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ON THE FINSLER GEOMETRY OF THE HEISENBERG GROUP H_{2n+1} AND ITS EXTENSION

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Abstract. We first classify left invariant Douglas (α, β)-metrics on the Heisenberg group H_{2n+1} of dimension $2n+1$ and its extension i.e., oscillator group. Then we explicitly give the flag curvature formulas and geodesic vectors for these spaces, when equipped with these metrics. We also explicitly obtain S-curvature formulas of left invariant Randers metrics of Douglas type on these spaces and obtain a comparison on geometry of these spaces, when equipped with left invariant Douglas (α, β)-metrics. More exactly, we show that although the results concerning bi-invariant Douglas (α, β)-metrics on these spaces are similar, several results concerning left invariant Douglas (α, β)-metrics on these spaces are different. For example we prove that the existence of left-invariant Matsumoto, Kropina and Randers metrics of Berwald type on oscillator groups can not extend to Heisenberg groups. Also we prove that oscillator groups have always vanishing S-curvature, whereas this can not occur on Heisenberg groups. Moreover, we prove that there exist new geodesic vectors on oscillator groups which can not extend to the Heisenberg groups.

1. Introduction

The search for left-invariant metrics on Lie groups in differential geometry leads to develop our understanding of which geometrical properties are strictly related to the metric signature and which ones are more general. This study began with the classical work of Milnor [19], who investigated the curvature properties of left-invariant Riemannian metrics on three-dimensional Lie groups. Among these Lie groups, the only two-step nilpotent Lie group with a 1-dimensional center is a Heisenberg group.

Heisenberg groups play an important role in various areas, such as physics, quantum mechanics, signal theory and number theory [4]. The study of left-invariant metrics on Heisenberg groups has attracted a considerable attention of geometers. For example, it is proved that up to homothety, there is a unique left-invariant Riemannian metric on the Heisenberg group H_3, whereas there are three metrics in the Lorentzian case on these spaces [24]. In [25] this study generalized to Riemannian

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and Lorentzian metrics on the Heisenberg group H_{2n+1} of dimension $2n+1$ and a classification of left-invariant Riemannian and Lorentzian metrics on this group is given.

On the other hand Finsler metrics which are a generalization of Riemannian metrics on these spaces have been studied in the recent years. For example, recently left-invariant Randers metrics on three and five dimensional Heisenberg groups have been investigated in [16, 17]. Also in [20] we investigated left-invariant Randers metrics of Douglas type on two-step nilpotent Lie groups of dimension five and in [22] and [12] this study is extended to a $(2n+1)$-dimensional Heisenberg group with a special left-invariant Randers metric.

These findings motivated us to extend this study for left-invariant Douglas (α, β)-metrics on this Lie group and its extension i.e., oscillator group and obtain a classification of these metrics on these spaces. We show although there is a unique left-invariant Douglas (α, β)-metric on the Heisenberg group H_{2n+1}, there are four left-invariant Douglas (α, β)-metric on its extension. We also obtain several similarities and differences on the results concerning to left-invariant Douglas (α, β)-metrics on these two Lie groups and show that although there exist some similarities between these spaces, there exist several results concerning the left-invariant Douglas (α, β)-metrics on oscillator groups that cannot exist on Heisenberg groups.

The structure of the paper is as follows. In Section 2 we remind some facts about Heisenberg groups and oscillator groups. Then we obtain (α, β)-metrics of Berwald type and explicitly obtain left-invariant Randers, Matsumoto and Kropina metrics of Berwald type on these spaces. We also give a complete description of left invariant Randers metrics of Douglas type on these Lie groups and apply them to obtain a classification of left-invariant Douglas (α, β)-metrics on these spaces. In Section 3 we first investigate the existence of bi-invariant Douglas (α, β)-metrics on these spaces and prove that neither Heisenberg groups nor oscillator groups can admit any bi-invariant Douglas (α, β)-metric. We also explicitly obtain the flag curvature formulas and geodesic vectors on these spaces when equipped with left-invariant Douglas (α, β)-metrics and obtain some new geodesic vectors on oscillator groups equipped with left-invariant Douglas (α, β)-metrics which do not exist on Heisenberg groups. In Section 4 we explicitly give the S-curvature formulas on these spaces when these spaces are equipped with left-invariant Randers metrics of Douglas type. We also prove that oscillator groups have always vanishing S-curvature, whereas this can not occur on Heisenberg groups. Moreover, we show that oscillator groups which are equipped with left-invariant Randers metrics of Douglas type are Ricci quadratic, whereas Heisenberg groups are not Ricci quadratic. Then we apply this result to prove that Heisenberg groups which are equipped with left-invariant Randers metrics of Douglas type can be non-Berwardian generalized Berwald spaces, whereas oscillator groups are generalized Berward spaces which are Berwardian.

2. Left-invariant Douglas (α, β)-metrics on the Heisenberg group H_{2n+1} and its extension

Heisenberg group: The Heisenberg group H_{2n+1} is a 2-step nilpotent Lie group. Moreover, any 2-step nilpotent Lie group of odd dimension with a one-dimensional center is locally isomorphic to the Heisenberg group H_{2n+1} [25]. By Theorem 3.1 in [25] any left-invariant Riemannian metric on $\mathcal{H}_{2n+1}$ is given by $g^{\sigma}_{\lambda,1}$ and by [21] with respect to this metric we get the orthonormal basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n, t\}$ and $[x_i, y_i] = \frac{t\sqrt{\lambda}}{\sigma_i}$. Then by adapting the convention $R(e_i, e_j) = \nabla_{[e_i, e_j]} - [\nabla_{e_i}, \nabla_{e_j}]$, up to symmetry the non-vanishing components of curvature are given by

$$R(x_i, y_i)x_i = \frac{-3\lambda}{4{\sigma_i}^2} y_i\,, \qquad R(x_i, t)x_i = \frac{\lambda}{4{\sigma_i}^2} t\,, \qquad R(y_i, t)y_i = \frac{\lambda}{4{\sigma_i}^2} t\,.$$

Oscillator group: The oscillator group $G_n(\lambda) = G_n(\lambda_1, \ldots, \lambda_n)$ is the connected, simply connected Lie group with the Lie algebra $\mathcal{G}_n(\lambda)$. This Lie algebra decomposes as a semi-direct product of the Heisenberg algebra $\mathcal{H}_{2n+1}$ (generated by $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n, T$) and the one-dimensional abelian Lie algebra (for more details see [13]). Thus it can be realized on $\mathcal{R} \times \mathcal{C}^m \times \mathcal{R}$ with group product given by

$$\begin{aligned}&(t, z_1, \ldots, z_n, q)(t', z'_1, \ldots, z'_n, q')\\ &= \Big(t + t' + \frac{1}{2}\sum_{j=1}^{n} \Im(\bar{z}_j e^{i\lambda_j q} z'_j), z_1 + e^{i\lambda_1 q} z'_1, \ldots, z_n + e^{i\lambda_n q} z'_n, q + q'\Big)\,,\end{aligned}$$

and is the real $(2n+2)$-dimensional solvable Lie algebra with non-zero commutators $[X_i, Y_i] = T, [X_i, Q] = -\lambda_i Y_i$ and $[Y_i, Q] = \lambda_i X_i$, where λ_i are positive real numbers [5]. If we consider left-invariant Riemannian metric g on $G_n(\lambda)$, which is defined by the inner product

$$\langle x_i, x_i\rangle = \delta_{ij}\,, \quad \langle y_i, y_i\rangle = \delta_{ij}\,, \quad \langle q, q\rangle = 1\,, \quad \langle t, t\rangle = 1\,,$$

where $x_i = X_i$, $y_i = Y_i$, $t = P$ and $q = Q$, then we obtain the orthonormal basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n, t, q\}$ and non-zero Lie brackets $[x_i, y_i] = t, [x_i, q] = -\lambda_i y_i$, and $[y_i, q] = \lambda_i x_i$. Thus we obtain the following non-zero Levi-Civita connection components:

$$\begin{aligned}&\nabla_{x_i} y_i = \frac{1}{2} t = -\nabla_{y_i} x_i \qquad \nabla_{y_i} t = \frac{1}{2} x_i = \nabla_t y_i, \qquad \nabla_q x_i = \lambda_i y_i\,,\\ &\nabla_t x_i = -\frac{1}{2} y_i = \nabla_{x_i} t, \qquad \nabla_q y_i = -\lambda_i x_i\,,\end{aligned}$$

where $i = 1, \ldots, n$. Thus up to symmetry the non-zero curvature components are given by

$$R(x_i, y_i)x_i = \frac{-3}{4} y_i\,, \qquad R(x_i, t)x_i = \frac{t}{4}\,, \qquad R(y_i, t)y_i = \frac{t}{4}\,.$$

Recall that a (α, β)-metric on a manifold M is a Finsler metric of the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where $\alpha(x, y) = \sqrt{\mathbf{a}(y, y)}$ and $\phi\colon (-b_0, b_0) \longrightarrow \mathbb{R}^+$ is a C^∞ function satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0\,, \qquad \forall\ |s| \leq b < b_0\,,$$

and $\|\beta\|_\alpha < b_0$ [6]. In the case that $M = G$ is a Lie group, then F is said to be a left-invariant (α, β)-metric, when for any $x \in G$ and $y \in T_xG$ satisfying the condition $F(x, y) = F(e, dl_{\,x^{-1}}y)$, where e is the unit element of G and l_x denotes the left translation [10]. To obtain all left-invariant (α, β)-metrics of Berwald type on these spaces we use Theorem 4.1 in [18] which says that a left-invariant (α, β) metric F on a Lie group G, arising from a left-invariant Riemannian metric g and a left-invariant vector field X is of Berwald type if and only if for all y, $z \in \mathcal{G}$, the following conditions are valid

$$g([y, X], z) + g([z, X], y) = 0\,, \qquad g([y, z], X) = 0\,. \tag{2.1}$$

Using these conditions we obtain the following result.

Theorem 2.1. *Oscillator groups admit left-invariant (α, β)-metrics of Berwald type, whereas this can not occur for Heisenberg groups.*

Proof. It is sufficient that for the oscillator group $G_n(\lambda)$ in the equation (2.1) we consider $X = b_{2n+2}q$. □

By Theorem 3.1 in [9] there is not any non-Berwaldian Matsumoto and Kropina metrics of Douglas type. Recall that left-invariant Randers, Kropina and Matsumoto metrics on Lie groups are defined as follows:
$F(x, y) = \sqrt{g(y, y)} + g(X(x), y)$, $F(x, y) = \frac{g(y,y)}{g(X(x),y)}$, $F(x, y) = \frac{g(y,y)}{\sqrt{g(y,y)} - g(X(x),y)}$, where the left-invariant vector field X on G corresponds to 1-form β i.e. $g(X(x), y) = \beta(x, y)$ and Randers and Matsumoto metrics are Finsler metrics if and only if we have $\|\beta_x\|_\alpha = \sqrt{g^{ij}(x)b_i(x)b_j(x)} < 1$ and $\|\beta_x\|_\alpha = \sqrt{g^{ij}(x)b_i(x)b_j(x)} < \frac{1}{2}$. Using these facts and Theorem 2.1 we obtain the following result.

Corollary 2.2. a) *There does not exist left-invariant Randers, Matsumoto and Kropina metrics of Berwald type on the Heisenberg group H_{2n+1}.*
b) *Left invariant Randers, Matsumoto and Kropina metrics of Berwald type on the oscillator group $G_n(\lambda)$ are respectively given by*

$$F(y) = \sqrt{k_1^2 + \cdots + k_{2n+2}^2} + k_{2n+2}b_{2n+2}\,, \quad \textit{with} \quad |\, b_{2n+2}\,| < 1\,, \tag{2.2}$$

$$F(y) = \frac{k_1^2 + \cdots + k_{2n+2}^2}{\sqrt{k_1^2 + \cdots + k_{2n+2}^2} - k_{2n+2}b_{2n+2}}\,, \quad \textit{with} \quad |\, b_{2n+2}\,| < \frac{1}{2}\,, \tag{2.3}$$

and

$$F(y) = \frac{\sqrt{k_1^2 + \cdots + k_{2n+2}^2}}{k_{2n+2}b_{2n+2}}\,, \tag{2.4}$$

where $y = \sum_{i=1}^{n}(k_ix_i + k_{n+i}y_i) + k_{2n+1}t + k_{2n+2}q$ and $X = b_{2n+2}q$.

Now to give a complete description of left invariant Randers metrics of Douglas type on these Lie groups, we recall that if $F = \alpha + \beta$ is a left invariant Randers metric on the Lie group G where F is defined by an inner product $\langle \, , \, \rangle$ on $\mathcal{G}$ and a left-invariant vector field V, then F is of Douglas type if and only if V satisfies the following condition $\langle [m, n], V\rangle = 0$, for all m, $n \in \mathcal{G}$. For more details see Proposition 7.4 in [8]. Thus we obtain the following result.

Theorem 2.3. a) *Any left-invariant Randers metric of Douglas type on H_{2n+1} is given by*

$$F(y) = \sqrt{k_1^2 + \cdots + k_{2n+1}^2} + \sum_{i=1}^{2n} k_i d_i \,, \tag{2.5}$$

where $y = \sum_{i=1}^{n}(k_i x_i + k_{n+i} y_i) + k_{2n+1} t$, $\sqrt{d_1^2 + \cdots + d_{2n}^2} < 1$ and $V = \sum_{i=1}^{n}(d_i x_i + d_{n+i} y_i)$.

b) *Any left-invariant Randers metric of Douglas type on $G_n(\lambda)$ is given by*

$$F(y) = \sqrt{k_1^2 + k_2^2 + \cdots + k_{2n+2}^2} + k_{2n+2} d_{2n+2} \,, \tag{2.6}$$

where $y = \sum_{i=1}^{n}(k_i x_i + k_{n+i} y_i) + k_{2n+1} t + k_{2n+2} q$, $|d_{2n+2}| < 1$ and $V = d_{2n+2} q$. In all above cases $d_1, \ldots, d_{2n+2}$ are real constants.

It is proved in [18] that a Douglas homogeneous (α, β)-metric must be a Berwaldian metric or a Douglas Randers metric. Thus by using Corollary 2.2 and Theorem 2.3, we obtain the following result which gives us a classification of left-invariant Douglas (α, β)-metrics on Heisenberg groups and oscillator groups.

Theorem 2.4. *Any left-invariant Douglas (α, β) metric for H_{2n+1} is given by* (2.5)*, while for $G_n(\lambda)$ is given by one of the cases* (2.2), (2.3), (2.4) *and* (2.6).

Thus we can state the following result.

Corollary 2.5. *Left-invariant Douglas (α, β)-metrics for H_{2n+1} coincide with left-invariant Randers metrics of Douglas type, while this property does not occur for $G_n(\lambda)$.*

3. Flag curvatures and geodesic vectors on the Heisenberg group H_{2n+1} and its extension

Flag curvature is the most important quantity in Finsler geometry, as it generalizes sectional curvature in Riemannian geometry. To explicitly obtain the flag curvature formulas of left-invariant Douglas (α, β)-metrics on Heisenberg and oscillator groups, we notice that if these metrics are bi-invariant, then we will have simple computations to obtain these curvatures. Thus we investigate the existence of these metrics by the following result.

Theorem 3.1. *None of the left-invariant Douglas (α, β)-metrics which are given in Theorem 2.4, on Heisenberg groups and oscillator groups are bi-invariant.*

Proof. Assume that left-invariant metrics given in Theorem 2.4 are also right-invariant. Then by Theorem 3.2 in [15] for any $x, y, z \in \mathcal{G}$, we must have $\langle [x,y], z\rangle + \langle [x,z], y\rangle = 0$. By replacing x_i, y_i, t for the Heisenberg group H_{2n+1} we have the contradiction $\lambda = 0$ and for the oscillator group $G_n(\lambda)$ we have the contradiction $1 = \lambda_i = 0$. □

Thus we can state the following result.

Corollary 3.2. *Neither Heisenberg groups which are equipped with the metric $g^{\sigma}_{\lambda,1}$, nor oscillator groups which are equipped with the metric g, can admit any bi-invariant Douglas (α,β)-metric.*

Theorem 3.1 shows that to explicitly obtain flag curvature formulas on these spaces we need to do some computations. Thus we first recall that the flag curvature of left-invariant Randers, Matsumoto and Kropina metrics of Berwald type and the flag curvature of left invariant Randers metrics of Douglas type are respectively given by

$$K^F(P,y) = \frac{g^2 K^g(P)}{F^2}, \tag{3.1}$$

$$K^F(P,y) = \frac{(2-F)K^g(P)}{F^2(2F^2 g^2(X,v) + 2 - F)}, \tag{3.2}$$

$$K^F(P,y) = \frac{K^g(P)}{F^4 g^2(X,v) + F^2}, \tag{3.3}$$

and

$$K(P,y) = \frac{\alpha^2(y) K^g(P)}{F^2(y)} + \frac{(3\langle U(y,y), V\rangle^2 - 4F(y)\langle U(y, U(y,y)), V\rangle)}{4F^4(y)}, \tag{3.4}$$

where $K^g(P)$ is the sectional curvature of a left-invariant Riemannian metric g and is given by $K^g(P) = \frac{g(R(y,t)y,t)}{g^2(y,t) - g(y,y)g(t,t)}$, when $P = \mathrm{span}\{y,t\}$ and $\{y,t\}$ is an orthonormal set with respect to g [11, 9]. Thus to explicitly obtain the flag curvature on these spaces we first need to obtain the following result.

Theorem 3.3. a) *Consider the Heisenberg group H_{2n+1} with the left-invariant Riemannian metric $g^{\sigma}_{\lambda,1}$. If $y = \sum_{i=1}^{n}(k_i x_i + k_{n+i} y_i) + k_{2n+1} t$ and $s = \sum_{i=1}^{n}(k'_i x_i + k'_{n+i} y_i) + k'_{2n+1} t$ are arbitrary vectors in the Lie algebra $\mathcal{H}_{2n+1}$, then the sectional curvature $K^g(P)$ is given by*

$$K^{g^{\sigma}_{\lambda,1}}(P) = \sum \frac{-3\lambda}{4{\sigma_i}^2}(k_i k'_{n+i} + k'_i k_{n+i})^2 + \sum \frac{\lambda}{4{\sigma_i}^2}(k_i k'_{2n+1} - k_{2n+1} k'_i)^2 + \sum \frac{\lambda}{4{\sigma_i}^2}(k'_{2n+1} k_{n+i} - k_{2n+1} k'_{n+i})^2 .$$

b) *Consider the oscillator group $G_n(\lambda)$ with the left-invariant Riemannian metric g. If $y = \sum_{i=1}^{n}(k_i x_i + k_{n+i} y_i) + k_{2n+1} t + k_{2n+2} q$ and $s = \sum_{i=1}^{n}(k'_i x_i + k'_{n+i} y_i) + k'_{2n+1} t + k'_{2n+2} q$ are arbitrary vectors in the Lie algebra $\mathcal{G}_n(\lambda)$, then the sectional* *curvature* $K^g(P)$ *is given by*

$$K^g(P) = \sum \frac{-3}{4}(k_i k'_{n+i} + k'_i k_{n+i})^2 + \sum \frac{1}{4}(k_i k'_{2n+1} - k_{2n+1}k'_i)^2 + \sum \frac{1}{4}(k'_{2n+1}k_{n+i} - k_{2n+1}k'_{n+i})^2 .$$

Where $P = \mathrm{span}\{s, y\}$ *and* $\{s, y\}$ *is an orthonormal basis with respect to the metric* g.

In [20] we obtained the flag curvature and S-curvature formulas on two-step nilmanifolds of dimension five. Here we extend these formulas for an arbitrary $(2n+1)$-dimensional case of these spaces, i.e., Heisenberg groups and their semi-direct product with a line i.e., oscillator groups as follows.

Theorem 3.4. a) *Consider the Heisenberg group* H_{2n+1} *with the left-invariant Douglas* (α,β)*-metric* (2.5). *If* $y = \sum_{i=1}^n (k_i x_i + k_{n+i}y_i) + k_{2n+1}t$ *and* $s = \sum_{i=1}^n (k'_i x_i + k'_{n+i}y_i) + k'_{2n+1}t$ *are arbitrary vectors in the Lie algebra* $\mathcal{H}_{2n+1}$ *such that* $\{s,t\}$ *is an orthonormal set, then the flag curvature* $K^F(P,y)$ *is given by*

$$K^F(P,y) = \frac{4\sigma_i \sum_{i=1}^{2n+1} k_i^2 K^g(P)(\sqrt{\sum_{i=1}^{2n+1} k_i^2} + \sum_{i=1}^{2n} d_i k_i)^2 + 3\sum \sqrt{\lambda} k_{2n+1}(k_{n+i}d_i - k_i d_{n+i})}{4\sigma_i(\sqrt{\sum_{i=1}^{2n+1} k_i^2} + \sum_{i=1}^{2n} d_i k_i)^4} - \frac{\sum \lambda k_{2n+1}^2 (k_i d_i + k_{n+i}d_{n+i})}{\sigma_i^2(\sqrt{\sum_{i=1}^{2n+1} k_i^2} + \sum_{i=1}^{2n} d_i k_i)^3} .$$

b) *Consider the oscillator group* $G_n(\lambda)$ *with left-invariant Douglas* (α,β) *metrics given in Theorem 2.4. If* $y = \sum_{i=1}^n (k_i x_i + k_{n+i}y_i) + k_{2n+1}t + k_{2n+2}q$ *and* $s = \sum_{i=1}^n (k'_i x_i + k'_{n+i}y_i) + k'_{2n+1}t + k'_{2n+2}q$ *are arbitrary vectors in the Lie algebra* $\mathcal{G}_n(\lambda)$*, then the flag curvature* $K^F(P,y)$ *has one of the following forms*

(1) $K^F(P,y) = \frac{K^g(P)\sum_{i=1}^{2n+2} k_i^2}{(\sqrt{\sum_{i=1}^{2n+2} k_i^2} + b_{2n+2}k_{2n+2})^2}$, *when* F *is given in* (2.2).

(2) $K^F(P,y) = \frac{(2\sqrt{\sum_i k_i^2} - 2k_{2n+2}b_{2n+2} - \sum_i k_i^2)(\sqrt{\sum_i k_i^2} - k_{2n+2}b_{2n+2})^3 K^g(P)}{(\sum_i k_i^2)^2[2(\sum_i k_i^2)^2 b_{2n+2}^2 k'^2{}_{2n+2} + (2\sqrt{\sum_i k_i^2} - 2k_{2n+2}b_{2n+2} - \sum_i k_i^2)(\sqrt{\sum_i k_i^2} - k_{2n+2}b_{2n+2})}$,
when $i = 1, \cdots, 2n+2$ *and* F *is given in* (2.3).

(3) $K^F(P,y) = \frac{K^g(P) b_{2n+2}^2 k_{2n+2}^4}{(\sum_{i=1}^{2n+2} k_i^2)^2 (k'^2{}_{2n+2} + k_{2n+2}^2)}$, *when* F *is given in* (2.4).

(4) $K^F(P,y) = \frac{K^g(P)(\sum_{i=1}^{2n+2} k_i^2)}{(\sqrt{\sum_i k_i^2} + d_{2n+2}k_{2n+2})^2}$, *when* F *is given in* (2.6).

In all above cases $K^g(P)$ *is given in Theorem 3.3.*

Proof. For $G_n(\lambda)$ by Theorem 2.4 we have four left-invariant Douglas (α,β)-metrics. For the metric (2.2) we use the formula (3.1) and for the metrics (2.3) and

(2.4) we use formulas (3.2) and (3.3). Thus by some computations we obtain the cases (1), (2) and (3) of theorem. For the metric (2.6), we assume that $z = \sum_{i=1}^{n}(k''_i x_i + k''_{n+i} y_i) + k''_{2n+1} t + k''_{2n+2} q$ is an arbitrary vector in the Lie algebra $\mathcal{G}_n(\lambda)$. Then by using the equation $2\langle U(y,s), z\rangle = \langle [z,s], y\rangle + \langle [z,y], s\rangle$, we obtain

$$\begin{aligned} U(y,s) &= \frac{1}{2}\sum [k_{n+i}(k'_{2n+1} - \lambda_i k'_{2n+2}) + k'_{n+i}(k_{2n+1} - \lambda_i k_{2n+2}]x_i \\ &- \frac{1}{2}\sum [k_i(k'_{2n+1} - \lambda_i k'_{2n+2}) + k'_i(k_{2n+1} - \lambda_i k_{2n+2})y_i \end{aligned}$$

which implies that $\langle U(y,y), V\rangle = 0$ and $\langle U(y, U(y,y)), V\rangle = 0$, where V is given in (2.3). Then by replacing these equations in the equation (3.4) and using Theorem 3.3 we obtain the result. For H_{2n+1} we have a similar argument. □

Thus as an immediate consequence we obtain the following result.

Corollary 3.5. *The flag curvature and sectional curvature of $G_n(\lambda)$ equipped with a left-invariant Randers metric of Douglas type have the same sing, whereas this may not occur for H_{2n+1}.*

Geodesics of left-invariant Riemannian metrics on Lie groups were studied by Arnold, extending Euler's theory of rigid-body motion [2] and have important applications to mechanics. Here to obtain geodesic vectors on Heisenberg and oscillator groups which are equipped with left-invariant Douglas (α,β)-metrics, we recall that by [23] the vector X is a geodesic vector of (α,β)-metric if and only if X is a geodesic vector of a Riemannian metric g. Thus we obtain the following result which gives us some new geodesic vectors on oscillator groups which do not exist on Heisenberg groups.

Theorem 3.6. *Let G be a Heisenberg group H_{2n+1} or an oscillator group $G_n(\lambda)$ which is equipped with a left-invariant Douglas (α,β) metric given in Theorem 2.4.*

(a) *If $G = H_{2n+1}$, then y is a geodesic vector of (H_{2n+1}, F) if and only if y has one of the following forms*

(1) $y \in V = \operatorname{span}\{x_1, \cdots, x_n, y_1, \ldots, y_n\}$;

(2) $y \in V = \operatorname{span}\{t\}$.

(b) *If $G = G_n(\lambda)$, then y is a geodesic vector of $(G_n(\lambda), F)$ if and only if y has one of the following forms*

(1) $y \in V = \operatorname{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, t, q\}$ *satisfying* $k_{2n+1} = \lambda_i k_{2n+2}$;

(2) $y \in V = \operatorname{span}\{y_1, \ldots, y_n, t, q\}$ *satisfying* $k_{2n+1} = \lambda_i k_{2n+2}$;

(3) $y \in V = \operatorname{span}\{x_1, \ldots, x_n, t, q\}$ *satisfying* $k_{2n+1} = \lambda_i k_{2n+2}$;

(4) $y \in V = \operatorname{span}\{t, q\}$;

Proof. For $G_n(\lambda)$ by geodesic lemma in [14] y is a geodesic vector of (G,g) if and only if $\langle [y,z], y\rangle = 0$ for any $z \in \mathcal{G}_n(\lambda)$. Hence $y = \sum_{i=1}^{n}(k_i x_i + k_{n+i} y_i) + k_{2n+1} t + k_{2n+2} q$ is a geodesic vector of $(G_n(\lambda), F)$ if and only if we have $\langle \sum_{i=1}^{n}(k_i x_i +$

$k_{n+i}y_i)+k_{2n+1}t+k_{2n+2}q, e_i], \sum_{i=1}^{n}(k_ix_i+k_{n+i}y_i)+k_{2n+1}t+k_{2n+2}q\rangle = 0$, when $e_i = x_i, y_i, t, q$, with $i = 1, \ldots, n$. Thus we obtain the system of equations $k_{n+i}k_{2n+1} - \lambda_i k_{2n+2}k_{n+i} = 0$ and $k_ik_{2n+1} - \lambda_i k_{2n+2}k_i = 0$. To solve these equations we consider the following cases:

a) $k_i \neq 0 \neq k_{n+i}$: this yields the case (1).

b) $k_i = 0 \neq k_{n+i}$: this implies the case (2).

c) $k_i \neq 0 = k_{n+i}$: this gives us the case (3).

d) $k_i = 0 = k_{n+i}$: this yields the case (4).

For H_{2n+1} we have a similar proof. □

Comparing the results on oscillator groups and Heisenberg groups given in the above theorem, we obtain the following result.

Corollary 3.7. *There exist some new geodesic vectors on oscillator groups which are equipped with left-invariant Douglas (α, β)-metrics while these geodesics do not exist on Heisenberg groups.*

4. S-curvature on the Heisenberg group H_{2n+1} and its extension

The concepts of S-curvatures are fundamental quantities in Finsler geometry which vanish for the Riemannian metrics. Here to explicitly obtain the S-curvature formulas on Heisenberg groups and their extension i.e., oscillator groups we recall some facts from [8]. Assume that G is a Lie group with a left-invariant Randers metric F which is defined by an inner product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$ of G and a left-invariant vector field V. Then the S-curvature is given by

$$S(e, y) = \frac{n+1}{2}\Big\{\frac{\langle [V, y], \langle y, V\rangle V - y\rangle}{F(y)} - \langle [V, y], V\rangle\Big\}. \tag{4.1}$$

see [7]. Using these facts and Theorem 2.3, we can obtain the following result.

Theorem 4.1. *Let G be the Heisenberg group H_{2n+1} or the oscillator group $G_n(\lambda)$ which is equipped with a left-invariant Randers metric of Douglas type given in Theorem 2.3. Then the S-curvature on $G = H_{2n+1}$ is given by*

$$S(e, y) = -\sqrt{\lambda}(n+1)\left\{\frac{k_{2n+1}\sum_{i=1}^{n}(\frac{d_ik_{n+i}-d_{n+i}k_i}{\sigma_i})}{\sqrt{\sum_{i=1}^{2n+1}k_i^2 + \sum_{i=1}^{2n}d_ik_i}}\right\} \tag{4.2}$$

and for $G = G_n(\lambda)$ is given by $S(e, y) = 0$. Where $y = \sum_{i=1}^{n}(k_ix_i+k_{n+i}y_i)+k_{2n+1}t$ is an arbitrary vector in $\mathcal{H}_{2n+1}$ and e is the identity element of G.

Proof. For H_{2n+1} by Theorem 2.3 we obtain $V = \sum_{i=1}^{n}(d_ix_i + d_{n+i}y_i)$. Then we get $[V, y] = \sum_{i=1}^{n}\frac{t\sqrt{\lambda}}{\sigma_i}(d_ik_{n+i} - d_{n+i}k_i)$ which implies that $\langle [V, y], \langle y, V\rangle V - y\rangle = -k_{2n+1}\sum_{i=1}^{n}\frac{\sqrt{\lambda}}{\sigma_i}(d_ik_{n+i} - d_{n+i}k_i)$ and $\langle [V, y], V\rangle = 0$. Thus by replacing these equations in (4.1), we get the formula (4.2). For $G_n(\lambda)$ we have $[V, y] = \sum_{i=1}^{n}a_{2n+1}\lambda_i(k_iy_i - k_{n+i}x_i)$ which gives us $\langle [V, y], \langle y, V\rangle V - y\rangle = 0$ and $\langle [V, y], V\rangle = 0$. Thus by replacing these equations in (4.1) we obtain the result. □

Using the above result we obtain the following corollary.

Corollary 4.2. *Oscillator groups equipped with left-invariant Randres metrics of Douglas type given in Theorem 4.1 have always vanishing S-curvature, whereas this can not occur on Heisenberg groups.*

The spaces with vanishing S-curvature have a close relationship with generalized Berwald spaces and Ricci-quadratic spaces [3, 8]. Recall that a Finsler metric is called Ricci-quadratic, if its Ricci curvature $Ricc(x, y)$ is quadratic in y. By Theorem 7.9 in [8] a homogeneous Randers space is Ricci quadratic if and only if it is of Berwald type. Thus by Theorem 2.1 we obtain the following result which gives us a property on oscillator groups which does not exist on Heisenberg groups.

Corollary 4.3. *Heisenberg groups equipped with left-invariant Randres metrics of Douglas type given in Theorem 4.1 are never Ricci-quadratic spaces, while oscillator groups can be considered as Ricci-quadratic spaces.*

Recall that a Finsler manifold (M, F) is a generalized Berwald manifold if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F. By [1] any left invariant Finsler metric on a Lie group is a generalized Berwald space. Thus by Theorem 2.1 we obtain the following result which gives us the existence of generalized Berwald spaces which are non-Berwardian.

Corollary 4.4. *Heisenberg groups equipped with left-invariant Randres metrics of Douglas type given in Theorem 4.1 are non-Berwardian generalized Berwald spaces, whereas oscillator groups are of Berward type.*

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