$

If $|sp(t_i)| = 1$ and the only non-zero entry is 1, then it is clear that we have an even PM zero-subsum. Next, consider the case where there exists a row of T that contains at least two zero entries,

$$T = \begin{bmatrix} a & 0 & 0 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Let $S' = \{1, b_2, b_3\}$. Since $De_{\pm}(C_5) = 3$, S' contains an even PM zero-subsum. Therefore, M contains an even PM zero-subsum.

Next, we consider the case where T has two zero entries in different rows. Since we have shown that if $|sp(t_i)| = 1$ and the non-zero entry is ± 1, then we have an even PM zero-subsum, we can assume without loss of generality that T has the following form,

$$T = \begin{bmatrix} 2 & 0 & a \\ 0 & 2 & b \end{bmatrix},$$

where $a, b \in \{1, 2, 3, 4\}$. Let us first consider all possible PM zero-subsums of $U = \{1, 2, s\}$. For for any choice of $s \in \{1, 2, 3, 4\}$, one can verify that U contains a PM zero-subsum of length two and of length three. Since U always contain two PM zero-subsums, one of length two and another of length three for any choice of s, then there exists a set of $\alpha_i \in \{0, \pm 1\}$ such that

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 t_1 + \alpha_4 t_2 + \alpha_5 t_3 = 0$$

where $\alpha_5 \neq 0$ and exactly three addition $\alpha_i \neq 0$ for $1 \leq i \leq 4$.

Next, we consider the case where T has only one zero entry. Since we have shown that if $|sp(t_i)| = 1$ and the non-zero entry is ± 1, then we have an even PM zero-subsum, we can assume without loss of generality that T has the following form,

$$T = \begin{bmatrix} 2 & a_2 & a_3 \\ 0 & b_2 & b_3 \end{bmatrix},$$

where $a_i, b_i \in \{1, 2, 3, 4\}$. First assume that $b_2 = \pm 1$. Since we can find a PM zero-subsum of length three including a_2, there exists a set of $\alpha_i \in \{\pm 1\}$ such that

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 t_1 + \alpha_4 t_2 = 0.$$

Next assume that neither $b_i = \pm 1$, so $b_i \in \{2, 3\}$ This implies that

$$\alpha_4 b_2 + \alpha_5 b_3 \equiv 0 \mod 5. \tag{3.5}$$

If $\alpha_4 a_2 + \alpha_5 a_3 \equiv 0 \mod 5$ than we are done. Assume that $\alpha_4 a_2 + \alpha_5 a_3 \not\equiv 0 \mod 5$. Since $D_{\pm}(C_5) = 3$ and we have shown for any $a \in C_5 \setminus \{0\}$ the subset $\{1, 2, a\} \subset C_5$ contains a PM zero-subsum of length two and three, there exists a set of $\alpha_i \in \{\pm 1\}$ such that

$$\alpha_1 e_1 + \alpha_3 t_1 + \alpha_4 t_2 + \alpha_5 t_3 = 0$$

where α_4, α_5 are given by equation 3.5.

Furthermore, we need to consider the case where T has no zero entries. Let T have the following form,

$$T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Case 1. Assume each a_i are equal up to sign. If there exists a $b_i = b_j$, then we have an even PM zero-subsum of length two. Assume that $b_i \neq b_j$ for any $i, j \in \{1, 2, 3\}$, So, without loss of generality,

$$T = \begin{bmatrix} a & a & a \\ 1 & 2 & b \end{bmatrix},$$

where $b \in \{3, 4\}$. In the case where $a \in \{1, 4\}$, then find a set of $\alpha_i \in \{\pm 1\}$ such that

$$\alpha_3 t_1 + \alpha_4 t_2 + \alpha_5 t_3 = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}.$$

For example, let $b = 4$ and notice that

$$-\begin{bmatrix} a \\ 1 \end{bmatrix} + \begin{bmatrix} a \\ 2 \end{bmatrix} + \begin{bmatrix} a \\ 4 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix},$$

since $a \in \{1, 4\} = \{\pm 1\}$ this is the result we are looking for. Hence,

$$\alpha_1 e_1 + \alpha_3 t_1 + \alpha_4 t_2 + \alpha_5 t_3 = 0$$

for some $\alpha_1 \in \{\pm 1\}$. In the case where $a \in \{2, 3\}$, then find a set of $\alpha_i \in \{0, \pm 1\}$ such that

$$\alpha_3 t_1 + \alpha_4 t_2 + \alpha_5 t_3 = \begin{bmatrix} \pm 1 \\ \pm 1 \end{bmatrix},$$

where $\alpha_j = 0$ for $j \in \{3, 4\}$ and otherwise $\alpha_i \neq 0$. For example, let $a = 3$ and $b = 4$ and notice that

$$0 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

thus, this is the result we are looking for. This will always occur since $2a = \pm 1$ for $a \in \{2, 3\}$ and since $b \in \{3, 4\}$ so either $1 + b \equiv 0 \bmod 5$ or $2 + b \equiv 0 \bmod 5$. Hence,

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 t_1 + \alpha_4 t_2 + \alpha_5 t_3 = 0$$

where exactly four α_i are non-zero.

Case 2. Assume there exist $a_i = \pm a_j$, then without loss of generality,

$$T = \begin{bmatrix} a & a & a' \\ b_1 & b_2 & b_3 \end{bmatrix},$$

where $a' \neq \pm a$. Notice that if $b_1 = b_2 = b_3$ then this falls into the previous case by interchanging the rows. So we can assume that not all b_i are equal. We choose to write $a, a' \in \{\pm 1, \pm 2\}$. Since the coefficients for our even PM zero-subsum are ± 1, then without loss of generality we can assume that $a, a' \in \{1, 2\}$.

Case 2.1. First assume that T has the following form

$$T = \begin{bmatrix} 1 & 1 & 2 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

If $b_1 = b_2$, then we have an even PM zero-subsum of length two. So, we assume that $b_1 \neq b_2$. Also, if $|b_i - b_3| = 1$ for $i \in \{1, 2\}$, then there exists $\{\alpha, \beta\} = \{1, -1\}$ such that

$$\alpha t_i + \beta t_3 = \begin{bmatrix} \pm 1 \\ \pm 1 \end{bmatrix}.$$

Thus,

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha t_2 + \beta t_3 = 0,$$

for some $\alpha_i \in \{\pm 1\}$. In the case where $b_3 \in \{2, 3\}$, one can conclude that if $|b_i - b_3| \neq 1$ then $b_1 = b_2$. This contradicts our assumption that $b_1 \neq b_2$. In the case where $b_3 \in \{1, 4\}$ then

$$[b_1, b_2, b_3] \in \{[1, 2, 4], [3, 4, 1]\}.$$

Note that these are the only subsets to consider when each b_i is distinct. Now, let $x^T = (1, 1, -1)$. Then in either case

$$Tx = \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}.$$

Thus, there exists an $\alpha \in \{\pm 1\}$ such that

$$\alpha e_2 + t_1 + t_2 - t_3 = 0$$

provides an even PM zero-subsum.

Now, assume there exist an i such that $b_i = b_3$. Without loss of generality, let $i = 2$. If $\{b_1, b_2, b_3\} \in \{\pm\{1, 1, 3\}, \pm\{1, 2, 2\}\}$, then

$$-e_1 + t_1 + t_2 + t_3 = 0,$$

is an even PM zero-subsum. Recall that if $|b_1 - b_3| = 1$ then we have shown that we are able to attain an even PM zero-subsum. We are left to consider the following subsets,

$$[b_1, b_2, b_3] \in \{[4, 1, 1], [4, 2, 2], [1, 3, 3], [1, 4, 4]\}.$$

In each case, there exists an $\alpha \in \{\pm 1\}$ such that

$$\alpha e_2 + t_1 + t_2 - t_3 = 0$$

provides an even PM zero-subsum.

Case 2.2. Now assume that T has the following form

$$T = \begin{bmatrix} 1 & 2 & 2 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Assume there exists an $x^T = (x_1, x_2, x_3)$ such that $Tx = 0$ that provides an even PM zero-subsum. Now, notice that

$$0 \equiv \begin{bmatrix} 1 & 2 & 2 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ b_1 & -b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ b_1 & -b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix}.$$

Since b_1, b_2, b_3 are arbitrary, this reduces to the previous case. Hence such an x exists that provides an even PM zero-subsum. $\square$

In Section 2.4, we have shown that we can row reduce our matrix representation of $S \subset C_p^r$, for when p is prime, where the left most $r \times r$ column make an identity matrix, I_r. Since we are able to reduce M in this way, we are able to compare lower and upper bounds for when $r < m$.

Lemma 3.2.9. *If*

$$r + \alpha \leq De_{\pm}(C_p^r)$$

then

$$m + \alpha \leq De_{\pm}(C_p^m)$$

for all $m \geq r$.

Proof. Assume $r + \alpha \leq De_{\pm}(C_p^r)$. Then there exist an $r \times (r + \alpha - 1)$ matrix N that does not contain an even PM zero-subsum and by row reduction,

$$N = [I_r \mid \widetilde{T}] = [e_1, \dots, e_r \mid \tilde{t}_1, \dots, \tilde{t}_{\alpha-1}].$$

Let t_i be the an m-tuple where the first r entries are equal to $\tilde{t}_i$ and the remaining entries are zero. Then, we construct M to be an $m \times (m + \alpha - 1)$ matrix where N is a submatrix of M where

$$M = [I_m \mid T] = [e_1, \dots, e_m \mid t_1, \dots, t_{\alpha-1}].$$

Then by Lemma 3.2.1, since N does not contain an even PM zero-subsum, M also does not contain an even PM zero-subsum. $\square$

Lemma 3.2.10. *Let* $r < m$. *If*

$$De_{\pm}(C_p^m) \leq m + \ell,$$

then

$$De_{\pm}(C_p^r) \leq r + \ell.$$

Proof. Assume $De_{\pm}(C_p^r) \geq r + \ell + 1$ and $De_{\pm}(C_p^m) = m + \ell$. Then there exists a $r \times (r + \ell)$ matrix M that does not contain an even PM zero-subsum where M has the following form

$$M = [e_1, \dots, e_r \mid t_1, \dots, t_\ell] = [I_r \mid T].$$

We will enlarge our matrix T to matrix $\widetilde{T}$. Let $\widetilde{T} = [\tilde{t_1}, \dots, \tilde{t_\ell}]$ be the $m \times \ell$ matrix where the last $m - r$ rows are zero rows, and so the last $m - r$ entries of $\tilde{t_i}$ are zero, for $1 \leq i \leq \ell$. Now, we define

$$\widetilde{M} = [e_1, \dots, e_m \mid \tilde{t_1}, \dots, \tilde{t_\ell}] = [I_m \mid \widetilde{T}],$$

to be an $m \times (m + \ell)$ matrix. Notice that since M does not contain an even PM zero-subsum, then neither does $\widetilde{M}$. This contradicts that $De_{\pm}(C_p^m) \leq m + \ell$. Therefore,

$$De_{\pm}(C_p^r) \leq r + \ell.$$

$\square$

3.3 Even Length PM zero-subsums in C_3^r

By Theorem 2.3.12, $D_{\pm}(C_3^r) = r + 1$. This guarantees there exists a PM zero-subsum for any collection of $r+1$ elements of G. In this section, let $k > r$ and $S = \{s_1, \ldots, s_k\} \subset C_3^r$, where each s_i is non-zero and $M = [I_r \mid T]$ is the $r \times k$ matrix corresponding to S. For more detail on this matrix M, refer to the beginning of Section 2.4. We begin by studying when $S \subset C_3^r$ contains an even PM zero-subsum.

Lemma 3.3.1. *Let $S \subset G$ where $|S| = k$ and $M = [I_r \mid T]$ is the $r \times k$ matrix corresponding to S where $T = [t_1, \ldots, t_{k-r}]$ is a $r \times (k-r)$ matrix with entries in C_3 and let x be a column vector of length $k - r$. If $|sp(Tx)|$ and $|sp(x)|$ have the same parity, then S contains an even PM zero-subsum.*

Proof. Let $sp(Tx) = \{e_{i_1}, \ldots, e_{i_\ell}\}$ where $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, r\}$. Without loss of generality, assume that the first $m \leq r$ entries x are non-zero and the rest of the entries of x are zero. Therefore, $\ell \equiv m \mod 2$ by hypothesis. Then there exist $\alpha_j \in \{\pm 1\}$ such that

$$\sum_{j=1}^{\ell} \alpha_j e_{i_j} = Tx = \sum_{n=1}^{m} x_n t_n.$$

By subtracting one side from the other and since $\ell \equiv m \mod 2$ this provides an even PM zero-subsum. □

When computing $De_{\pm}(C_3^r)$, we need to show that for every $S = \{s_1, \ldots, s_k\} \subset C_3^r$ where $k \geq De_{\pm}(C_3^r)$ there exists an even PM zero-subsum. Since Lemma 3.3.1 provides a condition when S contains an even PM zero-subsum, we can exclude the cases where $|sp(x)|$ and $|sp(Tx)|$ have the same parity. For example, assume $|sp(x)| = 1$, then we exclude the case when $|sp(Tx)|$ is odd. Thus, we can assume that every column t_i of T has even support. In the next section, we compute $De_{\pm}(C_3^r)$ when $r \leq 9$. Leading up to this computation, we are most interested in when T has at most three columns.

We now focus on when $|sp(x)| \equiv |sp(Tx)| \mod 2$. We first introduce some notation. Let

$$T = [a, b] = \begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_r & b_r \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}.$$

We define the following sets to partition some of the rows of T. Let

$$V^{ab} = V_1^{ab} \cup V_2^{ab}$$

where

$$V_1^{ab} = \{u_i \in V^{ab} \mid a_i = b_i; a_i, b_i \neq 0\}$$
$$V_2^{ab} = \{u_i \in V^{ab} \mid a_i \neq b_i; a_i, b_i \neq 0\}.$$

If $T = [t_1, \ldots, t_{k-r}]$ for $k - r \geq 3$, then we select two columns of T, t_i, t_j to define V^{ij}. In the case when our choice of columns are clear, we omit the superscript. For example, when $T = [a, b]$ we denote our subsets simply as V_1 and V_2.

We further investigate the simple case when $T = [a, b]$. Define $sp(a, b)$ to be the set $\{e_i\}$ such that either $e_i \in sp(a)$ or $e_i \in sp(b)$. This provides

$$|sp(a, b)| = |sp(a)| + |sp(b)| - |sp(a) \cap sp(b)|,$$

and

$$\Big| \; sp(a, b) \setminus sp(a) \cap sp(b) \; \Big| = |sp(a)| + |sp(b)| - 2|sp(a) \cap sp(b)|.$$

Then $sp(Tx) \subset sp(a, b)$ for every vector x. By construction of V^{ab}, $|V^{ab}| = |sp(a) \cap sp(b)|$. If there exists $i \in \{1, 2\}$ such that $V_i \subseteq Ann_T(x)$, then for $j \neq i$

$$\begin{aligned} |sp(Tx)| &= \Big| \; sp(a, b) \setminus sp(a) \cap sp(b) \; \Big| + |V_j| \\ &= |sp(a)| + |sp(b)| - 2|sp(a) \cap sp(b)| + |V_j|. \end{aligned}$$

Lemma 3.3.2. *Let $T = [a, b]$. Suppose $|sp(a)|, |sp(b)|$ are both even. If $|V_1|$ or $|V_2|$ is even, then there exists an x where $x^T = [x_1, x_2]$ and $x_i \in \{\pm 1\}$ such that $|sp(Tx)|$ is even.*

Proof. Suppose that $|V_i|$ is even. For $j \neq i$, choose x such that $V_j \subseteq Ann_T(x)$. Notice that

$$|sp(Tx)| = |sp(a)| + |sp(b)| - 2|sp(a) \cap sp(b)| + |V_i|.$$

Then

$$|sp(a)| + |sp(b)| - 2|sp(a) \cap sp(b)|$$

is even since $|sp(a)|, |sp(b)|$ are both even. Thus, $|sp(Tx)|$ is even. □

More generally, let $T = [t_1, \ldots, t_{k-r}]$ and let $1 \leq i < j \leq k - r$. It follows that if $|sp(t_i)|, |sp(t_j)|$ are both even and there exists a $\ell \in \{1, 2\}$ such that $|V_\ell^{ij}|$ is even, then there exists a column vector x, where $|sp(x)| = 2$ and $|sp(Tx)|$ is even. Thus by Lemma 3.3.1, for $M = [I_r \mid T]$, M contains an even PM zero-subsum. Hence, we can now assume that every $|V_\ell^{ij}|$ is odd, for $1 \leq i < j \leq k - r$ and $\ell \in \{1, 2\}$.

Corollary 3.3.3. *Let $T = [a, b]$. Suppose $|sp(a)|, |sp(b)|$ are both even. If both $|V_i|$ are odd, then $|sp(Tx)|$ is odd for every x where $x^T = [x_1, x_2]$ and $x_i \in \{\pm 1\}$.*

Proof. Recall

$$|sp(Tx)| = |sp(a)| + |sp(b)| - 2|sp(a) \cap sp(b)| + |V_i|.$$

Since

$$|sp(a)| + |sp(b)| - 2|sp(a) \cap sp(b)|$$

is even and each $|V_i|$ is odd, it follows that $|sp(Tx)|$ is odd for any x. □

Lemma 3.3.4. *Let $T = [t_1, t_2]$ be an $r \times 2$ matrix where every entry is in $\{\pm 1\}$ and r is even. Assume each t_i have an odd number of entries with the value -1 and an odd number of entries with the value 1. Then there exists an $x^T = (x_1, x_2)$, where $x_i \neq 0$, such that $|sp(Tx)|$ is even.*

Proof. The rows of $T = [t_1, t_2]$ must be one of the following $\{(1,1), (-1,-1), (1,-1), (-1,1)\}$. Let a, b, c, d denote the number of rows of $[t_1, t_2]$ that are $(1,1), (-1,-1), (1,-1), (-1,1)$ respectively. Since t_1, t_2 have an odd number -1 entries, we know that $b+c$ and $b+d$ are both odd. Then this tells us that $b+c+b+d = 2b+c+d$ is even. Thus $c + d = |V_2^{12}|$ is even. Thus, by Lemma 3.3.2, we know there exists an x such that $|sp(Tx)|$ is even. □

Lemma 3.3.5. *Let S be a subset of C_3^r where $|S| = \ell$ such that $\ell - r \geq 3$ and M be the corresponding $r \times \ell$ matrix such that*

$$M = [e_1, \dots, e_r \mid t_1, \dots, t_{\ell-r}] = [I_r \mid T].$$

If $|sp(t_i)| = |sp(t_j)| = |sp(t_k)| = r$ for some i, j, k where $1 \leq i < j < k \leq \ell - r$, then M contains an even PM zero-subsum.

Proof. If r is odd, then by Lemma 3.3.1, we have an even PM zero-subsum. Now assume r is even. It is sufficient to prove the result when $\ell - r = 3$. So $T = [t_1, t_2, t_3]$ which is an $n \times 3$ matrix such that $|sp(t_i)| = r$ and every entry t_{ij} in T is ± 1. By an agrument in the introduction of Section 2.4, we can assume t_1 contains only ones. For $j \neq 1$, if t_j has an even number of entries that are -1, then $|V_\ell^{1j}|$ is even for $\ell \in \{1, 2\}$. Hence, by Lemma 3.3.2, we have our even PM zero-subsum.

Now, we can assume that both t_2, t_3 have an odd number of entries that are -1 and so it also has an odd number of entires that are 1. Then, by Lemma 3.3.4, $|V_2^{23}|$ is even. Hence, by Lemma 3.3.2, we have our even PM zero-subsum. □

Corollary 3.3.6. *Let S be a subset of C_3^r where $|S| = \ell$ such that $\ell - r \geq 3$ and M be the corresponding $r \times \ell$ matrix such that*

$$M = [e_1, \dots, e_r \mid t_1, \dots, t_{\ell-r}] = [I_r \mid T].$$

If $sp(t_i) = sp(t_j) = sp(t_k)$ where $|sp(t_i)| = \ell$, for $\ell < r$ and for some i, j, k where $1 \leq i < j < k \leq \ell - r$, then M contains an even PM zero-subsum.

Proof. Let $sp(t_i) = sp(t_j) = sp(t_k) = \ell$. Then without loss of generality, we can assume the first ℓ entries of t_i, t_j and t_k are non-zero and the rest are zero. Next, we take the following submatrix of M,

$$N = [e_1, \dots, e_\ell \mid \tilde{t}_i, \tilde{t}_j, \tilde{t}_k]$$

where $\tilde{t}_m$ contains the first ℓ entries of t_m for $m \in \{i, j, k\}$. Then we apply Lemma 3.3.5 to N to show that N contains an even PM zero-subsum,

$$\sum_{n=1}^{\ell} \alpha_n e_n + \alpha_i \tilde{t}_i + \alpha_j \tilde{t}_j + \alpha_k \tilde{t}_k = 0,$$

where $\alpha_n \in \{0, \pm 1\}$ for $n \in \{1, \dots, \ell, i, j, k\}$ and an even number of α_n are non-zero. Since the last $r - \ell$ entries of each t_m are zero, then

$$\sum_{n=1}^{\ell} \alpha_n e_n + \alpha_i t_i + \alpha_j t_j + \alpha_k t_k = 0$$

provides an even PM zero-subsum. $\square$

To help us in our future proofs, we describe a partition for the rows of T. Assume $k - r \geq 2$. We can partition the rows of T by the number of zeros that each row u_i contains, where

$$T = [t_1, \dots, t_{k-r}] = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}.$$

Let

$$Z^m = \{u_i \in T \mid u_i \text{ contains exactly } m \text{ zeros}\},$$

where $0 \leq m \leq k - r$. Without loss of generality, we choose to organize the rows of T so that

$$T = \begin{bmatrix} Z^0 \\ Z^1 \\ \vdots \\ Z^{k-r} \end{bmatrix}.$$

If $u_i \in Z^m$, then there are $\binom{k-r}{m}$ ways to choose which entries of u_i are zero.

Let $z = \{i_1, \dots, i_{k-r-m}\} \subset \{1, \dots, k-r\}$ of cardinality $k - r - m$. Let W^z be the set of rows of Z^m for which the entries in columns $\{i_1, \dots, i_{k-r-m}\}$ are non-zero. For example, let $z = \{1, 2\}$. Then we are selecting t_1 and t_2 from T. So $W^{1,2}$ is the set of $u_i \in Z^{k-r-2}$ where only the first and second entries are non-zero. Thus, $m = k-r-2$ and u_i contains $k - r - 2$ entries that are zero.

We later use the following matrix T to show a lower bound improvement for $De_{\pm}(C_3^r)$ where $r \geq 10$.

$$T = [t_1, t_2, t_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix}, \tag{3.6}$$

where $Z^0 = \{u_1, u_2, u_3, u_4\}$ and $Z^1 = \{u_5, \ldots, u_{10}\}$. In this case, $k-r = 3$ and $m = 1$. Then

$$\begin{aligned} Z^1 &= W^{12} \cup W^{13} \cup W^{23} \\ &= \{u_5, u_6\} \cup \{u_7, u_8\} \cup \{u_9, u_{10}\}. \end{aligned}$$

Let $x^T = (x_1, \ldots, x_{k-r})$ where $x_i \in \{0, \pm 1\}$. Define

$$W^z(x) = W^z \cap Ann_T(x).$$

Continuing our example above for the 10×3 matrix T, let $x^T = (1, 1, 1)$. Then

$$Ann_T(x) = \{u_1, u_6, u_8, u_{10}\}$$

$$W^{12}(x) = \{u_6\} \qquad W^{13}(x) = \{u_8\} \qquad W^{23}(x) = \{u_{10}\}.$$

Assume that $0 \leq j < k - r$. Let

$$X_j = \{x \mid x^T = (x_1, \ldots, x_{k-r}), x_i \in \{0, \pm 1\} \text{ where } j \text{ entries of } x \text{ are zero}\}.$$

Define $\varphi : X_j \to \{\text{Subsets of } W^z\}$ by $\varphi(x) = W^z(x)$. Notice $\varphi(x) = \varphi(-x)$ by Lemma 2.4.2. So ,

$$|\operatorname{Im}(\varphi)| \leq \frac{1}{2}|X_j| = \frac{1}{2}\binom{k-r}{j}2^{k-r-j} = \binom{k-r}{j}2^{k-r-j-1}.$$

The following results use our subsets $W^z \subset Z^m$.

Lemma 3.3.7. *Let $T = [t_1, t_2, t_3]$ be a $r \times 3$ matrix. If $|Z^0| + |Z^1|$ is odd, then there exists $|V_\ell^{ij}|$ that is even.*

Proof. Recall the subsets $W^{ij} \subset Z^1$ where

$$Z^1 = W^{12} \cup W^{13} \cup W^{23}.$$

Now, recall that our set $V^{ij} = V_1^{ij} \cup V_2^{ij}$. Notice that if $|V^{ij}|$ is odd then there exists an $\ell \in \{1, 2\}$ such that $|V_\ell^{ij}|$ is even. With these subsets one can see that

$$V^{ij} = Z^0 \cup W^{ij}.$$

We first assume that $|Z^0|$ is odd, so $|Z^1|$ is even. This tells us that either every $|W^{ij}|$ is even or there exists only one pair i, j such that $|W^{ij}|$ is even and the rest are odd. In either case, there exists a pair i, j such that $|W^{ij}|$ is even. Thus,

$$|V^{ij}| = |Z^0| + |W^{ij}|$$

is odd. Hence, there exists an $\ell \in \{1, 2\}$ such that $|V_\ell^{ij}|$ is even.

Similarly, when $|Z^0|$ is even, we are able to find a pair i, j where $|W^{ij}|$ is odd since $|Z^1|$ is odd. Thus, this again provides that $|V^{ij}|$ is odd. $\square$

This tells us that if we assume $|V_\ell^{ij}|$ is odd for every pair i, j and $\ell \in \{1, 2\}$, then Z^0 and Z^1 must both have even cardinality or both have odd cardinality. Notice the matrix T from Equation 3.6 satisfies the conditions where every $|V_\ell^{ij}|$ is odd. For example,

$$V^{12} = V_1^{12} \cup V_2^{12} = \{u_1, u_2, u_5\} \cup \{u_3, u_4, u_6\}.$$

One can verify that each $|V_\ell^{ij}| = 3$ for every $1 \leq i < j \leq 3$ and $\ell \in \{1, 2\}$. Thus, this shows that $|Z^0| + |Z^1|$ is even.

Lemma 3.3.8. *Let $T = [t_1, t_2, t_3]$ be an $n \times 3$ matrix with entries in C_3. Assume $|V^{ij}|$ and $|Z^0|$ are even and $|Z^2|$ is odd. Then there exists a column t_i in T such that $|sp(t_i)|$ is odd.*

Proof. Given $T = [t_1, t_2, t_3]$ then we know that

$$Z^1 = W^{12} \cup W^{13} \cup W^{23},$$
$$Z^2 = W^1 \cup W^2 \cup W^3.$$

Since we are assuming that $|Z^2|$ is odd then there exists an i such that $|W^i|$ is odd, for $1 \leq i \leq 3$. Notice that

$$|V^{ij}| = |Z^0| + |W^{ij}|.$$

Since $|V^{ij}|$ is even, this tells us that $|W^{ij}|$ must each be even, for $1 \leq i < j \leq 3$. Let $\{i, j, k\} = \{1, 2, 3\}$. Since there exists an i such that $|W^i|$ is odd, then

$$|sp(t_i)| = |Z^0| + |W^{ij}| + |W^{ik}| + |W^i|$$

is odd. □

Recall our set $S = \{s_1, \ldots, s_k\} \subset C_3^r$ where $k > r$ and $M = [I_r \mid T]$ be the $r \times k$ corresponding to S, where $T = [t_1, \ldots, t_{k-r}]$. Let x be a column vector of length $k - r$ with entries in $\{\pm 1, 0\}$. If there exists an x such that $|sp(x)| \equiv |sp(Tx)| \mod 2$, then by Lemma 3.3.1 S contains an even PM zero-subsum. In conclusion, when finding if S contains an even PM zero-subsum, this leaves us to consider the cases when $|sp(x)| \not\equiv |sp(Tx)| \mod 2$ for any possible x. This assumption provides the following conditions on the columns of $T = [t_1, \ldots, t_{r-k}]$

- by Lemma 3.3.1, we can assume that every column t_i of T has even support,
- by Lemma 3.3.2, we can assume that each $|V_\ell^{ij}|$ is odd,
- if $r - k \geq 3$, then by Lemma 3.3.5 and Corollary 3.3.6, we can assume that no three distinct columns of T have the same support.

3.4 Special Cases of $De_{\pm}(C_3^r)$

Lemma 3.4.1. *Let $k \leq 8$.*

$$De_{\pm}(C_3^k) \leq 11.$$

Proof. By Lemma 3.2.10, it is sufficient to show

$$De_{\pm}(C_3^8) \leq 11.$$

Let $S \subset C_3^8$ with non-zero elements such that $|S| = 11$. Let M be the corresponding 8×11 matrix with the elements of S where $M = [I_8 \mid T]$. So, $T = [t_1, t_2, t_3]$ is a 8×3 matrix where every entry is in C_3. We assume that S does not contain an even PM zero-subsum and we will reach a contradiction. By the remarks at the end of Section 3.3, if the $|sp(x)| \leq 3$ and if S does not contain an even PM zero-subsum then we can assume that the following statements holds

1. by Lemma 3.3.1, we can assume that every column t_i of T has even support,
2. by Lemma 3.3.2, we can assume that each $|V_\ell^{ij}|$ is odd,
3. a consequence of Lemma 3.3.1, if $|sp(x)| = 3$, then $|sp(Tx)|$ is even.

We will show there exists an x such that $|sp(x)| = 3$ and $|sp(Tx)|$ is odd, which contradicts (3) and therefore contradicts our assumption that S does not contain an even PM zero-subsum. In the introduction of Section 2.4 we show

$$|sp(Tx)| = 8 - |Ann_T(x)|.$$

So, it is equivalent to show that $|Ann_T(x)|$ is odd. Each case is determined by the value of

$$\min_{1 \leq i \leq 3} \{|sp(t_i)|\}.$$

Without loss of generality, we assume that

$$|sp(t_1)| = \min_{1 \leq i \leq 3} \{|sp(t_i)|\} = k,$$

where the first k entries are t_1 are 1 and the others are 0. Let u_j by a row of T, for $1 \leq j \leq 8$. Then, we let $\widetilde{T}$ be the $k \times 3$ submatrix of T that contains the first k rows of T.

In each case below, we let $x^T = (0, x_2, x_3)$ be a column vector where $|Ann_T(x)|$ is odd. Then, we let $y^T = (y_1, x_2, x_3)$ be a column vector and show that $|Ann_T(y)|$ is also odd. Thus, a contradiction to our assumption (3).

First, consider when

$$\min_{1 \leq j \leq 3} \{|sp(t_i)|\} = 2$$

Assume $|sp(t_1)| = 2$. By (2), since $|V_\ell^{1j}|$ is odd, for $j \in \{2, 3\}$ and $\ell \in \{1, 2\}$, then without loss of generality $\widetilde{T}$ has the following form

$$\widetilde{T} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Let $x^T = (0,1,-1)$ and notice that $u_1, u_2 \in Ann_T(x)$. Let $y^T = (1,1,-1)$ and notice that $u_1, u_2 \notin Ann_T(y)$.

$$|Ann_T(y)| = |Ann_T(x)| - 2,$$

thus $|Ann_T(y)|$ is odd.

Second, consider when

$$\min_{1\le j\le 3}\{|sp(t_i)|\} = 4$$

Assume $|sp(t_1)| = 4$. Since $|V_\ell^{1j}|$ is odd, for $\ell \in \{1,2\}$ and $j \in \{2,3\}$, then each column of $\widetilde{T}$ must have an even number of zero entries. Thus, our possible $\widetilde{T}$ are the following

$$\widetilde{T} \in \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\}$$

Let

$$\widetilde{T} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Let $x^T = (0,1,-1)$ so then $u_1, u_2, u_3 \in Ann_T(x)$. Let $y^T = (-1,1,-1)$ so then $u_4 \in Ann_T(y)$ but $u_1, u_2, u_3 \notin Ann_T(y)$.

$$|Ann_T(y)| = |Ann_T(x)| - 3 + 1,$$

thus $|Ann_T(y)|$ is odd.

Let

$$\widetilde{T} \in \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\}.$$

Let $x^T = (0,1,-1)$ so $u_1, \ldots, u_4 \in Ann_T(x)$. Let $y^T = (1,1,-1)$ so then $u_1, \ldots, u_4 \notin Ann_T(x)$.

$$|Ann_T(y)| = |Ann_T(x)| - 4,$$

thus, $|Ann_T(y)|$ is odd.

Let

$$\widetilde{T} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Let $x^T = (0,1,1)$ so then $u_1, u_2 \in Ann_T(x)$. Let $y^T = (1,1,1)$ so then $u_3, u_4 \in Ann_T(y)$ but $u_1, u_2 \notin Ann_T(y)$.

$$|Ann_T(y)| = |Ann_T(x)| - 2 + 2,$$

thus $|Ann_T(y)|$ is odd.

Let

$$\widetilde{T} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Let $x^T = (0, 1, -1)$ so then $u_1, u_2 \in Ann_T(x)$. Let $y^T = (1, 1, 1)$ so then $u_1, u_2, u_3, u_4 \notin Ann_T(y)$.

$$|Ann_T(y)| = |Ann_T(x)| - 2 + 0,$$

thus $|Ann_T(y)|$ is odd.

Third, consider when

$$\min_{1 \le j \le 3} \{|sp(t_i)|\} = 6$$

Assume $|sp(t_1)| = 6$. We initially assume that every entry of $\widetilde{T}$ is non-zero and the last two entries of each t_i, for $1 \le i \le 3$ are zero. Hence, $u_7, u_8 \in Z^3$. Let $N = [I_6 \mid \widetilde{T}]$ be a submatrix of M. Then by Lemma 3.3.5 N contains an even PM zero-subsum. Hence by Lemma 3.2.1, M contains an even PM zero-subsum. Therefore we can assume that $u_7, u_8 \notin Z^3$, thus containing at least one non-zero entry.

Now, we consider the case where $u_7, u_8 \in Z^2$. Recall that since $\min_{1 \le j \le 3}\{|sp(t_i)|\} = 6$ and every column of $\widetilde{T}$ must contain an even number zeros, we can assume that T has the following form,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ 1 & * & * \\ 1 & * & * \\ 1 & * & * \\ 1 & * & * \\ 1 & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix},$$

where $* \in \{\pm 1\}$. By (2), since $|V_\ell^{1j}|$ is odd for $2 \le j \le 3$, each t_j has an odd number of entries that are -1 and an odd number of entries that are 1. Since we have assumed that every entry of $\widetilde{T}$ is non-zero, then by Lemma 3.3.4, there exists an $x^T = (0, x_2, x_3)$, such that $|sp(\widetilde{T}x)|$ is even. Notice that $u_7, u_8 \notin Ann_T(x)$, thus $|sp(Tx)|$ is also even. This contradicts (2).

Next, we consider the case when $\widetilde{T}$ has zero entries. By (2), since $|V_\ell^{13}|$ is odd, for $\ell \in \{1, 2\}$, without loss of generality, we can assume

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & a_1 & -1 \\ 1 & a_2 & 1 \\ 1 & a_3 & 1 \\ 1 & a_4 & 1 \\ 1 & a_5 & 0 \\ 1 & a_6 & 0 \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix},$$

where $a_i \in \{\pm 1\}$ and an odd number of $a_i = -1$. For $1 \leq j \leq 4$, if only one $a_j = -1$, then there exists an $\ell \in \{1, 2\}$ such that $|V_\ell^{23}|$ is even. This also holds if exactly three $a_j = -1$. We can assume that either all $a_j = 1$ or there are exactly two $a_j = -1$. First consider that each $a_j = 1$. Since $|V_\ell^{12}|$ is odd, $a_5 \neq a_6$. So, one of a_5, a_6 is equal to 1 and the other is -1. Let $x^T = (1, -1, 1)$ and observe that $|Ann_T(x)| = |\{u_m\}| = 1$, for $m \in \{5, 6\}$ which contradicts (3). If $a_1 = 1$ then let $x^T = (-1, 1, 1)$ and if $a_1 = -1$, then let $x^T = (1, 1, -1)$. In either case, $|Ann_T(x)| = |\{u_m\}| = 1$, for $m \in \{5, 6\}$ which again contradicts (3).

Next we assume that $u_7, u_8 \in Z^1$. Assume at least one t_i has $|sp(t_i)| = 8$, so let $|sp(t_3)| = 8$. Since $|V_\ell^{13}|$ and $|V_\ell^{23}|$ are both odd, then without loss of generality, we can assume that T has the following form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & b_1 \\ 1 & 1 & b_2 \\ 1 & 1 & b_3 \\ 1 & 1 & b_4 \\ 1 & 0 & b_5 \\ 1 & 0 & b_6 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

where $b_i \in \{\pm 1\}$ where an odd number of $b_i = -1$. First assume that $b_1 = 1$ and let $x^T = (1, 1, 1)$. It is clear that $u_1, u_8 \notin Ann_T(x)$ and $u_7 \in Ann_T(x)$. Consider the case where an even number of $b_j = -1$, for $j \in \{2, 3, 4\}$. This tells us that there is an odd number $u_j = (1, 1, 1) \in Ann_T(x)$ and $b_5 \neq b_6$. So,

$$|Ann_T(x)| = |\{u_j \mid u_j = (1, 1, 1, 0)\} \cup \{u_k \mid b_k = -1, k \in \{5, 6\}\} \cup \{u_7\}|,$$

which is odd. Now, consider the case where an odd number of $b_j = -1$. First let $b_2 = -1$ and $b_j = 1$ for $j \in \{3, 4\}$. This tells us that $b_5 = b_6$. Then

$$|Ann_T(x)| = |\{u_3, u_4, u_7\} \cup \{u_k \mid b_k = -1, k \in \{5, 6\}\}|.$$

Since $|\{u_k \mid b_k = -1, k \in \{5, 6\}\}|$ is even, then $|Ann_T(x)|$ is odd. Next, let $b_j = -1$, for $j \in \{2, 3, 4\}$. Again, this tell us that $b_5 = b_6$. Then

$$|Ann_T(x)| = |\{u_7\} \cup \{u_k \mid b_k = -1, k \in \{5, 6\}\}|.$$

Since $|\{u_k \mid b_k = -1, k \in \{5, 6\}\}|$ is even, then $|Ann_T(x)|$ is odd. In each case, we found $|Ann_T(x)|$ to be odd which contradicts (3).

Now, let $b_1 = -1$ and $x^T = (1, 1, -1)$. It is clear that $u_1, u_7 \notin Ann_T(x)$ and $u_8 \in Ann_T(x)$. Here we need to consider if $b_5 = b_6$ or $b_5 \neq b_6$. First, let $b_5 = b_6$. Then we know that an odd number $b_j = 1$. So,

$$|Ann_T(x)| = |\{u_j \mid u_j = (1, 1, 1)\} \cup \{u_k \mid b_k = 1, k \in \{5, 6\} \cup \{u_8\}|,$$

which is odd, since $b_5 = b_6$. Second, let $b_5 \neq b_6$. Then an even number of $b_j = 1$. So,

$$|Ann_T(x)| = |\{u_j \mid u_j = (1, 1, 1)\} \cup \{u_k \mid b_k = 1, k \in \{5, 6\} \cup \{u_8\}|,$$

which is odd, since $b_5 \neq b_6$. In each case, we found $|Ann_T(x)|$ to be odd which contradicts (3).

Now, we can assume that every t_i in T has $|sp(t_i)| = 6$ and that $u_7, u_8 \in Z^1$. First assume there exists a 2×2 zero matrix of $\widetilde{T}$. Since $|V_\ell^{1j}|$ is odd for $j \in \{2, 3, 4\}$ and $|V_\ell^{23}|$ is also odd, then without loss of generality, T has the following form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & b_1 \\ 1 & 1 & b_2 \\ 1 & 1 & b_3 \\ 1 & 1 & b_4 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

where only one $b_k = -1$ and all others are equal to 1, for $1 \leq k \leq 4$. Let $x^T = (1, 1, b_1)$. It is clear that $u_1 \notin Ann_T(x)$. Since only one $b_k = -1$, then there are an even number $u_j = (1, 1, b_1) \in Ann_T(x)$. The only other $u_i \in Ann_T(x)$ is either u_7 or u_8 but not both. Thus $|Ann_T(x)|$ is odd which contradicts (3).

Next, we consider the case where $\widetilde{T}$ does not contain a 2×2 zero submatrix. Let $a_k \in \{\pm 1\}$ for $1 \leq k \leq 6$. Then, without loss of generality, we can assume that T has the following form,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & a_1 \\ 1 & 1 & a_2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & a_3 \\ 1 & 0 & a_4 \\ 0 & 1 & a_5 \\ 0 & 1 & a_6 \end{bmatrix}$$

Without loss of generality, we can assume that for $1 \leq k \leq 4$, one $a_k = -1$ while the others are equal to 1. Let $a_1 = -1$. Since $|V_\ell^{23}|$ is odd, then $a_5 \neq a_6$. We can assume $a_5 = 1$. So T has the following form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Now, let $x^T = (1, -1, 1)$, then $|Ann_T(x)| = |\{u_2, u_3, u_7\}| = 3$, a contradiction to (3). Next, consider the case when $a_2 = -1$. Since $|V_\ell^{23}|$ is odd, then $a_5 \neq a_6$.We can assume $a_5 = 1$. So T has the following form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Let $x^T = (1, 1, 1)$. Then $|Ann_T(x)| = |\{u_8\}| = 1$, a contradiction to (3). Next, let $a_3 = -1$. Since $|V_\ell^{23}|$ is odd, then $a_5 = a_6$. So, we can assume $a_5 = a_6 = 1$. So T has the following form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, let $x^T = (1, 1, -1)$. Then $|Ann_T(x)| = |\{u_6, u_7, u_8\}|$, a contradiction to (3).

Next, we consider the case where $|sp(t_i)| = 8$ for $i \in \{2, 3\}$. By (2), since $|V_\ell^{13}|$ and $|V_\ell^{23}|$ are both odd, for $\ell \in \{1, 2\}$, then without loss of generality, we can assume that T has the following form

$$T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \\ 1 & a_4 & b_4 \\ 1 & a_5 & b_5 \\ 1 & a_6 & b_6 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

where $a_i, b_i \in \{\pm 1\}$ and an odd number of $a_i, b_i = -1$, not necessarily $a_i = b_i$. Since $|V_\ell^{1j}|$ is odd for $2 \leq j \leq 3$, each t_j has an odd number of entries that are -1 and an odd number of entries that are 1. Since we have assumed that every entry of $\widetilde{T}$ is non-zero, then by Lemma 3.3.4, there exists an $x^T = (0, x_2, x_3)$, such that $|sp(\widetilde{T}x)|$ is even. Notice that $u_7, u_8 \notin Ann_T(x)$, thus $|sp(Tx)|$ is also even. This contradicts (2). This concludes the case when

$$\min_{1 \leq i \leq 4} \{|sp(t_i)|\} = 6.$$

In the case where $\min_i\{|sp(t_i)|\} = 8$ then for each t_i, $|sp(t_i)| = 8$. Again, by Lemma 3.3.5 and Lemma 3.3.2, there exists an even PM zero-subsum. Hence, we can conclude that

$$De_{\pm}(C_3^8) \leq 11.$$

□

We want to recall some notation from Section 3.3, after Corollary 3.3.6 we introduced some structure to help us break down our matrix T. Let $T = [t_1, t_2, t_3]$ where every row is non-zero. Then we can organize the rows into subsets which depend on the number ℓ zeros in the row, for $\ell \in \{0, 1, 2\}$. So, we partition T into the following subsets,

$$T = Z^0 \cup Z^1 \cup Z^2.$$

Next, we then partition each Z^ℓ by which entries are non-zero,

$$\begin{aligned} Z^0 &= W^{123} \\ Z^1 &= W^{12} \cup W^{13} \cup W^{23} \\ Z^2 &= W^1 \cup W^2 \cup W^3. \end{aligned}$$

Recall that $V^{ij} = V_1^{ij} \cup V_2^{ij}$ where V^{ij} contains the rows where both the i and j entry are non-empty. So

$$V^{ij} = Z^0 \cup W^{ij}.$$

Since $sp(t_i)$ accounts for every non-zero entry of t_i, then

$$sp(t_i) = Z^0 \cup W^{ij} \cup W^{ik} \cup W^i,$$

where $\{i, j, k\} = \{1, 2, 3\}$. These structures will be used in the result below.

Theorem 3.4.2. *For $2 \leq r \leq 9$,*

$$De_{\pm}(C_3^r) = r + 3.$$

Proof. If , for $r = 9$,

$$De_{\pm}(C_3^9) \leq 12 = r + 3,$$

then by Lemma 3.2.10 and Lemma 3.2.7, the lower bound equals the upper bound. Therefore, for $2 \leq k \leq 9$,

$$De_{\pm}(C_3^k) = k + 3.$$

Thus, we achieve the result by showing that $De_{\pm}(C_3^9) \leq 12$. Let S be a subset of non-zero elements of C_3^9 where $|S| = 12$ and M be the corresponding 9×12 matrix,

$$M = [e_1, e_2, \ldots, e_8, e_9 \mid t_1, t_2, t_3] = [I_9 \mid T].$$

Let $u_1, u_2, u_3, \ldots, u_9$ be the rows of T. We will show that every possible M contains an even PM zero-subsum. So, we consider the cases where we currently do not have an even PM zero-subsum. The following results allows us to make some additional assumptions on the conditions of T. By Lemma 3.4.1, we can assume that every row of T is non-zero. Then every row of T must be contained in Z^ℓ, for $0 \leq \ell \leq 2$. The results listed below provide additional simplication of T,

1. by Lemma 3.3.1, we can assume that every column t_i of T has even support,
2. by Lemma 3.3.2, we can assume that each $|V_\ell^{ij}|$ is odd,
3. a consequence of Lemma 3.3.1, if $|sp(x)| = 3$, then $|sp(Tx)|$ is even.

In the proof of Lemma 3.4.1, when $min_i\{|sp(t_i)|\} \in \{2, 4\}$, we did not use the fact that T contains 8 rows. Thus, we can assume that $min_i\{|sp(t_i)|\} \in \{6, 8\}$, since, by (1), $|sp(t_i)|$ is even for $1 \leq i \leq 3$.

Since T has 9 non-zero rows, then $|Z^0| + |Z^1| + |Z^2| = 9$. Then by Lemma 3.3.7, we know that $|Z^0| + |Z^1|$ is even. Thus, $|Z^2|$ is odd. Since each $|sp(t_i)|$ is even, by Lemma 3.3.8, we can assume that $|Z^0|, |Z^1|$ must both be odd.

Recall that

$$|V^{ij}| = |Z^0| + |W^{ij}|, \text{ where } W^{ij} \subseteq Z^1.$$

Since each V_ℓ^{ij} has odd cardinalty by (2), then $|V^{ij}| = |V_1^{ij} \cup V_2^{ij}|$ is even. Above we have shown that $|Z^0|$ is odd, so $|W^{ij}|$ must also be odd for each pair $1 \leq i < j \leq 3$. Thus,

$$|Z^1| = |W^{12} \cup W^{13} \cup W^{23}| \geq 3.$$

Similarly, recall that

$$|sp(t_i)| = |Z^0| + |W^{ij}| + |W^{ik}| + |W^i|. \tag{3.7}$$

Since $|sp(t_i)|$ is even and $|Z^0|, |W^{ij}|, |W^{ik}|$ are odd, then $|W^i|$ is also odd for each $1 \leq i \leq 3$. Hence,

$$|Z^2| = |W^1 \cup W^2 \cup W^3| \geq 3.$$

Since each $|W^i| \geq 1$, then each column t_i of T will contain at least two zero entries. Thus, $|sp(t_i)| = 6$ for each i.

Since each $|W^i| \geq 1$ and $|Z^2| \geq 3$, then without loss of generality, we assume that $u_7, u_8, u_9 \in Z^2$ such that

$$\begin{bmatrix} u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that rows u_7, u_8, u_9 provide two zero-entries for each t_i. Since each $|W^{ij}| \geq 1$ and $|Z^1| \geq 3$, then without loss of generality, we assume that $u_4, u_5, u_6 \in Z^1$ such that

$$\begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & * \\ 1 & 0 & * \end{bmatrix},$$

where $* \in \{\pm 1\}$. Notice that rows u_4, u_5, u_6 provide one zero-entry for each t_i. Since $|sp(t_i)| = 6$, for each i, then each t_i contains exactly three zero entries. Hence

$$|Z^0| = |Z^1| = |Z^2| = 3.$$

Without loss of generality, we can assume that T has the following form,

$$T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ 1 & * & * \\ 1 & * & * \\ 1 & * & 0 \\ 0 & 1 & * \\ 1 & 0 & * \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $* \in \{\pm 1\}$. Since $|V_\ell^{1j}|$ is odd for $j \in \{2, 3\}$ and $\ell \in \{1, 2\}$, then

$$T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 1 & -1 & b_1 \\ 1 & 1 & b_2 \\ 1 & 1 & b_3 \\ 1 & 1 & 0 \\ 0 & 1 & c \\ 1 & 0 & b_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where an odd number of $b_i = -1$ and $c \in \{\pm 1\}$. It is equivalent to assume that exactly one $b_i = -1$ and all others are 1. First, assume that $b_4 = -1$ and all other $b_i = 1$. Then for either choice $c \in \{\pm 1\}$ there exists an ℓ such that $|V_\ell^{23}|$ is even. This contradicts (2). Next, assume that one $b_i = -1$ for $1 \leq i \leq 3$. For any choice i and since $|V_\ell^{23}|$ is odd, then $c = -1$. So,

$$T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 1 & -1 & b_1 \\ 1 & 1 & b_2 \\ 1 & 1 & b_3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $x^T = (x_1, 1, 1)$ where $x_1 \in \{\pm 1\}$. It is clear that $u_5 \in Ann_T(x)$ for either choice of x_1 and that $u_4, u_6, u_7, u_8, u_9 \notin Ann_T(x)$. First, let $b_1 = -1$ and $x_1 = 1$. Then $Ann_T(x) = \{u_2, u_3, u_5\}$. Hence, $|Ann_T(x)|$ is odd, contradicting (3). Second, let $b_1 = 1$ and $x_1 = -1$. Then $Ann_T(x) = \{u_5\}$. Thus, $|Ann_T(x)|$ is odd, contradicting (3).

We have shown that for all possible M, generated by S where $|S| = 12$, there exists an even PM zero-subsum. Therefore, we have that

$$De_{\pm}(C_3^9) = 12.$$

$\square$

Continuing on the structure given before Theorem 3.4.2, we define

$$W^z(x) = W^z \cap Ann_T(x),$$

when $z \subset \{1, 2, 3\}$ and $x^T = (x_1, x_2, x_3)$ where $x_i \in \{0, \pm 1\}$. Given an x and matrix T, we use $W^z(x)$ to help us show the result below.

Proposition 3.4.3.

$$De_{\pm}(C_3^{10}) > 13.$$

Proof. We provide an example to show the lower bound. Let $M = [e_1, \ldots, e_{10} \mid T]$ where

$$T = [t_1, t_2, t_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix},$$

where $Z^0 = \{u_1, u_2, u_3, u_4\}$ and $Z^1 = \{u_5, \ldots, u_{10}\}$. More specifically,

$$Z^1 = W^{12} \cup W^{13} \cup W^{23},$$

where

$$W^{12} = \{u_5, u_6\} \qquad W^{13} = \{u_7, u_8\} \qquad W^{23} = \{u_9, u_{10}\}.$$

We will show that T satisfies the two conditions below. It then follows easily that M does not have an even PM zero-subsum.

1. Every plus-minus linear combination of an odd number of columns of T has even support.

2. Every plus-minus linear combination of an even number of columns of T has odd support.

Let $x^T = (x_1, x_2, x_3)$ and let $|sp(x)|$ be odd. Notice that $|sp(t_i)| = 8$ for $1 \leq i \leq 3$. So, if $|sp(x)| = 1$, then $|sp(Tx)| = 8$. Next, assume that each x_j is non-zero, so

$|sp(x)| = 3$. First, we consider $u_k = (u_{k_1}, u_{k_2}, u_{k_3})$ where every entry is non-zero. Since each entry in u_k and x are non-zero, then

$$0 \equiv u_k \cdot x = u_{k_1}x_1 + u_{k_2}x_2 + u_{k_3}x_3 \mod 3,$$

if and only if each $u_{k_i}x_i$ are all equal to 1 or all equal to -1. Since $Ann_T(x) = Ann_T(-x)$ and each u_k is distinct, then $|Z^0(x)| = 1$ for every possible x where $|sp(x)| = 3$.

Second, we consider u_k that contains exactly one zero-entry. So, $u_k \in W^{ij}$ for some $1 \leq i < j \leq 3$. Notice that $|W^{ij}| = 2$ and each one is distinct. This tells us that $|\{u_k, u_{k+1}\} \cap W^{ij}(x)| = 1$ for $k \in \{5, 7, 9\}$. Since each entry of x is non-zero, then $|W^{ij}(x)| = 1$ for each $1 \leq i < j \leq 3$ pair. For example, let $x^T = (1, 1, 1)$. Then,

$$\begin{aligned} Ann_T(x) &= Z^0(x) \cup W^{12}(x) \cup W^{13}(x) \cup W^{23}(x) \\ &= \{u_1\} \cup \{u_6\} \cup \{u_8\} \cup \{u_{10}\}. \end{aligned}$$

Or if we let $x^T = (-1, 1, 1)$, then

$$\begin{aligned} Ann_T(x) &= Z^0(x) \cup W^{12}(x) \cup W^{13}(x) \cup W^{23}(x) \\ &= \{u_4\} \cup \{u_5\} \cup \{u_7\} \cup \{u_{10}\}. \end{aligned}$$

Since we have shown that when $|sp(x)| = 3$ then $|Z^0(x)| = 1$ and each $|W^{ij}(x)| = 1$. Thus $|Ann_T(x)| = 4$. Therefore, we have met the condition of (1).

Next, notice that by the columns of T show that, $|V_\ell^{ij}|$ is odd for $\ell \in \{1, 2\}$ and $1 \leq i < j \leq 3$. For example,

$$V_1^{12} = \{u_1, u_2, u_5\} \qquad V_2^{12} = \{u_3, u_4, u_6\}.$$

One can verify that $|V_\ell^{ij}| = 3$ for all $\ell \in \{1, 2\}$ and $1 \leq i < j \leq 3$. Then by Corollary 3.3.3, for every x such that $|sp(x)| = 2$, we know that $|sp(Tx)|$ is odd. Hence, condition (2) is satisfied.

Since our matrix M satisfies the two conditions above, we can conclude that M does not contain an even PM zero-subsum. Thus $De_{\pm}(C_3^{10}) > 13$. $\square$

Corollary 3.4.4. *Let $n \geq 10$. Then*

$$De_{\pm}(C_3^n) \geq n + 4.$$

Proof. Since we have shown that $De_{\pm}(C_3^{10}) \geq 14$, then by Lemma 3.2.9, $De_{\pm}(C_3^n) \geq n + 4$ for all $n \geq 10$. $\square$

We generalize Corollary 3.4.4 in Theorem 3.5.1.

3.5 Lower Bound for $De_{\pm}(C_3^r)$ for Large r

Theorem 3.5.1. *Let $S \subset C_3^n$ such that $|S| = n + q$. If*

$$n \geq \sum_{m=0}^{q-2} 2^{q-m-1} \binom{q}{m}$$

then
$$De_{\pm}(C_3^n) \geq n + q + 1.$$

We provide some examples for when $n \in \{10, 36\}$.

- When $q = 3$,
$$\sum_{m=0}^{1} 2^{2-m}\binom{3}{m} = 2^2\binom{3}{0} + 2\binom{3}{1} = 4 \cdot 1 + 2 \cdot 3 = 10.$$
So by Theorem 3.5.1, $De_{\pm}(C_3^{10}) \geq 14$. This proof is also provided by Proposition 3.4.3.

- When $q = 4$,
$$\sum_{m=0}^{2} 2^{3-m}\binom{4}{m} = 2^3\binom{4}{0} + 2^2\binom{4}{1} + 2\binom{4}{2} = 8 \cdot 1 + 4 \cdot 4 + 2 \cdot 6 = 36.$$
So by Theorem 3.5.1, $De_{\pm}(C_3^{36}) \geq 41$.

Lemma 3.5.2. *Let x, y be vectors such that $x^T = (x_1, \ldots, x_q)$ and $y^T = (y_1, \ldots, y_q)$ where $x_i, y_i \neq 0$ for $1 \leq i \leq q$. Let $k = 2^{q-1}$ and T be a $k \times q$ matrix where every entry is ± 1 and each row is unique up to sign. Then $|Ann_T(x)| = |Ann_T(y)|$.*

Recall that Z^0 are the rows of T that contain no zero-entries. The construction of subsets of the rows of T is provided after Lemma 3.3.5. Notice that Z^0 consists of all the row of T and Z^0 is maximal.

Proof. It is sufficient to prove the result when x is a vector of all ones and y is arbitrary. Let $u = (u_1, \ldots, u_q)$. Note that the rows of T contain each such vector u, up to sign, exactly once. Let $\varphi : Ann_T(y) \to Ann_T(x)$, where
$$(u_1, \ldots, u_q) \mapsto (u_1 y_1, \ldots, u_q y_q)$$
It is straightforward to check that φ is injective and surjective. This provides that $|Ann_T(x)| = |Ann_T(y)|$. □

The result below considers all possible rows with entries ± 1. For maximal Z^0, we are considering $2|Z^0| = 2^q$ possible rows.

Lemma 3.5.3. *Let x be a vector of all ones with length $q \geq 1$. Let $k = 2^q$ and T_q be a $k \times q$ matrix where every entry is ± 1 and each row is unique. Let $A_q = |Ann_{T_q}(x)|$. Then*
$$1 + \binom{q}{q-1}A_1 + \binom{q}{q-2}A_2 + \cdots + \binom{q}{1}A_{q-1} + \binom{q}{0}A_q = 3^{q-1}.$$

Proof. Let $u \in C_3^q$. Consider the linear map $\varphi : C_3^q \to C_3$ given by $\varphi(u) = u \cdot x$. Since φ is surjective, the rank-nullity theorem implies that the dimension of $ker(\varphi)$ is $q-1$. Thus, $|ker(\varphi)| = 3^{q-1}$.

For $1 \leq j \leq q$, the number of vectors $u \in C_3^q$ that have j non-zero entries and satisify $u \cdot x = 0$ is given by $\binom{q}{q-j} A_j$. By summing over all possible j and including the zero vector, we get the result above. $\square$

Lemma 3.5.4. *If $q \geq 2$, then $\frac{1}{2}A_q$ is odd.*

Proof. If $x \cdot u = 0$, then $x \cdot (-u) = 0$, and $u \neq -u \in C_3^q$ when $u \neq 0$. Thus A_q is even for all $q \geq 2$.

The proof is by induction on q. The result certainly holds for $q = 2$. Since $A_1 = 0$, Lemma 3.5.3, implies that

$$\binom{q}{q-2} A_2 + \cdots + \binom{q}{1} A_{n-1} + \binom{q}{0} A_q = 3^{q-1} - 1.$$

Then

$$\binom{q}{q-2} \frac{1}{2} A_2 + \cdots + \binom{q}{1} \frac{1}{2} A_{q-1} + \binom{q}{0} \frac{1}{2} A_q = \frac{3^{q-1} - 1}{2}.$$

We now examine this last equation modulo 2. We have

$$\frac{3^{q-1} - 1}{2} \equiv \begin{cases} 1 \bmod 2 & \text{if } q \text{ is even,} \\ 0 \bmod 2 & \text{if } q \text{ is odd.} \end{cases}$$

By induction, $\frac{1}{2}A_2, \ldots, \frac{1}{2}A_{q-1}$ are each odd. Since

$$2^q = (1+1)^q = \binom{q}{q} + \binom{q}{q-1} + \binom{q}{q-2} + \cdots + \binom{q}{1} + \binom{q}{0},$$

we have

$$\binom{q}{q-2} + \cdots + \binom{q}{1} + \binom{q}{0} = 2^q - 1 - q \equiv \begin{cases} 1 \bmod 2 & \text{if } q \text{ is even,} \\ 0 \bmod 2 & \text{if } q \text{ is odd.} \end{cases}$$

It follows that $\frac{1}{2}A_q$ is odd. $\square$

In the previous two lemmas, x was a vector of all ones with length q. Now, let x^j be a vector of length q where the first $q-j$ entries are ones and the last j entries are zero. Notice that $x^0 = x$ is a vector of all ones. Let y^{q-j} be a vector of length $q-j$ of all ones. These vectors will be used together in the coming lemmas. They will each correspond to their own maximal matrix T_q and T_{q-j} respectively and will each have their own set Z^0. To help destinguish these sets, we add a subscript to Z^0. Let Z_q^0 contain the rows of T_q and Z_{q-j}^0 contain the rows of T_{q-j} where each row has non-zero entries.

Lemma 3.5.5. *Let x^j and y^{q-j} be the vectors of ones and zeros as defined above. Let T_q be maximal and let each Z^0_{q-j} be maximal, for $0 \leq j \leq q-2$. Then*

$$\left|Ann_{T_q}(x^0)\right| = \sum_{j=0}^{q-2} \binom{q}{j} \left|Ann_{Z^0_{q-j}}(y^{q-j})\right|$$

Proof. Let $u \in Ann_{T_q}(x^0)$. Then $u \cdot x^0 = 0 \mod 3$. Let j be the number of zero entries contained in u. Let u' be the vector of length $q-j$ that contains all non-zero entries of u. Then, by construction, $u' \cdot y^{q-j} = 0 \mod 3$. Since u' has only non-zero entires then $u' \in Z^0_{q-j}$, so $u' \in Ann_{Z^0_{q-j}}(y^{q-j})$. There are $\binom{q}{j}$ ways to place the zero entries for each $u \in Ann_{T_q}(x^0)$ that reduces to u'. Hence, we have the desired result. □

Corollary 3.5.6. *Let T_q be maximal and x be a vector of all ones with length q.*

$$\left|Ann_{T_q}(x)\right| = \frac{3^{q-1}-1}{2}.$$

Proof. By Lemma 3.5.3,

$$3^{q-1} = \left[\sum_{j=0}^{q-1} A_{q-j}\binom{q}{j}\right] + 1.$$

Since $A_1 = 0$, then

$$3^{q-1} - 1 = \sum_{j=0}^{q-2} A_{q-j}\binom{q}{j}.$$

Notice that

$$A_{q-j} = 2\left|Ann_{Z^0_{q-j}}(y^{q-j})\right|.$$

This tells us that

$$3^{q-1} - 1 = \sum_{j=0}^{q-2} 2\left|Ann_{Z^0_{q-j}}(y^{q-j})\right|\binom{q}{j}.$$

Therefore

$$\begin{aligned}\frac{3^{q-1}-1}{2} &= \sum_{j=0}^{q-2} \left|Ann_{Z^0_{q-j}}(y^{q-j})\right|\binom{q}{j} \\ &= \left|Ann_{T_q}(x)\right|,\end{aligned}$$

by Lemma 3.5.5. □

Corollary 3.5.7. *Let $q \geq 3$. Then*

$$\left|Ann_{T_q}(x^0)\right| = \begin{cases} \textit{even} & \textit{when } q \textit{ is odd} \\ \textit{odd} & \textit{when } q \textit{ is even.} \end{cases}$$

Proof. By Lemma 3.5.4, we know that $|Ann_{Z^0_{q-j}}(y^{q-j})|$ is odd. Thus $|Ann_{Z^0_{q-j}}(y^{q-j})| \equiv 1 \mod 2$. By Lemma 3.5.5,

$$\left|Ann_{T_q}(x^0)\right| = \sum_{j=0}^{q-2} \binom{q}{j} \left|Ann_{Z^0_{q-j}}(y^{q-j})\right|.$$

Hence, to show that $|Ann_{T_q}(x^0)|$ is even or odd, we need to know when

$$2^q - (q+1) = \sum_{j=0}^{q-2} \binom{q}{j}$$

is even or odd. Since 2^q is even, the summand is even or odd when q is odd or even respectively. Thus providing us with the desired result. □

Lemma 3.5.8. *For $1 \leq j \leq q$, let y^{q-j} be the vector of all ones with length $q-j$. Let x^j be the vector whose first $q-j$ entries are 1 and the remaining j entries are 0. Let T_q and T_{q-j} be maximal. Then*

$$|Ann_{T_q}(x^j)| = \begin{cases} 3^j|Ann_{T_{q-j}}(y^{q-j})| & \textit{when } q-j=1, \\ 3^j|Ann_{T_{q-j}}(y^{q-j})| + \sum_{k=0}^{j-2} 2^{j-k}\binom{j}{k} & \textit{when } 2 \leq q-j \leq q-2. \end{cases}$$

Proof. Let $u \in Ann_{T_q}(x^j)$ where $u = (u_1, \dots, u_q)$. Then, $u \cdot x^j = 0 \mod 3$. Let $u' = (u_1, \dots, u_{q-j})$. Since x^j has non-zero entries for the first $q-j$ entries, then we know that $u' \cdot y^{q-j} = 0 \mod 3$. Hence $u' \in Ann_{T_{q-j}}(y^{q-j})$. Since the last j entries of x^j are zero, then u has 3^j choices for the last j entries.

Notice that when $u = (0, \dots, 0, u_i, \dots, u_q)$ where $i = q-j+1$, then $u \in Ann_{T_q}(x^j)$ but $u' = (0, \cdots, 0) \notin Ann_{T_{q-j}}(y^{q-j})$, since T_{q-j} is maximal. Note that when $q-j=1$ this case can not occur, since T_q is maximal and thus has at least two non-zero entries in each row. For $i \leq \ell \leq q$, suppose that $u_\ell = 0$ for exactly k values of ℓ. Since u contains at most $q-2$ zeros , then $q-j+k \leq q-2$. Hence, $0 \leq k \leq q-2-(q-j) = j-2$. There are $\binom{j}{k}$ ways to place our zero entries. The remaining entries have two choices, ± 1, and there are $j-k$ remaining non-zero entries in u. □

Corollary 3.5.9. *Let $q \geq 3$. Then*

$$\left|Ann_{T_q}(x^j)\right| = \begin{cases} \textit{even} & \textit{when } q-j \textit{ is odd} \\ \textit{odd} & \textit{when } q-j \textit{ is even} \end{cases}$$

Proof. By Lemma 3.5.8, we know that

$$|Ann_{T_q}(x^j)| = \begin{cases} 3^j|Ann_{T_{q-j}}(y^{q-j})| & \text{when } q-j=1, \\ 3^j|Ann_{T_{q-j}}(y^{q-j})| + \sum_{k=0}^{j-2} 2^{j-k}\binom{j}{k} & \text{when } 2 \leq q-j \leq q-2. \end{cases}$$

In either of the cases above, we only need to know if $|Ann_{T_{q-j}}(y^{q-j})|$ is even or odd, since the second summand is even. Assume $q-j$ is odd. By Corollary 3.5.7, we know that $|Ann_{T_{q-j}}(y^{q-j})|$ must be even. Hence $|Ann_{T_q}(x^j)|$ is even.

Next, assume $q-j$ is even. Then, by Corollary 3.5.7, $|Ann_{T_{q-j}}(y^{q-j})|$ is odd. Thus, $|Ann_{T_q}(x^j)|$ is odd. □

Proof of Theorem 3.5.1. Let T_q be maximal, so T_q is an $m \times q$ matrix such that

$$m = \sum_{k=0}^{q-2} 2^{q-k-1} \binom{q}{k}$$

and each Z^k is maximal, for $0 \leq k \leq q-2$. It is clear that m is even.

We introduced $Ann_T(x)$ to help us better understand $sp(Tx)$ at the beginning of Section 2.4, so

$$|sp(Tx)| = m - |Ann_T(x)|.$$

We will show the following statements are true.

1. Every plus-minus sum of an odd number of columns of T_q has even support.

2. Every plus-minus sum of an even number of columns of T_q has odd support.

Case (1) Let $q-j$ be odd. Then x^j has an odd number of non-zero entries. By Corollary 3.5.9, $\left|Ann_{T_q}(x^j)\right|$ is even. Thus,

$$|sp(T_q x^j)| = m - \left|Ann_{T_q}(x^j)\right|$$

is even.

Case (2) Let $q-j$ be even. Then x^j has an even number of non-zero entries. By Corollary 3.5.9, $\left|Ann_{T_q}(x^j)\right|$ is odd. Thus,

$$|sp(T_q x^j)| = m - \left|Ann_{T_q}(x^j)\right|$$

is odd.

Now we let $M = [I_m \mid T_q]$. By the construction of T_q, we are not able to find an even PM zero-subsum for columns of M. Therefore ,

$$De_{\pm}(C_3^m) \geq m + q + 1.$$

Next, let $n \geq m$. By Proposition 3.2.9, we know that

$$De_{\pm}(C_3^n) \geq n + q + 1.$$

□

Recall by Lemma 3.2.7, $De_{\pm}(C_3^n) \geq n+3$. For when $q \geq 2$, this improves the lower bound for $De_{\pm}(C_3^n)$ by $q-1$.

3.6 Connections between $De_{\pm}(G)$ and $D_{\pm}(C_2 \oplus G)$

Let G be a finite abelian group such that $D_{\pm}(G) = \lfloor \log_2 |G| \rfloor + 1$, i.e. $D_{\pm}(G)$ obtains the basic upper bound.

Lemma 3.6.1. *Let $q_0 \geq 0$. If*

$$D_{\pm}(C_2^{q_0} \oplus G) = \lfloor \log_2 2^{q_0}|G| \rfloor + 1 = \lfloor \log_2 |G| \rfloor + q_0 + 1,$$

and $q > q_0$, then

$$D_{\pm}(C_2^{q} \oplus G) = \lfloor \log_2 2^{q}|G| \rfloor + 1 = \lfloor \log_2 |G| \rfloor + q + 1.$$

Proof. Assume $D_{\pm}(C_2^{q_0} \oplus G) = \lfloor \log_2 2^{q_0}|G| \rfloor + 1 = \lfloor \log_2 |G| \rfloor + q_0 + 1$. Let $K = C_2^q \oplus G$ and $H = C_2^{q_0} \oplus G$. By the assumption that $D_{\pm}(H)$ attains the basic upper bound and K/H is a 2-group, then, by Lemma 2.3.10, $D_{\pm}(K)$ also attains the upper bound. Hence

$$D_{\pm}(C_2^{q} \oplus G) = \lfloor \log_2 2^{q}|G| \rfloor + 1 = \lfloor \log_2 |G| \rfloor + q + 1.$$

□

Corollary 3.6.2.

$$D_{\pm}(C_2^{q} \oplus C_n) = \lfloor \log_2 2^{q}n \rfloor + 1.$$

Proof. Let $q = 1$. By Proposition 2.3.2, we know that $D_{\pm}(C_n) = \lfloor \log_2 n \rfloor + 1$. Then by Lemma 2.3.10,

$$D_{\pm}(C_2 \oplus C_n) = \lfloor \log_2 2n \rfloor + 1.$$

Thus, the result follows by Lemma 3.6.1. □

Proposition 3.6.3. *Let $q \geq q_0$. If $D_{\pm}(C_2^{q_0} \oplus G) = \lfloor \log_2 2^{q_0}|G| \rfloor + 1$ and*

$$De_{\pm}(C_2^{q_0} \oplus G) = D_{\pm}(C_2^{q_0+1} \oplus G),$$

then

$$De_{\pm}(C_2^{q} \oplus G) = D_{\pm}(C_2^{q+1} \oplus G).$$

Proof. By Lemma 3.6.1 $D_{\pm}(C_2^q \oplus G) = \lfloor \log_2 2^q|G| \rfloor + 1$ for all $q \geq q_0$. Then by Proposition 2.4.5 and the assumption above

$$\begin{gathered} D_{\pm}(C_2^q \oplus G) \leq De_{\pm}(C_2^q \oplus G) \leq D_{\pm}(C_2^{q+1} \oplus G) \\ \lfloor \log_2 2^q|G| \rfloor + 1 \leq De_{\pm}(C_2^q \oplus G) \leq \lfloor \log_2 2^{q+1}|G| \rfloor + 1 \\ \lfloor \log_2 |G| \rfloor + q + 1 \leq De_{\pm}(C_2^q \oplus G) \leq \lfloor \log_2 |G| \rfloor + q + 2. \end{gathered}$$

Let $n = De_{\pm}(C_2^{q_0} \oplus G) - 1$. Let $S = \{s_1, \ldots, s_n\} \subset C_2^{q_0} \oplus G$ where each $s_i \in S$ is non-zero such that S does not contain an even PM zero-subsum. Let N be the corresponding $q_0 \times n$ matrix of S. Next, we construct the following $q \times (n + q - q_0)$ block matrix M,

$$M = \left[\begin{array}{c|c} 0 & I_{q-q_0} \\ \hline N & 0 \end{array}\right]$$

where 0 is the corresponding zero-matrix. Since N does not contain an even PM zero-subsum, then neither does M. Since

$$n+q-q_0=\lfloor \log_2\ 2^{q_0+1}|G|\rfloor+q-q_0=\lfloor \log_2\ |G|\rfloor+q+1,$$

therefore,

$$\lfloor \log_2\ |G|\rfloor+q+1<De_{\pm}(C_2^q\oplus G)\leq \lfloor \log_2\ |G|\rfloor+q+2.$$

Thus, the lower and upper bound are equal and hence

$$De_{\pm}(C_2^q\oplus G)=D_{\pm}(C_2^{q+1}\oplus G).$$

□

Corollary 3.6.4. *If $De_{\pm}(C_n)=\lfloor \log_2\ n\rfloor+2$, then*

$$De_{\pm}(C_2^q\oplus C_n)=\lfloor \log_2\ n\rfloor+q+2.$$

Proof. Assume $De_{\pm}(C_n)=\lfloor \log_2\ n\rfloor+2$. By Lemma 2.3.10, $D_{\pm}(C_2\oplus C_n)=\lfloor \log_2\ n\rfloor+2$. Then by Lemma 3.6.1, for $q\geq 0$,

$$D_{\pm}(C_2^q\oplus C_n)=\lfloor \log_2\ n\rfloor+q+1.$$

Since $De_{\pm}(C_n)=D_{\pm}(C_2\oplus C_n)=\lfloor \log_2\ n\rfloor+2$, by Proposition 2.3.2, and by Proposition 3.6.3

$$De_{\pm}(C_2^q\oplus C_n)=D_{\pm}(C_2^{q+1}\oplus C_n)=\lfloor \log_2\ n\rfloor+q+2.$$

□

3.6.1 Connections between $De_{\pm}(C_3^r)$ and $D_{\pm}(C_2\oplus C_3^r)$

In this section, we bridge the connection between $D_{\pm}(C_2\oplus C_3^r)$ and $De_{\pm}(C_3^r)$. In the proof of Theorem 3.6.7, we break down the cases by consider the order of an element in our group, $C_2\oplus C_3^r$. Notice that for every $g\in C_2\oplus C_3^r$, then $ord(g)\in\{2,3,6\}$ for when g is non-trivial. Since $G=C_2\oplus C_3^r$ is a finite abelian group then for $h\in G$, $H=\langle h\rangle$ is the subgroup of G that is generated by h.

Proposition 3.6.5. *Suppose $G=C_n\oplus H$. Let $S=\{s_1,\dots s_k,t_1,\dots,t_\ell\}\subset G$ such that $k=D_{\pm}(H)$ and $\ell=\lfloor \log_2\ n\rfloor$ where $t_i\in C_n$. Then S contains a PM zero-subsum.*

Proof. Let $S=\{s_1,\dots,s_k,t_1,\dots,t_\ell\}$ where $t_i\in C_n$ and k,ℓ have the values provides above. Consider the canonical map $\pi:G\to G/C_n\cong H$. Since $k=D_{\pm}(H)$, $\pi(S)$ contains a PM zero-subsum. Thus, we can write,

$$t=\sum_{i=1}^{k}\alpha_i s_i\in C_n$$

where $\alpha_i \in \{\pm 1, 0\}$. By Lemma 2.3.2, $D_{\pm}(C_n) = \lfloor \log_2 \ n \rfloor + 1$. Since $t \in C_n$ and $D_{\pm}(C_n) = \ell + 1$, then

$$0 \equiv \beta \sum_{i=1}^{k} \alpha_i s_i + \sum_{j=1}^{\ell} \beta_j t_j = \beta t + \sum_{j=1}^{\ell} \beta_j t_j \mod n$$

where $\beta, \beta_j \in \{0, \pm 1\}$. Thus, S contains a PM zero-subsum. □

Corollary 3.6.6. *Let $G = C_2 \oplus C_3^r$ for $r \geq 1$. Let S be a subset of elements of G such that $|S| = r + 2$. If S contains an element of order two, then S has a PM zero-subsum.*

Proof. Observe that G has a unique element h of order two. Then $S = \{s_1, \ldots, s_{r+1}, h\}$. By Theorem 2.3.12, $D_{\pm}(C_3^r) = r + 1$; and by Lemma 2.3.2, $D_{\pm}(C_2) = 2$. Thus, we satisfy the conditions of Proposition 3.6.5. Hence, S contains a PM zero-subsum. □

Theorem 3.6.7. *Let $r \geq 1$.*

$$De_{\pm}(C_3^r) = D_{\pm}(C_2 \oplus C_3^r).$$

Proof. First notice by Proposition 2.4.5,

$$De_{\pm}(C_3^r) \leq D_{\pm}(C_2 \oplus C_3^r).$$

This leaves us to show

$$D_{\pm}(C_2 \oplus C_3^r) \leq De_{\pm}(C_3^r).$$

Let $S \subseteq C_2 \oplus C_3^r$ such that $|S| = De_{\pm}(C_3^r)$. We want to show S contains a PM zero-subsum. Lemma 3.2.7 states that $De_{\pm}(C_3^r) \geq r + 3$. If there exists $s \in S$ such that $ord(s) = 2$ then by Corollary 3.6.6, S contains a PM zero-subsum. Now, we can assume S contains only elements of order three and six.

Assume every element of S has order six. We write $s_j \in S$ as

$$s_j = \begin{bmatrix} m_{0j} \\ m_{1j} \\ \vdots \\ m_{rj} \end{bmatrix}$$

where $m_{0j} \in C_2$ and $m_{ij} \in C_3$ for $1 \leq i \leq r$, for $1 \leq j \leq |S|$. Since we assume $ord(s_j) = 6$ then $m_{0j} = 1$ for all s_j. Let $De_{\pm}(C_3^r) = n$. Then let $\widetilde{M}$ be the corresponding $(r+1) \times n$ matrix that contains all $s_j \in S$. So

$$\widetilde{M} = [s_1, \ldots, s_n] = \begin{bmatrix} u \\ M \end{bmatrix}$$

where u is a row vector of all ones and M is a $r \times n$ matrix such that every $m_{ij} \in C_3$. So,

$$M = [m_1 \ m_2 \ldots m_n] = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & \ddots & & m_{2n} \\ \vdots & & \ddots & \vdots \\ m_{r1} & \ldots & \ldots & m_{rn} \end{bmatrix}.$$

Since we assumed that $De_{\pm}(C_3^r) = n$, we are guaranteed to find an even PM zero-subsum in M. So there exists a set of $\alpha_i \in \{0, \pm 1\}$ such that

$$\sum_{i=1}^{n} \alpha_i m_i = 0,$$

where an even number of $\alpha_i \neq 0$. Then since $ord(s_j) = 6$ for every $s_j \in S$, then each $m_{0j} = 1$ for the first entry of s_j. Hence

$$\sum_{i=1}^{n} \alpha_i s_i = 0.$$

This provides a PM zero-subsum.

Next, assume S contains elements of order three. We can assume that the elements in S are distinct. Let T be the set of $t_i \in S$ such that the order of t_i is three, so $T = \{t_i \in S \mid ord(t_i) = 3\}$. We denote the other elements of $s_i \in S$ as the elements of order six. Since $D_{\pm}(C_3^r) = r+1$, if $|T| \geq r+1$, then S contains a PM zero-subsum. Notice that for $s_i, s_j \in S \backslash T$, $ord(s_i + s_j) \in \{0, 3\}$. When the $ord(s_i + s_j) = 0$, we have an even PM zero-subsum. Now, assume that $ord(s_i + s_j) = 3$. So if $\frac{|S \backslash T|}{2} + |T| \geq r+1$, then S contains a PM zero-subsum.

We are left to consider the case when

$$\frac{|S \setminus T|}{2} + |T| < r + 1.$$

Recall $|S| = De_{\pm}(C_3^r) = n$. Then

$$\begin{aligned} \frac{|S \setminus T|}{2} + |T| &< r + 1 \\ \frac{n - |T|}{2} + |T| &< r + 1 \\ \frac{n}{2} + \frac{|T|}{2} &< r + 1 \\ |T| &< 2(r+1) - n \end{aligned}$$

Assume that T does not contain a PM zero-subsum where $|T| = k < 2(r+1) - n$. Then $|S \setminus T| = n - k$. Let $\widetilde{M}$ be the following $(r+1) \times n$ matrix where $t_i \in T$ and $s_j \in S \setminus T$,

$$\widetilde{M} = [t_1 \ldots t_k \mid s_1 \ldots s_{n-k}] = \begin{bmatrix} u \\ M \end{bmatrix},$$

where $u = (0, \ldots, 0, 1, \ldots, 1)$ contains k zeros and $n-k$ ones and M is the $r \times n$ matrix where every $m_{ij} \in C_3$. Suppose the first k columns of M have rank j where $j \leq k$. Then we row reduce M,

$$M = [t_1, \ldots, t_k, \bar{s}_1, \ldots \bar{s}_{n-k}] = [e_1, \ldots, e_j, n_{j+1}, \ldots, n_k, m_1, \ldots, m_{n-k}],$$

where $\bar{s}_i$ is constructed by removing the C_2 entry (the first entry) from s_i. If $j < k$, then the first $j+1$ columns have a PM zero-subsum in T. This contradicts our assumption that T does not contain a PM zero-subsum. Hence,

$$M = [e_1, \ldots, e_k, m_1, \ldots, m_{n-k}].$$

We partition M into a block matrix,

$$M = \left[\begin{array}{c|c} I_k & M_1 \\ \hline 0 & M_2 \end{array}\right],$$

where I_k is the $k \times k$ identity matrix, 0 is the corresponding zero matrix, M_1 is a $k \times (n-k)$ matrix, and M_2 is a $(r-k) \times (n-k)$ matrix. Our goal is to find an even PM zero-subsum in M_2. By Lemma 3.2.7, $De_{\pm}(C_3^r) > r$, so let $De_{\pm}(C_3^r) = n = r + \alpha$, for some $\alpha \in \mathbb{N}$. By Lemma 3.2.10,

$$De_{\pm}(C_3^{r-k}) \leq r - k + \alpha = n - k.$$

Thus, we can find an even PM zero-subsum in M_2. Hence, there exists a set of $\gamma_i \in \{0, \pm 1\}$, such that

$$\sum_{i=1}^{n-k} \gamma_i m_i = \begin{bmatrix} \hat{m}_1 \\ \hat{m}_2 \\ \vdots \\ \hat{m}_k \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where an even number of $\gamma_i \neq 0$. Since each $m_{0j} = 1$, for $s_i \in S \setminus T$, then

$$\sum_{i=1}^{n-k} \gamma_i s_i = \begin{bmatrix} 0 \\ \hat{m}_1 \\ \hat{m}_2 \\ \vdots \\ \hat{m}_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, there exists a set of $\beta_i \in \{0, \pm 1\}$ such that

$$\sum_{j=1}^{k} \beta_j e_j + \sum_{i=1}^{n-k} \gamma_i m_i = 0.$$

Hence, since row reduction does not change our non-trivial solution,

$$\sum_{j=1}^{k} \beta_j t_j + \sum_{i=1}^{n-k} \gamma_i s_i = 0$$

and

$$D_{\pm}(C_2 \oplus C_3^r) \leq De_{\pm}(C_3^r).$$

Therefore the lower and upper bound are equal and we have the desired result. □

The following corollary directly follows from the results of $De_{\pm}(C_3^r)$ in the previous section.

Corollary 3.6.8.

$$\begin{aligned} D_{\pm}(C_2 \oplus C_3^r) &= r + 3 && \textit{for } r < 10 \\ D_{\pm}(C_2 \oplus C_3^r) &\geq r + 4 && \textit{for } r \geq 10 \end{aligned}$$

Proof. Theorem 3.4.2 provides that

$$De_{\pm}(C_3^n) = n + 3,$$

for $n < 10$. By Theorem 3.5.1, when $n \geq 10$,

$$De_{\pm}(C_3^n) \geq n + 4.$$

Then by Theorem 3.6.7, we have that

$$De_{\pm}(C_3^n) = D_{\pm}(C_2 \oplus C_3^n).$$

Hence, we get the desired results. □

Conjecture 3.6.9. *For $r \geq 3$,*

$$D_{\pm}(C_2 \oplus C_3^r) < \lfloor \log_2\ n \rfloor + 2.$$

By Corollary 3.6.8, the conjecture holds for $3 \leq r \leq 9$. We strongly suspect that $D_{\pm}(C_2 \oplus C_3^r)$ will tend to the lower bound shown in Theorem 3.5.1. This would be the first class of groups that consistantly lie between the basic lower bound and basic upper bound.

Chapter 4 Plus-Minus Davenport Constant

4.1 Plus-Minus Davenport for $G = C_2^q \oplus C_3^r$

Let G be a finite abelian group of the form $C_2^q \oplus C_3^r$.

Proposition 4.1.1. *Let $q \geq 1$ and $r \leq 3$. Then,*

$$D_\pm(C_2^q \oplus C_3^r) = \lfloor \log_2 \ 2^q 3^r \rfloor + 1 = \lfloor \log_2 3^r \rfloor + q + 1.$$

Proof. First, let $q = 1$ and let $r \leq 3$. Notice that $|C_2 \oplus C_3^r| < 100$ for $r \leq 3$. Then, by Theorem 2.3.21, we know that

$$D_\pm(C_2 \oplus C_3^r) = \lfloor \log_2 \ 2 \cdot 3^r \rfloor + 1 = \lfloor r \log_2 3 \rfloor + 2.$$

Now, we can decompose $C_2^q \oplus C_3^r$ such that

$$C_2^q \oplus C_3^r \cong C_2^{q-1} \oplus (C_2 \oplus C_3^r).$$

Let $H = C_2 \oplus C_3^r$. We have shown that $D_\pm(H)$ attains the basic upper bound for $r \leq 3$. Since G/H is a 2-group, we apply Lemma 2.3.10 to achieve the desired result. $\square$

Note that $D_\pm(C_3^r)$ does not attain the basic upper bound when $r > 1$ by Theorem 2.3.12. Also by Corollary 3.6.8, we know, for $q = 1$ and $4 \leq r \leq 9$,

$$D_\pm(C_2 \oplus C_3^r) = r + 3 < \lfloor \log_2 \ 3^r \rfloor + 2,$$

which is smaller than the basic upper bound. Recall, by Theorem 3.4.2, $De_\pm(C_3^r) = r + 3$ for $2 \leq r \leq 9$.

Lemma 4.1.2. *Let*

$$H_1 \cong C_2^{q_1} \oplus C_3^{r_1} \qquad \text{and} \qquad H_2 \cong C_2^{q_2} \oplus C_3^{r_2}.$$

If $\{\log_2 \ 3^{r_1}\} + \{\log_2 \ 3^{r_2}\} < 1$ and $D_\pm(H_i) = \lfloor \log_2 \ |H_i| \rfloor + 1$, for $i \in \{1, 2\}$, then when $G = H_1 \oplus H_2$,

$$D_\pm(G) = \lfloor \log_2 \ |G| \rfloor + 1.$$

Proof. This directly follows from Lemma 2.3.10. $\square$

Lemma 4.1.3. *Let*

$$H_1 = C_2^{q_1} \oplus C_3^2 \qquad \text{and} \qquad H_2 = C_2^{q_2} \oplus C_3^{r_2},$$

and let $q = q_1 + q_2 \geq 2$ and $r = 2 + r_2$. If $q \geq 2$ and $r_2 \leq 3$, then

$$D_\pm(C_2^q \oplus C_3^r) = \lfloor \log_2 \ 2^q 3^r \rfloor + 1.$$

Proof. Let

$$H_1 = C_2^{q_1} \oplus C_3^2 \qquad \text{and} \qquad H_2 = C_2^{q_2} \oplus C_3^{r_2},$$

and

$$G = H_1 \oplus H_2 \cong C_2^q \oplus C_3^r.$$

Since $r_2 \leq 3$, then by Proposition 4.1.1,

$$D_{\pm}(H_i) = \lfloor \log_2 \ |H_i| \rfloor + 1,$$

for $i \in \{1, 2\}$. Notice that

$$\{\log_2 \ 3\} < \frac{3}{5}, \quad \{\log_2 \ 3^2\} < \frac{1}{5}, \quad \text{and} \quad \{\log_2 \ 3^3\} < \frac{4}{5}.$$

Hence, for $r_2 \leq 3$,

$$\{\log_2 \ 3^2\} + \{\log_2 \ 3^{r_2}\} < 1.$$

Then by Lemma 4.1.2,

$$D_{\pm}(G) = \lfloor \log_2 \ 2^q 3^r \rfloor + 1.$$

□

Corollary 4.1.4. *If $q \geq 3$, then*

$$D_{\pm}(C_2^q \oplus C_3^6) = \lfloor \log_2 \ 2^q 3^6 \rfloor + 1.$$

Proof. Let $H = C_2 \oplus C_3^2$. By Proposition 4.1.1, $D_{\pm}(H) = \lfloor \log_2 \ |H| \rfloor + 1$. Let $q_0 \geq 2$. By Lemma 4.1.3, $D_{\pm}(C_2^{q_0} \oplus C_3^4) = \lfloor \log_2 \ 2^{q_0} \cdot 3^4 \rfloor + 1$. Let $q = q_0 + 1 \geq 3$. Since $\{\log_2 \ 3^2\} + \{\log_2 \ 3^4\} < 1$ and $G \cong H \oplus C_2^{q_0} \oplus C_3^4$, by Lemma 4.1.3,

$$D_{\pm}(C_2^q \oplus C_3^6) = \lfloor \log_2 \ 2^q 3^6 \rfloor + 1.$$

□

By Corollay 4.1.4, using similar proof methods, for $q \geq 4$, one can show that

$$D_{\pm}(C_2^q \oplus C_3^8) = \lfloor \log_2 \ 2^q 3^8 \rfloor + 1.$$

Using the result directly above, one can then show that

$$D_{\pm}(C_2^q \oplus C_3^{10}) = \lfloor \log_2 \ 2^q 3^{10} \rfloor + 1.$$

From data, we have found that this is where this pattern ends. For when $G = C_2^q \oplus C_3^7$, we conjecture that $D_{\pm}(G)$ will not obtain the upper bound for $q \geq 1$.

Table 4.1: Known values of $D_{\pm}(C_2^q \oplus C_3^r)$.

q	r	$D_{\pm}(C_2^q \oplus C_3^r)$
$q = 1$	$1 \leq r \leq 3$	$\lfloor \log_2 \ 3^r \rfloor + q + 1$
$q = 1$	$4 \leq r \leq 9$	$r + 3 < \lfloor \log_2 \ 3^r \rfloor + q + 1$
$q \geq 2$	$r \leq 5$	$\lfloor \log_2 \ 3^r \rfloor + q + 1$
$q \geq 3$	$r = 6$	$\lfloor \log_2 \ 3^r \rfloor + q + 1$
$q \geq 4$	$r \in \{8, 10\}$	$\lfloor \log_2 \ 3^r \rfloor + q + 1$

This table summarizes the values that are computed in this section. In the second row, $De_{\pm}(C_2 \oplus C_3^r) = r + 3$, as shown in Theorem 3.4.2. Outside of these values, $D_{\pm}(C_2^q \oplus C_3^r)$ is unknown. We believe for $4 \leq r \leq 9$ and $q \geq 2$ that $D_{\pm}(C_2^q \oplus C_3^r) = r + q + 2$ but we currently have no proof of this.

Conjecture 4.1.5. *Let* $q \geq 1$.

$$D_{\pm}(C_2^q \oplus C_3^7) < \lfloor \log_2 \ 2^q 3^r \rfloor + 1.$$

For $q \geq 4$, let $G = C_2^q \oplus C_3^7$ and let $H = C_2^3 \oplus C_3^6$. Notice $G/H \cong C_2^{q-3} \oplus C_3$. We have previously shown that H and G/H attain the upper bound. Then, by Lemma 2.3.10, we can increase the lower bound, so

$$\begin{aligned} D_{\pm}(H) + D_{\pm}(G/H) - 1 \leq D_{\pm}(C_2^q \oplus C_3^7) &\leq \lfloor \log_2 \ 2^q 3^7 \rfloor + 1 \\ 13 + (2 + q - 3) - 1 \leq D_{\pm}(C_2^q \oplus C_3^7) &\leq \lfloor \log_2 \ 2^q 3^7 \rfloor + 1 \\ 11 + q \leq D_{\pm}(C_2^q \oplus C_3^7) &\leq 12 + q. \end{aligned}$$

The following results follows from Proposition 3.6.3.

Corollary 4.1.6. *Let* $q \geq 1$. *Then*

$$De_{\pm}(C_2^q \oplus C_3^2) = q + 5.$$

Proof. Let $q \geq q_0 = 1$. By Proposition 4.1.1,

$$D_{\pm}(C_2^q \oplus C_3^2) = \lfloor \log_2 \ 2^q 3^2 \rfloor + 1 = q + 4,$$

for all $q \geq 1$. Then by Proposition 2.4.5,

$$\begin{aligned} D_{\pm}(C_2^q \oplus C_3^2) \leq De_{\pm}(C_2^q \oplus C_3^2) &\leq D_{\pm}(C_2^{q+1} \oplus C_3^2) \\ q + 4 \leq De_{\pm}(C_2^q \oplus C_3^2) &\leq q + 5 \end{aligned}$$

We will show that

$$De_{\pm}(C_2 \oplus C_3^2) = 6.$$

Let

$$M = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = [m_1, m_2, \ldots, m_5],$$

where u_1 has entries in C_2 and u_2, u_3 has entries in C_3. Since the only PM zero-subsum of the first four columns of M have length three, if M contains an even PM zero-subsum, it must use m_5. Since the first entry of m_5 is the only entry of u_1 that is non-zero, M does not contain an even PM zero-subsum. Therefore

$$5 < De_{\pm}(C_2 \oplus C_3^2) \leq 6,$$

and hence $De_{\pm}(C_2 \oplus C_3^2) = 6$. For $q \geq 1$, we meet the conditions of Proposition 3.6.3 and thus,

$$De_{\pm}(C_2^q \oplus C_3^2) = q + 5.$$

□

Conjecture 4.1.7. *If*

$$D_{\pm}(C_2^q \oplus C_3^r) = \lfloor \log_2 \ 3^r \rfloor + q + 1,$$

then

$$De_{\pm}(C_2^q \oplus C_3^r) = \lfloor \log_2 \ 3^r \rfloor + q + 2.$$

4.2 Plus-Minus Davenport for $G = C_p^m \oplus C_q^n$

Let $G \cong C_p^m \oplus C_q^n$, where p, q are primes and $m, n \geq 1$.

Lemma 4.2.1. *Let* $k = \min\{p, q\}$*. Then*

$$D_{\pm}^*(G) = k\lfloor \log_2 \ pq \rfloor + (m-k)\lfloor \log_2 \ p \rfloor + (n-k)\lfloor \log_2 \ q \rfloor + 1.$$

Proof. Let $k = \min\{p, q\}$. Let $0 \leq j \leq k$. Then all possible decompositions of G have the form

$$G \cong C_{pq}^j \oplus C_p^{m-j} \oplus C_q^{n-j}.$$

Then

$$D_{\pm}^*(G) \geq j\lfloor \log_2 \ pq \rfloor + (m-j)\lfloor \log_2 \ p \rfloor + (n-j)\lfloor \log_2 \ q \rfloor + 1.$$

In general,

$$\lfloor \log_2 \ pq \rfloor \geq \lfloor \log_2 \ p \rfloor + \lfloor \log_2 \ q \rfloor.$$

Hence, by defintion of $D_{\pm}^*(G)$,

$$\begin{aligned} D_{\pm}^*(G) &= \max_{0 \leq j \leq k} \{j\lfloor \log_2 \ pq \rfloor + (m-j)\lfloor \log_2 \ p \rfloor + (n-j)\lfloor \log_2 \ q \rfloor + 1\} \\ &= k\lfloor \log_2 \ pq \rfloor + (m-k)\lfloor \log_2 \ p \rfloor + (n-k)\lfloor \log_2 \ q \rfloor + 1. \end{aligned}$$

□

Corollary 4.2.2. *If* $\{\log_2 \ p\} + \{\log_2 \ q\} < 1$*, then*

$$D_{\pm}^*(G) = m\lfloor \log_2 \ p \rfloor + n\lfloor \log_2 \ q \rfloor + 1.$$

Proof. Since $\{\log_2\ p\} + \{\log_2\ q\} < 1$, then

$$\lfloor \log_2\ pq \rfloor = \lfloor \log_2\ p \rfloor + \lfloor \log_2\ q \rfloor.$$

Hence, for $0 \le j \le k$,

$$D^*_\pm(G) = j\lfloor \log_2\ pq \rfloor + (m-j)\lfloor \log_2\ p \rfloor + (n-j)\lfloor \log_2\ q \rfloor + 1.$$

When $j = 0$, we have the desired result. $\square$

Lemma 4.2.3. *Let $G = C_p^m \oplus C_q^n$ and let $H = C_p^r \oplus C_q^s$ be a subgroup of G. So, $r \le m$ and $s \le n$. Assume $\{\log_2\ p\} + \{\log_2\ q\} < 1$. Suppose $D_\pm(H) - D^*_\pm(H) = \ell$. Then*

$$D_\pm(G) \ge D^*_\pm(G) + \ell.$$

Proof. Assume $D_\pm(H) - D^*_\pm(H) = \ell$, for some $\ell \in \mathbb{N}$. Notice that $G/H \cong C_p^{m-r} \oplus C_q^{n-s}$. Thus, by Lemma 2.3.3 and Corollary 4.2.2 applied to both G and H,

$$\begin{aligned}
D_\pm(G) &\ge D_\pm(H) + D_\pm(G/H) - 1 \\
&\ge D_\pm(H) + (m-r)\lfloor \log_2\ p \rfloor + (n-s)\lfloor \log_2\ q \rfloor + 1 - 1 \\
&= D_\pm(H) + m\lfloor \log_2\ p \rfloor + n\lfloor \log_2\ q \rfloor - \{r\lfloor \log_2\ p \rfloor + s\lfloor \log_2\ q \rfloor + 1\} \\
&= m\lfloor \log_2\ p \rfloor + n\lfloor \log_2\ q \rfloor + 1 + D_\pm(H) - D^*_\pm(H) \\
&= D^*_\pm(G) + \ell.
\end{aligned}$$

$\square$

Corollary 4.2.4. *Let $k = \min\left\{ \left\lfloor \frac{m}{r} \right\rfloor, \left\lfloor \frac{n}{s} \right\rfloor \right\}$. Let $G = C_p^m \oplus C_q^n$ and let $H = C_p^r \oplus C_q^s$ be a subgroup of G. So, $r \le m$ and $s \le n$. Assume $\{\log_2\ p\} + \{\log_2\ q\} < 1$. Suppose $D_\pm(H) - D^*_\pm(H) = \ell$. Then*

$$D_\pm(G) \ge D^*_\pm(G) + k\ell.$$

Proof. Notice that

$$G \cong H^k \oplus \left(C_p^{m-rk} \oplus C_q^{n-sk} \right).$$

Then we can apply Lemma 2.3.3 k times to get

$$\begin{aligned}
D_\pm(G) &\ge kD_\pm(H) + D_\pm(C_p^{m-rk} \oplus C_q^{n-sk}) - k \\
&\ge kD_\pm(H) + (m-rk)\lfloor \log_2\ p \rfloor + (n-sk)\lfloor \log_2\ q \rfloor + 1 - k \\
&= D^*_\pm(G) + k\left(D_\pm(H) - r\lfloor \log_2\ p \rfloor - s\lfloor \log_2\ q \rfloor - 1\right) \\
&= D^*_\pm(G) + k\left(D_\pm(H) - D^*_\pm(H)\right) \\
&= D^*_\pm(G) + k\ell.
\end{aligned}$$

$\square$

Table 4.2: Values for 4.2.5.

p	q	N
3	7	2
3	11	22
3	13	3
5	7	7
5	13	44
7	11	3
7	13	1

In the case when $\{\log_2\ p\} + \{\log_2\ q\} > 1$, we do not have an improvement on the lower bound. When $m = n$, then $C_p^m \oplus C_q^m \cong C_{pq}^m$. After some computations, we found that there exists a maximal N such that for every $m \leq N$,

$$D_\pm(C_{pq}^m) = \lfloor \log_2\ (pq)^m \rfloor + 1.$$

Since values of N differ between different prime pairs, this lead us to the following question:

Question 4.2.5. Let $G = C_p^m \oplus C_q^m$ and assume $\{\log_2\ p\} + \{\log_2\ q\} > 1$. What is the largest N such that the basic lower bound equals the basic upper bound for all $m \leq N$? This is asking when

$$m\left(\lfloor \log_2\ p\rfloor + \lfloor \log_2\ q\rfloor\right) = \lfloor \log_2\ (pq)^m \rfloor,$$

for all $m \leq N$.

Table 4.2 shows some computational values of N for primes p, q. We are interested in knowing how much $\{\log_2\ p^m\} + \{\log_2\ q^n\}$ comes into play with the value of N. In the next section, we provide applications of the general results above for when p and q are small.

4.2.1 Values when p and q are small

Let G be the finite abelian group $C_3^m \oplus C_5^n$ for this section. In the introduction, Theorem 2.3.21, $D_\pm(C_5 \oplus C_{15})$ was not known at the time of publication [20]. We were able to verify computationally that $D_\pm(C_5 \oplus C_{15}) = 6$ and a proof for this result can be found in Appendix A, Theorem A.5 using different methods than presented in 20. Even though this group is relatively small, there were some challenges in providing a proof for $D_\pm(C_5 \oplus C_{15})$. In this section, we work through some small results to improve the lower bound for our general finite abelian group $G = C_3^m \oplus C_5^n$.

Recall that

$$\begin{aligned} D_\pm(C_3^m) &= m + 1 && \text{Theorem 2.3.12} \\ D_\pm(C_5^n) &= 2n + 1 && \text{Theorem 2.3.13.} \end{aligned}$$

Currently, the only known results of $D_\pm(G)$, for $m, n \geq 1$, are

$$\begin{aligned} D_\pm(C_3 \oplus C_5) &= D_\pm(C_{15}) = 4 && \text{By Lemma 2.3.2} \\ D_\pm(C_3^2 \oplus C_5) &= D_\pm(C_3 \oplus C_{15}) = 6 && \text{By Lemma 2.3.19} \\ D_\pm(C_3 \oplus C_5^2) &= D_\pm(C_5 \oplus C_{15}) = 6 && \text{Proof provided in Section 4.3.} \end{aligned}$$

We begin by showing the basic lower bound with respect to m and n.

Lemma 4.2.6.

$$D_\pm(C_3^m \oplus C_5^n) \geq D_\pm^*(C_3^m \oplus C_5^n) = m + 2n + 1.$$

Table 4.3: Known values and bounds for $D_{\pm}(C_3^m \oplus C_5^n)$.

m	n	m + 2n +1	Lower Bound	Upper Bound	$D_{\pm}(C_3^m \oplus C_5^n)$
m	0	m+1	-	$m \cdot \log_2\ 3 + 1$	m+1
0	n	2n + 1	-	$n \cdot \log_2\ 5 + 1$	2n + 1
1	1	4	-	4	4
2	1	5	5	6	6
1	2	6		7	6
2	2	7	8	8	8
3	1	6	7	8	
1	3	8	-	9	
3	2	8	9	10	
2	3	9	10	11	
3	3	10	11	12	
4	1	7	8	9	
1	4	10	-	11	

Proof. Since $\{\log_2\ 3\} + \{\log_2\ 5\} < 1$, Corollary 4.2.2 implies

$$D_{\pm}^*(C_3^m \oplus C_5^n) = m\lfloor \log_2\ 3 \rfloor + n\lfloor \log_2\ 5 \rfloor + 1 = m + 2n + 1.$$

Then by Corollary 2.3.8,

$$D_{\pm}(C_3^m \oplus C_5^n) \geq m + 2n + 1.$$

□

These results above, along with others in this section, are provided in Table 4.3. The third column provides the basic lower bound, $D_{\pm}^*(C_3^m \oplus C_5^n)$ given in Lemma 4.2.6. The fourth column provides an improvement to the lower bound given by Lemma 4.2.8. The hash mark implies that there was no improvement to the basic lower bound. The fifth column is the value of the basic upper bound. The sixth column provides the known values of $D_{\pm}(C_3^m \oplus C_5^n)$. Spaces are left blank for where the value is not currently known.

Lemma 4.2.7. *Let $G = C_3^m \oplus C_5^n$ where $m \geq 2$ and $n \geq 1$.*

$$D_{\pm}(C_3^m \oplus C_5^n) \geq D_{\pm}^*(C_3^m \oplus C_5^n) + 1.$$

Proof. Let $H = C_3^2 \oplus C_5$. By Lemma 2.3.19, we know that $D_{\pm}(H) = 6$ and from above $D_{\pm}^*(H) = 5$. Therefore, $D_{\pm}(H) - D_{\pm}^*(H) = 1$. Then by Lemma 4.2.3,

$$D_{\pm}(G) \geq D_{\pm}^*(G) + 1.$$

□

Lemma 4.2.8. *Let $G = C_3^m \oplus C_5^n$ where $m \geq 2$ and $n \geq 1$. Let $k = \min\left\{\left\lfloor \frac{m}{2} \right\rfloor, n\right\}$. Then*

$$D_{\pm}(C_3^m \oplus C_5^n) \geq m + 2n + k + 1 = D_{\pm}^*(C_3^m \oplus C_5^n) + k.$$

Proof. Let $H = C_3^2 \oplus C_5$. Then $D_{\pm}(H) - D_{\pm}^*(H) = 1$. By Corollary 4.2.4,

$$D_{\pm}(G) \geq D_{\pm}^*(G) + k = m + 2n + k + 1.$$

□

Corollary 4.2.9.

$$D_{\pm}(C_3^2 \oplus C_5^2) = 8.$$

Proof. Notice that our basic upper bound is

$$D_{\pm}(C_3^2 \oplus C_5^2) \leq \lfloor \log_2\ 3^2 5^2 \rfloor + 1 = 8.$$

Let $k = \min\left\{\left\lfloor \frac{m}{2} \right\rfloor, n\right\} = 1$. Then by Lemma 4.2.8, then

$$2 + 2 \cdot 2 + 1 + 1 = 8 \leq D_{\pm}(C_3^4 \oplus C_5^2).$$

Notice the lower bound equals the basic upper bound. Therefore

$$D_{\pm}(C_3^2 \oplus C_5^2) = 8.$$

□

Corollary 4.2.10.

$$D_{\pm}(C_3^4 \oplus C_5^2) = 11.$$

Proof. Notice that our basic upper bound is

$$D_{\pm}(C_3^4 \oplus C_5^2) \leq \lfloor \log_2\ 3^4 5^2 \rfloor + 1 = 11.$$

Let $k = \min\left\{\left\lfloor \frac{m}{2} \right\rfloor, n\right\} = 2$. Then by Lemma 4.2.8, then

$$4 + 2 \cdot 2 + 2 + 1 = 11 \leq D_{\pm}(C_3^4 \oplus C_5^2).$$

Notice the lower bound equals the basic upper bound. Therefore

$$D_{\pm}(C_3^4 \oplus C_5^2) = 11.$$

□

Suppose $m + n \in \{4, 5\}$. The basic upper bound minus the lower bound in Lemma 4.2.8 equals one or zero. The cases where the difference equals zero are shown above. While this gives us an improvement, we currently do not know $D_{\pm}(C_3^m \oplus C_5^n)$ when the difference is one.

Conjecture 4.2.11.

$$D_{\pm}(C_3^3 \oplus C_5) = 7$$
$$D_{\pm}(C_3 \oplus C_5^3) = 8$$

both attaining their corresponding lower bounds. While

$$D_{\pm}(C_3^4 \oplus C_5^5) = 17$$
$$D_{\pm}(C_3^4 \oplus C_5^8) = 23$$

both attaining their corresponding basic upper bounds.

By Proposition 2.3.2 and 2.3.20, when $G = C_7^r$ and $r \in \{1, 2\}$, then

$$D_{\pm}(C_7^r) = \lfloor \log_2\ 7^r \rfloor + 1.$$

Currently, the values of $D_{\pm}(C_7^r)$ are currently unknown when $r \geq 3$. After computations, we have the following conjecture.

Conjecture 4.2.12.

$$D_{\pm}(C_7^3) = 8.$$

Also, more generally,

$$D_{\pm}(C_7^r) < \lfloor \log_2\ 7^r \rfloor + 1.$$

This would break the pattern of the previous primes, $p \in \{3, 5\}$.

Lemma 4.2.13. *Let $k = \lfloor \frac{r}{2} \rfloor$ for $r \geq 4$, then*

$$D_{\pm}(C_7^r) \geq \begin{cases} 5k+1 & \text{when } r \text{ is even} \\ 5k+3 & \text{when } r \text{ is odd} \end{cases}$$

Proof. Let $H = C_7^2$. From Lemma 2.3.3,

$$D_{\pm}(G) \geq D_{\pm}(H) + D_{\pm}(G/H) - 1.$$

Let $G_1 = G/H$. We know $D_{\pm}(H) = 6$ ([20], Prop 2.3.15) and $G_1 \cong C_7^{r-2}$. Next, we get a lower bound for $G_1 \cong C_7^{r-2}$ and use the same result to get

$$D_{\pm}(G_1) \geq D_{\pm}(H) + D_{\pm}(G_1/H) - 1.$$

This provides,

$$\begin{aligned} D_{\pm}(G) &\geq D_{\pm}(H) + D_{\pm}(G/H) - 1 \\ &\geq 2D_{\pm}(H) + D_{\pm}(G_1/H) - 2 \end{aligned}$$

Let $G_2 = G_1/H$. In general, $G_i = G_{i-1}/H$ and we see that $G_i = C_7^{r-2i}$. Continue this process inductively. Next, we look at the the cases when r is even or odd. First consider the case when r is even.

We continue to formulate the lower bound G_i for $i \in 1, \ldots, k-1$ for each i. Notice in the $k-1$, since r is even, $G_{k-1}/H = C_7^2$. This provides at most $kD_{\pm}(H)$ under this operation, including G_{k-1}/H. Hence,

$$D_{\pm}(G) \geq kD_{\pm}(H) - (k-1).$$

Thus, we get the even result,

$$D_{\pm}(G) \geq 6k - (k-1) = 5k + 1.$$

Now, for the case when r is odd.

Similar to the even case we continue to formulate the lower bound G_i for $i \in 1, \ldots, k$ for each i. Notice, in this case we will have to do this k times, verses $k-1$, since $G_{k-1}/H \cong C_7^3$. This provides at most k $D_\pm(H)$ under this operation. Thus,

$$D_\pm(G) \geq kD_\pm(H) + D_\pm(C_7) - k.$$

Thus, we get the even result,

$$D_\pm(G) \geq 6k - k + 3 = 5k + 3.$$

Therefore, we have the result for both the even and odd case.

$\square$

Next, we let $G = C_3^m \oplus C_7^r$. Since $D_\pm(C_3^m) = m+1$ and $\{\log_2\ 3\} + \{\log_2\ 7\} > 1$, this is a slightly different problem then when considering $C_3^m \oplus C_5^n$. Let $k = \min\{m, r\}$. By Lemma 4.2.1 and using the basic upper bound, the PM Davenport bounds are

$$\begin{aligned} k\lfloor \log_2\ 21 \rfloor + (m-k)\lfloor \log_2\ 3 \rfloor + (n-k)\lfloor \log_2\ 7 \rfloor + 1 \leq D_\pm(C_3^m \oplus C_7^r) &\leq \lfloor \log_2\ 3^m 7^r \rfloor + 1 \\ k + m + 2n + 1 \leq D_\pm(C_3^m \oplus C_7^r) &\leq \lfloor \log_2\ 3^m 7^r \rfloor + 1 \end{aligned}$$

Lemma 4.2.14.

$$D_\pm(C_3^2 \oplus C_7) = 6.$$

Proof. Notice that $C_3^2 \oplus C_7 \cong C_{21} \oplus C_3$. Let

$$H_1 = C_{21} \qquad H_2 = C_3.$$

We know that $D_\pm(H_i) = \lfloor \log_2\ |H_i| \rfloor + 1$ and one can verify that $\{\log_2\ 21\} + \{\log_2\ 3\} < 1$. Then by Lemma 2.3.10,

$$\begin{aligned} D_\pm(C_{21} \oplus C_3) &\geq D_\pm(C_{21}) + D_\pm(C_3) - 1 \\ &= 5 + 2 - 1 = 6. \end{aligned}$$

Notice that the basic upper bound is

$$D_\pm(C_3^2 \oplus C_7) = \lfloor \log_2\ 3^2 7 \rfloor + 1 = 6.$$

Since the lower and upper bound are equal, we have the desired result. $\square$

Lemma 4.2.15.

$$D_\pm(C_3^2 \oplus C_7^2) = 9.$$

Proof. Let $H = C_{21}$ and notice that $G/H \cong H$. Since $2\{\log_2\ 21\} < 1$, by Lemma 2.3.10 ,

$$D_\pm(C_3^2 \oplus C_7^2) = \lfloor \log_2\ 21^2 \rfloor + 1 = \lfloor \log_2\ 3^2 7^2 \rfloor + 1 = 9.$$

Thus, $D_\pm(C_3^2 \oplus C_7^2)$ attains the upper bound. $\square$

Lemma 4.2.16.
$$D_\pm(C_3^4 \oplus C_7^3) = \lfloor \log_2\ 3^4 7^3 \rfloor + 1 = 15.$$

Proof. Let $H_1 = C_3^2 \oplus C_7$ then $G/H = H_2 \cong C_3^2 \oplus C_7^2$. By Lemma 4.2.14 and 4.2.15 , $D_\pm(H_i)$ both attain the basic upper bound. One can verify that $\{\log_2\ |H_1|\} + \{\log_2\ |H_2|\} < 1$. Then by Lemma 2.3.10, $D_\pm(G)$ also attains the upper bound. $\square$

Question 4.2.17. Does there exists an $N \in \mathbb{N}$ where

$$D_\pm(C_3^m \oplus C_7^r) < \lfloor \log_2\ 3^m 7^r \rfloor + 1,$$

for all $m, r \geq N$?

4.3 Plus-Minus Davenport for when $100 < |G| \leq 200$.

Let G be a finite abelian group. Theorem 2.3.21 shows that the majority of groups G attain the basic upper bound when $1 \leq |G| \leq 100$, with only four exceptional cases, where each has been shown to obtain the basic lower bound provided in Definition 2.3.6. Proposition 2.3.2 provides that all cyclic groups attain the basic upper bound. Also, let H and K be subgroups of G such that $G \cong H \oplus K$. If H is a 2-group and K is cyclic, one can use Lemma 2.3.10 to show that $D_\pm(G)$ attains the basic upper bound. By Theorems 2.3.12 and 2.3.13, if G is a p-group where $p \in \{3, 5\}$, then

$$D_\pm(C_p^r) = \begin{cases} r+1 & \text{if } p = 3 \\ 2r+1 & \text{if } p = 5 \end{cases}.$$

Notice that both of these p-group both attain the basic lower bound.

In this section, we are interested in calculating $D_\pm(G)$ when $100 < |G| \leq 200$. We begin by understanding the prime factorizations that fall between these two bounds. Since $|G| \leq 200$, one can verify that $|G|$ is the product of at most three prime factors with an exponent greater than one. Let $|G| = n$ where $100 < n \leq 200$. Let $p_i^{\alpha_i}$ be a prime with exponent α_i. If $n = p_0 p_1^{\alpha_1} p_2^{\alpha_2}$, where $\alpha_1, \alpha_2 \geq 1$ and $p_0 < p_1 < p_2$, then $p_0 \in \{2, 3\}$. To show this assume that $p_0 \geq 5$, one can see that $n \geq 5 \cdot 7 \cdot 11 > 200$. So, the statement above holds.

Recall the following result provided in Chapter 2, Proposition 2.3.20.

Proposition 4.3.1. *Let $n \geq 2$ be an integer with either $\{\log_2\ (p^j n)\} < 1 - \{\log_2 p^k\}$ or $\{\log_2\ (p^j n)\} \geq \{\log_2 p^k\} \geq \{\log_2 3\}$ for prime $p \geq 3$. Then*

$$D_\pm(C_{p^k} \oplus C_{p^j n}) = \lfloor \log_2(p^{k+j} n) \rfloor + 1.$$

Notice that when $2 \nmid n$ and G is not cyclic, one can verify that n can only be product of at most 2 primes. Now, assume that $2 \nmid n$ and $3 \mid n$, so $n = 3^{\alpha_0} p^\alpha$. Since $100 < n \leq 200$, we divide both sides by 3^{α_0}, which provides that

$$p \in \{5, 7, 13, 17, 19\} \qquad \text{and} \qquad 3 \leq \alpha_0 + \alpha \leq 4.$$

We can apply Proposition 4.3.1 to compute the following $D_{\pm}(G)$. Assume $n = 3^2 p$. Then $p \in \{13, 17, 19\}$. By the sum of the fractional parts of $\{\log_2\ 3\} + \{\log_2\ 3 \cdot p\}$ and by Proposition 4.3.1,

$$D_{\pm}(C_3 \oplus C_{3p}) = \lfloor \log_2\ 9p \rfloor + 1,$$

which is the basic upper bound. From the list above we are left to checking when

$$n \in \{3 \cdot 7^2, 3^3 \cdot 5, 3^3 \cdot 7\}.$$

If $6 \nmid n$ and n has exactly two prime factors then $n = 5^2 7$. By our bounds of n and if $n = p^\alpha$, then $p \in \{11, 13\}$. Recall the following result provided in Proposition 2.3.20. Similarly, when $n = 5^2 \cdot 7$, $D_{\pm}(C_5 \oplus C_{35}) = \lfloor \log_2\ 5^2 \cdot 7 \rfloor + 1$. When $G \cong C_5 \oplus C_{5p}$, the first prime where this Proposition does not imply the basic upper bound is obtain is when $p = 23$. When $G \cong C_3 \oplus C_{3p}$, the first prime where this Proposition does not imply the basic upper bound is obtain is when $p = 29$.

Next, let $n = 2 \cdot 3^2 \cdot p$ when $p \in \{7, 11\}$. Notice that $G \cong C_3 \oplus C_{3k}$ where $k = 2p$. By Proposition 4.3.1, we get that $D_{\pm}(G) = \lfloor \log_2\ |G| \rfloor + 1$ for each value of p. Furthermore, let $n = 3^3 p$, so $p \in \{5, 7\}$. Using the basic bounds, we know that

$$\begin{aligned} 6 \leq & D_{\pm}(C_3^3 \oplus C_5) \leq 8 \\ 7 \leq & D_{\pm}(C_3^3 \oplus C_7) \leq 8 \end{aligned}$$

Conjecture 4.3.2.

$$D_{\pm}(C_3^3 \oplus C_p) = 7,$$

for $p \in \{5, 7\}$.

Now, we continue to consider when $2 \mid n$. By our assumptions for when $G \cong H \oplus K$ where H is a 2-group, then $|K|$ has at most two distinct prime factors with exponent greater than one. Let $n = 2^\alpha 3^\beta$ where $\alpha, \beta \geq 1$. One can verify that $1 \leq \alpha, \beta \leq 4$. Due to the results from $De_{\pm}(C_3^r)$, for when $\beta \leq 3$, we know that

$$D_{\pm}(C_2^\alpha \oplus C_3^\beta) = \lfloor \log_2\ 2^\alpha 3^\beta \rfloor + 1.$$

If $\beta = 4$, then $\alpha = 1$ we have previously shown

$$D_{\pm}(C_2 \oplus C_3^4) = 7 < \lfloor \log_2\ 2 \cdot 3^4 \rfloor + 1 = 8.$$

This is the first example where $D_{\pm}(G)$, for $100 < |G| \leq 200$, neither attains the basic upper bound or basic lower bound.

We are left with the case where $n = 2^\alpha p^\beta$ where $p \neq 3$. In this case, one can verify that this leaves us with only two cases, $n \in \{2^2 \cdot 7^2, 2^3 \cdot 5^2\}$. In both cases, we know that for $K = C_p^2$, where $p \in 5, 7$, then $D_{\pm}(K)$ obtains the basic upper bound. Hence by Lemma 2.3.10, $D_{\pm}(C_2^\alpha \oplus K)$ also obtains the basic upper bound.

Now, we consider the case where n has three distinct primes factors and $2 \mid n$, so $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}$ where there exists an $\alpha_i > 1$. We first consider the case where $p_1 = 3$ and the sum of $\alpha_1 + \alpha_2 = 3$. When $\alpha_0 > 0$, then $p_1^{\alpha_1} p_2^{\alpha_2} = 3^2 \cdot 5$. For previous results we know for all groups $|K| = 3^2 \cdot 5$, $D_\pm(K) = \lfloor \log_2 \ |K| \rfloor + 1$. Then $G \cong H \oplus K$ where H is a 2-group and K is cyclic and thus we have shown this also attains the basic upper bound. Next let $\alpha_0 = 1$, then

$$p_1^{\alpha_1} p_2^{\alpha_2} \in \{3^2 \cdot 7, 3^2 \cdot 11, 3 \cdot 5^2\}.$$

Then by Proposition 4.3.1, we know that each $K = C_{p_1}^{\alpha_1} \oplus C_{p_2}^{\alpha_2}$ attains the basic upper bound. Then by Lemma 2.3.10, $D_\pm(C_2 \oplus K)$ obtains the basic upper bound.

We have proved the following theorem.

Theorem 4.3.3. *Let G be a finite abelian group. If $100 < |G| \leq 200$, then*

$$D_\pm(G) = \lfloor \log_2 \ G \rfloor + 1,$$

with the possible exception of the following cases,

$$G \in \left\{C_5^3, C_2 \oplus C_3^4, C_3 \oplus C_7^2, C_3^3 \oplus C_5, C_3^3 \oplus C_7\right\},$$

where

$$\begin{aligned} D_\pm(C_5^3) =& 7 = \lfloor \log_2 \ 5^3 \rfloor + 1 \\ D_\pm(C_2 \oplus C_3^4) =& 7 < \{\log_2 \ 3^4\} + 2 = 8 \\ 7 \leq& D_\pm(C_3 \oplus C_7^2) \leq 8 \\ 7 \leq& D_\pm(C_3^3 \oplus C_5) \leq 8 \\ 7 \leq& D_\pm(C_3^3 \oplus C_7) \leq 8 \end{aligned}$$

From the results above, we have shown that the majority of the groups, G, where $100 < |G| \leq 200$ obtain the basic upper bound. The few groups that are conjectured or we have shown that they do not obtain the basic upper bound all have in common that $3^3 \mid n$. One could ask the following question.

Question 4.3.4. Let H be a subgroup of G where $G \cong C_3^3 \oplus H$ and $|H| = n$. Does there exist a subgroup H such that $D_\pm(G)$ attains the basic upper bound?

If the trends continues to fall below the basic upper bound then the next question would be to show that $C_3^r \oplus H$, for $r \geq 3$, also does not attain the basic upper bound. This would provide another class of groups that does not attain the basic upper bound. It would be interesting to know for which groups H does $C_3^r \oplus H$ attain the basic lower bound? Currently, we conjecture (3.6.9) that $C_2 \oplus C_3^r$ lies between the basic upper bound and basic lower bound.

Appendices

Appendix A: Proof of $D_{\pm}(C_{15} \oplus C_5)$

Let p be a prime and $\mathbb{F}_p$ be the field with $|\mathbb{F}_p| = p$. Let $a_1, \ldots, a_n \in \mathbb{F}_p$ and let $\tau = (a_1, \ldots, a_n)$. Let $b \in \mathbb{F}_p$. Let $N(\tau, b) = |\{(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n \mid \sum_{i=1}^{n} \epsilon_i a_i = b\}|$. Thus,

$$\sum_{b \in \mathbb{F}_p} N(\tau, b) = 2^n.$$

Let $\tau_i = (a_1, \ldots, a_i), 1 \leq i \leq n$. Thus, $\tau_n = \tau$.

Lemma A.1.

$$N(\tau_n, b) = N(\tau_{n-1}, b) + N(\tau_{n-1}, b - a_n).$$

Example A.2. Suppose $a_n = 0$. Then

$$N(\tau_n, b) = 2N(\tau_{n-1}, b)$$

Example A.3.

$$N(\tau_n, b) = N(\tau_{n-2}, b) + N(\tau_{n-2}, b - a_{n-1}) + N(\tau_{n-2}, b - a_n) + N(\tau_{n-2}, b - a_{n-1} - a_n).$$

Example A.4. Assuming τ contains k zeros, then

$$N(\tau_n, b) = 2^k N(\tau_{n-k}, b).$$

Theorem A.5.

$$D_{\pm}(C_3 \oplus C_5^2) = 6$$

Proof. We have $D_{\pm}(C_3 \oplus C_5^2) \geq D_{\pm}(C_5^2) + D_{\pm}(C_3) - 1 = 5 + 2 - 1 = 6$. We need to show that $D_{\pm}(C_3 \oplus C_5^2) \leq 6$. Let

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \cdots, \begin{bmatrix} a_6 \\ b_6 \\ c_6 \end{bmatrix}$$

be given where $a_i, b_i \in C_5$ and $c_i \in C_3$ for $1 \leq i \leq 6$. We can perform standard row operations on the first two rows whenever it is convenient. Let $\tau = (a_1, \ldots, a_6)$. Suppose there exists $d \in C_5$ such that $N(\tau, d) \geq 16$. Since $16 > |C_5 \oplus C_3|$, there exists $\epsilon_i, \epsilon'_j \in \{0,1\}$ for $1 \leq i, j \leq 6$ such that

$$\sum_{i=1}^{6} \epsilon_i a_i = \sum_{i=1}^{6} \epsilon'_i a_i = d \in C_5$$
$$\sum_{i=1}^{6} \epsilon_i b_i = \sum_{i=1}^{6} \epsilon'_i b_i \in C_5 \text{ and}$$
$$\sum_{i=1}^{6} \epsilon_i c_i = \sum_{i=1}^{6} \epsilon'_i c_i \in C_3$$

where $(\epsilon_1, \ldots, \epsilon_6) \neq (\epsilon'_1, \ldots, \epsilon'_6)$.

Thus

$$\sum_{li=1}^{6}(\epsilon_i - \epsilon_i') \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in C_3 \oplus C_5^2.$$

Note that $\epsilon_i - \epsilon_i' \neq 0$ for at least one i. Let $\vec{v} = (a_1, \ldots, a_6)$ and $\vec{w} = (b_1, \ldots, b_6)$. A similar statement holds for each τ of the form $\lambda\vec{v} + \mu\vec{w}$ where $\lambda, \mu \in \mathbb{F}_5$ and $(\lambda, \mu) \neq (0, 0)$.

Let

$$M = \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} a_1 \ldots a_6 \\ b_1 \ldots b_6 \end{bmatrix}.$$

Note that columns can be permuted without changing the result. We need to consider the following two cases:
Case 1: Suppose some 2×2 minor of M is zero.
Case 2: Suppose each 2×2 minor of M is nonzero.

Case 1: By using row operations and the assumption, we can assume that

$$= \begin{bmatrix} a_1 \ldots a_4 & 0 & 0 \\ b_1 \ldots b_4 & b_5 & b_6 \end{bmatrix}.$$

Let $\tau = (\sigma\ 0\ 0)$ where $\sigma = (a_1 \ldots a_4)$. Then for each $d \in C_5$ we have $N(\tau, d) = 4N(\sigma, d)$. We have $\sum_{d \in \mathbb{F}_5} N(\sigma, d) = 2^4 = 16$. Since $\frac{16}{5} = 3.2$, there exists a $d_0 \in C_5$ such that $N(\sigma, d_0) \geq 4$. Then $N(\tau, d_0) \geq 16$. We are done by a previous agrument.

Case 2: Since $|\mathbb{P}^1(\mathbb{F}_5)| = 6$, we have that $a_i = 0$ for a unique value of i. The same holds for each $\lambda\vec{v} + \mu\vec{w}$. That is, each nonzero row in the row space of M has exactly one entry equal to zero. We need two calculations here:

$$N((1,1,1,1,1),3) = \binom{5}{3} = 10$$

$$N((1,1,1,1,2),3) = N((1,1,1,1),3) + N((1,1,1,1),1) = \binom{4}{3} + \binom{4}{1} = 8.$$

It follows that

$$N((1,1,1,1,1,0),3) = 2 \cdot 10 = 20 \geq 16 \quad \text{and}$$
$$N((1,1,1,1,2,0),3) = 2 \cdot 8 = 16 \geq 16.$$

Observed that we can multiply any column of M by -1 without changing the problem of computing $D_{\pm}(C_3 \oplus C_5^2)$. Recall that any row of M can be multiplied by any element of $\mathbb{F}_5^{\times}$. It follows that a row of M can always be brought to one of the following:

$$(1,1,1,1,1,0), (1,1,1,1,2,0), (1,1,1,2,2,0)$$

Notice that

- 4 can be changed to a 1;

- 3 can be changed to a 2; and
- we can multiply a row by 2 to assume the number of quadratic residue $\{1,4\}$ is greater than the number of quadratic nonresidues, $\{2,3\}$.

If either $(1,1,1,1,1,0)$ or $(1,1,1,1,2,0)$ occurs then we have $N(\tau,d) \geq 16$ and we are done. Let $d \in \{0,1,\ldots,4\}$. We find that

$$N((1,1,1,2,2),d) < 8 = N(\tau,d)$$

for each $d \in \mathbb{F}_5$. This proof will be complete unless $\lambda\vec{v}+\mu\vec{w}$ has at least two quadratic residues and at least two quadratic nonresidues for each $\lambda,\mu \in \mathbb{F}_5, (\lambda,\mu) \neq (0,0)$, i.e. $\lambda\vec{v}+\mu\vec{w} = (1,1,1,1,2,2)$.

We now show that this cannot occur. Let R denote the set quadratic residues and Q denote the set quadratic nonresidues. We are focusing on how many elements in a specific row are contained in R or in Q. Let $[|R|,|Q|]$ be the number of elements in a row that are quadratic residues and nonresidues. For example, $(1,1,1,1,2,2)$ corresponds to $[4,2]$.

We now write

$$\begin{aligned}\vec{v} &= \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & 0 & f_1 \end{pmatrix}\\ \vec{w} &= \begin{pmatrix} a_2 & b_2 & c_2 & d_2 & e_2 & 0 \end{pmatrix}\end{aligned}$$

where each element of $\vec{v},\vec{w} \neq 0$ unless stated otherwise. We first fix $\vec{v}$ and get four additional equations:

$$\begin{array}{rclllllll}
\vec{v}+\vec{w} &=& (a_1+a_2 & b_1+b_2 & c_1+c_2 & d_1+d_2 & e_2 & f_1)\\
\vec{v}+2\vec{w} &=& (a_1+2a_2 & b_1+2b_2 & c_1+2c_2 & d_1+2d_2 & 2e_2 & f_1)\\
\vec{v}+3\vec{w} &=& (a_1+3a_2 & b_1+3b_2 & c_1+3c_2 & d_1+3d_2 & 3e_2 & f_1)\\
\vec{v}+4\vec{w} &=& (a_1+4a_2 & b_1+4b_2 & c_1+4c_2 & d_1+4d_2 & 4e_2 & f_1)
\end{array}$$

Suppose each one is either $[3,2]$ or $[2,3]$. Recall, we can preform row operations without changing the $D_{\pm}(C_3 \oplus C_5^2)$, so we multiply by 2 to either $\vec{v}$ or $\vec{w}$ so they are both $[3,2]$. Hence, we can assume $\vec{v},\vec{w}$ are both the same.

Now, we have six different equations including $\vec{v},\vec{w}$. Let us focus on the elements containing a_1 and a_2,

$$\{a_1, a_2, a_1+a_2, a_1+2a_2, a_1+3a_2, a_1+4a_2\}.$$

If we ignore the element a_2, then the remaining elements will provide all elements of C_5. Hence, we can view this set simply as $\{0,1,2,3,4,a_2\}$. Similarly, we get the same result for each of the elements with b_i, c_i, d_i, and e_i. Note, all five f_1 remains the same.

Counting the number of solutions over all six equations, we get six $0's$ and five of all other elements of C_5. This tells us there are 13 elements of R, 12 elements of Q, plus the 5 $f_1's$. So we either have $[18,12]$ which implies that each set of equations contain $[3,2]$; or if we have $[13,17]$, then there exists on equation that has $[3,2]$ while the remain have $[2,3]$. Notice that the latter contradicts the assumption that $\vec{v}, \vec{w}$ are both $[3,2]$. Thus $f_1 \in R$.

When considering the case where we fix $\vec{w}$ instead of $\vec{v}$, we get a similar result where $e_2 \in R$. Thus every row must have $[3,2]$. Let us look more closely to the six equations:

$$\begin{aligned} &\{\vec{v}, \quad \vec{w}, \quad \vec{v}+\vec{w}, \quad 2\vec{v}+\vec{w}, \quad 3\vec{v}+\vec{w}, \quad 4\vec{v}+\vec{w}\} \\ = &\{\vec{v}, \quad \vec{w}, \quad \vec{v}+\vec{w}, \quad 2(\vec{v}+3\vec{w}), \quad 3(\vec{v}+2\vec{w}), \quad 4(\vec{v}+4\vec{w})\}. \end{aligned}$$

This shows that $2\vec{v}+\vec{w}$ has $[2,3]$ which is a contradiction to the assumption that every row must be $[3,2]$. Therefore, $\lambda\vec{v}+\mu\vec{w}$ can be brought to either $(1,1,1,1,1,0)$ or $(1,1,1,1,2,0)$ where we are always guarenteed to have a zero subsum. □

Appendix B: $De_{\pm}(C_2^n)$ and connections to Coding Theory

Let G be 2-group, so $G \cong C_2^n$ for some n and $|G| = 2^n$. Notice that each element has order two, i.e. for $g \in G$, $g^2 = \bar{0}$. Since 2 is prime, we know that $C_2^n \cong \mathbb{F}_2^n$ as vector spaces. So $\mathbb{F}_2^n$ is a vector space of dimension n where

$$\mathbb{F}_2^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{F}_2\}.$$

For $x \in \mathbb{F}_2^n$, x is commonly known as a binary number since it is a vector of length n that contains only zeros and ones. There is extensive work done on the set of binary numbers in coding theory. We present some of these basic coding theory results with our group notation.

Definition B.1. Let $x, y \in \mathbb{F}_2^n$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The *hamming distance* between x and y is the number of $x_i \neq y_i$, denoted by $d(x,y)$.

The *hamming distance* is defined over general $\mathbb{F}_p^n$ for prime p, not just $\mathbb{F}_2^n$.

Lemma B.2. *Let $x, y \in \mathbb{F}_2^n$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Let $g_x, g_y \in C_2^n$ be the group elements corresponding to $x, y \in \mathbb{F}_2^n$.*

$$d(x,y) = |sp(g_x + g_y)|.$$

Proof. Assume $d(x,y) = k$. By the definition of hamming distance, this tells us that the are exactly k $x_i \neq y_i$ for some $1 \leq i \leq n$. Without loss of generality, we can assume that the first k entries of x and y are not equal and all others are equal. Then

$$x_i + y_i = \begin{cases} \bar{1} & \text{for } 1 \leq i \leq k \\ \bar{0} & \text{for } k+1 \leq i \leq n \end{cases}$$

Since g_x, g_y have similar form to x and y, this also follows for the entires of g_x, g_y. Recall the $|sp(g)|$ is the number of non-zero entries for some g. Hence

$$|sp(g_x + g_y)| = k = d(x,y).$$

□

Lemma B.3. *Let $a, b \in C_2^n$ where $a \neq b$. Let $k \in \mathbb{N}$.*

1. *If $|sp(a)| = |sp(b)|$, then $|sp(a+b)| = 2k$.*
2. *If $\big||sp(a)| - |sp(b)|\big|$ is even, then $|sp(a+b)|$ is even.*

Proof. Before we begin the proofs for statements (1) and (2), we first show how one can generally compute $sp(g)$ for $g \in C_2^n$. Let $g \in C_2^n$ be an arbitrary element where $g = (g_1, g_2, \ldots, g_n)$. By using the definition of $sp(g)$, $|sp(g)|$ essencially counts the number of non-zero entries of g. Since every entry of g is zero or one, then

$$|sp(g)| = \sum_{i=1}^{n} g_i.$$

1. Let $1 \leq i \leq n$. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. The following statements are equivalent.

$$\begin{aligned} |sp(a)| = \sum_i a_i &= \sum_i b_i = |sp(b)| \\ \sum_i a_i - \sum_i b_i &\equiv 0 \mod 2 \quad \text{since we assumed } |sp(a)| = |sp(b)| \\ \sum_i a_i - b_i &\equiv 0 \mod 2 \\ \sum_i a_i + b_i &\equiv 0 \mod 2 \\ |sp(a+b)| &\equiv 0 \mod 2. \end{aligned}$$

   This tells us that $|sp(a+b)| = 2k$ for some k.

2. We use a similar method as above to get that

$$\sum_i a_i - \sum_i b_i \equiv \sum_i a_i + b_i \mod 2.$$

   Without loss of generality, assume that $|sp(a)| - |sp(b)| = 2k$ for some k. Then

$$2k = |sp(a)| - |sp(b)| = \sum_i a_i - \sum_i b_i \equiv \sum_i a_i + b_i \equiv |sp(a+b)| \mod 2.$$

   Thus, we get our desired result.

∎

Let $x, y, z \in \mathbb{N}$. Then there are least two numbers such that their difference is even. This follows from the fact that the sum of two even numbers is even and the sum of two odd numbers is even.

Lemma B.4. *Let $g_1, g_2, g_3 \in C_2^n$. Then there exists $i, j \in \{1, 2, 3\}$ such that*

$$|sp(g_i)| - |sp(g_j)| = 2k$$

for some $k \in \mathbb{N}$.

Proof. This follows from the statement above since $|sp(g_i)| \in \mathbb{N}$ for $1 \leq i \leq 3$. $\square$

Corollary B.5.

$$De_{\pm}(C_2^n) \leq n + 3.$$

Proof. Let $S \subset C_2^n \setminus \{0\}$ where $|S| = n + 3$. Then we can get the corresponding matrix M

$$M = [e_1, \ldots, e_n \mid g_1, g_2, g_3]$$

for some $g_i \in C_2^n$. Then from the lemma above it follows that there exists g_i and g_j such that $|sp(g_i + g_j)|$ is even. Thus, we have our even PM zero-subsum. $\square$

For when $n = 2$, we can simply use the pigeon hole principle to get our result. This method does not work for $n \geq 3$.

Proposition B.6.

$$De_{\pm}(C_2^2) = 4.$$

Proof. Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Notice that M does not contain an even PM zero-subsum. This tells us that $De_{\pm}(C_2^n) \geq 3$. Notice that C_2^2 contains only four elements. If we remove the zero element and select four random elements from C_2^2, by pigeon hole principle, this guarentees that two will be the same. Thus we have a PM zero-subsum with two element when we have at least four non-zero elements of C_2^2. Hence,

$$De_{\pm}(C_2^2) = 4.$$

$\square$

Theorem B.7. *For $n \geq 3$,*

$$De_{\pm}(C_2^n) = n + 2.$$

Using the results above, we have an alternative proof for Theorem 3.2.6.

Proof. Let $x \in C_2^n$ such that $|sp(x)|$ is even. Then consider

$$M_0 = [e_1, e_2, \ldots, e_n \mid x].$$

Since the first n columns are linearly independent and $|sp(x)|$ is even, M_0 does not contain an even PM zero-subsum. Thus,

$$De_{\pm}(C_2^n) > n + 1$$

which implies

$$De_{\pm}(C_2^n) \geq n+2.$$

Let $S \subset C_2^n \setminus \{0\}$ such that $|S| = n+2$. We construction our matrix M

$$M = [e_1, \ldots, e_n \mid a, b].$$

Notice that if either the $|sp(a)|$ or $|sp(b)|$ is odd, then have an even PM zero-subsum. We can assume that both $|sp(a)|$ and $|sp(b)|$ is even. Then by Lemma B.3.2, we have that $|sp(a+b)|$ is also even. Hence, we have our PM zero-subsum. □

Appendix C: Equal Arcs and Free Arc Pair Sets

Let $n \in \mathbb{N}$ where $n \geq 5$. Let $4 \leq k < n$. Denote $[n] = \{1, \ldots, n\}$. Let $S = \{s_1, \ldots, s_k\} \subset [n]$ where $|S| = k$ and $s_1 < s_2 < \cdots < s_k$. Define

$$S - S \setminus \{0\} = \{s_i - s_j \mid s_i, s_j \in S, i < j\}$$

as our non-zero difference set of S. Let $\{a, b, c, d\} \subset [k]$ where each one is distinct.

Question C.1. Given $S \subset [n]$ where $|S| = k$, what subsets S of size k does the following property hold

$$|s_a - s_b| \neq |s_c - s_d| \tag{1}$$

for any choice of $\{a, b, c, d\}$?

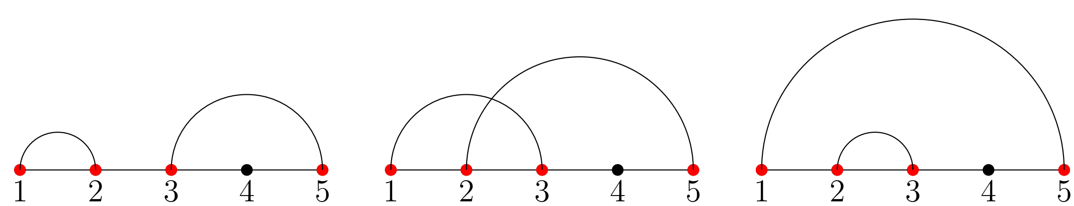
Notice that S must contain at least four elements to satisfy equation 1, hence why the lower bound on k is 4. Also, when $n = 4$ then $S = [4]$, thus $n \geq 5$. We can phrase this question by using symmetric arcs on a number line. An *arc* on a number line is the semicircle created when picking two points on the line. *Two arcs are disjoint* if they do not share a common node. The *arc diameter* is the distance between two nodes; i.e. let a, b be nodes on a number line that form an arc, then $|a - b|$ is the arc diameter. An *arc pair is symmetric* if the arc diameter is equal between two disjoint arches.

Question C.2. Let L_n be the number line from 1 to n and $S \subset [n]$ where $|S| = k < n$. Using nodes in S, we consider all possible disjoint arc pairs. What subsets $S \subset [n]$ do not contain any symmetric arc pairs?

Define S' to be the *reflective set of S* for when

$$S' = \{n + 1 - s_i \mid s_i \in S\}.$$

Example C.3. Let $n = 5$ and $k = 4$. Let $S = \{1, 2, 3, 5\}$. Then we have the following arc pairs.



Notice that there does not exist a symmetric arc pair. Namely, S is one such subset of $[5]$ that follows the property of equation 1; i.e.

$$|s_a - s_b| \neq |s_c - s_d|$$

for any $\{a, b, c, d\} \subset [5]$. S has a reflection set $S' = \{1, 3, 4, 5\}$ that also follows this property. When $n = 5$, S and S' are the only two sets that hold the property of equation 1.

Let $P(n, k)$ is set of $S \subset [n]$ where $|S| = k$ such that S does not contain a symmetric arch pair. It is clear that $P(n, k) \subset P(n + 1, k)$. To begin answering Question C.4.1, we first need to compute values of $|P(n, k)|$. The smallest values to consider are when $k = 4$ and $5 \leq n \leq 7$. To play with something slightly bigger, let $k = 5$ and $8 \leq n \leq 11$.

Question C.4.

1. Can we construct a recursion for fixed n or k for $P(n, k)$?
2. What is the relationship between when $|S| = k$ and $|S'| = k + 1$ for a fixed n.
3. Can we improve the upper bound on k when this is no longer possible? Right now we know $k < n$, can this get better?

Bibliography

[1] S. D. Adhikari, R. Balasubramanian, F. Pappalardi, and P. Rath. Some zero-sum constants with weights. *Proc. Indian Acad. Sci. Math. Sci.*, 118(2):183–188, 2008. ISSN 0253-4142. doi: 10.1007/s12044-008-0010-z. URL `https://doi-org.ezproxy.uky.edu/10.1007/s12044-008-0010-z`.

[2] S. D. Adhikari, Bidisha Roy, and Subha Sarkar. Weighted zero-sums for some finite abelian groups of higher ranks. In *Combinatorial and additive number theory. III*, volume 297 of *Springer Proc. Math. Stat.*, pages 1–12. Springer, Cham, 2020. doi: 10.1007/978-3-030-31106-3_1. URL `https://doi-org.ezproxy.uky.edu/10.1007/978-3-030-31106-3_1`.

[3] Sukumar Das Adhikari and Purusottam Rath. Davenport constant with weights and some related questions. *Integers*, 6:A30, 2006.

[4] Sukumar Das Adhikari and Purusottam Rath. Davenport constant with weights and some related questions. *Integers*, 6:A30, 6, 2006. ISSN 1553-1732.

[5] Sukumar Das Adhikari, David J. Grynkiewicz, and Zhi-Wei Sun. On weighted zero-sum sequences. *Adv. in Appl. Math.*, 48(3):506–527, 2012. ISSN 0196-8858. doi: 10.1016/j.aam.2011.11.007. URL `https://doi-org.ezproxy.uky.edu/10.1016/j.aam.2011.11.007`.

[6] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that really count*, volume 27 of *The Dolciani Mathematical Expositions*. Mathematical Association of America, Washington, DC, 2003. ISBN 0-88385-333-7. The art of combinatorial proof.

[7] Yair Caro. Zero-sum problems—a survey. *Discrete Math.*, 152(1-3):93–113, 1996. ISSN 0012-365X. doi: 10.1016/0012-365X(94)00308-6. URL `https://doi-org.ezproxy.uky.edu/10.1016/0012-365X(94)00308-6`.

[8] Yair Caro. Problems in zero-sum combinatorics. *J. London Math. Soc. (2)*, 55 (3):427–434, 1997. ISSN 0024-6107. doi: 10.1112/S0024610797005152. URL `https://doi-org.ezproxy.uky.edu/10.1112/S0024610797005152`.

[9] Scott T. Chapman, Michael Freeze, and William W. Smith. Minimal zero-sequences and the strong Davenport constant. *Discrete Math.*, 203(1-3):271–277, 1999. ISSN 0012-365X. doi: 10.1016/S0012-365X(99)00027-8. URL `https://doi-org.ezproxy.uky.edu/10.1016/S0012-365X(99)00027-8`.

[10] J. A. Dias da Silva and Hemar Godinho. Generalized derivations and additive theory. *Linear Algebra Appl.*, 342:1–15, 2002. ISSN 0024-3795. doi: 10.1016/S0024-3795(01)00406-2. URL `https://doi-org.ezproxy.uky.edu/10.1016/S0024-3795(01)00406-2`.

[11] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004. ISBN 0-471-43334-9.

[12] Yves Edel, Christian Elsholtz, Alfred Geroldinger, Silke Kubertin, and Laurence Rackham. Zero-sum problems in finite abelian groups and affine caps. *Q. J. Math.*, 58(2):159–186, 2007. ISSN 0033-5606. doi: 10.1093/qmath/ham003. URL `https://doi-org.ezproxy.uky.edu/10.1093/qmath/ham003`.

[13] P. Erdős and H. Heilbronn. On the addition of residue classes mod p. *Acta Arith.*, 9:149–159, 1964. ISSN 0065-1036. doi: 10.4064/aa-9-2-149-159. URL `https://doi-org.ezproxy.uky.edu/10.4064/aa-9-2-149-159`.

[14] Weidong Gao and Alfred Geroldinger. Zero-sum problems in finite abelian groups: a survey. *Expo. Math.*, 24(4):337–369, 2006. ISSN 0723-0869. doi: 10.1016/j.exmath.2006.07.002. URL `https://doi-org.ezproxy.uky.edu/10.1016/j.exmath.2006.07.002`.

[15] Hemar Godinho, Abílio Lemos, and Diego Marques. Weighted zero-sum problems over C_3^r. *Algebra Discrete Math.*, 15(2):201–212, 2013. ISSN 1726-3255.

[16] David J. Grynkiewicz, Luz Elimar Marchan, and Oscar Ordaz. A weighted generalization of two theorems of Gao. *Ramanujan J.*, 28(3):323–340, 2012. ISSN 1382-4090. doi: 10.1007/s11139-011-9350-x. URL `https://doi-org.ezproxy.uky.edu/10.1007/s11139-011-9350-x`.

[17] Franz Halter-Koch. Arithmetical interpretation of weighted Davenport constants. *Arch. Math. (Basel)*, 103(2):125–131, 2014. ISSN 0003-889X. doi: 10.1007/s00013-014-0677-6. URL `https://doi-org.ezproxy.uky.edu/10.1007/s00013-014-0677-6`.

[18] Yahya Ould Hamidoune and Gilles Zémor. On zero-free subset sums. *Acta Arith.*, 78(2):143–152, 1996. ISSN 0065-1036. doi: 10.4064/aa-78-2-143-152. URL `https://doi-org.ezproxy.uky.edu/10.4064/aa-78-2-143-152`.

[19] Luz E. Marchan, Oscar Ordaz, Dennys Ramos, and Wolfgang A. Schmid. Some exact values of the Harborth constant and its plus-minus weighted analogue. *Arch. Math. (Basel)*, 101(6):501–512, 2013. ISSN 0003-889X. doi: 10.1007/s00013-013-0590-4. URL `https://doi-org.ezproxy.uky.edu/10.1007/s00013-013-0590-4`.

[20] Luz E. Marchan, Oscar Ordaz, and Wolfgang A. Schmid. Remarks on the plus-minus weighted Davenport constant. *Int. J. Number Theory*, 10(5):1219–1239, 2014. ISSN 1793-0421. doi: 10.1142/S1793042114500225. URL `https://doi-org.ezproxy.uky.edu/10.1142/S1793042114500225`.

[21] Luz E. Marchan, Oscar Ordaz, Irene Santos, and Wolfgang A. Schmid. Multi-wise and constrained fully weighted Davenport constants and interactions with coding theory. *J. Combin. Theory Ser. A*, 135:237–267, 2015. ISSN 0097-3165.

doi: 10.1016/j.jcta.2015.05.004. URL `https://doi-org.ezproxy.uky.edu/10.1016/j.jcta.2015.05.004`.

[22] Luz E. Marchan, Oscar Ordaz, Dennys Ramos, and Wolfgang A. Schmid. Inverse results for weighted Harborth constants. *Int. J. Number Theory*, 12(7):1845–1861, 2016. ISSN 1793-0421. doi: 10.1142/S1793042116501141. URL `https://doi-org.ezproxy.uky.edu/10.1142/S1793042116501141`.

[23] D. G. Mead and W. Narkiewicz. The capacity of C_5 and free sets in C_m^2. *Proc. Amer. Math. Soc.*, 84(2):308–310, 1982. ISSN 0002-9939. doi: 10.2307/2043685. URL `https://doi-org.ezproxy.uky.edu/10.2307/2043685`.

[24] Filipe A. A. Oliveira, Abílio Lemos, and Hemar T. Godinho. Weighted EGZ-constant for p-groups of rank 2. *Int. J. Number Theory*, 15(10):2163–2177, 2019. ISSN 1793-0421. doi: 10.1142/S1793042119501197. URL `https://doi-org.ezproxy.uky.edu/10.1142/S1793042119501197`.

[25] John E. Olson. An addition theorem modulo p. *J. Combinatorial Theory*, 5:45–52, 1968. ISSN 0021-9800.

[26] John E. Olson. A combinatorial problem on finite Abelian groups. I. *J. Number Theory*, 1:8–10, 1969. ISSN 0022-314X. doi: 10.1016/0022-314X(69)90021-3. URL `https://doi-org.ezproxy.uky.edu/10.1016/0022-314X(69)90021-3`.

[27] John E. Olson. A combinatorial problem on finite Abelian groups. II. *J. Number Theory*, 1:195–199, 1969. ISSN 0022-314X. doi: 10.1016/0022-314X(69)90037-7. URL `https://doi-org.ezproxy.uky.edu/10.1016/0022-314X(69)90037-7`.

[28] Kenneth Rogers. A combinatorial problem in Abelian groups. *Proc. Cambridge Philos. Soc.*, 59:559–562, 1963. ISSN 0008-1981.

[29] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[30] Neil J. A. Sloane and The OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2020. URL `http://oeis.org/?language=english`.

[31] J. Sonn and H. Zassenhaus. On the theorem on the primitive element. *Amer. Math. Monthly*, 74:407–410, 1967. ISSN 0002-9890. doi: 10.2307/2314573. URL `https://doi-org.ezproxy.uky.edu/10.2307/2314573`.

[32] R. Thangadurai. A variant of Davenport's constant. *Proc. Indian Acad. Sci. Math. Sci.*, 117(2):147–158, 2007. ISSN 0253-4142. doi: 10.1007/s12044-007-0010-4. URL `https://doi-org.ezproxy.uky.edu/10.1007/s12044-007-0010-4`.

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