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To Professor RNDr. Jan Kratochvíl, CSc.

Report on the Habilitation thesis of Dr. Marek Cúth, entitled "Lipschitz-free Banach spaces."

The habilitation under review focuses on some of the results obtained by Marek Cúth after his PhD thesis. The common ground of these works is the study of the linear structure of the so-called Lipschitz free spaces. Both the isometric and isomorphic aspects are addressed. This memoir is built around eight important papers by M. Cúth and coauthors (references [A] to [F] in the thesis). Let us now recall briefly the necessary definitions. Let (M, d) be a metric space equipped with a distinguished point 0. We denote by $\mathrm{Lip}_0(\mathrm{M})$ the Banach space of all real-valued Lipschitz functions on M that vanish at 0 endowed with its natural norm:

$$\forall f \in \mathrm{Lip}_0(\mathrm{M}), \quad \|\mathrm{f}\|_{\mathrm{Lip}} = \sup\left\{\frac{|\mathrm{f(x)} - \mathrm{f(y)}|}{\mathrm{d(x, y)}},\ \mathrm{x, y} \in \mathrm{M},\ \mathrm{x} \neq \mathrm{y}\right\}.$$

The Dirac map $\delta : M \to \mathrm{Lip}_0(\mathrm{M})^*$ defined by $\langle f, \delta(x)\rangle = f(x)$ for $f \in \mathrm{Lip}_0(\mathrm{M})$ and $x \in M$ is an isometric embedding from M into $\mathrm{Lip}_0(\mathrm{M})^*$. The closed linear span of $\{\delta(x),\ x \in M\}$ in $\mathrm{Lip}_0(\mathrm{M})^*$ is denoted $\mathcal{F}(M)$ and called the Lipschitz-free space over M (free space in short). The space $\mathcal{F}(M)$ is an isometric predual of $\mathrm{Lip}_0(\mathrm{M})$. The fundamental property of the free space is that any Lipschitz function from M to a Banach space Z (sending 0 to 0) extends, via the map δ, to a continuous linear map from $\mathcal{F}(M)$ to Z. In fact, the author chooses to present $\mathcal{F}(M)$ to be the unique (up to isometry) Banach space satisfying this universal property (see Proposition 1 for details). I believe, that it is the best way to introduce this object, at least in the context of Banach space geometry. This automatic linearization of Lipschitz maps makes it very natural to study the linear structure of free spaces, but it is a difficult task, as most of the complexity of Lipschitz maps is now carried by this linear structure. This line of research was initiated by two fundamental papers by Godefroy and Kalton [1] and Kalton [2]. It took a few years to the community of the geometry of Banach spaces to digest some (possibly not all) of the ideas of [1] and [2]. Then, very naturally, the subject became very active and is now growing rapidly and attracting many researchers. It is clear that Marek Cúth is one of the important contributors to the recent progress made in this direction of research.

The thesis is organized in three chapters and a given paper may be relevant to more that one chapter. This choice, makes the thesis very pleasant to read.

The first chapter is called "Lipschitz free Banach spaces and their ℓ_1-like behavior". It essentially addresses two questions : when is a free space isomorphic to ℓ_1? To what extent does a free space always contain ℓ_1? Marek Cúth contributed to give a complete answer to the first question for ultrametric spaces. Indeed in paper [A] he showed with Doucha that for every infinite separable ultrametric space M, $\mathcal{F}(M)$ is isomorphic to ℓ_1 and has a monotone Schauder basis. This was completed in [H] with Albiac, Ansorena and Doucha, where it is shown that the Banach Mazur distance from $\mathcal{F}(M)$ to ℓ_1 can be arbitrarily close to 1 (which is optimal by another result of Dalet, Kaufman and Procházka). In paper [B], with Doucha and Wojtaszczyk, we can find the first answer to the second question: for any infinite metric space M, $\mathcal{F}(M)$ contains a complemented copy of ℓ_1. This is completed by paper [C] (with Johanis), where it is proved that $\mathrm{Lip}_0(\mathrm{M})$ always contains an isometric copy of ℓ_∞. Then the result from [B] is made even more precise in [H]: every infinite metric space M contains a subset N such that the image of the canonical injection from $\mathcal{F}(N)$ into $\mathcal{F}(M)$ is isomorphic to ℓ_1 and complemented in $\mathcal{F}(M)$.

The second chapter is devoted to the important study of the linear structure of $\mathcal{F}(\mathbb{R}^d)$. As it is explained in this memoir, it is rather easy to see that $\mathcal{F}(\mathbb{R})$ is isometric to L_1. On the other hand, Naor and Schechtman proved that $\mathcal{F}(\mathbb{R}^2)$ does not even bi-Lipschitz embed into L_1. However it is still unknown whether $\mathcal{F}(\mathbb{R}^2)$ is linearly isomorphic to $\mathcal{F}(\mathbb{R}^3)$. The first result of this part, is taken from paper [D] with Kalenda and Kaplický and is a concrete representation of $\mathcal{F}(M)$ when M is a convex open subset of a finite dimensional normed space E as the quotient of $L_1(M, E)$ by the space of functions with zero divergence in the sense of distributions. This description is quite delicate and proved to be extremely useful in the next work (paper [E]), with the same coauthors, where they prove that, for a finite dimensional normed space, $\mathcal{F}(E)$ is λ-complemented into its bidual, with λ being at most equal to the Banach-Mazur distance from E to $\ell_2^{dim\,E}$. The question of the complementation of a free space in its bidual is a crucial one. For instance if one could prove that $\mathcal{F}(\ell_1)$ is complemented in its bidual, one could immediately deduce that a Banach space Lipschitz equivalent to ℓ_1 is linearly isomorphic to it (a major open question in non linear classification of Banach spaces). This result, which is an important step towards the understanding of this question, is based, among other things, on a sophisticated analysis of vector valued finitely additive measures. I must mention that the question solved here by Cúth, Kalenda and Kaplický was known to be important by all the specialists of the subject and that, to the best of knowledge, it has not been improved since. Still, related to the structure of $\mathcal{F}(\mathbb{R}^d)$ is another result, obtained with Candido and Doucha (paper [F]), where they show that $\mathrm{Lip}_0(\mathbb{R}^\mathrm{d})$ is linearly isomorphic to $\mathrm{Lip}_0(\mathbb{Z}^\mathrm{d})$ (it is actually a particular case of a more general result on Lipschitz spaces over certain finitely generated groups). This is especially interesting as the corresponding free spaces are not isomorphic. Indeed, $\mathcal{F}(\mathbb{R}^d)$ clearly contains an isometric copy of L_1 and therefore fails to have the Radon-Nikodým property, unlike $\mathcal{F}(\mathbb{Z}^d)$ which is known to be a separable dual space (a result of A. Dalet). This paper [F] contains other interesting results, for instance on Lipschitz spaces over infinite dimensional L_p spaces. It also provides many well chosen open questions.

The last chapter of this thesis focuses on recent works of the author, with Albiac, Ansorena and Doucha (papers [G] and [H]) on Lipschitz-free p-spaces. For $0 < p \leq 1$, (M, d) is a p-metric space if (M, d^p) is a metric space. Then a p-norm on a vector space is a functional which satisfies the definition of a norm, except for the triangle inequality, which is replaced by its p-version : $\|x + y\|^p \leq \|x\|^p + \|y\|^p$; and then the map $(x, y) \mapsto \|x - y\|$ is naturally a p-metric. If (M, d) is p-metric and $(Z, \|\ \|)$ is a p-Banach space, $\mathrm{Lip}_0(\mathrm{M}, \mathrm{E})$

denotes the space of maps $f : M \to Z$ satisfying $\|f(x) - f(y)\| \le Cd(x, y)$. Then, $\mathcal{F}_p(M)$, the Lipschitz-free p-space associated with M, is the unique (up to isometry) p-Banach space such that there is an isometry δ_M from M onto a linearly dense subset of $\mathcal{F}_p(M)$ with the property that any Lipschitz map from M to a p-Banach space Z is equal to $T_f \circ \delta_M$, for some linear operator $T_f : \mathcal{F}_p(M) \to Z$ such that $\|T_f\| = \|f\|_{\mathrm{Lip}}$. These spaces were previously introduced by Albiac and Kalton to provide examples of separable quasi-Banach spaces that are Lipschitz equivalent but not linearly isomorphic. The systematic study of the spaces $\mathcal{F}_p(M)$ is however initiated in [G], where the ground is carefully settled for further work. Some special difficulties arise. It is shown for instance that the inclusion map from a subset of a p-metric space does not necessarily induce an isometric embedding between the corresponding p-free spaces. Note that if (M, d) is ultrametric then (M, d^p) is a metric space for all $p > 0$. It therefore makes sense to look at it as a p-metric space. Then the authors show in [G] that $\mathcal{F}_p(M)$ is isomorphic to ℓ_p, whenever M is an infinite separable ultrametric space. The same set of authors continues his work on $\mathcal{F}_p(M)$ in [H], by studying, in the spirit of the first part, the presence of ℓ_p in $\mathcal{F}_p(M)$. In particular, they show : if M is a p-metric space containing as many isolated points as its density character Γ, then $\ell_p(\Gamma)$ is complemented in $\mathcal{F}_p(M)$; if M is an infinite p-metric space then $\mathcal{F}_p(M)$ contains a subspace isomorphic to ℓ_p (but not always complemented, unlike the case $p = 1$). Many interesting results are also obtained on Schauder bases in p-free spaces. For instance if M is a net of c_0, then $\mathcal{F}_p(M)$ has a Schauder basis. It is especially interesting to note that the methods developed in this non locally convex setting allowed the authors to obtain new results in the case $p = 1$ (see section 3.1 of the thesis).

It is also important to mention that the eight papers that I tried to describe and that are covered by this thesis represent only one aspect of the mathematical work achieved by Marek Cúth after his PhD. His spectrum of research is actually very wide and diverse. Let me cite here, just as an example, his very recent and important work (with Doležal, Doucha and Kurka) on the complexity of classes of Banach or metric spaces.

Marek Cúth has presented in this memoir an important and coherent set of very interesting results. He has developed various new ways to attack difficult questions. The proofs, although often very involved, are natural and elegant and allow the author to obtain many important new results in directions where the competition is intense. Each of the papers described in this report contains significant advances. The text of the memoir is very well written: clear, precise and organized. I think that Marek Cúth is a very talented young researcher, working with great success at the international level. He has obtained many important results in a quite short amount of time. In conclusion, I believe that the work presented in this memoir is of high quality and reaches the standards of an excellent habilitation thesis.

References

[1] G. Godefroy and N. J. Kalton, Lipschitz-free Banach spaces, Studia Math. 159 (2003), no. 1, 121–141.

[2] N. J. Kalton, Spaces of Lipschitz and Hölder functions and their applications, Collect. Math. 55 (2004), 171–217.

I declare that I have read the control of originality report from the Turnitin system. I have found no element to change my conviction that this presentation is entirely honest and original.

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