Optimal prediction problems and the last zero of spectrally negative Lévy processes.

A thesis presented for the degree of
Doctor of Philosophy

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May 2021

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it). The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgement is made. This thesis may not be reproduced without my prior written consent. I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

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I confirm that a version of Chapters 2, 3 and 4 were jointly co-authored with Dr Erik J. Baurdoux.

I confirm that a version of Chapter 5 was jointly co-authored with Dr Erik J. Baurdoux and Professor Angelos Dassios.

Acknowledgments

First and foremost, I would like to thank my supervisor, Dr Erik Baurdoux. Not only did he provide the greatest supervision during my PhD studies, but he also offered continuous guidance and support on many aspects of my life. Thanks to him, I have also rediscovered one of my biggest passions, cycling. Erik has been always a big source of inspiration and I feel very fortunate to have met him in Guanajuato, Mexico. I would also like to express my deep gratitude to my second supervisor, Professor Beatrice Acciaio, for her unending encouragement, support and trust in me to the point of recommending me for amazing projects during and after this stage of my career. I would also like to extend a special thank you to Professor Angelos Dassios for his availability, valuable research discussions and suggestions as well as his helpful assistance as PhD programme director. Finally, I would like to express deep gratitude to Professor Kostas Kardaras and Dr Tiziano De Angelis for the careful reading of this thesis and the invaluable comments that made this work to have a substantial improvement.

I gratefully acknowledge the support of the London School of Economics and Political Science and the Department of Statistics for providing the "LSE PhD Studentship", funding that has allowed me to undertake this research. Particularly, I would like to thank all the staff in the department, all their invaluable help and support not only helped me to successfully conclude this period but they made my PhD study at the LSE a smooth and enjoyable journey. Special thanks to Penny Montague and Imelda Noble for their patience and aid with numerous administrative matters.

I would also like to express my sincerest appreciation to my colleagues and friends in the department, especially to my office mates, JingHan Tee, Davide de Santis, LuTing Li, KaiFang Zhou, Sahoko Ishida, Anica Kostic, Gianluca Giudice, Sasha Tsimbalyuk, Konstantinos Vamourellis, YiRui Liu and Viet Dang. I spent most of my time at the 7th-floor office and I could not have asked for better office mates. I am particularly grateful to my former teaching partner Phoenix (Huang) Feng who taught me everything I needed to know when I started my teaching experience at LSE, and Ragvir Sabharwal, who is one of the most talented teachers I know, for his constructive advice. I also extend my appreciation to great friends and fellow PhD students, Alice Pignatelli di Cerchiara, TianLin Xu, XiaoLin Zhu and Christine Yuen for their companionship.

Finally, I wish to thank my parents, Bernabe and Guadalupe, and my siblings for always believing in me and supporting all my pursuits. My special thanks are extended to Ana de Graaf Willems, Leandro Sánchez Betancourt, Karen Zárate Jiménez and JingHan Tee (again!), who are more than friends, for their constant encouragement and unconditional support.

Abstract

In recent years the study of Lévy processes has received considerable attention in the literature. In particular, spectrally negative Lévy processes have applications in insurance, finance, reliability and risk theory. For instance, in risk theory, the capital of an insurance company over time is studied. A key quantity of interest is the moment of ruin, which is classically defined as the first passage time below zero. Consider instead the situation where after the moment of ruin the company may have funds to endure a negative capital for some time. In that case, the last time below zero becomes an important quantity to be studied.

An important characteristic of last passage times is that they are random times which are not stopping times. This means that the information available at any time is not enough to determine its value and only with the whole realisation of the process that it can be determined. On the other hand, stopping times are random times such that its realisation can be derived only with the past information. Suppose that at any time period there is a need to know the value of a last passage time for some appropriate actions to be taken. It is then clear that an alternative to this problem is to approximate the last passage time with a stopping time such that they are close in some sense.

In this work, we consider the optimal prediction to the last zero of a spectrally negative Lévy process. This is equivalent to find a stopping time that minimises its distance with respect to the last time the process goes below zero. In order to fulfil this goal, we also study the last zero before at any fixed time and its dynamics as a process. Moreover, having in mind some applications in the insurance sector, we study the joint distribution of the number of downcrossings by jump and the local time before an exponential time.

Contents

Introduction

Every decision we make in daily life has a certain degree of risk associated with it. Since taking on risk is an integral part of our lives, it is therefore indisputable that selecting the best time to stop and act is essential. A decision-maker who observes a process evolving in time that involves some randomness, arrives at a strategy to either maximise reward or minimise cost based only on what is known.

The optimal stopping theory is concerned with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximise an expected pay-off or to minimise an expected cost. Problems of this type have many applications, particularly in the following areas:

1. Statistics: The action to test a hypothesis or to find a parameter as quickly and accurately as possible.

2. Quickest detection problem: When a natural phenomenon threatens to destroy a town, one needs to decide when to send out an alarm to avoid disaster based on observable data.

3. Operation research: One has to decide when it is optimal to replace a machine, hire a secretary, or reorder stock.

4. Finance: The non-arbitrage price of an American option has to be established.

For an overview of the general theory of optimal stopping, the reader can refer to Peskir and Shiryaev (2006) and Shiryaev (2007) or Hill (2009) for recreational reading. In this work, we deal with optimal prediction problems. These problems can be described as optimal stopping problems for which the gain process depends on the future (hence, standard techniques of the optimal stopping theory cannot be applied directly). Problems of this kind are becoming of increasing interest to many sectors, especially financial engineering. Indeed, suppose that we have a random variable that depends on the realisation of the whole process X, up to some time $T \geq 0$, and that we are interested to know its value at any time $t < T$ so some decisions can be taken with that information. Hence, given that stopping times are random times such that their realisation is determined with the present and past information, it becomes natural to approximate random times by stopping times in some sense. In the present literature, we can find two main ways of doing that, the first being in space and the second in time. That is, optimal prediction problems are of the form:

$$V = \inf_{\tau \in \mathcal{T}_T} \mathbb{E}(\varphi(H, X_\tau)) \qquad \text{and} \qquad V^* = \inf_{\tau \in \mathcal{T}_T} \mathbb{E}(d(\Theta, \tau)),$$

where φ and d are functions to be determined and H and Θ are random variables determined by the information at time T taking values in $\mathbb{R}$ and $[0, \infty)$, respectively.

In what follows, we give a short review of some optimal prediction problems studied in the literature together with a short description of the methodology used to find their solutions. We denote X as a stochastic process, $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ its running supremum and its running infimum, respectively. We also denote $\mathcal{T}_t$ as the set of stopping times bounded by $t \geq 0$ and the random times

$$\begin{aligned}
\overline{\theta}_T &= \inf\{0 \leq t \leq T : X_t = \overline{X}_T\} \\
\underline{\theta}_T &= \inf\{0 \leq t \leq T : X_t = \underline{X}_T\} \\
\xi_T &= \sup\{0 \leq t \leq T : X_t = 0\}.
\end{aligned}$$

Graversen et al. (2001) predicted the value of the ultimate maximum of a standard Brownian motion at a time 1, where the prediction is made by using a mean-square distance, that is,

$$\inf_{\tau \in \mathcal{T}_1} \mathbb{E}\left((X_\tau - \overline{X}_1)^2\right),$$

where X is a standard Brownian motion. They solved this problem by relying on a stochastic integral representation of the ultimate maximum so that the optimal prediction problem is equivalent to an optimal stopping problem. The latter is then solved by using time-change arguments and finding the solution to a free boundary problem. Lastly, this aforementioned solution is formally verified to be the value function and the optimal stopping time of the optimal stopping problem

Pedersen (2003) generalises the problem above by predicting the position of the ultimate maximum by a q-mean distance and a probability distance. That is, for fixed $q > 0$ and $\varepsilon > 0$,

$$\inf_{\tau\in\mathcal{T}_1} \mathbb{E}\left((X_\tau - \overline{X}_1)^q\right),$$

$$\inf_{\tau\in\mathcal{T}_1} \mathbb{P}\left(\overline{X}_1 - X_\tau \le \varepsilon\right),$$

where X is a standard Brownian motion. It is shown that both optimal prediction problems are equivalent to optimal stopping problems. The former is further simplified by using the fact that the Brownian motion reflected in its maximum has the same law as the reflected Brownian motion. This optimal stopping problem is then solved by using a deterministic change of time, solving a free boundary problem and a verification approach. The optimal stopping problem associated with the probability distance problem is solved by guessing the solution by heuristic arguments based on the smooth fit property and the verification theorem.

Shiryaev (2002) proposed that instead of using the closeness of $\overline{X}_1$ with X_τ, the closeness of τ with $\overline{\theta}$ could be used. For example,

$$\inf_{\tau\in\mathcal{T}_1} \mathbb{E}(|\tau - \overline{\theta}_1|^p)$$

or more generally,

$$\inf_{\tau\in\mathcal{T}_1} \mathbb{E}[G_1((\tau - \overline{\theta}_1)^+) + G_2((\tau - \overline{\theta}_1)^-)],$$

for some risk functions G_1 and G_2.

Urusov (2005) showed that the optimal prediction problem in Graversen et al. (2001) is equivalent to predicting the time of the ultimate maximum of the Brownian motion by stopping times using a L_1 distance. That is,

$$\inf_{\tau \in \mathcal{T}_1} \mathbb{E}\left((X_\tau - \overline{X}_1)^2\right) = \inf_{\tau \in \mathcal{T}_1} \mathbb{E}(|\tau - \overline{\theta}_1|) + 1/2.$$

Moreover, two additional optimal prediction problems were solved:

$$\inf_{\tau \in \mathcal{M}_\alpha} \mathbb{E}((\tau - \overline{\theta}_1)^+) \qquad \text{and} \quad \inf_{\tau \in \mathcal{N}_\alpha} \mathbb{E}((\tau - \overline{\theta}_1)^-),$$

where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are subclasses of $\mathcal{T}_1$ such that the penalty of stopping too late and stopping prematurely, respectively, are bounded by α. The methods of solution rely on using a "Lagrange multiplier method" and finding equivalent optimal stopping problems which are solved by a deterministic time-change and the solution of a free boundary problem followed by a verification method argument.

du Toit and Peskir (2007) predicted the position of the ultimate maximum with drift in a mean-square sense for a Brownian motion with drift and in a finite horizon setting, that is,

$$\inf_{\tau \in \mathcal{T}_T} \mathbb{E}((X_\tau - \overline{X}_T)^2)$$

where X is a Brownian motion with drift μ. This problem generalised the work of Graversen et al. (2001), but the method of time change cannot be extended to the case $\mu \neq 0$. The optimal prediction problem is reduced to an equivalent optimal stopping problem in terms of time and the process reflected at its maximum. Hence, by deriving some properties of the value function, the shape of the stopping set D is deduced (being in terms of two boundaries dependent on time) and, with that, a parabolic free boundary problem for the value function is stated. Thus, by using local time-space calculus, a coupled system of nonlinear Volterra integral equations is derived, a system that characterises uniquely the two boundaries and determines an optimal stopping time.

In du Toit and Peskir (2008) and du Toit et al. (2008), the time of the ultimate maximum at a time 1 and the time of the last zero before time 1, respectively, were predicted (in a L_1

sense) for a Brownian motion with drift $\mu \neq 0$, that is,

$$\inf_{\tau \in \mathcal{T}_1} \mathbb{E}(|\overline{\theta}_1 - \tau|) \qquad \text{and} \inf_{\tau \in \mathcal{T}_1} \mathbb{E}(|\xi_1 - \tau|).$$

The first problem is equivalent to an optimal stopping problem in terms of time and the Brownian motion (with drift) reflected at its maximum whereas the second is equivalent to an optimal stopping problem in terms of time and the Brownian motion with drift. The method of solution of both problems is similar to the one described above for du Toit and Peskir (2007).

Shiryaev et al. (2008) predicted the ultimate supremum of a geometric Brownian motion X using an approach different from the most of existing literature at the moment. That is,

$$\sup_{\tau \in \mathcal{T}_T} \mathbb{E}\left(\frac{X_\tau}{\overline{X}_T}\right).$$

After finding an equivalent optimal stopping problem dependent on the reflected Brownian motion at its maximum, a trivial solution was found by using a direct probability approach for some cases depending on the parameters of the process, while the rest of the cases were tackled by du Toit and Peskir (2009).

Shiryaev (2009) focused on the last time of the attainment of the ultimate maximum of a (driftless) Brownian motion and proceeded to show that it is equivalent to predicting the last zero of the process in this setting. Moreover, the optimal predicting problems were focused on minimising the positive part of the difference of the ultimate maximum within the class of stopping times for which the probability of early stopping is below a fixed value α , i.e.

$$\inf_{\tau \in \mathcal{M}_\alpha} \mathbb{E}((\tau - \overline{\theta}_1)^+) \qquad \text{and} \inf_{\tau \in \mathcal{M}'_\alpha} \mathbb{E}((\tau - \xi_1)^+),$$

where $\mathcal{M}_\alpha$ and $\mathcal{M}'_\alpha$ are sub-family of stopping times bounded by 1 such that the probability of stopping early is bounded by α. The method of solution is based on finding and equivalent optimal stopping problem, by using a “Lagrange multipliers method”, performing a deterministic time-change and finding the solution of a free boundary problem.

In du Toit and Peskir (2009), a similar approach as the one in Shiryaev et al. (2008) was used to predict the ultimate maximum of a geometric Brownian motion X, with drift μ and volatility $\sigma > 0$. That is, the proposed optimal prediction problem is

$$\inf_{\tau \in \mathcal{T}_T} \mathbb{E}\left(\frac{\overline{X}_T}{X_\tau}\right).$$

Using standard arguments, they find an equivalent optimal stopping problem in terms of a Brownian motion with drift reflected at its maximum. Then the optimal stopping problem is solved by deducing the shape of D. It is found that the optimal stopping time is trivial for some choices of the parameters whereas, in the remaining cases, it is shown that the optimal stopping is in terms of a moving boundary. The latter case is solved, with the help of the local-time space calculus, by characterising the boundary as the unique solution to a nonlinear Volterra integral equation.

Bernyk et al. (2011) predicted the ultimate supremum of a stable spectrally positive Lévy process of index $\alpha \in (1,2)$ in an L_p sense, that is,

$$\inf_{\tau \in \mathcal{T}_t} \mathbb{E}((X_\tau - \overline{X}_T)^p),$$

where $p \in (1, \alpha)$. Using standard arguments, they find an equivalent two-dimensional optimal stopping problem driven by time and the process reflected at its maximum. The problem is then solved by using a deterministic time-change thus reducing (and solving explicitly) the problem to a free boundary problem given in terms of an integro-diferential equation.

Glover et al. (2013) predicted the time of the ultimate minimum (in an infinite horizon setting) of a mean reverting diffusion that drifts to infinity, that is,

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}(|\underline{\theta} - \tau| - \underline{\theta}).$$

The method of the solution relies on guessing the shape of the stopping set, restricting the analysis to a subclass of stopping times. Hence, together with the free boundary problem, they are able to generate a set of candidate solutions to the value function for which a condition of optimality can be extracted by invoking the subharmonic characterization of the value function.

Baurdoux and van Schaik (2014) predicted the time of the ultimate maximum (in an infinite horizon setting) for a general Lévy process drifting to infinity, that is

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}(|\theta - \tau|).$$

Using standard methods, they show that the optimal prediction problem is equivalent to an infinite horizon optimal stopping problem driven by the Lévy process reflected at its maximum. Then the problems were solved by using a direct probability approach where the shape of the stopping set is deduced from properties of the value function. Moreover, an explicit characterisation is given in the spectrally negative case and the smooth (continuous) pasting property of V when the process is of infinite (finite) variation.

Glover and Hulley (2014) predicted the last time a transient diffusion hits a level $z > 0$. By using standard arguments, the optimal prediction problem is reduced to an optimal stopping problem. By analysing the value function, a solution of a restricted optimal stopping problem is first solved (by using a semi-explicit expression in terms of the scale function and speed measure so the problem can be easily minimised). Finally, by using a verification argument, it is shown that the solution of the restricted problem is also the solution of the original optimal stopping problem.

Baurdoux et al. (2016) predicted the time of the ultimate maximum and the time of the ultimate minimum of a positive self-similar Markov process in a infinite horizon setting. That is, they solved the problems

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}(|\overline{\theta} - \tau| - \overline{\theta}) \qquad \text{and} \qquad \inf_{\tau \in \mathcal{T}} \mathbb{E}(|\underline{\theta} - \tau| - \underline{\theta}).$$

Using standard arguments, both problems are found to be equivalent to optimal stopping problems which are further reduced, via a time change, to optimal stopping problems in terms of spectrally negative Lévy processes reflected on its maximum. These optimal stopping problems are then solved by finding optimal stopping times that minimise a restricted problem within a subfamily of stopping times. Then, by using a verification argument, they showed that the solution of the restricted problem is also the solution of the unrestricted problem.

Finally, Baurdoux and Pedraza (2020b) predicted the last zero of a spectrally negative Lévy process in an infinite horizon setting. Using standard arguments, it is shown that the optimal prediction problem is equivalent to an optimal stopping problem. This is then solved by deducing the shape of the stopping set D (from properties of the value function) and hence restricting the minimisation problem to a subclass of stopping times. The restricted optimal stopping problem is then solved by obtaining a semi-explicit form of the value function in terms of the scale functions for spectrally negative Lévy processes and using standard techniques of calculus to solve the problem.

Note that, as mentioned above, every optimal prediction problem is equivalent to an optimal stopping problem, in other words, optimal prediction problems and optimal stopping problems are intimately related.

On the other hand, Lévy processes are stochastic processes with independent and stationary increments. They can be seen as the continuous-time version of random walks and they form a wide class of stochastic processes that includes some known processes such as Brownian motion, Poisson processes and stable processes. Their applications appear in many areas of classical and modern stochastic processes, including storage models, renewal processes, insurance risk models, optimal stopping problems and mathematical finance. For a detailed overview of Lévy processes, the reader can refer to Bertoin (1998), Sato (1999), Doney (2007), Applebaum (2009) or Kyprianou (2014).

In particular, a special class of Lévy processes called the spectrally negative Lévy processes, which is a subclass of Lévy processes with only negative jumps and non-monotone paths, plays a central role in applied probability such as risk theory, degradation models, queuing theory, finance, etc. This is fundamentally due to the existence of the so-called scale functions and the fact that many fluctuation identities are derived in terms of them since spectrally negative processes can only move upwards in a continuous way.

For instance, the classical risk process (also known as the Cramér–Lundberg process) which consists of a deterministic, positive drift plus a compound Poisson process with only negative jumps, is used to model the capital of an insurance company. The drift can be viewed as a premium rate that is continuously collected and the compound Poisson process represents the claims made to the insurance company. A quantity of interest is the moment of ruin, i.e. the first time that the company has negative capital. Instead of going bankrupt when the risk process becomes negative, suppose that the company has funds to support the negative capital for a while. Then another quantity of interest is the last time that the process is below level zero, that is, the final recovery time in which after the company will have only a positive capital. Indeed Chiu and Yin (2005) proposes the following situation: suppose that the insurance company has many portfolios, so then, when any of them has a negative capital, the others will allow the insurance company to avoid bankruptcy with the hope that, in the long term, such portfolio will have a positive capital. In their work, Chiu and Yin (2005), find (among other things) the Laplace transform of the last time a spectrally negative Lévy process is below any level x. Moreover, as an application to risk theory, they find the joint Laplace transform of the difference of the first and last passage time and their difference for the classical risk process perturbed by a Brownian motion. This approach can be extended to include a general spectrally negative Lévy process.

For the past several decades, degradation data have been used to understand the ageing of a device alongside failure data. Lévy processes turn out to be useful tools for degradation models (see Figure 1). In particular, three models are mainly used: Brownian motion with positive drift, gamma process and compound Poisson process (see Park and Padgett (2005)). More generally, we can consider a spectrally positive Lévy process. The failure time of a component or system can traditionally be derived from a degradation model by considering the first hitting time of a critical level. Recent findings see a new approach being considered as a failure time (see Barker and Newby (2009) and Paroissin and Rabehasaina (2013)) by taking the last passage time below a pre-determined critical level.

The examples mentioned above suggest that the last passage time plays an important role in the applications of spectrally negative Lévy processes. It is however a challenging task to determine the value of the last passage time as it is necessary to be able to observe the whole process. In contrast, stopping times are random times such that the decision of whether to stop or not depends only on the past and present information. It is therefore of interest to predict the last passage times using stopping times. This can be done by finding a stopping time that is as close as possible (in some sense) to the last passage time.

Let us define g_t as the last time a spectrally negative Lévy process X is below the level

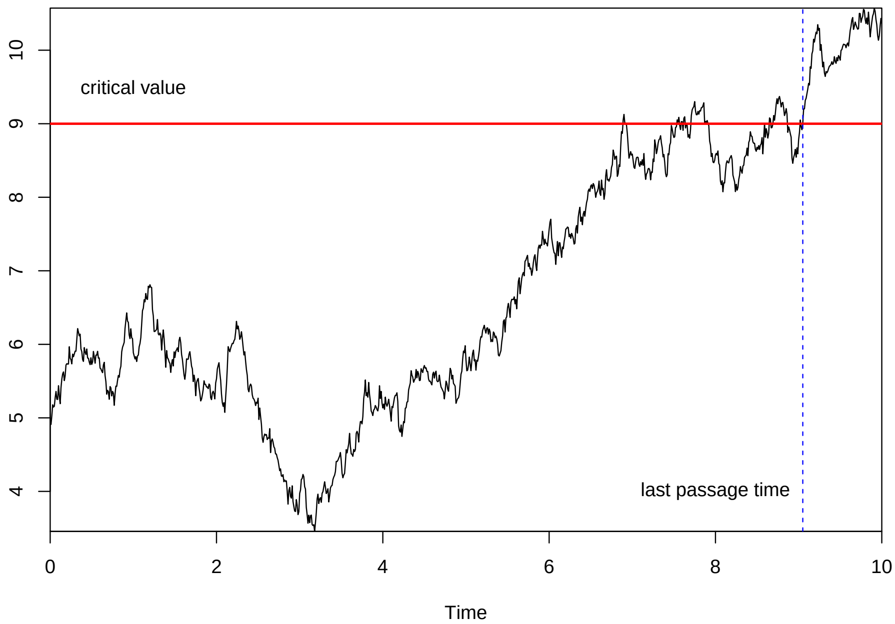

Figure 1: Aging of a Device.

zero before time $t \geq 0$, i.e.

$$g_t = \sup\{0 \leq s \leq t : X_s \leq 0\},$$

where we understand that $\sup \emptyset = 0$. There are some work in the literature related to this last passage time. For example, to mention a few: Chiu and Yin (2005) found the Laplace transform of g_∞ when X drifts to infinity; Baurdoux (2009) generalised the latter result by finding the Laplace transform of the last exit time before an independent exponential time; Li et al. (2017) found the joint Laplace transforms involving the last exit time (from a semi-infinite interval), the value of the process at the last exit time, and the associated occupation time; Cai and Li (2018) derived the Laplace transform of occupation times of intervals until last passage times for spectrally negative Lévy processes. A similar version of g_t is studied in Revuz and Yor (1999) (see Chapter XII.3), namely the last hitting time at zero of a Brownian motion, before any time $t \geq 0$, to describe excursions straddling at a given time. It is also shown that this random time at time $t = 1$ follows the arcsine distribution. The last-hitting time to zero has some applications in the study of Azéma's martingale (see Azéma and Yor

(1989)). In Salminen (1988) the distribution of the last hitting time of a moving boundary is found.

But, what can we say about the dynamics of the process $\{g_t, t \geq 0\}$? It turns out (see Chapter 3) that the three dimensional process (t, g_t, X_t) preserves the Markovian structure of X and it is a semi-martingale. Hence, there is a known (general) expression for the Itô formula (see e.g. Protter (2005), Theorem 33) and its infinitesimal generator (see Dynkin (1965)). However, given the strong dependence between the processes $\{g_t, t \geq 0\}$ and X, more explicit formulas can be obtained in terms of the dynamics of X. Moreover, it is also of mathematical interest and in applications to find formulas involving $U_t = t - g_t$, the length of the current positive excursion, such as the joint Laplace transform of (U, X) (before an exponential time) and the joint q-potential measure.

Let $\mathbf{e}_\theta$ be an independent exponential random variable with parameter $\theta \geq 0$ (here we understand that $\mathbf{e}_\theta$ is infinity when $\theta = 0$). It is of interest to know the value of $g_{\mathbf{e}_\theta}$ at any given time $t \geq 0$, so some early decisions can be taken with that information. However, to know the value of the random variable $g_{\mathbf{e}_\theta}$, we need to know the entire trajectory of the stochastic process X. Hence, there is a need to approximate or predict $g_{\mathbf{e}_\theta}$ with the information available at any moment in time. On the other hand, stopping times are random times such that their realisation can be determined with past and present information. Hence, it becomes natural to predict $g_{\mathbf{e}_\theta}$ with stopping times in some sense. Indeed, for any $p \geq 1$, we can predict the random variable $g_{\mathbf{e}_\theta}$ in an L_p sense with stopping times, that is, we aim to find a stopping time that attains the infimum in

$$V_* = \inf_{\tau \in \mathcal{T}} \mathbb{E}(|g_{\mathbf{e}_\theta} - \tau|^p), \tag{1}$$

where $\mathcal{T}$ is the set of all stopping times of X. In Baurdoux and Pedraza (2020b), where the case for $p = 1$ and $\theta = 0$ when X drifts to infinity is solved. It is shown that the stopping time that minimises the L_1 distance with respect to the last zero is the first time the process crosses above a fixed level $a^* > 0$. This value is characterised as a solution to a non-linear equation involving the cumulative distribution function of the overall infimum of the process. The aim of this study is to solve two more particular cases of this general optimal prediction problem. It is important to note that, to the best of our knowledge, the case $p > 1$ or $\theta > 0$

was never studied before. In Chapters 2 and 4, we dedicate our attention to solve the cases when $p = 1$ and $\theta > 0$; and $p > 1$ and $\theta = 0$, respectively. The main methods of proofs are based on the work of du Toit et al. (2008) and du Toit et al. (2008) (where the underlying process is a Brownian motion with drift) in which the shape of the stopping set is deduced by deriving properties of the value function and then the optimal boundaries are characterised by a system of Volterra integral equations. However, it is important to mention that adding jumps to the underlying process adds an important level of difficulty. For example, in our study, the system of integral equations incorporates a term in which the value function itself is included.

This thesis is divided into 5 chapters. We give a short description of each below:

Chapter 1: In this chapter we list known results related to Spectrally negative Lévy process and Optimal stopping that are needed throughout the thesis. This chapter does not contain any new result and is included to make this a self-contained work.

Chapter 2: In this Chapter, we solve the optimal prediction problem (1) for the case $p = 1$ where we predict the last time, before an exponential time, a spectrally negative Lévy process is below level zero. We show that the optimal prediction problem is equivalent to an optimal stopping problem driven by the two-dimensional process $\{(t, X_t), t \geq 0\}$. We then show that the optimal stopping time is the first time the process crosses above a non-negative, continuous and non-decreasing curve that depends on time. We show that there is smooth pasting on the points for which the curve is strictly positive. Moreover, the aforementioned curve and the value function of the optimal stopping problem are then characterised as the only solutions to a system of non-linear integral equations within a certain family of functions (see Theorem 2.3.13).

Chapter 3: The study in this chapter is mainly aimed at developing the necessary tools to solve the optimal prediction problem in Chapter 4. We derive some important properties of the three-dimensional process $\{(t, g_t, X_t), t \geq 0\}$. In particular, we derive a version of Itô formula and its infinitesimal generator. Moreover, considering the length of the current positive excursion, $U_t = t - g_t$, we obtain a formula for a functional that depends on the whole path of the two dimensional process $(U, X) = \{(U_t, X_t), t \geq 0\}$. As a direct consequence, we find the Laplace transform of $(U_{\mathbf{e}_q}, X_{\mathbf{e}_q})$, where $\mathbf{e}_q$ is an independent exponential random variable and a formula for the density of the q-potential measure of (U, X). The method of proof of the aforementioned results relies on a perturbation method for Lévy process (inspired by the work of Dassios and Wu (2011)) which makes the set of zeros of the perturbed Lévy process a countable set.

Chapter 4: The main contribution of this thesis is presented in this Chapter. We solve the optimal prediction problem (1) for the case in which $\theta = 0$ and $p > 1$ where X drifts to infinity, i.e, we find the stopping time that minimises the L_p distance with the last zero of a spectrally negative Lévy process. We show that the optimal prediction problem is equivalent to an optimal stopping problem driven by the two-dimensional process $(U, X) = \{(U_t, X_t), t \geq 0\}$ (see Lemma 4.3.1), where $U_t = t - g_t$ is the length of the current positive excursion away from zero at time $t \geq 0$. We show that there exists a continuous, non-increasing and non-negative function b such that the optimal stopping time is given by $\tau_D = \inf\{t \geq 0 : X_t \geq b(U_t)\}$. The function b is such that it is infinity at zero and tends to zero at infinity. Thus, it is optimal to stop when we have a sufficiently large positive excursion and the "clock" restarts when the process visits the negative half-line. This feature tells us that it is also important to characterise the value function at the origin, $V(0, 0)$. We show that there is a smooth fit at the boundary for those values where the function b is strictly positive. Moreover, in Theorem 4.4.23 we uniquely characterise the value function V, the curve b and the value function at the origin as the solution of a system of non-linear integral equations within a special class of functions. It is worth mentioning that, to the best of our knowledge, this optimal prediction problem has never been studied before.

Chapter 5: In this Chapter, we use the same perturbation method as in Chapter 3 to find the joint distribution of the number of downcrossings below level zero by a jump from the positive half-line and the local time at zero before an independent exponential time (see Theorem 5.2.1). As a direct result, we are able to calculate the joint Laplace transform of the time of the i-th downcrossing by jump and its overshot. Considering a Lévy insurance risk process, we use these results to calculate the expected present value of all the economic costs from all the downcrossing by jump before an exponential time.

Chapter 1

Preliminaries

1.1 Spectrally negative Lévy processes

A Lévy process $X = \{X_t, t \geq 0\}$ is an almost surely càdlàg process that has independent and stationary increments such that $\mathbb{P}(X_0 = 0) = 1$. We take it to be defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see Definition 1.3.38 of Bichteler (2002)) From the stationary and independent increments property the law of X is characterised by the distribution of X_1. We hence define the characteristic exponent of X, $\Psi(\theta) := -\log(\mathbb{E}(e^{i\theta X_1}))$, $\theta \in \mathbb{R}$. The Lévy–Khintchine formula guarantees the existence of constants, $\mu \in \mathbb{R}$, $\sigma \geq 0$ and a measure Π concentrated in $\mathbb{R} \setminus \{0\}$ with the property that $\int_{\mathbb{R}} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$ (called the Lévy measure) such that for any $\theta \in \mathbb{R}$,

$$\Psi(\theta) = i\mu\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y \mathbb{I}_{\{|y|<1\}})\Pi(\mathrm{d}y).$$

Moreover, from the Lévy–Itô decomposition we can write

$$X_t = \sigma B_t - \mu t + \int_{[0,t]} \int_{(-\infty,-1)} xN(\mathrm{d}s \times \mathrm{d}x) + \int_{[0,t]} \int_{(-1,0)} x(N(\mathrm{d}s \times \mathrm{d}x) - \mathrm{d}s\Pi(\mathrm{d}x)), \quad (1.1)$$

where B is a standard Brownian motion and N is an independent Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}$ with intensity $\mathrm{d}t \times \Pi(\mathrm{d}x)$. We state now some properties and facts about Lévy processes. The reader can refer, for example, to Bertoin (1998), Sato (1999) and Kyprianou (2014) for more details. Every Lévy process X is also a strong Markov $\mathbb{F}$-adapted process.

For all $x \in \mathbb{R}$, denote $\mathbb{P}_x$ as the law of X when started at the point $x \in \mathbb{R}$, that is, $\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot|X_0 = x)$. Due to the spatial homogeneity of Lévy processes, the law of X under $\mathbb{P}_x$ is the same as that of $X + x$ under $\mathbb{P}$.

The process X is a spectrally negative Lévy process if it has no negative jumps ($\Pi(0,\infty) = 0$) with no monotone paths. We state now some important properties and fluctuation identities of spectrally negative Lévy processes which will be of use to us later in this paper. We refer to Chapter 8 in Kyprianou (2014) or Chapter VII in Bertoin (1998) for details.

Due to the absence of positive jumps, we can define the Laplace transform of X_1. We denote $\psi(\beta)$ as the Laplace exponent of the process, that is, $\psi(\beta) = \log(\mathbb{E}(e^{\beta X_1}))$ for $\beta \geq 0$. For such β we have that

$$\psi(\beta) = -\mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)} (e^{\beta y} - 1 - \beta y \mathbb{I}_{\{y>-1\}})\Pi(\mathrm{d}y).$$

The function ψ is infinitely often differentiable and strictly convex function on $(0,\infty)$ with $\psi(\infty) = \infty$. In particular, $\psi'(0+) = \mathbb{E}(X_1) \in [-\infty,\infty)$ determines the behaviour of X at infinity. When $\psi'(0+) > 0$ the process X drifts to infinity, i.e., $\lim_{t\to\infty} X_t = \infty$; when $\psi'(0+) < 0$, X drifts to minus infinity and the condition $\psi'(0+) = 0$ implies that X oscillates, that is, $\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty$. We denote by Φ the right-inverse of ψ, i.e.

$$\Phi(q) = \sup\{\beta \geq 0 : \psi(\beta) = q\}, \qquad q \geq 0.$$

In the particular case that X drifts to infinity, we have that $\psi'(0+) > 0$ which implies that ψ is strictly increasing and then Φ is the usual inverse with $\Phi(0) = 0$.

The path variation of any Lévy process can be determined by σ and the Lévy measure Π. Indeed, the process X has paths of finite variation if and only if $\sigma = 0$ and $\int_{(-1,0)} |x|\Pi(\mathrm{d}x) < \infty$, otherwise X has paths of infinite variation. If X is of finite variation we can rewrite (1.1) as

$$X_t = \delta t + \int_{[0,t]} \int_{(-\infty,0)} x N(\mathrm{d}s \times \mathrm{d}x),$$

where

$$\delta := -\mu - \int_{(-1,0)} x\Pi(\mathrm{d}x) \tag{1.2}$$

Note that processes with monotone paths are excluded from the definition of spectrally negative Lévy processes, so we assume that $\delta > 0$ when X is of finite variation.

Denote by τ_a^+ the first passage time above the level $a > 0$,

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}.$$

The Laplace transform of τ_a^+ is given by

$$\mathbb{E}(e^{-q\tau_a^+}) = e^{-\Phi(q)a} \qquad a > 0. \tag{1.3}$$

An important family of functions for spectrally negative Lévy processes consists of the scale functions, usually denoted by $W^{(q)}$ and $Z^{(q)}$. There are many fluctuation identities in terms of these functions. The reader can refer for example to Bertoin (1998) (Chapter VII), Kuznetsov et al. (2013),Kyprianou (2014) (Chapter 8) and Avram et al. (2019) for an extensive review of them. We mention those identities which will be useful in forthcoming chapters.

For all $q \geq 0$, the scale function $W^{(q)} : \mathbb{R} \mapsto \mathbb{R}_+$ is such that $W^{(q)}(x) = 0$ for all $x < 0$ and it is characterised on the interval $(0, \infty)$ as the strictly and continuous function with Laplace transform given by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \text{for } \beta > \Phi(q). \tag{1.4}$$

The function $Z^{(q)}$ is defined for all $q \geq 0$ by

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y, \qquad \text{for } x \in \mathbb{R}.$$

For the case $q = 0$ we simply denote $W = W^{(0)}$. The behaviour of $W^{(q)}$ at infinity is the following. For $q \geq 0$, we have

$$\lim_{x \to \infty} e^{-\Phi(q)x} W^{(q)}(x) = \Phi'(q).$$

For all $q \geq 0$, the function $W^{(q)}$ has left and right derivatives. Moreover, from Kuznetsov et al. (2013) (Theorem 3.10) we know that if $\sigma^2 > 0$, $W^{(q)} \in C^2(0, \infty)$. When X is of finite variation $W^{(q)} \in C^1(0, \infty)$ when Π has no atoms. For all $q \geq 0$, the values of $W^{(q)}$ in the neighbourhood of zero can be deduced from (1.4):

$$W^{(q)}(0) = \begin{cases} \frac{1}{\delta} & \text{if } X \text{ is of finite variation} \\ 0 & \text{if } X \text{ is of infinite variation} \end{cases}.$$

The equation above implies that $W^{(q)}$ is continuous on $\mathbb{R}$ when X has paths of infinite variation. The right derivative at the origin is

$$W^{(q)\prime}(0+) = \begin{cases} \frac{\Pi(-\infty,0)+q}{\delta^2} & \text{if } X \text{ is of finite variation} \\ \frac{2}{\sigma^2} & \text{if } X \text{ is of infinite variation} \end{cases}, \tag{1.5}$$

where we understand $1/\infty = 0$ when $\sigma = 0$. Moreover, the second right-derivative at zero of $W^{(q)}$ can be found (see e.g. Avram et al. (2019)). In particular, when $\sigma > 0$, we have that

$$W^{(q)\prime\prime}(0+) = -\delta \left(\frac{2}{\sigma^2} \right)^2 \tag{1.6}$$

where δ is defined in (1.2) and we understand that $\delta = \infty$ when the jumps of X are of infinite variation.

For each $x \geq 0$ and $q \geq 0$, $W^{(q)}$ has the following alternative representation

$$W^{(q)}(x) = \sum_{k=0}^{\infty} q^k W^{*(k+1)}(x), \tag{1.7}$$

where $W^{*(k+1)}$ is the $(k+1)$-th convolution of W with itself. Various fluctuation identities for spectrally negative Lévy processes have been found in terms of the scale functions. Here we list some that will be useful in later chapters. Denote by τ_x^- the first passage time below the level $x \leq 0$, i.e.,

$$\tau_x^- = \inf\{t > 0 : X_t < x\}.$$

Then for any $q \geq 0$ and $x \leq a$ we have

$$\mathbb{E}_x\left(e^{-q\tau_a^+}\mathbb{I}_{\{\tau_0^- > \tau_a^+\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{1.8}$$

For any $x \in \mathbb{R}$ and $q \geq 0$,

$$\mathbb{E}_x(e^{-q\tau_0^-}\mathbb{I}_{\{\tau_0^- < \infty\}}) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x), \tag{1.9}$$

where we understand $q/\Phi(q)$ in the limiting sense when $q = 0$ so that

$$\mathbb{P}_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+)W(x) & \text{if } \psi'(0+) \geq 0 \\ 1 & \text{if } \psi'(0+) < 0 \end{cases}. \tag{1.10}$$

More generally, the joint Laplace transform of τ_0^- and $X_{\tau_0^-}$ is

$$\mathbb{E}_x(e^{-q\tau_0^- + \beta X_{\tau_0^-}}\mathbb{I}_{\{\tau_0^- < \infty\}}) = e^{\beta x}\mathcal{I}^{(q,\beta)}(x) \tag{1.11}$$

for all $x \in \mathbb{R}$ and $q > \psi(\beta) \vee 0$, where the function $\mathcal{I}^{(q,\beta)}$ is given by

$$\mathcal{I}^{(q,\beta)}(x) := 1 + (q - \psi(\beta))\int_0^x e^{-\beta y}W^{(q)}(y)\mathrm{d}y - \frac{q - \psi(\beta)}{\Phi(q) - \beta}e^{-\beta x}W^{(q)}(x) \qquad x \in \mathbb{R}. \tag{1.12}$$

In particular, for any $p \geq 0$, taking $\beta = \Phi(p)$ and $q = p + h$ and letting $h \downarrow 0$ (here we use that $\psi(\Phi(p)) = p$), we obtain that for any $x \in \mathbb{R}$,

$$\mathbb{E}_x(e^{-p\tau_0^- + \Phi(p)X_{\tau_0^-}}\mathbb{I}_{\{\tau_0^- < \infty\}}) = e^{\Phi(p)x}\mathcal{I}^{(q,\Phi(q))}(x) = e^{\Phi(p)x}(1 - \psi'(\Phi(p))e^{-\Phi(p)x}W^{(p)}(x)). \tag{1.13}$$

Since X has only negative jumps we have that the process X creeps upwards, that is

$$\mathbb{P}(X_{\tau_x^+} = x | \tau_x^+ < \infty) = 1.$$

Moreover, X creeps downwards if and only if $\sigma > 0$. The latter fact can be easily deduced from the Laplace transform of τ_0^- (see Theorem 2.6 in Kuznetsov et al. (2013)) in the event of creeping:

$$\mathbb{E}_x(e^{-p\tau_0^-}\mathbb{I}_{\{X_{\tau_0^-}=0\}}) = \mathcal{C}^{(p)}(x), \tag{1.14}$$

where for all $p \geq 0$ the function $\mathcal{C}^{(p)}$ is given by

$$\mathcal{C}^{(p)}(x) := \frac{\sigma^2}{2}\{W^{(p)\prime}(x) - \Phi(p)W^{(p)}(x)\}, \qquad x \in \mathbb{R}. \tag{1.15}$$

In particular we have that for any $x \in \mathbb{R}$,

$$\mathbb{P}_x(X_{\tau_0^-} = 0, \tau_0^- < \infty) = \frac{\sigma^2}{2}\left[W'(x) - \Phi(0)W(x)\right]. \tag{1.16}$$

Denote by $\underline{X}_t = \inf_{0\leq s\leq t} X_s$ and $\overline{X}_t = \sup_{0\leq s\leq t} X_s$ the running infimum and running maximum of the process X up to time $t > 0$, respectively. For $q \geq 0$, let $\mathbf{e}_q$ be an exponential random variable with mean $1/q$ independent of X, where we understand that $\mathbf{e}_q = \infty$ almost surely when $q = 0$. Then $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$ and the Laplace transform of $\underline{X}_{\mathbf{e}_q}$ is given by (see e.g. equation (8.4) in Kyprianou (2014), pp 233)

$$\mathbb{E}(e^{\beta \underline{X}_{\mathbf{e}_q}}) = \frac{q}{\Phi(q)}\frac{\Phi(q)-\beta}{q-\psi(\beta)}, \qquad \beta \geq 0. \tag{1.17}$$

Moreover, it turns out that the density of the random variable $-\underline{X}_{\mathbf{e}_q}$ can be written in terms of the scale function $W^{(q)}$ (see e.g. equation (8.24) in Kyprianou (2014), pp 239). Indeed, for all $x \geq 0$,

$$\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}x) = \frac{q}{\Phi(q)}W^{(q)}(\mathrm{d}x) - qW^{(q)}\mathrm{d}x.$$

Then, given the continuity of W on $(0,\infty)$, we have that the cumulative distribution function of the random variable $-\underline{X}_{\mathbf{e}_q}$ is continuous on $(0,\infty)$; with a possible discontinuity at 0 when X is of finite variation (due to the discontinuity of $W^{(q)}$ at zero in this case).

Denote by σ_x^- the first time the process X is below or equal to the level x, i.e.

$$\sigma_x^- = \inf\{t > 0 : X_t \leq x\}.$$

It is easy to show that the mapping $x \mapsto \sigma_x^-$ is non-increasing, right-continuous with left limits. The left limit is given by $\lim_{h\downarrow 0} \sigma_{x-h}^- = \tau_x^-$ for all $x \in \mathbb{R}$. Moreover, it is easy to show that σ_x^- and τ_x^- have the same distribution for all $x < 0$. Indeed, for any $q \geq 0$ and $x < 0$ we have that

$$\mathbb{E}(e^{-q\sigma_x^-}\mathbb{I}_{\{\sigma_x^- < \infty\}}) = \mathbb{P}(\sigma_x^- < \mathbf{e}_q) = \mathbb{P}(-\underline{X}_{\mathbf{e}_q} \geq -x) = \mathbb{P}(-\underline{X}_{\mathbf{e}_q} > -x) = \mathbb{E}(e^{-q\tau_x^-}\mathbb{I}_{\{\sigma_x^- < \infty\}}),$$

where we used the fact that the distribution function of $-\underline{X}_{\mathbf{e}_q}$ is continuous on $(0, \infty)$. When X is of infinite variation, X enters instantly to the set $(-\infty, 0)$ whilst in the finite variation case there is a positive time before the process enters it. That implies that in the infinite variation case $\tau_0^- = \sigma_0^- = 0$ a.s. whereas in the finite variation case, $\sigma_0^- = 0$ and $\tau_0^- > 0$.

Let $q > 0$ and $a \in \mathbb{R}$. The q-potential measure of X killed on exiting $[0, a]$ is absolutely continuous with respect to Lebesgue measure and it has a density given by

$$\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y, t < \tau_a^+ \wedge \tau_0^-)\mathrm{d}t = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \qquad x, y \in [0, a]. \tag{1.18}$$

The q-potential measure of X killed on exiting $[0, \infty)$ is absolutely continuous with respect to Lebesgue measure and it has a density given by

$$\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y, t < \tau_0^-)\mathrm{d}t = e^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x-y) \qquad x, y \geq 0. \tag{1.19}$$

Similarly, the q-potential measure of X killed on exiting $(-\infty, a]$ and the q-potential measure of X are absolutely continuous with respect to Lebesgue measure with a density given by

$$\int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y, t < \tau_a^+)\mathrm{d}t = e^{-\Phi(q)(a-x)}W^{(q)}(a-y) - W^{(q)}(x-y), \qquad x, y \leq a, \tag{1.20}$$

and

$$\int_0^\infty e^{-qt}\mathbb{P}_x(X_t\in \mathrm{d}y)\mathrm{d}t = \Phi'(q)e^{-\Phi(q)(y-x)} - W^{(q)}(x-y), \qquad x,y\in\mathbb{R}, \tag{1.21}$$

respectively. In the case when X drifts to infinity these expression are also valid for $q=0$.

Let $\beta\geq 0$, the process given by $\{e^{\beta X_t-\psi(\beta)t}, t\geq 0\}$ is a martingale. Then for each such β, we can define a change of measure given by

$$\left.\frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\beta X_t-\psi(\beta)t}. \tag{1.22}$$

Under the measure $\mathbb{P}^\beta$, X is a Lévy process with Laplace exponent given by $\psi_\beta(\lambda)=\psi(\lambda+\beta)-\psi(\beta)$ for $\lambda\geq-\beta$ and hence $\Phi_\beta(q):=\sup\{\lambda\geq -\beta : \psi_\beta(\lambda)=q\}=\Phi(q+\psi(\beta))-\beta$ for $q\geq-\psi(\beta)$. In other words, under $\mathbb{P}^\beta$, X has Lévy triplet

$$\left(\mu-\sigma^2\beta-\int_{(-1,0)} y(e^{\beta y}-1)\Pi(\mathrm{d}y), \sigma^2, e^{\beta y}\Pi(\mathrm{d}y)\right).$$

In the particular case when $\beta=\Phi(q)$ for $q\geq 0$ we have that $\psi_{\Phi(q)}(\lambda)=\psi(\lambda+\Phi(q))-q$. That implies that for any $q>0$, $\psi'_{\Phi(q)}(0+)=\psi'(\Phi(q))\geq 0$ and then the process X drifts to infinity under the measure $\mathbb{P}^{\Phi(q)}$. Moreover, denote $W_{\Phi(q)}$ the 0-scale function under the measure $\mathbb{P}^{\Phi(q)}$, we have that $W^{(q)}(x)=e^{\Phi(q)x}W_{\Phi(q)}(x)$ for all $x\in\mathbb{R}$ and $q>0$.

Note that by a change of measure we can show that for all $x<0$, the vector $(\tau_x^-, X_{\tau_x^-})$ has the same distribution as $(\sigma_x^-, X_{\sigma_x^-})$ under the measure $\mathbb{P}$. Indeed take $q>\psi(\beta)\vee 0$, then we have that

$$\begin{aligned}
\mathbb{E}\left(e^{-q\sigma_x^- +\beta X_{\sigma_x^-}}\mathbb{I}_{\{\sigma_x^-<\infty\}}\right) &= \mathbb{E}^\beta\left(e^{-(q-\psi(\beta))\sigma_x^-}\mathbb{I}_{\{\sigma_x^-<\infty\}}\right)\\
&= \mathbb{E}^\beta\left(e^{-(q-\psi(\beta))\tau_x^-}\mathbb{I}_{\{\tau_x^-<\infty\}}\right)\\
&= \mathbb{E}\left(e^{-q\tau_x^- +\beta X_{\tau_x^-}}\mathbb{I}_{\{\tau_x^-<\infty\}}\right).
\end{aligned}$$

The assertion then follows. Another important family of martingales is the following. Let

$q \geq 0$, then the process

$$\{e^{-q(t\wedge\tau_0^-)}W^{(q)}(X_{t\wedge\tau_0^-}), t \geq 0\}$$

is a $\mathbb{P}_x$-martingale for all $x \in \mathbb{R}$. Having the above martingale in mind, we are able to define the process conditioned to stay positive (see Bertoin (1998), Section VII.3). For any $x > 0$, we can define the probability measure $\mathbb{P}_x^{\uparrow}$ by

$$\mathbb{P}_x^{\uparrow}(A) = \frac{1}{W(x)}\mathbb{E}_x\left(W(X_t)\mathbb{I}_{A\cap\{t<\tau_0^-\}}\right) \tag{1.23}$$

for any $A \in \mathcal{F}_t$ and $t > 0$. It is shown that, for any $x > 0$, X is a Markov process under $\mathbb{P}_x^{\uparrow}$ and that

$$\mathbb{P}_x^{\uparrow}(X_t \in \mathrm{d}y) = \frac{W(y)}{W(x)}\mathbb{P}_x(X_t \in \mathrm{d}y, t < \tau_0^-)$$

for any $y \in \mathbb{R}$ and $t > 0$. Moreover, it is shown that (see Bertoin (1998), Proposition VII.3.14) that the probability $\mathbb{P}_x^{\uparrow}$ converges as $x \downarrow 0$ in the sense of finite-dimensional to distribution to a limit which is defined as $\mathbb{P}_0^{\uparrow} = \mathbb{P}^{\uparrow}$ and that X is a Markov process under $\mathbb{P}^{\uparrow}$. Moreover, we have the following formula

$$\mathbb{P}^{\uparrow}(X_t \in \mathrm{d}y) = \frac{yW(y)}{t}\mathbb{P}(X_t \in \mathrm{d}y) \tag{1.24}$$

for any $x, t > 0$. Furthermore, (see Corollary VII.4.19) in Bertoin (1998)) we have that for any $x > 0$, the process $\{X_{g^{(x)}+t} - x, t \geq 0\}$ has law $\mathbb{P}^{\uparrow}$, where

$$g^{(x)} = \sup\{t > 0 : X_t \leq 0\}.$$

Next, we state the compensation formula for Poisson random measures which is valid for any Lévy process X with Poisson random measure N with intensity $\mathrm{d}t \times \Pi(\mathrm{d}x)$. Suppose that $\phi : [0,\infty)\times\mathbb{R}\times\Omega \mapsto [0,\infty)$ is a function such that $(t, x, \omega) \mapsto \phi(t, x)(\omega)$ is $\mathbb{B}([0,\infty))\times\mathbb{B}(\mathbb{R})\times\mathcal{F}$ measurable, for each fixed $t \geq 0$, the function $(x, \omega) \mapsto \phi(t, x)(\omega)$ is $\mathbb{B}(\mathbb{R}) \times \mathcal{F}_t$ measurable and for any $x \in \mathbb{R}$, the process $\{\phi(t, x), t \geq 0\}$ is almost surely a left-continuous process.

Then for any $t \geq 0$ holds that

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \phi(s,x) N(\mathrm{d}s, \mathrm{d}x)\right) = \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \phi(s,x) \mathrm{d}s \Pi(\mathrm{d}x)\right). \tag{1.25}$$

In particular, if the right hand side of the equation above is finite we have that the process $\{M_t, t \geq 0\}$ is martingale, where

$$M_t = \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \phi(s,x) N(\mathrm{d}s, \mathrm{d}x)\right) - \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \phi(s,x) \mathrm{d}s \Pi(\mathrm{d}x)\right).$$

It can be easily seen from the Lévy Itô decomposition that Lévy processes are semimartingales and hence the Itô formula is well known (see e.g. Protter (2005), Theorem 32 pp 78). In particular, for the spectrally negative case, takes the form

$$\begin{aligned} f(t, X_t) = f(0, X_t) &+ \int_0^t \frac{\partial f}{\partial t}(s, X_s) \mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) \mathrm{d}X_s + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) \mathrm{d}s \\ &+ \int_0^t \int_{(-\infty,0)} \left[f(s, X_{s-} + y) - f(s, X_{s-}) - y \frac{\partial f}{\partial x}(s, X_{s-}) \right] N(\mathrm{d}s, \mathrm{d}y) \end{aligned}$$

for any $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. The infinitesimal generator of the process X (see e.g. Applebaum (2009), Theorem 3.3.3) takes the form

$$\begin{aligned} \mathcal{A}_X(f)(t,x) = &-\mu \frac{\partial}{\partial x} f(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f(t,x) \\ &+ \int_{(-\infty,0)} \left(f(t, x+y) - f(t,x) - y \mathbb{I}_{\{y > -1\}} \frac{\partial}{\partial x} f(t,x) \right) \Pi(\mathrm{d}y). \end{aligned} \tag{1.26}$$

where $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, the set of all bounded $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ functions with bounded derivatives. It is also well known that a continuous version of the local time can be defined (see Protter (2005), Section IV.7). Specifically, there exists an adapted, right continuous and increasing process $\{A_t^a, t \geq 0\}$ such that the following equation is satisfied:

$$|X_t - a| = |X_0 - a| + \int_0^t \mathrm{sign}(X_{s-} - a) \mathrm{d}X_s + A_t^a$$

for all $t \geq 0$, where the sign function is given by

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}.$$

The local time at the level a, $\{L_t^a, t \geq 0\}$, is defined as the continuous part of $\{A_t^a, t \geq 0\}$, i.e.

$$L_t^a = A_t^a - \sum_{0<s\leq t} \{|X_s - a| - |X_{s-} - a| - \text{sign}(X_{s-} - a)\Delta X_s\}.$$

The measure $\mathrm{d}L_t^a$ is carried by the set $\{t > 0 : X_{t-} = X_t = a\}$. Moreover, we have the occupation time density formula given by

$$\int_{-\infty}^{\infty} L_t^a g(a)\mathrm{d}a = \sigma^2 \int_0^t g(X_s)\mathrm{d}s$$

for all $t \geq 0$. Then it can be shown (see Bertoin (1998) Proposition V.2) that

$$L_t^a = \sigma^2 \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}_{\{|X_s - a| < \varepsilon\}} \mathrm{d}s$$

uniformly on compact intervals of time in L^2. Furthermore, for each stopping time τ such that $X_\tau = 0$ a.s. on $\{\tau < \infty\}$, the process $(X_{\tau+t}, L_{\tau+t} - L_\tau)$ is independent of $\mathcal{F}_\tau$ and has the same law as (X, L). For ease of notation, we simply denote $L = \{L_t, t \geq 0\}$ as the local time at zero, i.e., $L_t = L_t^0$ for all $t \geq 0$.

1.2 Optimal stopping

The theory of optimal stopping is concerned with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximise an expected payoff or to minimise an expected cost. Problems of this type are found in the area of statistics, where the action taken may be to test a hypothesis or to estimate a parameter, in the area of operations research, where the action may be to replace a machine, hire a secretary, or reorder stock and in applications to finance, valuation of American options.

The aim of the present section is to introduce basic results of general theory of optimal stopping. First we study the martingale approach in continuous time and then the Markovian approach, both only in an infinite horizon of time. This section is mainly based on Peskir and Shiryaev (2006).

1.2.1 Essential Supremum

Recall that if we take the supremum over an uncountable set of random variables then this does not necessarily defines a measurable function. To overcome this difficulty the concept of essential supremum proves to be useful. Let $\{Z_\alpha, \alpha \in I\}$ be a collection of real-valued random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with I an arbitrary index set. Then there exists a countable subset $J \subseteq I$ such that the random variable $Z^* : \Omega \mapsto \mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$Z^* = \sup_{\alpha \in J} Z_\alpha, \tag{1.27}$$

satisfies

i) $\mathbb{P}(Z_\alpha \leq Z^*) = 1$ for all $\alpha \in I$.

ii) If $Y : \Omega \mapsto \mathbb{R} \cup \{-\infty, \infty\}$ is another random variable satisfying *i)* then,

$$\mathbb{P}(Z^* \leq Y) = 1.$$

We call Z^* the essential supremum of $\{Z_\alpha, \alpha \in I\}$, and write

$$Z^* = \operatorname*{ess\,sup}_{\alpha \in I} Z_\alpha.$$

It is defined uniquely $\mathbb{P}$-almost surely.

Moreover, if the family $\{Z_\alpha, \alpha \in I\}$ is upwards directed, that is, for any $\alpha, \beta \in I$ there exists $\gamma \in I$ such that

$$Z_\alpha \vee Z_\beta \leq Z_\gamma \qquad \mathbb{P}\text{-a.s.}$$

Then there exists a countable set $J = \{\alpha_n, n \geq 1\}$ such that $Z_{\alpha_n} \leq Z_{\alpha_{n+1}}$ for any $n \geq 1$ and

$$Z^* = \lim_{n \to \infty} Z_{\alpha_n} \qquad \mathbb{P}\text{-a.s.}$$

1.2.2 Martingale Approach

Let $G = \{G_t, t \geq 0\}$ a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is a filtration of $\mathcal{F}$. Suppose that the filtration $\mathbb{F}$ satisfies the natural conditions (see Definition 1.3.38 of Bichteler (2002)), also assume that G is adapted to the filtration $\mathcal{F}$. We interpret G_t as the gain if the observation of G is stopped at time t.

We will assume that the process G is right-continuous and left-continuous over stopping times (if τ_n and τ are stopping times such that $\tau_n \to \tau$ as $n \to \infty$ then $G_{\tau_n} \to G_\tau$ $\mathbb{P}$-a.s. as $n \to \infty$). We will also assume that the following condition is satisfied,

$$\mathbb{E}\left(\sup_{t\geq 0} |G_t|\right) < \infty. \tag{1.28}$$

Define for all $t \geq 0$,

$$\mathcal{T}_t = \{\tau \geq t : \tau \text{ is stopping time}\},$$

the set of all stopping times greater or equal to t. For simplicity we only write $\mathcal{T}$ instead of $\mathcal{T}_0$, i.e. we denote by $\mathcal{T}$ the set of all stopping times.
Consider the optimal stopping problem

$$V_t = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau). \tag{1.29}$$

To solve the problem (1.29), consider the process $S = \{S_t, t \geq 0\}$ defined as follows:

$$S_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}(G_\tau|\mathcal{F}_t), \tag{1.30}$$

the process S is often called the Snell envelope of G. Note that by the definition of S_t we have that if we take $\tau = t$ then $S_t \geq G_t$ $\mathbb{P}$-a.s. Consider the following stopping time for $t \geq 0$

$$\tau_t = \inf\{s \geq t : S_s = G_s\},$$

where we define $\inf \emptyset = \infty$. It turns out that the process $\{S_t, t \geq 0\}$ defined in (1.30) is a supermartingale and admits a càdlàg modification. Moreover, the following relation holds,

$$\mathbb{E}(S_t) = V_t. \tag{1.31}$$

If $\mathbb{P}(\tau_t < \infty) = 1$ for all $t \geq 0$ we have,

$$S_t \geq \mathbb{E}(G_\tau|\mathcal{F}_t) \qquad \text{for each stopping time } \tau \in \mathcal{T}_t \tag{1.32}$$

$$S_t = \mathbb{E}(G_{\tau_t}|\mathcal{F}_t). \tag{1.33}$$

Moreover, if $t \geq 0$ is given and fixed, we have:

i) The stopping time τ_t is optimal in (1.29).

ii) If τ_* is an optimal stopping time in (1.29) then $\tau_t \leq \tau_*$ $\mathbb{P}$-a.s.

iii) The process $\{S_s, s \geq t\}$ is the smallest right-continuous supermartingale which dominates $\{G_s, s \geq t\}$.

iv) The stopped process $\{S_{s\wedge\tau_t}, s \geq t\}$ is a right-continuous martingale.

1.2.3 Markovian Approach

In this subsection we will consider a strong Markov process $X = \{X_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_x)$ and taking values in $(E, \mathcal{B}) = (\mathbb{R}, \mathbb{B}(\mathbb{R}))$. It is assumed that the process X starts at x under the probability measure $\mathbb{P}_x$ for $x \in \mathbb{R}$ and the sample paths of X are right-continuous and left-continuous over stopping times. It is also assumed that the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfies the natural conditions. In addition, it is assumed that the mapping $x \mapsto \mathbb{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. Finally, without loss of generality we will assume that $(\Omega, \mathcal{F})$ is equal to the canonical space $(E^{[0,\infty)}), \mathcal{B}^{[0,\infty)})$ so that the shift operator $\theta_t : \Omega \mapsto \Omega$ is well defined by $\theta_t(\omega)(s) = \omega(t+s)$ for $\omega = \{\omega(s), s \geq 0\} \in \Omega$ and $s, t \geq 0$.

Suppose that $G : E \mapsto \mathbb{R}$ is a measurable function which satisfies the condition

$$\mathbb{E}_x\left(\sup_{t\geq 0} |G(X_t)|\right) < \infty, \tag{1.34}$$

where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x$ and $x \in \mathbb{E}$. We consider the optimal stopping problem

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(G(X_\tau)), \tag{1.35}$$

where $x \in E$ and $\mathcal{T}$ is the set of all stopping times of $\mathbb{F}$. The function V is called the value function and G is called the gain function. Solving the optimal stopping problem (1.35) means two things. Firstly, we need to find an optimal stopping time, i.e. a stopping time τ_* at which the supremum is attained. Secondly, we need to compute the value $V(x)$ for $x \in E$ as explicitly as possible.

Note that if we take $\tau \equiv 0$ we have that from definition of V given in (1.35),

$$V(x) \geq \mathbb{E}_x(G(X_0)) = G(x) \tag{1.36}$$

The Markovian structure of X means that the process always starts afresh. Then for a fixed sample path we shall be able to decide whether to continue with the observation or to stop it. Thinking in this way we split the set E into two disjoint subsets, the continuation set C and the stopping set $D = E \setminus C$. It follows that as soon as the process enters into D, the observation should be stopped and an optimal stopping time is obtained. It turns out that the continuation set is given by

$$C = \{x \in E : V(x) > G(x)\} \tag{1.37}$$

and the stopping set

$$D = \{x \in E : V(x) = G(x)\}. \tag{1.38}$$

Formally, we define the process $\{G_t, t \geq 0\}$ where

$$G_t = G(X_t), \qquad t \geq 0.$$

Then the Snell envelope process of $\{G_t, t \geq 0\}$ under the measure $\mathbb{P}_x$ for $x \in E$ is given by

$\{S_t, t \geq 0\}$ where

$$\begin{aligned} S_t &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_x(G(X_\tau)|\mathcal{F}_t) \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E}_x(G(X_{\tau+t})|\mathcal{F}_t) \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E}_{X_t}(G(X_\tau)) \\ &= V(X_t). \end{aligned}$$

Hence an optimal stopping time is given by

$$\begin{aligned} \tau_0^* &= \inf\{t \geq 0 : S_t = G_t\} \\ &= \inf\{t \geq 0 : V(X_t) = G(X_t)\} \\ &= \inf\{t \geq 0 : X_t \in D\}. \end{aligned}$$

Proving that we have to stop when the process enters for the first time into the set D and continue otherwise. Let $f : E \mapsto \mathbb{R}$ be a function and take $c \in E$. The function f is said to be upper semi-continuous at a point c when

$$f(c) \geq \limsup_{x \to c} f(x).$$

It is said to be upper semi-continuous (usc) on E if it is upper semi-continuous at every point of E. In a similar way, f is said to be lower semi-continuous at a point c when

$$f(c) \leq \liminf_{x \to c} f(x).$$

It is said to be lower semi-continuous (lsc) on E if it is lower semi-continuous at every point of E. When $E = \mathbb{R}$ upper semi-continuity in $c \in E$ can be written in the following way. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all x such that $|x - c| < \delta$ then $f(x) \leq f(c) + \varepsilon$. Lower semi-continuity can be written, for all $\varepsilon > 0$ exists $\delta > 0$ such that for all x such that $|x - c| < \delta$ then $f(x) \geq f(c) - \varepsilon$. It can be shown that if V is lower semi-continuous and G upper semi-continuous then C is open and D is closed. Introduce the first entry time τ_D of

X into D by setting

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}. \tag{1.39}$$

Let us assume that there exists an optimal stopping time τ_* in (1.35), i.e.,

$$V(x) = \mathbb{E}_x(G(X_{\tau_*}))$$

for all $x \in E$. Then we have that if V is lsc and G is usc, then

ii) The process $\{V(X_t), t \geq 0\}$ is a right-continuous supermartingale.

iii) The stopping time τ_D satisfies $\tau_D \leq \tau_*$ $\mathbb{P}_x$-a.s. for all $x \in E$ and is optimal in (1.35).

iv) The stopped process $\{V(X_{t\wedge\tau_D}), t \geq 0\}$ is a right-continuous martingale under $\mathbb{P}_x$ for every $x \in E$.

The following result (extracted from Peskir and Shiryaev (2006), Corollary 2.9) is a very useful result when we are able to prove directly that V is lsc.

> Consider the optimal stopping problem (1.35) upon assuming that the condition (1.34) is satisfied. Suppose that V is lsc and G is usc. If $\mathbb{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, then τ_D is optimal in (1.35).

In this thesis we consider optimal stopping problems of the form

$$V_t = \inf_{\tau \in \mathcal{T}_t} \mathbb{E}(G_\tau).$$

The theory studied in this chapter also applies for these problems. We only have to consider the process $G' = \{G'_t, t \geq 0\}$ where $G'_t = -G_t$ for all $t \geq 0$.

Chapter 2

Predicting the Last Zero before an exponential time of a Spectrally Negative Lévy Process

Abstract

Given a spectrally negative Lévy process, we predict, in a L_1 sense, the last passage time of the process below zero before an independent exponential time. Using a similar argument as that in Urusov (2005), we show that this optimal prediction problem is equivalent to solving an optimal prediction problem in a finite horizon setting. The optimal stopping time is the first time the process crosses above a non-negative, continuous and non-increasing curve depending on time. This curve and the value function are characterised as a solution of a system of non-linear integral equations which can be understood as a generalisation of the free boundary equations (see e.g. Peskir and Shiryaev (2006) Chapter IV.14.1) in the presence of jumps.

2.1 Introduction

The study of last exit times has received much attention in several areas of applied probability, e.g. risk theory, finance and reliability in the past few years. Consider the Cramér–Lundberg process, a process consisting of a deterministic drift and a compound Poisson process with only negative jumps (see Figure 2.1), which is typically used to model the capital of an insurance company. Of particular interest is the moment of ruin, τ_0 which is defined to refer to the first moment when the process becomes negative. Within the framework of the insurance company having sufficient funds to endure negative capital for a considerale amount of time, another quantity of interest is the last time, g that the process is below zero. In a more general setting, we can consider a spectrally negative Lévy process instead of the classical risk process. Several studies, for example Baurdoux (2009) and Chiu and Yin (2005) studied the Laplace transform of the last time before an exponential time that a spectrally negative Lévy process is below some given level.

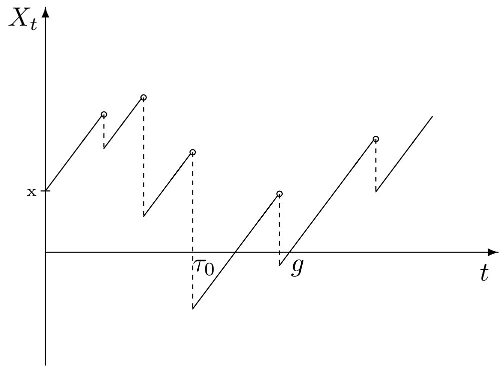

Figure 2.1: Cramér–Lundberg process with τ_0, the moment of ruin and g, the last zero.

Last passage time is increasingly becoming a vital factor in financial modeling as shown in Madan et al. (2008a) and Madan et al. (2008b) where the authors concludes that the price of a European put and call options, modelled by non-negative and continuous martingales that vanish at infinity, can be expressed in terms of the probability distributions of some last passage times.

Another application of last passage times is in degradation models. Paroissin and Rabehasaina (2013) propose a spectrally positive Lévy process to model the ageing of a device in which they consider a subordinator perturbed by an independent Brownian motion. A motivation for considering this model is that the presence of a Brownian motion can model small repairs of the device and the jumps represent major deterioration. In the literature, the failure time of a device is defined as the first hitting time of a critical level b. An alternative approach is to consider instead, the last time that the process is under the level b since the paths of this process are not necessarily monotone and this allows the process to return below the level b after it goes above b.

The aim of this work is to predict the last time a spectrally negative Lévy process is below zero before an independent exponential time where the terms "to predict" are understood to mean to find a stopping time that is closest (in L^1 sense) to this random time. This problem is an example of the optimal prediction problems which have been widely investigated by many. Graversen et al. (2001) predicted the value of the ultimate maximum of a Brownian motion in a finite horizon setting whereas Shiryaev (2009) focused on the last time of the attainment of the ultimate maximum of a (driftless) Brownian motion and proceeded to show that it is equivalent to predicting the last zero of the process in this setting. The work of the latter was generalised by du Toit et al. (2008) for a linear Brownian motion. Bernyk et al. (2011) studied the time at which a stable spectrally negative Lévy process attains its ultimate supremum in a finite horizon of time and this was later generalised by Baurdoux and van Schaik (2014) for any Lévy process in infinite horizon of time. Investigations on the time of the ultimate minimum and the last zero of a transient diffusion process were carried out by Glover et al. (2013) and Glover and Hulley (2014) respectively. More recent studies by Baurdoux and Pedraza (2020b) predicted the last zero of a spectrally negative Lévy process in a infinite horizon setting. It can be shown that the aforementioned problems are equivalent to optimal stopping problems, in other words, optimal prediction problems and optimal stopping problems are intimately related.

This chapter is organised as follows. In Section 2.2 we formulate the optimal prediction problem and we prove that it is equivalent to an optimal stopping problem. Section 2.3 is dedicated to the solution of the optimal stopping problem. The main result of this paper is stated in Theorem 2.3.13 and its proof is detailed in Section 2.4. The last section makes use of Theorem 2.3.13 to find numerical solution of the optimal stopping problem for the Brownian motion with drift case.

2.2 Formulation of the Problem

Throughout this chapter we use the notation and the preliminary results presented in Section 1.1. Let X be a spectrally negative Lévy process, that is, a Lévy process starting from 0 with only negative jumps and non-monotone paths, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see Definition 1.3.38 in Bichteler (2002)). We suppose that X has Lévy triplet (μ, σ, Π) where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (Lévy measure) concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

Let g_θ be the last passage time below zero before an exponential time, i.e.

$$g_\theta = \sup\{0 \leq t \leq \widetilde{e}_\theta : X_t \leq 0\}, \tag{2.1}$$

where $\widetilde{e}$ is an exponential random variable with parameter $\theta \geq 0$. Here, we use the convention that an exponential random variable with parameter 0 is taken to be infinite with probability 1. In the case of $\theta = 0$, we simply denote $g = g_0$.

Note that $g_\theta \leq \widetilde{e}_\theta < \infty$ $\mathbb{P}$-a.s. for all $\theta > 0$. However, in the case where $\theta = 0$, g could be infinite. Therefore, we assume that $\theta > 0$ throughout this paper. Moreover, we have that g_θ has finite moments for all $\theta > 0$.

Remark 2.2.1. *Since X is a spectrally negative Lévy process, we can exclude the case of a compound Poisson process and hence the only way of exiting the set $(-\infty, 0]$ is by creeping upwards. This tells us that $X_{g_\theta -} = X_{g_\theta} = 0$ in the event of $\{g_\theta < \widetilde{e}_\theta\}$ and that $g_\theta = \sup\{0 \leq t \leq \widetilde{e}_\theta : X_t < 0\}$ holds $\mathbb{P}$-a.s.*

Clearly, up to any time $t \geq 0$ the value of g_θ is unknown (unless X is trivial), and it is only with the realisation of the whole process that we know that the last passage time below 0 has occurred. However, this is often too late: typically, at any time $t \geq 0$, we would like to know how close we are to the time g_θ so we can take some actions based on this information. We search for a stopping time τ_* of X that is as "close" as possible to g_θ. Consider the optimal prediction problem

$$V_* = \inf_{\tau \in \mathcal{T}} \mathbb{E}(|g_\theta - \tau|), \tag{2.2}$$

where $\mathcal{T}$ is the set of all stopping times.

We state an equivalence between the optimal prediction problem (2.2) and an optimal stopping problem. This equivalence is mainly based on the work of Urusov (2005).

Lemma 2.2.2. *Suppose that $\{X_t, t \geq 0\}$ is a spectrally negative Lévy process. Let g_θ be the* *last time that X is below the level zero before an exponential time $\widetilde{e}_\theta$ with $\theta > 0$, as defined in (2.1). Consider the optimal stopping problem given by*

$$V = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G^{(\theta)}(s, X_s) \mathrm{d}s \right), \tag{2.3}$$

where the function $G^{(\theta)}$ is given by $G^{(\theta)}(s,x) = 1 + 2e^{-\theta s}\left[\frac{\theta}{\Phi(\theta)} W^{(\theta)}(x) - Z^{(\theta)}(x)\right]$ for all $x \in \mathbb{R}$. Then the stopping time which minimises (2.2) is the same which minimises (2.3). In particular,

$$V_* = V + \mathbb{E}(g_\theta). \tag{2.4}$$

Proof. Fix any stopping time $\tau \in \mathcal{T}$. We have that

$$|g_\theta - \tau| = \int_0^\tau [2\mathbb{I}_{\{g_\theta \le s\}} - 1] \mathrm{d}s + g_\theta.$$

From Fubini's theorem and the tower property of conditional expectations, we obtain

$$\begin{aligned} \mathbb{E}\left[\int_0^\tau \mathbb{I}_{\{g_\theta \le s\}} \mathrm{d}s \right] &= \mathbb{E}\left[\int_0^\infty \mathbb{I}_{\{s < \tau\}} \mathbb{E}[\mathbb{I}_{\{g_\theta \le s\}} | \mathcal{F}_s] \mathrm{d}s \right] \\ &= \mathbb{E}\left[\int_0^\tau \mathbb{P}(g_\theta \le s | \mathcal{F}_s) \mathrm{d}s \right]. \end{aligned}$$

Note that in the event of $\{\widetilde{e}_\theta \le s\}$, we have $g_\theta \le s$ so that

$$\mathbb{P}(g_\theta \le s | \mathcal{F}_s) = 1 - e^{-\theta s} + \mathbb{P}(g_\theta \le s, \widetilde{e}_\theta > s | \mathcal{F}_s).$$

On the other hand for $\{\widetilde{e}_\theta > s\}$, as a consequence of Remark 2.2.1, the event $\{g_\theta \le s\}$ is equal to $\{X_u \ge 0 \text{ for all } u \in [s, \widetilde{e}_\theta]\}$ (up to a $\mathbb{P}$-null set). Hence, we get that for all $s \ge 0$ that

$$\begin{aligned} \mathbb{P}(g_\theta \le s, \widetilde{e}_\theta > s | \mathcal{F}_s) &= \mathbb{P}(X_u \ge 0 \text{ for all } u \in [s, \widetilde{e}_\theta], \widetilde{e}_\theta > s | \mathcal{F}_s) \\ &= \mathbb{P}\left(\inf_{0 \le u \le \widetilde{e}_\theta - s} X_{u+s} \ge 0, \widetilde{e}_\theta > s | \mathcal{F}_s \right) \\ &= e^{-\theta s} \mathbb{P}_{X_s}\left(\underline{X}_{\widetilde{e}_\theta} \ge 0 \right), \end{aligned}$$

where the last equality follows from the lack of memory property of the exponential distribution and the Markov property for Lévy process. Hence, we have that

$$\mathbb{P}(g_\theta \le s, \widetilde{e}_\theta > s|\mathcal{F}_s) = e^{-\theta s}F^{(\theta)}(X_s),$$

where for all $x \in \mathbb{R}$, $F^{(\theta)}(x) = \mathbb{P}_x(\underline{X}_{\widetilde{e}_\theta} \ge 0)$. Then, since $\widetilde{e}_\theta$ is independent of X, we have that for $x \in \mathbb{R}$,

$$\begin{aligned}
F^{(\theta)}(x) &= \mathbb{P}_x(\underline{X}_{\widetilde{e}_\theta} \ge 0) \\
&= \mathbb{P}_x(\widetilde{e}_\theta < \tau_0^-) \\
&= 1 - \mathbb{E}_x(e^{-\theta\tau_0^-}\mathbb{I}_{\{\tau_0^- < \infty\}}) \\
&= \frac{\theta}{\Phi(\theta)}W^{(\theta)}(x) - Z^{(\theta)}(x) + 1,
\end{aligned}$$

where the last equality follows from equation (1.9). Thus,

$$\begin{aligned}
\mathbb{P}(g_\theta \le s|\mathcal{F}_s) &= 1 - e^{-\theta s} + e^{-\theta s}\left[\frac{\theta}{\Phi(\theta)}W^{(\theta)}(X_s) - Z^{(\theta)}(X_s) + 1\right] \\
&= 1 + e^{-\theta s}\left[\frac{\theta}{\Phi(\theta)}W^{(\theta)}(X_s) - Z^{(\theta)}(X_s)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_* &= \inf_{\tau\in\mathcal{T}} \mathbb{E}(|g_\theta - \tau|) \\
&= \mathbb{E}(g_\theta) + \inf_{\tau\in\mathcal{T}} \mathbb{E}\left(\int_0^\tau [2\mathbb{P}(g_\theta \le s|\mathcal{F}_s) - 1]\mathrm{d}s\right) \\
&= \mathbb{E}(g_\theta) + \inf_{\tau\in\mathcal{T}} \mathbb{E}\left(\int_0^\tau \left(1 + 2e^{-\theta s}\left[\frac{\theta}{\Phi(\theta)}W^{(\theta)}(X_s) - Z^{(\theta)}(X_s)\right]\right)\mathrm{d}s\right).
\end{aligned}$$

The conclusion holds. □

2.3 Optimal stopping problem

In order to find the solution to the optimal stopping problem (2.3), we extend its definition to Lévy process (and hence strong Markov process) $\{(t, X_t), t \ge 0\}$ in the following way. Define the function $V : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ as

$$V^{(\theta)}(t,x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{t,x} \left(\int_0^\tau G^{(\theta)}(s+t, X_{s+t}) \mathrm{d}s \right) = \inf_{\tau \in \mathcal{T}} \mathbb{E} \left(\int_0^\tau G^{(\theta)}(s+t, X_s + x) \mathrm{d}s \right). \quad (2.5)$$

So that

$$V_* = V^{(\theta)}(0,0) + \mathbb{E}(g_\theta).$$

Remark 2.3.1. *We can see from the proof above that $G^{(\theta)}$ can be written as,*

$$G^{(\theta)}(s,x) = 1 + 2e^{-\theta s}[F^{(\theta)}(x) - 1],$$

where $F^{(\theta)}$ is the distribution function of the positive random variable $-\underline{X}_{\widetilde{e}_\theta}$ given by

$$F^{(\theta)}(x) = \frac{\theta}{\Phi(\theta)} W^{(\theta)}(x) - Z^{(\theta)}(x) + 1. \quad (2.6)$$

Moreover, evaluating $\theta = 0$, the function $G^{(0)}$ coincides with the gain function found in Baurdoux and Pedraza (2020b) (see Lemma 3.2 and Remark 3.3).

Now we give some intuitions about the function $G^{(\theta)}$. Recall that for all $\theta \geq 0$, W^θ and $Z^{(\theta)}$ are continuous and strictly increasing functions on $[0,\infty)$ such that $W^{(\theta)}(x) = 0$ and $Z^{(\theta)}(x) = 1$ for $x \in (-\infty, 0)$. From the above, equation (2.6) and from the fact that $F^{(\theta)}$ is a distribution function, we have that for a fixed $t \geq 0$, the function $x \mapsto G^{(\theta)}(t,x)$ is strictly increasing and continuous in $[0,\infty)$ with a possible discontinuity at 0 depending on the path variation of X. Moreover, we have that $\lim_{x\to\infty} G^{(\theta)}(t,x) = 1$ for all $t \geq 0$. For $x < 0$ and $t \geq 0$, we have that the function $G^{(\theta)}$ takes the form $G^{(\theta)}(t,x) = 1 - 2e^{-\theta s}$. Similarly, from the fact that $F^{(\theta)}(x) - 1 \leq 0$ for all $x \in \mathbb{R}$, we have that for a fixed $x \in \mathbb{R}$ the function $t \mapsto G^{(\theta)}(t,x)$ is continuous and strictly increasing on $[0,\infty)$. Furthermore, from the fact that $0 \leq F^{(\theta)}(x) \leq 1$, we have that the function G is bounded by

$$1 - 2e^{-\theta t} \leq G^{(\theta)}(x,t) \leq 1 \quad (2.7)$$

which implies that $|G^{(\theta)}| \leq 1$. Define the value m_θ as the median of the random variable $\widetilde{e}_\theta$, in other words, m_θ is given by

$$m_\theta = \frac{\log(2)}{\theta}.$$

Hence from (2.7) we have that $G^{(\theta)}(t,x) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq m_\theta$. The above observations tell us that, to solve the optimal stopping problem (2.5), we are interested in a stopping time such that before stopping, the process X spends most of its time in the region where $G^{(\theta)}$ is negative, taking into account that (t, X) can live in the set $\{(s,x) \in \mathbb{R}_+ \times \mathbb{R} : G^{(\theta)}(s,x) > 0\}$ and then return back to the set $\{(s,x) \in \mathbb{R}_+ \times \mathbb{R} : G^{(\theta)}(s,x) \leq 0\}$. The only restriction that applies is that if a considerable amount of time has passed, then $\{x \in \mathbb{R} : G^{(\theta)}(s,x) > 0\} = \mathbb{R}$ for all $s \geq m_\theta$.

We then define the function $h^{(\theta)} : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$h^{(\theta)}(t) := \inf\{x \in \mathbb{R} : G^{(\theta)}(x,t) \geq 0\} = \inf\{x \in \mathbb{R} : F^{(\theta)}(x) \geq 1 - \frac{1}{2}e^{\theta t}\},$$

for all $t \geq 0$. Hence, we can see that the function $h^{(\theta)}$ is a non-increasing continuous function on $[0, m_\theta)$ such that $\lim_{t \uparrow m_\theta} h^{(\theta)}(t) = 0$ and $h^{(\theta)}(t) = -\infty$ for $t \in [m_\theta, \infty)$. Moreover, from the fact that $G^{(\theta)}(t,x) < 0$ for $(t,x) \in [0, m_\theta) \times (-\infty, 0)$, we have that $h^{(\theta)}(t) \geq 0$ for all $t \in [0, m_\theta)$.

In order to characterise the stopping time that minimises (2.5), we first derive some properties of the function $V^{(\theta)}$.

Lemma 2.3.2. *Let $\theta > 0$. The function $V^{(\theta)}$ is non-drecreasing in each argument. Moreover, $V^{(\theta)}(t,x) \in (-m_\theta, 0]$ for all $x \in \mathbb{R}$ and $t \geq 0$. In particular, $V^{(\theta)}(t,x) < 0$ for any $t \geq 0$ with $x < h^{(\theta)}(t)$ and $V^{(\theta)}(t,x) = 0$ for all $(t,x) \in [m_\theta, \infty) \times \mathbb{R}$.*

Proof. First, note that $V^{(\theta)}(t,x) \leq 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Indeed, if we take the stopping time $\tau \equiv 0$, we obtain that

$$V^{(\theta)}(t,x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{t,x} \left(\int_0^\tau G^{(\theta)}(s+t, X_{s+t}) \mathrm{d}s \right) \leq 0.$$

Now take $(t,x) \in [m_\theta, \infty) \times \mathbb{R}$, then for any $r \geq 0$, we have that $G^{(\theta)}(r+t, X_r + x) \geq 0$, which implies that for any $\tau \in \mathcal{T}$

$$0 \leq \mathbb{E}\left(\int_0^\tau G^{(\theta)}(r+t, X_r + x)\mathrm{d}r\right)$$

and hence $V^{(\theta)}(t,x) = 0$.

The fact that $V^{(\theta)}$ is non-decreasing in each argument follows from the non-decreasing property of the functions $t \mapsto G^{(\theta)}(t,x)$ and $x \mapsto G^{(\theta)}(t,x)$ as well as the monotonicity of the expectation. Moreover, we have that $\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : x < h^{(\theta)}(t)\} = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : G^{(\theta)}(t,x) < 0\} \subset \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : V^{(\theta)}(t,x) < 0\}$. Indeed, let $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$ such that $s < h(y)$ and take $U \subset \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : G^{(\theta)}(t,x) < 0\}$ be any neighbourhood of (s,y). Define the stopping time τ_U as the first exit time from the set U, that is

$$\tau_U = \inf\{r \geq 0 : (r, X_r) \notin U\}.$$

Then we have that $\tau_U > 0$ a.s. and

$$V^{(\theta)}(s,y) \leq \mathbb{E}_{s,y}\left(\int_0^{\tau_U} G^{(\theta)}(r+s, X_{r+s})\mathrm{d}r\right) < 0,$$

where the strict inequality follows since $(r, X_r) \in \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : G^{(\theta)}(t,x) < 0\}$ for all $r < \tau_U$.

Next we will show that $V^{(\theta)}(t,x) > -\infty$ for all $(t,x) \in [0, m_\theta) \times \mathbb{R}$ and for all $\theta > 0$. Note that $t < m_\theta$ if and only if $1 - 2e^{-\theta t} < 0$. Then for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$ we have that

$$G^{(\theta)}(s,x) \geq 1 - 2e^{-\theta s} \geq (1 - 2e^{-\theta s})\mathbb{I}_{\{s < m_\theta\}}.$$

Hence, for all $x \in \mathbb{R}$ and $t < m_\theta$

$$\begin{aligned}
V^{(\theta)}(t,x) &= \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G^{(\theta)}(s+t, X_s + x)\mathrm{d}s\right) \\
&\geq \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau (1 - 2e^{-\theta(s+t)})\mathbb{I}_{\{t+s < m_\theta\}}\mathrm{d}s\right) \\
&= -\sup_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau (2e^{-\theta(s+t)} - 1)\mathbb{I}_{\{t+s < m_\theta\}}\mathrm{d}s\right).
\end{aligned}$$

The term in the last integral is non-negative, so we obtain for all $t < m_\theta$ and $x \in \mathbb{R}$ that

$$\begin{aligned} V^{(\theta)}(t,x) &\geq -\left(\int_0^\infty (2e^{-\theta(s+t)} - 1)\mathbb{I}_{\{t+s<m_\theta\}}\mathrm{d}s\right) \\ &= -\left(\int_0^{m_\theta - t} (2e^{-\theta(s+t)} - 1)\mathrm{d}s\right) \\ &> -m_\theta. \end{aligned}$$

□

In the next lemma we use the general theory of optimal stopping to find an optimal stopping time for (2.3).

Lemma 2.3.3. *For any $\theta > 0$ we have that an optimal stopping time for (2.5) is given by*

$$\tau_D = \inf\{t \geq 0 : (t, X_t) \in D\}, \tag{2.8}$$

where $D = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : V^{(\theta)}(t,x) = 0\}$.

Proof. As a consequence of the fact that $V^{(\theta)}$ vanishes on the set $[m_\theta, \infty) \times \mathbb{R}$ (see Lemma 2.3.2) we have that for any $(t,x) \in [0, m_\theta) \times \mathbb{R}$,

$$V^{(\theta)}(t,x) = \inf_{\tau \in \mathcal{T}_{m_\theta - t}} \mathbb{E}_{t,x}\left(\int_0^\tau G^{(\theta)}(s+t, X_{s+t})\mathrm{d}s\right),$$

where $\mathcal{T}_{m_\theta - t}$ is the set of stopping times bounded by $m_\theta - t$. Indeed, since $\mathcal{T}_{m_\theta - t} \subset \mathcal{T}$ we have the inequality,

$$V^{(\theta)}(t,x) \leq \inf_{\tau \in \mathcal{T}_{m_\theta - t}} \mathbb{E}_{t,x}\left(\int_0^\tau G^{(\theta)}(s+t, X_{s+t})\mathrm{d}s\right).$$

On the other hand, for any $(t,x) \in [0, m_\theta] \times \mathbb{R}$ we have that from the Markov property at time m_θ,

$$\begin{aligned}
V^{(\theta)}(t,x) &= \inf_{\tau\in\mathcal{T}}\left[\mathbb{E}_{t,x}\left(\mathbb{I}_{\{\tau<m_\theta-t\}}\int_0^\tau G^{(\theta)}(s+t,X_{s+t})\mathrm{d}s\right)\right.\\
&\qquad\left.+\mathbb{E}_{t,x}\left(\mathbb{I}_{\{\tau\geq m_\theta-t\}}\int_0^\tau G^{(\theta)}(s+t,X_{s+t})\mathrm{d}s\right)\right]\\
&= \inf_{\tau\in\mathcal{T}}\left[\mathbb{E}_{t,x}\left(\int_0^{\tau\wedge(m_\theta-t)} G^{(\theta)}(s+t,X_{s+t})\mathrm{d}s\right)\right.\\
&\qquad\left.+\mathbb{E}_{t,x}\left(\mathbb{I}_{\{\tau\geq m_\theta-t\}}\mathbb{E}_{m_\theta,X_{m_\theta}}\left(\int_0^\tau G^{(\theta)}(s+m_\theta,X_{s+m_\theta})\mathrm{d}s\right)\right)\right]\\
&\geq \inf_{\tau\in\mathcal{T}}\left[\mathbb{E}_{t,x}\left(\int_0^{\tau\wedge(m_\theta-t)} G^{(\theta)}(s+t,X_{s+t})\mathrm{d}s+\mathbb{I}_{\{\tau\geq m_\theta-t\}}V(m_\theta,X_{m_\theta})\right)\right]\\
&= \inf_{\tau\in\mathcal{T}}\mathbb{E}_{t,x}\left(\int_0^{\tau\wedge(m_\theta-t)} G^{(\theta)}(s+t,X_{s+t})\mathrm{d}s\right),
\end{aligned}$$

where the inequality follows from the definition of $V^{(\theta)}$ and the last equality holds since $V^{(\theta)}(m_\theta,x)=0$ for all $x\in\mathbb{R}$. Then the assertion holds.

Hence, since $|G^{(\theta)}|\leq 1$ we have that for all $t\geq 0$ and $x\in\mathbb{R}$,

$$\mathbb{E}_{t,x}\left(\sup_{s\geq 0}\left|\int_0^{s\wedge(m_\theta-t)} G^{(\theta)}(r+t,X_{r+t})\mathrm{d}r\right|\right)<\infty$$

Next, we show that the function $V^{(\theta)}$ is upper semi-continuous. Recall that the function $F^{(\theta)}$ is strictly increasing and continuous on $[0,\infty)$ such that $F^{(\theta)}=0$ for $x<0$. This implies that $F^{(\theta)}$ is upper semi-continuous and then the function $G^{(\theta)}$ is upper semi-continuous (since $t\mapsto G^{(\theta)}(t,x)$ is continuous for all $x\in\mathbb{R}$). Hence for any stopping time τ, by using Fatou's lemma (since $G^{(\theta)}$ is bounded), we have that for any $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}$,

$$\begin{aligned}
\limsup_{(t,x)\to(t_0,x_0)}\mathbb{E}\left[\int_0^\tau G^{(\theta)}(s+t,X_s+x)\mathrm{d}s\right] &\leq \mathbb{E}\left[\int_0^\tau \limsup_{(t,x)\to(t_0,x_0)} G^{(\theta)}(s+t,X_s+x)\mathrm{d}s\right]\\
&\leq \mathbb{E}\left[\int_0^\tau G^{(\theta)}(s+t_0,X_s+x_0)\mathrm{d}s\right].
\end{aligned}$$

Showing that for any $\tau\in\mathcal{T}$, the mapping $(t,x)\mapsto\mathbb{E}\left[\int_0^\tau G^{(\theta)}(s+t,X_s+x)\mathrm{d}s\right]$ is upper semi-continuous. Hence, $V^{(\theta)}$ is upper semi-continuous (since $V^{(\theta)}$ is the infimum of upper semi-continuous functions).

Therefore, by general results of optimal stopping (see Peskir and Shiryaev (2006) Corollary 2.9 or Section 1.2.3) we conclude that an optimal stopping time for (2.5) exists and is given by (2.8). □

Hence, from Lemma 2.3.2, we derive that $D = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : x \geq b^{(\theta)}(t)\}$, where the function $b^{(\theta)} : \mathbb{R}_+ \mapsto \mathbb{R}$ is given by

$$b^{(\theta)}(t) = \inf\{x \in \mathbb{R} : (t, x) \in D\},$$

for each $t \geq 0$. It follows from Lemma 2.3.2 that $b^{(\theta)}$ is non-increasing and $b^{(\theta)}(t) \geq h^{(\theta)}(t) \geq 0$ for all $t \geq 0$. Moreover, $b^{(\theta)}(t) = -\infty$ for $t \in [m_\theta, \infty)$, since $V^{(\theta)}(t, x) = 0$ for all $t \geq m_\theta$ and $x \in \mathbb{R}$, giving us $\tau_D \leq m_\theta$. In the case that $t < m_\theta$, we have that $b^{(\theta)}(t)$ is finitely valued as we will prove in the following Lemma.

Lemma 2.3.4. *Let $\theta > 0$. The function $b^{(\theta)}$ is finitely valued for all $t \in [0, m_\theta)$.*

Proof. For any $\theta > 0$ and fix $t \geq 0$, consider the optimal stopping problem,

$$\mathcal{V}_t^{(\theta)}(x) = \inf_{\tau \in \mathcal{T}_{m_\theta - t}} \mathbb{E}_x \left(\int_0^\tau [1 + 2e^{-\theta t}(F^{(\theta)}(X_s) - 1)] \mathrm{d}s \right), \qquad x \in \mathbb{R},$$

where $\mathcal{T}_{m_\theta - t}$ is the set of all stopping times bounded by $m_\theta - t$. From the fact that for all $s \geq 0$ and $x \in \mathbb{R}$, $G(s + t, x) \geq 1 + 2e^{-\theta t}(F^{(\theta)}(x) - 1)$ and that $\tau_D \in \mathcal{T}_{m_\theta - t}$ (under $\mathbb{P}_{t,x}$ for all $x \in \mathbb{R}$), we have that

$$V^{(\theta)}(t, x) \geq \mathcal{V}_t^{(\theta)}(x) \tag{2.9}$$

for all $x \in \mathbb{R}$. Hence it suffices to show that there exists $\tilde{x}_t$ (finite) sufficiently large such that $\mathcal{V}_t^{(\theta)}(x) = 0$ for all $x \geq \tilde{x}_t$. Since $F^{(\theta)}$ is a distribution function, it can be easily shown that $\mathcal{V}_t^{(\theta)}$ is a non-decreasing function and that for all $x \in \mathbb{R}$, $\mathcal{V}_t^{(\theta)}(x) \in (-m_\theta, 0]$. Moreover, an optimal stopping time for $\mathcal{V}_t^{(\theta)}$ is $\tau_{\mathcal{D}_t}$, the first entry time before $m_\theta - t$ to the set $\mathcal{D}_t = \{x \in \mathbb{R} : \mathcal{V}_t^{(\theta)}(x) = 0\}$. We proceed by contradiction, assume that $\mathcal{D}_t = \emptyset$, then $\tau_{\mathcal{D}_t} = m_\theta - t$ and

$$\mathcal{V}_t^{(\theta)}(x) = \mathbb{E}_x \left(\int_0^{m_\theta - t} [1 + 2e^{-\theta t}(F^{(\theta)}(X_s) - 1)] \mathrm{d}s \right).$$

Hence, by the dominated convergence theorem and the spatial homogeneity of Lévy processes we have that

$$0 \geq \lim_{x\to\infty} \mathcal{V}_t^{(\theta)}(x) = \mathbb{E}\left(\int_0^{m_\theta - t} \lim_{x\to\infty} [1 + 2e^{-\theta t}(F^{(\theta)}(X_s + x) - 1)]\mathrm{d}s\right) = m_\theta - t > 0$$

which is a contradiction. Therefore, we conclude that for each $t \geq 0$, there exists a finite value $\tilde{x}_t$ such that $b^{(\theta)}(t) \leq \tilde{x}_t$. □

Remark 2.3.5. *From the proof of Lemma 2.3.4, we find an upper bound of the boundary $b^{(\theta)}$. Define, for each $t \in [0, m_\theta)$, $u^{(\theta)}(t) = \inf\{x \in \mathbb{R} : \mathcal{V}_t^{(\theta)}(x) = 0\}$. Then it follows that $u^{(\theta)}$ is a non-increasing finite function such that*

$$u^{(\theta)}(t) \geq b^{(\theta)}(t)$$

for all $t \in [0, m_\theta)$.

Next we show that the function $V^{(\theta)}$ is continuous.

Lemma 2.3.6. *The function $V^{(\theta)}$ is continuous. Moreover, for each $x \in \mathbb{R}$, $t \mapsto V^{(\theta)}(t, x)$ is Lipschitz on $\mathbb{R}_+$ and for every $t \in \mathbb{R}_+$, $x \mapsto V^{(\theta)}(t, x)$ is Lipschitz on $\mathbb{R}$.*

Proof. First, we are showing that, for a fixed $t \geq 0$, the function $x \mapsto V^{(\theta)}(t, x)$ is Lipschitz on $\mathbb{R}$. Note that if $t \geq m_\theta$, then we have that $V^{(\theta)}(t, x) = 0$ for all $x \in \mathbb{R}$. Suppose that $t < m_\theta$. Let $x, y \in \mathbb{R}$ and define $\tau_x^* = \tau_{D(t,x)} = \inf\{s \geq 0 : X_s + x \geq b^{(\theta)}(s + t)\}$. Since τ_x^* is optimal in $V^{(\theta)}(t, x)$ (under $\mathbb{P}$) we have that

$$\begin{aligned} V^{(\theta)}(t, y) - V^{(\theta)}(t, x) &\leq \mathbb{E}\left(\int_0^{\tau_x^*} G^{(\theta)}(s + t, X_s + y)\mathrm{d}s\right) - \mathbb{E}\left(\int_0^{\tau_x^*} G^{(\theta)}(s + t, X_s + x)\mathrm{d}s\right) \\ &= \mathbb{E}\left(\int_0^{\tau_x^*} 2e^{-\theta(s+t)}[F^{(\theta)}(X_s + y) - F^{(\theta)}(X_s + x)]\mathrm{d}s\right). \end{aligned}$$

Define the stopping time

$$\tau^+_{b(0)-x} = \inf\{t \geq 0 : X_t \geq b^{(\theta)}(0) - x\}.$$

Then we have that $\tau_x^* \leq \tau^+_{b^{(\theta)}(0)-x}$ (since $b^{(\theta)}$ is a non-increasing function). From the fact that

$F^{(\theta)}$ is non-decreasing, we obtain that for $b^{(\theta)}(0) \geq y \geq x$,

$$V^{(\theta)}(t,y) - V^{(\theta)}(t,x) \leq 2\mathbb{E}\left(\int_0^{\tau_x^*} e^{-\theta s}[F^{(\theta)}(X_s+y) - F^{(\theta)}(X_s+x)]\mathrm{d}s\right)$$
$$\leq 2\mathbb{E}\left(\int_0^{\tau_{b(0)-x}^+} e^{-\theta s}[F^{(\theta)}(X_s+y) - F^{(\theta)}(X_s+x)]\mathrm{d}s\right).$$

Using Fubini's theorem and a density of the potential measure of the process killed upon exiting $(-\infty, b^{(\theta)}(0)]$ (see equation (1.20)) we get that

$$\begin{aligned}
&V^{(\theta)}(t,y) - V^{(\theta)}(t,x) \\
&\leq 2\int_{-\infty}^{b^{(\theta)}(0)} [F^{(\theta)}(z+y-x) - F^{(\theta)}(z)] \int_0^\infty e^{-\theta s}\mathbb{P}_x(X_s \in \mathrm{d}z, \tau_{b(0)}^+ > s)\mathrm{d}s \\
&= 2\int_{-\infty}^{b^{(\theta)}(0)} [F^{(\theta)}(z+y-x) - F^{(\theta)}(z)] \left[e^{-\Phi(\theta)(b^{(\theta)}(0)-x)}W^{(\theta)}(b^{(\theta)}(0)-z) - W^{(\theta)}(x-z)\right]\mathrm{d}z \\
&\leq 2e^{-\Phi(\theta)(b^{(\theta)}(0)-x)}W^{(\theta)}(b^{(\theta)}(0)-x+y)\int_{x-y}^{b^{(\theta)}(0)} [F^{(\theta)}(z+y-x) - F^{(\theta)}(z)]\mathrm{d}z,
\end{aligned}$$

where in the last inequality, we used the fact that $W^{(\theta)}$ is strictly increasing and non-negative and that $F^{(\theta)}$ vanishes at $(-\infty, 0)$. By an integration by parts argument, we obtain that

$$\int_{x-y}^{b^{(\theta)}(0)} [F^{(\theta)}(z+y-x) - F^{(\theta)}(z)]\mathrm{d}z = (y-x)F^{(\theta)}(b^{(\theta)}(0)+y-x).$$

Moreover, it can be checked that (see Kuznetsov et al. (2013) lemma 3.3) the function $z \mapsto e^{-\Phi(\theta)(z)}W^{(\theta)}(z)$ is a continuous function in the interval $[0,\infty)$ such that

$$\lim_{z\to\infty} e^{-\Phi(\theta)(z)}W^{(\theta)}(z) = \frac{1}{\psi'(\Phi(\theta))} < \infty.$$

This implies that there exist a constant $M > 0$ such that for every $z \in \mathbb{R}$, we have the inequality $0 \leq e^{-\Phi(\theta)(z)}W^{(\theta)}(z) < M$. Then we obtain that for all $x \leq y \leq b^{(\theta)}(0)$,

$$0 \leq V^{(\theta)}(t,y) - V^{(\theta)}(t,x) \leq 2M(y-x)e^{\Phi(\theta)y} \leq 2M(y-x)e^{\Phi(\theta)b^{(\theta)}(0)}.$$

On the other hand, since $b^{(\theta)}(0) \geq b^{(\theta)}(t)$ for all $t \in [0, m_\theta)$ we have that for all $(t,x) \in$

$[0, m_\theta) \times [b^{(\theta)}(0), \infty)$, $V^{(\theta)}(t, x) = 0$. Hence we obtain that for all $x, y \in \mathbb{R}$ and $t \geq 0$,

$$|V^{(\theta)}(t, y) - V^{(\theta)}(t, x)| \leq 2M|y - x|e^{\Phi(\theta)b^{(\theta)}(0)}. \tag{2.10}$$

Therefore we conclude that for a fixed $t \geq 0$, the function $x \mapsto V^{(\theta)}(t, x)$ is Lipschitz on $\mathbb{R}$.

It remains to show that $t \mapsto V^{(\theta)}(t, x)$ is Lipschitz on $[0, \infty)$ for every $x \in \mathbb{R}$. We know that for all $x \in \mathbb{R}$, $V^{(\theta)}(t, x) = 0$ for all $t \geq m_\theta$ so $t \mapsto V(t, x)$ is Lipschitz on $[m_\theta, \infty)$ for all $x \in \mathbb{R}$. On the other hand, recall that the function $t \mapsto e^{-\theta t}$ is Lipschitz continuous on $[0, \infty)$. Indeed, using the fact that $e^{-\theta t} \leq 1$ for all $t \geq 0$ we have that for all $s, t \in [0, \infty)$,

$$\left| e^{-\theta s} - e^{-\theta t} \right| = \left| \int_s^t \theta e^{-\theta u} du \right| \leq \theta |t - s|.$$

Take $s, t \in [0, m_\theta]$ and suppose without loss of generality that $s \geq t$. Then, since $\tau_{D(t,x)}$ is optimal for $V^{(\theta)}(t, x)$, we have that for all $x \in \mathbb{R}$,

$$\begin{aligned}
0 \leq V^{(\theta)}(s, x) &- V^{(\theta)}(t, x) \\
&\leq \mathbb{E}\left(\int_0^{\tau_{D(t,x)}} G^{(\theta)}(r + s, X_r + x) \mathrm{d}r \right) - \mathbb{E}\left(\int_0^{\tau_{D(t,x)}} G^{(\theta)}(r + t, X_r + x) \mathrm{d}r \right) \\
&\leq \mathbb{E}\left(\int_0^{\tau_{D(t,x)}} 2[e^{-\theta(r+t)} - e^{-\theta(r+s)}] \mathrm{d}r \right) \\
&\leq 2\theta(s - t) m_\theta,
\end{aligned}$$

where the second inequality follows from the fact that $0 \leq F^{(\theta)} \leq 1$ and the last inequality results from $\tau_{D(t,x)} \leq m_\theta - t \leq m_\theta$. Therefore we conclude that

$$|V^{(\theta)}(s, x) - V^{(\theta)}(t, x)| \leq 2\theta m_\theta |s - t|$$

and therefore $t \mapsto V^{(\theta)}(t, x)$ is Lipschitz continuous for all $x \in \mathbb{R}$. □

In order to derive more properties of the boundary $b^{(\theta)}$, we first state some auxiliary results. Recall that if $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, the set of real bounded $C^{1,2}$ functions on $\mathbb{R}_+ \times \mathbb{R}$

with bounded derivatives, the infinitesimal generator of (t, X) is given by

$$\mathcal{A}_{(t,X)}(f)(t,x) = \frac{\partial}{\partial t} f(t,x) - \mu \frac{\partial}{\partial x} f(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f(t,x) + \int_{(-\infty,0)} [f(t,x+y) - f(t,x) - y\mathbb{I}_{\{y>-1\}} \frac{\partial}{\partial x} f(t,x)]\Pi(\mathrm{d}y). \tag{2.11}$$

However, sometimes is not easy to show that f has continuous derivatives and then a definition of the generator in a broader sense is needed. It turns out that the generator of a Lévy process can be defined in the sense of distributions. Indeed, it is shown in Lamberton and Mikou (2008) that when f is a bounded continuous function, the generator can be defined in the sense of distributions (see Proposition 2.1 and Remark 2.2). The reader can also refer to the Appendix A for further details on this. In particular, we show that, if f is a locally integrable function $\mathbb{R}_+ \times \mathbb{R}$ such that $(u,x) \mapsto \int_{(-\infty,-1)} |f(u,x+y)|\Pi(\mathrm{d}y)$ is locally integrable, we can define the distribution $\mathcal{A}_{(t,X)}(f)$ by $\mathcal{A}_{(t,X)}(f) = \mathcal{A}_X(f) + \frac{\partial}{\partial t} f$, where

$$\begin{aligned}\langle \mathcal{A}_X(f), \psi\rangle &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\left[\mu\frac{\partial}{\partial x}\varphi(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\varphi(t,x)\right]\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)B_X^*(\varphi)(t,x)\mathrm{d}x\mathrm{d}t \\ \langle \frac{\partial}{\partial t} f, \psi\rangle &= -\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t,x)\frac{\partial}{\partial t}\varphi(t,x)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

for any $\varphi \in C^\infty$ function with compact support on $\mathbb{R}_+ \times \mathbb{R}$ and

$$B_X^*(\varphi)(t,x) = \int_{(-\infty,0)} [\varphi(t,x-y) - \varphi(t,x) + y\frac{\partial}{\partial x}\varphi(t,x)\mathbb{I}_{\{y>-1\}}]\Pi(\mathrm{d}y).$$

Let $C = \mathbb{R}_+ \times \mathbb{R} \setminus D = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : x < b^{(\theta)}(t)\}$ be the continuation region. Then we have that the value function $V^{(\theta)}$ satisfies a variational inequality in the sense of distributions. The proof is analogous to the one presented in Lamberton and Mikou (2008) (see Proposition 2.5).

Lemma 2.3.7. *Fix $\theta > 0$. The distribution $\mathcal{A}_{(t,X)}V^{(\theta)} + G^{(\theta)}$ is non-negative on $\mathbb{R}_+ \times \mathbb{R}$. Moreover, we have that $\mathcal{A}_{(t,X)}V^{(\theta)} + G^{(\theta)} = 0$ on C.*

Proof. By means of Proposition A.6 the result follows since for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ that the process $\{Z_s, s \geq 0\}$ is a $\mathbb{P}_{t,x}$-submartingale and $\{Z_{s\wedge\tau_D}, s \geq 0\}$ is $\mathbb{P}_{t,x}$-martingale, where for any $s \geq 0$,

$$Z_s = V^{(\theta)}(t+s, X_{t+s}) + \int_0^s G^{(\theta)}(r+t, X_{r+t})\mathrm{d}r.$$

Indeed, it is a consequence of the fact that, under the measure $\mathbb{P}_{t,x}$, the Snell envelope of the process $\{\int_0^s G^{(\theta)}(r+t, X_{r+t})\mathrm{d}r, s \geq 0\}$ is given by Z_s (due to the Markovian structure of (t, X), see Section 1.2.3 or Theorem 2.2 of Peskir and Shiryaev (2006)). □

We define a special function which is useful to prove the left-continuity of the boundary $b^{(\theta)}$. For $\theta > 0$, let

$$\varphi^{(\theta)}(t,x) = \left[\int_{(-\infty,0)} V^{(\theta)}(t, x+y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,x)\right]\mathbb{I}_{\{x > b^{(\theta)}(t)\}}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.12}$$

Provided that $\varphi^{(\theta)}$ is locally integrable, we can define the distribution $\varphi^{(\theta)}$ by

$$\langle \varphi^{(\theta)}, \phi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \varphi^{(\theta)}(t,x)\phi(t,x)\mathrm{d}x\mathrm{d}t$$

for any $\phi \in C^\infty$ with compact support in $\mathbb{R}_+ \times \mathbb{R}$. The next Lemma states some properties of $\varphi^{(\theta)}$.

Lemma 2.3.8. *On the interior of D, the function $\varphi^{(\theta)}$ is strictly increasing on each argument, strictly positive and continuous whereas, on C, it vanishes. Moreover, we have that $\mathcal{A}_{(t,X)}(V^{(\theta)}) + G^{(\theta)} = \varphi^{(\theta)}$ on the interior of D in the sense of distributions.*

Proof. Let $t \in [0, m_\theta)$ and $x \in \mathbb{R}$. Note that if $x > b^{(\theta)}(t)$, we have that for all $y \in (b^{(\theta)}(t) - x, 0)$, $V^{(\theta)}(t, x+y) = 0$. Then from the fact that $V^{(\theta)}$ is bounded (see Lemma 2.3.2) and $|G| \leq 1$, we obtain that

$$\begin{aligned}
|\varphi^{(\theta)}(t,x)| &= \left|\int_{(-\infty, b^{(\theta)}(t)-x)} V^{(\theta)}(t, x+y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,x)\right| \\
&\leq \int_{(-\infty, b^{(\theta)}(t)-x)} \left|V^{(\theta)}(t, x+y)\right| \Pi(\mathrm{d}y) + |G^{(\theta)}(t,x)| \\
&\leq \int_{(-\infty, b^{(\theta)}(t)-x)} m_\theta \Pi(\mathrm{d}y) + 1 \\
&< \infty,
\end{aligned}$$

where the last inequality follows from the fact that Π is a finite measure on the interval $(-\infty, -\varepsilon)$, for all $\varepsilon > 0$. Then $\varphi^{(\theta)}(t, x)$ is finite for all $t \geq 0$ and $x \in \mathbb{R}$. Recall that the function $G^{(\theta)}$ is continuous and strictly increasing in each argument on the set $\mathbb{R}_+ \times (0, \infty)$. Then from the fact that $b^{(\theta)}$ is non-negative, $V^{(\theta)}$ is continuous and non-decreasing in each argument (see Lemmas 2.3.2 and 2.3.6) and the dominated convergence theorem, we conclude that $\varphi^{(\theta)}$ is continuous and strictly increasing on D. Then $\varphi^{(\theta)}$ is locally integrable and hence $\varphi^{(\theta)}$ can be defined as a distribution.

Next, we show that $\mathcal{A}_{(t,X)}(V^{(\theta)}) + G^{(\theta)} = \varphi^{(\theta)}$ on the interior of D in the sense of distributions. Take $\phi \in C^\infty$ with compact support on the interior of D, then

$$\begin{aligned}
\langle \mathcal{A}_{(t,X)}(V^{(\theta)}), \phi \rangle &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} V^{(\theta)}(t, x) B_X^*(\phi)(t, x) \mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty \int_{-\infty}^{b^{(\theta)}(t)} V^{(\theta)}(t, x) \int_{(-\infty,0)} \phi(t, x-y)\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty \int_{b^{(\theta)}(t)}^{\infty} \phi(t, x) \int_{(-\infty,0)} V^{(\theta)}(t, x+y)\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Then we conclude that $\mathcal{A}_{(t,X)}(V^{(\theta)}) + G^{(\theta)} = \varphi^{(\theta)}$ holds on D in the sense of distributions.

Lastly, we show that $\varphi^{(\theta)}$ is strictly positive on D. From Lemma 2.3.7, we have that φ is a non-negative distribution. Then by continuity, we have that $\varphi^{(\theta)}(t, x) \geq 0$ for all $(t, x) \in D$. Indeed, assume that there exists (t_0, x_0) such that $\varphi^{(\theta)}(t_0, x_0) < 0$. By continuity we have that there exists an open set A, such that $(t_0, x_0) \in A$ and $\varphi^{(\theta)}(t, x) < 0$ for all $(t, x) \in A$. Then, if we take any non-negative function $\phi \in C^\infty$ with compact support in A we have that

$$\langle \varphi^{(\theta)}, \phi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \varphi^{(\theta)}(t, x)\phi(t, x)\mathrm{d}x\mathrm{d}t < 0$$

which contradicts the fact that $\varphi^{(\theta)}$ is a non-negative distribution. Fix $t \in [0, m_\theta)$ and suppose that there exists $y > b^{(\theta)}(t)$ such that $\varphi^{(\theta)}(t, y) = 0$. Let $x \in (b^{(\theta)}(t), y)$ then since $\varphi^{(\theta)}$ is strictly increasing in each argument, we have

$$0 = \varphi^{(\theta)}(t, y) > \varphi^{(\theta)}(t, x) \geq 0$$

which is a contradiction. Then $\varphi^{(\theta)}$ is strictly positive on the interior of D. □

Now we are ready to show that the optimal boundary is continuous on the set $[0, m_\theta)$. The method proof is based on Lamberton and Mikou (2008) (see Theorem 4.2) where the continuity of the boundary is shown in the American option context.

Lemma 2.3.9. *The function $b^{(\theta)}$ is continuous on $[0, m_\theta)$.*

Proof. From the continuity of $V^{(\theta)}$, we deduce that the set D is closed. Let $t \in [0, m_\theta)$ and let $\{t_n\}_{n\geq 0}$ be a sequence of numbers such that $t_n \downarrow t$, and consider the limit $b^{(\theta)}(t+) = \lim_{n\to\infty} b^{(\theta)}(t_n)$ (which exists since $b^{(\theta)}$ is non-increasing). Note that from the fact that $b^{(\theta)}$ is non-increasing, we have that $b^{(\theta)}(t) \geq b^{(\theta)}(t+)$. On the other hand, we have that $(b^{(\theta)}(t_n), t_n) \in D$ and from the fact that D is closed, $\lim_{n\to\infty}(t_n, b^{(\theta)}(t_n)) = (t, b^{(\theta)}(t+)) \in D$. Hence we conclude that $b^{(\theta)}(t) \leq b^{(\theta)}(t+)$ and therefore $b^{(\theta)}$ is right-continuous.

We now show that $b^{(\theta)}$ is left-continuous. For this, suppose that there exists some $t_d \in (0, m_\theta)$ such that $\lim_{h\downarrow 0} b^{(\theta)}(t_d - h) =: b^{(\theta)}(t_d-) > b^{(\theta)}(t_d)$ and choose any $(s, x) \in [0, t_d) \times (b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$. We then have that $x < b^{(\theta)}(t_d-) \leq b^{(\theta)}(s)$, so that $V^{(\theta)}(s, x) < 0$ and then $[0, t_d) \times (b^{(\theta)}(t_d), b^{(\theta)}(t_d-)) \subset C$. From Lemma 2.3.7, we deduce that $\mathcal{A}_{(t,X)}(V^{(\theta)}) + G^{(\theta)} = 0$ on $(0, t_d) \times (b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$. Then, if we take any non-negative function $\phi \in C^\infty$ we have that

$$\langle \mathcal{A}_X(V^{(\theta)}) + G^{(\theta)}, \phi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} V^{(\theta)}(t, x) \frac{\partial}{\partial t}\phi(t, x) \mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}_+} \int_{\mathbb{R}} V^{(\theta)}(\mathrm{d}t, x)\phi(t, x)\mathrm{d}x \leq 0,$$

where the last inequality follows from the fact for all $x \in \mathbb{R}$, $t \mapsto V^{(\theta)}(t, x)$ is non-decreasing and then for any $x \in \mathbb{R}$, the measure $V^{(\theta)}(\mathrm{d}t, x)$ is well defined. Note that the equation above means that $\mathcal{A}_X(V^{(\theta)}) + G^{(\theta)}$ is a non positive distribution. By continuity, we have that for any $t \in (0, t_d)$, the distribution $\mathcal{A}_X(V^{(\theta)})(t, \cdot) + G^{(\theta)}(t, \cdot)$ is a non positive distribution on $(b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$. Indeed, suppose that there exists $t_0 \in (0, t_d)$ and a non-negative function $\phi \in C^\infty$ with compact support on $(b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$ such that

$$\int_{\mathbb{R}} V^{(\theta)}(t_0, x) \left[\mu \frac{\partial}{\partial x}\phi(x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\phi(x) + B_X^*(\phi)(x) \right] \mathrm{d}x + \int_{\mathbb{R}} G^{(\theta)}(t_0, x)\phi(x)\mathrm{d}x > 0$$

Then by the continuity of the functions $t \mapsto V^{(\theta)}(t, x)$ and $t \mapsto G^{(\theta)}(t, x)$ (for any $x > 0$), we have that there exists an open set $A \subset (0, t_d)$ such that for any nonnegative function $\varphi \in C^\infty$ with compact support on A,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} V^{(\theta)}(t,x)\left[\mu\frac{\partial}{\partial x}\phi(x)\varphi(t) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi(x)\varphi(t) + B_X^*(\phi\cdot\varphi)(t,x)\right]\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}_+}\int_{\mathbb{R}} G^{(\theta)}(t,x)\phi(x)\varphi(t)\mathrm{d}x\mathrm{d}t > 0.$$

Note that the equation above contradicts the fact that $\mathcal{A}_X(V^{(\theta)}) + G^{(\theta)}$ is a non positive distribution on $(0, t_d) \times (b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$. Hence, for any $t \in (0, t_d)$, the distribution $\mathcal{A}_X(V^{(\theta)})(t,\cdot) + G^{(\theta)}(t,\cdot)$ non positive on $(b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$. Then, once again by continuity, we have that for any non-negative function $\phi \in C^\infty$ with compact support on $(b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$,

$$\begin{aligned}
0 &\geq \lim_{t\uparrow t_d}\langle \mathcal{A}_X(V^{(\theta)})(t,\cdot) + G^{(\theta)}(t,\cdot), \phi\rangle \\
&= \lim_{t\uparrow t_d}\left\{\int_{\mathbb{R}} V^{(\theta)}(t,x)\left[\mu\frac{\partial}{\partial x}\phi(x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi(x) + B_X^*(\phi)(x)\right]\mathrm{d}x + \int_{\mathbb{R}} G^{(\theta)}(t,x)\phi(x)\mathrm{d}x\right\} \\
&= \int_{\mathbb{R}} V^{(\theta)}(t_d,x)\left[\mu\frac{\partial}{\partial x}\phi(x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi(x) + B_X^*(\phi)(x)\right]\mathrm{d}x + \int_{\mathbb{R}} G^{(\theta)}(t_d,x)\phi(x)\mathrm{d}x \\
&= \int_{-\infty}^{b^{(\theta)}(t_d)} V^{(\theta)}(t_d,x)\int_{(-\infty,0)}\phi(x-y)\Pi(\mathrm{d}y)\mathrm{d}x + \int_{\mathbb{R}} G^{(\theta)}(t_d,x)\phi(x)\mathrm{d}x \\
&= \langle \varphi^{(\theta)}, \phi\rangle,
\end{aligned}$$

where the second last equality follows from the fact that $V^{(\theta)}(t_d,\cdot)$ vanishes on the set $(b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$. Note that that the equation above contradicts the fact that $\varphi^{(\theta)})(t_d,\cdot)$ is a strictly positive function on $(b^{(\theta)}(t_d), b^{(\theta)}(t_d-))$ (see Lemma 2.3.8). Therefore we conclude that $b^{(\theta)}$ is also left-continuous and the proof is complete. □

Recall that we have that $b^{(\theta)}(t) = -\infty$ for $t \in [m_\theta, \infty)$. The next Lemma describes the limit behaviour of $b^{(\theta)}$ around m_θ.

Lemma 2.3.10. *We have that* $\lim_{t\uparrow m_\theta} b^{(\theta)}(t) = 0$.

Proof. Define $b^{(\theta)}(m_\theta-) := \lim_{t\uparrow m_\theta} b^{(\theta)}(t)$. We obtain $b^{(\theta)}(m_\theta-) \geq 0$ since $b^{(\theta)}(t) \geq h^{(\theta)}(t) \geq 0$ for all $t \in [0, m_\theta)$. The proof is by contradiction so we assume that $b^{(\theta)}(m_\theta-) > 0$.

Note that for all $x \in \mathbb{R}$, we have that $V^{(\theta)}(m_\theta, 0) = 0$ and

$$G^{(\theta)}(m_\theta, x) = 1 + 2e^{-\theta\frac{\log(2)}{\theta}}[F^{(\theta)}(x) - 1] = F^{(\theta)}(x).$$

Moreover, following an analogous argument as in Lemma 2.3.9 we have that,

$$\mathcal{A}_X(V^{(\theta)}) + G^{(\theta)} = -\partial_t V^{(\theta)} \le 0$$

in the sense of distributions on $(0, m_\theta) \times (0, b^{(\theta)}(m_\theta-))$. Hence by continuity, we can derive, for $t \in [0, m_\theta)$, that $\mathcal{A}_X(V^{(\theta)})(t, \cdot) + G^{(\theta)}(t, \cdot) \le 0$ on the interval $(0, b^{(\theta)}(m_\theta-))$. Hence, for any non-negative function $\phi \in C^\infty$ with compact support on $(0, b^{(\theta)}(m_\theta-))$, we have that

$$\begin{aligned}
0 &\ge \lim_{t\uparrow m_\theta} \langle \mathcal{A}_X(V^{(\theta)})(t,\cdot) + G^{(\theta)}(t,\cdot), \phi\rangle \\
&= \lim_{t\uparrow m_\theta} \left\{ \int_{\mathbb{R}} V^{(\theta)}(t,x) \left[\mu\frac{\partial}{\partial x}\phi(x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi(x) + B_X^*(\phi)(x)\right] \mathrm{d}x + \int_{\mathbb{R}} G^{(\theta)}(t,x)\phi(x)\mathrm{d}x \right\} \\
&= \int_{\mathbb{R}} F^{(\theta)}(x)\phi(x)\mathrm{d}x \\
&> 0,
\end{aligned}$$

where we used in third equality we used the continuity of $V^{(\theta)}$ and $G^{(\theta)}$ on the first argument and the fact that $V^{(\theta)}(m_\theta, x) = 0$ for all $x \in \mathbb{R}$ and in the last inequality we used that $F^{(\theta)}(x) > 0$ for all $x > 0$. Note that we have got a contradiction and we conclude that $b^{(\theta)}(m_\theta) = 0$. □

Define the value

$$t_b := \inf\{t \ge 0 : b^{(\theta)}(t) \le 0\}. \tag{2.13}$$

Note that in the case where X is a process of infinite variation, we have that the distribution function of $-\underline{X}_{\widetilde{e}_\theta}$, $F^{(\theta)}$ is continuous in $\mathbb{R}$, strictly increasing and strictly positive in the open set $(0, \infty)$ with $F^{(\theta)}(0) = 0$. This fact implies that the inverse function of $F^{(\theta)}$ exists on

$(0,\infty)$ and then function $h^{(\theta)}$ can be written for $t \in [0, m_\theta)$ as

$$h^{(\theta)}(t) = (F^{(\theta)})^{-1}\left(1 - \frac{1}{2}e^{\theta t}\right).$$

Hence we conclude that $h^{(\theta)}(t) > 0$ for all $t \in [0, m_\theta)$. Therefore, when X is a process of infinite variation, we have $b^{(\theta)}(t) > 0$ for all $t \in [0, m_\theta)$ and hence $t_b = m_\theta$. For the case of finite variation, we have that $t_b \in [0, m_\theta)$ which implies that $b^{(\theta)}(t) = 0$ for all $t \in [t_b, m_\theta)$ and $b^{(\theta)}(t) > 0$ for all $t \in [0, t_b)$. In the next lemma, we characterise its value.

Lemma 2.3.11. *Let $\theta > 0$ and X be a process of finite variation. We have that for all $t \geq 0$ and $x \in \mathbb{R}$,*

$$\int_{(-\infty,0)} [V^{(\theta)}(t, x + y) - V^{(\theta)}(t, x)]\Pi(\mathrm{d}y) > -\infty.$$

Moreover, for any Lévy process, t_b is given by

$$t_b = \inf\left\{t \in [0, m_\theta] : \int_{(-\infty,0)} V_B^{(\theta)}(t, y)\Pi(\mathrm{d}y) + G^{(\theta)}(t, 0) \geq 0\right\}, \tag{2.14}$$

where $V_B^{(\theta)}$ is given by

$$V_B^{(\theta)}(t, y) = \mathbb{E}_y(\tau_0^+ \wedge (m_\theta - t)) - \frac{2}{\theta}e^{-\theta t}[1 - \mathbb{E}_y(e^{-\theta(\tau_0^+ \wedge (m_\theta - t))})]$$

for all $t \in [0, m_\theta)$ and $y \in \mathbb{R}$.

Proof. Assume that X is a process of finite variation. We first show that

$$\int_{(-\infty,0)} [V^{(\theta)}(t, x + y) - V^{(\theta)}(t, x)]\Pi(\mathrm{d}y) > -\infty$$

for all $t \geq 0$ and $x \in \mathbb{R}$. The case $t \geq m_\theta$ is straightforward since $V^{(\theta)}(t, x) = 0$ for all $x \in \mathbb{R}$. Assuming that $t \in [0, m_\theta)$, if $x > b^{(\theta)}(0) \geq b^{(\theta)}(t)$, we have $V^{(\theta)}(t, x) = 0$ resulting in

$$\begin{aligned}\int_{(-\infty,0)} [V^{(\theta)}(t, x + y) - V^{(\theta)}(t, x)]\Pi(\mathrm{d}y) &= \int_{(-\infty, b^{(\theta)}(t)-x)} V^{(\theta)}(t, x + y)\Pi(\mathrm{d}y) \\ &\geq -(m_\theta - t)\Pi(-\infty, b^{(\theta)}(t) - x) \\ &> -\infty,\end{aligned}$$

where the last equality follows since $|V^{(\theta)}| \le m_\theta - t$ and Π is finite on intervals away from zero. If $x \le b^{(\theta)}(0)$, we have by equation (2.10) that

$$\begin{aligned}
&\int_{(-\infty,0)} [V^{(\theta)}(t,x+y) - V^{(\theta)}(t,x)]\Pi(\mathrm{d}y) \\
&\quad= \int_{(-1,0)} [V^{(\theta)}(t,x+y) - V^{(\theta)}(t,x)]\Pi(\mathrm{d}y) + \int_{(-\infty,-1)} [V^{(\theta)}(t,x+y) - V^{(\theta)}(t,x)]\Pi(\mathrm{d}y) \\
&\quad\ge 2Me^{\Phi(\theta)b^{(\theta)}(0)} \int_{(-1,0)} y\Pi(\mathrm{d}y) - (m_\theta - t)\Pi(-\infty,-1) \\
&\quad> -\infty,
\end{aligned}$$

where the last quantity is finite since X is of finite variation and then $\int_{(-1,0)} y\Pi(\mathrm{d}y) > -\infty$. Moreover, from Lemma 2.3.7, we obtain that

$$\int_{(-\infty,0)} [V^{(\theta)}(t,x+y) - V^{(\theta)}(t,x)]\Pi(\mathrm{d}y) + G^{(\theta)}(t,x) = -\frac{\partial}{\partial t}V^{(\theta)}(t,x) - \delta\frac{\partial}{\partial x}V^{(\theta)}(t,x) \le 0$$

on C in the sense of distributions, where the last inequality follows since $V^{(\theta)}$ is non-decreasing in each argument and $\delta > 0$. Next, we show that the set $\{t \in [0, m_\theta) : b^{(\theta)}(t) = 0\}$ is non empty. We proceed by contradiction, assume that $b^{(\theta)}(t) > 0$ for all $t \in [0, m_\theta)$. Then by continuity of the functions $t \mapsto V^{(\theta)}(t,y)$ for all $y \le 0$ and $t \mapsto G^{(\theta)}(t,0)$, we can derive

$$\int_{(-\infty,0)} [V^{(\theta)}(t,y) - V^{(\theta)}(t,0)]\Pi(\mathrm{d}y) + G^{(\theta)}(t,0) \le 0 \tag{2.15}$$

for all $t \in [0, m_\theta)$. Taking $t \uparrow m_\theta$ and applying dominated convergence theorem, we obtain that

$$0 \ge \lim_{t\uparrow m_\theta} \left\{ \int_{(-\infty,0)} [V^{(\theta)}(t,y) - V^{(\theta)}(t,0)]\Pi(\mathrm{d}y) + G^{(\theta)}(t,0) \right\} = G^{(\theta)}(m_\theta, 0) = F^{(\theta)}(0) > 0,$$

where the strict inequality follows from $F^{(\theta)}(0) = \frac{\theta}{\Phi(\theta)}W^{(\theta)}(0) = \frac{\theta}{\delta\Phi(\theta)} > 0$ since X is of finite variation. Therefore, we observe a contradiction which shows that $\{t \in [0, m_\theta) : b^{(\theta)}(t) = 0\} \ne \emptyset$. Moreover, by the definition, we have that $t_b = \inf\{t \in [0, m_\theta) : b^{(\theta)}(t) = 0\}$.

Next we find an expression for $V^{(\theta)}$ for $x \in (-\infty, 0)$. Take any $t \in (0, m_\theta)$ and $x < 0$.

Since $b^{(\theta)}(t) \geq 0$ for all $t \in [0, m_\theta)$, we have that

$$\begin{aligned}
V^{(\theta)}(t,x) &= \mathbb{E}_x\left(\int_0^{\tau_0^+\wedge(m_\theta - t)} (1-2e^{-\theta(t+s)})\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+ < m_\theta - t\}} V^{(\theta)}(t+\tau_0^+, 0)\right) \\
&= \mathbb{E}_x(\tau_0^+ \wedge (m_\theta - t)) - \frac{2}{\theta} e^{-\theta t}[1 - \mathbb{E}_x(e^{-\theta(\tau_0^+\wedge(m_\theta - t))})] \\
&\quad + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+ < m_\theta - t\}} V^{(\theta)}(t+\tau_0^+, 0)\right) \\
&= V_B^{(\theta)}(t,x) + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+ < m_\theta - t\}} V^{(\theta)}(t+\tau_0^+, 0)\right), \qquad (2.16)
\end{aligned}$$

where the first equality follows since $X_s \leq 0$ for all $s \leq \tau_0^+$ and $G(t,x) = 1 - 2e^{-\theta t}$ for all $x < 0$. Hence, in particular, we have that $V^{(\theta)}(t,x) = V_B^{(\theta)}(t,x)$ for all $t \in [t_b, m_\theta)$ and $x \in \mathbb{R}$. We show that (2.14) holds. By Lemma 2.3.8, we obtain that

$$\int_{(-\infty,0)} V^{(\theta)}(t, x+y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,x) \geq 0$$

for all $x > 0$ and $t \geq t_b$. Then by taking $x \downarrow 0$, making use of the right continuity of $x \mapsto G(t,x)$, continuity of $V^{(\theta)}$ (see Lemma 2.3.6) and applying dominated convergence theorem, we derive that

$$\int_{(-\infty,0)} V^{(\theta)}(t_b, y)\Pi(\mathrm{d}y) + G^{(\theta)}(t_b, 0) \geq 0.$$

In particular, if $t_b = 0$, (2.14) holds since $V^{(\theta)}(t,0)$ and $G^{(\theta)}(t,0)$ are non-decreasing functions. If $t_b > 0$, taking $t \uparrow t_b$ in (2.15) gives us

$$\int_{(-\infty,0)} V^{(\theta)}(t_b, y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,0) \leq 0.$$

Hence, we have that $\int_{(-\infty,0)} V_B^{(\theta)}(t_b, y)\Pi(\mathrm{d}y) + G^{(\theta)}(t_b, 0) = 0$ with (2.14) becoming clear due to the fact that $t \mapsto V_B^{(\theta)}(t,x)$ is non-decreasing. If X is a process of infinite variation, we have that $h^{(\theta)}(t) > 0$ for all $t \in [0, m_\theta)$ and therefore $G^{(\theta)}(t,x) < 0$ for all $t \in [0, m_\theta)$ and $x \leq 0$ which implies that

$$t_b = m_\theta = \inf\left\{ t \in [0, m_\theta] : \int_{(-\infty,0)} V_B^{(\theta)}(t,y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,0) \geq 0 \right\}.$$

□

Now we prove that the partial derivatives of V are equal to zero on the curve $b^{(\theta)}$ for those values for which $b^{(\theta)}$ is strictly positive.

Lemma 2.3.12. *For all $t \in [0, t_b)$, the partial derivatives of $V^{(\theta)}(t,x)$ at the point $(t, b^{(\theta)}(t))$ exist and are equal to zero, i.e.,*

$$\frac{\partial}{\partial t} V^{(\theta)}(t, b^{(\theta)}(t)) = 0 \qquad \textit{and} \qquad \frac{\partial}{\partial x} V^{(\theta)}(t, b^{(\theta)}(t)) = 0.$$

Proof. First, we prove that the assertion in the first argument. Using a similar idea as in Lemma 2.3.6, we have that for any $t < t_b$, $x \in \mathbb{R}$ and $h > 0$,

$$\begin{aligned}
0 &\leq \frac{V^{(\theta)}(t, b^{(\theta)}(t)) - V^{(\theta)}(t-h, b^{(\theta)}(t))}{h} \\
&\leq 2\mathbb{E}_{b^{(\theta)}(t)}\left(\int_0^{\tau_h^*} \frac{[e^{-\theta(r+t-h)} - e^{-\theta(r+t)}]}{h} \mathrm{d}r \right) \\
&\leq 2\mathbb{E}_{b^{(\theta)}(t)}\left(\int_0^{\tau^+_{b^{(\theta)}(t-h)}} \frac{[e^{-\theta(r+t-h)} - e^{-\theta(r+t)}]}{h} \mathrm{d}r \right) \\
&= \frac{2}{\theta} \frac{e^{-\theta(t-h)} - e^{-\theta t}}{h} \mathbb{E}_{b^{(\theta)}(t)}\left(1 - e^{-\theta \tau^+_{b^{(\theta)}(t-h)}} \mathbb{I}_{\{\tau^+_{b^{(\theta)}(t-h)} < \infty\}} \right) \\
&= \frac{2}{\theta} \frac{e^{-\theta(t-h)} - e^{-\theta t}}{h} [1 - e^{-\Phi(\theta)[b^{(\theta)}(t-h) - b^{(\theta)}(t)]}],
\end{aligned}$$

where $\tau_h^* = \inf\{r \in [0, m_\theta - t + h] : X_r \geq b^{(\theta)}(r + t - h)\}$ is the optimal stopping time for $V^{(\theta)}(t-h, x)$, the second inequality follows since b is non increasing and the last equality by equation (1.3). Since $b^{(\theta)}$ is continuous, we have that $b^{(\theta)}(t-h) \downarrow b^{(\theta)}(t)$ when $h \downarrow 0$. Hence, we obtain that

$$\lim_{h \downarrow 0} \frac{V^{(\theta)}(t, b^{(\theta)}(t)) - V^{(\theta)}(t-h, b^{(\theta)}(t))}{h} = 0.$$

Now we show that the partial derivative of the second argument exists at $b^{(\theta)}(t)$ and is equal to zero. Fix any time $t \in [0, t_b)$, $\varepsilon > 0$ and $x \leq b^{(\theta)}(t)$ (without loss of generality, we assume that $\varepsilon < x$). By a similar argument as in Lemma 2.3.6, we obtain that

$$\begin{aligned}
&V^{(\theta)}(t,x) - V^{(\theta)}(t,x-\varepsilon) \\
&\leq 2\int_{-\infty}^{b^{(\theta)}(t)} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)] \left[e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(x-\varepsilon-z)\right] \mathrm{d}z \\
&= 2e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} \int_{x-\varepsilon}^{b^{(\theta)}(t)} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)] W^{(\theta)}(b^{(\theta)}(t)-z)\mathrm{d}z \\
&\quad + 2\int_{0}^{x-\varepsilon} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)] \left[e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(x-\varepsilon-z)\right] \mathrm{d}z \\
&\quad + 2\int_{-\varepsilon}^{0} F^{(\theta)}(z+\varepsilon) \left[e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(x-\varepsilon-z)\right] \mathrm{d}z.
\end{aligned}$$

Dividing by ε, we have that for $t \in [0, t_b)$ and $\varepsilon < x$ that

$$0 \leq \frac{V^{(\theta)}(t,x) - V^{(\theta)}(t,x-\varepsilon)}{\varepsilon} \leq R_1^{(\varepsilon)}(t,x) + R_2^{(\varepsilon)}(t,x) + R_3^{(\varepsilon)}(t,x),$$

where

$$\begin{aligned}
R_1^{(\varepsilon)}(t,x) &= 2e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} \frac{1}{\varepsilon} \int_{x-\varepsilon}^{b^{(\theta)}(t)} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)] W^{(\theta)}(b^{(\theta)}(t)-z)\mathrm{d}z, \\
R_2^{(\varepsilon)}(t,x) &= 2\frac{1}{\varepsilon} \int_{0}^{x-\varepsilon} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)] \\
&\qquad \times \left[e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(x-\varepsilon-z)\right] \mathrm{d}z, \\
R_3^{(\varepsilon)}(t,x) &= 2\frac{1}{\varepsilon} \int_{-\varepsilon}^{0} F^{(\theta)}(z+\varepsilon) \left[e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)} W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(x-\varepsilon-z)\right] \mathrm{d}z.
\end{aligned}$$

Then we show that for $t \in [0, t_b)$, $\lim_{\varepsilon\downarrow 0} R_i^{(\varepsilon)}(t, b^{(\theta)}(t)) = 0$ for each $i = 1, 2, 3$. Using the fact that W and F are non-decreasing, we derive that

$$\begin{aligned}
0 \leq \lim_{\varepsilon\downarrow 0} R_1^{(\varepsilon)}(t, b^{(\theta)}(t)) &\leq \lim_{\varepsilon\downarrow 0} 2e^{-\Phi(\theta)\varepsilon} W^{(\theta)}(b^{(\theta)}(t)) \frac{1}{\varepsilon} \int_{b^{(\theta)}(t)-\varepsilon}^{b^{(\theta)}(t)} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)]\mathrm{d}z \\
&\leq \lim_{\varepsilon\downarrow 0} 2e^{-\Phi(\theta)\varepsilon} W^{(\theta)}(b^{(\theta)}(t)) [F(b^{(\theta)}(t)+\varepsilon) - F(b^{(\theta)}(t)-\varepsilon)] \\
&= 0,
\end{aligned}$$

where in the last equality, we used the fact that $b^{(\theta)}(t) > 0$ and that $F^{(\theta)}$ is continuous on

$(0,\infty)$. In a similar way, we obtain that

$$
\begin{aligned}
0 \le \lim_{\varepsilon\downarrow 0} & R_3^{(\varepsilon)}(t, b^{(\theta)}(t)) \\
&= \lim_{\varepsilon\downarrow 0} 2\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} F^{(\theta)}(z+\varepsilon)\left[e^{-\Phi(\theta)\varepsilon}W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(b^{(\theta)}(t)-\varepsilon-z)\right]\mathrm{d}z \\
&\le \lim_{\varepsilon\downarrow 0} 2F^{(\theta)}(\varepsilon)\left[W^{(\theta)}(b^{(\theta)}(t)+\varepsilon) - W^{(\theta)}(b^{(\theta)}(t)-\varepsilon)\right] \\
&= 0.
\end{aligned}
$$

To show that $\lim_{\varepsilon} R_2^{(\varepsilon)}(t, b^{(\theta)}(t)) = 0$, we first note that for all $z \in (0, b^{(\theta)}(t))$,

$$
\begin{aligned}
[F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)]&\left[e^{-\Phi(\theta)(b^{(\theta)}(t)-x+\varepsilon)}W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(x-\varepsilon-z)\right] \\
&\le W^{(\theta)}(b^{(\theta)}(t))[F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)].
\end{aligned}
$$

Moreover, using Fubini's theorem, it can be shown that

$$
\int_0^{b^{(\theta)}(t)} [F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)]\mathrm{d}z \le \varepsilon[F^{(\theta)}(b^{(\theta)}(t)+\varepsilon) - F^{(\theta)}(0)].
$$

Using the dominated convergence theorem, we get

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0} & R_2^{(\theta)}(t, b^{(\theta)}(t)) \\
&\le 2e^{-\theta t}\int_0^{b^{(\theta)}(t)-\varepsilon} \lim_{\varepsilon\downarrow 0}\frac{F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)}{\varepsilon} \\
&\qquad\qquad \times\left[e^{-\Phi(\theta)\varepsilon}W^{(\theta)}(b^{(\theta)}(t)-z) - W^{(\theta)}(b^{(\theta)}(t)-\varepsilon-z)\right]\mathrm{d}z \\
&= 0,
\end{aligned}
$$

where we used the fact that $W^{(\theta)}$ is left continuous on $\mathbb{R}$ and

$$
\lim_{\varepsilon\downarrow 0}\frac{F^{(\theta)}(z+\varepsilon) - F^{(\theta)}(z)}{\varepsilon} = (F^{(\theta)})'_+(z) < \infty
$$

where $(F^{(\theta)})'_+$ is the right derivative of $F^{(\theta)}$ which exists since $W^{(\theta)}$ has right and left derivatives. We can then conclude that

$$\lim_{\varepsilon\downarrow 0}\frac{V^{(\theta)}(t,b^{(\theta)}(t))-V^{(\theta)}(t,b^{(\theta)}(t)-\varepsilon)}{\varepsilon}=0$$

proving that $x\mapsto V^{(\theta)}(x,t)$ is differentiable at $b^{(\theta)}(t)$ with $\partial/\partial x V^{(\theta)}(t,b^{(\theta)}(t))=0$ for $t\in[0,t_b)$. □

The next theorem looks at how the value function $V^{(\theta)}$ and the curve $b^{(\theta)}$ can be characterised as a solution of non-linear integral equations within a certain family of functions. These equations are in fact generalisations of the free boundary equation (see e.g. Peskir and Shiryaev (2006) Section 14.1 in a diffusion setting) in the presence of jumps. It is important to mention that the proof of Theorem 2.3.13 is mainly inspired by the ideas of du Toit et al. (2008) with some extensions to allow for the presence of jumps.

Theorem 2.3.13. *Let X be a spectrally negative Lévy process and let t_b be as characterised in (2.14). For all $t\in[0,t_b)$ and $x\in\mathbb{R}$, we have that*

$$\begin{aligned}V^{(\theta)}(t,x)&=\mathbb{E}_x\left(\int_0^{m_\theta-t}G^{(\theta)}(r+t,X_r)\mathbb{I}_{\{X_r<b^{(\theta)}(r+t)\}}\mathrm{d}r\right)\\&\quad-\mathbb{E}_x\left(\int_0^{m_\theta-t}\int_{(-\infty,b^{(\theta)}(r+t)-X_r)}V^{(\theta)}(r+t,X_r+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r>b^{(\theta)}(r+t)\}}\mathrm{d}r\right)\end{aligned}\tag{2.17}$$

and $b^{(\theta)}(t)$ solves the equation

$$\begin{aligned}0&=\mathbb{E}_{b^{(\theta)}(t)}\left(\int_0^{m_\theta-t}G^{(\theta)}(r+t,X_r)\mathbb{I}_{\{X_r<b^{(\theta)}(r+t)\}}\mathrm{d}r\right)\\&\quad-\mathbb{E}_{b^{(\theta)}(t)}\left(\int_0^{m_\theta-t}\int_{(-\infty,b^{(\theta)}(r+t)-X_r)}V^{(\theta)}(r+t,X_r+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r>b^{(\theta)}(r+t)\}}\mathrm{d}r\right).\end{aligned}\tag{2.18}$$

If $t\in[t_b,m_\theta)$, we have that $b^{(\theta)}(t)=0$ and

$$V^{(\theta)}(t,x)=\mathbb{E}_x(\tau_0^+\wedge(m_\theta-t))-\frac{2}{\theta}e^{-\theta t}[1-\mathbb{E}_x(e^{-\theta(\tau_0^+\wedge(m_\theta-t))})]\tag{2.19}$$

for all $x\in\mathbb{R}$. Moreover, the pair $(V^{(\theta)},b^{(\theta)})$ is uniquely characterised as the solutions to equations (2.17)-(2.19) in the class of continuous functions in $\mathbb{R}_+\times\mathbb{R}$ and $\mathbb{R}_+$, respectively, *such that* $b^{(\theta)} \geq h^{(\theta)}$, $V^{(\theta)} \leq 0$ *and* $\int_{(-\infty,0)} V^{(\theta)}(t, x+y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,x) \geq 0$ *for all* $t \in [0, t_b)$ *and* $x \geq b^{(\theta)}(t)$.

2.4 Proof of Theorem 2.3.13

Since the proof of Theorem 2.3.13 is rather long, we split it into a series of Lemmas. This section is entirely dedicated for this purpose. With the help of Itô formula and following an analogous argument as in Lamberton and Mikou (2013) (in the infinite variation case), we prove that $V^{(\theta)}$ and $b^{(\theta)}$ are solutions to the integral equations listed above. The finite variation case is proved using an argument that considers the consecutive times in which X hits the curve $b^{(\theta)}$.

Lemma 2.4.1. *The pair* $(V^{(\theta)}, b^{(\theta)})$ *are solutions to the equations* (2.17)-(2.19).

Proof. Recall from Lemma 2.3.11 that, when $t_b < m_\theta$, the value function $V^{(\theta)}$ satisfies equation (2.19). We also have that equation (2.18) follows from (2.17) by letting $x = b^{(\theta)}(t)$ and using that $V^{(\theta)}(t, b^{(\theta)}(t)) = 0$.

We proceed to show that $(V^{(\theta)}, b^{(\theta)})$ solves equation (2.17). First, we assume that X is a process of infinite variation. We follow an analogous argument as Lamberton and Mikou (2013) (see Theorem 3.2). Let ρ be a positive C^∞ function with support in $[0,1] \times [0,1]$ and $\int_0^\infty \int_0^\infty \rho(v,y)\mathrm{d}v\mathrm{d}y = 1$. For each $n \geq 1$, define $\rho_n(v,y) = n^2\rho(nv, ny)$. Then ρ_n is a non-negative $C^\infty(\mathbb{R}_+ \times \mathbb{R})$ function with support in $[0, 1/n] \times [0, 1/n]$ such that $\int_0^\infty \int_0^\infty \rho_n(s,y)\mathrm{d}s\mathrm{d}y = 1$. For every $n \geq 1$, define the function $V_n^{(\theta)}$ by

$$V_n^{(\theta)}(t,x) = (V^{(\theta)} * \rho_n)(t,x) = \int_0^\infty \int_0^\infty V^{(\theta)}(t-s, x-y)\rho_n(s,y)\mathrm{d}s\mathrm{d}y.$$

for any $(t,x) \in [-1/n, \infty) \times \mathbb{R}$. Then for each $n \geq 1$, the function $V_n^{(\theta)}$ is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ bounded function (since $V^{(\theta)}$ is bounded) with bounded derivatives. Moreover, we have that $V_n^{(\theta)}(t,x) \leq V_{n+1}(t,x)$ for all $(t,x) \in [1/n, \infty) \times \mathbb{R}$ and $V_n^{(\theta)} \uparrow V$ on $\mathbb{R}_+ \times \mathbb{R}$ when $n \to \infty$. Indeed, take $(t,x) \in [1/n, \infty) \times \mathbb{R}$, we have that

$$
\begin{aligned}
V_n^{(\theta)}(t,x) &= \int_0^\infty \int_0^\infty V^{(\theta)}(t-s,x-y)n^2\rho(ns,ny)\mathrm{d}s\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty V^{(\theta)}(t-s/n,x-y/n)\rho(s,y)\mathrm{d}s\mathrm{d}y \\
&\leq \int_0^\infty \int_0^\infty V^{(\theta)}(t-s/(n+1),x-y/(n+1))\rho(s,y)\mathrm{d}s\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty V^{(\theta)}(t-s,x-y)(n+1)^2\rho((n+1)s,(n+1)y)\mathrm{d}s\mathrm{d}y \\
&= V_{n+1}^{(\theta)}(t,x),
\end{aligned}
$$

where in the inequality we used that $s, y \geq 0$ and that $V^{(\theta)}$ is non-decreasing in each argument. The convergence of $V_n^{(\theta)}$ to $V^{(\theta)}$ follows from

$$
\begin{aligned}
|V_n^{(\theta)}(t,x) - V^{(\theta)}(t,x)| &\leq \int_0^\infty \int_0^\infty |V^{(\theta)}(t-s,x-y) - V^{(\theta)}(t,x)|\rho_n(s,y)\mathrm{d}s\mathrm{d}y \\
&\leq \sup_{s,y\in[0,1/n]} |V^{(\theta)}(t-s,x-y) - V^{(\theta)}(t,x)|,
\end{aligned}
$$

where we used the fact that the integral of ρ_n is equal to 1. Taking $n \to \infty$ we obtain the desired convergence by using the fact that $V^{(\theta)}$ is continuous. Furthermore, using a similar argument as in Lamberton and Mikou (2008) (see the proof of Proposition 2.5) we have that for a fixed $n \geq 1$,

$$
\frac{\partial}{\partial t}V_n^{(\theta)}(t,x) + \mathcal{A}_X(V_n^{(\theta)})(t,x) = -(G^{(\theta)} * \rho_n)(u,x) \qquad \text{for all } (t,x) \in (1/n,\infty)\times\mathbb{R}\cap C, \tag{2.20}
$$

where $\mathcal{A}_X$ is the infinitesimal generator of X given in (2.11) and $C = \mathbb{R}_+ \times \mathbb{R} \setminus D$. Indeed, take φ a non-negative C^∞ function with compact support in $[(1/n,\infty)\times\mathbb{R}]\cap C$ then we have that the function $\varphi * \check{\rho}_n$ is C^∞ and has compact support in C, where $\check{\rho}(v,y) = \rho_n(-v,-y)$ for all $(v,y) \in \mathbb{R}\times\mathbb{R}$. Hence, from Proposition A.5 we get that

$$
\langle \frac{\partial}{\partial t}V_n^{(\theta)} + \mathcal{A}_X(V_n^{(\theta)}) + G^{(\theta)} * \rho_n, \varphi\rangle = \langle \frac{\partial}{\partial t}V^{(\theta)} + \mathcal{A}_X(V^{(\theta)}) + G^{(\theta)}, \varphi * \check{\rho}_n\rangle = 0,
$$

where the last equality follows from Lemma 2.3.7. Therefore we have that by integration by parts formula and Lemma A.4 that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\left[\frac{\partial}{\partial t}V_n^{(\theta)}(t,x)+\mathcal{A}_X(V_n^{(\theta)})(t,x)+G^{(\theta)}*\rho_n(t,x)\right]\varphi(t,x)\mathrm{d}x\mathrm{d}t=0$$

for any φ non-negative and C^∞ function with compact support in $[(1/n,\infty)\times\mathbb{R}]\cap C$. Therefore (2.20) follows by continuity. On the other hand, note that if $(t,x)\in D$ we have that $V^{(\theta)}(t,x)=0$ and hence $V_n^{(\theta)}(t,x)=0$ for $n\geq 1$ sufficiently large. Hence,

$$\frac{\partial}{\partial t}V_n^{(\theta)}(t,x)+\mathcal{A}_X(V_n^{(\theta)})(t,x)=\int_{(-\infty,0)}V_n^{(\theta)}(t,x+y)\Pi(\mathrm{d}y).$$

Therefore, by the dominated convergence theorem we have that,

$$\lim_{n\to\infty}\left[\frac{\partial}{\partial t}V_n^{(\theta)}(t,x)+\mathcal{A}_X(V_n^{(\theta)})(t,x)\right]=\int_{(-\infty,0)}V^{(\theta)}(t,x+y)\Pi(\mathrm{d}y).$$

for any $(t,x)\in D$.

Let $t\in(0,t_b]$, $m>0$ such that $t>1/m$ and $x\in\mathbb{R}$, applying Itô formula to $V_n^{(\theta)}(t+s,X_s+x)$, for $s\in[0,m_\theta-t]$, we obtain that for any $n\geq m$,

$$\begin{aligned}V_n^{(\theta)}(s+t,X_s+x)&=V_n^{(\theta)}(t,x)+M_s^{t,n}\\&\quad+\int_0^s\left[\frac{\partial}{\partial t}V_n^{(\theta)}(r+t,X_r+x)+\mathcal{A}_X(V_n^{(\theta)})(r+t,X_r+x)\right]\mathrm{d}r,\end{aligned}$$

where $\{M_s^{t,n},t\geq 0\}$ is a zero mean martingale (see Lemma A.2). Hence, taking expectations and using (2.20), we derive that

$$\begin{aligned}&\mathbb{E}(V_n^{(\theta)}(s+t,X_s+x))\\&\quad=V_n^{(\theta)}(t,x)+\mathbb{E}\left(\int_0^s\left[\frac{\partial}{\partial t}V_n^{(\theta)}(r+t,X_r+x)+\mathcal{A}_X(V_n^{(\theta)})(r+t,X_r+x)\right]\mathrm{d}r\right)\\&\quad=V_n^{(\theta)}(t,x)-\mathbb{E}\left(\int_0^s(G^{(\theta)}*\rho_n)(r+t,X_r+x)\mathbb{I}_{\{X_r<b^{(\theta)}(r+t)\}}\mathrm{d}r\right)\\&\qquad+\mathbb{E}\left(\int_0^s\left[\frac{\partial}{\partial t}V_n^{(\theta)}(r+t,X_r+x)+\mathcal{A}_X(V_n^{(\theta)})(r+t,X_r+x)\right]\mathbb{I}_{\{X_r>b^{(\theta)}(r+t)\}}\mathrm{d}r\right),\end{aligned}$$

where we used the fact that $b^{(\theta)}(s)$ is finite for all $s\geq 0$ and that $\mathbb{P}(X_s+x=b(t+s))=0$ for all $s>0$ and $x\in\mathbb{R}$ when X is of infinite variation (see Sato (1999)). Taking $s=m_\theta-t$, using the fact that $V^{(\theta)}(m_\theta, x) = 0$ for all $x \in \mathbb{R}$ and letting $n \to \infty$ (by the dominated convergence theorem), we obtain that (2.17) holds for any $(t,x) \in (0,t_b) \times \mathbb{R}$. The case when $t = 0$ follows by continuity.

For the finite variation case, we define the auxiliary function

$$\begin{aligned} R^{(\theta)}(t,x) = \mathbb{E}_x &\left(\int_0^{m_\theta - t} G^{(\theta)}(r+t, X_r) \mathbb{I}_{\{X_r < b^{(\theta)}(r+t)\}} \mathrm{d}r \right) \\ &- \mathbb{E}_x \left(\int_0^{m_\theta - t} \int_{(-\infty,0)} V^{(\theta)}(r+t, X_r + y) \Pi(\mathrm{d}y) \mathbb{I}_{\{X_r > b^{(\theta)}(r+t)\}} \mathrm{d}r \right) \end{aligned}$$

for all $(t,x) \in [0, m_\theta] \times \mathbb{R}$. We then prove that $R^{(\theta)} = V^{(\theta)}$. First, note that from Lemma 2.3.8 we have that $\int_{(-\infty,0)} V^{(\theta)}(t, x+y) + G^{(\theta)}(t,x) \geq 0$ for all $(t,x) \in D$. Then we have that for all $(t,x) \in [0, m_\theta] \times \mathbb{R}$,

$$\begin{aligned} |R^{(\theta)}(t,x)| &\leq \mathbb{E}_x \left(\int_0^{m_\theta - t} |G^{(\theta)}(r+t, X_r)| \mathbb{I}_{\{X_r < b^{(\theta)}(r+t)\}} \mathrm{d}r \right) \\ &\quad - \mathbb{E}_x \left(\int_0^{m_\theta - t} \int_{(-\infty,0)} V^{(\theta)}(r+t, X_r + y) \Pi(\mathrm{d}y) \mathbb{I}_{\{X_r > b^{(\theta)}(r+t)\}} \mathrm{d}r \right) \\ &\leq \mathbb{E}_x \left(\int_0^{m_\theta - t} |G^{(\theta)}(r+t, X_r)| \mathrm{d}r \right) \\ &\leq m_\theta - t, \end{aligned} \tag{2.21}$$

where we used the triangle inequality and the fact that $V^{(\theta)} \leq 0$ in the first inequality and that $|G^{(\theta)}| \leq 1$ in the last. For each $(t,x) \in [0, m_\theta] \times \mathbb{R}$, we define the times at which the process X hits the curve $b^{(\theta)}$. Let $\tau_b^{(1)} = \inf\{s \in [0, m_\theta - t] : X_s \geq b^{(\theta)}(s+t)\}$ and for $k \geq 1$,

$$\begin{aligned} \sigma_b^{(k)} &= \inf\{s \in [\tau_b^k, m_\theta - t] : X_s < b^{(\theta)}(s+t)\} \\ \tau_b^{(k+1)} &= \inf\{s \in [\sigma_b^k, m_\theta - t] : X_s \geq b^{(\theta)}(s+t)\}, \end{aligned}$$

where in this context, we understand that $\inf \emptyset = m_\theta - t$. Taking $t \in [0, m_\theta]$ and $x > b^{(\theta)}(t)$ gives us

$$\begin{aligned}
R^{(\theta)}(t,x) &= -\mathbb{E}_x\left(\int_0^{\sigma_b^{(1)}}\int_{(-\infty,0)} V^{(\theta)}(r+t,X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right)\\
&\quad + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_b^{(1)}<m_\theta-t\}}\int_{\sigma_b^{(1)}}^{\tau_b^{(2)}} G^{(\theta)}(r+t,X_r)\mathrm{d}r\right)\\
&\quad + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_b^{(2)}<m_\theta-t\}}\int_{\tau_b^{(2)}}^{m_\theta-t} G^{(\theta)}(r+t,X_r)\mathbb{I}_{\{X_r<b^{(\theta)}(r+t)\}}\mathrm{d}r\right)\\
&\quad - \mathbb{E}_x\left(\mathbb{I}_{\{\tau_b^{(2)}<m_\theta-t\}}\int_{\tau_b^{(2)}}^{m_\theta-t}\int_{(-\infty,0)} V^{(\theta)}(r+t,X_r+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r>b^{(\theta)}(r+t)\}}\mathrm{d}r\right)\\
&= -\mathbb{E}_x\left(\int_0^{\sigma_b^{(1)}}\int_{(-\infty,0)} V^{(\theta)}(r+t,X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right)\\
&\quad + \mathbb{E}_x(V^{(\theta)}(t+\sigma_b^{(1)},X_{\sigma_b^{(1)}})\mathbb{I}_{\{\sigma_b^{(1)}<m_\theta-t\}})\\
&\quad + \mathbb{E}_x(R^{(\theta)}(t+\tau_b^{(2)},X_{\tau_b^{(2)}})\mathbb{I}_{\{\tau_b^{(2)}<m_\theta-t\}}),
\end{aligned}$$

where the last equality follows from the strong Markov property applied at time $\sigma_b^{(1)}$ and $\tau_b^{(2)}$, respectively, and the fact that τ_D is optimal for $V^{(\theta)}$. Using the compensation formula for Poisson random measures (see (1.25)), it can be shown that

$$\mathbb{E}_x\left(\int_0^{\sigma_b^{(1)}}\int_{(-\infty,0)} V^{(\theta)}(r+t,X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right) = \mathbb{E}_x(V^{(\theta)}(t+\sigma_b^{(1)},X_{\sigma_b^{(1)}})\mathbb{I}_{\{\sigma_b^{(1)}<m_\theta-t\}}).$$

Hence, for all $(t,x)\in D$, we have that

$$R^{(\theta)}(t,x) = \mathbb{E}_x(R^{(\theta)}(t+\tau_b^{(2)},X_{\tau_b^{(2)}})\mathbb{I}_{\{\tau_b^{(2)}<m_\theta-t\}}).$$

Using an induction argument, it can be shown that for all $(t,x)\in D$ and $n\geq 2$,

$$R^{(\theta)}(t,x) = \mathbb{E}_x(R^{(\theta)}(t+\tau_b^{(n)},X_{\tau_b^{(n)}})\mathbb{I}_{\{\tau_b^{(n)}<m_\theta-t\}}) = \mathbb{E}_x(R^{(\theta)}(t+\tau_b^{(n)},X_{\tau_b^{(n)}})), \tag{2.22}$$

where the last equality follows since $R^{(\theta)}(m_\theta,x)=0$ for all $x\in\mathbb{R}$. We next show that $\lim_{n\to\infty}\tau_b^{(n)} = m_\theta - t$ $\mathbb{P}_x$-a.s. First, note that for any $n\geq 2$ and $x\in\mathbb{R}$, under the measure

$\mathbb{P}_x$,

$$\begin{aligned}
\sigma_b^{(n)} - \tau_b^{(n)} &= \inf\{s \in [\tau_b^{(n)}, m_\theta - t] : X_s < b^{(\theta)}(s+t)\} - \tau_b^{(n)} \\
&= \inf\{s \in [0, m_\theta - t - \tau_b^{(n)}] : X_{s+\tau_b^{(n)}} < b^{(\theta)}(s + \tau_b^{(n)} + t)\} \\
&\geq \inf\{s \in [0, m_\theta - t - \tau_b^{(n)}] : X_{s+\tau_b^{(n)}} < b^{(\theta)}(\tau_b^{(n)} + t)\} \\
&= \inf\{s \in [0, m_\theta - t - \tau_b^{(n)}] : X_{s+\tau_b^{(n)}} - X_{s+\tau_b^{(n)}} + x < x\} \\
&= \widetilde{\tau}_x^- \wedge (m_\theta - t - \tau_b^{(n)}),
\end{aligned}$$

where $\widetilde{\tau}_x^-$ (due to the strong Markov property of Lévy processes) is a copy of τ_x^-, under $\mathbb{P}_x$, independent of $\mathcal{F}_{\tau_b^{(n)}}$. Hence, we obtain that for any $n \geq 2$,

$$\begin{aligned}
\mathbb{P}_x(\sigma_b^{(n)} < m_\theta - t) &= \mathbb{P}_x(\sigma_b^{(n)} - \tau_b^{(n)} < m_\theta - t - \tau_b^{(n)}, \tau_b^{(n)} < m_\theta - t) \\
&\leq \mathbb{P}_x(\widetilde{\tau}_x^- \wedge (m_\theta - t - \tau_b^{(n)}) < m_\theta - t - \tau_b^{(n)}, \tau_b^{(n)} < m_\theta - t) \\
&= \mathbb{P}_x(\widetilde{\tau}_x^- < m_\theta - t - \tau_b^{(n)}, \tau_b^{(n)} < m_\theta - t) \\
&\leq \mathbb{P}_x(\widetilde{\tau}_x^- < m_\theta - t, \tau_b^{(n)} < m_\theta - t) \\
&= \mathbb{P}(\tau_0^- < m_\theta - t)\mathbb{P}_x(\tau_b^{(n)} < m_\theta - t) \\
&\leq \mathbb{P}(\tau_0^- < m_\theta - t)\mathbb{P}_x(\sigma_b^{(n-1)} < m_\theta - t),
\end{aligned}$$

where in the last equality we used the fact that $\sigma_b^{(n-1)} \leq \tau_b^{(n)}$. Therefore, by an induction argument we obtain that for any $x \in \mathbb{R}$ and $n \geq 1$,

$$\mathbb{P}_x(\sigma_b^{(n)} < m_\theta - t) \leq \left[\mathbb{P}(\tau_0^- < m_\theta - t)\right]^{n-1} \mathbb{P}_x(\sigma_b^{(1)} < m_\theta - t).$$

Since X is of finite variation we have that $\mathbb{P}(\tau_0^- < m_\theta - t) \in (0,1)$. Taking $n \to \infty$ in the equation above, we see that $\sigma_b^{(n)}$ converges in distribution (under the measure $\mathbb{P}_x$) to $m_\theta - t$. Moreover, since $m_\theta - t$ is a constant, we have that the convergence also holds in probability. Furthermore, the sequence $\{\sigma_b^{(n)}, n \geq 1\}$ is non decreasing implying that $\lim_{n\to\infty} \sigma_b^{(n)} = m_\theta - t$ $\mathbb{P}_x$- a.s. From the fact that for each $n \geq 1$, $\sigma_b^{(n)} \leq \tau_b^{(n+1)}$, the convergence for $\tau_b^{(n)}$ also holds. Therefore, taking $n \to \infty$ in (2.22), we conclude that for all $(t,x) \in D$,

$$|R^{(\theta)}(t,x)| \leq \lim_{n\to\infty} \mathbb{E}_x\left(|R^{(\theta)}(t + \tau_b^{(n)}, X_{\tau_b^{(n)}})|\right) \leq \lim_{n\to\infty} \mathbb{E}_x(m_\theta - t - \tau_b^{(n)}) = 0,$$

where the second inequality follows from (2.21) and the last equality from the dominated convergence theorem. On the other hand, if we take $t \in [0, m_\theta]$ and $x < b^{(\theta)}(t)$ we have, by the strong Markov property applied to the filtration at time $\tau_b^{(1)}$, that

$$R^{(\theta)}(t,x) = \mathbb{E}_x\left(\int_0^{\tau_b^{(1)}} G^{(\theta)}(r+t, X_r)\mathrm{d}r\right) + \mathbb{E}_x(R^{(\theta)}(t+\tau_b^{(1)}, X_{\tau_b^{(1)}})) = V^{(\theta)}(t,x),$$

where we used the fact that $\tau_b^{(1)}$ is an optimal stopping time for $V^{(\theta)}$ and that $R^{(\theta)}$ vanishes on D. So then (2.17) also holds in the finite variation case. □

Next we proceed to show the uniqueness result. Suppose that there exist a non-positive continuous function $U^{(\theta)} : [0, m_\theta] \times \mathbb{R} \mapsto (-\infty, 0]$ and a continuous function $c^{(\theta)}$ on $[0, m_\theta)$ such that $c^{(\theta)} \geq h^{(\theta)}$ and $c^{(\theta)}(t) = 0$ for all $t \in [t_b, m_\theta)$. We assume that the pair $(U^{(\theta)}, c^{(\theta)})$ solves the equations

$$\begin{aligned} U^{(\theta)}(t,x) = \mathbb{E}_x&\left(\int_0^{m_\theta - t} G^{(\theta)}(r+t, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(r+t)\}}\mathrm{d}r\right) \\ &- \mathbb{E}_x\left(\int_0^{m_\theta - t}\int_{(-\infty, c^{(\theta)}(r+t)-X_r)} U^{(\theta)}(r+t, X_r+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+t)\}}\mathrm{d}r\right) \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} 0 = \mathbb{E}_{c^{(\theta)}(t)}&\left(\int_0^{m_\theta - t} G^{(\theta)}(r+t, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(r+t)\}}\mathrm{d}r\right) \\ &- \mathbb{E}_{c^{(\theta)}(t)}\left(\int_0^{m_\theta - t}\int_{(-\infty, c^{(\theta)}(r+t)-X_r)} U^{(\theta)}(r+t, X_r+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+t)\}}\mathrm{d}r\right) \end{aligned} \tag{2.24}$$

when $t \in [0, t_b)$ and $x \in \mathbb{R}$. For $t \in [t_b, m_\theta)$ and $x \in \mathbb{R}$, we assume that

$$U^{(\theta)}(t,x) = \mathbb{E}_x(\tau_0^+ \wedge (m_\theta - t)) - \frac{2}{\theta}e^{-\theta t}[1 - \mathbb{E}_x(e^{-\theta(\tau_0^+ \wedge (m_\theta - t))})]. \tag{2.25}$$

In addition, we assume that

$$\int_{(-\infty,c^{(\theta)}(t)-x)} U^{(\theta)}(t,x+y)\Pi(\mathrm{d}y) + G^{(\theta)}(t,x) \geq 0 \qquad \text{for all } t \in [0,t_b) \text{ and } x > c^{(\theta)}(t). \tag{2.26}$$

Note that $(U^{(\theta)}, c^{(\theta)})$ solving the above equations means that $U^{(\theta)}(t, c^{(\theta)}(t)) = 0$ for all $t \in [0, m_\theta)$ and $U^{(\theta)}(m_\theta, x) = 0$ for all $x \in \mathbb{R}$. Denote D_c as the "stopping region" under the curve $c^{(\theta)}$, i.e., $D_c = \{(t,x) \in [0,m_\theta] \times \mathbb{R} : x \geq c^{(\theta)}(t)\}$ and recall that $D = \{(t,x) \in [0,m_\theta] \times \mathbb{R} : x \geq b^{(\theta)}(t)\}$ is the "stopping region" under the curve $b^{(\theta)}$. We show that $U^{(\theta)}$ vanishes on D_c in the next Lemma.

Lemma 2.4.2. *We have that $U^{(\theta)}(t,x) = 0$ for all $(t,x) \in D_c$.*

Proof. Since the statement is clear for $(t,x) \in [t_b, m_\theta) \times [0,\infty)$, we take $t \in [0,t_b)$ and $x \geq c^{(\theta)}(t)$. Define σ_c to be the first time that the process is outside D_c before time $m_\theta - t$, i.e.,

$$\sigma_c = \inf\{0 \leq s \leq m_\theta - t : X_s < c^{(\theta)}(t+s)\},$$

where in this context, we understand that $\inf \emptyset = m_\theta - t$. From the fact that $X_r \geq c^{(\theta)}(t+r)$ for all $r < \sigma_c$ and the strong Markov property at time σ_c, we obtain that

$$\begin{aligned}
U^{(\theta)}(t,x) &= \mathbb{E}_x\left(\int_0^{m_\theta - t} G^{(\theta)}(r+t, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(r+t)\}}\mathrm{d}r\right) \\
&\qquad - \mathbb{E}_x\left(\int_0^{m_\theta - t}\int_{(-\infty,c^{(\theta)}(r+t)-X_r)} U^{(\theta)}(r+t, X_r+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+t)\}}\mathrm{d}r\right) \\
&= \mathbb{E}_x(U^{(\theta)}(t+\sigma_c, X_{\sigma_c})) - \mathbb{E}_x\left(\int_0^{\sigma_c}\int_{(-\infty,c^{(\theta)}(r+t)-X_r)} U^{(\theta)}(r+t, X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right) \\
&= \mathbb{E}_x(U^{(\theta)}(t+\sigma_c, X_{\sigma_c})\mathbb{I}_{\{\sigma_c < m_\theta - t, X_{\sigma_c} < c^{(\theta)}(t+\sigma_c)\}}) \\
&\qquad - \mathbb{E}_x\left(\int_0^{\sigma_c}\int_{(-\infty,c^{(\theta)}(r+t)-X_r)} U^{(\theta)}(r+t, X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right),
\end{aligned}$$

where the last equality follows since $U^{(\theta)}(m_\theta, x) = 0$ for all $x \in \mathbb{R}$ and $U^{(\theta)}(t, c^{(\theta)}(t)) = 0$ for all $t \in [0,t_b)$. Then, since $U^{(\theta)} \leq 0$ and applying the compensation formula for Poisson random measures (see equation (1.25)) we get

$$
\begin{aligned}
&\mathbb{E}_x(U^{(\theta)}(t+\sigma_c, X_{\sigma_c})\mathbb{I}_{\{\sigma_c<m_\theta-t, X_{\sigma_c}<c^{(\theta)}(t+\sigma_c)\}}) \\
&= \mathbb{E}_x\left(\int_0^{m_\theta-t}\int_{(-\infty,0)} \mathbb{I}_{\{X_u\geq c^{(\theta)}(t+u) \text{ for all } u<r, X_{r-}+y<c^{(\theta)}(t+r)\}} U^{(\theta)}(t+r, X_{r-}+y)N(\mathrm{d}r,\mathrm{d}y)\right) \\
&= \mathbb{E}_x\left(\int_0^{m_\theta-t}\int_{(-\infty,0)} \mathbb{I}_{\{X_u\geq c^{(\theta)}(t+u) \text{ for all } u<r\}}\mathbb{I}_{\{X_r+y<c^{(\theta)}(t+r)\}} U^{(\theta)}(t+r, X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right) \\
&= \mathbb{E}_x\left(\int_0^{\sigma_c}\int_{(-\infty,c^{(\theta)}(t+r)-X_{t+r})} U^{(\theta)}(t+r, X_r+y)\Pi(\mathrm{d}y)\mathrm{d}r\right).
\end{aligned}
$$

Hence $U^{(\theta)}(t,x)=0$ for all $(t,x)\in D_c$ as we claimed. □

The next lemma shows that $U^{(\theta)}$ can be expressed as an integral involving only the gain function $G^{(\theta)}$ stopped at the first time the process enters the set D_c. As a consequence, $U^{(\theta)}$ dominates the function $V^{(\theta)}$.

Lemma 2.4.3. *We have that $U^{(\theta)}(t,x)\geq V^{(\theta)}(t,x)$ for all $(x,t)\in\mathbb{R}\times[0,m_\theta]$,*

Proof. Note that we can assume that $t\in[0,t_b)$ because for $(t,x)\in D_c$, we have that $U^{(\theta)}(t,x)=0\geq V^{(\theta)}(t,x)$ and for $t\in[t_b,m_\theta)$, $U^{(\theta)}(t,x)=V^{(\theta)}(t,x)$ for all $x\in\mathbb{R}$. Consider the stopping time

$$\tau_c=\inf\{s\in[0,m_\theta-t]:X_s\geq c^{(\theta)}(t+s)\}.$$

Let $x\leq c^{(\theta)}(t)$, using the fact that $X_r<c^{(\theta)}(t+r)$ for all $r\leq\tau_c$ and the strong Markov property at time τ_c, we obtain that

$$
\begin{aligned}
U^{(\theta)}(t,x) &= \mathbb{E}_x\left(\int_0^{\tau_c} G^{(\theta)}(r+t,X_r)\mathrm{d}r\right)+\mathbb{E}_x(U^{(\theta)}(t+\tau_c,X_{\tau_c})) \\
&= \mathbb{E}_x\left(\int_0^{\tau_c} G^{(\theta)}(r+t,X_r)\mathrm{d}r\right),
\end{aligned}
\tag{2.27}
$$

where the second equality follows since X creeps upwards and therefore $X_{\tau_c}=c^{(\theta)}(t+\tau_c)$ for $\{\tau_c<m_\theta-t\}$ and $U^{(\theta)}(m_\theta,x)=0$ for all $x\in\mathbb{R}$. Then from the definition of $V^{(\theta)}$ (see (2.5)), we have that

$$U^{(\theta)}(t,x)\geq\inf_{\tau\in\mathcal{T}}\mathbb{E}_{t,x}\left(\int_0^\tau G^{(\theta)}(X_{t+r},t+r)dr\right)=V^{(\theta)}(t,x).$$

Therefore $U^{(\theta)} \geq V^{(\theta)}$ on $[0, m_\theta] \times \mathbb{R}$. □

We proceed by showing that the function $c^{(\theta)}$ is dominated by $b^{(\theta)}$. In the upcoming lemmas, we show that equality indeed holds.

Lemma 2.4.4. *We have that $b^{(\theta)}(t) \geq c^{(\theta)}(t)$ for all $t \in [0, m_\theta)$.*

Proof. The statement is clear for $t \in [t_b, m_\theta)$. We prove the statement by contradiction. Suppose that there exists a value $t_0 \in [0, t_b)$ such that $b^{(\theta)}(t_0) < c^{(\theta)}(t_0)$ and take $x \in (b^{(\theta)}(t_0), c^{(\theta)}(t_0))$. Consider the stopping time

$$\sigma_b = \inf\{s \in [0, m_\theta - t_0] : X_s < b^{(\theta)}(t_0 + s)\}.$$

Applying the strong Markov property to the filtration at time σ_b, we obtain that

$$\begin{aligned} U^{(\theta)}(t_0, x) = \mathbb{E}_x(U^{(\theta)}(t_0 + \sigma_b, X_{\sigma_b})) + \mathbb{E}_x\left(\int_0^{\sigma_b} G^{(\theta)}(t_0 + r, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(t_0+r)\}}\mathrm{d}r\right) \\ - \mathbb{E}_x\left(\int_0^{\sigma_b}\int_{(-\infty,0)} U^{(\theta)}(t + r, X_r + y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r > c^{(\theta)}(t_0+r)\}}\mathrm{d}r\right), \end{aligned}$$

where we used the fact that $U^{(\theta)}(t,x) = 0$ for all $(t,x) \in D_c$. From Lemma 2.4.3 and the fact that $U^{(\theta)} \leq 0$ (by assumption), we have that for all $t \in [0, m_\theta)$ and $x > b^{(\theta)}(t)$, $U^{(\theta)}(t,x) = 0$. Then,

$$\begin{aligned} \mathbb{E}_x(U^{(\theta)}(t_0 + \sigma_b, X_{\sigma_b})) &= \mathbb{E}_x(U^{(\theta)}(t_0 + \sigma_b, X_{\sigma_b})\mathbb{I}_{\{\sigma_b < m_\theta - t, X_{\sigma_b} < b^{(\theta)}(t_0+\sigma_b)\}}) \\ &= \mathbb{E}_x\left(\int_0^\infty \int_{(-\infty,0)} U^{(\theta)}(t_0 + r, X_r + y)\mathbb{I}_{\{r < \sigma_b\}} N(\mathrm{d}r, \mathrm{d}y)\right) \end{aligned}$$

Hence, by the compensation formula for Poisson random measures (see equation (1.25)), we obtain that

$$\mathbb{E}_x(U^{(\theta)}(t_0 + \sigma_b, X_{\sigma_b})) = \mathbb{E}_x\left(\int_0^{\sigma_b}\int_{(-\infty,0)} U^{(\theta)}(t_0 + r, X_r + y)\Pi(\mathrm{d}y)\mathrm{d}r\right).$$

Therefore,

$$
\begin{aligned}
0 &\geq U^{(\theta)}(t_0, x) \\
&= \mathbb{E}_x \left(\int_0^{\sigma_b} \int_{(-\infty,0)} U^{(\theta)}(t_0 + r, X_r + y)\Pi(\mathrm{d}y)\mathrm{d}r \right) \\
&\quad + \mathbb{E}_x \left(\int_0^{\sigma_b} G^{(\theta)}(t_0 + r, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(t_0+r)\}}\mathrm{d}r \right) \\
&\quad - \mathbb{E}_x \left(\int_0^{\sigma_b} \int_{(-\infty,0)} U^{(\theta)}(t + r, X_r + y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_r > c^{(\theta)}(t_0+r)\}}\mathrm{d}r \right) \\
&= \mathbb{E}_x \left(\int_0^{\sigma_b} \left[\int_{(-\infty,0)} U^{(\theta)}(t_0 + r, X_r + y)\Pi(\mathrm{d}y) + G^{(\theta)}(t_0 + r, X_r) \right] \mathbb{I}_{\{X_r < c^{(\theta)}(t_0+r)\}}\mathrm{d}r \right).
\end{aligned}
$$

Recall from Lemma 2.3.8 that the function $\varphi_t^{(\theta)}$ is strictly positive on D. Hence, we obtain that for all $(t, x) \in D$,

$$
\begin{aligned}
\int_{(-\infty,0)} U^{(\theta)}(t, x + y)\Pi(\mathrm{d}y) + G^{(\theta)}(t, x) &\geq \int_{(-\infty,0)} V^{(\theta)}(t, x + y)\Pi(\mathrm{d}y) + G^{(\theta)}(t, x) \\
&= \varphi_t^{(\theta)}(t, x) \\
&> 0.
\end{aligned}
$$

The assumption that $b^{(\theta)}(t_0) < c^{(\theta)}(t_0)$ together with the continuity of the functions $b^{(\theta)}$ and $c^{(\theta)}$ mean that there exists $s_0 \in (t_0, m_\theta)$ such that $b^{(\theta)}(r) < c^{(\theta)}(r)$ for all $r \in [t_0, s_0]$. Consequently, the $\mathbb{P}_x$ probability of X spending a strictly positive amount of time (with respect to Lebesgue measure) in this region is strictly positive. We can then conclude that

$$
0 \geq \mathbb{E}_x \left(\int_0^{\sigma_b} \left[\int_{(-\infty,0)} U^{(\theta)}(t_0 + r, X_r + y)\Pi(\mathrm{d}y) + G^{(\theta)}(t_0 + r, X_r) \right] \mathbb{I}_{\{X_r < c^{(\theta)}(t_0+r)\}}\mathrm{d}r \right) > 0.
$$

This is a contradiction and therefore we conclude that $b^{(\theta)}(t) \geq c^{(\theta)}(t)$ for all $t \in [0, m_\theta)$. $\square$

Note that the definition of $U^{(\theta)}$ on $[t_b, m_\theta) \times \mathbb{R}$ (see equation (2.25)) together with condition (2.26) imply that

$$
\int_{(-\infty,0)} U^{(\theta)}(t, x + y)\Pi(\mathrm{d}y) + G^{(\theta)}(t, x) \geq 0
$$

for all $t \in [0, m_\theta)$ and $x > c^{(\theta)}(t)$. The next Lemma shows that $U^{(\theta)}$ and $V^{(\theta)}$ coincide.

Lemma 2.4.5. *We have that* $b^{(\theta)}(t) = c^{(\theta)}(t)$ *for all* $t \geq 0$ *and hence* $V^{(\theta)} = U^{(\theta)}$*.*

Proof. We prove that $b^{(\theta)} = c^{(\theta)}$ by contradiction. Assume that there exists s_0 such that $b^{(\theta)}(s_0) > c^{(\theta)}(s_0)$. Since $c^{(\theta)}(t) = b^{(\theta)}(t) = 0$ for all $t \in [t_b, m_\theta)$, we deduce that $s_0 \in [0, t_b)$. Let τ_b be the stopping time

$$\tau_b = \inf\{t \geq 0 : X_s \geq b^{(\theta)}(s_0 + t)\}.$$

With the Markov property applied to the filtration at time τ_b, we obtain that for any $x \in (c^{(\theta)}(s_0), b^{(\theta)}(s_0))$

$$\begin{aligned}
\mathbb{E}_x(U^{(\theta)}(s_0 + \tau_b, X_{\tau_b})) &= U^{(\theta)}(s_0, x) - \mathbb{E}_x\left(\int_0^{\tau_b} G^{(\theta)}(r + s_0, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(r+s_0)\}}\mathrm{d}r\right) \\
&\quad + \mathbb{E}_x\left(\int_0^{\tau_b}\int_{(-\infty,0)} U^{(\theta)}(r + s_0, X_r + y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+s_0)\}}\Pi(\mathrm{d}y)\mathrm{d}r\right) \\
&\geq V^{(\theta)}(s_0, x) - \mathbb{E}_x\left(\int_0^{\tau_b} G^{(\theta)}(r + s_0, X_r)\mathbb{I}_{\{X_r < c^{(\theta)}(r+s_0)\}}\mathrm{d}r\right) \\
&\quad + \mathbb{E}_x\left(\int_0^{\tau_b}\int_{(-\infty,0)} U^{(\theta)}(r + s_0, X_r + y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+s_0)\}}\Pi(\mathrm{d}y)\mathrm{d}r\right) \\
&= \mathbb{E}_x\left(\int_0^{\tau_b} G^{(\theta)}(r + s_0, X_r)\mathbb{I}_{\{X_r \geq c^{(\theta)}(r+s_0)\}}\mathrm{d}r\right) \\
&\quad + \mathbb{E}_x\left(\int_0^{\tau_b}\int_{(-\infty,0)} U^{(\theta)}(r + s_0, X_r + y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+s_0)\}}\Pi(\mathrm{d}y)\mathrm{d}r\right),
\end{aligned}$$

where the second inequality follows from the fact that $U^{(\theta)} \geq V^{(\theta)}$ (see Lemma 2.4.3) and the last equality follows as τ_b is the optimal stopping time for $V^{(\theta)}(s_0, x)$. Note that since X creeps upwards, we have that $U^{(\theta)}(s_0 + \tau_b, X_{\tau_b}) = U^{(\theta)}(s_0 + \tau_b, b^{(\theta)}(s_0 + \tau_b)) = 0$. Hence,

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^{\tau_b} G^{(\theta)}(r + s_0, X_r)\mathbb{I}_{\{X_r \geq c^{(\theta)}(r+s_0)\}}\mathrm{d}r\right) \\
&\qquad + \mathbb{E}_x\left(\int_0^{\tau_b}\int_{(-\infty,0)} U^{(\theta)}(r + s_0, X_r + y)\mathbb{I}_{\{X_r > c^{(\theta)}(r+s_0)\}}\Pi(\mathrm{d}y)\mathrm{d}r\right) \leq 0.
\end{aligned}$$

However, the continuity of the functions $b^{(\theta)}$ and $c^{(\theta)}$ gives the existence of $s_1 \in (s_0, m_\theta)$ such that $c^{(\theta)}(r) < b^{(\theta)}(r)$ for all $r \in [s_0, s_1]$. Hence, together with the fact that $\int_{(-\infty,0)} U^{(\theta)}(x + y, t)\Pi(\mathrm{d}y) + G^{(\theta)}(x, t) > 0$ for all $(t, x) \in D_c$ we can conclude that

$$\mathbb{E}_x\left(\int_0^{\tau_b} G^{(\theta)}(r+s_0,X_r)\mathbb{I}_{\{X_r\geq c^{(\theta)}(r+s_0)\}}\mathrm{d}r\right)$$
$$+\mathbb{E}_x\left(\int_0^{\tau_b}\int_{(-\infty,0)} U^{(\theta)}(r+s_0,X_r+y)\mathbb{I}_{\{X_r>c^{(\theta)}(r+s_0)\}}\Pi(\mathrm{d}y)\mathrm{d}r\right)>0,$$

which shows a contradiction.

□

2.5 Examples

2.5.1 Brownian motion with drift

Suppose that $X=\{X_t,t\geq 0\}$ is a Brownian motion with drift. That is for any $t\geq 0$, $X_t=\mu t+\sigma B_t$, where $\sigma>0$ and $\mu\in\mathbb{R}$. In this case, we have that

$$\psi(\beta)=\mu\beta+\frac{1}{2}\sigma^2\beta^2$$

for all $\beta\geq 0$. Then

$$\Phi(q)=\frac{1}{\sigma^2}\left[\sqrt{\mu^2+2\sigma^2 q}-\mu\right].$$

It is well known that $-\underline{X}_{\widetilde{e}_\theta}$ has exponential distribution (see e.g. Borodin and Salminen (2002) pp251 or Kyprianou (2014) pp 233) with distribution function given by

$$F^{(\theta)}(x)=1-\exp\left(-\frac{x}{\sigma^2}\left[\sqrt{\mu^2+2\sigma^2\theta}+\mu\right]\right)\qquad\text{for } x>0.$$

Denote $\Phi(x;a,b^2)$ as the distribution function of a Normal random variable with mean $a\in\mathbb{R}$ and variance b^2, i.e., for any $x\in\mathbb{R}$,

$$\Phi(x;a,b^2)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi b^2}}e^{-\frac{1}{2b^2}(y-a)^2}\mathrm{d}y.$$

For any $b,s,t\geq 0$ and $x\in\mathbb{R}$, define the function

$$K(t,x,s,b)=\mathbb{E}\left(G^{(\theta)}(s+t,X_s+x)\mathbb{I}_{\{X_s+x\leq b\}}\right).$$

Then it can be easily shown that

$$
\begin{aligned}
&K(t,x,s,b)\\
&\quad = \Phi(b-x;\mu s,\sigma^2 s) - 2e^{-\theta(s+t)}\Phi(-x;\mu s,\sigma^2 s) - 2e^{-\theta t}\exp\left(-\frac{x}{\sigma^2}\left[\sqrt{\mu^2+2\sigma^2\theta}+\mu\right]\right)\\
&\qquad\times\left[\Phi(b-x,-s\sqrt{\mu^2+2\sigma^2\theta},s\sigma^2) - \Phi(-x,-s\sqrt{\mu^2+2\sigma^2\theta},s\sigma^2)\right].
\end{aligned}
$$

Thus, we have that $b^{(\theta)}$ satisfies the non-linear integral equation

$$\int_0^{m_\theta - t} K(t, b^{(t)}(t), s, b^{(\theta)}(t+s))\mathrm{d}s = 0$$

for all $t \in [0, m_\theta)$ and the value function $V^{(\theta)}$ is given by

$$V^{(\theta)}(t,x) = \int_0^{m_\theta - t} K(t, x, s, b^{(\theta)}(t+s))\mathrm{d}s$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Note that we can approximate the integrals above by Riemann sums so a numerical approximation can be implement. Indeed, take $n \in \mathbb{Z}_+$ sufficiently large and define $h = m_\theta/n$. For each $k \in \{0,1,2,\dots,n\}$, we define $t_k = kh$. Then the sequence of times $\{t_k, k = 0,1,\dots,n\}$ is a partition of the interval $[0, m_\theta]$. Then, for any $x \in \mathbb{R}$ and $t \in [t_k, t_{k+1})$ for $k \in \{0,1,\dots,n-1\}$ we approximate $V^{(\theta)}(t,x)$ by

$$V_h^{(\theta)}(t_k, x) = \sum_{i=k}^{n-1} K(t_k, x, t_{i-k+1}, b_i)h,$$

where the sequence $\{b_k, k = 0,1,\dots,n-1\}$ is a solution to

$$\sum_{i=k}^{n-1} K(t_k, x, t_{i-k+1}, b_i) = 0$$

for each $k \in \{0,1,\dots,n-1\}$. Note that the sequence $\{b_k, k = 0,1,\dots,n\}$ is a numerical approximation to the sequence $\{b^{(\theta)}(t_k), k = 0,1,\dots,n-1\}$ (for n sufficiently large) and can be calculated by using backwards induction. In the Figure 2.2, we show a numerical calculation of the equations above. The parameters used are $\mu = 2$ and $\sigma = 1$, whereas we chose $m_\theta = 10$.

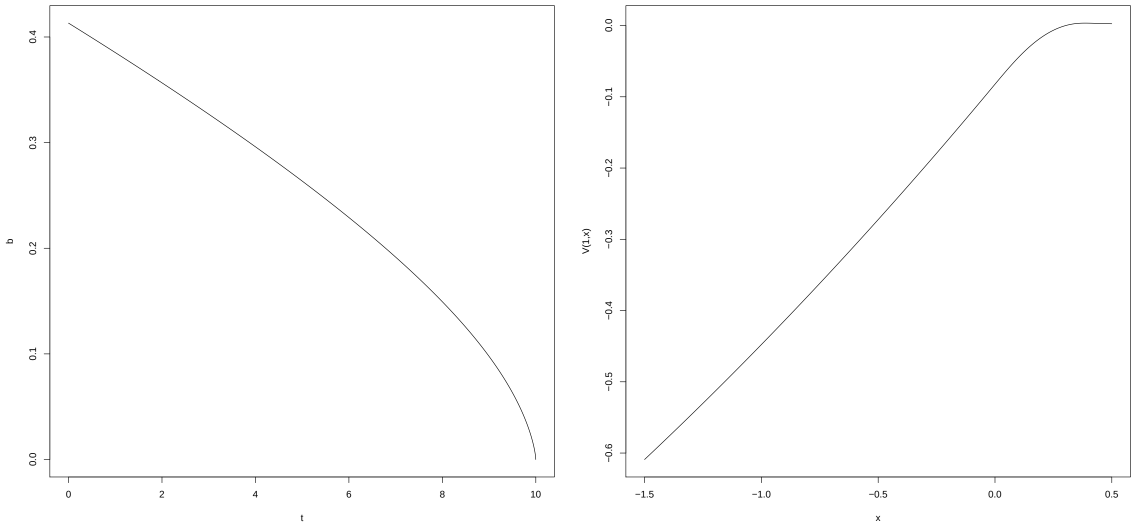

Figure 2.2: Brownian motion with drift $\mu = 2$ and $\sigma = 1$. Left hand side: Optimal boundary; Right hand side: Value function fixing $t = 1$.

2.5.2 Brownian motion with exponential jumps

Let $X = \{X_t, t \geq 0\}$ be a compound Poisson process perturbed by a Brownian motion, that is

$$X_t = \sigma B_t + \mu t - \sum_{i=1}^{N_t} Y_i, \tag{2.28}$$

where $B = \{B_t, t \geq 0\}$ is a standard Brownian motion, $N = \{N_t, t \geq 0\}$ is Poisson process with rate λ independent of B, $\mu \in \mathbb{R}$, $\sigma > 0$ and the sequence $\{Y_1, Y_2, \ldots\}$ is a sequence of independent random variables exponentially distributed with mean $1/\rho > 0$. Then in this case, the Laplace exponent is derived as

$$\varphi(\beta) = \frac{\sigma^2}{2}\beta^2 + \mu\beta - \frac{\lambda\beta}{\rho + \beta}.$$

Its Lévy measure, given by $\Pi(\mathrm{d}y) = \lambda\rho e^{\rho y}\mathbb{I}_{\{y<0\}}\mathrm{d}y$ is a finite measure and X is a process of infinite variation. According to Kuznetsov et al. (2013), the scale function in this case is given for $q \geq 0$ and $x \geq 0$ by,

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{\zeta_1(q)x}}{\psi'(\zeta_1(q))} + \frac{e^{\zeta_2(q)x}}{\psi'(\zeta_2(q))},$$

where $\zeta_2(q), \zeta_1(q)$ and $\Phi(q)$ are the three real solutions to the equation $\psi(\beta) = q$, which satisfy $\zeta_2(q) < -\rho < \zeta_1(q) < 0 < \Phi(q)$. The second scale function, $Z^{(q)}$, takes the form

$$Z^{(q)}(x) = 1 + q\left[\frac{e^{\Phi(q)x} - 1}{\Phi(q)\psi'(\Phi(q))} + \frac{e^{\zeta_1(q)x} - 1}{\zeta_1(q)\psi'(\zeta_1(q))} + \frac{e^{\zeta_2(q)x} - 1}{\zeta_2(q)\psi'(\zeta_2(q))}\right].$$

Note that since we have exponential jumps (and hence $\Pi(\mathrm{d}y) = \lambda\rho e^{\rho y}\mathbb{I}_{\{y<0\}}$, we have that for all $t \in [0, m_\theta)$ and $x > 0$,

$$\int_{(-\infty,-x)} V^{(\theta)}(t, b^{(\theta)}(t) + x + y)\Pi(\mathrm{d}y) = e^{-\rho x}\int_{(-\infty,0)} V^{(\theta)}(t, b^{(\theta)}(t) + y)\Pi(\mathrm{d}y).$$

Then, for any $(t, x) \in [0, m_\theta] \times \mathbb{R}$, equation (2.17) reads as

$$\begin{aligned} V^{(\theta)}(t,x) = \mathbb{E}_x &\left(\int_0^{m_\theta - t} G^{(\theta)}(r+t, X_r)\mathbb{I}_{\{X_r < b^{(\theta)}(r+t)\}}\mathrm{d}r\right) \\ &- \mathbb{E}_x\left(\int_0^{m_\theta - t} e^{-\rho(X_r - b^{(\theta)}(r+t))}\mathcal{V}(r+t)\mathbb{I}_{\{X_r > b^{(\theta)}(r+t)\}}\mathrm{d}r\right) \end{aligned}$$

where for any $r, s \in [0, m_\theta)$, $b \geq 0$ and $x \in \mathbb{R}$,

$$\mathcal{V}(t) = \int_{(-\infty,0)} V^{(\theta)}(t, b^{(\theta)}(t) + y)\Pi(\mathrm{d}y).$$

Note that the equation above suggest that in order to find a numerical value of $b^{(\theta)}$ using Theorem 2.3.13 we only need to know the values of the function $\mathcal{V}$ and not the values of $\int_{(-\infty,0)} V^{(\theta)}(t, x+y)\Pi(\mathrm{d}y)$ for all $t \in [0, m_\theta]$ and $x > b^{(\theta)}(t)$. Note that using Fubini's theorem, we can write

$$V^{(\theta)}(t,x) = \int_0^{m_\theta - t} K_1(t, x, r, b^{(\theta)}(r+t))\mathrm{d}r - \int_0^{m_\theta - t} \mathcal{V}(r+t)K_2(x, r, b^{(\theta)}(r+t))\mathrm{d}r,$$

where for any $r, s \in [0, m_\theta)$, $b \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} \mathcal{V}(t) &= \int_{(-\infty,0)} V^{(\theta)}(t, b^{(\theta)}(t) + y)\Pi(\mathrm{d}y) \\ K_1(t,x,s,b) &= \mathbb{E}\left(G^{(\theta)}(s+t, X_s + x)\mathbb{I}_{\{X_s < b - x\}}\right) \\ K_2(x,s,b) &= \mathbb{E}\left(e^{-\rho(X_s + x - b)}\mathbb{I}_{\{X_s > b - x\}}\right). \end{aligned}$$

Then we have that $V^{(\theta)}$ and $b^{(\theta)}$ and satisfy the equations

$$V^{(\theta)}(t,x) = \int_0^{m_\theta - t} K_1(t,x,r,b^{(\theta)}(r+t))\mathrm{d}r - \int_0^{m_\theta - t} \mathcal{V}(r+t)K_2(x,r,b^{(\theta)}(r+t))\mathrm{d}r,$$
$$0 = \int_0^{m_\theta - t} K_1(t,b^{(\theta)}(t),r,b^{(\theta)}(r+t))\mathrm{d}r - \int_0^{m_\theta - t} \mathcal{V}(r+t)K_2(b^{(\theta)}(t),r,b^{(\theta)}(r+t))\mathrm{d}r,$$

for all $t \in [0, m_\theta]$ and $x \in \mathbb{R}$. We can approximate the integrals above by Riemann sums so a numerical approximation can be implement. Indeed, take $n \in \mathbb{Z}_+$ sufficiently large and define $h = m_\theta/n$. For each $k \in \{0,1,2,\dots,n\}$, we define $t_k = kh$. Then the sequence of times $\{t_k, k = 0,1,\dots,n\}$ is a partition of the interval $[0, m_\theta]$. Then, for any $x \in \mathbb{R}$ and $t \in [t_k, t_{k+1})$ for $k \in \{0,1,\dots,n-1\}$ we approximate $V^{(\theta)}(t,x)$ by

$$V_h(u_k,x) = \sum_{i=k}^{n-1} [K_1(t_{i-k+1},x,t_k,b_i) - \mathcal{V}_h(t_{i+1})K_2(x,t_{i-k+1},b_i)]h,$$

where $\mathcal{V}(t_n) = 0$ and

$$\mathcal{V}_h(t_i) = \sum_{j=-N}^{\lfloor b_i/h \rfloor} V_h(t_i, jh)\lambda\rho e^{\rho jh} h$$

for any $i \in \{1,2,\dots,n-1\}$ and N sufficiently large. The sequence $\{b_k, k = 1,\dots,n-1\}$ is solution to

$$\sum_{i=k}^{n-1} [K_1(t_{i-k+1},b_k,t_k,b_i) - \mathcal{V}_h(t_{i+1})F_2(b_k,t_{i-k+1},b_i)]h = 0 \tag{2.29}$$

for each $k \in \{0,1,\dots,n-1\}$. The functions K_1 and K_2 can be estimated by simulating the process $\{X_t, t \geq 0\}$ (see e.g. Kuznetsov et al. (2011), Theorem 4 and Remark 3). Note that, for n sufficiently large, the sequence $\{b_k, k = 1,\dots,n\}$ is a numerical approximation to the sequence $\{b(t_k), k = 1,\dots,n\}$, provided that $V_h \leq 0$ and $\mathcal{V}(t_i) + G^{(\theta)}(t_k,b_k) \geq 0$, and can be calculated by using backwards induction. Indeed, using the condition $\mathcal{V}(t_n,b_n) = 0$, we can first obtain b_{n-1} using equation (2.29). This allows us to compute $V_h(t_{n-1},x)$ which in turn gives us $\mathcal{V}_h(t_{n-1})$. We can then finally obtain $b_{n-2}, \mathcal{V}_h(t_{n-2}), b_{n-3}, \mathcal{V}_h(t_{n-3}), \dots, b_1$ by repeating the aforementioned steps.

2.6 Conclusions

In this chapter, we have managed to solve the problem of predicting the last zero of a spectrally negative Lévy process before an exponential time in a L_1 sense. It is shown that this optimal prediction problem is equivalent to an optimal stopping problem which means that finding an optimal stopping time that solves (2.5) also solves (2.2) (taking $t = x = 0$). The rest of this chapter is focused on solving such optimal stopping problem.

The first important finding of this problem is that it is always optimal to stop when the elapsed time (if we have not stopped) has reached the value m_θ, the median of the exponential random variable. This is most likely due to the fact that when $X_{\mathbf{e}_p} \leq 0$ we have that $g_{\mathbf{e}_p} = \mathbf{e}_p$ and thus the best predictor of $\mathbf{e}_p$ is its median. Therefore, the optimal stopping problem (2.5) can be treated as a finite horizon problem.

In contrast to Baurdoux and Pedraza (2020b) (where the last zero in an infinite horizon is predicted for a spectrally negative Lévy process) in which an optimal stopping is given as the first time the process crosses below a level $a^* > 0$. It is shown that an optimal stopping time for (2.5) is the first time the process crosses above a non-increasing, non-negative and continuous curve which depends on time. The curve and the value function are characterised as in Theorem 4.4.23 as a solution of non-linear integral equations in a special class of functions. These equations can be regarded as a generalisation of the free boundary equations (see Peskir and Shiryaev (2006) Chapter IV.14). We have presented the proof of the uniqueness part of Theorem 4.4.23 using the inequality

$$\int_{(-\infty,0)} V^{(\theta)}(t, x + y) + G(t, x) \geq 0, \qquad (t, x) \in D.$$

However, we believe that in the presence of a Brownian component ($\sigma > 0$), such an assumption regarding the inequality can be removed. This conjecture will be explored in future research.

Therefore, we conclude that the stopping time that minimises the L_1 distance with respect to $g_{\mathbf{e}_p}$ is given by

$$\tau_D = \inf\{t \in [0, m_\theta] : X_t \geq b^{(\theta)}(t)\},$$

where the curve $b^{(\theta)}$ is as characterised in Theorem 4.4.23. That is,

$$V_* = \mathbb{E}(|\tau_D - g_{\mathbf{e}_p}|).$$

A drawback of this solution is that, since $b^{(\theta)}$ is non-negative, at the moment of stopping by hitting the curve $b^{(\theta)}$ the value of the process can be away from zero which implies that τ_D and $g_{\mathbf{e}_p}$ can never take the same value.

Chapter 3

On the last zero process of a spectrally negative Lévy process

Abstract

Let X be a spectrally negative Lévy process and consider g_t the last time X is below the level zero before time $t \geq 0$. We derive an Itô formula for the three dimensional process $\{(g_t, t, X_t), t \geq 0\}$ and its infinitesimal generator using a perturbation method for Lévy processes. We also find an explicit formula for calculating functionals that include the whole path of the length of current positive excursion at time $t \geq 0$, $U_t := t - g_t$. These results are applied to optimal prediction problems for the last zero $g := \lim_{t\to\infty} g_t$, when X drifts to infinity. Moreover, the joint Laplace transform of $(U_{\mathbf{e}_q}, X_{\mathbf{e}_q})$, where $\mathbf{e}_q$ is an independent exponential time is found, and a formula for a density of the q-potential measure of the process $\{(U_t, X_t,), t \geq 0\}$ is derived.

3.1 Introduction

Last passage times have received considerable attention in the recent literature. For instance, in the classic ruin theory (which describes the capital of an insurance company), the moment of ruin is considered as the first time the process is below the level zero. However, in more recent literature the last passage time below zero is treated as the moment of ruin and the Cramér–Lundberg has been generalised to spectrally negative Lévy processes (see e.g. Chiu and Yin (2005)). Moreover, in Paroissin and Rabehasaina (2013) spectrally positive Lévy processes are considered as degradation models and the last passage time above a certain fixed boundary is considered as the failure time.

Let X be a spectrally negative Lévy process. For any $t \geq 0$ and $x \in \mathbb{R}$, we define as $g_t^{(x)}$ as the last time that the process is below x before time t, i.e.,

$$g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\},$$

with the convention $\sup \emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. Note that

A similar version of this random time is studied in Revuz and Yor (1999) (see Chapter XII.3), namely the last hitting time at zero, before any time $t \geq 0$, to describe excursions straddling at a given time. It is also shown that this random time at time $t = 1$ follows the arcsine distribution. The last hitting time to zero has some applications in the study of Azéma's martingale (see Azéma and Yor (1989)). In Salminen (1988) the distribution of the the last hitting time of a moving boundary is found.

Note that the process $\{g_t, t \geq 0\}$ is non-decreasing and hence is a process of finite variation implying that belongs to the class of semi-martingales. Then Itô formula for the process $\{(g_t, t, X_t), t \geq 0\}$ is well known (see e.g. Protter (2005), Theorem 33) and is given for any function F in $C^{1,1,i}(E_g)$, where $i = 2$ if X is of infinite variation and $i = 1$ otherwise, by

$$\begin{aligned}
&F(g_t, t, X_t)\\
&= F(g_0, 0, X_0) + \int_0^t \frac{\partial}{\partial \gamma} F(g_{s-}, s, X_{s-}) \mathrm{d}g_s + \int_0^t \frac{\partial}{\partial t} F(g_s, s, X_s) \mathrm{d}s\\
&\quad + \int_0^t \frac{\partial}{\partial x} F(g_{s-}, s, X_{s-}) \mathrm{d}X_s + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2}{\partial x^2} F(g_s, s, X_s) \mathrm{d}s\\
&\quad + \sum_{0<s\leq t} \left(F(g_s, s, X_s) - F(g_{s-}, s, X_{s-}) - \frac{\partial}{\partial \gamma} F(g_{s-}, s, X_{s-}) \Delta g_s - \frac{\partial}{\partial x} F(g_{s-}, s, X_{s-}) \Delta X_s \right).
\end{aligned}$$

Note that the formula above is given in terms of the jumps of the process $\{g_t, t \geq 0\}$ and it does not reflect the connection on its behaviour with the process X. Indeed, note that some of the jumps of $\{g_t, t \geq 0\}$ occur when X jumps to $(-\infty, 0)$ from the positive half line. Moreover, when a Brownian motion component is included in the dynamics of X, the stochastic process $\{g_t, t \geq 0\}$ has infinity many (small) number of jumps as a consequence of creeping to the level zero from the positive half line. These facts imply that, in order to obtain a more explicit version of Itô formula, a careful study of the trajectory of $t \mapsto g_t$ needs to be done in terms of the excursions of X .

On the other hand, it turns out that $\{(g_t, t, X_t), t \geq 0\}$ belongs to the family of strong Markov processes (see Proposition 3.2.1) and then a general form of its infinitesimal generator is known. For instance, from the general theory of Markov processes (see Dynkin (1965)), we know that if Z is a strong Markov process in $\mathbb{R}^d$, with d a positive integer. Then for any relative compact set $B \subset \mathbb{R}^d$, there exist functions σ_{ij}, b_i and c on B and a kernel ν such that for any function $F \in C^2$ with compact support and $z \in B$,

$$\mathcal{A}_Z F(z) = c(z)F(z) + \sum_{i=1}^{d} b_i(z)\frac{\partial}{\partial z_i}F(z) + \sum_{i,j=1}^{d} \sigma_{ij}(z)\frac{\partial^2}{\partial z_i \partial z_j}F(z) + \int_{\mathbb{R}^d\setminus\{0\}} \left(F(y) - F(z) - \sum_{i=1}^{d}(y_i - z_i)\frac{\partial}{\partial z_i}F(z) \right) \nu(z, \mathrm{d}y).$$

However, in applications (for example, in optimal stopping and free boundary problems) an explicit expression for Itô formula and the infinitesimal generator are required in terms of the dynamics of X. In this work (see Theorem 3.2.3 and Corollary 3.2.5) we give and expression for Itô formula and the infinitesimal generator of the process $\{(t, g_t, X_t), t \geq 0\}$ in terms of the dynamics of X only.

We also consider, for any $t \geq 0$, the random variable $U_t = t - g_t$, the time of the positive current positive excursion away from zero. Having in mind the derivation of expressions for the potential measure of $(U, X)) = \{(U_t, X_t), t \geq 0\}$ and its join Laplace transform at an exponential time, we also derive an explicit formulae for additive functionals of the process (U, X) of the form

$$\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} K(U_r, X_r)\mathrm{d}r \right),$$

for some function K, where $q \geq 0$ and $\mathbb{P}_{u,x}$ is the measure for which $(U_0, X_0) = (u, x)$ in views of the Markov property of (U, X).

This Chapter is organised as follows. Section 3.2 is dedicated to the definition of the last zero process in which basic properties of this process are shown. Moreover, a derivation of

Itô formula, infinitesimal generator and formula for the expectation of a functional of U_t are the main results of this section (see Theorems 3.2.3 and 3.2.6 and Corollary 3.2.5). Then the aforementioned results are applied to find formulas for the joint Laplace transform of (U, X) at an exponential time and the q-potential measures are found. Lastly, Section 3.3 is exclusively dedicated to introduce a perturbated Lévy process and Theorems 3.2.3 and 3.2.6 are proven.

3.2 The last zero process

Throughout this chapter we use the notation and the preliminary results presented in Section 1.1. Let X be a spectrally negative Lévy process, that is, a Lévy process starting from 0 with only negative jumps and non-monotone paths, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see Definition 1.3.38 in Bichteler (2002)). We suppose that X has Lévy triplet (μ, σ, Π) where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (Lévy measure) concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

Recall that $g_t^{(x)}$ is the last time that the process is below x before time t, i.e.,

$$g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\},$$

with the convention $\sup \emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. For any stopping time τ, the random variable $g_\tau^{(x)}$ is $\mathcal{F}_\tau$ measurable. In particular we get that $\{g_t^{(x)}, t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$. Moroever, It is easy to show that for a fixed $x \in \mathbb{R}$, the stochastic process $\{g_t, t \geq 0\}$ is non-decreasing, right-continuous with left limits. Similarly, for a fixed $t \geq 0$ the mapping $x \mapsto g_t^{(x)}$ is non-decreasing and almost surely right-continuous with left limits.

It turns out that for all $x \in \mathbb{R}$ the process $\{g_t^{(x)}, t \geq 0\}$ is not a Markov process, in particular not Lévy process. However, the strong Markov property holds for the three dimensional process $\{(g_t, t, X_t), t \geq 0\}$.

Proposition 3.2.1. *The process* $\{(g_t, t, X_t), t \geq 0\}$ *is a strong Markov process with respect to the filtration* $\{\mathcal{F}_t, t \geq 0\}$ *with state space given by* $E_g = \{(\gamma, t, x) : 0 \leq \gamma < t$ *and* $x > 0\} \cup \{(\gamma, t, x) : 0 \leq \gamma = t$ *and* $x \leq 0\}$.

Proof. From the definition of g_t it easy to note that for all $t \geq 0$ we have that $X_t \leq 0$ if and only if $g_t = t$ from which we obtain that (g_t, t, X_t) can take only values in E_g. Now we proceed to show the strong Markov property. Consider a measurable positive function $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$. Then we have for any stopping time τ and $s \geq 0$,

$$\begin{aligned}
&\mathbb{E}(h(g_{\tau+s}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) \\
&\qquad = \mathbb{E}(h(g_\tau \vee \sup\{r \in [\tau, s + \tau] : X_r \leq 0\}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) \\
&\qquad = \mathbb{E}(h(g_\tau \vee \sup\{r \in [\tau, s + \tau] : \widetilde{X}_{r-\tau} + X_\tau \leq 0\}, \tau + s, \widetilde{X}_s + X_\tau)|\mathcal{F}_\tau),
\end{aligned}$$

where $\widetilde{X}_u = X_{u+\tau} - X_\tau$ for any $u \geq 0$ and $a \vee b := \max(a, b)$ for any $a, b \in \mathbb{R}$. From the strong Markov property of Lévy processes we deduce that the process $\widetilde{X}$ is independent of $\mathcal{F}_\tau$ and has the same law as of X. Then, together with the fact that g_τ and X_τ are $\mathcal{F}_\tau$ measurable, we obtain that

$$\mathbb{E}(h(g_{\tau+s}, \tau + s, X_{\tau+s})|\mathcal{F}_\tau) = f_s(g_\tau, \tau, X_\tau),$$

where for any $x \in \mathbb{R}$ and $0 \leq \gamma \leq t$, the function f_s is given by

$$f_s(\gamma, t, x) = \mathbb{E}(h(\gamma \vee \sup\{r \in [t, s + t] : X_{r-t} + x \leq 0\}, t + s, X_s + x)).$$

Note that in the event $\{\sigma_0^- > s\}$ we have that the set $\{r \in [t, s + t] : X_{r-t} + x \leq 0\} = \emptyset$ so then $\gamma \vee \sup\{r \in [t, s + t] : X_{r-t} + x \leq 0\} = \gamma$, where we used the convention that $\sup \emptyset = 0$. Otherwise, in the event $\{\sigma_0^- \leq s\}$, we have that $\{r \in [t, s + t] : X_{r-t} + x \leq 0\} \neq \emptyset$ and then $\sup\{r \in [t, s + t] : X_{r-t} + x \leq 0\} \geq t \geq \gamma$. Hence,

$$\begin{aligned}
\gamma \vee \sup\{r \in [t, s + t] : X_{r-t} + x \leq 0\} &= \sup\{r \in [t, s + t] : X_{r-t} + x \leq 0\} \\
&= t + \sup\{r \in [0, s] : X_r + x \leq 0\} \\
&= t + g_s.
\end{aligned}$$

Therefore, for any $x \in \mathbb{R}$ and $0 \leq \gamma \leq t$, f_s takes the form

$$f_s(\gamma, t, x) = \mathbb{E}_x(h(\gamma, t+s, X_s)\mathbb{I}_{\{\sigma_0^- > s\}}) + \mathbb{E}_x(h(g_s + t, t+s, X_s)\mathbb{I}_{\{\sigma_0^- \leq s\}}). \tag{3.1}$$

On the other hand, similar calculations lead us to

$$\mathbb{E}(h(g_{\tau+s}, \tau+s, X_{\tau+s})|\sigma(g_\tau, \tau, X_\tau)) = f_s(g_\tau, \tau, X_\tau).$$

Hence, for any measurable positive function h we obtain

$$\mathbb{E}(h(g_{\tau+s}, \tau+s, X_{\tau+s})|\mathcal{F}_\tau) = \mathbb{E}(h(g_{\tau+s}, \tau+s, X_{\tau+s})|\sigma(g_\tau, \tau, X_\tau)).$$

Therefore the process $\{(g_t, t, X_t), t \geq 0\}$ is a strong Markov process. □

In the spirit of the above Proposition we define for all $(\gamma, t, x) \in E_g$ the probability measure $\mathbb{P}_{\gamma,t,x}$ in the following way: for every measurable and positive function h we define

$$\mathbb{E}_{\gamma,t,x}(h(g_{t+s}, t+s, X_{t+s})) := \mathbb{E}\left(h(g_{t+s}, t+s, X_{t+s})\middle|(g_t, t, X_t) = (\gamma, t, x)\right) = f_s(\gamma, t, x),$$

where f_s is given in (3.1). Then we can write $\mathbb{P}_{\gamma,t,x}$ in terms of $\mathbb{P}_x$ by

$$\mathbb{E}_{\gamma,t,x}(h(g_{t+s}, t+s, X_{t+s})) = \mathbb{E}_x(h(\gamma, t+s, X_s)\mathbb{I}_{\{\sigma_0^- > s\}}) + \mathbb{E}_x(h(g_s + t, t+s, X_s)\mathbb{I}_{\{\sigma_0^- \leq s\}}). \tag{3.2}$$

Define $U_t = t - g_t$ as the length of the current excursion above the level zero. As a direct consequence we have that the process $\{(U_t, X_t), t \geq 0\}$ is also a strong Markov process with state space given by $E = \{(u, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : u > 0 \text{ and } x > 0\} \cup \{(0, x) \in \mathbb{R}^2 : x \leq 0\}$. We hence can define a probability measure $\mathbb{P}_{u,x}$, for all $(u, x) \in E$, by

$$\mathbb{E}_{u,x}(f(U_t, X_t)) = \mathbb{E}_x(f(u+t, X_t)\mathbb{I}_{\{\sigma_0^- > t\}}) + \mathbb{E}_x(f(U_t, X_t)\mathbb{I}_{\{\sigma_0^- \leq t\}}). \tag{3.3}$$

for any positive and measurable function f.

Remark 3.2.2. *We know that for any $x \in \mathbb{R}$, $g_t^{(x)}$ is a non-decreasing process. That directly implies that $g_t^{(x)}$ is a process of finite variation and then it has a countable number of jumps. Moreover, with a close inspection to the definition of $g_t^{(x)}$ we notice that $g_t^{(x)} = t$ on the set*

$\{t \geq 0 : X_t \leq x\}$, it is flat when X is in the set (x, ∞) and it has a jump when X enters the set $(-\infty, x]$. Moreover, if X is a process of infinite variation we know that the set of times in which X visits the level x from above is infinite. That implies that when X is of infinite variation, $t \mapsto g_t^{(x)}$ has an infinite number of arbitrary small jumps.

Note that the process (g_t, t, X_t) is a semi-martingale so its Itô formula is well known (see e.g. Protter (2005), Theorem 33). In the next Theorem we give a more explicit expression for the Itô formula for the process (g_t, t, X_t) in terms of the random measure N. Note that this formula will be useful later to derive the infinitesimal generator of (g_t, t, X_t). The proof can be found in Section 3.3.2.

Theorem 3.2.3 (Itô formula)**.** *Let X be any spectrally negative and F a $C^{1,1,i}(E_g)$ real-valued function, where $i = 2$ if X is of infinite variation and $i = 1$ otherwise. In addition, in the case that $\sigma > 0$, assume that $\lim_{h \downarrow 0} F(\gamma, t, h) = F(t, t, 0)$ for all $\gamma \leq t$. Then we have the following version of Itô formula for the three dimensional process $\{(g_t, t, X_t), t \geq 0\}$.*

$$
\begin{aligned}
&F(g_t, t, X_t) \\
&= F(g_0, 0, X_0) + \int_0^t \frac{\partial}{\partial \gamma} F(g_s, s, X_s) \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s + \int_0^t \frac{\partial}{\partial t} F(g_s, s, X_s) \mathrm{d}s \\
&\quad + \int_0^t \frac{\partial}{\partial x} F(g_{s-}, s, X_{s-}) \mathrm{d}X_s + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2}{\partial x^2} F(g_s, s, X_s) \mathrm{d}s \\
&\quad + \int_{[0,t]} \int_{(-\infty,0)} \left(F(g_s, s, X_{s^-} + y) - F(g_{s-}, s, X_{s-}) - y \frac{\partial}{\partial x} F(g_{s-}, s, X_{s-}) \right) N(\mathrm{d}s \times \mathrm{d}y).
\end{aligned}
$$

Remark 3.2.4. *When $\sigma > 0$, the Brownian motion part of X implies that X can visit the interval $(-\infty, 0]$ by creeping. That implies that $t \mapsto g_t$ has two types of jumps: those as a consequence of X jumping from the positive half line to $(-\infty, 0)$ which is finite (since $\Pi(-\infty, -\varepsilon) < \infty$ for all $\varepsilon > 0$) and those as a consequence of creeping. The limit condition imposed for F (when $\sigma > 0$) ensures that the jumps due to the Brownian component vanish, otherwise a more careful analysis involving the local time needs to be done.*

Now that we have an Itô's formula for the three dimensional process (g_t, t, X_t) in terms of the Poisson random measure N, we are ready to state an explicit formula for its infinitesimal generator. Denote by $C_b^{1,1,2}(E_g)$ the set of bounded $C^{1,1,2}(E_g)$ functions with bounded derivatives. We have the following Corollary which proof follows directly from equation (3.16) and using standard arguments so it is omitted.

Corollary 3.2.5. *Let X be any spectrally negative Lévy process and F a $C_b^{1,1,2}(E_g)$ function such that when $\sigma > 0$, $\lim_{h\downarrow 0} F(\gamma,t,h) = F(t,t,0)$ for all $\gamma \leq t$. Then the infinitesimal generator $\mathcal{A}_Z$ of the process $Z_t = (g_t, t, X_t)$ satisfies*

$$
\begin{aligned}
&\mathcal{A}_Z F(\gamma,t,x) \\
&\quad = \frac{\partial}{\partial\gamma}F(\gamma,t,x)\mathbb{I}_{\{x\leq 0\}} + \frac{\partial}{\partial t}F(\gamma,t,x) - \mu\frac{\partial}{\partial x}F(\gamma,t,x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(\gamma,t,x) \\
&\qquad + \int_{(-\infty,0)}\left(F(\gamma,t,x+y) - F(\gamma,t,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}F(\gamma,t,x)\right)\mathbb{I}_{\{x+y>0\}}\Pi(\mathrm{d}y) \\
&\qquad + \int_{(-\infty,0)}\left(F(t,t,x+y) - F(t,t,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}F(t,t,x)\right)\mathbb{I}_{\{x\leq 0\}}\Pi(\mathrm{d}y) \\
&\qquad + \int_{(-\infty,0)}\left(F(t,t,x+y) - F(\gamma,t,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}F(\gamma,t,x)\right)\mathbb{I}_{\{x>0\}}\mathbb{I}_{\{x+y<0\}}\Pi(\mathrm{d}y)
\end{aligned}
\tag{3.4}
$$

Recall from Remark 3.2.2 that the behaviour of g_t and then U_t can be determined from the excursions of X away from zero. Then, using that fact, we are able to derive a formula for a functional that involves the whole trajectory of the process U_t. The next theorem provides a formula to calculate an integral involving the process $\{(U_t, X_t), t \geq 0\}$ with respect of time in terms of the excursions of X above and below zero.

Theorem 3.2.6. *Let $q \geq 0$ and X be a spectrally negative Lévy process and $K : E \mapsto \mathbb{R}$ a left-continuous function in each argument. Assume that there exists a non-negative function $C : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ such that $u \mapsto C(u,x)$ is a monotone function for all $x \in \mathbb{R}$, $|K(u,x)| \leq C(u,x)$ and $\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr}C(U_r, X_r + y)\mathrm{d}r\right) < \infty$ for all $(u,x) \in E$ and $y \in \mathbb{R}$. Then we have that for any $(u,x) \in E$ that*

$$
\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr}K(U_r,X_r)\mathrm{d}r\right) \\
&\quad = K^+(u,x) + \int_{-\infty}^0 K(0,y)\left[e^{\Phi(q)(x-y)}\Phi'(q) - W^{(q)}(x-y)\right]\mathrm{d}y \\
&\qquad + e^{\Phi(q)x}\left[1 - \psi'(\Phi(q)+)e^{-\Phi(q)x}W^{(q)}(x)\right]\lim_{\varepsilon\downarrow 0}\frac{K^+(0,\varepsilon)}{\psi'(\Phi(q)+)W^{(q)}(\varepsilon)},
\end{aligned}
\tag{3.5}
$$

where K^+ is given by

$$
K^+(u,x) = \mathbb{E}_x\left(\int_0^{\tau_0^-} e^{-qr}K(u+r,X_r)\mathrm{d}r\right), \qquad (u,x) \in E.
$$

In particular, when $u = x = 0$ we have that

$$\mathbb{E}\left(\int_0^\infty e^{-qr}K(U_r,X_r)\mathrm{d}r\right) = \int_{-\infty}^0 K(0,y)\left[e^{-\Phi(q)y}\Phi'(q) - W^{(q)}(-y)\right]\mathrm{d}y + \lim_{\varepsilon\downarrow 0}\frac{K^+(0,\varepsilon)}{\psi'(\Phi(q)+)W^{(q)}(\varepsilon)}.$$

Remark 3.2.7. *Note that from the proof of Theorem 3.2.6 we can find an alternative representation for formula (3.5) as a limit in terms of excursions of X above and below zero divided by a normalisation term. Indeed, for $(u,x) \in E$,*

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr}K(U_r,X_r)\mathrm{d}r\right)\\
&= K^+(u,x) + \lim_{\varepsilon\downarrow 0}\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K^-(X_{\tau_0^-}-\varepsilon)\right)\\
&\quad + e^{\Phi(q)x}\left[1-\psi'(\Phi(q)+)e^{-\Phi(q)x}W^{(q)}(x)\right]\lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K^-(X_{\tau_0^-}-\varepsilon)\right)+K^+(0,\varepsilon)}{\psi'(\Phi(q)+)W^{(q)}(\varepsilon)},
\end{aligned} \tag{3.6}$$

where K^- is given by

$$K^-(x) = \mathbb{E}_x\left(\int_0^{\tau_0^+} e^{-qr}K(0,X_r)\mathrm{d}r\right),$$

for all $x \in \mathbb{R}$.

3.2.1 Applications of Theorem 3.2.6

In this section we consider applications of Theorem (3.2.6). We first calculate the joint Laplace transform of $(U_{\mathbf{e}_q}, X_{\mathbf{e}_q})$ where $\mathbf{e}_q$ is an exponential time with parameter $q > 0$ independent of X.

Corollary 3.2.8. *Let X be a spectrally negative Lévy process. Let $q > 0$ and $\alpha \in \mathbb{R}$, $\beta \geq 0$*

such that $q > \psi(\beta) \vee (\psi(\beta) - \alpha)$. *We have that for all* $(u, x) \in E$,

$$\mathbb{E}_{u,x}\left(e^{-\alpha U_{e_q} + \beta X_{e_q}}\right) = \frac{qe^{\beta x}}{q - \psi(\beta)} + e^{\Phi(q)x}\Phi'(q)\left[\frac{q}{\Phi(q+\alpha) - \beta} - \frac{q}{\Phi(q) - \beta}\right] + e^{\beta x} q \int_0^x e^{-\beta y}[W^{(q)}(y) - e^{-\alpha u} W^{(q+\alpha)}(y)]\mathrm{d}y + \frac{q}{\Phi(q+\alpha) - \beta}\left[e^{-\alpha u} W^{(q+\alpha)}(x) - W^{(q)}(x)\right]. \tag{3.7}$$

Proof. Consider the function $K(u, x) = e^{-\alpha u + \beta x}$ for all $(u, x) \in E$. We have that K is a continuous function and $K(u, x) \leq e^{-(\alpha \wedge 0)u + \beta x}$ for all $(u, x) \in E$. Moreover we have for all $q > 0$ such that $q > \psi(\beta) \vee (\psi(\beta) - \alpha) = \psi(\beta) - (\alpha \wedge 0)$ that

$$\mathbb{E}_x\left(\int_0^\infty e^{-qr} e^{-(\alpha \wedge 0)r + \beta X_r}\mathrm{d}r\right) = e^{\beta x}\int_0^\infty e^{-(q + (\alpha \wedge 0) - \psi(\beta))r}\mathrm{d}r = \frac{e^{\beta x}}{q + (\alpha \wedge 0) - \psi(\beta)} < \infty$$

for all $x \in \mathbb{R}$. Then for all $u > 0$ and $x > 0$ we have, by Fubini's theorem and from equation (1.19), that

$$\begin{aligned}
K^+(u, x) &= \mathbb{E}_x\left(\int_0^{\tau_0^-} e^{-qr} e^{-\alpha(u+r) + \beta X_r}\mathrm{d}r\right) \\
&= e^{-\alpha u}\int_{(0,\infty)} e^{\beta y}\int_0^\infty e^{-(q+\alpha)r}\mathbb{P}_x(X_r \in \mathrm{d}y, r < \tau_0^-)\mathrm{d}r \\
&= e^{-\alpha u}\int_0^\infty e^{\beta y}\left[e^{-\Phi(q+\alpha)y} W^{(q+\alpha)}(x) - W^{(q+\alpha)}(x - y)\right]\mathrm{d}y \\
&= \frac{e^{-\alpha u} W^{(q+\alpha)}(x)}{\Phi(q+\alpha) - \beta} - e^{-\alpha u} e^{\beta x}\int_0^x e^{-\beta y} W^{(q+\alpha)}(y)\mathrm{d}y.
\end{aligned}$$

Similarly, we calculate for any $x \in \mathbb{R}$

$$\begin{aligned}
&\int_{-\infty}^0 e^{\beta y}\left[e^{\Phi(q)(x-y)}\Phi'(q) - W^{(q)}(x - y)\right]\mathrm{d}y \\
&\quad = \Phi'(q)e^{\Phi(q)x}\int_0^\infty e^{-(\beta - \Phi(q))y}\mathrm{d}y - e^{\beta x}\int_x^\infty e^{-\beta y} W^{(q)}(y)\mathrm{d}y \\
&\quad = \frac{\Phi'(q)e^{\Phi(q)x}}{\beta - \Phi(q)} - \frac{e^{\beta x}}{\psi(\beta) - q} + e^{\beta x}\int_0^x e^{-\beta y} W^{(q)}(y)\mathrm{d}y,
\end{aligned}$$

where the last equality follows from equation (1.4) and the last integral is understood like 0 when $x < 0$. Then from (3.5) we get that for all $(u, x) \in E$,

$$
\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} e^{-\alpha U_r + \beta X_r} \mathrm{d}r\right) \\
&= \frac{e^{-\alpha u} W^{(q+\alpha)}(x)}{\Phi(q+\alpha) - \beta} - e^{-\alpha u} e^{\beta x} \int_0^x e^{-\beta y} W^{(q+\alpha)}(y) \mathrm{d}y \\
&\quad + \frac{\Phi'(q) e^{\Phi(q)x}}{\beta - \Phi(q)} - \frac{e^{\beta x}}{\psi(\beta) - q} + e^{\beta x} \int_0^x e^{-\beta y} W^{(q)}(y) \mathrm{d}y \\
&\quad + e^{\Phi(q)x} \mathcal{I}^{(q,\Phi(q))}(x) \lim_{\varepsilon \downarrow 0} \frac{1}{\psi'(\Phi(q)+) W^{(q)}(\varepsilon)} \left[\frac{W^{(q+\alpha)}(\varepsilon)}{\Phi(q+\alpha) - \beta} - e^{\beta \varepsilon} \int_0^\varepsilon e^{-\beta y} W^{(q+\alpha)}(y) \mathrm{d}y\right] \\
&= \frac{e^{-\alpha u} W^{(q+\alpha)}(x)}{\Phi(q+\alpha) - \beta} - e^{-\alpha u} e^{\beta x} \int_0^x e^{-\beta y} W^{(q+\alpha)}(y) \mathrm{d}y \\
&\quad + \frac{\Phi'(q) e^{\Phi(q)x}}{\beta - \Phi(q)} - \frac{e^{\beta x}}{\psi(\beta) - q} + e^{\beta x} \int_0^x e^{-\beta y} W^{(q)}(y) \mathrm{d}y \\
&\quad + e^{\Phi(q)x} \left[1 - \psi'(\Phi(q)+) e^{-\Phi(q)x} W^{(q)}(x)\right] \frac{\Phi'(q)}{\Phi(q+\alpha) - \beta},
\end{aligned}
$$

where in the last equality we used the fact that $\Phi'(q) = 1/\psi'(\Phi(q)+)$, $W^{(q)}(x)$ is non-negative and strictly increasing on $[0, \infty)$ for all $q \geq 0$ and that

$$
\lim_{\varepsilon \downarrow 0} \frac{W^{(q+\alpha)}(\varepsilon)}{W^{(q)}(\varepsilon)} = 1
$$

for all $\alpha, q \geq 0$. The latter fact follows from the representation $W^{(q)}(x) = \sum_{k=0}^\infty q^k W^{*(k+1)}(x)$ and the estimate $W^{*(k+1)}(x) \leq x^k / k! W(x)^{k+1}$ (see equations (8.28) and (8.29) in Kyprianou (2014), pp 241-242). Rearranging the terms and using the fact that

$$
\mathbb{E}_{u,x}\left(e^{-\alpha U_{\mathbf{e}_q} + \beta X_{\mathbf{e}_q}}\right) = q \mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} e^{-\alpha U_r + \beta X_r} \mathrm{d}r\right)
$$

for all $(u, x) \in E$, we obtain the desired result. □

Remark 3.2.9. *Note that from formula (3.7) we can recover some known expressions for Lévy processes. If we take $\alpha = 0$ we obtain for all $\beta \geq 0$ and $q > \psi(\beta) \vee 0$ and $x \in \mathbb{R}$,*

$$
\mathbb{E}_x(e^{\beta X_{\mathbf{e}_q}}) = \frac{q e^{\beta x}}{q - \psi(\beta)}.
$$

On the other for any $\theta \geq 0$, $q \geq 0$ and $x \in \mathbb{R}$ we have that

$$\mathbb{E}_x(e^{-\theta g_{\mathbf{e}_q}}) = \int_0^\infty q e^{-qt} \mathbb{E}_x(e^{-\theta g_t}) \mathrm{d}t = \int_0^\infty q e^{-(q+\theta)t} \mathbb{E}_x(e^{\theta U_t}) \mathrm{d}t = \frac{q}{q+\theta} \mathbb{E}_x(e^{\theta U_{\mathbf{e}_{q+\theta}}}),$$

where $\mathbf{e}_{q+\theta}$ is an exponential random variable with parameter $q+\theta$. This result coincides with the one found in Baurdoux (2009) (see Theorem 2).

Let $q > 0$ we consider the q-potential measure of (U, X) given by

$$\int_0^\infty e^{-qr} \mathbb{P}_{u,x}(U_r \in \mathrm{d}v, X_r \in \mathrm{d}y) \mathrm{d}r$$

for $(u, x), (v, y) \in E$. From the fact that for all $t > 0$, $U_t = 0$ if and only if $X_t \leq 0$ we have that for any $(u, x) \in E$ and $y \leq 0$

$$\int_0^\infty e^{-qr} \mathbb{P}_{u,x}(U_r = 0, X_r \in \mathrm{d}y) \mathrm{d}r = \int_0^\infty e^{-qr} \mathbb{P}_x(X_r \in \mathrm{d}y) \mathrm{d}r$$

In the next corollary we find the an expression for a density when $v, x > 0$.

Corollary 3.2.10. *Let $q > 0$. The q-potential measure of (U, X) has a density given by*

$$\begin{aligned} \int_0^\infty e^{-qr} \mathbb{P}_{u,x}(U_r \in \mathrm{d}v, X_r \in \mathrm{d}y) \mathrm{d}r &= e^{-q(v-u)} \mathbb{P}_x(X_{v-u} \in \mathrm{d}y, v - u < \tau_0^-) \mathbb{I}_{\{v>u\}} \mathrm{d}v \\ &\quad + \left[e^{\Phi(q)x} \Phi'(q) - W^{(q)}(x) \right] \frac{y}{v} e^{-qv} \mathbb{P}(X_v \in \mathrm{d}y) \mathrm{d}v \end{aligned} \tag{3.8}$$

for all $(u, x) \in E$ and $v, y > 0$. In particular, when $u = x = 0$ we have that

$$\int_0^\infty e^{-qr} \mathbb{P}(U_r \in \mathrm{d}v, X_r \in \mathrm{d}y) \mathrm{d}r = e^{\Phi(q)x} \Phi'(q) \frac{y}{v} e^{-qv} \mathbb{P}(X_v \in \mathrm{d}y) \mathrm{d}v \tag{3.9}$$

Proof. Let $0 < u_1 < u_2$ and $0 < x_1 < x_2$ and define the sets $A = (u_1, u_2]$ and $Y = (x_1, x_2]$. Then the function $K(u, x) = \mathbb{I}_{\{u \in A, x \in Y\}}$ is left-continuous and bounded by above by $C(x) = \mathbb{I}_{\{x \in Y\}}$. Moreover, we have that for all $q > 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}_x \left(\int_0^\infty e^{-qr} \mathbb{I}_{\{X_r \in Y\}} \mathrm{d}r \right) < \infty.$$

First we calculate for all $u, x > 0$ such that $u < u_1$,

$$K^+(u,x) = \mathbb{E}_{u,x}\left(\int_0^{\tau_0^-} e^{-qr}\mathbb{I}_{\{U_r\in A, X_r\in Y\}}\mathrm{d}r\right) = \int_A\int_Y e^{-q(r-u)}\mathbb{P}_x(X_{r-u}\in \mathrm{d}y, r-u<\tau_0^-)\mathrm{d}r$$

and for every $x \leq 0$ we have that

$$K^-(x) = \mathbb{E}_{u,x}\left(\int_0^{\tau_0^+} e^{-qr}\mathbb{I}_{\{U_r\in A, X_r\in Y\}}\mathrm{d}r\right) = 0$$

Hence, for all $(u, x) \in E$ we obtain that

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^{\infty} e^{-qr}\mathbb{I}_{\{U_r\in A, X_r\in Y\}}\mathrm{d}r\right)\\
&\quad= \int_A\int_Y e^{-q(r-u)}\mathbb{P}_x(X_{r-u}\in \mathrm{d}y, r-u<\tau_0^-)\mathrm{d}r\\
&\qquad+ e^{\Phi(q)x}\left[1-\psi'(\Phi(q)+)e^{-\Phi(q)x}W^{(q)}(x)\right]\lim_{\varepsilon\downarrow 0}\int_A\int_Y \frac{e^{-qr}\mathbb{P}_\varepsilon(X_r\in \mathrm{d}y, r<\tau_0^-)}{\psi'(\Phi(q)+)W^{(q)}(\varepsilon)}\mathrm{d}r.
\end{aligned}$$

We calculate the limit on the right-hand side of the equation above. Denote $\mathbb{P}_\varepsilon^\uparrow$ as the law of X starting from ε conditioned to stay positive (see (1.23)). We have that for all $x \in \mathbb{R}$ and $y > 0$ that

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\int_A\int_Y \frac{e^{-qr}\mathbb{P}_\varepsilon(X_r\in \mathrm{d}y, r<\tau_0^-)}{\psi'(\Phi(q)+)W^{(q)}(\varepsilon)}\mathrm{d}r &= \frac{W(\varepsilon)}{\psi'(\Phi(q)+)W^{(q)}(\varepsilon)}\lim_{\varepsilon\downarrow 0}\int_A\int_Y \frac{e^{-qr}\mathbb{P}_\varepsilon^\uparrow(X_r\in \mathrm{d}y)}{W(y)}\mathrm{d}r\\
&= \frac{1}{\psi'(\Phi(q)+)}\int_A\int_Y \frac{e^{-qr}\mathbb{P}^\uparrow(X_r\in \mathrm{d}y)}{W(y)}\mathrm{d}r,
\end{aligned}$$

where the first equality follows from the definition of $\mathbb{P}_\varepsilon^\uparrow$ and the last equality follows since $\lim_{\varepsilon\downarrow 0} W(\varepsilon)/W^{(q)}(\varepsilon) = 1$ and $\mathbb{P}_\varepsilon^\uparrow$ converges to $\mathbb{P}^\uparrow$ in the sense of finite-dimensional distributions (see Proposition VII.3.14 in Bertoin (1998)). Therefore from (1.24) we have that for all $(u, x) \in E$,

$$\begin{aligned}
\mathbb{E}_{u,x}\left(\int_0^{\infty} e^{-qr}\mathbb{I}_{\{U_r\in A, X_r\in Y\}}\mathrm{d}r\right) &= \int_A\int_Y e^{-q(r-u)}\mathbb{P}_x(X_{r-u}\in \mathrm{d}y, r-u<\tau_0^-)\mathrm{d}r\\
&\quad+\left[\Phi'(q)e^{\Phi(q)x}-W^{(q)}(x)\right]\int_A\int_Y \frac{y}{r}e^{-qr}\mathbb{P}(X_r\in \mathrm{d}y)\mathrm{d}r,
\end{aligned}$$

where we also used the fact that $\Phi'(q) = 1/\psi'(\Phi(q)+)$. Using the same arguments one can easily see that when X is of finite variation

$$\int_A \int_Y e^{-q(r-u)} \mathbb{P}(X_{r-u} \in \mathrm{d}y, r-u < \tau_0^-)\mathrm{d}r = W^{(q)}(0) \int_A \int_Y \frac{y}{r} e^{-qr} \mathbb{P}(X_r \in \mathrm{d}y)\mathrm{d}r.$$

The proof is now complete. □

Remark 3.2.11. *Bingham (1975) showed that the q-potential measure of X has a density that is absolutely continuous with respect to the Lebesgue measure. This can be shown moving the killing barrier on the q-potential measure killed on entering the set $(-\infty, 0]$ (see (1.19)) and taking limits. Alternatively, it can deduced taking limits on (3.5). Moreover, Corollary 3.2.10 provides an alternative method for finding the aforementioned density. For this, we use Kendall's identity (see e.g. Bertoin (1998), Corollary VII.3) given by*

$$r\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z = z\mathbb{P}(X_r \in \mathrm{d}z)\mathrm{d}r \tag{3.10}$$

for all $r, z \geq 0$. Indeed, let $u, y > 0$ and $x \in \mathbb{R}$, integrating (3.8) with respect to the variable v, we obtain that

$$\begin{aligned}
\int_0^\infty e^{-qr} \mathbb{P}_x(X_r \in \mathrm{d}y)\mathrm{d}r &= \int_{(0,\infty)} \int_0^\infty e^{-qr} \mathbb{P}_{u,x}(U_r \in \mathrm{d}v, X_r \in \mathrm{d}y)\mathrm{d}r \\
&= \int_0^\infty e^{-qv} \mathbb{P}_x(X_v \in \mathrm{d}y, v < \tau_0^-)\mathrm{d}v \\
&\quad + \int_0^\infty \left[e^{\Phi(q)x} \Phi'(q) - W^{(q)}(x) \right] \frac{y}{v} e^{-qv} \mathbb{P}(X_v \in \mathrm{d}y)\mathrm{d}v \\
&= [e^{-\Phi(q)y} W^{(q)}(x) - W^{(q)}(x-y)]\mathrm{d}y \\
&\quad + \left[e^{\Phi(q)x} \Phi'(q) - W^{(q)}(x) \right] \int_0^\infty e^{-qv} \mathbb{P}(\tau_y^+ \in \mathrm{d}v)\mathrm{d}y,
\end{aligned}$$

where the last equality follows from (1.19) and (3.10). Hence, using the formula for the Laplace transform of τ_y^+ (see equation (1.3)) we have that

$$\int_0^\infty e^{-qr} \mathbb{P}_x(X_r \in \mathrm{d}y)\mathrm{d}r = \left(e^{\Phi(q)(x-y)} \Phi'(q) - W^{(q)}(x-y) \right) \mathrm{d}y.$$

3.3 Main proofs

Suppose that X is a spectrally negative Lévy process of finite variation. Then with positive probability it takes a positive amount of time to cross below 0, i.e. $\tau_0^- > 0$ $\mathbb{P}$-a.s. Hence, stopping at the consecutive times in which X is below zero and the ideas mentioned in Remark 3.2.2 we can fully describe the behaviour of g_t and then derive the results mentioned in Theorems 3.2.3 and 3.2.6. However, in the case X is of infinite variation it is well known that the closed zero set of X is perfect and nowhere dense and the mentioned approach proves to be no longer useful (since we have that $\tau_0^- = 0$ a.s). Therefore, in order to exploit the idea applicable for finite variation processes we make use of a perturbation method. This method is mainly based on the work of Dassios and Wu (2011) and Revuz and Yor (1999) (see Theorem VI.1.10) which consists in construct a new "perturbed" process $X^{(\varepsilon)}$ (for ε sufficiently small) that approximates X with the property that $X^{(\varepsilon)}$ visits the level zero a finite number of times before any time $t \geq 0$. Then we approximate g_t by the corresponding last zero process of $X^{(\varepsilon)}$.

3.3.1 Perturbed Lévy process

We describe formally the construction of the "perturbed" process $X^{(\varepsilon)}$. Let $\varepsilon > 0$, define the stopping times $\sigma_{1,\varepsilon}^- = 0$ and for any $k \geq 1$,

$$\sigma_{k,\varepsilon}^+ := \inf\{t > \sigma_{k,\varepsilon}^- : X_t \geq \varepsilon\}$$
$$\sigma_{k+1,\varepsilon}^- := \inf\{t > \sigma_{k,\varepsilon}^+ : X_t < 0\}$$

and define the auxiliary process $X^{(\varepsilon)} = \{X_t^{(\varepsilon)}, t \geq 0\}$ where

$$X_t^{(\varepsilon)} = \begin{cases} X_t - \varepsilon & \text{if} \quad \sigma_{k,\varepsilon}^- \leq t < \sigma_{k,\varepsilon}^+ \\ X_t & \text{if} \quad \sigma_{k,\varepsilon}^+ \leq t < \sigma_{k+1,\varepsilon}^-. \end{cases}$$

In Figure 3.1 we include a sample path of the process $X^{(\varepsilon)}$ compared with the original process X.

Lemma 3.3.1. *We have that $X_r^{(\varepsilon)} < 0$ if and only if there exists $k \geq 1$ such that $r \in [\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^+)$. Moreover, for each $t \geq 0$, $X_t^{(\varepsilon)}$ increases when $\varepsilon \downarrow 0$ and $X^{(\varepsilon)}$ converges uniformly to X when $\varepsilon \downarrow 0$, that is,*

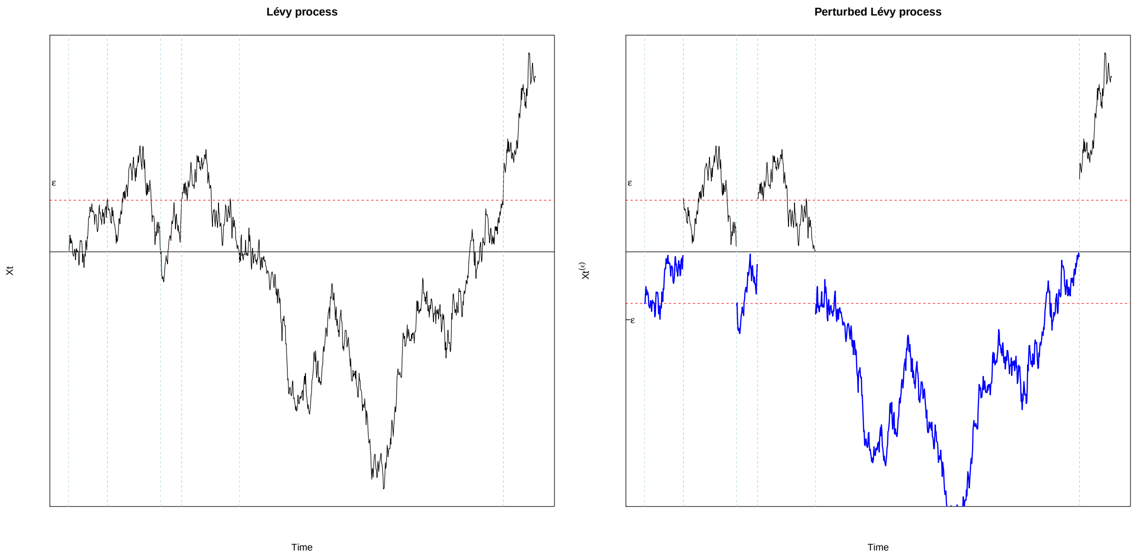

Figure 3.1: Left: Sample path of X. Right: Sample path of the perturbed process $X^{(\varepsilon)}$. The lightblue vertical lines correspond to the sequence of stopping times $\{\sigma^-_{k,\varepsilon}, k \geq 1\}$, whereas the gray vertical lines correspond to $\{\sigma^+_{k,\varepsilon}, k \geq 1\}$.

$$\lim_{\varepsilon \downarrow 0} \sup_{t \geq 0} |X^{(\varepsilon)}_t - X_t| = 0.$$

Proof. From the definition of the stopping times $\sigma^-_{k,\varepsilon}$ and $\sigma^+_{k,\varepsilon}$ we have that for some $k \geq 1$, if $r \in [\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon})$, then $X^{(\varepsilon)}_r < 0$ whereas if $r \in [\sigma^+_{k,\varepsilon}, \sigma^-_{k+1,\varepsilon})$, then $X^{(\varepsilon)}_r \geq 0$. Moreover. it is easy to see that for all $t \geq 0$

$$X_t - \varepsilon \leq X^{(\varepsilon)}_t \leq X_t. \tag{3.11}$$

We deduce that $\sup_{t \geq 0} |X^{(\varepsilon)}_t - X_t| < \varepsilon$ and hence

$$\lim_{\varepsilon \downarrow 0} \sup_{t \geq 0} |X^{(\varepsilon)}_t - X_t| = 0.$$

It is only left to show that for $0 < \varepsilon_1 \leq \varepsilon_2$ and all $t \geq 0$ that $X^{(\varepsilon_2)}_t \leq X^{(\varepsilon_1)}_t$. By the definition of $X^{(\varepsilon)}_t$, we have to check the cases in which $X^{(\varepsilon_i)}_t = X_t$ or $X^{(\varepsilon_i)}_t = X_t - \varepsilon_i$ for

$i = 1, 2$. Thus, since $\varepsilon_1 \le \varepsilon_2$, it suffices to prove that is not possible that $t \in [\sigma^-_{i,\varepsilon_1}, \sigma^+_{i,\varepsilon_1})$ and $t \in [\sigma^+_{j,\varepsilon_2}, \sigma^-_{j+1,\varepsilon_2})$ for some $i, j \ge 1$. Indeed, suppose that there exists $j \ge 1$ such that $t \in [\sigma^+_{j,\varepsilon_2}, \sigma^-_{j+1,\varepsilon_2})$ and define $i := \max\{k \ge 1 : \sigma^-_{k,\varepsilon_1} < \sigma^+_{j,\varepsilon_2}\}$. Since $\varepsilon_1 \le \varepsilon_2$, we have that $\sigma^+_{i,\varepsilon_1} < \sigma^+_{j,\varepsilon_2}$. Moreover, from the definition of i, we have that $\sigma^-_{i+1,\varepsilon_1} > \sigma^+_{j,\varepsilon_2}$ which implies that $X_r \ge 0$ for all $r \in [\sigma^+_{j,\varepsilon_2}, \sigma^-_{i+1,\varepsilon_1})$ and hence $\sigma^-_{i+1,\varepsilon_1} = \sigma^-_{j,\varepsilon_2}$. We conclude that if $t \in [\sigma^+_{j,\varepsilon_2}, \sigma^-_{j+1,\varepsilon_2})$ for some $j \ge 1$, there exists $i \ge 1$ such that $t \in [\sigma^+_{i,\varepsilon_1}, \sigma^-_{i+1,\varepsilon_1})$ and the proof is complete. □

In addition we define the last zero process $g_{\varepsilon,t}$ associated to the process $X^{(\varepsilon)}$, i.e.

$$g_{\varepsilon,t} = \sup\{0 \le s \le t : X^{(\varepsilon)}_s \le 0\}.$$

The inequality $g_t \le g_{\varepsilon,t} \le g^{(\varepsilon)}_t$ holds for all $t \ge 0$. Taking $\varepsilon \downarrow 0$ and by right continuity of $x \mapsto g^x_t$ we obtain that $g_{\varepsilon,t} \downarrow g_t$ when $\varepsilon \downarrow 0$ for all $t \ge 0$. Therefore we have that $t - U_{\varepsilon,t} =: U_{\varepsilon,t} \uparrow U_t$ when $\varepsilon \downarrow 0$ for all $t \ge 0$.

Recall that the local time at zero, $L = \{L_t, t \ge 0\}$, is a continuous process defined in terms of the Itô–Tanaka formula (see Protter (2005) Chapter IV) and its measure $\mathrm{d}L_t$ is carried by the set $\{s \ge 0 : X_{s-} = X_s = 0\}$.

Denote $M^{(\varepsilon)}_t$ as the number of downcrossings of the level zero at time $t \ge 0$ of the process $X^{(\varepsilon)}_t$, i.e.

$$M^{(\varepsilon)}_t = \sum_{k=1}^{\infty} \mathbb{I}_{\{\sigma^-_{k,\varepsilon} < t\}}. \tag{3.12}$$

We simply denote $M^{(\varepsilon)} = \lim_{t\to\infty} M^{(\varepsilon)}_t$ for all $\varepsilon > 0$. It turns out that $M^{(\varepsilon)}_t$ works as an approximation of the local time at zero in some sense. We have the following lemma; its proof follows an analogous argument than Revuz and Yor (1999) (see Exercise VI.1.19).

Lemma 3.3.2. *Suppose that X is a spectrally negative Lévy process. Then for all $t \ge 0$,*

$$\lim_{\varepsilon\downarrow 0} \varepsilon M^{(\varepsilon)}_t = \frac{1}{2} L_t \qquad a.s.$$

Proof. Using the Meyer–Itô formula (see Protter (2005) Theorem 70), we have for any $t \ge 0$

that

$$X_t^+ = X_0^+ + \int_{(0,t]} \mathbb{I}_{\{X_{s-}>0\}} \mathrm{d}X_s + \int_{(0,t]} \int_{(-\infty,0)} (X_{s-}+y)^- \mathbb{I}_{\{X_{s-}>0\}} N(\mathrm{d}s \times \mathrm{d}y) + \frac{1}{2} L_t,$$

where x^+ and x^- are the positive and negative parts, respectively, of x defined by $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. Hence, for all $1 \leq k \leq M_t^{(\varepsilon)}$,

$$\begin{aligned} X^+_{\sigma^+_{k,\varepsilon}\wedge t} - X^+_{\sigma^-_{k,\varepsilon}} \\ &= \int_{(\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon}\wedge t]} \mathbb{I}_{\{X_{s-}>0\}} \mathrm{d}X_s + \int_{(\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon}\wedge t]} \int_{(-\infty,0)} (X_{s-}+y)^- \mathbb{I}_{\{X_{s-}>0\}} N(\mathrm{d}s \times \mathrm{d}y) \\ &\quad + \frac{1}{2}(L_{\sigma^+_{k,\varepsilon}\wedge t} - L_{\sigma^-_{k,\varepsilon}}). \end{aligned}$$

From the definition of the stopping times $\{\sigma^-_{k,\varepsilon}, k \geq 1\}$, we have that for every $k \geq 1$, $X_r > 0$ for all $r \in [\sigma^+_{k,\varepsilon}, \sigma^-_{k+1,\varepsilon})$ and since L is continuous and only charge points in the set of zeros of X, we have that $L_{\sigma^+_{k,\varepsilon}} = L_{\sigma^-_{k+1,\varepsilon}}$ and $L_{t\wedge\sigma^+_{M_t^{(\varepsilon)},\varepsilon}} = L_t$. Hence, we have that using a telescopic sum and the fact that $X^{(\varepsilon)}_{r-} \leq 0$ if and only if $r \in (\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon}]$ for some $k \geq 1$,

$$\begin{aligned} X^+_{t\wedge\sigma^+_{M_t^{(\varepsilon)},\varepsilon}} - X^+_{\sigma^-_{M_t^{(\varepsilon)},\varepsilon}} + \sum_{k=1}^{M_t^{(\varepsilon)}-1} X^+_{\sigma^+_{k,\varepsilon}} - X^+_{\sigma^-_{k,\varepsilon}} \\ &= \int_{(0,t]} \mathbb{I}_{\{X^{(\varepsilon)}_{s-}\leq 0\}} \mathbb{I}_{\{X_{s-}>0\}} \mathrm{d}X_s \\ &\quad + \int_{(0,t]} \int_{(-\infty,0)} (X_{s-}+y)^- \mathbb{I}_{\{X^{(\varepsilon)}_{s-}\leq 0\}} \mathbb{I}_{\{X_{s-}>0\}} N(\mathrm{d}s \times \mathrm{d}y) + \frac{1}{2} L_t. \end{aligned}$$

Thus, using the fact that $X_{\sigma^-_{k,\varepsilon}} \leq 0$ and $X_{\sigma^+_{k,\varepsilon}} = \varepsilon$ for all $k \geq 1$, we obtain

$$\begin{aligned} X^+_{t\wedge\sigma^+_{M_t^{(\varepsilon)},\varepsilon}} + \varepsilon(M_t^{(\varepsilon)} - 1) \\ &= \int_{(0,t]} \mathbb{I}_{\{X^{(\varepsilon)}_{s-}\leq 0\}} \mathbb{I}_{\{X_{s-}>0\}} \mathrm{d}X_s \\ &\quad + \int_{(0,t]} \int_{(-\infty,0)} (X_{s-}+y)^- \mathbb{I}_{\{X^{(\varepsilon)}_{s-}\leq 0\}} \mathbb{I}_{\{X_{s-}>0\}} N(\mathrm{d}s \times \mathrm{d}y) + \frac{1}{2} L_t. \end{aligned}$$

Note that $0 \leq X^+_{t\wedge\sigma^+_{M_t^{(\varepsilon)},\varepsilon}} \leq \varepsilon$ and then $\lim_{\varepsilon\downarrow 0} X^+_{t\wedge\sigma^+_{M_t^{(\varepsilon)},\varepsilon}} = 0$. Moreover, from the dominated convergence theorem for stochastic integrals (see for example Theorem 32 Chapter IV of

Protter (2005)), we have that the first term in the right hand side of the equation above converges to 0 uniformly on compacts in probability, i.e. for all $t > 0$,

$$\sup_{0\leq s\leq t}\left|\int_{(0,s]}\mathbb{I}_{\{X^{(\varepsilon)}_{r-}\leq 0\}}\mathbb{I}_{\{X_{r-}>0\}}\mathrm{d}X_r\right|$$

converges to 0 in probability. Note that for all $s \geq 0$, $(X_{s-}+y)^{-}\mathbb{I}_{\{X^{(\varepsilon)}_{s-}\leq 0\}}\mathbb{I}_{\{X_{s-}>0\}} \leq (X_{s-}+y)^{-}\mathbb{I}_{\{X_{s-}>0\}}$ and

$$\int_{(0,t]}\int_{(-\infty,0)}(X_{s-}+y)^{-}\mathbb{I}_{\{X_{s-}>0\}}N(\mathrm{d}s\times\mathrm{d}y) < \infty.$$

Then, by the dominated convergence theorem

$$\lim_{\varepsilon\downarrow 0}\int_{(0,t]}\int_{(-\infty,0)}(X_{s-}+y)^{-}\mathbb{I}_{\{X^{(\varepsilon)}_{s-}\leq 0\}}\mathbb{I}_{\{X_{s-}>0\}}N(\mathrm{d}s\times\mathrm{d}y) = 0.$$

Hence, we have that $\varepsilon M^{(\varepsilon)}_t$ converges to $L_t/2$ in probability when $\varepsilon \downarrow 0$. We know that there exists a subsequence $\{\varepsilon_n, n \geq 1\}$ converging to 0 such that $\lim_{n\to\infty}\varepsilon_n M^{(\varepsilon_n)}_t = L_t/2$ a.s. From the fact that $M^{(\varepsilon)}_t$ increases when ε decreases, we have that for each $\varepsilon \in [\varepsilon_{n+1}, \varepsilon_n]$

$$\varepsilon_{n+1}M^{(\varepsilon_n)}_t \leq \varepsilon M^{(\varepsilon)}_t \leq \varepsilon_n M^{(\varepsilon_{n+1})}_t$$

and we conclude that $\lim_{\varepsilon\downarrow 0}\varepsilon M^{(\varepsilon)}_t = L_t/2$ a.s. as claimed. □

In the next Lemma we calculate explicitly the probability mass function of the random variable $M_{\mathbf{e}_p}$.

Lemma 3.3.3. *Let $\mathbf{e}_p$ an independent exponential random variable with parameter $p \geq 0$. For all $\varepsilon > 0$ we have that the probability mass function of the random variable $M^{(\varepsilon)}_{\mathbf{e}_p}$ is given by*

$$\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} = n) = \begin{cases} 1 - \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon-x)} & n = 1 \\ \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon-x)}[\mathcal{I}^{(p,\Phi(p))}(\varepsilon)]^{n-2}[1-\mathcal{I}^{(p,\Phi(p))}(\varepsilon)] & n \geq 2 \end{cases} \tag{3.13}$$

for all $x < \varepsilon$.

Proof. We calculate the probability of the event $\{M^{(\varepsilon)}_{\mathbf{e}_p} \geq n\}$ for $n \geq 2$ which happens if and only if $\{\sigma^-_{n,\varepsilon} < \mathbf{e}_p\}$. First, for any $x < \varepsilon$ we calculate

$$\begin{aligned}
\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} \geq 2) &= \mathbb{P}_x(\sigma^-_{2,\varepsilon} < \mathbf{e}_p) \\
&= \mathbb{E}_x(\mathbb{P}_x(\sigma^-_{2,\varepsilon} < \mathbf{e}_p, \sigma^+_{1,\varepsilon} < \mathbf{e}_p | \mathcal{F}_{\sigma^+_{1,\varepsilon}})) \\
&= \mathbb{E}_\varepsilon(e^{-p\tau^-_0}\mathbb{I}_{\{\tau^-_0<\infty\}})\mathbb{E}_x(e^{-p\tau^+_\varepsilon}\mathbb{I}_{\{\tau^+_\varepsilon<\infty\}}) \\
&= \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon-x)},
\end{aligned}$$

where the second last equality follows from the strong Markov property and the lack of memory property of the exponential distribution, the last by equations (1.3) and (1.11). Next assume that $n \geq 3$, we have that for any $x < \varepsilon$

$$\begin{aligned}
\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} \geq n) &= \mathbb{P}_x(\sigma^-_{n,\varepsilon} < \mathbf{e}_p) \\
&= \mathbb{E}_x(\mathbb{P}_x(\sigma^-_{n,\varepsilon} < \mathbf{e}_p, \sigma^+_{n-1,\varepsilon} < \mathbf{e}_p | \mathcal{F}_{\sigma^+_{n-1,\varepsilon}})) \\
&= \mathbb{E}_\varepsilon(e^{-p\tau^-_0}\mathbb{I}_{\{\tau^-_0<\infty\}})\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}) \\
&= \mathcal{I}^{(p,0)}(\varepsilon)\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}),
\end{aligned} \tag{3.14}$$

where the third equality follows from the strong Markov property and the lack of memory property of the exponential distribution and the last equality by equation (1.11). Applying the strong Markov property at the stopping time $\sigma^-_{n-1,\varepsilon}$ we get

$$\begin{aligned}
\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} \geq n) &= \mathcal{I}^{(p,0)}(\varepsilon)\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}) \\
&= \mathcal{I}^{(p,0)}(\varepsilon)\mathbb{E}_x(e^{-p\sigma^-_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^-_{n-1,\varepsilon}<\infty\}}\mathbb{E}_{X_{\sigma^-_{n-1,\varepsilon}}}(e^{-p\tau^+_\varepsilon}\mathbb{I}_{\{\tau^+_0<\infty\}})) \\
&= \mathcal{I}^{(p,0)}e^{-\Phi(p)\varepsilon}(\varepsilon)\mathbb{E}_x(e^{-p\sigma^-_{n-1,\varepsilon}+\Phi(p)X_{\sigma^-_{n-1,\varepsilon}}}\mathbb{I}_{\{\sigma^-_{n-1,\varepsilon}<\infty\}}),
\end{aligned}$$

where the last equality follows from equation (1.3). We apply the strong Markov property at $\sigma^+_{n-2,\varepsilon}$ and we use the fact that for all $k \geq 2$, $X_{\sigma^+_{k,\varepsilon}} = \varepsilon$ on the event $\{0 < \sigma^+_{k,\varepsilon} < \infty\}$ to deduce for all $n \geq 3$ that

$$\begin{aligned}\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} \geq n) &= \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)\varepsilon}\mathbb{E}_\varepsilon(e^{-p\tau_0^- + \Phi(p)X_{\tau_0^-}}\mathbb{I}_{\{\tau_0^- < \infty\}})\mathbb{E}_x(e^{-p\sigma^+_{n-2,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-2,\varepsilon} < \infty\}}) \\ &= \mathcal{I}^{(p,\Phi(p))}(\varepsilon)\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} \geq n-1),\end{aligned}$$

where last equality follows from equations (1.11) and (3.14). Then by an induction argument we get that for all $n \geq 2$ and $x < \varepsilon$

$$\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} \geq n) = \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon - x)}[\mathcal{I}^{(p,\Phi(p))}(\varepsilon)]^{n-2}. \tag{3.15}$$

□

Remark 3.3.4. *For all $\varepsilon > 0$ we can describe the paths of the process $\{g_{\varepsilon,t}, t \geq 0\}$ in terms of the stopping times $\{(\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon}), k \geq 1\}$. When $X^{(\varepsilon)}_t \leq 0$ we have that $\sigma^-_{k,\varepsilon} \leq t < \sigma^+_{k,\varepsilon}$ for some $k \geq 1$ and then $g_{\varepsilon,t} = t$. Similarly, when $X^{(\varepsilon)}_t > 0$ there exists $k \geq 1$ such that $\sigma^+_{k,\varepsilon} \leq t < \sigma^-_{k+1,\varepsilon}$ and hence $g_{\varepsilon,t} = \sigma^+_{k,\varepsilon}$. The reader can refer to Figure 3.1 for a graphical representation of this fact.*

3.3.2 Proof of Theorem 3.2.3

Suppose that $X_t > 0$ and choose $\varepsilon < X_t$. For any function $F \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R})$ we have that

$$\begin{aligned}F(g_{\varepsilon,t}, t, X^{(\varepsilon)}_t) &= F(g_{\varepsilon,0}, 0, X^{(\varepsilon)}_0) + \sum_{k=1}^{M^{(\varepsilon)}_t}[F(g_{\varepsilon,(\sigma^+_{k,\varepsilon})-}, \sigma^+_{k,\varepsilon}, X^{(\varepsilon)}_{(\sigma^+_{k,\varepsilon})-}) - F(g_{\varepsilon,\sigma^-_{k,\varepsilon}}, \sigma^-_{k,\varepsilon}, X^{(\varepsilon)}_{\sigma^-_{k,\varepsilon}})] \\ &+ \sum_{k=1}^{M^{(\varepsilon)}_t}[F(g_{\varepsilon,\sigma^+_{k,\varepsilon}}, \sigma^+_{k,\varepsilon}, X^{(\varepsilon)}_{\sigma^+_{k,\varepsilon}}) - F(g_{\varepsilon,(\sigma^+_{k,\varepsilon})-}, \sigma^+_{k,\varepsilon}, X^{(\varepsilon)}_{(\sigma^+_{k,\varepsilon})-})] \\ &+ \sum_{k=1}^{M^{(\varepsilon)}_t - 1}[F(g_{\varepsilon,(\sigma^-_{k+1,\varepsilon})-}, \sigma^-_{k+1,\varepsilon}, X^{(\varepsilon)}_{(\sigma^-_{k+1,\varepsilon})-}) - F(g_{\varepsilon,\sigma^+_{k,\varepsilon}}, \sigma^+_{k,\varepsilon}, X^{(\varepsilon)}_{\sigma^+_{k,\varepsilon}})] \\ &+ \sum_{k=1}^{M^{(\varepsilon)}_t - 1}[F(g_{\varepsilon,\sigma^-_{k+1,\varepsilon}}, \sigma^-_{k+1,\varepsilon}, X^{(\varepsilon)}_{\sigma^-_{k+1,\varepsilon}}) - F(g_{\varepsilon,(\sigma^-_{k+1,\varepsilon})-}, \sigma^-_{k+1,\varepsilon}, X^{(\varepsilon)}_{(\sigma^-_{k+1,\varepsilon})-})] \\ &+ [F(g_{\varepsilon,t}, t, X^{(\varepsilon)}_t) - F(g_{\varepsilon,\sigma^+_{M^{(\varepsilon)}_t,\varepsilon}}, \sigma^+_{M^{(\varepsilon)}_t,\varepsilon}, X^{(\varepsilon)}_{\sigma^+_{M^{(\varepsilon)}_t,\varepsilon}})].\end{aligned}$$

Note that $g_{\varepsilon,(\sigma^-_{k+1,\varepsilon})^-} = \sigma^+_{k,\varepsilon}$, $g_{\varepsilon,\sigma^-_{k,\varepsilon}} = \sigma^-_{k,\varepsilon}$ and $g_{\varepsilon,(\sigma^+_{k,\varepsilon})^-} = \sigma^+_{k,\varepsilon}$ for all $k \geq 1$. Thus,

$$\begin{aligned} F(g_{\varepsilon,t}, t, X_t^{(\varepsilon)}) = F(g_{\varepsilon,0}, 0, X_0^{(\varepsilon)}) &+ \sum_{k=1}^{M_t^{(\varepsilon)}} [F(\sigma^+_{k,\varepsilon}, \sigma^+_{k,\varepsilon}, X_{\sigma^+_{k,\varepsilon}} - \varepsilon) - F(\sigma^-_{k,\varepsilon}, \sigma^-_{k,\varepsilon}, X_{\sigma^-_{k,\varepsilon}} - \varepsilon)] \\ &+ \sum_{k=1}^{M_t^{(\varepsilon)}} [F(\sigma^+_{k,\varepsilon}, \sigma^+_{k,\varepsilon}, X_{\sigma^+_{k,\varepsilon}}) - F(\sigma^+_{k,\varepsilon}, \sigma^+_{k,\varepsilon}, X_{\sigma^+_{k,\varepsilon}} - \varepsilon)] \\ &+ \sum_{k=1}^{M_t^{(\varepsilon)}-1} [F(\sigma^+_{k,\varepsilon}, \sigma^-_{k+1,\varepsilon}, X_{(\sigma^-_{k+1,\varepsilon})^-}) - F(\sigma^+_{k,\varepsilon}, \sigma^+_{k,\varepsilon}, X_{\sigma^+_{k,\varepsilon}})] \\ &+ \sum_{k=1}^{M_t^{(\varepsilon)}-1} [F(\sigma^-_{k+1,\varepsilon}, \sigma^-_{k+1,\varepsilon}, X_{\sigma^-_{k+1,\varepsilon}} - \varepsilon) - F(\sigma^+_{k,\varepsilon}, \sigma^-_{k+1,\varepsilon}, X_{(\sigma^-_{k+1,\varepsilon})^-})] \\ &+ [F(\sigma^+_{M_t^{(\varepsilon)},\varepsilon}, t, X_t) - F(\sigma^+_{M_t^{(\varepsilon)},\varepsilon}, \sigma^+_{M_t^{(\varepsilon)},\varepsilon}, X_{\sigma^+_{M_t^{(\varepsilon)},\varepsilon}})], \end{aligned}$$

where we also used that $X_s^{(\varepsilon)} = X_s$ when $s \in [\sigma^+_{k,\varepsilon}, \sigma^-_{k+1,\varepsilon})$ and $X_s^{(\varepsilon)} = X_s - \varepsilon$ when $s \in [\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon})$ for all $k \geq 1$. Applying Itô formula on intervals of the form $(\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon}]$ for $k \geq 1$

we have that

$$
\begin{aligned}
&\sum_{k=1}^{M_t^{(\varepsilon)}}[F(\sigma_{k,\varepsilon}^+,\sigma_{k,\varepsilon}^+,X_{\sigma_{k,\varepsilon}^+}-\varepsilon)-F(\sigma_{k,\varepsilon}^-,\sigma_{k,\varepsilon}^-,X_{\sigma_{k,\varepsilon}^-}-\varepsilon)]\\
&=\sum_{k=1}^{M_t^{(\varepsilon)}}\left[\int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+}\frac{\partial}{\partial\gamma}F(s,s,X_s-\varepsilon)\mathrm{d}s+\int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+}\frac{\partial}{\partial t}F(s,s,X_s-\varepsilon)\mathrm{d}s\right]\\
&\quad+\sum_{k=1}^{M_t^{(\varepsilon)}}\left[\frac{1}{2}\sigma^2\int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+}\frac{\partial^2}{\partial x^2}F(s,s,X_s-\varepsilon)\mathrm{d}s+\int_{(\sigma_{k,\varepsilon}^-,\sigma_{k,\varepsilon}^+]}\frac{\partial}{\partial x}F(s,s,X_{s-}-\varepsilon)\mathrm{d}X_s\right]\\
&\quad+\sum_{k=1}^{M_t^{(\varepsilon)}}\int_{(\sigma_{k,\varepsilon}^-,\sigma_{k,\varepsilon}^+]}\int_{(-\infty,0)}N(\mathrm{d}s\times\mathrm{d}y)\\
&\qquad\qquad\times\left(F(s,s,X_{s-}+y-\varepsilon)-F(s,s,X_{s-}-\varepsilon)-y\frac{\partial}{\partial x}F(s,s,X_{s-}-\varepsilon)\right)\\
&=\int_0^t\frac{\partial}{\partial\gamma}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}\le 0\}}\mathrm{d}s+\int_0^t\frac{\partial}{\partial t}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}\le 0\}}\mathrm{d}s\\
&\quad+\frac{1}{2}\sigma^2\int_0^t\frac{\partial^2}{\partial x^2}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}\le 0\}}\mathrm{d}s+\int_{[0,t]}\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}\le 0\}}\mathrm{d}X_s\\
&\quad-\int_{[0,t]}y\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\mathbb{I}_{\{X_{s-}^{(\varepsilon)}>0\}}\mathbb{I}_{\{X_{s-}^{(\varepsilon)}+y<0\}}N(\mathrm{d}s\times\mathrm{d}y)\\
&\quad+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{X_{s-}^{(\varepsilon)}\le 0\}}N(\mathrm{d}s\times\mathrm{d}y)\\
&\qquad\qquad\times\left(F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)}+y)-F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})-y\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\right),
\end{aligned}
$$

where the last equality follows since $X_s^{(\varepsilon)}\le 0$ if and only if $s\in[\sigma_{k,\varepsilon}^-,\sigma_{k,\varepsilon}^+)$ for some $k\ge 1$ (and hence $g_{\varepsilon,s}=s$), X has a jump at time s on the event $\{X_{s-}^{(\varepsilon)}>0\}\cap\{X_s^{(\varepsilon)}<0\}$ and there are no jumps at time $\sigma_{k,\varepsilon}^+$ for all $k\ge 1$. Similarly, applying Itô formula on intervals of the form $(\sigma_{k,\varepsilon}^+,\sigma_{k+1}^-)$ for $k\ge 1$, there are no jumps at time $\sigma_{k,\varepsilon}^+$ for all $k\ge 1$ and the fact that $X_s^{(\varepsilon)}>0$ if and only if $s\in[\sigma_{k,\varepsilon}^+,\sigma_{k+1,\varepsilon}^-)$ for some $k\ge 1$ (and hence $g_{\varepsilon,s}=g_{\varepsilon,s-}=\sigma_{k,\varepsilon}^+$) we have that

$$\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma_{k,\varepsilon}^+,\sigma_{k+1,\varepsilon}^-,X_{(\sigma_{k+1,\varepsilon}^-)^-})-F(\sigma_{k,\varepsilon}^+,\sigma_{k,\varepsilon}^+,X_{\sigma_{k,\varepsilon}^+})]$$

$$+[F(\sigma^+_{M_t^{(\varepsilon)},\varepsilon},t,X_t)-F(\sigma^+_{M_t^{(\varepsilon)},\varepsilon},\sigma^+_{M_t^{(\varepsilon)},\varepsilon},X_{\sigma^+_{M_t^{(\varepsilon)},\varepsilon}})]$$

$$=\int_0^t\frac{\partial}{\partial t}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}>0\}}\mathrm{d}s$$

$$+\frac{1}{2}\sigma^2\int_0^t\frac{\partial^2}{\partial x^2}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}>0\}}\mathrm{d}s+\int_{[0,t]}\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}>0\}}\mathrm{d}X_s$$

$$+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{X_s^{(\varepsilon)}>0\}}N(\mathrm{d}s\times\mathrm{d}y)$$

$$\times\left(F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)}+y)-F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})-y\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\right).$$

Hence, we obtain that

$$F(g_{\varepsilon,t},t,X_t^{(\varepsilon)})=F(g_{\varepsilon,0},0,X_0^{(\varepsilon)})+\int_0^t\frac{\partial}{\partial\gamma}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathbb{I}_{\{X_s^{(\varepsilon)}\leq 0\}}\mathrm{d}s+\int_0^t\frac{\partial}{\partial t}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathrm{d}s$$

$$+\frac{1}{2}\sigma^2\int_0^t\frac{\partial^2}{\partial x^2}F(g_{\varepsilon,s},s,X_s^{(\varepsilon)})\mathrm{d}s+\int_0^t\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\mathrm{d}X_s$$

$$-\int_{[0,t]}\int_{(-\infty,0)}y\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\mathbb{I}_{\{X_{s-}^{(\varepsilon)}>0\}}\mathbb{I}_{\{X_{s-}^{(\varepsilon)}+y<0\}}N(\mathrm{d}s\times\mathrm{d}y)$$

$$+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{X_s^{(\varepsilon)}>0\}}N(\mathrm{d}s\times\mathrm{d}y)$$

$$\times\left(F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)}+y)-F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})-y\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\right)$$

$$+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{X_{s-}^{(\varepsilon)}\leq 0\}}N(\mathrm{d}s\times\mathrm{d}y)$$

$$\times\left(F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)}+y)-F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})-y\frac{\partial}{\partial x}F(g_{\varepsilon,s-},s,X_{s-}^{(\varepsilon)})\right)$$

$$+\sum_{k=1}^{M_t^{(\varepsilon)}}[F(\sigma_{k,\varepsilon}^+,\sigma_{k,\varepsilon}^+,X_{\sigma_{k,\varepsilon}^+})-F(\sigma_{k,\varepsilon}^+,\sigma_{k,\varepsilon}^+,X_{\sigma_{k,\varepsilon}^+}-\varepsilon)]$$

$$+\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma_{k+1,\varepsilon}^-,\sigma_{k+1,\varepsilon}^-,X_{\sigma_{k+1,\varepsilon}^-}-\varepsilon)-F(\sigma_{k,\varepsilon}^+,\sigma_{k+1,\varepsilon}^-,X_{(\sigma_{k+1,\varepsilon}^-)^-})].$$

Since $X_{\sigma_{k,\varepsilon}^+}=\varepsilon$ and that X can cross below 0 either by creeping or by a jump we have that the last two terms in the expression above become

$$\sum_{k=1}^{M_t^{(\varepsilon)}}[F(\sigma^+_{k,\varepsilon},\sigma^+_{k,\varepsilon},\varepsilon)-F(\sigma^+_{k,\varepsilon},\sigma^+_{k,\varepsilon},0)]$$

$$+\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},X_{\sigma^-_{k+1,\varepsilon}}-\varepsilon)-F(\sigma^+_{k,\varepsilon},\sigma^-_{k+1,\varepsilon},X_{(\sigma^-_{k+1,\varepsilon})^-})]$$

$$=\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},\varepsilon)-F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},0)]+F(\sigma^+_{1,\varepsilon},\sigma^+_{1,\varepsilon},\varepsilon)-F(\sigma^+_{1,\varepsilon},\sigma^+_{1,\varepsilon},0)$$

$$+\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},-\varepsilon)-F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},0)]\mathbb{I}_{\{X_{\sigma^-_{k+1,\varepsilon}}=0\}}$$

$$+\int_{[0,t]}\int_{(-\infty,0)}[F(s,s,X_{s-}+y-\varepsilon)-F(g_{\varepsilon,s-},s,X_{s-})]\mathbb{I}_{\{X^{(\varepsilon)}_{s-}>0\}}\mathbb{I}_{\{X^{(\varepsilon)}_{s-}+y\le 0\}}N(\mathrm{d}s\times\mathrm{d}y),$$

where we used the fact that when $\sigma>0$, $\lim_{h\downarrow 0}F(\gamma,t,h)=F(t,t,0)$ for all $0\le\gamma\le t$ by assumption, F is continuous and that $X_{(\sigma^-_{k+1,\varepsilon})^-}=0$ on the event of creeping. Without loss of generality assume that $\varepsilon<1$. By the mean value theorem we have that, for each $k\ge 1$, there exist $c_{1,k}\in(0,\varepsilon)$ and $c_{2,k}\in(-\varepsilon,0)$ such that

$$\Bigg|\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},\varepsilon)-F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},0)]$$

$$+\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},-\varepsilon)-F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},0)]\mathbb{I}_{\{X_{\sigma^-_{k+1,\varepsilon}}=0\}}\Bigg|$$

$$\le\sum_{k=1}^{M_t^{(\varepsilon)}-1}\left|\frac{\partial}{\partial x}F(\sigma^+_{k,\varepsilon},\sigma^+_{k,\varepsilon},c_{1,k})-\frac{\partial}{\partial x}F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},c_{2,k})\mathbb{I}_{\{X_{\sigma^-_{k+1,\varepsilon}}=0\}}\right|\varepsilon$$

$$\le 2K_t\varepsilon(M_t^{(\varepsilon)}-1),$$

where we used the fact that F is at least $C^{1,1,1}(E_g)$ and then $(s,x)\mapsto|\frac{\partial}{\partial x}F(s,s,x)|$ is bounded in the set $[0,t]\times[-1,1]$ by a constant, namely $K_t>0$. Moreover, we know that $\varepsilon M_t^{(\varepsilon)}\to L_t/2$ a.s. when $\varepsilon\downarrow 0$ (see Lemma 3.3.2). Hence, using the dominated convergence and the mean value theorem we deduce that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\Bigg|\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},\varepsilon)-F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},0)]\\
&\quad+\sum_{k=1}^{M_t^{(\varepsilon)}-1}[F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},-\varepsilon)-F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},0)]\mathbb{I}_{\{X_{\sigma^-_{k+1,\varepsilon}}=0\}}\Bigg|\\
&\le\sum_{k=1}^{\infty}\lim_{\varepsilon\downarrow 0}\left|F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},\varepsilon)-F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},0)]\right|\mathbb{I}_{\{k<M_t^{(\varepsilon)}\}}\\
&\quad+\sum_{k=1}^{\infty}\lim_{\varepsilon\downarrow 0}\left|[F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},-\varepsilon)-F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},0)]\right|\mathbb{I}_{\{X_{\sigma^-_{k+1,\varepsilon}}=0\}}\mathbb{I}_{\{k<M_t^{(\varepsilon)}\}}\\
&\le\sum_{k=1}^{\infty}\lim_{\varepsilon\downarrow 0}\varepsilon\left[\left|\frac{\partial}{\partial x}F(\sigma^+_{k+1,\varepsilon},\sigma^+_{k+1,\varepsilon},c_{1,k})\right|+\left|\frac{\partial}{\partial x}F(\sigma^-_{k+1,\varepsilon},\sigma^-_{k+1,\varepsilon},c_{2,k})\right|\right]\\
&\le\sum_{k=1}^{\infty}\lim_{\varepsilon\downarrow 0}\varepsilon 2K_t\\
&=0,
\end{aligned}
$$

almost surely. Therefore, by the dominated convergence theorem for stochastic integrals, we deduce that

$$
\begin{aligned}
F(g_t,t,X_t)&=F(g_0,0,X_0)+\int_0^t\frac{\partial}{\partial\gamma}F(g_s,s,X_s)\mathbb{I}_{\{X_s\le 0\}}\mathrm{d}s+\int_0^t\frac{\partial}{\partial t}F(g_s,s,X_s)\mathrm{d}s\\
&\quad+\int_0^t\frac{\partial}{\partial x}F(g_{s-},s,X_{s-})\mathrm{d}X_s+\frac{1}{2}\sigma^2\int_0^t\frac{\partial^2}{\partial x^2}F(g_s,s,X_s)\mathrm{d}s\\
&\quad+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{X_s>0\}}N(\mathrm{d}s\times\mathrm{d}y)\\
&\qquad\qquad\times\left(F(g_{s-},s,X_{s^-}+y)-F(g_{s-},s,X_{s^-})-y\frac{\partial}{\partial x}F(g_{s-},s,X_{s^-})\right)\\
&\quad+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{X_{s-}\le 0\}}N(\mathrm{d}s\times\mathrm{d}y)\\
&\qquad\qquad\times\left(F(g_{s-},s,X_{s-}+y)-F(g_{s-},s,X_{s-})-y\frac{\partial}{\partial x}F(g_{s-},s,X_{s-})\right)\\
&\quad+\int_{[0,t]}\int_{(-\infty,0)}\mathbb{I}_{\{0<X_{s-}<-y\}}N(\mathrm{d}s\times\mathrm{d}y)\\
&\qquad\qquad\times\left(F(s,s,X_{s-}+y)-F(g_{s-},s,X_{s-})-y\frac{\partial}{\partial x}F(g_{s-},s,X_{s-})\right).
\end{aligned}\tag{3.16}
$$

From the fact that g_t is continuous in the set $\{t \geq 0 : X_t > 0 \text{ or } X_{t-} \leq 0\}$ we obtain the desired result. The case when $X_t \leq 0$ is similar and proof is omitted.

3.3.3 Proof of Theorem 3.2.6

Note that, since $|K(U_s, X_s)| \leq C(U_s, X_s)$ for all $s \geq 0$ and $\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} C(U_r, X_r + y)\mathrm{d}r\right) < \infty$ for all $(u,x) \in E$ and $y \in \mathbb{R}$, we have that K^+ and K^- are finite. Moreover, since $u \mapsto C(u,x)$ is monotone for all $x \in \mathbb{R}$ and non-negative we have that for all $r \geq 0$ and $\varepsilon > 0$,

$$\begin{aligned}
|K(U_{\varepsilon,r}, X_r^{(\varepsilon)})| &\leq C(U_{\varepsilon,r}, X_r^{(\varepsilon)}) \\
&\leq C(U_r, X_r) + C(U_r, X_r - \varepsilon) + C(U_r^{(\varepsilon)}, X_r) + C(U_r^{(\varepsilon)}, X_r - \varepsilon),
\end{aligned}$$

where $U_r^{(\varepsilon)} = r - g_t^{(\varepsilon)} = r - \sup\{0 \leq s \leq r : X_s \leq \varepsilon\}$. It follows from integrability of $e^{-qr}C(U_r, X_r + y)$ with respect to the product measure $\mathbb{P}_{u,x} \times \mathrm{d}r$ for all $x, y \in \mathbb{R}$, by dominated convergence theorem and left-continuity in each argument of K that for all $(u,x) \in E$,

$$\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} K(U_r, X_r)\mathrm{d}r\right) = \lim_{\varepsilon \downarrow 0} \mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} K(U_{\varepsilon,r}, X_r^{(\varepsilon)})\mathrm{d}r\right).$$

Then we calculate the right-hand side of the equation above. Fix $\varepsilon > 0$, using the fact that $\{M^{(\varepsilon)} = n\} = \{\sigma_{n,\varepsilon}^- < \infty\} \cap \{\sigma_{n+1,\varepsilon}^- = \infty\}$ for $n = 1, 2, \ldots,$ we have for any $x \leq 0$ that

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qr} K(U_{\varepsilon,r}, X_r^{(\varepsilon)})\mathrm{d}r\right) \\
&= \sum_{n=1}^\infty \mathbb{E}_x\left(\mathbb{I}_{\{M^{(\varepsilon)}=n\}} \int_0^\infty e^{-qr} K(U_{\varepsilon,r}, X_r^{(\varepsilon)})\mathrm{d}r\right) \\
&= \sum_{n=1}^\infty \sum_{k=1}^n \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_{n,\varepsilon}^- < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon}^- = \infty\}} \int_{\sigma_{k,\varepsilon}^-}^{\sigma_{k,\varepsilon}^+} e^{-qr} K(0, X_r^{(\varepsilon)})\mathrm{d}r\right) \\
&\quad + \sum_{n=2}^\infty \sum_{k=1}^{n-1} \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_{n,\varepsilon}^- < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon}^- = \infty\}} \int_{\sigma_{k,\varepsilon}^+}^{\sigma_{k+1,\varepsilon}^-} e^{-qr} K(r - \sigma_{k,\varepsilon}^+, X_r^{(\varepsilon)})\mathrm{d}r\right) \\
&\quad + \sum_{n=1}^\infty \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_{n,\varepsilon}^+ < \infty\}} \mathbb{I}_{\{\sigma_{n+1,\varepsilon}^- = \infty\}} \int_{\sigma_{n,\varepsilon}^+}^{\sigma_{n+1,\varepsilon}^-} e^{-qr} K(r - \sigma_{n,\varepsilon}^+, X_r^{(\varepsilon)})\mathrm{d}r\right),
\end{aligned}$$

where the last equality follows from the fact that $g_{\varepsilon,r} = r$ when $r \in [\sigma_{k,\varepsilon}^-, \sigma_{k,\varepsilon}^+]$ and $g_{\varepsilon,r} = \sigma_{k,\varepsilon}^+$ when $r \in [\sigma_{k,\varepsilon}^+, \sigma_{k+1,\varepsilon}^-)$ for some $k \geq 1$. We first analyse the first double sum on the right hand side of the expression above. Conditioning with respect to the filtration at the stopping time $\sigma^+_{k,\varepsilon}$, the strong Markov property and the fact that X creeps upwards we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}=\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\quad=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\mathbb{E}\left(\mathbb{I}_{\{\sigma^+_{k,\varepsilon}<\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\cdot\mathbb{P}(\sigma^-_{n,\varepsilon}<\infty,\sigma^-_{n+1,\varepsilon}=\infty|\mathcal{F}_{\sigma^+_{k,\varepsilon}})\right)\\
&\qquad+\sum_{n=1}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}=\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\quad=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{k,\varepsilon}<\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n-k+1)\\
&\qquad+\sum_{n=1}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^+_{n,\varepsilon}=\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\qquad+\sum_{n=1}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{n,\varepsilon}<\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\cdot\mathbb{P}(\sigma^-_{n+1,\varepsilon}=\infty|\sigma^+_{n,\varepsilon})\right)\\
&\quad=\sum_{n=1}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{1,\varepsilon}<\infty\}}\int_{0}^{\sigma^+_{1,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n)\\
&\qquad+\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{1,\varepsilon}=\infty\}}\int_{0}^{\sigma^+_{1,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\qquad+\sum_{n=3}^{\infty}\sum_{k=2}^{n-1}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{k,\varepsilon}<\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n-k+1)\\
&\qquad+\sum_{n=2}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^+_{n,\varepsilon}=\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\qquad+\sum_{n=2}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{n,\varepsilon}<\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=1)\\
&\quad=K^-(x-\varepsilon)+\sum_{n=3}^{\infty}\sum_{k=2}^{n-1}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{k,\varepsilon}<\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X_r-\varepsilon)\mathrm{d}r\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n-k+1)\\
&\qquad+\sum_{n=2}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^+_{n,\varepsilon}=\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X_r-\varepsilon)\mathrm{d}r\right)\\
&\qquad+\sum_{n=2}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{n,\varepsilon}<\infty\}}\int_{\sigma^-_{n,\varepsilon}}^{\sigma^+_{n,\varepsilon}} e^{-qr}K(0,X_r-\varepsilon)\mathrm{d}r\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=1),
\end{aligned}
$$

where the second equality follows from splitting the second summation on the cases where $\sigma^+_{n,\varepsilon}$ is finite and infinity; the first term in the last equality corresponds to the first excursion of $X^{(\varepsilon)}$ below zero (case $k=1$) and we also used the fact that $X^{(\varepsilon)}_r = X_r - \varepsilon$ for $r \in [\sigma^-_{k,\varepsilon}, \sigma^+_{k,\varepsilon}]$ for any $k \geq 1$. We define the auxiliary functions

$$K^-(x) := \mathbb{E}_x\left(\int_0^{\tau_0^+} e^{-qr} K(0, X_r)\mathrm{d}r\right),$$
$$K_1^-(x) := \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+<\infty\}}\int_0^{\tau_0^+} e^{-qr} K(0, X_r)\mathrm{d}r\right),$$
$$K_2^-(x) := \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+=\infty\}}\int_0^{\tau_0^+} e^{-qr} K(0, X_r)\mathrm{d}r\right)$$

for all $x \in \mathbb{R}$. Then we have that $K^-(x) = K_1^-(x) + K_2^-(x)$ for all $x \in \mathbb{R}$. Conditioning again with respect to the filtration at time $\sigma^-_{k,\varepsilon}$ (resp. $\sigma^-_{n,\varepsilon}$) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}=\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&= K^-(x-\varepsilon) + \sum_{n=3}^{\infty}\sum_{k=2}^{n-1}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}}e^{-q\sigma^-_{k,\varepsilon}}K_1^-(X_{\sigma^-_{k,\varepsilon}}-\varepsilon)\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n-k+1)\\
&\quad + \sum_{n=2}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}e^{-q\sigma^-_{n,\varepsilon}}K_2^-(X_{\sigma^-_{k,\varepsilon}}-\varepsilon)\right)\\
&\quad + \sum_{n=2}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}e^{-q\sigma^-_{n,\varepsilon}}K_1^-(X_{\sigma^-_{k,\varepsilon}}-\varepsilon)\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=1)\\
&= K^-(x-\varepsilon) + \sum_{n=3}^{\infty}\sum_{k=2}^{n-1}\mathbb{E}_\varepsilon(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K_1^-(X_{\tau_0^-}-\varepsilon))\\
&\qquad\qquad\qquad\qquad \times \mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{k-1,\varepsilon}<\infty\}}e^{-q\sigma^+_{k-1,\varepsilon}}\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n-k+1)\\
&\quad + \sum_{n=2}^{\infty}\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K_2^-(X_{\tau_0^-}-\varepsilon)\right)\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}e^{-q\sigma^+_{n-1,\varepsilon}}\right)\\
&\quad + \sum_{n=2}^{\infty}\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K_1^-(X_{\tau_0^-}-\varepsilon)\right)\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}e^{-q\sigma^+_{n-1,\varepsilon}}\right)\mathbb{P}_\varepsilon(M^{(\varepsilon)}=1)\\
&= K^-(x-\varepsilon) + \sum_{n=3}^{\infty}\sum_{k=2}^{n-1}\mathbb{E}_\varepsilon(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K_1^-(X_{\tau_0^-}-\varepsilon))\frac{\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_q}\geq k)}{\mathcal{I}^{(q,0)}(\varepsilon)}\mathbb{P}_\varepsilon(M^{(\varepsilon)}=n-k+1)\\
&\quad + \sum_{n=2}^{\infty}\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K_2^-(X_{\tau_0^-}-\varepsilon)\right)\frac{\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_q}\geq n)}{\mathcal{I}^{(q,0)}(\varepsilon)}\\
&\quad + \sum_{n=2}^{\infty}\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K_1^-(X_{\tau_0^-}-\varepsilon)\right)\frac{\mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_q}\geq n)}{\mathcal{I}^{(q,0)}(\varepsilon)}\mathbb{P}_\varepsilon(M^{(\varepsilon)}=1),
\end{aligned}$$

where the second equality follows from conditioning with respect to time $\sigma^+_{k-1,\varepsilon}$ (resp. $\sigma^+_{n-1,\varepsilon}$) and the Markov property of X and the last from equation (3.14). From Lemma 3.3.3 and solving the corresponding series we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}=\infty\}}\int_{\sigma^-_{k,\varepsilon}}^{\sigma^+_{k,\varepsilon}} e^{-qr}K(0,X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\qquad = K^-(x-\varepsilon)+\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K^-(X_{\tau_0^-}-\varepsilon)\right)\frac{e^{-\Phi(q)(\varepsilon-x)}}{1-\mathcal{I}^{(q,\Phi(q))}(\varepsilon)}.
\end{aligned}$$

Using similar arguments we have, from the strong Markov property, the fact that X creeps upwards, equation (1.11) and Lemma 3.3.3, that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}=\infty\}}\int_{\sigma^+_{k,\varepsilon}}^{\sigma^-_{k+1,\varepsilon}} e^{-qr}K(r-\sigma^+_{k,\varepsilon},X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\qquad +\sum_{n=1}^{\infty}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^+_{n,\varepsilon}<\infty\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}=\infty\}}\int_{\sigma^+_{n,\varepsilon}}^{\sigma^-_{n+1,\varepsilon}} e^{-qr}K(r-\sigma^+_{n,\varepsilon},X^{(\varepsilon)}_r)\mathrm{d}r\right)\\
&\quad = K^+(0,\varepsilon)\frac{e^{-\Phi(q)(\varepsilon-x)}}{1-\mathcal{I}^{(q,\Phi(q))}(\varepsilon)}.
\end{aligned}$$

Therefore, by the dominated convergence theorem we have that for all $x\leq 0$

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qr}K(U_r,X_r)\mathrm{d}r\right)\\
&\quad=\lim_{\varepsilon\downarrow 0}\left\{K^-(x-\varepsilon)+\frac{e^{-\Phi(q)(\varepsilon-x)}}{1-\mathcal{I}^{(q,\Phi(q))}(\varepsilon)}\left[\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K^-(X_{\tau_0^-}-\varepsilon)\right)+K^+(0,\varepsilon)\right]\right\}.
\end{aligned}$$

When $u,x>0$ we have that

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr}K(U_r,X_r)\mathrm{d}r\right)\\
&\quad=\mathbb{E}_x\left(\int_0^{\tau_0^-} e^{-qr}K(u+r,X_r)\mathrm{d}r\right)+\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-<\infty\}}\int_{\tau_0^-}^\infty e^{-qr}K(U_r,X_r)\mathrm{d}r\right)\\
&\quad=K^+(u,x)+\lim_{\varepsilon\downarrow 0}\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K^-(X_{\tau_0^-}-\varepsilon)\right)\\
&\qquad+e^{\Phi(q)x}\mathcal{I}^{(q,\Phi(q))}(x)\lim_{\varepsilon\downarrow 0}\frac{e^{-\Phi(q)\varepsilon}}{1-\mathcal{I}^{(q,\Phi(q))}(\varepsilon)}\left[\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^-<\infty\}}e^{-q\tau_0^-}K^-(X_{\tau_0^-}-\varepsilon)\right)+K^+(0,\varepsilon)\right],
\end{aligned}\tag{3.17}$$

where the last equality follows from conditioning at time τ_0^- and the strong Markov property. Using Fubini's theorem and equation (1.20) we have that for all $x<0$,

$$
\begin{aligned}
K^-(x) &= \int_{(-\infty,0)} K(0,y) \int_0^\infty e^{-qr} \mathbb{P}_x(X_r \in \mathrm{d}y, r<\tau_0^+)\mathrm{d}r \\
&= \int_{-\infty}^0 K(0,y)[e^{\Phi(q)x} W^{(q)}(-y) - W^{(q)}(x-y)]\mathrm{d}y
\end{aligned}
$$

Then for any $x,\varepsilon>0$ we have that by Fubini's theorem and equation (1.11),

$$
\begin{aligned}
&\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-<\infty\}} e^{-q\tau_0^-} K^-(X_{\tau_0^-}-\varepsilon)\right) \\
&= e^{\Phi(q)(x-\varepsilon)} \mathcal{I}^{(q,\Phi(q))}(x) \int_{-\varepsilon}^0 K(0,y) W^{(q)}(-y)\mathrm{d}y \\
&\quad + \int_{-\infty}^{-\varepsilon} K(0,y) \\
&\qquad \times \left[e^{\Phi(q)(x-\varepsilon)} \mathcal{I}^{(q,\Phi(q))}(x) W^{(q)}(-y) - \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-<\infty\}} e^{-q\tau_0^-} W^{(q)}(X_{\tau_0^-}-\varepsilon-y)\right)\right] \mathrm{d}y.
\end{aligned}
$$

Let $x,\varepsilon>0$ and $y<-\varepsilon$, using a change of measure (see equation (1.22)) we obtain that

$$
\begin{aligned}
&\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-=\infty\}} e^{-q\tau_0^-} W^{(q)}(X_{\tau_0^-}-\varepsilon-y)\right) \\
&\qquad = e^{\Phi(q)(x-\varepsilon-y)} \mathbb{E}_x^{\Phi(q)}\left(\mathbb{I}_{\{\tau_0^-=\infty\}} e^{-\Phi(q)(X_{\tau_0^-}-\varepsilon-y)} W^{(q)}(X_{\tau_0^-}-\varepsilon-y)\right) \\
&\qquad = e^{\Phi(q)(x-\varepsilon-y)} \Phi'(q) \mathbb{P}_x^{\Phi(q)}(\tau_0^-=\infty) \\
&\qquad = e^{-\Phi(q)(\varepsilon+y)} W^{(q)}(x),
\end{aligned}
$$

where in the second equality we used that fact that X drifts to infinity under the measure $\mathbb{P}^{\Phi(q)}$ and the last follows from equation (1.11). Then from the fact that $e^{-q(t\wedge\tau_0^-)} W^{(q)}(X_{t\wedge\tau_0^-})$ is a martingale and since $\tau_{-\varepsilon-y}^- < \tau_0^-$ we have that

$$
\begin{aligned}
&\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-<\infty\}} e^{-q\tau_0^-} W^{(q)}(X_{\tau_0^-}-\varepsilon-y)\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_0^-} W^{(q)}(X_{\tau_0^-}-\varepsilon-y)\right) - \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^-=\infty\}} e^{-q\tau_0^-} W^{(q)}(X_{\tau_0^-}-\varepsilon-y)\right) \\
&= W^{(q)}(x-\varepsilon-y) - e^{-\Phi(q)(\varepsilon+y)} W^{(q)}(x).
\end{aligned}
$$

Hence we obtain that for any $x > 0$,

$$\begin{aligned}
&\mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^- < \infty\}} e^{-q\tau_0^-} K^-(X_{\tau_0^-} - \varepsilon)\right) \\
&= e^{\Phi(q)(x-\varepsilon)} \mathcal{I}^{(q,\Phi(q))}(x) \int_{-\varepsilon}^0 K(0,y) W^{(q)}(-y) \mathrm{d}y \\
&\quad + \int_{-\infty}^{-\varepsilon} K(0,y) \\
&\qquad\qquad \times \left[e^{\Phi(q)(x-\varepsilon)} \mathcal{I}^{(q,\Phi(q))}(x) W^{(q)}(-y) - W^{(q)}(x-\varepsilon-y) + e^{-\Phi(q)(\varepsilon+y)} W^{(q)}(x)\right] \mathrm{d}y.
\end{aligned}$$

Substituting the expression above in (3.17) and taking limits we obtain that for all $u, x > 0$,

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^\infty e^{-qr} K(U_r, X_r) \mathrm{d}r\right) \\
&\quad = K^+(u,x) + \int_{-\infty}^0 K(0,y) \left[e^{\Phi(q)(x-y)} \Phi'(q) - W^{(q)}(x-y)\right] \mathrm{d}y \\
&\qquad + e^{\Phi(q)x} \left[1 - \psi'(\Phi(q)+) e^{-\Phi(q)x} W^{(q)}(x)\right] \lim_{\varepsilon \downarrow 0} \frac{K^+(0,\varepsilon)}{\psi'(\Phi(q)+) W^{(q)}(\varepsilon)}.
\end{aligned}$$

The result follows from equation (1.11). The case when $x \leq 0$ is similar and the proof is omitted.

3.4 Conclusions

The focus of this chapter is on the study of the dynamics of the last zero before any fixed time which is denoted by g_t. We have derived some important identities of the three dimensional process (t, g_t, X_t) which will be useful in the next Chapter. For instance, we have computed a version of the Itô formula and its infinitesimal generator (see Theorem 3.2.3 and Corollary 3.2.5). They are particularly challenging to compute due to the infinite number of jumps of the process $\{g_t, t \geq 0\}$ in the infinite variation case. Indeed, the jumps of $t \mapsto g_t$ can occur when X crosses below the level zero which can happen either by a jump or by creeping (when $\sigma > 0$). The latter implies that there is an infinite number of jumps since the set of zeroes of X is perfect and nowhere dense (note that the limit condition on F on Theorem 3.2.3 makes these kind of jumps vanish).

The proof of Theorem 3.2.3 is based on a perturbation approach first presented by Dassios and Wu (2011) in which a perturbed version of X is proposed. An interesting feature of this process is that it visits the level zero a finite number of times and then its corresponding last zero process has only a finite number of jumps. Therefore, we can easily derive an Itô formula for the perturbed process and conclude via a limit argument.

Using the same approach, we have also derived a formula to calculate a functional that depends on the whole path of $U = \{U_t, t \geq 0\}$ (see Theorem 3.2.6), where U_t is the length of the current excursion above the level zero at time $t \geq 0$. This formula is then used to find the joint Laplace transform of $(U_{\mathbf{e}_p}, X_{\mathbf{e}_p})$, where $\mathbf{e}_p$ is an independent exponential time, and to compute the q-potential measure of (U, X) without killing. Moreover, the formula (3.6) is derived and we will learn how useful this is in the next chapter.

Chapter 4

L^p optimal prediction of the last zero of a spectrally negative Lévy process

Abstract

Given a spectrally negative Lévy process X drifting to infinity, we are interested in finding a stopping time which minimises the L^p distance ($p > 1$) with g, the last time X is negative. The solution is substantially more difficult compared to the $p = 1$ case for which it was shown in Baurdoux and Pedraza (2020b) that it is optimal to stop as soon as X exceeds a constant barrier. In the case of $p > 1$ treated here, we prove that solving this optimal prediction problem is equivalent to solving an optimal stopping problem in terms of a two-dimensional strong Markov process which incorporates the length of the current excursion away from 0. We show that an optimal stopping time is now given by the first time that X exceeds a non-increasing and non-negative curve depending on the length of the current excursion away from 0. We also show that the derivatives of the value function exist and are zero at the boundary.

4.1 Introduction

In recent years last passage times have received a considerable attention in the literature. For instance, in risk theory, the capital of an insurance company over time is studied. In the classical risk theory this is modelled by the Cramér–Lundberg process, defined as a compound Poisson process with drift. In more recent literature, this process has been replaced by a more general spectrally negative Lévy process. A key quantity of interest is the moment of ruin, which is classically defined as the first passage time below zero. Consider instead the situation where after the moment of ruin the company may have funds to endure a negative capital for some time. In that case, the last passage time below zero becomes an important quantity to be studied. In this framework, in Chiu and Yin (2005) the Laplace transform of the last passage time is derived.

Secondly, Paroissin and Rabehasaina (2013) consider spectrally positive Lévy processes as a degradation model. In a traditional setting, the failure time of a device is the first time the model hits a certain critical level b. However, another approach has been considered in the literature. For example, in Barker and Newby (2009) they considered the failure time as a last passage time. After the last passage time the process can never go back to this level meaning that the device is "beyond repair".

Thirdly, Egami and Kevkhishvili (2020) studied the last passage time of a general time-homogeneous transient diffusion with applications to credit risk management. They proposed the leverage process (the ratio of a company asset process over its debt) as a geometric Brownian motion over a process that grows at a risk free rate. It is shown there that the last passage time of the leverage ratio is equivalent to a last passage time of a Brownian motion with drift. In this setting the last passage represents the situation where the company cannot recover to normal business conditions after this time has occurred.

An important feature of last passage times is that they are random times which are not stopping times. In the recent literature the problem of finding a stopping time that approximates last passage times has been solved in various. There are for example various papers in which the approximation is in L_1 sense. To mention a few: du Toit et al. (2008) predicted the last zero of a Brownian motion with drift in a finite horizon setting; du Toit and Peskir (2008) predicted the time of the ultimate maximum at time $t = 1$ for a Brownian motion with drift is attained; Glover et al. (2013) predicted the time in which a transient difussion attains its ultimate minimum; Glover and Hulley (2014) predicted the last passage time of a level $z > 0$ for an arbitrary nonnegative time-homogeneous transient diffusion; Baurdoux and van Schaik (2014) predicted the time at which a Lévy process attains its ultimate supremum and Baurdoux et al. (2016) predicted when a positive self-similar Markov process attain its pathwise global supremum or infimum before hitting zero for the first time and Baurdoux and Pedraza (2020b) predicted the last zero of a spectrally negative Lévy process.

In this paper we consider the problem in an L^p sense, i.e. we are interested in solving

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}(|\tau - g|^p)$$

where $g = \sup\{t \geq 0 : X_t \leq 0\}$ is the last time a spectrally negative Lévy process drifting to infinity is below the level zero and $p > 1$. The case when $p = 1$ was solved in Baurdoux and Pedraza (2020b) for the spectrally negative case. An optimal stopping time in this case is the first time the process crosses above a fixed level $a^* > 0$ which is characterised in terms of the distribution function of the infimum of the process. The case $p > 1$ is substantially more complex, as an optimal stopping time now depends on the length of the current excursion above the level zero given by $U_t = t - \sup\{0 \leq s \leq t : X_s \leq 0\}$. Recall that U_t is the length of the current positive excursion which implies that $U_t = 0$ if and only if $X_t \leq 0$ and U_t has linear behaviour when $X_t > 0$. Moreover, as seen in Chapter 2, the process (U, X) is a Markov process taking values in $E = [(0, \infty) \times (0, \infty)] \cup [\{0\} \times (-\infty, 0)]$.

In this chapter, we show that an optimal stopping time (when $p > 1$) is given by $\tau_D = \inf\{t > 0 : (U_t, X_t) \in D\} = \inf\{t \geq 0 : X_t \geq b(U_t)\}$, where b is a non-negative, non-increasing and continuous curve. That is, is not optimal to stop when (U, X) is in the (continuation) set $C := E \setminus D$ whilst we should stop as soon as the process enters the (stopping) set D (see Figure 4.1). In other words, given the strong dependence of U on X, the latter has the following interpretation in terms of the sample paths of X: It is optimal to stop when X is sufficiently large or has stayed for a sufficiently large period of time above zero and we will never stop when X is in the negative half line (see Figure 4.1).

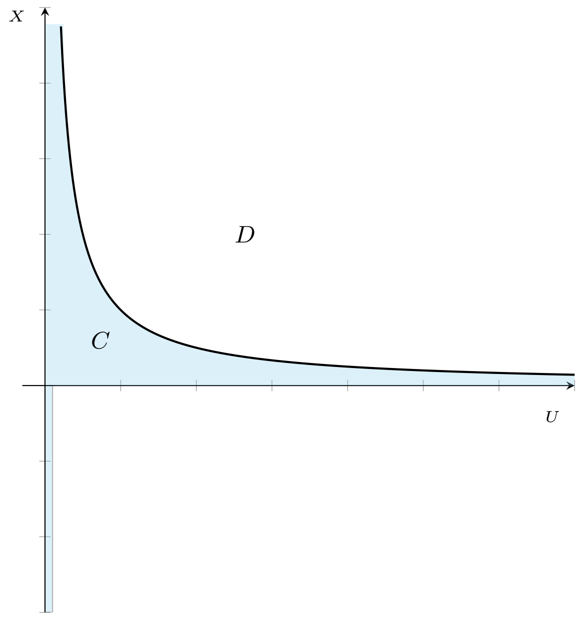

Figure 4.1: Stopping and continuation set in the (U, X) plane

In the figure below we include a plot of a sample path of X_t and $b(U_t)$, where we calculated numerically the function b for the Brownian motion with drift case (see Section 4.5.1 and Figure 4.3).

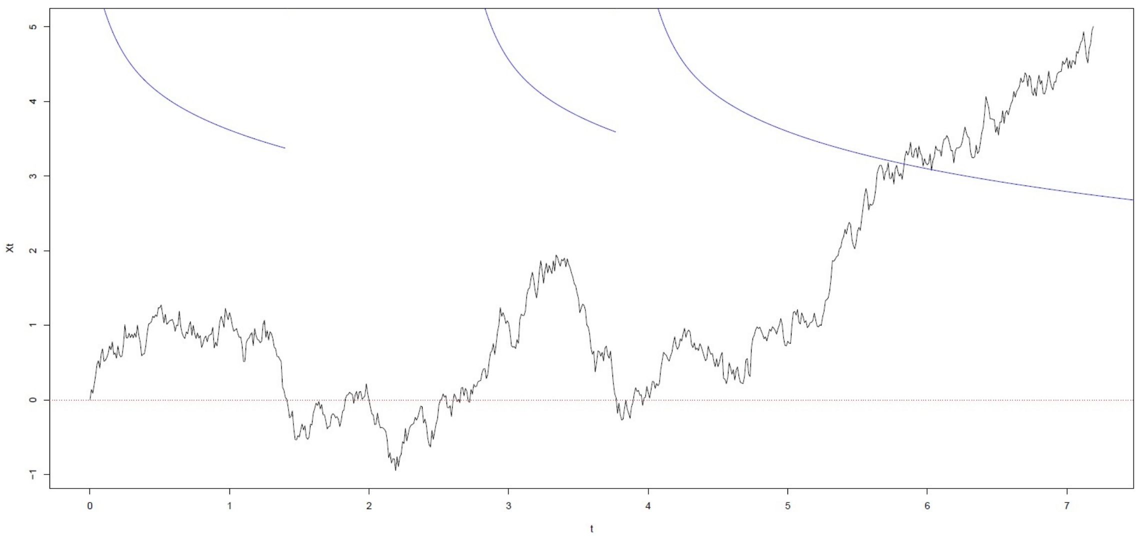

Figure 4.2: Black line: $t \mapsto X_t$; Blue line: $t \mapsto b(U_t)$.

This chapter is organised as follows. Section 4.2 gives a short overview of some results obtained in Chapter 3 but applied to the process $U_t = t - g_t$. We also state and derive some technical results related to the last zero of a Lévy processes drifting to infinite that will be useful in later sections. In Section 4.3 we formulate the optimal prediction problem and we show that it is equivalent to an optimal stopping problem which is solved in Section 4.4. In particular, we show an optimal stopping time is given by the first time X exceeds a boundary b which depends on the length of the current excursion above zero. We derive various properties of b. For example, in Lemma 4.4.20 we show that b is continuous and in Theorem 4.4.22 we show that the derivatives of the the value function exist and vanish at the boundary. The main result, Theorem 4.4.23, provides a characterisation of b and the value function of the optimal stopping problem. In Section 4.5 we provide two numerical examples: Firstly, when X is a Brownian motion with drift, and secondly when X is a Brownian motion with exponential jumps. Finally, some of the more technical proofs are deferred to the Appendix.

4.2 Length of the current positive excursion and the last zero

Throughout this chapter we use the notation and the preliminary results presented in Section 1.1. Let X be a spectrally negative Lévy process, that is, a Lévy process starting from 0 with only negative jumps and non-monotone paths, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see Definition 1.3.38 in Bichteler (2002)). We suppose that X has Lévy triplet (μ, σ, Π) where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (Lévy measure) concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)} (1 \wedge x^2) \Pi(\mathrm{d}x) < \infty$. We will assume that, when X is of finite variation, the Lévy measure Π has no atoms

For any $t \geq 0$ and $x \in \mathbb{R}$, we denote by $g_t^{(x)}$ the last time that the process is below x before time t, i.e.,

$$g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\}, \tag{4.1}$$

with the convention $\sup \emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. We also define, for each $t \geq 0$, $U_t^{(x)}$ as the time spent by X above the level zero before time t since the last visit to the interval $(-\infty, x]$, i.e.

$$U_t^{(x)} := t - g_t^{(x)} \qquad t \geq 0.$$

It turns out that in roder to solve our optimal prediction problem (see Section 4.3 below) the process $U_t = U_t^{(0)}$ plays a vital role so then we list a number of results from Chapter 3 applied to the process U.

Note the process U is not a Markov process. However, the strong Markov property holds for the two dimensional process $\{(U_t, X_t), t \geq 0\}$ (see Proposition 3.2.1) with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ and state space given by

$$E = \{(u, x) : u > 0 \text{ and } x > 0\} \cup \{(u, x) : u = 0 \text{ and } x \leq 0\}.$$

Then there exists a family of probability measures $\{\mathbb{P}_{u,x}, (u, x) \in E\}$ such that for any $A \in \mathbb{B}(E)$, Borel set of E, we have that $\mathbb{P}_{u,x}((U_{\tau+s}, X_{\tau+s}) \in A | \mathcal{F}_\tau) = \mathbb{P}_{U_\tau, X_\tau}((U_s, X_s) \in A)$. For each $(u, x) \in E$, $\mathbb{P}_{u,x}$ can be written in terms of $\mathbb{P}_x$ via

$$\mathbb{E}_{u,x}(h(U_s, X_s)) := \mathbb{E}_x(h(u + s, X_s)\mathbb{I}_{\{\sigma_0^- > s\}}) + \mathbb{E}_x(h(U_s, X_s)\mathbb{I}_{\{\sigma_0^- \leq s\}}), \tag{4.2}$$

for any positive measurable function h. Let F a $C^{1,2}(E)$ real-valued function. In addition, in the case that $\sigma > 0$ assume that $\lim_{h \downarrow 0} F(u, h) = F(0, 0)$ for all $u > 0$. Then we have the following version of Itô formula (see Theorem 3.2.3)

$$\begin{aligned} F(U_t, X_t) &= F(U_0, X_0) \\ &\quad + \int_0^t \frac{\partial}{\partial u} F(U_s, X_s)\mathbb{I}_{\{X_s > 0\}} \mathrm{d}s + \int_0^t \frac{\partial}{\partial x} F(U_{s-}, X_{s-}) \mathrm{d}X_s + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2}{\partial x^2} F(U_s, X_s) \mathrm{d}s \\ &\quad + \int_{[0,t]} \int_{(-\infty,0)} \left(F(U_s, X_{s-} + y) - F(U_{s-}, X_{s-}) - y \frac{\partial}{\partial x} F(U_{s-}, X_{s-}) \right) N(\mathrm{d}s \times \mathrm{d}y) \end{aligned} \tag{4.3}$$

Moreover, if f is a $C^{1,2}(E)$ bounded function with bounded derivatives, the infinitesimal generator $\mathcal{A}_{U,X}$ of the process (U,X) (see Corollary 3.2.5) is given by

$$\begin{aligned}\mathcal{A}_{U,X}(f)(u,x) = &\frac{\partial}{\partial u}f(u,x)\mathbb{I}_{\{x>0\}} - \mu\frac{\partial}{\partial x}f(u,x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}f(u,x)\\ &+ \int_{(-\infty,0)}\left(f(u,x+y) - f(u,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}f(u,x)\right)\mathbb{I}_{\{x+y>0\}}\Pi(\mathrm{d}y)\\ &+ \int_{(-\infty,0)}\left(f(0,x+y) - f(0,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}f(0,x)\right)\mathbb{I}_{\{x\leq 0\}}\Pi(\mathrm{d}y)\\ &+ \int_{(-\infty,0)}\left(f(0,x+y) - f(u,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}f(u,x)\right)\mathbb{I}_{\{0<x<-y\}}\Pi(\mathrm{d}y)\end{aligned}$$

Note that the equation above can be simplified by introducing the following notation. For any $(u,x)\in E$ we define,

$$\widetilde{f}(u,x) = \begin{cases} f(u,x) & u>0 \text{ and } x>0,\\ f(0,x) & u\geq 0 \text{ and } x\leq 0,\\ f(0,0) & u=0 \text{ and } x>0.\end{cases} \tag{4.4}$$

Note that $\tilde{f}$ extends the function f (defined only on E) to the set $\mathbb{R}_+\times\mathbb{R}$ and is such that $\partial f/\partial u\widetilde{f}(u,x) = 0$ when $x\leq 0$. Then the generator $\mathcal{A}_{(U,X)}$ applied to f can be written as

$$\begin{aligned}\mathcal{A}_{U,X}(f)(u,x) &= \frac{\partial}{\partial u}\widetilde{f}(u,x) - \mu\frac{\partial}{\partial x}\widetilde{f}(u,x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\widetilde{f}(u,x)\\ &\quad + \int_{(-\infty,0)}\left(\widetilde{f}(u,x+y) - \widetilde{f}(u,x) - y\mathbb{I}_{\{y>-1\}}\frac{\partial}{\partial x}\widetilde{f}(u,x)\right)\Pi(\mathrm{d}y)\\ &= \frac{\partial}{\partial u}\widetilde{f}(u,x) + \mathcal{A}_X(\widetilde{f}),\end{aligned} \tag{4.5}$$

where $\mathcal{A}_X$ is the infinitesimal generator of the process X. Note that the representation above tells us that the process (U,X) behaves (locally) as the process (t,X) when X is in the set $(0,\infty)$ and as $(0,X)$ when X is in the set $(-\infty,0)$.

We conclude this section by collecting some additional results about the last passage time

$$g = g_\infty = \sup\{t\geq 0 : X_t\leq 0\}. \tag{4.6}$$

The Laplace transform of g was found in Chiu and Yin (2005) as

$$\mathbb{E}_x(e^{-qg}) = e^{\Phi(q)x}\Phi'(q)\psi'(0+) + \psi'(0+)(W(x) - W^{(q)}(x)), \qquad q \geq 0. \tag{4.7}$$

The distribution function of g under $\mathbb{P}_x$ is found by observing that

$$\begin{aligned}\mathbb{P}_x(g \leq \gamma) &= \mathbb{P}_x(X_{u+\gamma} > 0 \text{ for all } u \in (0,\infty)) \\ &= \mathbb{E}_x(\mathbb{P}_x(X_{u+\gamma} > 0 \text{ for all } u \in (0,\infty)|\mathcal{F}_\gamma)) \\ &= \mathbb{E}_x(\mathbb{P}_{X_\gamma}(\tau_0^- = \infty)) \\ &= \mathbb{E}_x(\psi'(0+)W(X_\gamma)), \end{aligned} \tag{4.8}$$

where we used the tower property of conditional expectation in the second equality, the Markov property of Lévy processes in the third equality and equation (1.10) in the last. Note that the law of g under $\mathbb{P}_x$ may have an atom at zero given by

$$\mathbb{P}_x(g = 0) = \mathbb{P}_x(\tau_0^- = \infty) = \psi'(0+)W(x).$$

For our optimal prediction problem we require the p-th moment of g to be finite.The following result is from Doney and Maller (2004) (see Theorem 1, Theorem 4, Theorem 5 and Remark (ii)).

Lemma 4.2.1. *Let X be a spectrally negative Lévy process drifting to infinity. Then for a fixed $p > 0$ the following are equivalent.*

1. $\mathbb{E}_x(g^p) < \infty$ *for some (hence every)* $x \leq 0$.
2. $\int_{(-\infty,-1)} |x|^{1+p}\Pi(\mathrm{d}x) < \infty$.
3. $\mathbb{E}((-\underline{X}_\infty)^p) < \infty$.
4. $\mathbb{E}_x((\tau_0^+)^{p+1}) < \infty$ *for some (hence every)* $x \leq 0$.
5. $\mathbb{E}_x((\tau_0^-)^p\mathbb{I}_{\{\tau_0^- < \infty\}}) < \infty$ *for some (hence every)* $x \geq 0$.

The next lemma states that when τ_0^+ has finite p-th moment under $\mathbb{P}_x$, then the function $\mathbb{E}_x((\tau_0^+)^p)$ has a polynomial bound in x. It will be of use later to deduce a lower bound for our optimal prediction problem.

Lemma 4.2.2. *Let $p > 0$ and suppose $\mathbb{E}_x((\tau_0^+)^{p+1}) < \infty$ for some $x \le 0$. Then for each $0 \le r \le p$ there exist non-negative constants A_r and C_r such that*

$$\mathbb{E}_x((\tau_0^+)^r) \le A_r + C_r|x|^r \qquad \textit{and} \qquad \mathbb{E}_x(g^r) \le 2^r[\mathbb{E}(g^r) + A_r] + 2^r C_r|x|^r, \qquad x \le 0.$$

Here $\lfloor p \rfloor$ denotes the integer part of p.

Proof. From equation (1.3) we know that

$$F(\theta, x) := \mathbb{E}_x(e^{-\theta\tau_0^+}) = e^{\Phi(\theta)x}, \qquad x \le 0.$$

Then using Faà di Bruno's formula (see for example Spindler (2005)) we have that for any $n \ge 1$,

$$\frac{\partial^n}{\partial\theta^n}F(\theta,x) = \sum_{k=1}^{n} e^{\Phi(\theta)x}x^k \sum_{\substack{k_1+\cdots+k_n=k,\\ k_1+\cdots+nk_n=n}} \frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{\Phi'(\theta)}{1!}\right)^{k_1}\left(\frac{\Phi''(\theta)}{2!}\right)^{k_2}\cdots\left(\frac{\Phi^{(n)}(\theta)}{n!}\right)^{k_n}.$$

Then evaluating at zero the above equation, using $\Phi(0) = 0$ and the fact that $\Phi^{(i)}(0) < \infty$ for $i = 1, \ldots, \lfloor p \rfloor + 1$, we can find constants $A_r, C_r \ge 0$ such that $\mathbb{E}_x((\tau_0^+)^r) \le A_r + C_r|x|^r$ for any $r \in \{1, \ldots, \lfloor p \rfloor + 1\}$. For any non integer $r < \lfloor p \rfloor + 1$ we can use Hölder's inequality to obtain

$$\mathbb{E}_x((\tau_0^+)^r) \le [\mathbb{E}_x((\tau_0^+)^{\lfloor r \rfloor+1})]^{\frac{r}{\lfloor r \rfloor+1}} \le (A_{\lfloor r \rfloor+1} + C_{\lfloor r \rfloor+1}|x|^{\lfloor r \rfloor+1})^{\frac{r}{\lfloor r \rfloor+1}}.$$

The result follows from the inequality $(a+b)^q \le 2^q(a^q + b^q)$ which is true for any $q > 0$ and $a, b > 0$. Now we show that the second inequality holds. From the strong Markov property we get that for any $x < 0$

$$\mathbb{E}_x(g^r) \le 2^r\mathbb{E}(g^r) + 2^r\mathbb{E}_x((\tau_0^+)^r) \le 2^r[\mathbb{E}(g^r) + A_r] + 2^r C_r|x|^r.$$

□

In the next lemma we give some properties of the function $x \mapsto \mathbb{E}_x(g^p)$.

Lemma 4.2.3. *Let $p > 0$ and assume that $\int_{(-\infty,-1)} |x|^{p+1}\Pi(dx) < \infty$. Then $x \mapsto \mathbb{E}_x(g^p)$ is a non-increasing, non-negative and continuous function. Moreover,*

$$\lim_{x\to-\infty} \mathbb{E}_x(g^p) = \infty \qquad \textit{and} \qquad \lim_{x\to\infty} \mathbb{E}_x(g^p) = 0.$$

Proof. It follows from the definition of g that $x \mapsto \mathbb{E}_x(g^p) = \mathbb{E}(g^{(-x)})$ is non-negative and non-increasing. In order to check continuity notice that by integration by parts we get

$$\begin{aligned}\mathbb{E}_x(g^p) &= p\int_0^\infty s^{p-1}\mathbb{P}_x(g > s)\mathrm{d}s \\ &= p\int_0^\infty s^{p-1}\mathbb{E}_x(1 - \psi'(0+)W(X_s))\mathrm{d}s\end{aligned}$$

where the last equality follows from (4.8). Take $x \in \mathbb{R}$ and $\delta \in \mathbb{R}$. Then using the equation above we have that

$$|\mathbb{E}_x(g^p) - \mathbb{E}_{x+\delta}(g^p)| \leq p\psi'(0+)\mathbb{E}\left(\int_0^\infty s^{p-1}|W(X_s + x + \delta) - W(X_s + x)|\mathrm{d}s\right). \tag{4.9}$$

First, suppose that X is of infinite variation and thus W is continuous on $\mathbb{R}$. From the fact that X drifts to ∞ we know that $W(\infty) = 1/\psi'(0+)$ and therefore it follows that $s^{p-1}(1 - \psi'(0+)W(X_s))$ is integrable with respect to the product measure $\mathbb{P}_x \times \lambda([0,\infty))$, where λ denotes Lebesgue measure. We can now invoke the dominated convergence theorem to deduce that $x \mapsto \mathbb{E}_x(g^p)$ is continuous.

Next, in the case that X is of finite variation we have that W has a discontinuity at zero. However, the set $\{s \geq 0 : X_s = x\}$ is almost surely countable and thus has Lebesgue measure zero. We can again use the dominated convergence theorem to conclude that continuity also holds in this case.

We prove now the asymptotic behaviour of $\mathbb{E}_x(g^p)$. Note that when x tends to $-\infty$ the random variable $g^{(-x)} \to \infty$. Then using Fatou's lemma

$$\liminf_{x\to-\infty} \mathbb{E}_x(g^p) \geq \mathbb{E}(\liminf_{x\to-\infty}(g^{(-x)})^p) = \infty$$

Therefore, $\lim_{x\to-\infty}\mathbb{E}_x(g^p)=\infty$. In the other hand, note that for $x>0$

$$\mathbb{P}_x(g^p=0)=\mathbb{P}_x(g=0)=\mathbb{P}_x(\tau_0^-=\infty)=\psi'(0+)W(x)\xrightarrow{x\to\infty}1. \tag{4.10}$$

Hence we deduce that the sequence $\{(g^{(-n)})^p\}_{n\geq 1}$ converges in probability to 0 (under the measure $\mathbb{P}$) when n tends to infinity. Moreover, since the sequence $\{\mathbb{E}((g^{-n})^p)\}_{n\geq 1}$ is a non-increasing sequence of positive numbers we get that

$$\sup_{n\geq 1}\mathbb{E}((g^{-n})^p)\leq \mathbb{E}(g^p)<\infty,$$

where the last inequality holds due to Lemma 4.2.1 and assumption. Then $\{(g^{(-n)})^p\}_{n\geq 1}$ is an uniformly integrable family of random variables. The latter together with the convergence in probability allows us to conclude that $\mathbb{E}_x(g^p)\to 0$ when $x\to\infty$ as claimed. □

We conclude this section with a technical result extracted from Baurdoux and van Schaik (2014) (see Lemma 5) related to optimal stopping that will be useful later.

Lemma 4.2.4. *Let X be any Lévy process drifting to $-\infty$. Denote $T_+(0)=\inf\{t\geq 0: X_t\geq 0\}$ Consider, for $a>0$ and $b<0$, the optimal stopping problem*

$$P(x)=\inf_{\tau\in\mathcal{T}}\mathbb{E}_x[a\tau+\mathbb{I}_{\{\tau\geq T_+(0)\}}b] \qquad \text{for } x\in\mathbb{R}.$$

Then there is an $x_0\in(-\infty,0)$ so that $P(x)=0$ for all $x\leq x_0$.

4.3 Optimal prediction problem

Denote by V_* the value of the optimal prediction problem, i.e.

$$V_*=\inf_{\tau\in\mathcal{T}}\mathbb{E}(|\tau-g|^p), \tag{4.11}$$

where $\mathcal{T}$ is the set of all stopping times with respect to $\mathbb{F}$, $p>1$ and g is the last zero of X given in (4.6). Since g is only $\mathcal{F}$ measurable standard techniques of optimal stopping times are not directly applicable. However, there is an equivalence between the optimal prediction problem (4.11) and an optimal stopping problem. The next lemma, inspired in the work of Urusov (2005), states such equivalence.

Lemma 4.3.1. *Let $p > 1$ and let X be a spectrally negative Lévy process drifting to infinity such that $\int_{(-\infty,-1)} |x|^{p+1}\Pi(\mathrm{d}x) < \infty$. Consider the optimal stopping problem*

$$V = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau G(s - g_s, X_s)\mathrm{d}s\right), \tag{4.12}$$

where the function G is given by

$$G(u, x) = u^{p-1}\psi'(0+)W(x) - \mathbb{E}_x(g^{p-1}),$$

for $u \geq 0$ and $x \in \mathbb{R}$. Then we have that $V_* = pV + \mathbb{E}(g^p)$ and a stopping time minimises (4.11) if and only if it minimises (4.12).

Proof. Let $\tau \in \mathcal{T}$. Then the following equality holds

$$|\tau - g|^p = \int_0^\tau \varrho(s - g)\mathrm{d}s + g^p, \tag{4.13}$$

where the function ϱ is defined by

$$\varrho(x) = p\left[\frac{(-x)^p}{x}\mathbb{I}_{\{x<0\}} + x^{p-1}\mathbb{I}_{\{x\geq 0\}}\right].$$

Taking expectations in equation (4.13) and then using Fubini's theorem and the tower property for conditional expectation we obtain

$$\begin{aligned}\mathbb{E}(|\tau-g|^p) &= \int_0^\infty \mathbb{E}\left(\varrho(s-g)\mathbb{I}_{\{s\le\tau\}}\mathrm{d}s\right)+\mathbb{E}(g^p)\\ &= \int_0^\infty \mathbb{E}\left[\mathbb{I}_{\{s\le\tau\}}\mathbb{E}\left(\varrho(s-g)|\mathcal{F}_s\right)\mathrm{d}s\right]+\mathbb{E}(g^p)\\ &= \mathbb{E}\left(\int_0^\tau \mathbb{E}\left(\varrho(s-g)|\mathcal{F}_s\right)\mathrm{d}s\right)+\mathbb{E}(g^p).\end{aligned}$$

To evaluate the conditional expectation inside the last integral, note that for all $t \geq 0$ we can write the the time g as

$$g = g_t \vee \sup\{s \in (t,\infty) : X_s \leq 0,\}$$

recalling that $g_t = g_t^{(0)}$ defined in (4.1). Hence, using the Markov property for Lévy processes and the fact that g_s is $\mathcal{F}_s$ measurable we have that

$$\begin{aligned}\mathbb{E}(\varrho(s-g)|\mathcal{F}_s) &= \mathbb{E}\left(\varrho\left(s-[g_s\vee\sup\{r\in(s,\infty):X_r\le 0\}]\right)|\mathcal{F}_s\right)\\ &= \varrho(s-g_s)\mathbb{E}(\mathbb{I}_{\{X_r>0 \text{ for all } r\in(s,\infty)\}}|\mathcal{F}_s)\\ &\qquad + \mathbb{E}(\varrho(s-\sup\{r\in(s,\infty):X_r\le 0\})\mathbb{I}_{\{X_r\le 0 \text{ for some } r\in(s,\infty)\}}|\mathcal{F}_s)\\ &= \varrho(s-g_s)\mathbb{P}_{X_s}(g=0)+\mathbb{E}_{X_s}(\varrho(-g)\mathbb{I}_{\{g>0\}})\\ &= p(s-g_s)^{p-1}\psi'(0+)W(X_s)-p\mathbb{E}_{X_s}(g^{p-1}).\end{aligned}$$

Then we have that

$$\mathbb{E}(|\tau-g|^p) = p\mathbb{E}\left(\int_0^\tau G(s-g_s,X_s)\mathrm{d}s\right)+\mathbb{E}(g^p)$$

□

Remark 4.3.2. *A close inspection of the proof of Lemma 4.3.1 tells us that the function ϱ corresponds to the right derivative of the function $f(x)=|x|^p$. Therefore, using similar arguments we can actually extend the result to any convex function $d:\mathbb{R}_+\times\mathbb{R}_+\mapsto\mathbb{R}_+$. That* *is, under the assumption that* $\mathbb{E}(d(0,g)) < \infty$*, the optimal prediction problem*

$$V_d = \inf_{\tau \in \mathcal{T}} \mathbb{E}(d(\tau, g))$$

is equivalent to the optimal stopping problem

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau G_d(g_s, s, X_s)\mathrm{d}s\right]$$

where $G_d(\gamma, t, x) = \varrho_d(s, \gamma)\psi'(0+)W(x) + \mathbb{E}_x(\varrho_d(s, g+s)\mathbb{I}_{\{g>0\}})$ *and* ϱ_d *is the right derivative with respect the first argument of* d*.*

4.4 Solution to the optimal stopping problem

In order to solve the optimal stopping problem (4.12) using the general theory of optimal stopping (see e.g. Peskir and Shiryaev (2006) or Section 1.2.3) we have to extend it to an optimal stopping problem driven by a strong Markov process. For every $(u, x) \in E$, we define the optimal stopping problem

$$V(u, x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{u,x}\left[\int_0^\tau G(U_s, X_s)\mathrm{d}s\right], \tag{4.14}$$

where the function G is given by $G(u, x) = u^{p-1}\psi'(0+)W(x) - \mathbb{E}_x(g^{p-1})$ for any $u \geq 0$ and $x \in \mathbb{R}$. Therefore we have that $V_* = pV(0,0) + \mathbb{E}(g^p)$. Note that using the definition of $\mathbb{E}_{u,x}$ we have that (4.14) takes the form

$$V(u, x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left(\int_0^\tau \left\{G(u+s, X_s)\mathbb{I}_{\{\sigma_0^- > s\}} + G(U_s, X_s)\mathbb{I}_{\{\sigma_0^- \leq s\}}\right\}\mathrm{d}s\right). \tag{4.15}$$

The optimal stopping problem (4.14) is given in terms of a function G which involves the function $x \mapsto \mathbb{E}_x(g^{p-1})$. Recall that for a fixed $p > 1$, the function G is given by $G(u, x) = u^{p-1}\psi'(0+)W(x) - \mathbb{E}_x(g^{p-1})$ for all $(u, x) \in E$. Then as a consequence of Lemma 4.2.3 we have the following behaviour. For all $x \in \mathbb{R}$, the function $u \mapsto G(u, x)$ is non-decreasing. In particular when $x < 0$, $u \mapsto G(u, x) = -\mathbb{E}_x(g^{p-1})$ is a strictly negative constant. For fixed $u \geq 0$, $x \mapsto G(u, x)$ is a non-decreasing right-continuous function which is continuous everywhere apart from possibly at $x = 0$ (since W is discontinuous at zero when

X is of finite variation) such that for all $u \geq 0$,

$$\lim_{x \to -\infty} G(u,x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} G(u,x) = u^{p-1} \geq 0,$$

where we used that $\lim_{x\to\infty} \mathbb{E}_x(g^{p-1}) = 0$ (see Lemma 4.2.3) and $\lim_{x\to\infty} \psi(0+)W(x) = \lim_{x\to\infty} \mathbb{P}_x(\tau_0^- = \infty) = \lim_{x\to\infty} \mathbb{P}(-\underline{X}_\infty < x) = 1$, where $\underline{X}_\infty = \inf_{t\geq 0} X_t$. Moreover, we have that $\lim_{u\to\infty} G(u,x) = \infty$ and $G(0,x) = -\mathbb{E}_x(g^{p-1}) < 0$ for all $x \geq 0$. We then define the function

$$h(u) = \inf\{x \in \mathbb{R} : G(u,x) \geq 0\}. \tag{4.16}$$

From the description of G above we have that h is a non-negative and non-increasing function such that $h(u) < \infty$ for all $u \in (0,\infty)$, $h(0) = \infty$ and $\lim_{u\to\infty} h(u) = 0$. Moreover, since W is strictly increasing on $(0,\infty)$, the function

$$T(x) = \frac{\mathbb{E}_x(g^{p-1})}{\psi'(0+)W(x)}$$

is continuous and strictly decreasing on $[0,\infty)$. Then there exists an inverse function T^{-1} which is continuous and strictly decreasing on $(0, u_h^*]$ with

$$u_h^* := \frac{\mathbb{E}(g^{p-1})}{\psi'(0+)W(0)}, \tag{4.17}$$

where we understand $1/0 = \infty$ when X is of infinite variation. Hence we can write

$$h(u) = \begin{cases} T^{-1}(u^{p-1}) & u < (u_h^*)^{\frac{1}{p-1}} \\ 0 & u \geq (u_h^*)^{\frac{1}{p-1}} \end{cases}.$$

Therefore, since $T^{-1}(u_h^*-) = 0$, we conclude that h is a continuous function on $[0,\infty)$. From the definition of h we clearly have that $G(u,x) \geq 0$ if and only if $x \geq h(u)$.

The latter facts about give us some intuition about the optimal stopping rule for the optimal stopping problem (4.14). Since we are dealing with a minimisation problem, before stopping we want the process (U, X) to be in the set in which G is negative as much as possible. Then the fact that $G(U_t, X_t)$ is strictly negative when $X_t < h(U_t)$ suggests that it is never optimal to stop on this region. When $X_t > h(U_t)$ we have that $G(U_t, X_t) \geq 0$ but with strictly positive probability (U, X) can enter the set in which G is negative. Moreover, $t \mapsto U_t$ is strictly increasing when X is in the positive half line and then $t \mapsto h(U_t)$ gets closer to zero when the current excursion away from $(-\infty, 0]$ is sufficiently large and then $G(U_t, X_t) \geq 0$ even when X_t is relatively close to zero. That implies that it is optimal to stop when the current excursion away from $(-\infty, 0]$ is large and X takes a sufficiently large values. That suggest the existence of a non-negative curve $b \geq h$ such that it is optimal to stop when X crosses above $b(U_t)$. We will formally show in the next Lemmas the existence of such boundary.

Note that if there exists a stopping time τ for which the expectation of the right hand side of (4.14) is minus infinity then V would also be minus infinity. The next Lemma provides the finiteness of a lower bound of V that will ensure that V only takes finite values, its proof is included in the Appendix.

Lemma 4.4.1. *Let $p > 1$ and X be a spectrally negative Lévy process drifting to infinity. Assume that $\int_{(-\infty,-1)} |x|^{p+1}\Pi(dx) < \infty$. Then*

$$0 \leq \mathbb{E}_x \left(\int_0^\infty \mathbb{E}_{X_s}(g^{p-1}) \mathrm{d}s \right) < \infty \qquad \textit{for all } x \in \mathbb{R}.$$

We now prove the finiteness of the function V.

Lemma 4.4.2. *Let $p > 1$. For every $(u, x) \in E$ we have that $V(u, x) \in (-\infty, 0]$. In particular $V(u, x) < 0$ for $(u, x) \in B := \{(u, x) \in E : x < h(u)\}$, where h is defined in (4.16).*

Proof. Taking the stopping time $\tau = 0$ we deduce that for all $(u, x) \in E$, $V(u, x) \leq 0$. In order to check that $V(u, x) > -\infty$ we use that $G(u, x) \geq -\mathbb{E}_x(g^{p-1})$ to get

$$V(u, x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_{u,x} \left[\int_0^\tau G(U_s, X_s) \mathrm{d}s \right] \geq - \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\int_0^\tau \mathbb{E}_{X_s}(g^{p-1}) \mathrm{d}s \right],$$

for all $(u,x) \in E$. Hence by Lemma 4.4.1 we have that

$$V(u,x) \geq -\mathbb{E}_x\left[\int_0^\infty \mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right] > -\infty, \qquad (u,x) \in E. \tag{4.18}$$

Now we prove that $V(u,x) < 0$ when $(u,x) \in B$. Since h is continuous we have that B is an open set. Morever, from the definition of h we have that if $(u,x) \in B$ then $G(u,x) < 0$. Take $(u,x) \in B$ and consider the stopping time

$$\tau_{B^*} := \inf\{t \geq 0 : (U_t, X_t) \in E \setminus B\}.$$

Note that under the measure $\mathbb{P}_{u,x}$, $\tau_{B^*} > 0$. Then for all $s < \tau_{B^*}$, $(U_s, X_s) \in B$ which implies that $G(U_s, X_s) < 0$. Hence, by the definition of V, we have that

$$V(u,x) \leq \mathbb{E}_{u,x}\left[\int_0^{\tau_{B*}} G(U_s, X_s)\mathrm{d}s\right] < 0.$$

□

Remark 4.4.3. *Note that we have that $h(0) = \infty$ which implies that $(0,0) \in B$ and then, from the Lemma above, $V(0,0) < 0$. Moreover, from Lemma 4.3.1 we have that $pV(0,0) + \mathbb{E}(g^{p-1}) = V_* \geq 0$ which implies that*

$$-\frac{\mathbb{E}(g^{p-1})}{p} \leq V(0,0) < 0.$$

Now we prove some basic properties of V.

Lemma 4.4.4. *Let $p > 1$. We have the following monotonicity property of V. For all $(u,x),(v,y) \in E$ such that $u \leq v$ and $x \leq y$ we have that $V(u,x) \leq V(v,y)$.*

Proof. From equation (4.15) we have that

$$\begin{aligned} V(u,x) &= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left(\int_0^\tau \left\{G(u+s, X_s)\mathbb{I}_{\{\sigma_0^- > s\}} + G(U_s, X_s)\mathbb{I}_{\{\sigma_0^- \leq s\}}\right\}\mathrm{d}s\right) \\ &= \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^\tau \left\{G(u+s, X_s + x)\mathbb{I}_{\{\sigma_{-x}^- > s\}} + G(U_s^{(-x)}, X_s + x)\mathbb{I}_{\{\sigma_{-x}^- \leq s\}}\right\}\mathrm{d}s\right), \end{aligned}$$

where $\sigma_{-x}^- = \inf\{t \geq 0 : X_t \leq -x\}$ and $U_s^{(-x)} = s - \sup\{t \geq 0 : X_t \leq -x\}$. Recall that for all $s \geq 0$, $x \mapsto U_s^{(-x)}$ and $x \mapsto \sigma_{-x}^-$ are non-decreasing and that the function G is non-decreasing in each argument. Define the function

$$G^*(u,x) := G(u+s, X_s + x)\mathbb{I}_{\{\sigma^-_{-x} > s\}} + G(U_s^{(-x)}, X_s + x)\mathbb{I}_{\{\sigma^-_{-x} \leq s\}}.$$

We show by cases that the function G^* is non-decreasing in each argument. Take $x \leq y$ and $0 \leq u \leq v$. First we suppose that $\omega \in \{\sigma^-_{-x} > s\} \subset \{\sigma^-_{-y} > s\}$. Since G is non-decreasing in each argument we then have

$$G^*(u,x)(\omega) = G(u+s, X_s(\omega)+x) \leq G(v+s, X_s(\omega)+y) = G^*(v,y)(\omega).$$

Similarly, if $\omega \in \{\sigma^-_{-x} \leq s\} \cap \{\sigma^-_{-y} \leq s\}$ we have that

$$G^*(u,x)(\omega) = G(U_s^{(-x)}(\omega), X_s(\omega)+x) \leq G(U_s^{(-y)}(\omega), X_s(\omega)+y) = G^*(v,y)(\omega).$$

Lastly, take $\omega \in \{\sigma^-_{-x} \leq s\} \cap \{\sigma^-_{-y} > s\}$. Then using the fact that $U_s^{(-x)} = s - g_s^{(-x)} \leq s \leq v+s$ and the monotonicity of G we get

$$G^*(u,x)(\omega) = G(U_s^{(-x)}(\omega), X_s(\omega)+x) \leq G(v+s, X_s(\omega)+y) = G^*(v,y)(\omega).$$

All this together implies that the function $G^*(u,x)$ is non-decreasing in each argument for all $u \geq 0$ and $x \in \mathbb{R}$, in particular for all $(u,x) \in E$ and hence the claim on V holds. □

In the next Lemma we give an expression for $V(0,x)$ when $x < 0$ in terms of $V(0,0)$ and we use it to give a lower bound for V.

Lemma 4.4.5. *Let $p > 1$. For any $x \leq 0$ we have that*

$$V(0,x) = \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s\right) + V(0,0) = -\int_0^{-x}\int_{[0,\infty)} \mathbb{E}_{-u-z}(g^{p-1})W(\mathrm{d}u)\mathrm{d}z + V(0,0). \tag{4.19}$$

Moreover, for all $(u,x) \in E$ we have that there exist non-negative constants A'_{p-1} and C'_{p-1} such that

$$V(u,x) \geq -A'_{p-1} - C'_{p-1}|x|^p + V(0,0). \tag{4.20}$$

Proof. Let $x < 0$ and take $\tau \in \mathcal{T}$. Then we have that

$$\begin{aligned} V(0,x) &\leq \mathbb{E}_x\left(\int_0^{\tau\circ\theta_{\tau_0^+}+\tau_0^+} G(0,X_s)\mathrm{d}s\right) \\ &= \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s + \int_0^{\tau\circ\theta_{\tau_0^+}} G(U_{s+\tau_0^+},X_{s+\tau_0^+})\mathrm{d}s\right) \\ &= \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s\right) + \mathbb{E}\left(\int_0^{\tau} G(U_s,X_s)\mathrm{d}s\right), \end{aligned}$$

where θ_t is the shift operator and the last equality follows from the strong Markov property at time τ_0^+. Taking infimum over all $\tau \in \mathcal{T}$ we get that,

$$V(0,x) \leq \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s\right) + V(0,0).$$

Similarly, by the strong Markov property we have that for any $\tau \in \mathcal{T}$ and for any $x < 0$,

$$\begin{aligned} \mathbb{E}_x\left(\int_0^{\tau} G(0,X_s)\mathrm{d}s\right) &= \mathbb{E}_x\left(\int_0^{\tau\wedge\tau_0^+} G(0,X_s)\mathrm{d}s + \mathbb{I}_{\{\tau_0^+<\tau\}}\int_{\tau_0^+}^{\tau} G(U_s,X_s)\mathrm{d}s\right) \\ &= \mathbb{E}_x\left(\int_0^{\tau\wedge\tau_0^+} G(0,X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+<\tau\}}\mathbb{E}\left(\int_0^{\tau} G(U_s,X_s)\mathrm{d}s\right)\right) \\ &\geq \mathbb{E}_x\left(\int_0^{\tau\wedge\tau_0^+} G(0,X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_0^+<\tau\}}V(0,0)\right) \\ &\geq \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s\right) + V(0,0), \end{aligned}$$

where the second last equality follows from the definition of V and the last follows since $G(0,x) \leq 0$ for all $x \leq 0$ and $V(0,0) \leq 0$ and hence the infimum is attained for any $\tau \geq \tau_0^+$. Taking infimum over all stopping times in the equation above we conclude that

$$V(0,x) = \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s\right) + V(0,0).$$

Using the fact that $G(0,x) = -\mathbb{E}_x(g^{p-1})$ for all $x<0$ and Fubini's theorem we get that

$$\begin{aligned} V(0,x) &= -\mathbb{E}_x\left(\int_0^{\tau_0^+} \mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right) + V(0,0) \\ &= -\int_{(-\infty,0)} \mathbb{E}_z(g^{p-1}) \int_0^\infty \mathbb{P}_x(X_s \in \mathrm{d}z, s<\tau_0^+)\mathrm{d}s + V(0,0) \end{aligned}$$

Using the 0-potential measure of X killed on exiting the interval $(-\infty,0]$ (see equation (1.20)) and Fubini's theorem we obtain that

$$\begin{aligned} V(0,x) &= -\int_0^\infty \mathbb{E}_{-z}(g^{p-1})[W(z)-W(x+z)]\mathrm{d}z + V(0,0) \\ &= -\int_0^\infty \mathbb{E}_{-z}(g^{p-1}) \int_{(x+z,z]} W(\mathrm{d}u)\mathrm{d}z + V(0,0) \\ &= -\int_{[0,\infty)} W(\mathrm{d}u) \int_u^{u-x} \mathbb{E}_{-z}(g^{p-1})\mathrm{d}z + V(0,0) \\ &= -\int_0^{-x} \int_{[0,\infty)} \mathbb{E}_{-u-z}(g^{p-1})W(\mathrm{d}u)\mathrm{d}z + V(0,0). \end{aligned}$$

From equation (4.19) and the fact that $x \mapsto \mathbb{E}_x(g^{p-1})$ is is non-increasing and bounded from above by a polynomial (see Lemmas 4.2.2 and 4.2.3) we have the inequalities for $x<0$,

$$\begin{aligned} V(0,x) &\geq x\int_{[0,\infty]} \mathbb{E}_{x-u}(g^{p-1})W(\mathrm{d}u) + V(0,0) \\ &\geq \frac{1}{\psi'(0+)}2^{p-1}[\mathbb{E}(g^{p-1}) + A_{p-1}]x + \frac{1}{\psi'(0+)}2^{p-1}C_{p-1}x\mathbb{E}(|x+\underline{X}_\infty|^{p-1}) + V(0,0) \\ &\geq \frac{1}{\psi'(0+)}2^{p-1}[\mathbb{E}(g^{p-1}) + A_{p-1} + 2^{p-1}C_{p-1}\mathbb{E}((-\underline{X}_\infty)^{p-1})]x \\ &\quad - \frac{1}{\psi'(0+)}2^{p-1}C_{p-1}|x|^p + V(0,0). \end{aligned}$$

Hence (4.20) follows for $x<0$. The general statement holds since V is non-decreasing in each argument. □

Define the set $D := \{(u,x) \in E : V(u,x) = 0\}$. From Lemma 4.4.2 we know that $V(u,x) < 0$ for all $(u,x) \in E$ such that $x < h(u)$. Hence if $(u,x) \in D$ we have that $x \geq h(u) \geq 0$. We then define the function $b : (0,\infty) \mapsto \mathbb{R}$ by

$$b(u) = \inf\{x > 0 : V(u,x) = 0\},$$

where $\inf \emptyset = \infty$ and $\inf(0,\infty) = 0$. Then it directly follows that $b(u) \geq h(u) \geq 0$ for all $u > 0$. Moreover, since $h(0) = \infty$ we have that $\lim_{u\downarrow 0} b(u) = \infty$. Furthermore, since V is monotone in each argument we deduce that $u \mapsto b(u)$ is non-increasing and $V(u,x) = 0$ for all $x > b(u)$. We then have the following Lemma.

Lemma 4.4.6. *The function $b : \mathbb{R}_+ \mapsto \mathbb{R}$ is non-increasing with $0 \leq h(u) \leq b(u)$. We have that $\lim_{u\downarrow 0} b(u) = \infty$ and $b(u) < \infty$ for all $u > 0$.*

Proof. We show that for each $u > 0$, $b(u) < \infty$. Fix $u > 0$ and take $x > y > 0$. By using the strong Markov property and the definition of V we have that

$$\begin{aligned}
V(u,x) &= \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^\tau G(U_s,X_s)\mathrm{d}s\right)\\
&= \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_y^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_y^-<\tau\}}\mathbb{E}_{u,x}\left(\int_{\sigma_y^-}^\tau G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma_y^-}\right)\right)\\
&= \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_y^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_y^-<\tau\}}\mathbb{E}_{U_{\sigma_y^-},X_{\sigma_y^-}}\left(\int_0^\tau G(U_s,X_s)\mathrm{d}s\right)\right)\\
&\geq \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_y^-} G(u+s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_y^-<\tau\}}V(U_{\sigma_y^-},X_{\sigma_y^-})\right)\\
&\geq \inf_{\tau\in\mathcal{T}} \mathbb{E}_{x}\left(\int_0^{\tau\wedge\sigma_y^-} G(u+s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_y^-<\tau\}}V(0,0) + \mathbb{I}_{\{\sigma_y^-<\tau, X_{\sigma_y^-}<0\}}V(0,X_{\sigma_y^-})\right),
\end{aligned}$$

where the last inequality follows since V is non-positive and non decreasing. By the compensation formula for Poisson random measures (see (1.25)) we have that for any stopping time τ (we assume without loss of generality that $\tau < \infty$ a.s.),

$$\begin{aligned}
&\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_y^-<\tau, X_{\sigma_y^-}<0\}}V(0,X_{\sigma_y^-})\right)\\
&\quad= \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)} V(0,X_{s-}+z)\mathbb{I}_{\{X_{s-}+z<0\}}\mathbb{I}_{\{s<\tau\wedge\sigma_y^-\}}N(\mathrm{d}s,\mathrm{d}z)\right)\\
&\quad= \mathbb{E}_x\left(\int_0^{\tau\wedge\sigma_y^-}\int_{(-\infty,0)} V(0,X_s+z)\mathbb{I}_{\{X_s+z<0\}}\Pi(\mathrm{d}z)\mathrm{d}s\right).
\end{aligned}$$

Hence, from the equation above, since G and V are non-decreasing in each argument, $V \leq 0$ and $X_s \geq y$ for all $s \geq \sigma_y^-$ we have that

$$0 \geq V(u,x) \geq \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left((\tau\wedge\sigma_y^-)\left[G(u,y) + \int_{(-\infty,-y)} V(0,z)\Pi(\mathrm{d}z)\right] + \mathbb{I}_{\{\sigma_y^-<\tau\}}V(0,0)\right).$$

Note that from equation (4.20) and Lemma 4.2.1 the integral with respect to $\Pi(\mathrm{d}z)$ above is finite so we can take y sufficiently large such that $a := G(u,y) + \int_{(-\infty,-y)} V(0,z)\Pi(\mathrm{d}z) \geq 0$. Then from Lemma 4.2.4 we have that (since $V(0,0) \leq 0$ and $-X$ drifts to $-\infty$) there exists a value $x_0(u) < 0$ such that the right hand side of the equation above vanishes for all $y - x \leq x_0(u)$. Hence, we have that $V(u,x) = 0$ for all $x \geq y - x_0(u)$ and then $b(u) < \infty$.

□

Let $(u,x) \in E$. We define, under the measure $\mathbb{P}_{u,x}$, the stopping times

$$\begin{aligned} \tau_D &= \inf\{t \geq 0 : (U_t, X_t) \in D\} = \inf\{t \geq 0 : X_t \geq b(U_t)\}, \\ \tau_b^{v,y} &= \inf\{t > 0 : X_t + y \geq b(v+t)\} \qquad v \geq 0 \text{ and } y \in \mathbb{R}, \end{aligned} \tag{4.21}$$

and for any $x \in \mathbb{R}$, under the measure $\mathbb{P}_x$, the stopping time

$$\tau_b^{g,y} = \inf\{t > 0 : X_t + y \geq b(U_t^{(-y)})\} \qquad y \in \mathbb{R}. \tag{4.22}$$

Note that for any $y \in \mathbb{R}$ and $v \geq 0$, the stopping time $\tau_b^{v,y}$ does not depend on the process U and hence for any measurable function f, we have that $\mathbb{E}_{u,x}(f(\tau_b^{v,y})) = \mathbb{E}_x(f(\tau_b^{v,y}))$. The following lemma allows to write the stopping time τ_D in terms of the measure $\mathbb{P}_x$.

Lemma 4.4.7. *For any $(u,x) \in E$ and any measurable function f we have that*

$$\mathbb{E}_{u,x}(f(\tau_D)) = \mathbb{E}_x(f(\tau_b^{u,0})\mathbb{I}_{\{\tau_b^{u,0} \leq \sigma_0^-\}}) + \mathbb{E}_x(f(\tau_b^{g,0})\mathbb{I}_{\{\tau_b^{u,0} > \sigma_0^-\}}).$$

Proof. We have that

$$\mathbb{E}_{u,x}(f(\tau_D)) = \mathbb{E}_{u,x}(f(\tau_D)\mathbb{I}_{\{\tau_b^{u,0} \leq \sigma_0^-\}}) + \mathbb{E}_{u,x}(f(\tau_D)\mathbb{I}_{\{\tau_b^{u,0} > \sigma_0^-\}}).$$

Recall that under the measure $\mathbb{P}_{u,x}$, for any $u, x > 0$, we have that if $s < \sigma_0^-$, then $U_s = u + s$. Hence, under the measure $\mathbb{P}_{u,x}$,

$$\begin{aligned} \tau_D &= \inf\{t \geq 0 : X_t \geq b(U_t)\} \\ &= \inf\{0 \leq t \leq \sigma_0^- : X_t \geq b(U_t)\} \wedge \inf\{t \geq \sigma_0^- : X_t \geq b(U_t)\} \\ &= \tau_b^{u,0} \wedge \inf\{t \geq \sigma_0^- : X_t \geq b(U_t)\}. \end{aligned}$$

That implies that

$$\mathbb{E}_{u,x}(f(\tau_D)\mathbb{I}_{\{\tau_b^{u,0}\leq\sigma_0^-\}}) = \mathbb{E}_x(f(\tau_b^{u,0})\mathbb{I}_{\{\tau_b^{u,0}\leq\sigma_0^-\}})$$

and

$$\begin{aligned}
\mathbb{E}_{u,x}(f(\tau_D)\mathbb{I}_{\{\tau_b^{u,0}>\sigma_0^-\}}) &= \mathbb{E}_{u,x}(f(\inf\{t\geq\sigma_0^- : X_t\geq b(U_t)\})\mathbb{I}_{\{\tau_b^{u,0}>\sigma_0^-\}}) \\
&= \mathbb{E}_x(f(\inf\{t\geq\sigma_0^- : X_t\geq b(U_t)\})\mathbb{I}_{\{\tau_b^{u,0}>\sigma_0^-\}}) \\
&= \mathbb{E}_x(f(\tau_b^{g,0})\mathbb{I}_{\{\tau_b^{u,0}>\sigma_0^-\}}),
\end{aligned}$$

where in the last equality we used that under the measure $\mathbb{P}_x$, $\tau_b^{u,0}\leq\tau_b^{g,0}$ and then $\{0\leq t\leq\sigma_0^- : X_t\geq b(U_t)\}=\emptyset$ on the event $\{\tau_b^{u,0}>\sigma_0^-\}$. The result holds. □

Now we introduce a series of technical lemmas in order to show that the stopping time τ_D is optimal for V, their proof can be found in the Appendix 4.6. We first show that τ_D has moments of order p.

Lemma 4.4.8. *Let $p>1$ and assume that $\int_{(-\infty,0)}|x|^{p+1}\Pi(dx)$. Then for all $(u,x)\in E$,*

$$\mathbb{E}_{u,x}((\tau_D)^p)<\infty.$$

The next lemma contains a technical result related to convergence involving the stopping time σ_x^-.

Lemma 4.4.9. *Let X be any spectrally negative. Then for any $x<0$ we have that*

$$\lim_{h\to 0}\sigma_{x+h}^- = \sigma_x^- \text{ a.s.} \qquad \text{and} \qquad \lim_{h\to 0}X_{\sigma_{x+h}^-} = X_{\sigma_x^-} \text{ a.s.}$$

We show that the dynamic programming principle is satisfied for the stopping time σ_0^-. That is, we give an alternative expression for V.

Lemma 4.4.10. *For all $(u,x)\in E$, we have that*

$$V(u,x)=\inf_{\tau\in\mathcal{T}'}\mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-}G(U_s,X_s)\mathrm{d}s+\mathbb{I}_{\{\sigma_0^-<\tau\}}V(0,X_{\sigma_0^-})\right),$$

where $\mathcal{T}'$ is the family of finite stopping times of X.

Now we are ready to show (using the general theory of optimal stopping) that τ_D is an optimal stopping time for (4.14) in terms of the set D.

Lemma 4.4.11. *For any $p > 1$ assume that $\int_{(-\infty,0)} |x|^{p+1}\Pi(dx)$. Then we have that an optimal stopping time for (4.14) is given by τ_D, the first entrance of (U, X) to the closed set D, i.e.*

$$\tau_D = \inf\{t \geq 0 : (U_t, X_t) \in D\}.$$

Then the function V takes the form

$$V(u, x) = \mathbb{E}_{u,x}\left(\int_0^{\tau_D} G(U_s, X_s)\mathrm{d}s\right), \qquad (u, x) \in E.$$

Proof. Note that it follows from Lemma 4.4.8 that $\mathbb{P}_{u,x}(\tau_D < \infty) = 1$ for all $(u, x) \in E$. Moreover, from the strong Markov property and the definition of V we obtain that

$$\begin{aligned} V(u, x) &= \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau} G(U_s, X_s)\mathrm{d}s\right) \\ &= \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\tau_D} G(U_s, X_s)\mathrm{d}s + \mathbb{I}_{\{\tau_D<\tau\}}\int_{\tau_D}^{\tau} G(U_s, X_s)\mathrm{d}s\right) \\ &\geq \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\tau_D} G(U_s, X_s)\mathrm{d}s + \mathbb{I}_{\{\tau_D<\tau\}}V(U_{\tau_D}, X_{\tau_D})\right) \\ &= \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\tau_D} G(U_s, X_s)\mathrm{d}s\right), \end{aligned}$$

where in the last equality we used that $V(u, x) = 0$ on D. On the other hand we have that

$$V(u, x) \leq \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\tau_D} G(U_s, X_s)\mathrm{d}s\right),$$

since the inequality follows since the infimum of the right hand side is taken over all the stopping times $\tau \leq \tau_D$. Hence, we conclude that for any $(u, x) \in E$ that

$$V(u, x) = \inf_{\tau\in\mathcal{T}} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\tau_D} G(U_s, X_s)\mathrm{d}s\right).$$

Since $W(x) \leq 1/\psi'(0+)$ for all $x \in \mathbb{R}$ we have that $|G(u, x)| \leq u^{p-1} + \mathbb{E}_x(g^{p-1})$. Then for any $(u,x)\in E$ we deduce that

$$\begin{aligned}
\mathbb{E}_{u,x}\left[\sup_{t\geq 0}\left|\int_0^{t\wedge\tau_D} G(U_s,X_s)\right|\mathrm{d}s\right] &\leq \mathbb{E}_{u,x}\left[\int_0^{\tau_D}[(U_s)^{p-1}+\mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right] \\
&\leq \mathbb{E}_{u,x}\left(\int_0^{\tau_D}(u+s)^{p-1}\mathrm{d}s\right)+\mathbb{E}_{u,x}\left(\int_0^{\tau_D}\mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right) \\
&\leq 2^{p-1}[u^{p-1}+\frac{1}{p}\mathbb{E}_{u,x}[(\tau_D)^p]]+\mathbb{E}_x\left(\int_0^{\infty}\mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right) \\
&< \infty,
\end{aligned}$$

where the last equality follows from Lemmas 4.4.1 and 4.4.8.

Next we show that the function V is upper semi-continuous. Note that from equation (4.19) we have that V is continuous (and hence upper semi-continuous) on $(-\infty,0]$. Now we show that V is upper semi-continuous on $(0,\infty)\times(0,\infty)$. From Lemma 4.4.10 we know that that for any $u>0$ and $x>0$,

$$V(u,x)=\inf_{\tau\in\mathcal{T}}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_{-x}^-}G(u+s,X_s+x)\mathrm{d}s+\mathbb{I}_{\{\sigma_{-x}^-<\tau\}}V(0,X_{\sigma_{-x}^-}+x)\right).$$

Take any stopping time τ and $u>0$ and $x>0$, then by Fatou's lemma we have that

$$\begin{aligned}
\limsup_{(v,y)\to(u,x)} V(v,y) &\leq \limsup_{(v,y)\to(u,x)}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_{-y}^-}G(v+s,X_s+y)\mathrm{d}s+\mathbb{I}_{\{\sigma_{-y}^-<\tau\}}V(0,X_{\sigma_{-y}^-}+y)\right) \\
&= \mathbb{E}\left(\int_0^{\tau}\limsup_{(v,y)\to(u,x)} G(v+s,X_s+y)\mathbb{I}_{\{s<\sigma_{-y}^-\}}\mathrm{d}s\right. \\
&\qquad\left.+\limsup_{(v,y)\to(u,x)}\mathbb{I}_{\{\sigma_{-y}^-<\tau\}}V(0,X_{\sigma_{-y}^-}+y)\right) \\
&\leq \mathbb{E}\left(\int_0^{\tau\wedge\sigma_{-x}^-}G(u+s,X_s+x)\mathrm{d}s+\mathbb{I}_{\{\sigma_{-x}^-<\tau\}}V(0,X_{\sigma_{-x}^-}+x)\right),
\end{aligned}$$

where in the last equality we used Lemma 4.4.9, the fact that the function $s\mapsto\mathbb{I}_{\{s<\tau\}}$ is a right continuous function, that $\mathbb{I}_{\{\sigma_x^-<\tau\}}\leq\mathbb{I}_{\{\sigma_x^-\leq\tau\}}$, V is non-positive and the continuity of $V(0,x)$ on the set $(-\infty,0]$. Since the above inequality holds for any stopping time, we have that $\limsup_{(v,y)\to(u,x)}V(v,y)\leq V(u,x)$. Note that if X is of infinite variation (and hence $\lim_{h\downarrow 0}\sigma_{-h}^-=\tau_0^-=0$ a.s.) the same method used above shows that $\limsup_{(v,y)\downarrow(0,0)}V(v,y)\leq V(0,0)$.

If X is of finite variation we have that, $\tau_0^- > 0$ and $\mathbb{P}(X_{\tau_0^-} = 0) = 0$ so we have (under $\mathbb{P}$) that for any $s < \tau_0^-$, $U_s = s$. Hence, using an identical argument as the one used in the proof of Lemma 4.4.10 we obtain that

$$V(0,0) = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left(\int_0^{\tau \wedge \tau_0^-} G(s, X_s)\mathrm{d}s + \mathbb{I}_{\{\tau_0^- < \tau\}} V(0, X_{\tau_0^-}) \right).$$

Then for any stopping time τ,

$$\begin{aligned}
\limsup_{(u,h)\downarrow(0,0)} V(u,h) &\leq \limsup_{(u,h)\downarrow(0,0)} \mathbb{E}\left(\int_0^{\tau \wedge \sigma_{-h}^-} G(u+s, X_s+h)\mathrm{d}s + \mathbb{I}_{\{\sigma_{-h}^- < \tau\}} V(0, X_{\sigma_{-h}^-} + h) \right) \\
&\leq \mathbb{E}\left(\int_0^{\tau \wedge \tau_0^-} G(s, X_s)\mathrm{d}s + \mathbb{I}_{\{\tau_0^- < \tau\}} V(0, X_{\tau_0^-}) \right).
\end{aligned}$$

Since the above equality is true for any stopping time τ we have that by the definition of infimum that $\limsup_{(u,h)\downarrow(0,0)} V(u,h) \leq V(0,0)$. Therefore the function V is upper semi-continuous (hence D is a closed set) and from general results of optimal stopping (see Corollary 2.9 in Peskir and Shiryaev (2006) or Section 1.2.3) we have that τ_D is an optimal stopping time for V and the proof is complete. □

Using the fact that τ_D is optimal, and following the ideas as in Lemma 4.4.7 we can then give a representation of V in terms of the measure $\mathbb{P}$ and the stopping times $\tau_b^{u,x}$ and $\tau_b^{g,x}$ defined in (4.21) and (4.22), respectively.

$$\begin{aligned}
V(u,x) &= \mathbb{E}_{u,x}\left(\int_0^{\tau_D} G(U_s, X_s)\mathrm{d}s \right) \\
&= \mathbb{E}\left(\int_0^{\sigma_{-x}^- \wedge \tau_b^{u,x}} G(u+s, X_s+x)\mathrm{d}s + \mathbb{I}_{\{\sigma_{-x}^- \leq \tau_b^{u,x}\}} \int_{\sigma_{-x}^-}^{\tau_b^{g,x}} G(U_s^{(-x)}, X_s+x)\mathrm{d}s \right) \\
&= \mathbb{E}\left(\int_0^{\sigma_{-x}^- \wedge \tau_b^{u,x}} G(u+s, X_s+x)\mathrm{d}s + \mathbb{I}_{\{\sigma_{-x}^- \leq \tau_b^{u,x}\}} V(0, X_{\sigma_{-x}^-} + x) \right). \qquad (4.23)
\end{aligned}$$

Note that in the last equation we do not longer have explicitly the process $\{U_t^{(-x)}, t \geq 0\}$. So this alternative representation of V in terms of the original measure $\mathbb{P}$ will be useful to prove further properties of b and V.

The next lemma describes the limit behaviour of the function b.

Lemma 4.4.12. *We have that*

$$\lim_{u\to\infty} b(u) = 0.$$

Proof. Note that, since b is non-increasing and it is bounded from below by $\lim_{u\to\infty} h(u) = 0$, the limit $b^* := \lim_{u\to\infty} b(u)$ exists and $b^* \geq 0$. We prove by contradiction that $b^* = 0$. Suppose $b^* > 0$, define the stopping time

$$\sigma_* = \inf\{t \geq 0 : X_t \notin (0, b^*)\}.$$

Take $u > 0$ and $x \in (0, b^*)$. From the fact that $b(u) \geq b^* > 0$ we have that $\sigma_* \leq \tau_D \wedge \sigma_0^-$ under $\mathbb{P}_{u,x}$. Then we have that

$$\begin{aligned}
V(u,x) &= \mathbb{E}_{u,x}\left(\int_0^{\tau_D} G(U_s, X_s)\mathrm{d}s\right) \\
&= \mathbb{E}_x\left(\int_0^{\sigma_*} G(u+s, X_s)\mathrm{d}s\right) + \mathbb{E}_{u,x}\left(V(U_{\sigma_*}, X_{\sigma_*})\right) \\
&= \mathbb{E}_x\left(\int_0^{\sigma_*} G(u+s, X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(V(u+\sigma_*, X_{\sigma_*})\mathbb{I}_{\{X_{\sigma_*}>0\}}\right) \\
&\quad + \mathbb{E}_x\left(V(0, X_{\sigma_*})\mathbb{I}_{\{X_{\sigma_*}\leq 0\}}\right),
\end{aligned} \tag{4.24}$$

where in the last equality we used the Markov property of the two dimensional process $\{(U_t, X_t), t \geq 0\}$. For a fixed $x \in \mathbb{R}$, the function $u \mapsto V(u,x)$ is non-decreasing and bounded from above by zero, thus we have that $\lim_{u\to\infty} V(u,x)$ exists and $-\infty < \lim_{u\to\infty} V(u,x) \leq 0$ for all $x \in \mathbb{R}$. By the dominated convergence theorem we also conclude that $-\infty < \lim_{u\to\infty} \mathbb{E}_x\left(V(u+\sigma_*, X_{\sigma_*})\mathbb{I}_{\{X_{\sigma_*}>0\}}\right) \leq 0$. Moreover, using the general version of Fatou's lemma and the fact that $\lim_{u\to\infty} G(u,x) = \infty$ we have that

$$\lim_{u\to\infty} \mathbb{E}_x\left(\int_0^{\sigma_*} G(u+s, X_s)\mathrm{d}s\right) = \infty.$$

Therefore, taking $u \to \infty$ in (4.24) we get that

$$\lim_{u\to\infty} V(u,x) = \infty.$$

Which yields the desired contradiction. Therefore we conclude that $b^* = 0$. ☐

Now we prove right continuity of b. The proof follows an standard argument (see e.g. du Toit et al. (2008)) but it is included for completeness. It turns out that b is continuous, the proof of this fact makes use of a variational inequality and will be proved later.

Lemma 4.4.13. *The function b is right-continuous.*

Proof. Take $u \geq 0$ and consider for any $n \geq 1$, $u_n = t + 1/n$. Note that since the function b is non-increasing and bounded by below by zero, we have that the limit $\lim_{n\to\infty} b(u_n)$ exists and

$$0 \leq \lim_{n\to\infty} b(u_n) \leq b(u).$$

On the other hand, recall from Lemma 4.4.11 that the set D is closed. Since $(u_n, b(u_n)) \in D$ for all $n \geq 1$, we have that $(u, \lim_{n\to\infty} b(u_n)) \in D$. Hence, from the definition of $b(u)$ we have that $b(u) \leq \lim_{n\to\infty} b(u_n)$. Therefore we have that for any $u \geq 0$, $\lim_{n\to\infty} b(u + 1/n) = b(u)$ and then b is right-continuous. ☐

In order to prove continuity of the value function V we are in need of a technical result regarding convergence of the stopping time $\tau_b^{u,x}$. The proof can be found in the Appendix 4.6.

Lemma 4.4.14. *For any $u \geq 0$ and $x \in \mathbb{R}$ we have that*

$$\lim_{h\to 0} \tau_b^{u,x+h} = \tau_b^{u,x} \qquad a.s.$$

Moreover, we have that

$$\lim_{(h_1,h_2)\to(0,0)+} \tau_b^{u+h_1,x+h_2} = \tau_b^{u,x} \qquad a.s.$$

for all $u \geq 0$ and $x \in \mathbb{R}$.

Now we show the continuity of the value function V.

Lemma 4.4.15. *The function V is continuous on E. Moreover, in the case that X is of infinite variation we have that*

$$\lim_{h\downarrow 0} V(u,h) = V(0,0)$$

for all $u>0$.

Proof. First, we show that the function $u \mapsto V(u,x)$ is continuous for all $x>0$ fixed. Take $u_1,u_2>0$ and $x>0$, then since the stopping time $\tau^*_{(u_1,x)} := \tau_b^{u_1,x}\mathbb{I}_{\{\tau_b^{u_1,x}<\sigma^-_{-x}\}} + \tau_b^{g,x}\mathbb{I}_{\{\tau_b^{u_1,x}\geq\sigma^-_{-x}\}}$ is optimal for $V(u_1,x)$ (under $\mathbb{P}$) we have that

$$\begin{aligned}
V^{(\theta)}(u_1,x) &= \mathbb{E}\left(\int_0^{\sigma^-_{-x}\wedge\tau_b^{u_1,x}} G(u_1+s,X_s+x)\mathrm{d}s + \mathbb{I}_{\{\tau_b^{u_1,x}\geq\sigma^-_{-x}\}}V(0,X_{\sigma^-_{-x}}+x)\right)\\
&= \mathbb{E}_x\left(\int_0^{\sigma^-_0\wedge\tau_b^{u_1,0}} G(u_1+s,X_s)\mathrm{d}s + \mathbb{I}_{\{\tau_b^{u_1,0}\geq\sigma^-_0\}}V(0,X_{\sigma^-_0})\right).
\end{aligned}$$

On the other hand, from (4.15) we get

$$\begin{aligned}
V(u_2,x) &\leq \mathbb{E}\left(\int_0^{\tau^*_{(u_1,x)}} \left\{G(u_2+s,X_s+x)\mathbb{I}_{\{\sigma^-_{-x}>s\}} + G(U_s^{(-x)},X_s+x)\mathbb{I}_{\{\sigma^-_{-x}\leq s\}}\right\}\mathrm{d}s\right)\\
&= \mathbb{E}_x\left(\mathbb{I}_{\{\tau_b^{u_1,0}<\sigma^-_0\}}\int_0^{\tau_b^{u_1,0}} \left\{G(u_2+s,X_s)\mathbb{I}_{\{\sigma^-_0>s\}} + G(U_s,X_s)\mathbb{I}_{\{\sigma^-_0\leq s\}}\right\}\mathrm{d}s\right)\\
&\quad + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_b^{u_1,0}\geq\sigma^-_0\}}\int_0^{\tau_b^{g,0}} \left\{G(u_2+s,X_s)\mathbb{I}_{\{\sigma^-_0>s\}} + G(U_s,X_s)\mathbb{I}_{\{\sigma^-_0\leq s\}}\right\}\mathrm{d}s\right)\\
&= \mathbb{E}_x\left(\int_0^{\tau_b^{u_1,0}\wedge\sigma^-_0} G(u_2+s,X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_b^{u_1,0}\geq\sigma^-_0\}}\int_{\sigma^-_0}^{\tau_b^{g,0}} G(U_s,X_s)\mathrm{d}s\right)\\
&= \mathbb{E}_x\left(\int_0^{\tau_b^{u_1,0}\wedge\sigma^-_0} G(u_2+s,X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\tau_b^{u_1,0}\geq\sigma^-_0\}}V(0,X_{\sigma^-_0})\right),
\end{aligned}$$

where in the first equality we used the definition of $\tau^*_{(u_1,x)}$ given above, in the second equality that $\tau_b^{u_1,0}\leq\tau_b^{g,0}$ and the last equality follows from the strong Markov property applied at time σ^-_0 and since for any $x\leq 0$, the stopping time $\tau_b^{g,0}$ is optimal for $V(0,x)$ (under $\mathbb{P}_x$).

Hence, we have that for any $x > 0$ fixed and $u_1, u_2 > 0$,

$$\begin{aligned}
|V(u_2,x) - V(u_1,x)| &\leq \mathbb{E}_x\left(\int_0^{\sigma_0^- \wedge \tau_b^{u_1,0}} |G(u_2+s,X_s) - G(u_1+s,X_s)|\mathrm{d}s\right)\\
&\leq \mathbb{E}_x\left(\int_0^{\tau_{b(u_1)}^+} |G(u_2+s,X_s) - G(u_1+s,X_s)|\mathrm{d}s\right)\\
&\leq \mathbb{E}\left(\int_0^{\tau_{b(u_1)}^+} |(u_2+s)^{p-1} - (u_1+s)^{p-1}|\mathrm{d}s\right)\\
&= \frac{1}{p}|\mathbb{E}((\tau_{b(u_1)}^+ + u_2)^p) - \mathbb{E}((\tau_{b(u_1)}^+ + u_1)^p) - [u_2^p - u_1^p]|
\end{aligned}$$

where $\tau_{b(u_1)}^+ = \inf\{t \geq 0 : X_t > b(u_1)\}$. Thus tending $u_2 \mapsto u_1$, with the dominated convergence theorem and the fact that $\mathbb{E}((\tau_a^+ + u)^p) < \infty$ for all $u, a \geq 0$ we get that $u \mapsto V(u,x)$ is continuous uniformly over all $x > 0$.

Now we show that $x \mapsto V(u,x)$ is continuous. From equation (4.19) we easily deduce that $x \mapsto V(0,x)$ is a continuous function on $(-\infty, 0]$. Next, suppose that $u > 0$ and $x > 0$. Recall from equation (4.23) that we can write

$$V(u,x) = \mathbb{E}\left(\int_0^{\sigma_{-x}^- \wedge \tau_b^{u,x}} G(u+s, X_s + x)\mathrm{d}s\right) + \mathbb{E}(V(0, X_{\sigma_{-x}^-} + x)\mathbb{I}_{\{\sigma_{-x}^- \leq \tau_b^{u,x}\}}).$$

Note that for all $s \leq \tau_b^{u,x} \wedge \sigma_{-x}^-$, it holds that $0 < X_s + x \leq b(u+s) \leq b(u)$ and for all $x \in \mathbb{R}$ (see equation (4.20)), $V(0, X_{\sigma_{-x}^-} + x)\mathbb{I}_{\{\sigma_{-x}^- \leq \tau_b^{u,x}\}} \geq V(0, X_\infty + x) \geq -A'_{p-1} - C'_{p-1}|X_\infty + x|^p + V(0,0)$, where the latter expression is integrable from Lemma 4.2.1. Moreover, from Lemmas 4.4.9 and 4.4.14 we know that for any $x > 0$ we have that $\lim_{h\to 0} \sigma_{x+h}^- = \sigma_x^-$ a.s. and $\lim_{h\to 0} \tau_b^{u,x+h} = \tau_b^{u,x}$ a.s. Then by the dominated convergence theorem and the fact that V is continuous on $(-\infty, 0]$ and $x \mapsto G(u,x)$ is continuous on $(0,\infty)$ we conclude that $x \mapsto V(u,x)$ is continuous on $(0,\infty)$ for all $u > 0$. Note that when X is of infinite variation, $\lim_{h\downarrow 0} \sigma_{-h}^- = \tau_0^- = 0$ a.s. and the latter argument also tells us that for all $u > 0$,

$$\lim_{h\downarrow 0} V(u,h) = V(0,0).$$

Note that the limit above implies that $\lim_{(u,x)\to(0,0)^+} V(u,x) = V(0,0)$ in the infinite variation case. Then we proceed to prove that this also holds when X is of finite variation. In this case we know that $\sigma_0^- = 0$ and $\tau_0^- > 0$. Due to the strong Markov property,

$$V(0,0) = \mathbb{E}\left(\int_0^{\tau_b^{0,0}\wedge\tau_0^-} G(s,X_s)\mathrm{d}s\right) + \mathbb{E}(\mathbb{I}_{\{\tau_0^-<\tau_b^{0,0}\}}V(0,X_{\tau_0^-})),$$

where $\tau_b^{0,0} = \inf\{t > 0 : X_t \geq b(s)\}$. Taking limits in (4.23) we have from the dominated convergence theorem,

$$\begin{aligned}
\lim_{(u,x)\to(0,0)^+} V(u,x) &= \lim_{(u,x)\to(0,0)^+} \mathbb{E}\left(\int_0^{\sigma_{-x}^-\wedge\tau_b^{u,x}} G(u+s,X_s+x)\mathrm{d}s\right)\\
&\quad + \lim_{(u,x)\to(0,0)^+} \mathbb{E}(V(0,X_{\sigma_{-x}^-}+x)\mathbb{I}_{\{\sigma_{-x}^-\leq\tau_b^{u,x}\}})\\
&= \mathbb{E}\left(\int_0^{\tau_b^{0,0}\wedge\tau_0^-} G(s,X_s)\mathrm{d}s\right) + \mathbb{E}(\mathbb{I}_{\{\tau_0^-<\tau_b^{0,0}\}}V(0,X_{\tau_0^-}))\\
&= V(0,0),
\end{aligned}$$

where we used that $\lim_{x\downarrow 0}\sigma_{-x}^- = \tau_0^-$ (see proof of Lemma 4.4.9) and $\lim_{(u,x)\to(0,0)+}\tau_b^{u,x} = \tau_b^{0,0}$ a.s. (see Lemma 4.4.14). Therefore V is continuous on the set E. □

We know from Lemma 4.4.15 that the function b is a right-continuous function. In order to show left continuity we make use of a variational inequality that is satisfied by the value function V. The oncoming paragraphs will be dedicated on introducing that.

It is well known that for every optimal stopping problem there is a free boundary problem which is stated in terms of the infinitesimal generator (see e.g. Peskir and Shiryaev (2006) Chapter III). In this particular case, provided that the value function is smooth enough, we have that V solves the Dirichlet/Poisson problem. That is,

$$\mathcal{A}_{U,X}(V) = \frac{\partial}{\partial u}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) = -G \qquad \text{in } E\setminus D,$$

where $\mathcal{A}_{U,X}$ and $\mathcal{A}_X$ correspond to the infinitesimal generator of the process (U,X) and X, respectively, given in (4.5) and (1.26) whereas $\widetilde{V}$ is the extension of V to the set $\mathbb{R}_+\times\mathbb{R}$ given by

$$\widetilde{V}(u,x) = \begin{cases} V(u,x) & u > 0 \text{ and } x > 0, \\ V(0,x) & u \geq 0 \text{ and } x \leq 0, \\ V(0,0) & u = 0 \text{ and } x > 0. \end{cases} \tag{4.25}$$

However, in our setting turns out to be challenging to show that V is a $C^{1,2}$ function. Lamberton and Mikou (2008) showed that we can state an analogous (in)equality in the sense of distributions (see in particular section 2 for its definition). The reader can also refer to Appendix A for more details on how to define the infinitesimal generator of the process (t, X) in the sense of the distributions. In particular, since V is continuous on E we have that $\widetilde{V}$ is a locally integrable function in $\mathbb{R}_+ \times \mathbb{R}$ (note that $\widetilde{V}$ may be discontinuous at points of the form $(u, 0)$ for $u > 0$ when X is of finite variation) so we can define $\widetilde{V}$ as a distribution in any open set $O \subset \mathbb{R}_+ \times \mathbb{R}$ via

$$\langle \widetilde{V}, \varphi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \widetilde{V}(u,x)\varphi(u,x)\mathrm{d}x\mathrm{d}u$$

for any test function φ with compact support in O. Then the derivatives of the distribution $\widetilde{V}$ are defined as

$$\langle \frac{\partial^{i+j}}{\partial u^i \partial x^j} \widetilde{V}, \varphi \rangle = (-1)^{i+j} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \widetilde{V}(u,x) \frac{\partial^{i+j}}{\partial u^i \partial x^j} \varphi(u,x)\mathrm{d}x\mathrm{d}u.$$

Moreover, provided that the function $(u, x) \mapsto \int_{(-\infty,-1)} \widetilde{V}(u, x + y)\Pi(\mathrm{d}y)$ is locally integrable in $\mathbb{R}_+ \times \mathbb{R}$ the operator B_X, defined for any test function φ, with compact support in O, by

$$\langle B_X(\widetilde{V}), \varphi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \widetilde{V}(u,x) B_X^*(\varphi)(u,x)\mathrm{d}x\mathrm{d}y,$$

defines a distribution on O (see Lemma A.4), where

$$B_X^*(\varphi)(u,x) = \int_{(-\infty,0)} [\varphi(u, x-y) - \varphi(u,x) + y\frac{\partial}{\partial x}\varphi(u,x)\mathbb{I}_{\{y>-1\}}]\Pi(\mathrm{d}y).$$

We have the following Lemma that ensures that the integrability conditions for $\widetilde{V}$ are satisfied so then $B_X(\widetilde{V})$ is indeed a distribution.

Lemma 4.4.16. *The function*

$$(u,x) \mapsto \int_{(-\infty,-1)} \widetilde{V}(u,x+y)\Pi(\mathrm{d}y)$$

is locally integrable in $\mathbb{R}_+ \times \mathbb{R}$.

Proof. First note that from equation (4.20) we have that for any $x \le 0$,

$$\begin{aligned}\int_{(-\infty,-1)} V(0,x+y)\Pi(\mathrm{d}y) &\ge -A'_{p-1}\Pi(-\infty,-1] - C'_{p-1}\int_{(-\infty,-1)} |x+y|^p\Pi(\mathrm{d}y) \\ &\quad + V(0,0)\Pi(-\infty,-1) \\ &> -\infty,\end{aligned}$$

where we used the fact that $\Pi(-\infty,-1) < \infty$ and Lemma 4.2.1. Moreover, since V is non-decreasing in each argument we have that for any $u > 0$ and $x > 0$ that

$$\int_{(-\infty,-1)} \widetilde{V}(u,x+y)\Pi(\mathrm{d}y) \ge \int_{(-\infty,-1)} V(0,y)\Pi(\mathrm{d}y) > -\infty.$$

Hence we conclude that $\int_{(-\infty,-1)} \widetilde{V}(u,x+y)\Pi(\mathrm{d}y) > -\infty$ for any $(u,x) \in \mathbb{R}_+ \times \mathbb{R}$. Since V is continuous on E and the definition of $\widetilde{V}$ we have that the mapping $(u,x) \mapsto \int_{(-\infty,-1)} \widetilde{V}(u,x+y)\Pi(\mathrm{d}y)$ is locally integrable. □

Hence, we can define the operator $\mathcal{A}_X$ in the sense of distributions by

$$\mathcal{A}_X(\widetilde{V}) = -\mu\frac{\partial}{\partial x}\widetilde{V} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\widetilde{V} + B_X(\widetilde{V}).$$

The next lemma is an extension of Proposition 2.5 in Lamberton and Mikou (2008).

Lemma 4.4.17. *The distribution* $\frac{\partial}{\partial u}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G$ *is a non-negative distribution on* $(0,\infty)\times(0,\infty)$. *Moreover, we have* $\frac{\partial}{\partial u}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G = 0$ *on the set* $C^+ := \{(u,x) \in (0,\infty)\times(0,\infty) : 0 < x < b(u)\}$ *and* $\mathcal{A}_X(V(0,\cdot)) + G(0,\cdot) = 0$ *on* $(-\infty,0)$ *in the sense of distributions.*

Proof. From the general theory of optimal stopping we have that (see Peskir and Shiryaev (2006), Theorem 2.4 or Section 1.2.3) for every $(u,x) \in E$, the stochastic process $\{Z_t, t \ge 0\}$

is a sub-martingale under the measure $\mathbb{P}_{u,x}$, where

$$Z_t = V(U_t, X_t) + \int_0^t G(U_s, X_s)\mathrm{d}s.$$

Moreover, we have that the stopped process $\{Z_{t\wedge\tau_D}, t \geq 0\}$ is a martingale under $\mathbb{P}_{u,x}$ for all $(u,x) \in E$. Then from Doob's stopping time theorem we have that for every $(u,x) \in E$, the process $\{Z_{t\wedge\sigma_0^-}, t \geq 0\}$ is a sub-martingale and $\{Z_{t\wedge\tau_D\wedge\sigma_0^-}, t \geq 0\}$ is a martingale under $\mathbb{P}_{u,x}$. From the fact that $U_t = 0$ if and only if $X_t \leq 0$ we have that under $\mathbb{P}_{u,x}$,

$$\begin{aligned}
Z_{t\wedge\sigma_0^-} &= V(U_{t\wedge\sigma_0^-}, X_{t\wedge\sigma_0^-}) + \int_0^{t\wedge\sigma_0^-} G(U_s, X_s)\mathrm{d}s \\
&= V(u+t, X_t)\mathbb{I}_{\{t<\sigma_0^-\}} + V(0, X_{\sigma_0^-})\mathbb{I}_{\{\sigma_0^-\leq t\}} + \int_0^{t\wedge\sigma_0^-} G(u+s, X_s)\mathrm{d}s \\
&= \widetilde{V}(u+t, X_t)\mathbb{I}_{\{t<\sigma_0^-\}} + \widetilde{V}(u+\sigma_0^-, X_{\sigma_0^-})\mathbb{I}_{\{\sigma_0^-\leq t\}} + \int_0^{t\wedge\sigma_0^-} G(u+s, X_s)\mathrm{d}s \\
&= \widetilde{V}(u+t\wedge\sigma_0^-, X_{t\wedge\sigma_0^-}) + \int_0^{t\wedge\sigma_0^-} G(u+s, X_s)\mathrm{d}s
\end{aligned}$$

for every $u > 0$ and $x > 0$. Hence from Proposition A.6 we have that $\frac{\partial}{\partial u}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G$ is a non-negative distribution on $(0,\infty) \times (0,\infty)$. Similarly, we have that for any $u > 0$ and $x > 0$ such that $x < b(u)$,

$$Z_{t\wedge\sigma_0^-\wedge\tau_D} = \widetilde{V}(u+t\wedge\sigma_0^-\wedge\tau_D, X_{t\wedge\sigma_0^-\wedge\tau_D}) + \int_0^{t\wedge\sigma_0^-\wedge\tau_D} G(u+s, X_s)\mathrm{d}s.$$

Therefore, we have that (from Proposition A.6) that $\frac{\partial}{\partial u}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G = 0$ on C^+ in the sense of distributions. Lastly, since b is non-negative we have that $\tau_0^+ \leq \tau_D$. Hence, under the measure $\mathbb{P}_{0,x}$, for any $x < 0$, we have that $\{Z_{t\wedge\tau_0^+}, t \geq 0\}$ is a martingale. Moreover, we since $X_t \leq 0$ for all $t < \tau_0^+$ we have that

$$Z_{t\wedge\tau_0^+} = V(0, X_{t\wedge\tau_0^+}) + \int_0^{t\wedge\tau_0^+} G(0, X_s)\mathrm{d}s.$$

Then from Proposition A.6 we have that $\mathcal{A}_X(V(0,\cdot)) + G(0,\cdot) = 0$ in the sense of distributions on the set $(-\infty, 0)$. □

Remark 4.4.18. *i) In Lamberton and Mikou (2008) the definition of the infinitesimal* *generator in the sense of distributions assumes that the value function is a bounded Borel measurable function. In our setting such condition can be relaxed by the fact that* $(u,x) \mapsto \int_{(-\infty,-1)} |\widetilde{V}(u,x+y)|\Pi(\mathrm{d}y)$ *is a locally integrable function on* $\mathbb{R}_+ \times \mathbb{R}$*.*

ii) We note that similar as in (4.5) the infinitesimal generator of (U,X) *can be defined as* $\mathcal{A}_{U,X}(V) := \partial/\partial u \widetilde{V} + \mathcal{A}_X(\widetilde{V})$ *in the sense of distributions, where* $\mathcal{A}_X$ *corresponds to the infinitesimal generator of* X *(seen as a distribution).*

For the proof of left-continuity of b we define an auxiliary function. For $(u,x) \in E$,

$$\Lambda(u,x) := \begin{cases} \int_{(-\infty,0)} \widetilde{V}(u,x+y)\Pi(\mathrm{d}y) + G(u,x) & x > b(u) \\ 0 & x \leq b(u). \end{cases}$$

The next lemma states some basic properties of the function Λ.

Lemma 4.4.19. *The function* Λ *is a non-decreasing (in each argument) function such that* $0 < \Lambda(u,x) < \infty$ *for all* $x > b(u)$*. Moreover, is strictly increasing in each argument and continuous in the interior of the set* D*. Furthermore,* $\Lambda = \frac{\partial}{\partial u}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G$ *on the interior of* D *in the sense of distributions.*

Proof. It follows from Lemma 4.4.16 and the fact that V vanishes in D that $|\Lambda(u,x)| < \infty$ for all $(u,x) \in E$. The fact that Λ is continuous on D follows from the continuity of V and G, the dominated convergence theorem and the fact that Π has no atoms. Moreover, Λ is strictly increasing in each argument on D since V is non-decreasing in each argument and G is strictly increasing in each argument on D. Then we show that $\partial/\partial u \widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G = \Lambda$ on in the interior of D. Let φ be a C^∞ function with compact support on the interior of D.

Since V vanishes on D (and then also $\widetilde{V}$), we have that

$$\begin{aligned}\langle\frac{\partial}{\partial u}\widetilde{V}+\mathcal{A}_X(\widetilde{V})+G,\varphi\rangle &= \int_0^\infty\int_{-\infty}^{b(u)}\widetilde{V}(u,x)\int_{(-\infty,0)}\varphi(u,x-y)\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u\\ &\quad+\int_0^\infty\int_{b(u)}^\infty G(u,x)\varphi(u,x)\mathrm{d}x\mathrm{d}u\\ &=\int_0^\infty\int_{b(u)}^\infty\varphi(u,x)\int_{(-\infty,0)}\widetilde{V}(u,x+y)\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u\\ &\quad+\int_0^\infty\int_{b(u)}^\infty G(u,x)\varphi(u,x)\mathrm{d}x\mathrm{d}u\\ &=\int_0^\infty\int_{b(u)}^\infty\Lambda(u,x)\varphi(u,x)\mathrm{d}x\mathrm{d}u\\ &=\langle\Lambda,\varphi\rangle\end{aligned}$$

Then we have that $\partial/\partial u\widetilde{V}+\mathcal{A}_X(\widetilde{V})+G=\Lambda$ on in the interior of D.. Moreover from Lemma 4.4.17 and continuity of Λ we conclude that $\Lambda(u,x)\geq 0$ for all $(u,x)\in E$. In particular is strictly positive in the interior of D since it is strictly increasing in that set. □

Now we are ready to show that the function b is continuous, the proof is based on the ideas of Lamberton and Mikou (2008) (Theorem 4.2). We include the proof for completeness.

Lemma 4.4.20. *The function b is continuous.*

Proof. From Lemma 4.4.13 we already know that b is right continuous. We then show left continuity of b. We proceed by contradiction. Suppose there is a point $u_*>0$ such that $b(u_*-):=\lim_{h\downarrow 0}b(u_*-h)>b(u_*)$. Then since b is non-decreasing we have for all $(u,x)\in(0,u_*)\times(b(u_*),b(u_*-))$ that $V(u,x)<0$. Thus, $(0,u_*)\times(b(u_*),b(u_*-))\subset C^+$. From Lemma 4.4.17 we obtain that $\frac{\partial}{\partial u}\widetilde{V}+\mathcal{A}_X(\widetilde{V})+G=0$ in $(0,u_*)\times(b(u_*),b(u_*-))$. Hence, for any non-negative C^∞ function φ with compact support in $(0,u_*)\times(b(u_*),b(u_*-))$ we have that

$$\begin{aligned}\langle\mathcal{A}_X(\widetilde{V})+G,\varphi\rangle &= -\langle\frac{\partial}{\partial u}\widetilde{V},\varphi\rangle\\ &=\int_{\mathbb{R}}\int_{(0,\infty)}\widetilde{V}(u,x)\frac{\partial}{\partial u}\varphi(u,x)\mathrm{d}u\mathrm{d}x\\ &=-\int_{\mathbb{R}}\int_{(0,\infty)}\widetilde{V}(\mathrm{d}u,x)\varphi(u,x)\mathrm{d}x\\ &\leq 0,\end{aligned}$$

where we used the fact that for each $x > 0$, $u \mapsto \widetilde{V}(u,x) = V(u,x)$ is non-decreasing. Hence, we conclude that $\mathcal{A}_X(\widetilde{V}) + G$ is a non-positive distribution on $(0, u_*) \times (b(u_*), b(u_*-))$. Thus, by continuity of $\widetilde{V} = V$ and G on $(0,\infty) \times (0,\infty)$ we have that for any $u \in (0, u_*)$ and any non-negative test function ψ with compact support in $(b(u_*), b(u_*-))$

$$\int_{\mathbb{R}} \left\{ \widetilde{V}(u,x) \left[-\mu \frac{\partial}{\partial x}\psi(x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\psi(x) + B_X^*(\psi)(x) \right] + G(u,x)\psi(x) \right\} \mathrm{d}x \leq 0,$$

where $B_X^*(\psi)(x) = \int_{(-\infty,0)} (\psi(x-y) - \psi(x) + y\frac{d}{dx}\psi(x)\mathbb{I}_{\{|y|\leq 1\}})\Pi(\mathrm{d}y)$. Taking $u \uparrow u_*$ in the equation above, using the fact that $\widetilde{V}(u_*, x) = 0$ for all $x \geq b(u_*)$ and since ψ has compact support in $(b(u_*), b(u_*-))$ we get that

$$\begin{aligned}
0 &\geq \lim_{u \uparrow u_*} \int_{\mathbb{R}} \left\{ \widetilde{V}(u,x) \left[-\mu \frac{\partial}{\partial x}\psi(x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\psi(x) + B_X^*(\psi)(x) \right] + G(u,x)\psi(x) \right\} \mathrm{d}x \\
&= \int_{-\infty}^{b(u_*)} \widetilde{V}(u_*, x) \int_{(-\infty,0)} \psi(x-y)\Pi(\mathrm{d}y)\mathrm{d}x + \int_{b(u_*)}^{b(u_*-)} G(u_*, x)\psi(x)\mathrm{d}x \\
&= \int_{b(u_*)}^{b(u_*-)} \psi(x) \int_{(-\infty,0)} \widetilde{V}(u_*, x+y)\Pi(\mathrm{d}y)\mathrm{d}x + \int_{b(u_*)}^{b(u_*-)} G(u_*, x)\psi(x)\mathrm{d}x \\
&= \int_{b(u_*)}^{b(u_*-)} \psi(x)\Lambda(u_*, x)\mathrm{d}x \\
&> 0,
\end{aligned}$$

where the strict inequality follows from the fact that Λ is strictly positive in each argument in D (see Lemma 4.4.19). Hence we have got a contradiction and $b(u-) = b(u)$ for all $u > 0$. Therefore b is a continuous function. □

From Lemma 4.4.6 we know that b and h converge at the same limit when u tends to infinity. Moreover, from the discussion about h after Lemma 4.2.3 we know that in case that X is of finite variation there exists a value $u_h^* < \infty$ for which $h(u) = 0$ for all $u \geq u_h^*$. That suggests a similar behaviour for b, the next lemma addresses that conjecture.

Lemma 4.4.21. *Denote as $u_b = \inf\{u > 0 : b(u) = 0\}$. If X is of infinite variation or finite variation and infinite activity we have that $u_b = \infty$. Otherwise $u_b = u^*$, where u^* is the unique solution to*

$$G(u,0) + \int_{(-\infty,0)} V(0,y)\Pi(\mathrm{d}y) = 0. \tag{4.26}$$

Proof. From the fact that $h(u) > 0$ for all $u > 0$ when X is of infinite variation and inequality $b(u) \geq h(u)$ we have that assertion is true for this case. Suppose that X has finite variation with infinite activity, that is $\Pi(-\infty, 0) = \infty$, and assume that $u_b < \infty$. Then since b is non-increasing we have that $b(u) = 0$ for all $u > u_b$ and then $V(u, x) = 0$ for all $x > 0$ and $u > u_b$. From Lemma 4.4.19 we have that

$$G(u,x) + \int_{(-\infty,-x)} V(0, x+y)\Pi(dy) \geq 0 \text{ for all } x > 0$$

for all $u > u_b$. Taking $x \downarrow 0$ in the equation above and using the expression for $V(0, z)$ for $z < 0$ given in (4.19) we have that for any $u > u_b$,

$$\begin{aligned}
0 &\leq G(u,0) - \lim_{x\downarrow 0} \int_{(-\infty,0)} \int_0^{-x+y} \int_{[0,\infty)} \mathbb{E}_{-u-z}(g^{p-1}) W(\mathrm{d}u)\mathrm{d}z\Pi(\mathrm{d}y) + \lim_{x\downarrow 0} V(0,0)\Pi(-\infty,-x) \\
&= -\infty
\end{aligned}$$

which is a contradiction and then $u_b = \infty$. Now assume that X has finite variation with $\Pi(-\infty, 0) < \infty$. Assume that $b(u^*) > 0$, then $V(u^*, x) < 0$ for $x \in (0, b(u^*))$. Moreover, since $V \leq 0$ and using the compensation formula for Poisson random measures (see equation (1.25)) we have that for all $u > 0$ and $x < b(u)$,

$$\begin{aligned}
&\mathbb{E}_{u,x}(V(0, X_{\tau_0^-})\mathbb{I}_{\{\tau_0^- < \tau_D\}}) \\
&\quad = \mathbb{E}_{u,x}\left(\int_{[0,\infty)} \int_{(-\infty,0)} V(0, X_{s-}+y)\mathbb{I}_{\{\underline{X}_{s-}>0\}}\mathbb{I}_{\{X_{s-}+y<0\}}\mathbb{I}_{\{X_r \leq b(U_r) \text{ for all } r<s\}} N(\mathrm{d}s, \mathrm{d}y) \right) \\
&\quad = \mathbb{E}_{u,x}\left(\int_0^{\infty} \int_{(-\infty,0)} V(0, X_s+y)\mathbb{I}_{\{\underline{X}_{s-}>0\}}\mathbb{I}_{\{X_s+y<0\}}\mathbb{I}_{\{X_r \leq b(U_r) \text{ for all } r<s\}} \Pi(\mathrm{d}y)\mathrm{d}s \right) \\
&\quad = \mathbb{E}_{u,x}\left(\int_0^{\tau_0^- \wedge \tau_D} \int_{(-\infty,0)} V(0, X_s+y)\mathbb{I}_{\{X_s+y<0\}} \Pi(\mathrm{d}y)\mathrm{d}s \right).
\end{aligned}$$

Then from the Markov property we have that for all $x < b(u^*)$

$$
\begin{aligned}
V(u^*,x) &= \mathbb{E}_{u^*,x}\left(\int_0^{\tau_D\wedge\tau_0^-} G(u^*+s,X_s)\mathrm{d}s\right) + \mathbb{E}_{u^*,x}(V(0,X_{\tau_0^-})\mathbb{I}_{\{\tau_0^-<\tau_D\}}) \\
&= \mathbb{E}_{u^*,x}\left(\int_0^{\tau_D\wedge\tau_0^-} \left[G(u^*+s,X_s) + \int_{(-\infty,0)} V(0,X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(dy)\right]\mathrm{d}s\right) \\
&> 0,
\end{aligned}
$$

where the strict inequality follows from the fact that that X is of finite variation and then $\tau_D\wedge\tau_0^- > 0$, the definition of u^* and the fact that G and V are non-decreasing in each argument. Then we are contradicting the fact that $V(u^*,x) < 0$ and we conclude that $b(u^*) = 0$ and $u_b \leq u^*$. Moreover, from Lemma 4.4.19 we know that for all $u > u_b$

$$
G(u,x) + \int_{(-\infty,-x)} V(0,x+y)\Pi(\mathrm{d}y) \geq 0 \text{ for all } x > 0.
$$

Taking $x \downarrow 0$ we get that for all $u \geq u_b$, $G(u,0)+\int_{(-\infty,0)} V(0,y)\Pi(\mathrm{d}y) \geq 0$. The latter implies that $u^* \leq u_b$ (since $u \mapsto G(u,0)$ is strictly increasing). Therefore we conclude that $u^* = u_b$ and the proof is complete. □

As we mentioned before it is challenging to prove the existence of the derivatives of V. However, it is possible to show that the derivatives of V at the boundary exist and are equal to zero. Recall from Lemma 4.4.21 that when X is of infinite variation or finite variation with infinite activity we have that $b(u) > 0$ for all $u > 0$. In the case that X is of finite variation we have that $b(u) > 0$ only if $u < u_b$ where u_b is the solution to (4.26). In such cases we can guarantee that the derivatives of V exist at the boundary and are equal to zero which is proven in the following Theorem. Since the proof is rather long and technical it can be found in the Appendix.

Theorem 4.4.22. *Suppose that $u > 0$ is such that $b(u) > 0$. Then the first partial derivatives of $V(u,x)$ exist at the point $x = b(u)$ and*

$$
\frac{\partial}{\partial x}V(u,b(u)) = 0 \qquad \textit{and} \qquad \frac{\partial}{\partial u}V(u,b(u)) = 0.
$$

Recall from equation (4.19) that when $x < 0$,

$$V(0,x) = -\int_0^{-x} \int_{[0,\infty)} \mathbb{E}_{-u-z}(g^{p-1}) W(\mathrm{d}u)\mathrm{d}z + V(0,0).$$

Note that the first term on the right-hand side of the equation above does not depend on the boundary b. Then, for $x < 0$, the value function $V(0,x)$ is characterised by the value $V(0,0)$. Moreover, from Lemma 4.4.21 we know that when X is of finite variation with $\Pi(-\infty, 0) < \infty$, the value u_b is the unique solution to

$$G(u,0) - \int_{(-\infty,0)} \int_0^{-y} \int_{[0,\infty)} \mathbb{E}_{-u-z}(g^{p-1}) W(\mathrm{d}u)\mathrm{d}z\Pi(\mathrm{d}y) + V(0,0)\Pi(-\infty,0) = 0.$$

Otherwise, $u_b = \infty$. Then if X is of finite variation with finite activty, u_b is also characterised by the value $V(0,0)$, where we know from Remark 4.4.3 that

$$-\frac{\mathbb{E}(g^{p-1})}{p} \leq V(0,0) < 0.$$

The next theorem gives a characterisation of the value function V on the set $(0,\infty) \times (0,\infty)$, the boundary b and the values $V(0,0)$ and u_b as unique solutions of a system of non-linear integral equations. The method of proof is deeply inspired on the ideas of du Toit et al. (2008). However, the presence of jumps adds an important level of difficulty. In particular, when $\Pi \neq 0$, the inequality

$$G(u,x) + \int_{(-\infty,0)} \widetilde{V}(u, x+y)\Pi(\mathrm{d}y) > 0$$

for all $(u,x) \in D$ is a necessary condition for the process $\{V(U_t, X_t) + \int_0^t G(U_s, X_s)\mathrm{d}s, t \geq 0\}$ to be a submartingale.

Theorem 4.4.23. *Let $p > 1$ and X be a spectrally negative Lévy process drifting to infinity such that $\int_{(-\infty,-1)} |x|^{p+1}\Pi(\mathrm{d}x) < \infty$. For all $u > 0$ and $x > 0$, the function V can be written* *as*

$$\begin{aligned}
&V(u,x)\\
&\quad= V(0,0)\frac{\sigma^2}{2}W'(x)\\
&\qquad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} V(u+s,X_s+y)\mathbb{I}_{\{0<X_s+y<b(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right)\\
&\qquad + \mathbb{E}_x\left(\int_0^{\tau_0^-}\left[G(u+s,X_s)+\int_{(-\infty,0)} V(0,X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right),
\end{aligned} \tag{4.27}$$

the value $V(0,0)$ satisfies

$$\begin{aligned}
V(0,0) = &-\frac{1}{\psi'(0+)}\int_0^\infty \mathbb{E}_{-z}(g^{p-1})[1-\psi'(0+)W(z)]\mathrm{d}z\\
&+\frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty G(s,X_s)\frac{X_s}{s}\mathbb{I}_{\{0<X_s<b(s)\}}\mathrm{d}s\right)\\
&-\frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)} V(s,X_s+y)\mathbb{I}_{\{0<X_s+y<b(s)\}}\Pi(\mathrm{d}y)\frac{X_s}{s}\mathbb{I}_{\{X_s>b(s)\}}\mathrm{d}s\right)\\
&-\frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)} V(0,X_s+y)\mathbb{I}_{\{X_s+y\le 0\}}\Pi(\mathrm{d}y)\frac{X_s}{s}\mathbb{I}_{\{X_s>b(s)\}}\mathrm{d}s\right),
\end{aligned} \tag{4.28}$$

whilst the curve b satisfies the equation

$$\begin{aligned}
0 = &\,V(0,0)\frac{\sigma^2}{2}W'(b(u))\\
&- \mathbb{E}_{b(u)}\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} V(u+s,X_s+y)\mathbb{I}_{\{0<X_s+y<b(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right)\\
&+ \mathbb{E}_{b(u)}\left(\int_0^{\tau_0^-}\left[G(u+s,X_s)+\int_{(-\infty,0)} V(0,X_s+y)\mathbb{I}_{\{X_s+y\le 0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right)
\end{aligned} \tag{4.29}$$

for all $u < u_b$, where for $x \le 0$, the function $V(0,x)$ depends on $V(0,0)$ via (4.19). For $u \ge u_b$ we have $b(u) = 0$, where $u_b = \infty$ in the case X is of infinite variation or finite variation with $\Pi(-\infty,0) = \infty$. Otherwise u_b is the unique solution to

$$G(u,0) - \int_{(-\infty,0)}\int_0^{-y}\int_{[0,\infty)} \mathbb{E}_{-u-z}(g^{p-1})W(\mathrm{d}u)\mathrm{d}z\Pi(\mathrm{d}y) + V(0,0)\Pi(-\infty,0) = 0. \tag{4.30}$$

Moreover, in the case that there is a Brownian motion component (i.e. $\sigma > 0$) we have that (4.28) is equivalent to

$$\frac{\partial}{\partial x}V_+(0,0) = \frac{\partial}{\partial x}V_-(0,0), \tag{4.31}$$

where $\frac{\partial}{\partial x}V_+(u,0)$ and $\frac{\partial}{\partial x}V_-(0,0)$ are the right and left derivatives of $x \mapsto V(u,x)$ and $x \mapsto V(0,x)$ at zero, respectively and $\frac{\partial}{\partial x}V_+(0,0) = \lim_{u\downarrow 0} \frac{\partial}{\partial x}V_+(u,0)$.

Furthermore, the quadruplet $(V, b, V(0,0), u_b)$ is uniquely characterised by the equations above, where V is considered in the class of non-positive continuous functions such that

$$\int_{(-x-b(u),-x)} V(u, b(u)+x+y)\Pi(\mathrm{d}y) \\ + \int_{(-\infty,-x-b(u)]} V(0, b(u)+x+y)\Pi(\mathrm{d}y) + G(u, x+b(u)) \geq 0 \tag{4.32}$$

for all $u < u_b$ and $x > 0$ and b is considered in the class of non-increasing functions with $b \geq h$ whereas $-\frac{1}{p}\mathbb{E}(g^p) \leq V(0,0) < 0$.

Since the proof of Theorem 4.4.23 is rather long we break it in a series of Lemmas. Next subsection is entirely dedicated to that purpose.

4.4.1 Proof of Theorem 4.4.23

First, we show that the relevant quantities are integrable.

Lemma 4.4.24. *We have that for all $(u,x) \in E$,*

$$\mathbb{E}_{u,x}\left(\int_0^\infty |G(U_s, X_s)|\mathbb{I}_{\{X_s < b(U_s)\}}\mathrm{d}s\right) < \infty, \tag{4.33}$$

$$\mathbb{E}_{u,x}\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s > b(U_s)\}}\right) > -\infty. \tag{4.34}$$

Moreover, we have that

$$\lim_{u,x\to\infty} \mathbb{E}_{u,x}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right) = 0, \tag{4.35}$$

$$\lim_{u,x\to\infty} \mathbb{E}_{u,x}\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\right) = 0. \tag{4.36}$$

Proof. Let $(u,x)\in E$, we first show that (4.33) is satisfied. Indeed, using that $|G(u,x)| < u^{p-1} - \mathbb{E}_x(g^{p-1})$ and that $U_s \le s$ we have that

$$\begin{aligned}
\mathbb{E}_{u,x}&\left(\int_0^\infty |G(U_s,X_s)|\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right)\\
&\le \mathbb{E}_{u,x}\left(\int_0^\infty [U_s^{p-1}+\mathbb{E}_{X_s}(g^{p-1})]\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right)\\
&\le \mathbb{E}_{u,x}\left(\int_0^\infty s^{p-1}\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right) + \mathbb{E}_x\left(\int_0^\infty \mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right).
\end{aligned}$$

From Lemma 4.4.1 we know that the second integral above is finite. Now we check that the first integral above is also finite. Consider $\delta>0$ and consider $g^{(b(\delta))}$, the last time X is below the level $b(\delta)$, then $g^{(b(\delta))}\ge g$ and $X_{s+g^{(b(\delta))}+\delta}\ge b(\delta)$ for all $s\ge 0$. Hence, since b is non-increasing we get

$$\begin{aligned}
\mathbb{E}_{u,x}\left(\int_0^\infty s^{p-1}\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right) &= \mathbb{E}_x\left(\int_0^{g^{(b(\delta))}+\delta} s^{p-1}\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right)\\
&\le \mathbb{E}_x((g^{(b(\delta))}+\delta)^p) + \mathbb{E}_x\left(\int_0^\infty \mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right)\\
&< \infty,
\end{aligned}$$

where the last expectation is finite by Lemma 4.2.1. Therefore we conclude that (4.33) holds. Moreover, from the fact that $x\mapsto \mathbb{E}_x(g^p)$ is non increasing, the fact that $\lim_{x\to\infty}\mathbb{E}_x(g^p)=0$

(see Lemma 4.2.3) and the dominated convergence theorem that

$$\begin{aligned}
\lim_{u,x\to\infty} & \left|\mathbb{E}_{u,x}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right)\right| \\
&\leq \lim_{u,x\to\infty}\mathbb{E}_{u,x}\left(\int_0^\infty |G(U_s,X_s)|\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right) \\
&\leq \lim_{x\to\infty}\mathbb{E}_x((g^{(b(\delta))}+\delta)^p)+\lim_{x\to\infty}\mathbb{E}_x\left(\int_0^\infty \mathbb{E}_{X_s}(g^{p-1})\mathrm{d}s\right) \\
&= \delta^p
\end{aligned}$$

for any $\delta>0$. Hence we conclude that

$$\lim_{u,x\to\infty}\mathbb{E}_{u,x}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right)=0.$$

Next we prove that (4.34) also holds. Since V is non-decreasing in each argument we have that is enough to show that (4.34) holds for $u=0$ and $x\leq 0$. Let $N>0$ any positive number, then we have that

$$\begin{aligned}
\mathbb{E}_x & \left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right) \\
&= \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s\leq N\}}\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right) \\
&\quad + \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{X_s>b(N)\}}\mathrm{d}s\right) \\
&\quad + \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{b(N)\geq X_s>b(U_s)\}}\mathrm{d}s\right).
\end{aligned}$$

Hence, we next show that the three expectations above are finite. Using the fact that $\int_{(-\infty,0)}V(u,x+y)\Pi(\mathrm{d}y)+G(u,x)\geq 0$ for all $u>0$ and $x>b(u)$ (see Lemma 4.4.19), that $G(u,x) \le u^{p-1}$ for all $(u,x) \in E$ and that b is non increasing we get that

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s\le N\}}\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right)\\
&\qquad \ge -\mathbb{E}_x\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{U_s\le N, X_s>b(U_s)\}}\mathrm{d}s\right)\\
&\qquad \ge -N^{p-1}\mathbb{E}_x\left(\int_0^\infty \mathbb{I}_{\{U_s\le N\}}\mathrm{d}s\right)\\
&\qquad \ge -N^{p-1}[\mathbb{E}_x(g)+N]\\
&\qquad > -\infty,
\end{aligned}$$

where in the second last inequality we used the fact that $U_s > N$ for all $s \ge g+N$. In a similar way we have that

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{b(N)\ge X_s>b(U_s)\}}\mathrm{d}s\right)\\
&\ge -\mathbb{E}\left(\int_0^\infty s^{p-1}\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{b(N)\ge X_s>b(U_s)\}}\mathrm{d}s\right)\\
&= -\mathbb{E}\left(\int_0^{g^{(b(N))}} s^{p-1}\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{b(N)\ge X_s>b(U_s)\}}\mathrm{d}s\right)\\
&\ge -\frac{1}{p}\mathbb{E}((g^{(b(N))})^p)\\
&> -\infty,
\end{aligned}$$

where we used that $U_s \le s$ and that $g^{(b(N))} = \sup\{t \ge 0 : X_t \le b(N)\}$ has moments of order p (see Lemma 4.2.1). Lastly, since V is non increasing in each argument and b is non decreasing we have that by Fubini's theorem that

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{X_s>b(N)\}}\mathrm{d}s\right)\\
&\ge \mathbb{E}_x\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(N, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(N)\}}\mathrm{d}s\right)\\
&= \int_{(b(N),\infty)}\int_{(-\infty,0)} \widetilde{V}(N, z+y)\Pi(\mathrm{d}y)\int_0^\infty \mathbb{P}_x(X_s\in \mathrm{d}z)\mathrm{d}s\\
&= \Phi'(0)\int_{b(N)}^\infty \int_{(-\infty,0)} \widetilde{V}(N,z+y)\Pi(\mathrm{d}y)\mathrm{d}z,
\end{aligned}$$

where in the last equality we used a density of the 0-potential measure of X without killing (see (1.21)) and the fact that W vanishes on $(-\infty,0)$. From Fubini's theorem and since V is non-decreasing function in each argument that vanishes on D we obtain that

$$
\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{U_s>N\}}\mathbb{I}_{\{X_s>b(N)\}}\mathrm{d}s\right)\\
&\quad\geq \Phi'(0)\int_{b(N)}^{b(N)+1}\int_{(-\infty,0)}\widetilde{V}(N,z+y)\Pi(\mathrm{d}y)\mathrm{d}z+\Phi'(0)\int_{b(N)+1}^{\infty}\int_{(-\infty,0)}\widetilde{V}(N,z+y)\Pi(\mathrm{d}y)\mathrm{d}z\\
&\quad\geq \Phi'(0)\int_{(-\infty,0)}\widetilde{V}(N,b(N)+y)\Pi(\mathrm{d}y)+\Phi'(0)\int_{(-\infty,-1)}\int_{b(N)+1}^{b(N)-y}\widetilde{V}(N,z+y)\mathrm{d}z\Pi(\mathrm{d}y)\\
&\quad\geq \Phi'(0)\int_{(-\infty,0)}\widetilde{V}(N,b(N)+y)\Pi(\mathrm{d}y)-\Phi'(0)\int_{(-\infty,-1)}(y+1)\widetilde{V}(0,y)\Pi(\mathrm{d}y)\\
&\quad> -\infty,
\end{aligned}
$$

where the finiteness of the last integrals follow from Lemmas 4.2.1 and 4.4.19 and equation (4.20). Moreover, from the dominated convergence theorem we have that

$$
\begin{aligned}
&\lim_{u,x\to\infty}\mathbb{E}_{u,x}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\right)\\
&\qquad=\lim_{u,x\to\infty}\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(u+s,X_s+x+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s+x>b(u+s)\}}\mathbb{I}_{\{s<\sigma_{-x}^-\}}\right)\\
&\qquad\quad+\lim_{x\to\infty}\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s^{(-x)},X_s+x+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s+x>b(U_s^{(-x)})\}}\mathbb{I}_{\{s\geq\sigma_{-x}^-\}}\right)\\
&\qquad\geq\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)}\lim_{u,x\to\infty}\widetilde{V}(u+s,X_s+x+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s+x>b(u+s)\}}\right)\\
&\qquad\quad+\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)}\lim_{x\to\infty}\widetilde{V}(U_s^{(-x)},X_s+x+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s+x>b(U_s^{(-x)})\}}\right).
\end{aligned}
$$

Note that b is a decreasing function and then $\lim_{u,x\to\infty}V(u,x)=0$ and $\lim_{x\to\infty}V(u,x)=0$ for any $u>0$. Moreover, for any $s\geq 0$, $x\mapsto U_s^{(-x)}$ is increasing and bounded so then $\lim_{x\to\infty}U_s^{(-x)}$ exists. Then we have that

$$
\lim_{u,x\to\infty}\mathbb{E}_{u,x}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\right)=0
$$

as claimed. $\square$

Next, we show that V satisfies the alternative representation mentioned in the infinite variation case.

Lemma 4.4.25. *Suppose that X is of infinite variation. Then we have that V and b satisfy equations (4.27) and (4.29).*

Proof. We first prove that V satisfies equation (4.27) for the infinite variation case. Recall that V is continuous on E and, in this case, (see Lemma 4.4.15) we have that for any $u>0$, $\lim_{x\downarrow 0} V(u,x)=V(0,0)$ implying that $\widetilde{V}$ is continuous on $\mathbb{R}_+\times\mathbb{R}$. We follow an analogous argument as Lamberton and Mikou (2013) (see Theorem 3.2). Let ρ be a positive C^∞ function with support in $[0,1]\times[0,1]$ and $\int_0^\infty\int_0^\infty \rho(v,y)\mathrm{d}v\mathrm{d}y=1$. For $n\geq 1$, define $\rho_n(v,y)=n^2\rho(nv,ny)$, then ρ_n is C^∞ and has compact support in $[0,1/n]\times[0,1/n]$ and $\widetilde{V}_n(u,x):=(\widetilde{V}*\rho_n)(u,x)=\int_0^\infty\int_0^\infty \widetilde{V}(u-v,x-y)\rho_n(v,y)\mathrm{d}v\mathrm{d}y$ is a $C^{1,2}(\mathbb{R}_+\times\mathbb{R})$ function. Recall that since V is non-decreasing in each argument we have that $0\geq \widetilde{V}(u,x)=V(u,x)\geq V(0,-1)$ for all $u>0$ and $x>0$. Hence we have that for any $u>0$ and $x>0$,

$$\begin{aligned}\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}\widetilde{V}_n(u,x)\right| &\leq \int_0^\infty\int_0^\infty \left|\widetilde{V}(u-v,x-y)\right|\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}\rho_n(v,y)\right|\mathrm{d}v\mathrm{d}y\\ &\leq -V(0,-1)\int_0^\infty\int_0^\infty\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}n^2\rho(nv,ny)\right|\mathrm{d}v\mathrm{d}y\\ &= -V(0,-1)\int_0^\infty\int_0^\infty\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}\rho(v,y)\right|\mathrm{d}v\mathrm{d}y\end{aligned}$$

for any $i,j=\{0,1,2,\ldots\}$. Moreover, we have that

$$\begin{aligned}\int_{(-\infty,-1)}\widetilde{V}_n(u,x+y)\Pi(\mathrm{d}y) &\geq \int_{(-\infty,-1)}\widetilde{V}(u-1/n,x-1/n+y)\Pi(\mathrm{d}y)\\ &\geq \int_{(-\infty,-1)} V(0,y-1)\Pi(\mathrm{d}y)\\ &> -\infty\end{aligned}$$

for all $u>0$ and $x>0$, where the last inequality follows from Lemma 4.4.16. Hence, we conclude that the derivatives of $\widetilde{V}_n$ and the function $(u,x)\mapsto\int_{(-\infty,-1)}\widetilde{V}_n(u,x+y)\Pi(\mathrm{d}y)$ are bounded in the set $\mathbb{R}_+\times\mathbb{R}$. Furthermore, we that $\widetilde{V}_n\uparrow\widetilde{V}$ on $\mathbb{R}_+\times\mathbb{R}$ when $n\to\infty$. Indeed, for any $n\geq 1$ and $(u,x)\in\mathbb{R}_+\times\mathbb{R}$ we have that

$$
\begin{aligned}
\widetilde{V}_n(u,x) &= \int_0^\infty \int_0^\infty \widetilde{V}(u-v,x-y)n^2\rho(nv,ny)\mathrm{d}v\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty \widetilde{V}(u-v/n,x-y/n)\rho(v,y)\mathrm{d}v\mathrm{d}y \\
&\leq \int_0^\infty \int_0^\infty \widetilde{V}(u-v/(n+1),x-y/(n+1))\rho(v,y)\mathrm{d}v\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty \widetilde{V}(u-v,x-y)(n+1)^2\rho((n+1)v,(n+1)y)\mathrm{d}v\mathrm{d}y \\
&= V_{n+1}(u,x),
\end{aligned}
$$

where in the inequality we used that $v, y \geq 0$ and that V is non-decreasing in each argument. The convergence of $\widetilde{V}_n$ to $\widetilde{V}$ in $\mathbb{R}_+ \times \mathbb{R}$ follows from the inequality

$$
\begin{aligned}
|\widetilde{V}_n(u,x) - \widetilde{V}(u,x)| &\leq \int_0^\infty \int_0^\infty |\widetilde{V}(u-v,x-y) - \widetilde{V}(u,x)|\rho_n(v,y)\mathrm{d}v\mathrm{d}y \\
&\leq \sup_{v,y\in[0,1/n]} |\widetilde{V}(u-v,x-y) - \widetilde{V}(u,x)|,
\end{aligned}
$$

which is valid for any $(u,x) \in \mathbb{R}_+ \times \mathbb{R}$, where we used the fact that the integral of ρ_n is equal to 1. Taking $n \to \infty$ we obtain the desired convergence by using the fact that $\widetilde{V}$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$.

Next, we show (similar as in Lamberton and Mikou (2008), proof of Proposition 2.5) that for all $(u,x) \in [(1/n,\infty) \times (1/n,\infty)] \cap C^+$,

$$
\frac{\partial}{\partial u}\widetilde{V}_n(u,x) + \mathcal{A}_X(\widetilde{V}_n)(u,x) = -(G * \rho_n)(u,x), \tag{4.37}
$$

where $\mathcal{A}_X$ is the infinitesimal generator of the process X. Indeed, take φ a non-negative C^∞ function with compact support in $[(1/n,\infty) \times (1/n,\infty)] \cap C^+$ then we have that the function $\varphi * \check{\rho}_n$ is C^∞ and has compact support in C^+, where $\check{\rho}(v,y) = \rho_n(-v,-y)$ for all $(v,y) \in \mathbb{R} \times \mathbb{R}$. Hence, from Proposition A.5 we get that

$$
\langle \frac{\partial}{\partial t}\widetilde{V}_n + \mathcal{A}_X(\widetilde{V}_n) + G * \rho_n, \varphi \rangle = \langle \frac{\partial}{\partial t}\widetilde{V} + \mathcal{A}_X(\widetilde{V}) + G, \varphi * \check{\rho}_n \rangle = 0,
$$

where the last equality follows from Lemma 4.4.17. Therefore we have that by integration by parts formula and Lemma A.4 that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\left[\frac{\partial}{\partial t}\widetilde{V}_n(u,x)+\mathcal{A}_X(\widetilde{V}_n)(u,x)+G*\rho_n(u,x)\right]\varphi(u,x)\mathrm{d}x\mathrm{d}u=0$$

for any φ non-negative and C^∞ function with compact support in $[(1/n,\infty)\times(1/n,\infty)]\cap C^+$. Therefore (4.37) follows by continuity. On the other hand, note that if $(u,x)\in D$ we have that $V(u,x)=0$ and hence $\widetilde{V}_n(u,x)=0$ for n sufficiently large. Hence,

$$\frac{\partial}{\partial u}\widetilde{V}_n(u,x)+\mathcal{A}_X(\widetilde{V}_n)(u,x)=\int_{(-\infty,0)}\widetilde{V}_n(u,x+y)\Pi(\mathrm{d}y).$$

Therefore, by the dominated convergence theorem we have that,

$$\lim_{n\to\infty}\left[\frac{\partial}{\partial u}\widetilde{V}_n(u,x)+\mathcal{A}_X(\widetilde{V}_n)(u,x)\right]=\int_{(-\infty,0)}\widetilde{V}(u,x+y)\Pi(\mathrm{d}y).$$

for any $(u,x)\in D$.

Next, let $u>0$ and $x>0$ fixed and take $n>0$ and $k>0$ such that $u>1/n>0$ and $x>k\geq 1/n>0$. We apply Itô formula to $\widetilde{V}_n(u+t\wedge\tau_{k-x}^-,X_{t\wedge\tau_{k-x}^-}+x)$ to get that

$$\begin{aligned}\widetilde{V}_n(u+t\wedge\tau_{k-x}^-,X_{t\wedge\tau_{k-x}^-}+x)&=\widetilde{V}_n(u,x)+M_t\\&\quad+\int_0^{t\wedge\tau_{k-x}^-}\left[\frac{\partial}{\partial u}\widetilde{V}_n(u+s,X_s+x)+\mathcal{A}_X(\widetilde{V}_n)(u+s,X_s+x)\right]\mathrm{d}s,\end{aligned}$$

where $\{M_t,t\geq 0\}$ is a zero mean martingale (see Lemma A.2). Taking expectations we get that

$$\begin{aligned}&\mathbb{E}_x\left(\widetilde{V}_n(u+t\wedge\tau_k^-,X_{t\wedge\tau_k^-})\right)\\&\quad=\widetilde{V}_n(u,x)+\mathbb{E}_x\left(\int_0^{t\wedge\tau_k^-}\left[\frac{\partial}{\partial u}\widetilde{V}_n(u+s,X_s)+\mathcal{A}_X(\widetilde{V}_n)(u+s,X_s)\right]\mathrm{d}s\right)\\&\quad=\widetilde{V}_n(u,x)-\mathbb{E}_x\left(\int_0^{t\wedge\tau_k^-}(G*\rho_n)(u+s,X_s)\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right)\\&\qquad+\mathbb{E}_x\left(\int_0^{t\wedge\tau_k^-}\left[\frac{\partial}{\partial u}\widetilde{V}_n(u+s,X_s)+\mathcal{A}_X(\widetilde{V}_n)(u+s,X_s)\right]\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right),\end{aligned}$$

where we used the fact that b is finite for all $u > 0$ and that $\mathbb{P}_x(X_s = b(u+s)) = 0$ for all $s > 0$ and $x \in \mathbb{R}$ when X is of infinite variation (see Sato (1999)). Note that, since $X_t \geq \underline{X}_\infty$ for all $t > 0$ and V is non-decreasing in each argument, we have that

$$0 \geq \mathbb{E}_x\left(\widetilde{V}_n(u + t \wedge \tau_k^-, X_{t\wedge\tau_k^-})\right) \geq -A'_{p-1} - C'_{p-1}\mathbb{E}_{x-1}((-\underline{X}_\infty)^p) + V(0,0) > -\infty,$$

where the second inequality follows from equation (4.20) and the last quantity is finite by Lemma 4.2.1. Therefore by the dominated convergence theorem we have that letting $n, t \to \infty$ and $k \downarrow 0$,

$$\begin{aligned}\mathbb{E}_x\left(\widetilde{V}(u+\tau_0^-, X_{\tau_0^-})\right) = V(u,x) - \mathbb{E}_x\left(\int_0^{\tau_0^-} G(u+s, X_s)\mathbb{I}_{\{X_s < b(u+s)\}}\mathrm{d}s\right) \\ + \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} \widetilde{V}(u+s, X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s > b(u+s)\}}\mathrm{d}s\right)\end{aligned} \tag{4.38}$$

for all $u > 0$ and $x > 0$. Note that, since $\lim_{u\to\infty} b(u) = 0$, we have that $\lim_{u,x\to\infty}\widetilde{V}(u,x) = V(u,x) = 0$. Hence, since $\widetilde{V}(u,x) = V(0,x)$ for any $u \geq 0$ and $x \leq 0$ and X drifts to infinity we get that

$$\begin{aligned}\mathbb{E}_x\left(\widetilde{V}(u+\tau_0^-, X_{\tau_0^-})\right) &= \mathbb{E}_x\left(V(0, X_{\tau_0^-})\mathbb{I}_{\{\tau_0^- < \infty\}}\right) \\ &= V(0,0)\mathbb{P}_x(X_{\tau_0^-} = 0, \tau_0^- < \infty) + \mathbb{E}_x\left(V(0, X_{\tau_0^-})\mathbb{I}_{\{X_{\tau_0^-} < 0\}}\right) \\ &= V(0,0)\frac{\sigma^2}{2}W'(x) + \mathbb{E}_x\left(\int_0^{\tau_0^-} V(0, X_{s-}+y)\mathbb{I}_{\{X_{s-}+y<0\}}N(\mathrm{d}s,\mathrm{d}y)\right) \\ &= V(0,0)\frac{\sigma^2}{2}W'(x) + \mathbb{E}_x\left(\int_0^{\tau_0^-} V(0, X_s+y)\mathbb{I}_{\{X_s+y<0\}}\mathrm{d}s\Pi(\mathrm{d}y)\right),\end{aligned}$$

where in the second last equality we used the probability of creeping given in (1.16) (note that $\Phi(0) = 0$ since X drifts to infinity) and in the last the compensation formula for Poisson random measures (see equation (1.25)). Recall that for any $u > 0$ and $x > 0$,

$$\begin{aligned}\int_{(-\infty,0)} \widetilde{V}(u, x+y)\Pi(\mathrm{d}y) = \int_{(-\infty,0)} V(u,x+y)\mathbb{I}_{\{x+y>0\}}\Pi(\mathrm{d}y) \\ + \int_{(-\infty,0)} V(0,x+y)\mathbb{I}_{\{x+y\leq 0\}}\Pi(\mathrm{d}y).\end{aligned}$$

Then from above and equation (4.38) we see that

$$
\begin{aligned}
&V(u,x)\\
&= \mathbb{E}_x\left(\widetilde{V}(u+\tau_0^-, X_{\tau_0^-})\right) + \mathbb{E}_x\left(\int_0^{\tau_0^-} G(u+s,X_s)\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right)\\
&\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} \widetilde{V}(u+x,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right)\\
&= V(0,0)\frac{\sigma^2}{2}W'(x) + \mathbb{E}_x\left(\int_0^{\tau_0^-} V(0,X_s+y)\mathbb{I}_{\{X_s+y<0\}}\mathrm{d}s\Pi(\mathrm{d}y)\right)\\
&\quad + \mathbb{E}_x\left(\int_0^{\tau_0^-} G(u+s,X_s)\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right)\\
&\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} V(u+s,X_s+y)\mathbb{I}_{\{X_s+y>0\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right)\\
&\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} V(0,X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right)\\
&= V(0,0)\frac{\sigma^2}{2}W'(x)\\
&\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} V(u+s,X_s+y)\mathbb{I}_{\{0<X_s+y<b(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right)\\
&\quad + \mathbb{E}_x\left(\int_0^{\tau_0^-}\left[G(u+s,X_s)+\int_{(-\infty,0)} V(0,X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right),
\end{aligned}
$$

where in the last equality we used that $V(u+s,X_s+y)=0$ when $X_s+y\geq b(u+s)$. Moreover, we have that (4.29) follows directly from the equation above since $V(u,b(u))=0$ for all $u>0$. □

We define an auxiliary function. For all $(u,x)\in E$, we define

$$
\begin{aligned}
R(u,x) = \mathbb{E}_{u,x}&\left(\int_0^{\infty} G(U_s,X_s)\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right)\\
&- \mathbb{E}_{u,x}\left(\int_0^{\infty}\int_{(-\infty,0)} \widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right).
\end{aligned}
$$

Note from Lemma 4.4.24 that R is well defined and

$$\lim_{u,x\to\infty} R(u,x)=0.$$

Lemma 4.4.26. *For any $(u,x) \in E$ we have that*

$$\begin{aligned} V(u,x) &= R(u,x) \\ &= \mathbb{E}_{u,x}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right) \\ &\quad - \mathbb{E}_{u,x}\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right). \end{aligned} \tag{4.39}$$

Proof. First, we assume that X is of infinite variation. Let $(u,x) \in E$, from the Markov property applied to the stopping time τ_0^+, the fact that b is non-negative and equation (4.19) we get that for all $x<0$,

$$R(0,x) = \mathbb{E}_x\left(\int_0^{\tau_0^+} G(0,X_s)\mathrm{d}s\right) + R(0,0) = V(0,x) + R(0,0) - V(0,0).$$

Similarly, using the Markov property at time τ_0^- we get that for any $u>0$ and $x>0$ that

$$\begin{aligned} R(u,x) &= \mathbb{E}_x(R(0,X_{\tau_0^-})\mathbb{I}_{\{\tau_0^-<\infty\}}) + \mathbb{E}_x\left(\int_0^{\tau_0^-} G(u+s,X_s)\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right) \\ &\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)} \widetilde{V}(u+s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(u+s)\}}\mathrm{d}s\right) \\ &= V(u,x) + \mathbb{E}_x([R(0,X_{\tau_0^-}) - V(0,X_{\tau_0^-})]\mathbb{I}_{\{\tau_0^-<\infty\}}) \\ &= V(u,x) + [R(0,0)-V(0,0)]\mathbb{P}_x(\tau_0^-<\infty), \end{aligned}$$

where the second equality follows from equation (4.38) and the last from the expression for $R(u,x)$ deduced above. Then applying the strong Markov property at time τ_D, the fact that for any $s<\tau_D$ we have that $X_s<b(U_s)$ and the equation above we get that for $u \geq 0$ and $x<b(u)$

$$\begin{aligned} R(u,x) &= \mathbb{E}_{u,x}\left(\int_0^{\tau_D} G(U_s,X_s)\mathrm{d}s\right) + \mathbb{E}_{u,x}(R(U_{\tau_D},X_{\tau_D})) \\ &= V(u,x) + \mathbb{E}_{u,x}(R(U_{\tau_D},X_{\tau_D})) \\ &= V(u,x) + [R(0,0)-V(0,0)]\mathbb{E}_{u,x}(\mathbb{P}_{X_{\tau_D}}(\tau_0^-<\infty)), \end{aligned}$$

where in the first equality we used that τ_D is optimal for V and in the last we used that V

vanishes on D. Taking $u=0$ and $x=0$ we conclude that

$$0 = [R(0,0) - V(0,0)]\mathbb{E}(\mathbb{P}_{X_{\tau_D}}(\tau_0^- = \infty)).$$

Since $b(u) > 0$ for all $u > 0$ and $\mathbb{P}_x(\tau_0^- = \infty) > 0$ for all $x > 0$, the equation above implies that $R(0,0) = V(0,0)$ and then $V(u,x) = R(u,x)$ in the infinite variation case. For the finite variation case consider the sequence of stopping times,

$$\tau_b^{(1)} = \inf\{t \geq 0 : X_t \geq b(U_t)\}$$

and for $k = 1, 2, \ldots$

$$\begin{aligned}\sigma_b^{(k)} &= \inf\{t \geq \tau_b^{(k)} : X_t < b(U_t)\}\\ \tau_b^{(k+1)} &= \inf\{t \geq \sigma_b^{(k)} : X_t \geq b(U_t)\}.\end{aligned}$$

Since X is of finite variation we have that $\tau_b^{(k)} < \sigma_b^{(k)} < \tau_b^{(k+1)}$ for all $k = 1, 2, \ldots$. Let $u > 0$ and $x \geq b(u)$, by the Markov property applied to time $\tau_b^{(2)}$ we get that

$$\begin{aligned}R(u,x) &= -\mathbb{E}_{u,x}\left(\int_0^{\sigma_b^{(1)}}\int_{(-\infty,0)}\widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathrm{d}s\right)\\ &\quad + \mathbb{E}_{u,x}\left(\mathbb{I}_{\{\sigma_b^{(1)}<\infty\}}\int_{\sigma_b^{(1)}}^{\tau_b^{(2)}} G(U_s, X_s)\mathrm{d}s\right) + \mathbb{E}_{u,x}(R(U_{\tau_b^{(2)}}, X_{\tau_b^{(2)}})\mathbb{I}_{\{\tau_b^{(2)}<\infty\}})\\ &= -\mathbb{E}_{u,x}\left(\int_0^{\sigma_b^{(1)}}\int_{(-\infty,0)}\widetilde{V}(U_s, X_s+y)\Pi(\mathrm{d}y)\mathrm{d}s\right) + \mathbb{E}_{u,x}\left(\mathbb{I}_{\{\sigma_b^{(1)}<\infty\}}V(U_{\sigma_b^{(1)}}, X_{\sigma_b^{(1)}})\right)\\ &\quad + \mathbb{E}_{u,x}(R(U_{\tau_b^{(2)}}, X_{\tau_b^{(2)}})\mathbb{I}_{\{\tau_b^{(2)}<\infty\}})\\ &= \mathbb{E}_{u,x}(R(U_{\tau_b^{(2)}}, X_{\tau_b^{(2)}})\mathbb{I}_{\{\tau_b^{(2)}<\infty\}}),\end{aligned}$$

where in the second inequality we used the Markov property at time $\sigma_b^{(1)}$, the definition of V in terms of the stopping time τ_D and in the last equality we used the compensation formula for Poisson random measures. Using an induction argument we can verify that for all $x \geq b(u)$ and $n \geq 1$,

$$R(u,x) = \mathbb{E}_{u,x}(R(U_{\tau_b^{(n)}}, X_{\tau_b^{(n)}})\mathbb{I}_{\{\tau_b^{(n)}<\infty\}}).$$

Next, we show that for any $(u,x)\in E$, $\lim_{n\to\infty}\tau_b^{(n)}=\infty$ $\mathbb{P}_{u,x}$-a.s. First, note that since b is a non-negative function we have that for all $(u,x)\in D$, under the measure $\mathbb{P}_{u,x}$,

$$\sigma_b^{(1)}=\inf\{s\geq\tau_b^{(1)}:X_s<b(U_s)\}=\inf\{s\geq 0:X_s<b(U_s)\}\leq\tau_0^-$$

Hence, we obtain that under the measure $\mathbb{P}_{u,x}$, for any $(u,x)\in D$ that

$$\sigma_b^{(1)}=\inf\{s\geq 0:X_s<b(u+s)\}\geq\inf\{s\geq 0:X_s<b(u)\}=\tau_{b(u)}^-$$

Thus, for any $n\geq 2$, conditioning at time $\tau_b^{(n)}$ and the strong Markov property and using the fact that X creeps upwards we get that

$$\begin{aligned}\mathbb{P}_{u,x}(\sigma_b^{(n)}<\infty)&=\mathbb{P}_{u,x}(\tau_b^{(n)}<\infty,\sigma_b^{(n)}<\infty)\\&=\mathbb{E}_{u,x}(\mathbb{I}_{\{\tau_b^{(n)}<\infty\}}\mathbb{P}_{u,x}(\sigma_b^{(n)}-\tau_b^{(n)}<\infty|\mathcal{F}_{\tau_b^{(n)}})\\&=\mathbb{E}_x(\mathbb{I}_{\{\tau_b^{(n)}<\infty\}}\mathbb{P}_{U_{\tau_b(n)},X_{\tau_b(n)}}(\sigma_b^{(1)}<\infty))\\&\leq\mathbb{E}_x(\mathbb{I}_{\{\tau_b^{(n)}<\infty\}}f(U_{\tau_b(n)})),\end{aligned}$$

where $f(u)=\mathbb{P}_{b(u)}(\tau_{b(u)}^-<\infty)=\mathbb{P}(\tau_0^-<\infty)$. Therefore we have that for any $n\geq 2$,

$$\mathbb{P}_{u,x}(\sigma_b^{(n)}<\infty)\leq\mathbb{P}_{u,x}(\tau_b^{(n)}<\infty)\mathbb{P}(\tau_0^-<\infty)\leq\mathbb{P}_{u,x}(\sigma_b^{(n-1)}<\infty)\mathbb{P}(\tau_0^-<\infty),$$

where in the last inequality we used the fact that $\sigma_b^{(n-1)}\leq\tau_b^{(n)}$. Therefore, by an induction argument we obtain that for any $x\in\mathbb{R}$ and $n\geq 1$,

$$\mathbb{P}_{u,x}(\sigma_b^{(n)}<\infty)\leq\left[\mathbb{P}(\tau_0^-<\infty)\right]^{n-1}\mathbb{P}_{u,x}(\sigma_b^{(1)}<\infty).$$

Since X is of finite variation and drifts to infinity we have that $\mathbb{P}(\tau_0^-<\infty)\in(0,1)$. Then we have that for any $K>0$,

$$\sum_{n=1}^{\infty}\mathbb{P}_{u,x}(\sigma_b^{(n)}<K)\leq\sum_{n=1}^{\infty}\mathbb{P}_{u,x}(\sigma_b^{(n)}<\infty)\leq\mathbb{P}_{u,x}(\sigma_b^{(1)}<\infty)\sum_{n=1}^{\infty}\left[\mathbb{P}(\tau_0^-<\infty)\right]^{n-1}<\infty.$$

Hence, by the Borel–Cantelli Lemma we have that for all $K > 0$

$$\mathbb{P}_{u,x}(\limsup_{n\to\infty}\{\sigma_b^{(n)} < K\}) = 0$$

which implies that $\lim_{n\to\infty}\sigma_b^{(n)} = \infty$, $\mathbb{P}_{u,x}$-a.s. for all $(u,x) \in E$. Hence, by the dominated convergence theorem, the fact that $\lim_{u,x\to\infty} R(u,x) = 0$ (see (4.35) and (4.36)), that $\lim_{t\to\infty} U_t = t - g_t \geq \lim_{t\to\infty} t - g = \infty$ and that X drifts to infinity we get that

$$R(u,x) = \lim_{n\to\infty}\mathbb{E}_{u,x}(R(U_{\tau_b^{(n)}}, X_{\tau_b^{(n)}})\mathbb{I}_{\{\tau_b^{(n)}<\infty\}}) = 0,$$

for all $u > 0$ and $x \geq b(u)$. Now take $x < b(u)$, applying the strong Markov property and using that $\tau_b^{(1)}$ is optimal for V we get that

$$R(u,x) = \mathbb{E}_{u,x}\left(\int_0^{\tau_b^{(1)}} G(U_s, X_s)\mathrm{d}s\right) + \mathbb{E}_{u,x}(R(U_{\tau_b^{(1)}}, X_{\tau_b^{(1)}})) = V(u,x).$$

Hence, we conclude that for all $(u,x) \in E$,

$$V(u,x) = R(u,x).$$

□

Lemma 4.4.27. *The quadruplet $(V, b, V(0,0), u_b)$ satisfy equations (4.27)-(4.30) and equation (4.32).*

Proof. We know from Lemma 4.4.25 that equations (4.27) and (4.29) hold in the infinite variation case. Then suppose that X is of finite variation. The strong Markov property applied at time τ_0^- in (4.39) imply that (4.38) also holds in the finite variation case. Then proceeding as in Lemma 4.4.25 (see argument below equation (4.38)) we see that equations (4.27) and (4.29) also hold in the finite variation case. Moreover, the assertions about u_b and equation (4.30) follow from Lemma 4.4.21, the lower bound for $V(0,0)$ follows from Remark 4.4.3 and (4.32) holds due to Lemma 4.4.19.

We now proceed to show that (4.28) is satisfied for $V(0,0)$. Taking $u = x = 0$ in (4.39) and using Fubini's theorem we have that

$$
\begin{aligned}
&V(0,0)\\
&= \mathbb{E}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<b(U_s)\}}\mathrm{d}s\right) - \mathbb{E}\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right)\\
&= \mathbb{E}\left(\int_0^\infty G(0,X_s)\mathbb{I}_{\{X_s\leq 0\}}\mathrm{d}s\right) + \mathbb{E}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{0<X_s<b(U_s)\}}\mathrm{d}s\right)\\
&\quad - \mathbb{E}\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s,X_s+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>b(U_s)\}}\mathrm{d}s\right)\\
&= \int_{(-\infty,0]} G(0,z)\int_0^\infty \mathbb{P}(X_s\in \mathrm{d}z)\mathrm{d}s + \int_{(0,\infty)}\int_{(0,b(u))} G(u,z)\int_0^\infty \mathbb{P}(U_s\in \mathrm{d}u, X_s\in \mathrm{d}z)\mathrm{d}s\\
&\quad - \int_{(0,\infty)}\int_{(b(u),\infty)}\int_{(-\infty,0)} \widetilde{V}(u,z+y)\Pi(\mathrm{d}y)\int_0^\infty \mathbb{P}(U_s\in \mathrm{d}u, X_s\in \mathrm{d}z)\mathrm{d}s,
\end{aligned}
$$

where in the second equality we used the fact that b is non-negative and that $U_s=0$ if and only if $X_s\leq 0$. From (1.21) we know that 0-potential measure without killing is given by

$$
\int_0^\infty \mathbb{P}(X_s\in \mathrm{d}z)\mathrm{d}s = \frac{1}{\psi'(0+)} - W(-z) = \frac{1}{\psi'(0+)}\left[1-\psi'(0+)W(-z)\right]
$$

for any $z\leq 0$, where we used the $\Phi'(0+)=1/\psi'(0+)$. Hence, since $G(0,z)=-\mathbb{E}_z(g^{p-1})$ for any $z<0$ and the formula for the 0-potential density of (U,X) (see equation (3.9)) we have that

$$
\begin{aligned}
&V(0,0)\\
&= \int_{(-\infty,0]} G(0,z)\int_0^\infty \mathbb{P}(X_s\in \mathrm{d}z)\mathrm{d}s + \int_{(0,\infty)}\int_{(0,b(u))} G(u,z)\int_0^\infty \mathbb{P}(U_s\in \mathrm{d}u, X_s\in \mathrm{d}z)\mathrm{d}s\\
&\quad - \int_{(0,\infty)}\int_{(b(u),\infty)}\int_{(-\infty,0)} \widetilde{V}(u,z+y)\Pi(\mathrm{d}y)\int_0^\infty \mathbb{P}(U_s\in \mathrm{d}u, X_s\in \mathrm{d}z)\mathrm{d}s\\
&= -\frac{1}{\psi'(0+)}\int_0^\infty \mathbb{E}_{-z}(g^{p-1})[1-\psi'(0+)W(z)]\mathrm{d}z + \frac{1}{\psi'(0+)}\int_0^\infty\int_{(0,b(s))} G(s,z)\frac{z}{s}\mathbb{P}(X_s\in \mathrm{d}z)\mathrm{d}s\\
&\quad - \frac{1}{\psi'(0+)}\int_0^\infty\int_{(b(s),\infty)}\int_{(-\infty,0)} \widetilde{V}(s,z+y)\Pi(\mathrm{d}y)\frac{z}{s}\mathbb{P}(X_s\in \mathrm{d}z)\mathrm{d}s\\
&= -\frac{1}{\psi'(0+)}\int_0^\infty \mathbb{E}_{-z}(g^{p-1})[1-\psi'(0+)W(z)]\mathrm{d}z\\
&\quad + \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty G(s,X_s)\frac{X_s}{s}\mathbb{I}_{\{0<X_s<b(s)\}}\mathrm{d}s\right)\\
&\quad - \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)} \widetilde{V}(s,X_s+y)\Pi(\mathrm{d}y)\frac{X_s}{s}\mathbb{I}_{\{X_s>b(s)\}}\mathrm{d}s\right).
\end{aligned}
$$

Then equation (4.28) holds by recalling that $\widetilde{V}(u,x) = V(u,x)$ when $u > 0$ and $x > 0$ and $\widetilde{V}(u,x) = V(0,x)$ when $x \leq 0$ for any $u \geq 0$. □

We finish the first part of the proof by showing that the derivative of V at $(0,0)$ exists when there is a Brownian motion component.

Lemma 4.4.28. *The function V satisfies equation (4.31) when $\sigma > 0$*

Proof. Lastly we proceed to show that equation (4.31) holds when $\sigma > 0$. From equation (4.39) and the dominated convergence theorem we obtain that

$$\begin{aligned} V(0,0) &= \mathbb{E}\left(\int_0^\infty G(U_s, X_s)\mathbb{I}_{\{X_s < b(U_s)\}}\mathrm{d}s\right) \\ &\quad - \mathbb{E}\left(\int_0^\infty \int_{(-\infty,0)} \widetilde{V}(U_s, X_s + y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s > b(U_s)\}}\mathrm{d}s\right) \\ &= \lim_{\delta \downarrow 0}\left\{\mathbb{E}\left(\int_0^\infty K_1(U_s+\delta, X_s)\mathrm{d}s\right) - \mathbb{E}\left(\int_0^\infty K_2(U_s+\delta, X_s)\mathrm{d}s\right)\right\}, \end{aligned}$$

where $K_1(u,x) := G(u,x)\mathbb{I}_{\{x<b(u)\}}$ and $K_2(u,x) := \int_{(-\infty,0)} \widetilde{V}(u, x+y)\Pi(\mathrm{d}y)\mathbb{I}_{\{x>b(u)\}}$ for all $(u,x) \in E$. Note since b is non-increasing we have that $u \mapsto K_2(u,x)$ is non-decreasing for all $x \in \mathbb{R}$ and $\delta > 0$ and $|K_1(u+\delta,x)| \leq (u+\delta)^{p-1}\mathbb{I}_{\{u<b(\delta)\}} + \mathbb{E}_x(g^{p-1})$. Hence, from equations (4.33) and (4.34) and Theorem 3.2.6 applied to the functions K_1 and K_2 above we get that

$$V(0,0) = \lim_{\delta \downarrow 0}\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_\varepsilon\left(\mathbb{I}_{\{\tau_0^- < \infty\}}K^-(\delta, X_{\tau_0^-} - \varepsilon)\right) + K^+(\delta, \varepsilon)}{\psi'(0+)W(\varepsilon)},$$

where for all $\delta > 0$ and $x \leq 0$,

$$K^-(\delta, x) = \mathbb{E}_x\left(\int_0^{\tau_0^+} [K_1(\delta, X_r) - K_2(\delta, X_r)]\mathrm{d}r\right)$$

and for all $\delta, x > 0$,

$$K^+(\delta, x) = \mathbb{E}_x\left(\int_0^{\tau_0^-} [K_1(\delta+s, X_r) - K_2(\delta+s, X_r)]\mathrm{d}r\right).$$

Using the fact that b is non-negative and $W(x) = 0$ for all $x < 0$ (and then $G(\delta, x) = G(0,x)$

for all $x < 0$) we have that for all $x < 0$

$$K^{-}(\delta, x) = \mathbb{E}_x \left(\int_0^{\tau_0^+} G(\delta, X_s) \mathrm{d}s \right) = V(0,x) - V(0,0),$$

where the last equality follows from the expression of V in terms of the stopping time τ_D. Moreover for all $\delta > 0$ and $x > 0$ we have that from equation (4.38) that

$$K^{+}(\delta, \varepsilon) = V(\delta, \varepsilon) - \mathbb{E}_{\varepsilon}(V(0, X_{\tau_0^-}) \mathbb{I}_{\{\tau_0^- < \infty\}}).$$

Hence, rearranging the terms and by dominated convergence theorem we have that

$$\begin{aligned} V(0,0) &= \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}(V(0, X_{\tau_{-\varepsilon}^-}) \mathbb{I}_{\{\tau_{-\varepsilon}^- < \infty\}}) - \mathbb{E}(V(0, X_{\tau_{-\varepsilon}^-} + \varepsilon) \mathbb{I}_{\{\tau_{-\varepsilon}^- < \infty\}})}{\psi'(0+) W(\varepsilon)} \\ &\quad + \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{V(\delta, \varepsilon) - V(0,0) \mathbb{P}_{\varepsilon}(\tau_0^- < \infty)}{\psi'(0+) W(\varepsilon)} \\ &= \frac{\sigma^2}{2\psi'(0+)} \left[-\frac{\partial}{\partial x} V_{-}(0,0) + \frac{\partial}{\partial x} V_{+}(0,0) \right] + V(0,0), \end{aligned}$$

where in the last equality we used that $\mathbb{P}_{\varepsilon}(\tau_0^- < \infty) = 1 - \psi'(0+) W(\varepsilon)$ (see equation (1.9)) and the fact that $W'(0) = 2/\sigma^2$. Therefore we conclude that (4.31) holds. The proof is now complete. □

Now we show the uniqueness claim. Suppose that there exist continuous functions H and c on E and $\mathbb{R}_+$, respectively, and real numbers $H_0 < 0$ and $u_H > 0$ such that the conclusions of the theorem hold. Specifically, suppose that H is a non-positive continuous real valued function on E, c is a continuous real valued function on $(0, \infty)$ such that $c \geq h \geq 0$ and $H_0 \in (-\frac{1}{p} \mathbb{E}(g^p), 0)$ such that equations (4.27)-(4.29) hold. That is, we assume that H, H_0

and c are solutions to the equations

$$\begin{aligned}
&H(u,x)\\
&\quad = H_0 \frac{\sigma^2}{2} W'(x)\\
&\qquad - \mathbb{E}_x\left(\int_0^{\tau_0^-} \int_{(-\infty,0)} H(u+s, X_s+y)\mathbb{I}_{\{0<X_s+y<b(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(u+s)\}}\mathrm{d}s\right)\\
&\qquad + \mathbb{E}_x\left(\int_0^{\tau_0^-} \left[G(u+s, X_s) + \int_{(-\infty,0)} H(0, X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<c(u+s)\}}\mathrm{d}s\right),
\end{aligned} \tag{4.40}$$

for $u>0$ and $x>0$,

$$\begin{aligned}
H_0 = &-\frac{1}{\psi'(0+)}\int_0^\infty \mathbb{E}_{-z}(g^{p-1})[1-\psi'(0+)W(z)]\mathrm{d}z\\
&+ \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty G(s,X_s)\frac{X_s}{s}\mathbb{I}_{\{0<X_s<c(s)\}}\mathrm{d}s\right)\\
&- \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty \int_{(-\infty,0)} H(s,X_s+y)\mathbb{I}_{\{0<X_s+y<c(s)\}}\Pi(\mathrm{d}y)\frac{X_s}{s}\mathbb{I}_{\{X_s>c(s)\}}\mathrm{d}s\right)\\
&- \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty \int_{(-\infty,0)} H(0,X_s+y)\mathbb{I}_{\{X_s+y\le 0)\}}\Pi(\mathrm{d}y)\frac{X_s}{s}\mathbb{I}_{\{X_s>c(s)\}}\mathrm{d}s\right),
\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}
0 = &H_0 \frac{\sigma^2}{2} W'(c(u))\\
&- \mathbb{E}_{c(u)}\left(\int_0^{\tau_0^-} \int_{(-\infty,0)} H(u+s, X_s+y)\mathbb{I}_{\{0<X_s+y<c(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(u+s)\}}\mathrm{d}s\right)\\
&+ \mathbb{E}_{c(u)}\left(\int_0^{\tau_0^-} \left[G(u+s, X_s) + \int_{(-\infty,0)} H(0, X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<c(u+s)\}}\mathrm{d}s\right),
\end{aligned} \tag{4.42}$$

for $u<u_H$, where for any $x\le 0$,

$$H(0,x) = -\int_0^{-x}\int_{[0,\infty]} \mathbb{E}_{-u-z}(g^{p-1})W(\mathrm{d}u)\mathrm{d}z + H_0. \tag{4.43}$$

The value u_H is such that $u_H=\infty$ when X is of infinite variation or X is of finite variation with infinite activity. Otherwise, let u_H be the solution of equation (4.30), that is,

$$G(u,0)-\int_{(-\infty,0)}\int_0^{-y}\int_{[0,\infty]}\mathbb{E}_{-u-z}(g^{p-1})W(\mathrm{d}u)\mathrm{d}z\Pi(\mathrm{d}y)+H_0\Pi(-\infty,0)=0. \tag{4.44}$$

Moreover, assume that $c(u)>0$ for all $u<u_H$ and $c(u)=0$ for all $u\geq u_H$ and that

$$\int_{(-\infty,-x)}\widetilde{H}(u,x+c(u)+y)\Pi(\mathrm{d}y)+G(u,c(u)+x)\geq 0 \tag{4.45}$$

for all $u<u_H$ and $x>0$, where $\widetilde{H}$ is the extension of H to the set $\mathbb{R}_+\times\mathbb{R}$ as in (4.4). That is,

$$\widetilde{H}(u,x)=\begin{cases} H(u,x) & u>0 \text{ and } x>0,\\ H(0,x) & u\geq 0 \text{ and } x\leq 0,\\ H(0,0) & u=0 \text{ and } x>0.\end{cases} \tag{4.46}$$

Note that using the exact same arguments as the ones used in Lemma 4.4.25 (see argument below equation (4.38)) that (4.40) and (4.42) are equivalent to

$$\begin{aligned} H(u,x)&=\mathbb{E}_x(H(0,X_{\tau_0^-})\mathbb{I}_{\{\tau_0^-<\infty\}})+\mathbb{E}_x\left(\int_0^{\tau_0^-}G(u+s,X_s)\mathbb{I}_{\{X_s<c(u+s)\}}\mathrm{d}s\right)\\ &\quad-\mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)}\widetilde{H}(u+s,X_s+y)\mathbb{I}_{\{X_s+y<c(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(u+s)\}}\mathrm{d}s\right) \end{aligned} \tag{4.47}$$

for all $(u,x)\in E$ and

$$\begin{aligned} &\mathbb{E}_{c(u)}(H(0,X_{\tau_0^-})\mathbb{I}_{\{\tau_0^-<\infty\}})+\mathbb{E}_{c(u)}\left(\int_0^{\tau_0^-}G(u+s,X_s)\mathbb{I}_{\{X_s<c(u+s)\}}\mathrm{d}s\right)\\ &\quad=\mathbb{E}_{c(u)}\left(\int_0^{\tau_0^-}\int_{(-\infty,0)}\widetilde{H}(u+s,X_s+y)\mathbb{I}_{\{X_s+y<c(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(u+s)\}}\mathrm{d}s\right) \end{aligned} \tag{4.48}$$

for any $u<u_H$. Following a similar proof than du Toit and Peskir (2008) we are going to show that $c=b$ which implies that $H=V$, $H_0=V(0,0)$ and $u_H=u_b$.

First, we show that H has an alternative representation.

Lemma 4.4.29. *For all $(u,x)\in E$ we have that*

$$H(u,x)=\mathbb{E}_{u,x}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<c(U_s)\}}\mathrm{d}s\right) \\ -\mathbb{E}_{u,x}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{H}(U_s,X_s+y)\mathbb{I}_{\{X_s+y<c(U_s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(U_s)\}}\mathrm{d}s\right). \quad (4.49)$$

Moreover, the same conclusion holds if, in the case that $\sigma>0$, instead of (4.41) we assume that

$$\frac{\partial}{\partial x}H_+(0,0)=\frac{\partial}{\partial x}H_-(0,0), \quad (4.50)$$

where $\frac{\partial}{\partial x}H_+(u,0)$ and $\frac{\partial}{\partial x}H_-(0,0)$ are the right and left derivatives of $x\mapsto H(u,x)$ and $x\mapsto H(0,x)$ at zero, respectively and $\frac{\partial}{\partial x}H_+(0,0)=\lim_{u\downarrow 0}\frac{\partial}{\partial x}H_+(u,0)$.

Proof. Define for all $(u,x)\in E$ the function

$$K(u,x)=\mathbb{E}_{u,x}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<c(U_s)\}}\mathrm{d}s\right) \\ -\mathbb{E}_{u,x}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{H}(U_s,X_s+y)\mathbb{I}_{\{X_s+y<c(U_s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(U_s)\}}\mathrm{d}s\right).$$

In a analogous way than Lemma 4.4.27, from (3.9) and (1.21) and we have that for any spectrally negative Lévy process X,

$$
\begin{aligned}
K(0,0) &= \mathbb{E}\left(\int_0^\infty G(U_s,X_s)\mathbb{I}_{\{X_s<c(U_s)\}}\mathrm{d}s\right)\\
&\quad - \mathbb{E}_{u,x}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{H}(U_s,X_s+y)\mathbb{I}_{\{X_s+y<c(U_s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(U_s)\}}\mathrm{d}s\right)\\
&= \frac{1}{\psi'(0+)}\int_0^\infty \mathbb{E}_{-z}(g^{p-1})[1-\psi'(0+)W(z)]\mathrm{d}z\\
&\quad + \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty G(s,X_s)\frac{X_s}{s}\mathbb{I}_{\{0<X_s<c(s)\}}\mathrm{d}s\right)\\
&\quad - \frac{1}{\psi'(0+)}\mathbb{E}\left(\int_0^\infty\int_{(-\infty,0)}\widetilde{H}(s,X_s+y)\mathbb{I}_{\{X_s+y<c(s)\}}\Pi(\mathrm{d}y)\frac{X_s}{s}\mathbb{I}_{\{X_s>c(s)\}}\mathrm{d}s\right)\\
&= H_0\\
&= H(0,0).
\end{aligned}
$$

Moreover, for $u=0$ and $x<0$ we have that by the Markov property, the fact that X creeps upwards, c is a nonnegative curve and the definition of $H(0,x)$ for $x<0$ (see (4.43)) that

$$
K(0,x) = \mathbb{E}_x\left(\int_0^{\tau_0^+} G(U_s,X_s)\mathrm{d}s\right) + K(0,0) = H(0,x). \tag{4.51}
$$

Then, taking $u>0$ and $x>0$, by the strong Markov property at time τ_0^- and equation (4.47),

$$
\begin{aligned}
K(u,x) &= \mathbb{E}_x(K(0,X_{\tau_0^-})\mathbb{I}_{\{\tau_0^-<\infty\}}) + \mathbb{E}_x\left(\int_0^{\tau_0^-} G(u+s,X_s)\mathbb{I}_{\{X_s<c(u+s)\}}\mathrm{d}s\right)\\
&\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-}\int_{(-\infty,0)}\widetilde{H}(u+s,X_s+y)\mathbb{I}_{\{X_s+y<c(u+s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(u+s)\}}\mathrm{d}s\right)\\
&= H(u,x).
\end{aligned}
$$

If in the case that $\sigma>0$ we assume that H and c satisfy equations (4.43), (4.47), (4.48) and (4.50). From formula (3.6) (in a similar way than Lemma 4.4.27) we obtain that

$$
\begin{aligned}
K(0,0) &= \frac{\sigma^2}{2\psi'(0+)}\left[-\frac{\partial}{\partial x}H_-(0,0)+\frac{\partial}{\partial x}H_+(0,0)\right]+H(0,0)\\
&= H(0,0).
\end{aligned}
$$

The rest of the proof remains unchanged. □

Define the set $D_c = \{(u,x) \in E : x \geq c(u)\}$. We show in the following lemma that H vanishes in D_c so that D_c corresponds to the "stopping set" of H.

Lemma 4.4.30. *We have that $H(u,x) = 0$ for all $(u,t) \in D_c$.*

Proof. Note that from equations (4.47) and (4.48) we know that $H(u,c(u)) = 0$ for all $u \in (0,u_H)$. Let $(u,x) \in D_c$ such that $x > c(u)$ and define σ_c as the first time that (U,X) exits D_c, i.e.

$$\sigma_c = \inf\{s \geq 0 : X_s < c(U_s)\}.$$

From the fact that $X_r \geq c(U_r)$ for all $r < \sigma_c$ we have that from the Markov property and representation (4.49) of H,

$$\begin{aligned} H(u,x) &= \mathbb{E}_{u,x}(H(U_{\sigma_c}, X_{\sigma_c})\mathbb{I}_{\{\sigma_c<\infty\}}) + \mathbb{E}_{u,x}\left(\int_0^{\sigma_c} G(U_s,X_s)\mathbb{I}_{\{X_s<c(U_s)\}}\mathrm{d}s\right) \\ &\qquad - \mathbb{E}_{u,x}\left(\int_0^{\sigma_c}\int_{(-\infty,0)} \widetilde{H}(U_s,X_s+y)\mathbb{I}_{\{X_s+y<c(U_s)\}}\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s>c(U_s)\}}\mathrm{d}s\right) \\ &= \mathbb{E}_{u,x}(H(U_{\sigma_c}, X_{\sigma_c})\mathbb{I}_{\{\sigma_c<\infty, X_{\sigma_c}<c(U_s)\}}) \\ &\qquad - \mathbb{E}_{u,x}\left(\int_0^{\sigma_c}\int_{(-\infty,0)} \widetilde{H}(U_s,X_s+y)\mathbb{I}_{\{X_s+y<c(U_s)\}}\Pi(\mathrm{d}y)\mathrm{d}s\right), \end{aligned}$$

where the last equality follows from the fact that $\mathbb{P}_x(X_{\sigma_c} = c(u+\sigma_c)) > 0$ only when $\sigma > 0$ and then $U(u,c(u)) = 0$ for all $u > 0$ (since $u_H = \infty$). Then, since $H \leq 0$, applying the compensation formula for Poisson random measures (see equation (1.25)) and the fact that $\sigma_c \leq \tau_0^-$ (since $c(u) \geq 0$ for all $u > 0$) we get

$$\begin{aligned} &\mathbb{E}_{u,x}(H(U_{\sigma_c}, X_{\sigma_c})\mathbb{I}_{\{\sigma_c<\infty\}}\mathbb{I}_{\{X_{\sigma_c}<c(U_s)\}}) \\ &\quad = \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)} \mathbb{I}_{\{X_r\geq c(u+s) \text{ for all } r<s\}}\mathbb{I}_{\{X_{s-}+y<c(u+s)\}}\widetilde{H}(u+s, X_{s-}+y)N(\mathrm{d}s,\mathrm{d}y)\right) \\ &\quad = \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)} \mathbb{I}_{\{X_r\geq c(u+s) \text{ for all } r<s\}}\mathbb{I}_{\{X_{s-}+y<c(u+s)\}}\widetilde{H}(u+s, X_{s-}+y)\Pi(\mathrm{d}y)\mathrm{d}s\right) \\ &\quad = \mathbb{E}_{u,x}\left(\int_0^{\sigma_c}\int_{(-\infty,0)} \widetilde{H}(U_s,X_s+y)\mathbb{I}_{\{X_s<c(U_s)\}}\Pi(\mathrm{d}y)\mathrm{d}s\right). \end{aligned}$$

Hence we have that $H(u,x) = 0$ for all $(u,x) \in D_c$ as claimed. $\square$

The following Lemma states that H dominates the value function V. That suggest that H is the largest function with $H \leq 0$ that makes the process $\{H(U_t, X_t) + \int_0^t G(U_s, X_s)\mathrm{d}s, t \geq 0\}$ a $\mathbb{P}_{u,x}$-submartingale. The latter assertion will be shown indirectly on the upcoming lemmas.

Lemma 4.4.31. *We have that $H(u,x) \geq V(u,x)$ for all $(u,x) \in E$.*

Proof. If $(u,x) \in D_c$ we have the inequality

$$H(u,x) = 0 \geq V(u,x).$$

Now we show that the inequality also holds in $E \setminus D_c$. Consider the stopping time

$$\tau_c = \inf\{s \geq 0 : X_s \geq c(U_s)\}.$$

Then using the Markov property and equation (4.49) we get that for all $(u,x) \in E$ with $x < c(u)$ (here we take $c(0) := \lim_{u \downarrow 0} c(u) \geq \lim_{u \downarrow 0} h(u) = \infty$),

$$\begin{aligned}
H(u,x) &= \mathbb{E}_{u,x}\left(H(U_{\tau_c}, X_{\tau_c})\right) + \mathbb{E}_{u,x}\left(\int_0^{\tau_c} G(U_s, X_s)\mathbb{I}_{\{X_s < c(U_s)\}}\mathrm{d}s\right) \\
&\quad - \mathbb{E}_{u,x}\left(\int_0^{\tau_c}\int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s > c(U_s)\}}\mathrm{d}s\right) \\
&= \mathbb{E}_{u,x}\left(H(U_{\tau_c}, c(U_{\tau_c}))\right) + \mathbb{E}_{u,x}\left(\int_0^{\tau_c} G(U_s, X_s)\mathrm{d}s\right),
\end{aligned}$$

where in the second equality we used the fact X creeps upwards and $\tau_c < \infty$. Note that since $X_t > 0$ if and only if $U_t > 0$ for all $t > 0$ and that $c(u) > 0$ for all u sufficiently small we have that $c(U_{\tau_c}) > 0$ and hence $H(U_{\tau_c}, c(U_{\tau_c})) = 0$. Therefore

$$H(u,x) = \mathbb{E}_{u,x}\left(\int_0^{\tau_c} G(U_s, X_s)\mathrm{d}s\right) \geq V(u,x),$$

where the inequality follows from the definition of V as per (4.14). $\square$

It turns out that the fact that H dominates V implies that b dominates the curve c. This fact is shown in the following Lemma.

Lemma 4.4.32. *We have that* $b(u) \geq c(u)$ *for all* $u > 0$*.*

Proof. Note that in the case that X is of finite variation with $\Pi(-\infty, 0) < \infty$ we have that $c(u) = 0 \leq b(u)$ for all $u > u_H$. We proceed by contradiction. Suppose that there exists $u_0 > 0$ such that $b(u_0) < c(u_0)$. Then in the case that X is of finite variation with $\Pi(-\infty, 0) < \infty$, it holds that $u_0 < u_H$. Take $x > c(u_0)$ and consider the stopping time

$$\sigma_b = \inf\{s > 0 : X_s < b(U_s)\}.$$

Then from the Markov property and the representation of H given in (4.49) we have that

$$\begin{aligned} H(u_0, x) = \mathbb{E}_{u_0,x}\left(H(U_{\sigma_b^-}, X_{\sigma_b^-})\right) + \mathbb{E}_{u_0,x}\left(\int_0^{\sigma_b^-} G(U_s, X_s)\mathbb{I}_{\{X_s < c(U_s)\}} \mathrm{d}s\right) \\ - \mathbb{E}_{u_0,x}\left(\int_0^{\sigma_b^-} \int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s > c(U_s)\}} \mathrm{d}s\right). \end{aligned}$$

Moreover, since $V(u, x) = 0$ for $(u, x) \in D$ and $0 \geq H \geq V$ we have that

$$\begin{aligned} \mathbb{E}_{u_0,x}\left(H(U_{\sigma_b^-}, X_{\sigma_b^-})\right) &= \mathbb{E}_{u_0,x}\left(H(U_{\sigma_b^-}, X_{\sigma_b^-})\mathbb{I}_{\{X_{\sigma_b^-} < b(U_{\sigma_b^-})\}}\right) \\ &= \mathbb{E}_{u_0,x}\left(\int_0^{\sigma_b^-} \int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\mathbb{I}_{\{X_s + y \leq b(U_s)\}}\Pi(\mathrm{d}y)\mathrm{d}s\right) \\ &= \mathbb{E}_{u_0,x}\left(\int_0^{\sigma_b^-} \int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\Pi(\mathrm{d}y)\mathrm{d}s\right), \end{aligned}$$

where the second equality follows from the compensation formula for Poisson random measures. Hence, combining the two equations above and from the fact that $x > c(u_0)$ and then $H(u_0, x) = 0$ we get

$$0 = \mathbb{E}_{u_0,x}\left(\int_0^{\sigma_b^-} \left[G(U_s, X_s) + \int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\Pi(\mathrm{d}y)\right] \mathbb{I}_{\{X_s < c(U_s)\}} \mathrm{d}s\right).$$

Due to the continuity of b and c we have that there exists a value u_1 sufficiently small such that $c(v) > b(v)$ for all $v \in [u_0, u_1)$. Thus, from Lemma 4.4.19, the fact that $u \mapsto G(u,x)$ is strictly increasing when $x > 0$ and the inequality $U \geq V$ (see Lemma 4.4.31) we have that for all $u > 0$ and $x > b(u)$,

$$G(u,x) + \int_{(-\infty,0)} \widetilde{H}(u, x+y)\Pi(\mathrm{d}y) \geq G(u,x) + \int_{(-\infty,0)} \widetilde{V}(u, x+y)\Pi(\mathrm{d}y) > 0,$$

where the strict inequality follows from Lemma 4.4.19. Note that taking x sufficiently big we have that, under the measure $\mathbb{P}_{u_0,x}$, X spends a positive amount of time between the curves $b(u)$ and $c(u)$ for $u \in [u_0, u_1]$ with positive probability. Thus, since $\sigma_c < \tau_0^-$ the above facts imply that

$$0 = \mathbb{E}_{u_0,x}\left(\int_0^{\sigma_b^-} \left[G(U_s, X_s) + \int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\Pi(\mathrm{d}y)\right] \mathbb{I}_{\{X_s < c(U_s)\}}\mathrm{d}s\right) > 0,$$

which is a contradiction and then we have that $c(u) \leq b(u)$ for all $u > 0$. □

Note that (4.45) and the definition of u_H given in (4.44) imply the inequality $G(u,x) + \int_{(-\infty,0)} \widetilde{H}(u, x+y)\Pi(\mathrm{d}y) \geq 0$ for all $u > 0$ and $x > c(u)$. It can be shown that such inequality guarantees that the process $\{H(U_t, X_t) + \int_0^t G(U_s, X_s)\mathrm{d}s, t \geq 0\}$ is a $\mathbb{P}_{u,x}$-submartingale for all $(u,x) \in E$. We finish the proof showing that indeed c corresponds to b.

Lemma 4.4.33. *We have that then $c(u) = b(u)$ for all $u > 0$ and $V(u,x) = H(u,x)$ for all $(u,x) \in E$.*

Proof. Suppose that there exists $u > 0$ such that $c(u) < b(u)$ and take $x \in (c(u), b(u))$. Then we have by the Markov property and representation (4.49) that

$$\begin{aligned} H(u,x) = \mathbb{E}_{u,x}\left(H(U_{\tau_D}, X_{\tau_D})\right) + \mathbb{E}_{u,x}\left(\int_0^{\tau_D} G(U_s, X_s)\mathbb{I}_{\{X_s < c(U_s)\}}\mathrm{d}s\right) \\ - \mathbb{E}_{u,x}\left(\int_0^{\tau_D} \int_{(-\infty,0)} \widetilde{H}(U_s, X_s + y)\Pi(\mathrm{d}y)\mathbb{I}_{\{X_s > c(U_s)\}}\mathrm{d}s\right), \end{aligned}$$

where $\tau_D = \inf\{t > 0 : X_t \geq b(U_t)\}$. On the other hand, we have that

$$V(u,x)=\mathbb{E}_{u,x}\left(\int_0^{\tau_D}G(U_s,X_s)\mathrm{d}s\right).$$

Hence, since $X_{\tau_D}=b(U_{\tau_D})\geq c(U_{\tau_D})$ and Lemma 4.4.30 we have that $H(U_{\tau_D},X_{\tau_D})=0$. Moreover, using the inequality $H\geq V$ (see Lemma 4.4.31) we obtain that

$$0\geq\mathbb{E}_{u,x}\left(\int_0^{\tau_D}\left[G(U_s,X_s)+\int_{(-\infty,0)}\widetilde{H}(U_s,X_s+y)\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s>c(U_s)\}}\mathrm{d}s\right)>0,$$

where the strict inequality follows by the inequality (4.45) and the continuity of b and c. This contradiction allows us to conclude that $c(u)=b(u)$ for all $u>0$ and $H(u,x)=V(u,x)$ for all $(u,x)\in E$. □

Remark 4.4.34. *A close inspection of the proof tells us that the assumptions that $H\leq 0$ can be dropped when $\Pi\equiv 0$.*

4.5 Examples

4.5.1 Brownian Motion with drift example

Suppose that X_t is given by

$$X_t=\mu t+\sigma B_t,$$

where $\mu>0$, $\sigma>0$ and $B=\{B_t,t\geq 0\}$ is a standard Brownian motion. Here, we consider the case $p=2$. Then

$$G(u,x)=u\psi'(0+)W(x)-\mathbb{E}_x(g).$$

It is well known that for $\beta\geq 0$ and $q\geq 0$,

$$\psi(\beta)=\frac{\sigma^2}{2}\beta^2+\mu\beta\qquad\text{and}\qquad\Phi(q)=\frac{1}{\sigma^2}\left[\sqrt{\mu^2+2\sigma^2 q}-\mu\right].$$

Thus, $\psi'(0+) = \mu$, $\Phi'(0) = \frac{1}{\mu}$, $\Phi''(0) = -\frac{\sigma^2}{\mu^3}$ and $\Phi'''(0) = 3\sigma^4/\mu^5$. The scale function is (see e.g. Kuznetsov et al. (2013), pp 102) given by

$$W(x) = \frac{1}{\mu}(1 - \exp(-2\mu x/\sigma^2)), \qquad x \geq 0.$$

An easy calculation shows that $W^{*(2)}$ is given by

$$W^{*(2)}(x) = \frac{1}{\mu^2}x[1 + \exp(-2\mu/\sigma^2 x)] - \frac{\sigma^2}{\mu^2}\frac{1}{\mu}(1 - \exp(-2\mu/\sigma^2 x)), \qquad x \geq 0.$$

For all $x \in \mathbb{R}$, the value $\mathbb{E}_x(g)$ can be calculated from (4.7) via differentiation to have

$$\begin{aligned}\mathbb{E}_x(g) &= -\psi'(0+)[\Phi''(0+) + x\Phi'(0)^2] + \psi'(0+)W^{*2}(x) \\ &= \begin{cases} \frac{\sigma^2}{\mu^2} - \frac{x}{\mu} & x < 0 \\ \frac{\sigma^2}{\mu^2}\exp(-2\mu/\sigma^2 x) + \frac{x}{\mu}\exp(-2\mu/\sigma^2 x) & x \geq 0 \end{cases}.\end{aligned}$$

and $\mathbb{E}(g^2) = \Phi'''(0)\psi'(0+) = 3(\sigma/\mu)^4$. Moreover, we know that $X_r \sim N(\mu r, \sigma^2 r)$ and for any $x \geq y$ and $z \in \mathbb{R}$ that (see e.g. Salminen (1988), pp 154) that

$$\mathbb{P}_x\left(B_r^{(a)} \in \mathrm{d}z, \inf_{0\leq s\leq r} B_s^{(a)} \leq y\right) = \frac{1}{\sqrt{2\pi r}}e^{a(z-x)-a^2r/2-(|z-y|+x-y)^2/(2r)}\mathrm{d}z,$$

where $B_t^{(a)} = at + B_t$. Hence by noticing that $X_t + x = \sigma\left[\mu/\sigma t + B_t + x/\sigma\right] = \sigma[B_t^{(\mu/\sigma)} + x/\sigma]$ for any $t \geq 0$, we obtain that for any $x \geq 0$ and $z \geq 0$,

$$\begin{aligned}\mathbb{P}_x(X_r \in \mathrm{d}z, \underline{X}_r \leq 0) &= \mathbb{P}_{x/\sigma}(\sigma B_t^{(\mu/\sigma)} \in \mathrm{d}z, \sigma \underline{B}_r^{(\mu/\sigma)} \leq 0) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 r}}e^{\left(\frac{\mu}{\sigma^2}\right)(z-x)-\left(\frac{\mu}{\sigma}\right)^2 r/2-(z+x)^2/(2\sigma^2 r)}\mathrm{d}z \\ &= \frac{1}{\sqrt{2\pi\sigma^2 r}}e^{-\frac{2\mu}{\sigma^2}x}e^{\left(\frac{\mu}{\sigma^2}\right)(z+x)-\left(\frac{\mu}{\sigma}\right)^2 r/2-(z+x)^2/(2\sigma^2 r)}\mathrm{d}z \\ &= e^{-\frac{2\mu}{\sigma^2}x}\frac{1}{\sqrt{\sigma^2 r}}\phi\left(\frac{z+x-\mu r}{\sqrt{\sigma^2 r}}\right)\mathrm{d}z,\end{aligned}$$

where ϕ is the density of a standard normal distribution. Hence we have that for any $x \geq 0$ and $z \geq 0$,

$$\mathbb{P}_x(X_r \in \mathrm{d}z, \underline{X}_r \geq 0) = \frac{1}{\sqrt{\sigma^2 r}}\left[\phi\left(\frac{z-x-\mu r}{\sqrt{\sigma^2 r}}\right) - e^{-\frac{2\mu}{\sigma^2}x}\phi\left(\frac{z+x-\mu r}{\sqrt{\sigma^2 r}}\right)\right]\mathrm{d}z.$$

Then we calculate for any $u > 0$

$$\begin{aligned}
&\mathbb{E}_x \left(\int_0^{\tau_0^-} [(r+u)\psi'(0+)W(X_r) - \mathbb{E}_{X_r}(g)]\mathbb{I}_{\{X_r < b(r+u)\}} \mathrm{d}r \right) \\
&= \int_0^\infty \int_0^{b(r+u)} [(r+u)\psi'(0+)W(z) - \mathbb{E}_z(g)]\mathbb{P}_x(X_r \in \mathrm{d}z, \underline{X}_r \geq 0)\mathrm{d}r \\
&= \int_0^\infty \left\{ H(r,u,x,b(r+u)) - e^{-2\mu/\sigma^2 x} H(r,u,-x,b(r+u)) \right\} \mathrm{d}r,
\end{aligned}$$

where a lengthy but straightforward calculation gives

$$\begin{aligned}
H(r,t,x,b) &= \int_0^b [(r+t)\psi'(0+)W(z) - \mathbb{E}_z(g)] \frac{1}{\sqrt{\sigma^2 r}} \phi\left(\frac{z-x-\mu r}{\sqrt{\sigma^2 r}}\right) dz \\
&= (r+t)\left[\Psi\left(\frac{b-x-\mu r}{\sigma\sqrt{r}}\right) - \Psi\left(\frac{-x-\mu r}{\sigma\sqrt{r}}\right)\right] \\
&\quad - \left[\frac{x}{\mu} + t + \frac{\sigma^2}{\mu^2}\right] e^{-2\mu/\sigma^2 x}\left[\Psi\left(\frac{b-x+\mu r}{\sigma\sqrt{r}}\right) - \Psi\left(\frac{-x+\mu r}{\sigma\sqrt{r}}\right)\right] \\
&\quad + \frac{\sigma\sqrt{r}}{\mu} e^{-2\mu/\sigma^2 x}\left[\phi\left(\frac{b-x+\mu r}{\sigma\sqrt{r}}\right) - \phi\left(\frac{-x+\mu r}{\sigma\sqrt{r}}\right)\right].
\end{aligned}$$

From formula (4.19) we know that

$$\begin{aligned}
V(0,x) &= -\int_0^{-x} \int_{[0,\infty)} \mathbb{E}_{-u-z}(g) W(\mathrm{d}u)\mathrm{d}z + V(0,0) \\
&= \frac{3\sigma^2}{2\mu^3}x - \frac{1}{2\mu^2}x^2 + V(0,0).
\end{aligned}$$

Then,

$$\frac{\partial}{\partial x} V_-(0,0) = \frac{3\sigma^2}{2\mu^3}.$$

From Theorem 4.4.23 we have that for $u > 0$ and $x > 0$,

$$\begin{aligned}
V(u,x) &= V(0,0)[1 - \psi'(0+)W(x)] \\
&\quad + \int_0^\infty \left\{ H(r,u,x,b(r+u)) - e^{-2\mu/\sigma^2 x} H(r,u,-x,b(r+u)) \right\} \mathrm{d}r.
\end{aligned}$$

Therefore the curve $b(u)$ and $V(0,0)$ satisfy the equations

$$\int_0^\infty \left\{ H(r,u,x,b(r+u)) - e^{-2\mu/\sigma^2 x} H(r,u,-x,b(r+u)) \right\} \mathrm{d}r$$
$$+V(0,0)[1-\psi'(0+)W(b(u))] = 0,$$
$$\frac{3\sigma^2}{2\mu^3} - \frac{\partial}{\partial x} V_+(0,0) = 0,$$

for all $u > 0$, where

$$-\frac{3}{2}\frac{\sigma^4}{\mu^4} \leq V(0,0) < 0.$$

Note that $\frac{\partial}{\partial x} V_+(0,0)$ can be estimated via $[V(h_0,h_0) - V(0,0)]/h_0$ for h_0 sufficiently small.

We can approximate the integrals above by Riemann sums so a numerical approximation can be implement. Indeed, take $n \in \mathbb{Z}_+$ and $T > 0$ sufficiently large such that $h = T/n$ is small. For each $k \in \{0,1,2,\dots,n\}$, we define $u_k = kh$. Then the sequence of times $\{u_k, k = 0,1,\dots,n\}$ is a partition of the interval $[0,T]$. For any $x \in \mathbb{R}$ and $u \in [u_k, u_{k+1})$, we approximate $V(u,x)$ by

$$V_h(u_k,x) = V_0[1-\psi'(0+)W(x)] + \sum_{i=k}^{n-1} [H(u_{i-k+1},u_k,x,b_i) - e^{-2\mu/\sigma^2 x} H(u_{i-k+1},u_k,-x,b_i)]h,$$

where the sequence $\{b_k, k = 0,1,\dots,n-1\}$ and V_0 are solutions to

$$V_0[1-\psi'(0+)W(b_k)] + \sum_{i=k}^{n-1} [H(u_{i-k+1},u_k,b_k,b_i) - e^{-2\mu/\sigma^2 x} H(u_{i-k+1},u_k,-b_k,b_i)]h = 0$$
$$\frac{3\sigma^2}{2\mu^3} - \frac{V_h(h_0,h_0) - V_0}{h_0} = 0$$

for each $k \in \{0,1,\dots,n-1\}$. Note that, for T and n sufficiently large such that h is sufficiently small, the sequence $\{b_k, k = 0,1,\dots,n\}$ is a numerical approximation to the sequence $\{b(t_k), k = 0,1,\dots,n\}$ and can be calculated by using backwards for a fixed value V_0. Indeed, a method for solving the system is: fix V_0 and calculate the sequence $\{b_k^{V_0}, k = 0,1,\dots,n\}$ by using the first equation above. If the curve obtained and the value V_0 satisfy the second equation above then we have that $V_0 = V(0,0)$ and $\{b_k^{V_0}, k = 0,1,\dots,n\} =$

$\{b_k, k = 0, 1, \ldots, n\}$. Otherwise, vary the quantity V_0 and recalculate until both equations are satisfied. We show in Figure 4.3 a numerical calculation of the optimal boundary and the value function using the equations above. The case considered is when $\mu = 1/2$ and $\sigma = 1$.

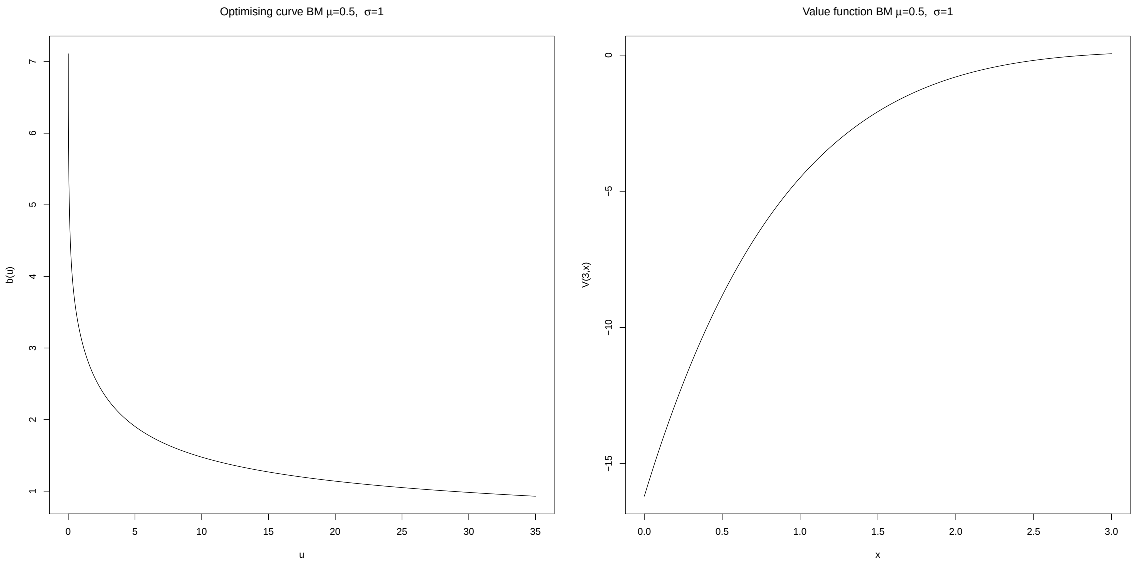

Figure 4.3: Numeric calculation of the optimal boundary and value function V for the Brownian motion with drift case.

4.5.2 Brownian motion with exponential jumps example

Consider the case in which $p = 2$ and X a Brownian motion with drift and exponential jumps, this is, $X = \{X_t, t \geq 0\}$ with

$$X_t = \mu t + \sigma B_t - \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

where $\sigma > 0$, $\mu > 0$, $B = \{B_t, t \geq 0\}$ is a standard Brownian motion, $N = \{N_t, t \geq 0\}$ is an independent Poisson process with rate $\lambda > 0$ and $\{Y_i, i \geq 1\}$ is a sequence of independent exponential distributed random variables with parameter $\rho > 0$ independent of B and N. We further assume that $\mu\rho > \lambda$ so X drifts to infinity. The Laplace exponent is given for $\beta \geq 0$ by

$$\psi(\beta) = \mu\beta + \frac{\sigma^2}{2}\beta^2 - \frac{\lambda\beta}{\rho + \beta},$$

where μ is a positive constant. In this case the Lévy measure is given by $\Pi(\mathrm{d}x) = \lambda\rho e^{\rho x}\mathrm{d}x$ for all $x < 0$. An easy calculation leads to $\psi'(0+) = \mu - \lambda/\rho$,

$$\Phi'(0+) = \frac{\rho}{\mu\rho - \lambda} \qquad \text{and} \qquad \Phi''(0+) = -\frac{\sigma^2\rho^3 + 2\lambda\rho}{[\mu\rho - \lambda]^3}.$$

It is know that (see e.g. Kuznetsov et al. (2013), pp 101) the scale function W is given by

$$W(x) = \frac{1}{\psi'(0+)} + \frac{e^{\zeta_1 x}}{\psi'(\zeta_1)} + \frac{e^{\zeta_2 x}}{\psi'(\zeta_2)}$$

for $x \geq 0$, where

$$\zeta_1 = \frac{-\left(\frac{\sigma^2}{2}\rho + \mu\right) + \sqrt{\left(\frac{\sigma^2}{2}\rho - \mu\right)^2 + 2\sigma^2\lambda}}{\sigma^2}$$

and

$$\zeta_2 = \frac{-\left(\frac{\sigma^2}{2}\rho + \mu\right) - \sqrt{\left(\frac{\sigma^2}{2}\rho - \mu\right)^2 + 2\sigma^2\lambda}}{\sigma^2}.$$

Then differentiating (4.7) we have that

$$\begin{aligned}
\mathbb{E}_x(g) &= -\psi'(0+)[\Phi''(0) + x\Phi'(0)^2] + \psi'(0+)W^{*2}(x) \\
&= \begin{cases} \frac{\sigma^2\rho^2+2\lambda}{[\mu\rho-\lambda]^2} - \frac{\rho}{\mu\rho-\lambda}x & x < 0 \\ \frac{\sigma^2\rho^2+2\lambda}{[\mu\rho-\lambda]^2} - \frac{\rho}{\mu\rho-\lambda}x + (\mu - \lambda/\rho)W^{*2}(x) & x \geq 0 \end{cases}.
\end{aligned}$$

For $x < 0$ the value function is then given by

$$\begin{aligned}
V(0,x) &= -\int_0^{-x}\int_{[0,\infty)} \mathbb{E}_{-u-z}(g)W(\mathrm{d}u)\mathrm{d}z + V(0,0) \\
&= \int_0^{-x}\int_{[0,\infty)} \left[\Phi''(0+) + \Phi'(0)^2(-u-z)\right]\psi'(0+)W(\mathrm{d}u)\mathrm{d}z + V(0,0) \\
&= \left[\Phi''(0)(-x) + \Phi'(0)^2\mathbb{E}(\underline{X}_\infty)(-x) - \Phi'(0)^2x^2/2\right] + V(0,0),
\end{aligned}$$

where in the last equality we used that $\psi(0+)W(x) = \mathbb{P}_x(\tau_0^- = \infty) = \mathbb{P}(-\underline{X}_\infty \leq x)$ and hence $\psi'(0+)W(\mathrm{d}u)$ is the density function of the random variable $\underline{X}_\infty$. From (1.17) we know that for any $\beta \geq 0$,

$$\mathbb{E}(e^{\beta \underline{X}_\infty}) = \psi'(0+)\frac{\beta}{\psi(\beta)}.$$

Hence, by differentiating and using the fact that $\Phi'(q) = 1/\psi'(\Phi(q))$ we can see that

$$\mathbb{E}(\underline{X}_\infty) = \frac{\Phi''(0)}{2\Phi'(0)^2}.$$

Hence,

$$V(0,x) = -\left[\frac{3}{2}\Phi''(0)x + \Phi'(0)^2x^2/2\right] + V(0,0)$$

for any $x < 0$. Next, we calculate for any $x > 0$,

$$\begin{aligned}
&\int_{(-\infty,0)} V(0,x+y)\mathbb{I}_{\{x+y<0\}}\Pi(\mathrm{d}y) \\
&\quad = \int_{-\infty}^{-x}\left[-\frac{3}{2}\Phi''(0)(x+y) - \Phi'(0)^2(x+y)^2/2 + V(0,0)\right]\lambda\rho e^{\rho y}\mathrm{d}y \\
&\quad = \lambda e^{-\rho x}\int_{-\infty}^{0}\left[-\frac{3}{2}\Phi''(0)y - \Phi'(0)^2y^2/2 + V(0,0)\right]\rho e^{\rho y}\mathrm{d}y \\
&\quad = \lambda e^{-\rho x}\left[\frac{3\Phi''(0)}{2\rho} - \frac{\Phi'(0)^2}{\rho^2} + V(0,0)\right].
\end{aligned}$$

Similarly, we have that for all $u > 0$ and $x > b(u)$,

$$\begin{aligned}
\int_{(-\infty,0)} V(u,x+y)\mathbb{I}_{\{0<x+y<b(u)\}}\Pi(\mathrm{d}y) &= e^{-\rho x}\int_0^{b(u)} V(u,y)\lambda\rho e^{\rho y}\mathrm{d}y \\
&= e^{-\rho(x-b(u))}\int_{(-b(u),0)} V(u,y+b(u))\Pi(\mathrm{d}y).
\end{aligned}$$

Hence, we have that for any $u, x > 0$, (4.27) reads as

$$\begin{aligned}
&V(u,x)\\
&= V(0,0)\frac{\sigma^2}{2}W'(x)\\
&\quad - \mathbb{E}_x\left(\int_0^{\tau_0^-} e^{-\rho(X_s - b(u+s))}\mathbb{I}_{\{X_s > b(u+s)\}}\int_{(-b(u+s),0)} V(u+s, y+b(u+s))\Pi(\mathrm{d}y)\mathrm{d}s\right)\\
&\quad + \mathbb{E}_x\left(\int_0^{\tau_0^-}\left[G(u+s, X_s) + \int_{(-\infty,0)} V(0, X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<b(u+s)\}}\mathrm{d}s\right)\\
&= V(0,0)\frac{\sigma^2}{2}W'(x)\\
&\quad - \int_0^{\infty}\int_{(-b(u+s),0)} V(u+s, y+b(u+s))\Pi(\mathrm{d}y)\mathbb{E}_x\left(e^{-\rho(X_s-b(u+s))}\mathbb{I}_{\{X_s>b(u+s), \underline{X}_s\geq 0\}}\right)\mathrm{d}s\\
&\quad + \int_0^{\infty}\mathbb{E}_x\left(\left[G(u+s, X_s) + \int_{(-\infty,0)} V(0, X_s+y)\mathbb{I}_{\{X_s+y<0\}}\Pi(\mathrm{d}y)\right]\mathbb{I}_{\{X_s<b(u+s), \underline{X}_s\geq 0\}}\mathrm{d}s\right)\\
&= V(0,0)\frac{\sigma^2}{2}W'(x) - \int_0^{\infty}\mathcal{V}(u+s, b(u+s))F_2(s,x,b(u+s))\mathrm{d}s\\
&\quad + \int_0^{\infty} F_1(s,u,x,b(u+s))\mathrm{d}s,
\end{aligned}$$

where for any $s, u, x, b > 0$

$$\begin{aligned}
F_1(s,u,x,b) &= \mathbb{E}\left(G(u+s, X_s+x)\mathbb{I}_{\{X_s+x<b, \underline{X}_s+x\geq 0\}}\right)\\
&\qquad + \mathbb{E}\left(\lambda e^{-\rho(X_s+x)}\left[\frac{3\Phi''(0)}{2\rho} - \frac{\Phi'(0)^2}{\rho^2} + V(0,0)\right]\mathbb{I}_{\{X_s+x<b, \underline{X}_s+x\geq 0\}}\right),\\
F_2(s,x,b) &= \mathbb{E}\left(e^{-\rho(X_s+x-b)}\mathbb{I}_{\{X_s+x>b, \underline{X}_s+x\geq 0\}}\right),\\
\mathcal{V}(u,b) &= \int_{(-b,0)} V(u, y+b)\Pi(\mathrm{d}y).
\end{aligned}$$

Then we have that V, b and $V(0,0)$ satisfy the equations

$$\begin{aligned}
V(u,x) &= V(0,0)\frac{\sigma^2}{2}W'(x) + \int_0^\infty F_1(s,u,x,b(u+s))\mathrm{d}s \\
&\quad - \int_0^\infty \mathcal{V}(u+s,b(u+s))F_2(s,x,b(u+s))\mathrm{d}s, \\
0 &= V(0,0)\frac{\sigma^2}{2}W'(b(u)) + \int_0^\infty F_1(s,u,b(u),b(u+s))\mathrm{d}s \\
&\quad - \int_0^\infty \mathcal{V}(u+s,b(u+s))F_2(s,b(u),b(u+s))\mathrm{d}s, \\
0 &= \frac{3}{2}\Phi''(0) + \frac{\partial}{\partial x}V_+(0,0)
\end{aligned}$$

for all $u,x>0$, where for any $b,s,u>0$ and $x\in\mathbb{R}$,

We can approximate the integrals above by Riemann sums so a numerical approximation can be implement. Indeed, take $n\in\mathbb{Z}_+$ and $T>0$ sufficiently large such that $h=T/n$ is small. For each $k\in\{0,1,2,\dots,n\}$, we define $u_k=kh$. Then the sequence of times $\{u_k, k=0,1,\dots,n\}$ is a partition of the interval $[0,T]$. For any $x\in\mathbb{R}$ and $u\in[u_k,u_{k+1})$, we approximate $V(u,x)$ by

$$V_h(u_k,x) = V_0\frac{\sigma^2}{2}W'(x) + \sum_{i=k}^{n-1}[F_1(u_{i-k+1},u_k,x,b_i) - \mathcal{V}_h(u_{i+1},b_{i+1})F_2(u_{i-k+1},x,b_i)]h,$$

where $\mathcal{V}(u_n,b_n)=0$ and

$$\mathcal{V}_h(u_i,b_i) = \sum_{j=1}^{\lfloor b_i/h\rfloor} V_h(u_i,jh)\lambda\rho e^{\rho jh}h$$

for any $i\in\{1,2,\dots,n-1\}$. The sequence $\{b_k, k=1,\dots,n-1\}$ and V_0 are solutions to

$$V_0\frac{\sigma^2}{2}W'(b_k) + \sum_{i=k}^{n-1}[F_1(u_{i-k+1},u_k,b_k,b_i) - \mathcal{V}_h(u_{i+1},b_{i+1})F_2(u_{i-k+1},b_k,b_i)]h = 0 \qquad (4.52)$$

$$\frac{3}{2}\Phi''(0) + \frac{V_h(h_0,h_0)-V_0}{h_0} = 0 \qquad (4.53)$$

for each $k\in\{0,1,\dots,n-1\}$. The functions F_1 and F_2 can be estimated by simulating the process $\{(X_t,\underline{X}_t),t\geq 0)\}$ (see e.g. Kuznetsov et al. (2011), Theorem 4 and Remark 3). Note that, for n and T sufficiently large, the sequence $\{b_k, k=1,\dots,n\}$ is a numerical approximation to the sequence $\{b(t_k), k=1,\dots,n\}$ and can be calculated by using backwards induction. Indeed, with a fixed value V_0 and the condition $\mathcal{V}(u_n, b_n) = 0$, we can first obtain b_{n-1} using equation (4.52). This allows us to compute $V_h(u_{n-1}, x)$ which in turn gives us $\mathcal{V}_h(u_{n-1}, b_{n-1})$. We can then finally obtain $b_{n-2}, \mathcal{V}_h(u_{n-2}, b_{n-2}), b_{n-3}, \mathcal{V}_h(u_{n-3}, b_{n-3}), \ldots, b_1$ by repeating the aforementioned steps. With these values, we can calculate $V_h(h_0, h_0)$ and repeat the procedure for different values of V_0 until (4.53) is satisfied.

4.6 Appendix

Proof of Lemma 4.4.1. First, notice that due to the spatial homogeneity of Lévy processes and that $x \mapsto \mathbb{E}_x(g^{p-1})$ is non-increasing it suffices to prove the assertion for $x \leq 0$. Using Fubini's theorem we have that for all $x \leq 0$,

$$\mathbb{E}_x \left(\int_0^\infty \mathbb{E}_{X_s}(g^{p-1}) \mathrm{d}s \right) = \int_{(-\infty,\infty)} \mathbb{E}_z(g^{p-1}) \int_0^\infty \mathbb{P}_x(X_s \in \mathrm{d}z).$$

Since X drifts to infinity we can use the density for the 0-potential measure of X without killing (see equation (1.21)) to obtain

$$\begin{aligned} \mathbb{E}_x \left(\int_0^\infty \mathbb{E}_{X_s}(g^{p-1}) \mathrm{d}s \right) &= \int_{-\infty}^\infty \mathbb{E}_z(g^{p-1}) \left[\frac{1}{\psi'(0+)} - W(x-z) \right] \mathrm{d}z \\ &= \frac{1}{\psi'(0+)} \int_{-\infty}^x \mathbb{E}_z(g^{p-1}) \left[1 - \psi'(0+) W(x-z) \right] \mathrm{d}z \\ &\quad + \frac{1}{\psi'(0+)} \int_x^\infty \mathbb{E}_z(g^{p-1}) \mathrm{d}z. \end{aligned} \tag{4.54}$$

Now we prove that the above two integrals are finite for all $x \leq 0$. From the fact that $z \mapsto \mathbb{E}_z(g^{p-1})$ is continuous on $\mathbb{R}$ and W is continuous on $(0,\infty)$ we can assume without of loss of generality that $x = 0$.

First, we show that the first integral on the right hand side of (4.54) is finite. From Lemma 4.2.2 we have that

$$\begin{aligned} &\int_0^\infty \mathbb{E}_{-z}(g^{p-1}) \left[1 - \psi'(0+) W(z) \right] \mathrm{d}z \\ &\qquad \leq 2^{p-1} \mathbb{E}(-\underline{X}_\infty) [\mathbb{E}(g^{p-1}) + A_{p-1}] + \frac{2^{p-1}}{p} C_{p-1} \mathbb{E}((-\underline{X}_\infty)^p), \end{aligned}$$

where A_{p-1} and C_{p-1} are non-negative constants. In the equality above we relied on the fact that $z \mapsto \psi(0+)W(z)$ corresponds to the distribution function of the random variable $-\underline{X}_\infty$. We conclude from Lemma 4.2.1 that

$$\int_0^\infty \mathbb{E}_{-z}(g^{p-1})\left[1-\psi'(0+)W(z)\right]\mathrm{d}z<\infty.$$

Now we proceed to check the finiteness of the second integral in (4.54) when $x=0$. Using the strong Markov property we have that

$$\begin{aligned}\int_0^\infty \mathbb{E}_z(g^{p-1})\mathrm{d}z &= \int_0^\infty \mathbb{E}_z(g^{p-1}\mathbb{I}_{\{\tau_0^-<\infty\}})\mathrm{d}z\\ &\leq 2^{p-1}\int_0^\infty \mathbb{E}_z((\tau_0^-)^{p-1}\mathbb{I}_{\{\tau_0^-<\infty\}})dz+2^{p-1}\int_0^\infty \mathbb{E}_z(\mathbb{E}_{X_{\tau_0^-}}(g^{p-1})\mathbb{I}_{\{\tau_0^-<\infty\}})\mathrm{d}z\\ &\leq 2^{p-1}\int_0^\infty \mathbb{E}_z((\tau_0^-)^{p-1}\mathbb{I}_{\{\tau_0^-<\infty\}})\mathrm{d}z+2^{p-1}\int_0^\infty \mathbb{E}_z(\mathbb{E}_{\underline{X}_\infty}(g^{p-1})\mathbb{I}_{\{\underline{X}_\infty<0\}})\mathrm{d}z\end{aligned}$$

where in the last inequality we used the fact that $\underline{X}_\infty \leq X_{\tau_0^-}$ and that $x\mapsto \mathbb{E}_x(g^{p-1})$ is a non-increasing function. Using Fubini's theorem we have that

$$\begin{aligned}\int_0^\infty \mathbb{E}_z(\mathbb{E}_{\underline{X}_\infty}(g^{p-1})\mathbb{I}_{\{\underline{X}_\infty<0\}})\mathrm{d}z &= \int_0^\infty\int_{(-\infty,0)} \mathbb{E}_y(g^{p-1})\mathbb{P}_z(\underline{X}_\infty\in \mathrm{d}y)\mathrm{d}z\\ &= \int_{(-\infty,0)} \mathbb{E}_y(g^{p-1})\int_0^\infty \mathbb{P}_z(\underline{X}_\infty\in \mathrm{d}y)\mathrm{d}z\\ &= \int_0^\infty \mathbb{E}_{-y}(g^{p-1})[1-\psi'(0+)W(y)]\mathrm{d}y\\ &< \infty.\end{aligned}$$

It thus only remains to show that

$$\int_0^\infty \mathbb{E}_z((\tau_0^-)^{p-1}\mathbb{I}_{\{\tau_0^-<\infty\}})\mathrm{d}z<\infty.$$

For this, define the function $F_1(q):=\int_0^\infty \mathbb{E}_z(e^{-q\tau_0^-}\mathbb{I}_{\{\tau_0^-<\infty\}})\mathrm{d}z$. Differentiating with respect to β the equation (1.17) and evaluating at zero we obtain that

$$F_1(q)=\int_0^\infty \mathbb{P}(-\underline{X}_{\mathbf{e}_q}>z)\mathrm{d}z=\mathbb{E}(-\underline{X}_{\mathbf{e}_q})=\frac{1}{\Phi(q)}-\frac{\psi'(0+)}{q},$$

where $\mathbf{e}_q$ is an independent exponential random variable with parameter $q > 0$. On the other hand, define the function $F_2(q) = \int_0^\infty \mathbb{E}_{-z}(e^{-q\tau_0^+})[1 - \psi'(0+)W(z)]\mathrm{d}z$. Using the expression for the Laplace transform of τ_0^+ the definition of W, we have that

$$F_2(q) = \int_0^\infty e^{-\Phi(q)z}[1 - \psi'(0+)W(z)]\mathrm{d}z = \frac{1}{\Phi(q)} - \frac{\psi'(0+)}{q} = F_1(q).$$

The fact that $F_2 = F_1$ implies that, when α is a natural number, we can take derivatives of order α (with the help of the dominated convergence theorem), at $q = 0$ and conclude that

$$\int_0^\infty \mathbb{E}_z((\tau_0^-)^\alpha \mathbb{I}_{\{\tau_0^- < \infty\}})\mathrm{d}z < \infty \quad \text{if and only if} \quad \int_0^\infty \mathbb{E}_{-z}((\tau_0^+)^\alpha)[1 - \psi'(0+)W(z)]\mathrm{d}z < \infty.$$

Furthermore, if $\alpha = k + \lambda$, with k a positive integer and $0 < \lambda < 1$, we can draw the same conclusion using the Marchaud derivative (see e.g. Laue (1980)). Using Lemma 4.2.2 we have that

$$\int_0^\infty \mathbb{E}_{-z}((\tau_0^+)^{p-1})[1 - \psi'(0+)W(z)]\mathrm{d}z < \infty.$$

and the proof is complete. □

Proof of Lemma 4.4.8. Let $x < 0$ and take $\delta > 0$. Then

$$\begin{aligned}
\mathbb{E}_{0,x}((\tau_D)^p) &= \mathbb{E}_x((\tau_b^{g,0})^p) \\
&\leq \mathbb{E}_x((\tau_b^{g,0})^p \mathbb{I}_{\{g+\delta < \tau_b^{g,0}\}}) + \mathbb{E}_x((g+\delta)^p \mathbb{I}_{\{g+\delta > \tau_b^{g,0}\}}) \\
&= \mathbb{E}((\tau_b^{g,x})^p \mathbb{I}_{\{g^{(-x)}+\delta < \tau_b^{g,x}\}}) + \mathbb{E}((g^{(-x)}+\delta)^p \mathbb{I}_{\{g^{(-x)}+\delta > \tau_b^{g,x}\}}).
\end{aligned}$$

Note that on the event $\{g^{(-x)} + \delta < \tau_b^{g,x}\}$ we have that

$$\begin{aligned}
\tau_b^{g,x} &= \inf\{t > g^{(-x)} + \delta : X_t + x \geq b(U_t^{(-x)})\} \\
&= \inf\{t > 0 : X_{t+g^{(-x)}+\delta} + x \geq b(t+\delta)\} + g^{(-x)} + \delta \\
&\leq \inf\{t > 0 : X_{t+g^{(-x)}+\delta} \geq b(\delta)\} + g^{(-x)} + \delta,
\end{aligned}$$

where the second equality follows from the fact that after $g^{(-x)}$, the process X never goes back below $-x$ and the last inequality holds since b is non-increasing. We have that the law of the process $\{X_{t+g^{(-x)}}+x, t\geq 0\}$ is the same as that of $\mathbb{P}^{\uparrow}$ where $\mathbb{P}^{\uparrow}=\mathbb{P}^{\uparrow}_0$ is the limit of $\mathbb{P}^{\uparrow}_x$ when $x\downarrow 0$ (see (1.23) for the definition of $\mathbb{P}^{\uparrow}_x$ and the lines below for the result stated). Using the Markov property and equation (1.24) we get

$$\begin{aligned}\mathbb{E}_x((\tau_b^{g,0})^p) &\leq 2^p\mathbb{E}^{\uparrow}(\mathbb{E}^{\uparrow}_{X_\delta}[(\tau^+_{b(\delta)})^p])+(2^p+1)\mathbb{E}_x((g+\delta)^p)\\ &= 2^p\mathbb{E}^{\uparrow}\left(\frac{W(b(\delta))}{W(X_\delta)}\mathbb{E}_{X_\delta}[(\tau^+_{b(\delta)})^p\mathbb{I}_{\{\tau_0^->\tau^+_{b(\delta)}\}}]\right)+(2^p+1)\mathbb{E}_x((g+\delta)^p)\\ &\leq 2^p\mathbb{E}[(\tau^+_{b(\delta)})^p]\mathbb{E}^{\uparrow}\left(\frac{W(b(\delta))}{W(X_\delta)}\right)+(2^p+1)\mathbb{E}_x((g+\delta)^p)\\ &= 2^p\mathbb{E}[(\tau^+_{b(\delta)})^p]\int_{(0,\infty)}\frac{W(b(\delta))}{W(z)}\mathbb{P}^{\uparrow}(X_\delta\in \mathrm{d}z)+(2^p+1)\mathbb{E}_x((g+\delta)^p),\end{aligned}$$

where the second inequality follows from the fact that $\mathbb{E}_x[(\tau_a^+)^p]\leq \mathbb{E}[(\tau_a^+)^p]$ for all $0\leq x\leq a$ and $X_\delta>0$ under $\mathbb{P}^{\uparrow}$. Thus, using (1.24) we have that

$$\begin{aligned}\mathbb{E}_x((\tau_b^{g,0})^p) &\leq 2^p\mathbb{E}[(\tau^+_{b(\delta)})^p]\int_{(0,\infty)}\frac{W(b(\delta))}{W(z)}\mathbb{P}^{\uparrow}(X_\delta\in \mathrm{d}z)+(2^p+1)\mathbb{E}_x((g+\delta)^p)\\ &= 2^p\mathbb{E}[(\tau^+_{b(\delta)})^p]\frac{W(b(\delta))}{\delta}\mathbb{E}(X_\delta^+)+2^p(2^p+1)\delta^p+2^p(2^p+1)\mathbb{E}_x((g)^p),\end{aligned}\tag{4.55}$$

where X_δ^+ is the positive part of X_δ. Thus from Lemma 4.2.1 we have that $\mathbb{E}_x((\tau_b^{g,0})^p)$ is finite for any $x<0$. Note that from the definition of $\tau_b^{g,x}$ we have that for any $x<0$, $\tau_b^{g,0}\leq \tau_b^{g,x}$ (since $U_t\geq U_t^{(-x)}$ for any $t>0$ and b is non increasing) and hence

$$\mathbb{E}((\tau_D)^p)=\mathbb{E}((\tau_b^{g,0})^p)\leq \mathbb{E}((\tau_b^{g,x})^p)=\mathbb{E}_{0,x}((\tau_D)^p)<\infty$$

Next, we show that $\mathbb{E}_{u,x}((\tau_D)^p)<\infty$ when $u,x>0$. From the Markov property of Lévy processes we have that

$$\begin{aligned}\mathbb{E}_{u,x}((\tau_D)^p) &= \mathbb{E}_x((\tau_b^{u,0})^p\mathbb{I}_{\{\tau_b^{u,0}<\sigma_0^-\}})+\mathbb{E}_x((\tau_b^{g,0})^p\mathbb{I}_{\{\tau_b^{u,0}>\sigma_0^-\}})\\ &\leq \mathbb{E}_x((\tau^+_{b(u)})^p)+2^p\mathbb{E}_x((\sigma_0^-)^p\mathbb{I}_{\{\sigma_0^-<\infty\}})+\mathbb{E}_x(\mathbb{I}_{\{\sigma_0^-<\infty\}}\mathbb{E}_{X_{\sigma_0^-}}[(\tau_b^{g,0})^p]).\end{aligned}$$

Using (4.55), the inequality $|X_{\sigma_0^-}|\leq|\underline{X}_\infty|$ under the event $\{\sigma_0^-<\infty\}$ and Lemmas 4.2.1 and

4.2.2 we deduce that $\mathbb{E}_{u,x}((\tau_D)^p) < \infty$ and the proof is complete. $\square$

Proof of Lemma 4.4.9. For any $x \leq 0$, we first show that $\lim_{h\downarrow 0} \sigma^-_{x+h} = \sigma^-_x$ and $\lim_{h\downarrow 0} \sigma^-_{x-h} = \tau^-_x$. Since $x \mapsto \sigma^-_x$ has non-increasing paths it follows that has right and left limits. Moreover, due the monotonicity property we can see that

$$\lim_{h\downarrow 0} \sigma^-_{x+h} \leq \sigma^-_x.$$

From the definition of σ^-_x we have that for all $x \leq 0$, under the event $\{\sigma^-_x < \infty\}$, $X_{\sigma^-_x} \leq x$. Then from the fact that X has only negative jumps we have that under $\{\sigma^-_x < \infty\}$,

$$X_{\lim_{h\downarrow 0} \sigma^-_{x+h}} \leq \lim_{h\downarrow 0} X_{\sigma^-_{x+h}} \leq x$$

which implies (from the definition of σ^-_x) that $\sigma^-_x \leq \lim_{h\downarrow 0} \sigma^-_{x+h}$ and then $\lim_{h\downarrow 0} \sigma^-_{x+h} = \sigma^-_x$ when $\{\sigma^-_x < \infty\}$. If $\omega \in \{\sigma^-_x = \infty\}$ we have that $\underline{X}_\infty(\omega) = \inf_{t\geq 0} X_t(\omega) > x$ and then there exists $h_0(\omega)$ sufficiently small such that $\underline{X}_\infty(\omega) > x + h$ for all $h < h_0(\omega)$. That directly implies that

$$\lim_{h\downarrow 0} \sigma^-_{x+h} = \infty = \sigma^-_x \qquad \text{under } \{\sigma^-_x = \infty\}.$$

Now we prove that $\lim_{h\downarrow 0} \sigma^-_{x-h} = \tau^-_x$ for all $x \leq 0$. From the definition of τ^-_x we can see that for all $x \leq 0$, $\tau^-_x \leq \lim_{h\downarrow 0} \sigma^-_{x-h}$. Then if $\tau^-_x = \infty$ the result follows. Take $\omega \in \{\tau^-_x < \infty\}$ and assume that $\tau^-_x(\omega) < \lim_{h\downarrow 0} \sigma^-_{x-h}(\omega)$. Note that for any $h > 0$ we have that $\lim_{h\downarrow 0} \sigma^-_{x-h}(\omega) \leq \sigma^-_{x-h}(\omega)$, implying that for all $s \in [0, \lim_{h\downarrow 0} \sigma^-_{x-h}(\omega))$, $X_s > x - h$ for all $h > 0$. Hence we conclude that $X_s \geq x$ for all $s \in [0, \lim_{h\downarrow 0} \sigma^-_{x-h})$, in particular holds for $s = \tau^-_x(\omega)$ which is a contradiction. Thus,

$$\lim_{h\downarrow 0} \sigma^-_{x-h} = \tau^-_x$$

Next, we show that for any $x < 0$, $\tau^-_x = \sigma^-_x$ a.s. Using the fact that $\sigma^-_x \leq \tau^-_x$ and the strong

Markov property we obtain that

$$\begin{aligned}\mathbb{P}(\tau_x^- > \sigma_x^-) &= \mathbb{E}\left(\mathbb{I}_{\{\sigma_x^-<\infty\}}\mathbb{P}_{X_{\sigma_x^-}}(\tau_x^- > 0)\right)\\ &= \mathbb{E}\left(\mathbb{I}_{\{\sigma_x^-<\infty\}}\mathbb{I}_{\{X_{\sigma_x^-}=0\}}\mathbb{P}_{X_{\sigma_x^-}}(\tau_x^- > 0)\right) + \mathbb{E}\left(\mathbb{I}_{\{\sigma_x^-<\infty\}}\mathbb{I}_{\{X_{\sigma_x^-}<0\}}\mathbb{P}_{X_{\sigma_x^-}}(\tau_x^- > 0)\right)\\ &= \mathbb{P}(\sigma_x^- < \infty, X_{\sigma_x^-} = 0)\mathbb{P}_x(\tau_x^- > 0).\end{aligned}$$

Note that if X is of infinite variation we have that $\mathbb{P}_x(\tau_x^- > 0) = 0$, otherwise $\mathbb{P}(\sigma_x^- < \infty, X_{\sigma_x^-} = x) = \mathbb{P}(\tau_x^- < \infty, X_{\tau_x^-} = x) = 0$, where we used the fact that the random vectors $(\tau_x^-, X_{\tau_0^-})$ and $(\sigma_x^-, X_{\sigma_x^-})$ have the same distribution and that X can only keep downwards when $\sigma > 0$ (see (1.14)). Hence $\mathbb{P}(\tau_x^- > \sigma_x^-) = 0$ for any $x < 0$. That implies that $\lim_{h\to 0}\sigma_{x+h}^- = \sigma_x^-$ a.s.

Now we proceed to show the second statement. Recall that $x \mapsto \sigma_x^-$ is non increasing, by the right continuity of X_t we have that for any $x < 0$,

$$\lim_{h\downarrow 0} X_{\sigma_{x-h}^-} = X_{\lim_{h\downarrow 0}\sigma_{x-h}^-} = X_{\sigma_x^-} \text{ a.s..}$$

Moreover, since $\sigma_{x+h}^- \uparrow \sigma_x^-$ a.s. when $h \downarrow 0$ we have by quasi-left continuity that for any $x < 0$,

$$\lim_{h\downarrow 0} X_{\sigma_{x+h}^-} = X_{\sigma_x^-} \text{ a.s.}$$

The proof is now complete. □

Proof of Lemma 4.4.10. First note that from Lemmas 4.2.1 and 4.20 we have that for any $x \in \mathbb{R}$,

$$\mathbb{E}_x\left(V(0, X_{\sigma_0^-})\mathbb{I}_{\{\sigma_0^-<\infty\}}\right) \geq \mathbb{E}_{u,x}\left(V(0, \underline{X}_\infty)\right) \geq -A'_{p-1} - C'_{p-1}\mathbb{E}_x(|\underline{X}_\infty|^p) + V(0,0) > \infty,$$

where $\underline{X}_\infty = \inf_{t\geq 0} X_t$. Next, since $\sigma_0^- = 0$ under the measure $\mathbb{P}_x$, for any $x \leq 0$, the assertion is satisfied for $V(0,x)$ when $x \leq 0$.

Assume that $u > 0$ and $x > 0$ and let $\tau \in \mathcal{T}$, and assume that $\tau < \infty$ $\mathbb{P}_{u,x}$-a.s. Then we have that by the strong Markov property that

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^{\tau} G(U_s,X_s)\mathrm{d}s\right)\\
&\quad= \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}\int_{\sigma_0^-}^{\tau} G(U_s,X_s)\mathrm{d}s\right)\\
&\quad= \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}\mathbb{E}_{X_{\sigma_0^-}}\left(\int_0^{\tau} G(U_s,X_s)\mathrm{d}s\right)\right)\\
&\quad\geq \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}V(0,X_{\sigma_0^-})\right),
\end{aligned}$$

where in the last inequality we used the definition of V. Hence, taking infimum over all $\tau\in\mathcal{T}'$, we have that for any $(u,x)\in E$,

$$\inf_{\tau\in\mathcal{T}}\mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}V(0,X_{\sigma_0^-})\right)\leq V(u,x).$$

On the other hand, from the definition of V, we obtain that for any $\tau\in\mathcal{T}'$ and $u,x>0$ that

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}V(0,X_{\sigma_0^-})\right)\\
&\quad\mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}\inf_{\tau'\in\mathcal{T}}\mathbb{E}_{X_{\sigma_0^-}}\left(\int_0^{\tau'} G(U_s,X_s)\mathrm{d}s\right)\right)\\
&\quad= \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma<\tau\}}\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}}\mathbb{E}_{u,x}\left(\int_0^{\tau'\circ\sigma_0^-+\sigma_0^-} G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma_0^-}\right)\right)\\
&\quad= \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma<\tau\}}\operatorname*{ess\,inf}_{\tau'\geq\sigma_0^-}\mathbb{E}_{u,x}\left(\int_{\sigma}^{\tau'} G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma}\right)\right)\\
&\quad= \mathbb{E}_{u,x}\left(\mathbb{I}_{\{\sigma_0^->\tau\}}\int_0^{\tau} G(U_s,X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^-<\tau\}}\operatorname*{ess\,inf}_{\tau'\geq\sigma_0^-}\mathbb{E}_{u,x}\left(\int_0^{\tau'} G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma_0^-}\right)\right),
\end{aligned}$$

where the second last equality follows since for any stopping time $\tau\in\mathcal{T}'$, we have that $\tau'\circ\sigma_0^-+\sigma_0^-\geq\sigma_0^-$ and the last since the term $\mathbb{I}_{\{\sigma_0^-<\tau\}}\int_0^{\sigma_0^-} G(U_s,X_s)\mathrm{d}s$ is $\mathcal{F}_{\sigma_0^-}$ measurable and does not depend on τ'. By using the definition of the essential infimum we have that for any stopping times τ and $\tau'\geq\sigma_0^-$,

$$\int_0^{\tau} G(U_s,X_s)\mathrm{d}s\geq\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}'}\int_0^{\tau} G(U_s,X_s)\mathrm{d}s$$

and

$$\operatorname*{ess\,inf}_{\tau'\geq\sigma_0^-}\mathbb{E}_{u,x}\left(\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma_0^-}\right)\geq\mathbb{E}_{u,x}\left(\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}'}\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma_0^-}\right).$$

Hence, we deduce that for any stopping time τ and any $u>0$ and $x>0$ that

$$\begin{aligned}
&\mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-}G(U_s,X_s)\mathrm{d}s+\mathbb{I}_{\{\sigma_0^-<\tau\}}V(0,X_{\sigma_0^-})\right)\\
&\geq\mathbb{E}_{u,x}\left(\mathbb{I}_{\{\sigma_0^->\tau\}}\int_0^{\tau}G(U_s,X_s)\mathrm{d}s+\mathbb{I}_{\{\sigma_0^-<\tau\}}\mathbb{E}_{u,x}\left(\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}}\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\middle|\mathcal{F}_{\sigma_0^-}\right)\right)\\
&=\mathbb{E}_{u,x}\left(\mathbb{I}_{\{\sigma_0^->\tau\}}\int_0^{\tau}G(U_s,X_s)\mathrm{d}s+\mathbb{I}_{\{\sigma_0^-<\tau\}}\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}}\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\right)\\
&\geq\mathbb{E}_{u,x}\left(\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}'}\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\right),
\end{aligned}$$

where in the equality we used the fact that for any stopping time the random variable $\mathbb{I}_{\{\sigma_0^->\tau\}}\int_0^\tau G(U_s,X_s)\mathrm{d}s$ is $\mathcal{F}_{\sigma_0^-}$ measurable. It is easy to show that the family of random variables $\{-\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s,\tau'\in\mathcal{T}'\}$ is upwards directed (see for example Peskir and Shiryaev (2006), pp 29) so that (see e.g. Peskir and Shiryaev (2006), Lemma 1.3 or Section 1.2.1) there exists a sequence of stopping times $\{\tau_k:k\geq1\}$ such that

$$\operatorname*{ess\,sup}_{\tau'\in\mathcal{T}'}\left[-\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\right]=\lim_{n\to\infty}\left[-\int_0^{\tau_k}G(U_s,X_s)\mathrm{d}s\right]$$

with $-\int_0^{\tau_k}G(U_s,X_s)\mathrm{d}s\leq-\int_0^{\tau_{k+1}}G(U_s,X_s)\mathrm{d}s$ for all $k\geq1$. Hence, by the monotone convergence theorem we have that

$$\begin{aligned}
\mathbb{E}_{u,x}\left(\operatorname*{ess\,inf}_{\tau'\in\mathcal{T}'}\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\right)&=-\mathbb{E}_{u,x}\left(\operatorname*{ess\,sup}_{\tau'\in\mathcal{T}'}\left[-\int_0^{\tau'}G(U_s,X_s)\mathrm{d}s\right]\right)\\
&=-\mathbb{E}_{u,x}\left(\lim_{k\to\infty}\left[-\int_0^{\tau_k}G(U_s,X_s)\mathrm{d}s\right]\right)\\
&=\lim_{k\to\infty}\mathbb{E}_{u,x}\left(\int_0^{\tau_k}G(U_s,X_s)\mathrm{d}s\right)\\
&\geq V(u,x).
\end{aligned}$$

Therefore we deduce that for any $u > 0$ and $x > 0$ and for any $\tau \in \mathcal{T}'$,

$$\mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s, X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^- < \tau\}} V(0, X_{\sigma_0^-})\right) \geq V(u, x)$$

which implies that

$$\inf_{\tau\in\mathcal{T}'} \mathbb{E}_{u,x}\left(\int_0^{\tau\wedge\sigma_0^-} G(U_s, X_s)\mathrm{d}s + \mathbb{I}_{\{\sigma_0^- < \tau\}} V(0, X_{\sigma_0^-})\right) \geq V(u, x)$$

for any $u > 0$ and $x > 0$. The proof is now complete. □

Proof of Lemma 4.4.14. Let $u \geq 0$ fixed. First we note that $x \mapsto \tau_b^{u,x}$ is non-increasing. That implies that for any $x \in \mathbb{R}$ the limits $\lim_{h\downarrow 0} \tau_b^{u,x+h}$ and $\lim_{h\downarrow 0} \tau_b^{u,x-h}$ exist and

$$\lim_{h\downarrow 0} \tau_b^{u,x+h} \leq \tau_b^{u,x} \leq \sigma_b^{u,x} \leq \lim_{h\downarrow 0} \tau_b^{u,x-h},$$

where $\sigma_b^{u,x} = \inf\{t > 0 : X_t + x > b(u + t)\}$.
First we show that for any $x \in \mathbb{R}$, $\lim_{h\downarrow 0} \tau_b^{u,x-h} = \sigma_b^{u,x}$. Note that the assertion is clear when $x > b(u)$, so we assume that $x \leq b(u)$. From the right continuity of X and the fact that $\tau_b^{u,x-h}$ decreases when $h \downarrow 0$ we have that

$$X_{\lim_{h\downarrow 0} \tau_b^{u,x-h}} = \lim_{h\downarrow 0} X_{\tau_b^{u,x-h}} = \lim_{h\downarrow 0}[b(u + \tau_b^{u,x-h}) - x + h] \leq b(u + \lim_{h\downarrow 0} \tau_b^{u,x-h}) - x,$$

where the last equality follows since b is non increasing. Moreover, we have that for all $s < \lim_{h\downarrow 0} \tau_b^{u,x-h}$ and $h > 0$, $X_s < b(u + s) - x + h$. The above facts imply that for all $s \in [0, \lim_{h\downarrow 0} X_{\tau_b^{u,x-h}}]$, $X_s \leq b(u + s) - x$ and then $\lim_{h\downarrow 0} X_{\tau_b^{u,x-h}} \leq \sigma_b^{u,x}$ establishing the claim. Furthermore, using the fact that $\tau_b^{u,x} \leq \sigma_b^{u,x}$ and the strong Markov property, we have that

$$\mathbb{P}(\tau_b^{u,x} < \sigma_b^{u,x}) = \mathbb{E}(f(u + \tau_b^{u,x}, X_{\tau_b^{u,x}})) = \mathbb{E}(f(u + \tau_b^{u,x}, b(u + \tau_b^{u,x}))),$$

where $f(v, y) = \mathbb{P}_y(\sigma_b^{v,0} > 0)$. Since b is non increasing we have that $\sigma_b^{v,0} \leq \tau_{b(v)}^+$, where $\tau_{b(v)}^+ = \inf\{t > 0 : X_t > b(v)\}$ so then for any $v \geq 0$ and $y \geq 0$, $f(v, y) \leq \mathbb{P}_y(\tau_{b(v)}^+ > 0)$. Therefore, since 0 is a regular point for $(0, \infty)$, we obtain that $f(v, b(v)) \leq \mathbb{P}_{b(v)}(\tau_{b(v)}^+ > 0) = 0$

for any $v > 0$ and hence

$$\mathbb{P}(\tau_b^{u,x} < \sigma_b^{u,x}) = 0.$$

Therefore we conclude that

$$\lim_{h\downarrow 0} \tau_b^{u,x-h} = \sigma_b^{u,x} = \tau_b^{u,x} \text{ a.s.}$$

Now we proceed to show that for any $u \geq 0$ and $x \in \mathbb{R}$, $\lim_{(h_1,h_2)\to(0,0)+} \tau_b^{u+h_1,x+h_2} = \tau_b^{u,x}$. Take $u \geq 0$ and $x \in \mathbb{R}$ fixed values. Note that since b is non-increasing we have for any $0 \leq h_1 < h$ and $0 \leq h_2 < h$ and $t \geq 0$ that,

$$b(u+h+t) - h \leq b(u+h_1+t) - h_2 \leq b(u+t).$$

Then we have that for $0 \leq h_1 < h$ and $0 \leq h_2 < h$,

$$\tau_b^{u+h,x+h} \leq \tau_b^{u+h_1,x+h_2} \leq \tau_b^{u,x}.$$

So it is enough to show that $\lim_{h\downarrow 0} \tau_b^{u+h,x+h} = \tau_b^{u,x}$. Note that $\tau_b^{u+h,x+h}$ increases when h decreases so the limit exists and

$$\lim_{h\downarrow 0} \tau_b^{u+h,x+h} \leq \tau_b^{u,x}.$$

Moreover, $\lim_{h\downarrow 0} \tau_b^{u+h,x+h} = \sup_{n\geq 0} \tau_b^{u+1/n,x+1/n}$ is a stopping time. Then by quasi-left continuity property of Lévy processes we have that

$$\begin{aligned}
X_{\lim_{h\downarrow 0} \tau_b^{u+h,x+h}} &= \lim_{h\downarrow 0} X_{\tau_b^{u+h,x+h}} \text{ a.s.} \\
&= \lim_{h\downarrow 0} [b(u+h+\tau_b^{u,x+h}) - x - h] \\
&\geq \lim_{h\downarrow 0} b(u+h+\lim_{h\downarrow 0} \tau_b^{u,x+h}) - x \\
&= b(u+\lim_{h\downarrow 0} \tau_b^{u,x+h}) - x,
\end{aligned}$$

where in the inequality we used the fact that b is non-increasing and that for any $h > 0$, $\tau_b^{u+h,x+h} \leq \lim_{h\downarrow 0} \tau_b^{u+h,x+h}$ and in the last equality that b is right-continuous. Hence we have that $\lim_{h\downarrow 0} \tau_b^{u+h,x+h} \in \{t > 0 : X_t + x \geq b(u+t)\}$ almost surely and then $\lim_{h\downarrow 0} \tau_b^{u+h,x+h} \geq \tau_b^{u,x}$ a.s. Hence, we conclude that for any $u \geq 0$ and $x \in \mathbb{R}$

$$\lim_{(h_1,h_2)\to(0,0)+} \tau_b^{u+h_1,x+h_2} = \tau_b^{u,x} \qquad \text{a.s.}$$

In particular we have that $\lim_{h\downarrow 0} \tau_b^{u,x+h} = \tau_b^{u,x}$ a.s. for any $u \geq 0$ and $x \in \mathbb{R}$ and then $\lim_{h\to 0} \tau_b^{u,x+h} = \tau_b^{u,x}$ a.s. holds.

□

Before proving Theorem 4.4.22 we first consider a technical lemma involving the derivative of the potential measure. More specifically, for fixed $a > 0$, $x \in (0,a)$ and $r \in \mathbb{N} \cup \{0\}$ denote by $U_r(a,x,\mathrm{d}y)$ as the measure

$$U_r(a,x,\mathrm{d}y) = \int_0^\infty t^r \mathbb{P}_x(X_t \in \mathrm{d}y, t < \sigma_0^- \wedge \tau_a^+)\mathrm{d}t.$$

Lemma 4.6.1. *Let $q \in \mathbb{N} \cup \{0\}$ such that $\int_{(-\infty,-1)} |x|^q \Pi(\mathrm{d}x) < \infty$. Fix $a > 0$ and $0 \leq x \leq a$. We have that for all $r \in \{0,1,\ldots,q\}$ the measure $U_r(a,x,\mathrm{d}y)$ is absolutely continuous with respect to the Lebesgue measure. It has a density $u_r(a,x,y)$ given by*

$$u_r(a,x,y) = \lim_{q\downarrow 0} (-1)^r \frac{\partial^r}{\partial q^r}\left[\frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(a)}(a)} - W^{(q)}(x-y)\right],$$

for $y \in (0,a]$. Moreover, for a fixed $a > 0$ the functions $x \mapsto \mathbb{E}_x((\tau_a^+)^r \mathbb{I}_{\{\sigma_0^- < \tau_a^+\}})$ and $x \mapsto u_r(a,x,y)$ are differentiable on $(0,a)$ and have finite left derivative at $x = a$ for all $y \in (0,a)$ and $r \in \{0,1,\ldots,q\}$.

Proof. Let $a > 0$ and $x \in (0,a)$. First we show that for all $r \in \{0,1,\ldots,q\}$ the measure $U_r(a,x,\mathrm{d}y)$ is absolutely continuous with respect to the Lebesgue measure. Take any measurable set $A \subset (0,a)$, thus by Fubini's theorem

$$\begin{aligned}\int_A U_r(a,x,\mathrm{d}y) &= \int_0^\infty t^r \mathbb{P}_x(X_t \in A, t < \sigma_0^- \wedge \tau_a^+)\mathrm{d}t \\ &= \mathbb{E}_x\left(\int_0^{\tau_a^+ \wedge \sigma_0^-} t^r \mathbb{I}_{\{X_t \in A\}}\mathrm{d}t\right).\end{aligned}$$

From Lemma 4.2.1 we know that $\mathbb{E}_x((\tau_a^+)^r) < \infty$ for all $r \in \{0, 1, \ldots, q\}$. Then by dominated convergence theorem we have that

$$\begin{aligned}\int_A U_r(a, x, \mathrm{d}y) &= \lim_{q \downarrow 0} \mathbb{E}_x\left(\int_0^{\tau_a^+ \wedge \sigma_0^-} t^r e^{-qt} \mathbb{I}_{\{X_t \in A\}} \mathrm{d}t\right) \\ &= \int_A \lim_{q \downarrow 0} (-1)^r \frac{\partial^r}{\partial q^r}\left[\frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(a)}(a)} - W^{(q)}(x-y)\right] \mathrm{d}y,\end{aligned}$$

where the last equality follows from (1.18). From the convolution representation of $W^{(q)}$ (see equation (1.7)) the derivatives in the last equation above exist and indeed $u_r(a, x, y)$ is a density of $U_r(a, x, \mathrm{d}y)$ for all $y \in (0, a)$. Now we proceed to show the differentiation statements. Note that from equations (1.3) and (1.8) we have that

$$f_x(q) := \mathbb{E}_x(e^{-q\tau_a^+} \mathbb{I}_{\{\sigma_0^- < \tau_a^+\}}) = e^{\Phi(q)(x-a)} - \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

for any $x \in (0, a)$. Since W is differentiable, the proof follows by induction and implicit differentiation. A similar argument works for the function $x \mapsto u_r(a, x, y)$. □

We also need a technical lemma regarding the convergence of the stopping time $\tau_b^{g,y}$ (defined in (4.22)). Recall that in this context we understand $b(0)$ as infinity.

Lemma 4.6.2. *For all $y \in \mathbb{R}$ we have that*

$$\lim_{h \downarrow 0} \tau_b^{g, y-h} = \tau_b^{g,y} \qquad a.s.$$

Proof. Recall that for all $t \geq 0$, the mapping $x \mapsto U_t^{(x)}$ is non-increasing. Then we have that, for any $y_1 \leq y_2$ and fixed $t \geq 0$, $U_t^{(-y_1)} \leq U_t^{(-y_2)}$ so that $b(U_t^{(-y_1)}) \geq b(U_t^{(-y_2)})$. Thus for any $y_1 \leq y_2$ we see that

$$\{t > 0 : X_t \geq b(U_t^{(-y_1)}) - y_1\} \subset \{t > 0 : X_t \geq b(U_t^{(-y_2)}) - y_2\}.$$

Therefore we conclude that $\tau_b^{g,y_2} \leq \tau_b^{g,y_1}$ when $y_1 \leq y_2$. That implies that for all $y \in \mathbb{R}$, $\lim_{h \downarrow 0} \tau_b^{g,y-h}$ exists and

$$\sigma_b^{g,y} \leq \lim_{h \downarrow 0} \tau_b^{g,y-h},$$

where $\sigma_b^{g,y} = \inf\{t > 0 : X_t > b(U_t^{(-y)}) - y\}$. Note that since the sequence $\tau_b^{g,y-h}$ decreases when $h \downarrow 0$ we have that for all $s \in [0, \lim_{h\downarrow 0} \tau_b^{g,y-h})$ that $X_s < b(U_s^{(-y+h)}) - y + h$ for all $h > 0$. By taking $h \downarrow 0$ and by right-continuity of the mapping $x \mapsto U_t^{(x)}$ and the continuity of b we conclude that $X_s \leq b(U_s^{(-y)}) - y$ for all $s \in [0, \lim_{h\downarrow 0} \tau_b^{g,y-h})$. Therefore we have that $\lim_{h\downarrow 0} \tau_b^{g,y-h} = \sigma_b^{g,y}$.

Hence it is only left to show that $\tau_b^{g,y} = \sigma_b^{g,y}$ a.s. Note that we have the inequality $\tau_b^{g,y} \leq \sigma_b^{g,y}$, then by the strong Markov property applied to the time $\tau_b^{g,y}$ we have that

$$\begin{aligned}\mathbb{P}(\tau_b^{g,y} < \sigma_b^{g,y}) &= \mathbb{E}_y(\mathbb{P}(\tau_b^{g,0} < \sigma_b^{g,0} | \tau_b^{g,0}))\\ &= \mathbb{E}_y(f(U_{\tau_b^{g,0}}, b(U_{\tau_b^{g,0}})),\end{aligned}$$

where $f(u,x) = \mathbb{P}_{u,x}(\sigma_b^{g,0} > 0)$. Note that for any $u > 0$ and $x > 0$,

$$\begin{aligned}\mathbb{P}_{u,x}(\sigma_b^{g,0} > 0) &= \mathbb{P}_{u,x}(\sigma_b^{g,0} > 0, \sigma_b^{u,0} \leq \sigma_0^-) + \mathbb{P}_{u,x}(\sigma_b^{g,0} > 0, \sigma_b^{u,0} > \sigma_0^-)\\ &= \mathbb{P}_x(\sigma_b^{u,0} > 0, \sigma_b^{u,0} \leq \sigma_0^-) + \mathbb{P}_{u,x}(\sigma_b^{g,0} > 0, \sigma_b^{u,0} > \sigma_0^-)\\ &\leq \mathbb{P}_x(\sigma_b^{u,0} > 0) + \mathbb{P}_x(\sigma_b^{u,0} > \sigma_0^-)\\ &\leq \mathbb{P}_x(\tau_{b(u)}^+ > 0) + \mathbb{P}_x(\tau_{b(u)}^+ > \sigma_0^-)\end{aligned}$$

where $\sigma_b^{u,0} = \inf\{t > 0 : X_t > b(u+t)\}$ and the last inequality follows since b is non-increasing and then $\sigma_b^{u,0} \leq \tau_{b(u)}^+$. Hence, since 0 is a regular point for $(0,\infty)$, we have that for any $u > 0$ such that $b(u) > 0$, $\mathbb{P}_{u,b(u)}(\sigma_b^{g,0} > 0) = 0$. Then we conclude that $\mathbb{P}(\tau_b^{g,y} < \sigma_b^{g,y}) = \mathbb{E}(f(U_{\tau_b^{g,0}}, b(U_{\tau_b^{g,0}})) = 0$, where we used that $b(U_{\tau_b^{g,0}}) > 0$. The proof is then complete.

□

Hence, we are ready to proof that the partial derivatives of V at $(u, b(u))$ exist and are equal to zero.

Proof of Theorem 4.4.22. We first show that for all $u > 0$ such that $b(u) > 0$,

$$\frac{\partial}{\partial u} V(u, b(u)) = 0.$$

From the proof of Lemma 4.4.15 we know that for any $h > 0$

$$0 \leq \frac{V(u,b(u)) - V(u-h,b(u))}{h} \leq \mathbb{E}_{b(u)}\left(\int_0^{\tau^+_{b(u-h)}} \frac{[(u+s)^{p-1} - (u-h+s)^{p-1}]}{h} \mathrm{d}s\right).$$

The result then follows taking $h \downarrow 0$ and from the fact that the function $u \mapsto u^p$ is differentiable on $[0,\infty)$ for all $p > 1$, the dominated convergence theorem and the fact that b is continuous.

Next we proceed to show that derivative on the spatial argument exists and is zero, i.e.

$$\frac{\partial}{\partial x} V(u, b(u)) = 0.$$

Let $x > 0$, $u > 0$ and $0 < \varepsilon < 1$ such that $x - \varepsilon > 0$ and $b(u) > 0$. From equation (4.23) we know that

$$\begin{aligned}
&V(u, x-\varepsilon)\\
&= \mathbb{E}\left(\int_0^{\tau_b^{u,x-\varepsilon} \wedge \sigma^-_{\varepsilon - x}} G(u+s, X_s + x - \varepsilon)\mathrm{d}s\right)\\
&\quad + \mathbb{E}\left(\mathbb{I}_{\{\sigma^-_{\varepsilon-x} < \tau_b^{u,x-\varepsilon}\}} \int_{\sigma^-_{\varepsilon-x}}^{\tau_b^{g,x-\varepsilon}} G(U_s^{(\varepsilon - x)}, X_s + x - \varepsilon)\mathrm{d}s\right)\\
&= \mathbb{E}_x\left(\int_0^{\tau_b^{u,-\varepsilon} \wedge \sigma^-_{\varepsilon}} G(u+s, X_s - \varepsilon)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{\varepsilon} < \tau_b^{u,-\varepsilon}\}} \int_{\sigma^-_{\varepsilon}}^{\tau_b^{g,-\varepsilon}} G(U_s^{(\varepsilon)}, X_s - \varepsilon)\mathrm{d}s\right)\\
&= \mathbb{E}_x\left(\int_0^{\tau_b^{u,-\varepsilon} \wedge \sigma^-_{\varepsilon}} G(u+s, X_s - \varepsilon)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_{\varepsilon} < \tau_b^{u,-\varepsilon}\}} \int_{\sigma^-_{\varepsilon}}^{\tau_b^{g,-\varepsilon} \wedge \sigma_0^-} G(U_s^{(\varepsilon)}, X_s - \varepsilon)\mathrm{d}s\right)\\
&\quad + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_0 < \tau_b^{g,-\varepsilon}\}} \mathbb{I}_{\{\sigma^-_{\varepsilon} < \tau_b^{u,-\varepsilon}\}} \int_{\sigma^-_0}^{\tau_b^{g,-\varepsilon}} G(U_s^{(\varepsilon)}, X_s - \varepsilon)\mathrm{d}s\right),
\end{aligned}$$

where in the last inequality we used that $\sigma_\varepsilon < \sigma_0^-$ under the measure $\mathbb{P}_x$. On the other hand, define the stopping time $\tau_* := \tau_b^{u,-\varepsilon}\mathbb{I}_{\{\sigma^-_\varepsilon > \tau_b^{u,-\varepsilon}\}} + \tau_b^{g,-\varepsilon}\mathbb{I}_{\{\sigma^-_\varepsilon < \tau_b^{u,-\varepsilon}\}}$. From equation (4.15) we have that

$$
\begin{aligned}
V(u,x) &\leq \mathbb{E}_x\left(\int_0^{\tau_*\wedge\sigma_0^-} G(u+s,X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_*\}}\int_{\sigma_0^-}^{\tau_*} G(U_s,X_s)\mathrm{d}s\right)\\
&= \mathbb{E}_x\left(\int_0^{\tau_b^{u,-\varepsilon}\wedge\sigma_\varepsilon^-} G(u+s,X_s)\mathrm{d}s\right) + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\int_{\sigma_\varepsilon^-}^{\tau_b^{g,-\varepsilon}\wedge\sigma_0^-} G(u+s,X_s)\mathrm{d}s\right)\\
&\quad + \mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_b^{g,-\varepsilon}\}}\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\int_{\sigma_0^-}^{\tau_b^{g,-\varepsilon}} G(U_s,X_s)\mathrm{d}s\right),
\end{aligned}
$$

where we again used that $\sigma_\varepsilon^- \leq \sigma_0^-$. Hence for any $u>0$, $0<x\leq b(u)$ and $0<\varepsilon<1$ such that $x-\varepsilon>0$ and $b(u)>0$,

$$
0 \leq \frac{V(u,x)-V(u,x-\varepsilon)}{\varepsilon} \leq R_1^{(\varepsilon)}(u,x) + R_2^{(\varepsilon)}(u,x) + R_3^{(\varepsilon)}(u,x),
$$

where

$$
\begin{aligned}
R_1^{(\varepsilon)}(u,x) &:= \frac{1}{\varepsilon}\mathbb{E}_x\left(\int_0^{\tau_b^{u,-\varepsilon}\wedge\sigma_\varepsilon^-}[G(u+s,X_s)-G(u+s,X_s-\varepsilon)]\mathrm{d}s\right) \geq 0,\\
R_2^{(\varepsilon)}(u,x) &:= \frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\int_{\sigma_\varepsilon^-}^{\tau_b^{g,-\varepsilon}\wedge\sigma_0^-}[G(u+s,X_s)-G(U_s^{(\varepsilon)},X_s-\varepsilon)]\mathrm{d}s\right) \geq 0,\\
R_3^{(\varepsilon)}(u,x) &:= \frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_b^{g,-\varepsilon}\}}\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\int_{\sigma_0^-}^{\tau_b^{g,-\varepsilon}}[G(U_s,X_s)-G(U_s^{(\varepsilon)},X_s-\varepsilon)]\mathrm{d}s\right) \geq 0.
\end{aligned}
$$

We will show that $\lim_{\varepsilon\downarrow 0} R_i^{(\varepsilon)}(u,x)=0$ for $i=1,2,3$. From the fact that b is non-increasing we have that $\tau_b^{u,-\varepsilon}\leq\tau_{b(u)+\varepsilon}^+$ and then for all $u>0$ and $x=b(u)$ we have that

$$
\begin{aligned}
R_1^{(\varepsilon)}(u,b(u)) &\leq \frac{1}{\varepsilon}\mathbb{E}_{b(u)}\left(\int_0^{\tau_{b(u)+\varepsilon}^+\wedge\sigma_\varepsilon^-}(u+s)^{p-1}\psi'(0+)[W(X_s)-W(X_s-\varepsilon)]\mathrm{d}s\right)\\
&\quad - \frac{1}{\varepsilon}\mathbb{E}_{b(u)-\varepsilon}\left(\int_0^{\tau_{b(u)}^+\wedge\sigma_0^-}[\mathbb{E}_{X_s+\varepsilon}(g^{p-1})-\mathbb{E}_{X_s}(g^{p-1})]\mathrm{d}s\right)\\
&= \frac{1}{\varepsilon}\mathbb{E}_{b(u)}\left(\int_0^{\tau_{b(u)+\varepsilon}^+\wedge\sigma_\varepsilon^-}(u+s)^{p-1}\psi'(0+)[W(X_s)-W(X_s-\varepsilon)]\mathrm{d}s\right)\\
&\quad - \frac{1}{\varepsilon}\int_{(0,b(u))}[\mathbb{E}_{z+\varepsilon}(g^{p-1})-\mathbb{E}_z(g^{p-1})]\int_0^\infty \mathbb{P}_{b(u)-\varepsilon}(X_s\in\mathrm{d}z, t<\tau_{b(u)}^+\wedge\sigma_0^-)\mathrm{d}s
\end{aligned}
$$

Using the density of the 0-potential measure of X exiting the interval $[0, b(u)]$ given in equation (1.18) we obtain that

$$
\begin{aligned}
& R_1^{(\varepsilon)}(u, b(u)) \\
& \leq \mathbb{E}_{b(u)}\left(\int_0^{\tau^+_{b(u)+\varepsilon}\wedge\sigma^-_\varepsilon} (u+s)^{p-1}\psi'(0+)\frac{W(X_s)-W(X_s-\varepsilon)}{\varepsilon}\mathrm{d}s\right) \\
& \quad - \int_0^{b(u)-\varepsilon}[\mathbb{E}_{z+\varepsilon}(g^{p-1})-\mathbb{E}_z(g^{p-1})]\frac{1}{\varepsilon}\left[\frac{W(b(u)-\varepsilon)W(b(u)-z)}{W(b(u))}-W(b(u)-\varepsilon-z)\right]\mathrm{d}z \\
& \quad - \frac{1}{\varepsilon}\int_{b(u)-\varepsilon}^{b(u)}[\mathbb{E}_{z+\varepsilon}(g^{p-1})-\mathbb{E}_z(g^{p-1})]\left[\frac{W(b(u)-\varepsilon)W(b(u)-z)}{W(b(u))}\right]\mathrm{d}z.
\end{aligned}
$$

Note that for all $s < \tau^+_{b(u)+\varepsilon}\wedge\sigma^-_\varepsilon$, we have $X_s \in (\varepsilon, b(u)+\varepsilon)$. Then using the fact that $W \in C^1((0,\infty))$, the function $z \mapsto \mathbb{E}_z(g^{p-1})$ is continuous, $\lim_{\varepsilon\downarrow 0}\tau^+_{b(u)+\varepsilon}\wedge\sigma^-_\varepsilon = \tau^+_{b(u)}\wedge\sigma^-_0 = 0$ a.s. under $\mathbb{P}_{b(u)}$ and the dominated convergence theorem we conclude that

$$\lim_{\varepsilon\downarrow 0} R_1^{(\varepsilon)}(u, b(u)) = 0.$$

Now we show that $\lim_{\varepsilon\downarrow 0} R_2^{(\varepsilon)}(u, b(u)) = 0$. Take $0 < x \leq b(u)$. Then using the inequality $G(u,x) \leq u^{p-1}$, the fact that for $s < \sigma_0^-$, $X_s > 0$ (then $-\mathbb{E}_{-1}(g^{p-1}) = G(0,-1) \leq G(U_s^{(\varepsilon)}, X_s-\varepsilon)$) and the strong Markov property at time σ^-_ε we get that

$$
\begin{aligned}
R_2^{(\varepsilon)}(u,x) &\leq \frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_\varepsilon < \tau_b^{u,-\varepsilon}\}}[\tau_b^{g,-\varepsilon}\wedge\sigma_0^- - \sigma_\varepsilon^+][(u+\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^{p-1}+\mathbb{E}_{-1}(g^{p-1})]\right) \\
&\leq \frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma^-_\varepsilon < \tau^+_{b(u)+\varepsilon}\}} f(\sigma^-_\varepsilon, X_{\sigma^-_\varepsilon})\right),
\end{aligned}
$$

where f is given for all $t \geq 0$ and $x \in \mathbb{R}$ by

$$f(t,x) := [2^{p-1}(u+t)^{p-1}+\mathbb{E}_{-1}(g^{p-1})]\mathbb{E}_x(\tau_b^{g,-\varepsilon}\wedge\sigma_0^-) + 2^{p-1}\mathbb{E}_x((\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^p) < \infty,$$

due to Lemma 4.4.8. Note that $\mathbb{E}_x(\tau_b^{g,-\varepsilon}\wedge\sigma_0^-) = \mathbb{E}_x((\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^p) = 0$ for all $x \leq 0$. Thus, from (4.55) there exists $M > 0$ such that

$$\max\{\mathbb{E}_x(\tau_b^{g,-\varepsilon}\wedge\sigma_0^-), \mathbb{E}_x((\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^p)\} \leq M$$

for all $x \leq \varepsilon$. Hence from the compensation formula for Poisson random measures we get that

$$
\begin{aligned}
&R_2^{(\varepsilon)}(u,x)\\
&\leq \max\{\mathbb{E}_\varepsilon(\tau_b^{g,-\varepsilon}\wedge\sigma_0^-),\mathbb{E}_\varepsilon((\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^p)\}\\
&\qquad\times\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_{b(u)+\varepsilon}^+\}}[2^{p-1}(u+\tau_{b(u)+\varepsilon}^+)^{p-1}+\mathbb{E}_{-1}(g^{p-1})+2^{p-1}]\right)\\
&\qquad+M\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_{b(u)+\varepsilon}^+\}}[2^{p-1}(u+\sigma_\varepsilon^-)^{p-1}+\mathbb{E}_{-1}(g^{p-1})+2^{p-1}]\mathbb{I}_{\{0<X_{\sigma_\varepsilon^-}<\varepsilon\}}\right)\\
&=\max\{\mathbb{E}_\varepsilon(\tau_b^{g,-\varepsilon}\wedge\sigma_0^-),\mathbb{E}_\varepsilon((\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^p)\}\\
&\qquad\times\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_{b(u)+\varepsilon}^+\}}[2^{p-1}(u+\tau_{b(u)+\varepsilon}^+)^{p-1}+\mathbb{E}_{-1}(g^{p-1})+2^{p-1}]\right)\\
&\qquad+\frac{M}{\varepsilon}\mathbb{E}_{x-\varepsilon}\left(\int_0^{\tau_{b(u)}^+\wedge\sigma_0^-}\int_{(-\infty,0)}[2^{p-1}(u+t)^{p-1}-G(0,-1)+2^{p-1}]\mathbb{I}_{\{-\varepsilon<X_t+y<0\}}\Pi(\mathrm{d}y)\mathrm{d}t\right)\\
&=\max\{\mathbb{E}_\varepsilon(\tau_b^{g,-\varepsilon}\wedge\sigma_0^-),\mathbb{E}_\varepsilon((\tau_b^{g,-\varepsilon}\wedge\sigma_0^-)^p)\}\\
&\qquad\times\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_{b(u)+\varepsilon}^+\}}[2^{p-1}(u+\tau_{b(u)+\varepsilon}^+)^{p-1}+\mathbb{E}_{-1}(g^{p-1})+2^{p-1}]\right)\\
&\qquad+\int_0^{b(u)}\int_{(-\varepsilon-z,-z)}\frac{M}{\varepsilon}\int_0^\infty[2^{p-1}(u+t)^{p-1}+\mathbb{E}_{-1}(g^{p-1})+2^{p-1}]\\
&\qquad\times\mathbb{P}_{x-\varepsilon}(X_t\in\mathrm{d}z,t<\tau_{b(u)}^+\wedge\sigma_0^-)\mathrm{d}t\Pi(\mathrm{d}y).
\end{aligned}
$$

Letting $x=b(u)$ and tending $\varepsilon\downarrow 0$ we get from Lemma 4.6.1 that

$$
\lim_{\varepsilon\downarrow 0}R_2^{(\varepsilon)}(u,b(u))=0.
$$

Lastly, using the Markov property at time σ_0^- and the fact that $\tau_b^{g,0}\leq\tau_b^{g,-\varepsilon}$ we get that

$$
\begin{aligned}
R_3^{(\varepsilon)}(u,x)&=\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_b^{g,-\varepsilon}\}}\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\mathbb{E}_{X_{\sigma_0^-}}\left[\int_0^{\tau_b^{g,-\varepsilon}}[G(U_s,X_s)-G(U_s^{(\varepsilon)},X_s-\varepsilon)]\mathrm{d}s\right]\right)\\
&=\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_b^{g,-\varepsilon}\}}\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}[V(0,X_{\sigma_0^-})-V(0,X_{\sigma_0^-}-\varepsilon)]\right)\\
&\qquad+\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_b^{g,-\varepsilon}\}}\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\mathbb{E}_{X_{\sigma_0^-}}\left[\int_{\tau_b^{g,0}}^{\tau_b^{g,-\varepsilon}}G(U_s,X_s)\mathrm{d}s\right]\right)\\
&\leq\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}[V(0,X_{\sigma_0^-})-V(0,X_{\sigma_0^-}-\varepsilon)]\right)\\
&\qquad+\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_0^-<\tau_b^{g,-\varepsilon}\}}\mathbb{I}_{\{\sigma_\varepsilon^-<\tau_b^{u,-\varepsilon}\}}\mathbb{E}_{X_{\sigma_0^-}}\left([\tau_b^{g,-\varepsilon}-\tau_b^{g,0}](\tau_b^{g,-\varepsilon})^{p-1}\right)\right),
\end{aligned}
$$

where we used the fact that $G(U_s, X_s) \leq s^{p-1} \leq (\tau_b^{g,-\varepsilon})^{p-1}$ for all $s \in [\tau_b^{g,0}, \tau_b^{g,-\varepsilon}]$. We can easily deduce from (4.19) that for any $x < 0$,

$$0 \leq \frac{\partial}{\partial x} V(0,x) = \int_{[0,\infty)} \mathbb{E}_{x-u}(g^{p-1}) W(\mathrm{d}u).$$

Then for all $x < 0$, $x \mapsto V(0,x)$ is differentiable and has left derivative at zero. Using Lemma 4.2.2 and the fact that $\mathbb{P}(-\underline{X}_\infty \in \mathrm{d}u) = \psi'(0+) W(\mathrm{d}u)$ we get that for all $x < 0$,

$$\frac{\partial}{\partial x} V(0,x) \leq \frac{2^{p-1}[\mathbb{E}(g^{p-1}) + A_{p-1}] + 4^{p-1} C_{p-1} \mathbb{E}((-\underline{X}_\infty)^{p-1})}{\psi'(0+)} + \frac{4^{p-1} C_{p-1}}{\psi'(0+)} |x|^{p-1}.$$

Thus since $|X_{\sigma_0^-}| \leq |\underline{X}_\infty|$ and $\mathbb{E}_x((-\underline{X}_\infty)^{p-1}) < \infty$ for all $x \in \mathbb{R}$ (see Lemma 4.2.1) we have that $\mathbb{E}_x(\frac{\partial}{\partial x} V(0, X_{\sigma_0^-}))$ is locally bounded. Moreover, by the dominated convergence theorem we can also conclude that for each $x < 0$, $\frac{\partial}{\partial x} V(0,x)$ is continuous. Hence, by the dominated convergence theorem and the right continuity of b we have that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_x \left(\mathbb{I}_{\{\sigma_\varepsilon^- < \tau_b^{u,-\varepsilon}\}} [V(0, X_{\sigma_0^-}) - V(0, X_{\sigma_0^-} - \varepsilon)] \right) = \mathbb{E}_x \left(\mathbb{I}_{\{\sigma_0^- < \tau_b^{u,0}\}} \frac{\partial}{\partial x} V(0, X_{\sigma_0^-}) \right).$$

In particular taking $x = b(u)$ we have that equation above is equal to zero. On the other hand, conditioning on σ_ε^- we have that

$$\begin{aligned} &\frac{1}{\varepsilon} \mathbb{E}_x \left(\mathbb{I}_{\{\sigma_0^- < \tau_b^{g,-\varepsilon}\}} \mathbb{I}_{\{\sigma_\varepsilon^- < \tau_b^{u,-\varepsilon}\}} \mathbb{E}_{X_{\sigma_0^-}} \left([\tau_b^{g,-\varepsilon} - \tau_b^{g,0}] (\tau_b^{g,-\varepsilon})^{p-1} \right) \right) \\ &\quad = \frac{1}{\varepsilon} \mathbb{E}_x \left(\mathbb{I}_{\{\sigma_\varepsilon^- < \tau_b^{u,-\varepsilon}\}} f_2(\varepsilon, X_{\sigma_\varepsilon^-}) \right), \end{aligned}$$

where

$$0 \leq f_2(\varepsilon, x) = \mathbb{E}_x \left(\mathbb{I}_{\{\sigma_0^- < \tau_b^{g,-\varepsilon}\}} \mathbb{E}_{X_{\sigma_0^-}} \left([\tau_b^{g,-\varepsilon} - \tau_b^{g,0}] (\tau_b^{g,-\varepsilon})^{p-1} \right) \right).$$

We show that f_2 is finite function. For all $y \leq 0$ we have that conditioning with respect to τ_0^+ and the strong Markov property of Lévy processes

$$\begin{aligned} \mathbb{E}_y \left([\tau_b^{g,-\varepsilon} - \tau_b^{g,0}] (\tau_b^{g,-\varepsilon})^{p-1} \right) &\leq 2^p \mathbb{E}((\tau_b^{g,-\varepsilon})^p) + 2^p \mathbb{E}_y((\tau_0^+)^p) \\ &\leq 2^p \mathbb{E}((\tau_b^{g,-\varepsilon})^p) + 2^p A_p + 2^p C_p |y|^p. \end{aligned}$$

where the last inequality follows from Lemma 4.2.2. Hence, since $|X_{\sigma_0^-}| \leq |\underline{X}_\infty|$ under the event $\{\sigma_0^- < \infty\}$ we have that

$$f_2(\varepsilon, x) \leq \begin{cases} 2^p\mathbb{E}((\tau_b^{g,-\varepsilon})^p) + 2^pA_p + 2^pC_p\mathbb{E}_x(|\underline{X}_\infty|^p) & x > 0, \\ 2^p\mathbb{E}((\tau_b^{g,-\varepsilon})^p) + 2^pA_p + 2^pC_p|x|^p & x \leq 0. \end{cases} \tag{4.56}$$

From Lemmas 4.2.1 and 4.4.8 we conclude that $f_2(\varepsilon, x)$ is a finite function. Moreover from the fact that b is continuous and $x \mapsto U_t^{(x)}$ is right continuous we deduce from Lemma 4.6.2 that $\lim_{\varepsilon\downarrow 0} \tau_b^{g,-\varepsilon} = \tau_b^{g,0}$ a.s. and then by the dominated convergence theorem, $\lim_{\varepsilon\downarrow 0} f_2(\varepsilon, x) = 0$ for all $x \in \mathbb{R}$. Moreover, using the compensation formula for Poisson random measures we get that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^- < \tau_b^{u,-\varepsilon}\}} f_2(\varepsilon, X_{\sigma_\varepsilon^-})\right) \\
&\qquad = \frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^- < \tau_{b(u)+\varepsilon}^+\}} f_2(\varepsilon, X_{\sigma_\varepsilon^-})\right) \\
&\qquad = f_2(\varepsilon, \varepsilon)\frac{\mathbb{P}_x(\sigma_\varepsilon^- < \tau_{b(u)+\varepsilon}^+, X_{\sigma_\varepsilon^-} = \varepsilon)}{\varepsilon} \\
&\qquad\qquad + \frac{1}{\varepsilon}\mathbb{E}_x\left(\int_{[0,\infty)}\int_{(-\infty,0)} f_2(\varepsilon, X_{t-} + y)\mathbb{I}_{\{\overline{X}_{t-} < b(u)+\varepsilon\}}\mathbb{I}_{\{\underline{X}_{t-} > \varepsilon\}}\mathbb{I}_{\{X_{t-}+y \leq \varepsilon\}} N(\mathrm{d}t, \mathrm{d}y)\right) \\
&\qquad \leq f_2(\varepsilon, \varepsilon)\frac{\mathbb{P}_x(\sigma_\varepsilon^- < \tau_{b(u)+\varepsilon}^+)}{\varepsilon} \\
&\qquad\qquad + \frac{1}{\varepsilon}\mathbb{E}_{x-\varepsilon}\left(\int_0^\infty\int_{(-\infty,0)} f_2(\varepsilon, X_t + \varepsilon + y)\mathbb{I}_{\{t < \tau_{b(u)}^+ \wedge \sigma_0^-\}}\mathbb{I}_{\{X_t + y \leq 0\}}\Pi(\mathrm{d}y)\mathrm{d}t\right).
\end{aligned}$$

From the 0-potential density of the process killed on exiting $[0, b(u)]$ (see equation (1.18)) and from equation (1.8) we obtain that for $x \geq \varepsilon$,

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{E}_x\left(\mathbb{I}_{\{\sigma_\varepsilon^- < \tau_b^{u,-\varepsilon}\}} f_2(\varepsilon, X_{\sigma_\varepsilon^-})\right) \\
&\leq f_2(\varepsilon,\varepsilon)\frac{W(b(u)) - W(x-\varepsilon)}{\varepsilon W(b(u))} \\
&\quad + \frac{1}{\varepsilon}\int_{(0,b(u))}\int_{(-\infty,0)} f_2(\varepsilon, z+\varepsilon+y)\mathbb{I}_{\{z+y\leq 0\}}\Pi(\mathrm{d}y)\int_0^\infty \mathbb{P}_{x-\varepsilon}(X_t \in \mathrm{d}z, t < \tau_{b(u)}^+ \wedge \sigma_0^-)\mathrm{d}t \\
&= f_2(\varepsilon,\varepsilon)\frac{W(b(u)) - W(x-\varepsilon)}{\varepsilon W(b(u))} \\
&\quad + \frac{1}{\varepsilon}\int_0^{x-\varepsilon}\left[\frac{W(x-\varepsilon)W(b(u)-z)}{W(b(u))} - W(x-\varepsilon-z)\right]\int_{(-\infty,-z)} f_2(\varepsilon, z+\varepsilon+y)\Pi(\mathrm{d}y)\mathrm{d}z \\
&\quad + \frac{1}{\varepsilon}\int_{x-\varepsilon}^{b(u)}\frac{W(x-\varepsilon)W(b(u)-z)}{W(b(u))}\int_{(-\infty,-z)} f_2(\varepsilon, z+\varepsilon+y)\Pi(\mathrm{d}y)\mathrm{d}z
\end{aligned}
$$

Note that since Π is finite on sets of the form $(-\infty,-\delta)$ for all $\delta > 0$, Lemma 4.2.1 and equation (4.56) we have that the integrals above with respect to Π are finite and bounded. Hence, taking $x = b(u)$ and from the dominated convergence theorem we conclude that

$$
\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}_{b(u)}\left(\mathbb{I}_{\{\sigma_\varepsilon^- < \tau_b^{u,-\varepsilon}\}} g(\varepsilon, X_{\sigma_\varepsilon^-})\right) \leq 0.
$$

Hence, we also have that

$$
\lim_{\varepsilon\downarrow 0} R_3(\varepsilon)(u, b(u)) = 0
$$

and the conclusion of the Lemma holds. □

4.7 Conclusions

The aim of this chapter is to predict the last zero, g, of a spectrally negative Lévy process drifting to infinity in a more general sense than the one studied in Baurdoux and Pedraza (2020b). For any $p > 1$, we have shown that a stopping time that minimises the L_p distance to g depends on $U_t = t - g_t$, the current excursion above the level zero at time $t \geq 0$, as studied in Chapter 2. That is, we have showed that

$$
V_* = \mathbb{E}\left(|\tau_D - g|^p\right),
$$

where $\tau_D = \inf\{t \geq 0 : X_t \geq b(U_t)\}$ and b is a non-increasing, non-negative and continuous function as characterised in Theorem 4.4.23. Moreover, the function b is infinity at the origin and tends to zero at infinity. Note that since $U_t = 0$ when $X_t \leq 0$, this means that the length of the current positive excursion keeps restarting whenever X visits the negative half line until there is a sufficiently large positive excursion. As we have in Chapter 1, an important drawback of this prediction is that $X_{\tau_D} > 0$ (since b is non-negative), implying that τ_D and g can never coincide in value.

A key feature of the optimal stopping problem (4.14) is that, since the process U restarts when X visits the set $(-\infty, 0)$, the value $V(0,0)$ plays an important role in its solution. Similarly of what we have in Chapter 1, $(V, b, V(0,0), u_b)$ is uniquely characterised by a system of non linear integral equations within a certain family of functions. For instance, the inequality

$$\int_{(-\infty,0)} \widetilde{V}(u, x+y)\Pi(\mathrm{d}y) + G(u,x) \geq 0 \qquad \text{on } D$$

is crucial for the submartingale property of the process $\{V(U_t, X_t) + \int_0^t G(U_s, X_s)\mathrm{d}s, t \geq 0\}$ to be satisfied. When X corresponds to a Brownian motion with drift (then $\Pi = 0$), such inequality is trivially satisfied. We have the conjecture that when $\sigma > 0$, such inequality can be disregarded for the uniqueness part of Theorem 4.4.23. This is left for future research.

Chapter 5

On the downcrossings by jump to the negative half line for spectrally negative Lévy processes

Abstract

For a spectrally negative Lévy process, using perturbation argument for Lévy processes (see Dassios and Wu (2011)), we find the joint Laplace transform of the local time at zero and the number of times that the process crosses below the level zero by a jump from the positive half line before an exponential time. We then find the joint Laplace transform of the i-th downcrossing by jump and its overshoot. For Lévy insurance risk processes, we use this result to find a formula for the expected present value of the total economic costs of the downcrossings by jump before an exponential time.

5.1 Introduction

Spectrally negative Lévy processes are popular in risk theory. In particular, they are used to model the capital of an insurance company. The Cramér–Lundberg model used in the classical risk theory assumes that the insurance company collects premium constantly at rate $c > 0$, the number of claims are modelled by a Poisson process $N = \{N_t, t \geq 0\}$ with rate $\lambda > 0$ whereas the size of the claims are modelled by a sequence of independent and identically distributed random variables $\{Y_i, i \geq 0\}$ which are independent of N. The capital of the insurance company at any time $t \geq 0$ is then given by

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$

where $x \in \mathbb{R}$ is the initial capital. More general models also include a stochastic perturbation modelled by a Brownian motion or processes that belong to the class of spectrally negative Lévy processes (see Section 2.7.1 in Kyprianou (2014)).

In the classical risk theory, the study of the first moment of ruin τ_0^-, i.e. the first time the process becomes negative is of interest. If we assume that the process has both a diffusion component and jumps, one question that arises naturally, is whether the moment of ruin is made by crossing the boundary continuously or as a consequence of a sufficiently large jump from the positive half line. The first event can happen as a consequence of an insufficient premium rate or a small initial capital whereas the second can be understood as an "unexpected" ruin which may be caused by a big claim or a catastrophic event. Moreover, assuming that the insurance company can support a negative capital for a while, then the insurance company can return rapidly to have solvency when the ruin occurs due to continuous crossing of the boundary and it is when the ruin occurs due to big jump that the process takes a strictly positive amount of time to have a positive capital. It is then useful study the distribution of the number of times the ruin occurs as a consequence of a sufficiently big jump in a finite time horizon.

A function that is of interest in the literature is the Gerber–Shiu function (see Gerber and Shiu (1997) and Gerber and Shiu (1998)) which is given by

$$\mathbb{E}_x \left(e^{-r\tau_0^-} \omega(-X_{\tau_0^-}, X_{\tau_0^- -}) \mathbb{I}_{\{\tau_0^- < \infty\}} \right),$$

where $r > 0$ is the force of interest, the term $-X_{\tau_0^-}$ is the deficit at ruin and $X_{\tau_0^- -} := \lim_{h \downarrow 0} X_{\tau_0^- - h}$ is the wealth prior ruin. The function ω is a measurable non-negative function chosen such that $\omega(-X_{\tau_0^-}, X_{\tau_0^- -})$ represents the economics costs of the insurer at the moment of ruin. Formulas for calculating this function have been derived in the literature for several models (see Asmussen and Albrecher (2010) for a review of them). For the spectrally negative case, Biffis and Morales (2010) derived a formula for the generalised penalty function which includes the last minimum before ruin. Cai et al. (2009) considered the expectation of the total discounted claim costs up to the time of ruin in a Poisson process with drift setting. Motivated by the latter, we derive a formula for the expected present value of the total economic costs of all the downcrossing by jumps below the level zero before an independent exponential time, i.e.

$$\mathbb{E}_x\left(\sum_{i=1}^{J_{\mathbf{e}_p}} e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\right),$$

where J_t is the number of downcrossings by jump below the level zero at time $t \geq 0$, $\{\kappa_i, i \geq 1\}$ are the consecutive times in which the downcrossing by jumps occur and $\mathbf{e}_p$ is an exponential distribution with parameter $p \geq 0$ independent of X.

The main contribution of this paper is the derivation the Laplace transform of the random variable $J_{\mathbf{e}_p}$ (see Theorem 5.2.1) for a spectrally negative Lévy process X of finite variation. In order to count the number of downcrossing by jumps, we can define a sequence of stopping times at which the process has positive and negative excursions away from zero. With the help of the Markov property, the fact that X creeps upwards and the lack of memory property of the exponential distribution, we can derive the distribution of $J_{\mathbf{e}_p}$. However, when X is a process of infinite variation, this method is no longer useful. Using a perturbation method as in Dassios and Wu (2011), we can "perturb" the process X in such a way that the number of zeroes is at most countable. We can then evaluate the distribution of $J_{\mathbf{e}_p}$ by a limit argument. It turns out that $J_{\mathbf{e}_p}$ is only finite when the jumps are of finite variation and its distribution can be studied as the product of a Bernoulli random variable and an independent geometric distribution (see Remark 5.2.2).

This chapter is organised as follows. In section 5.2, we define formally J_t as the number of downcrossing by jumps to the negative half line of a spectrally negative Lévy process in terms of the Poisson random measure. We then derive the joint Laplace transform of the local time at 0 and the number of downcrossings at an independent exponential time (see

Theorem 5.2.1). We then derive in Corollary 5.2.3, the joint Laplace transform of the i-th downcrossing and its overshoot. In Section 5.3, we derive a formula for the expected presented value of the total economic costs of all the downcrossing by jumps below the level zero before an independent exponential time (see Corollaries 5.3.1 and 5.3.2). The proof of Theorem 5.2.1 uses the perturbation method for Lévy processes that was studied in Section 3.3.1.

We finish this section by introducing some additional notations.

5.2 Downcrossings by jumps

Throughout this chapter we use the notation and the preliminary results presented in Chapter 1.1. Let X be a spectrally negative Lévy process, that is, a Lévy process starting from 0 with only negative jumps and non-monotone paths, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see Definition 1.3.38 in Bichteler (2002)). We suppose that X has Lévy triplet (μ, σ, Π) where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (Lévy measure) concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

For any $p, \beta \geq 0$ we define

$$\theta^{(p)}(\beta) = \frac{\delta - \frac{p-\psi(\beta)}{\Phi(p)-\beta} + \frac{\sigma^2}{2}(\Phi(p)+\beta)}{\delta + \sigma^2\Phi(p)}, \tag{5.1}$$

where δ is defined in (1.2). When $p = \psi(\beta)$ or $\delta = \infty$, the above quantity is understood in the limiting sense, i.e. when $\delta = \infty$, $\theta^{(p)}(\beta) = 1$ for all $p, \beta \geq 0$ and

$$\theta^{(p)}(\Phi(p)) = \frac{\delta - \psi'(\Phi(p)) + \sigma^2\Phi(p)}{\delta + \sigma^2\Phi(p)}$$

for any $p \geq 0$.

We denote by J_t the number of downcrossings below the level zero of the process made by jumps , i.e.

$$J_t := \int_{[0,t]} \int_{(-\infty,0)} \mathbb{I}_{\{X_{s-}>0\}} \mathbb{I}_{\{X_{s-}+y<0\}} N(\mathrm{d}s \times \mathrm{d}y). \tag{5.2}$$

Clearly $J_t = 0$ a.s. for all $t \geq 0$ when $\Pi = 0$. Henceforth, we will assume that $\Pi \neq 0$. Moreover, from the fact that N takes values in $\{0, 1, 2, \dots\}$ and from the strong Markov property of X, we can easily deduce that $J_t \in \{0, 1, 2, \dots\}$. In the next Theorem, we find the joint Laplace transform of the local time at zero and the number of downcrossings by jump at an exponential time. For ease of reading, the proof is presented in Section 5.4.

Theorem 5.2.1. *Let X be a spectrally negative Lévy process. Then the joint the Laplace transform of the local time and the number of downcrossings by jump at an exponential time is given for all $\alpha, \beta \geq 0$ and $x \leq 0$ by*

$$\mathbb{E}_x(e^{-\alpha L_{e_p} - \beta J_{e_p}}) = 1 + e^{\Phi(p)x} \frac{(e^{-\beta} - 1)\left(\delta + \Phi(p)\frac{\sigma^2}{2} - \frac{p}{\Phi(p)}\right) - \alpha\sigma^2}{\alpha\sigma^2 + \psi'(\Phi(p)) + (e^{-\beta} - 1)\left[\psi'(\Phi(p)) - \Phi(p)\sigma^2 - \delta\right]} \tag{5.3}$$

and for $x > 0$,

$$\begin{aligned} \mathbb{E}_x(e^{-\alpha L_{e_p} - \beta J_{e_p}}) = 1 - \mathcal{I}^{(p,0)}(x) + e^{-\beta}\left[\mathcal{I}^{(p,0)}(x) - e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)\right] \\ + \left[e^{-\beta}[e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x) - \mathcal{C}^{(p)}(x)] + \mathcal{C}^{(p)}(x)\right]\mathbb{E}(e^{-\alpha L_{e_p} - \beta J_{e_p}}), \end{aligned} \tag{5.4}$$

where $\boldsymbol{e}_p$ is an exponential random variable with parameter $p \geq 0$ and the functions $\mathcal{I}$ and $\mathcal{C}$ are given in (1.12) and (1.15) respectively. The terms δ and σ should be understood in the limiting sense when X has jumps of infinite variation or in the absence of Brownian component.

Remark 5.2.2. *From the proof of Theorem 5.2.1, we can simplify formulas (5.3) and (5.4) for specific cases depending on the characteristics of X. For instance, if X is of finite variation, we have that $L_t = 0$ for all $t \geq 0$ and*

$$\mathbb{E}_x(e^{-\beta J_{e_p}}) = \begin{cases} 1 + (e^{-\beta} - 1)\mathcal{I}^{(p,0)}(x) + e^{-\beta}e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)\frac{(e^{-\beta}-1)\left(\delta - \frac{p}{\Phi(p)}\right)}{\delta - e^{-\beta}[\delta - \psi'(\Phi(p))]}, & x > 0 \\ 1 + e^{\Phi(p)x}\frac{(e^{-\beta}-1)\left(\delta - \frac{p}{\Phi(p)}\right)}{\delta - e^{-\beta}[\delta - \psi'(\Phi(p))]}, & x \leq 0 \end{cases}.$$

When X has jumps of infinite variation, we have that $\delta = \infty$ so that when $\beta > 0$,

$$\mathbb{E}_x(e^{-\beta J_{e_p}}) = \begin{cases} 1 - \mathcal{I}^{(p,0)}(x) + e^{-\beta}\left[\mathcal{I}^{(p,0)}(x) - e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)\right], & x > 0 \\ 1 - e^{\Phi(p)x}, & x \leq 0 \end{cases}.$$

which implies that

$$\mathbb{P}_x(J_{e_p} = \infty) = \begin{cases} e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x), & x > 0 \\ e^{\Phi(p)x}, & x \leq 0 \end{cases}.$$

Moreover, when we take $\alpha = 0$, we can rewrite equation (5.3) as

$$\mathbb{E}_x(e^{-\beta J_{e_p}}) = 1 + e^{\Phi(p)x}\frac{(e^{-\beta} - 1)\theta^{(p)}(0)}{1 - e^{-\beta}\theta^{(p)}(\Phi(p))},$$

where the function $\theta^{(p)}$ is given in (5.1). A close inspection of the formula above tells us that, under the measure $\mathbb{P}$, the random variable J_{e_p} can be seen as the product of a Bernoulli random variable with success probability $\pi_B := \theta^{(p)}(0)$ and an independent geometric random variable with support on the set $\{1, 2, \ldots\}$ and success probability $\pi_G := 1 - \theta^{(p)}(\Phi(p))$, that is,

$$\mathbb{P}(J_{e_p} = 0) = 1 - \pi_B = \frac{\Phi(p)\frac{\sigma^2}{2} + \frac{p}{\Phi(p)}}{\Phi(p)\sigma^2 + \delta}$$

$$\begin{aligned}\mathbb{P}(J_{e_p} = n) &= \pi_B\pi_G[1 - \pi_G]^{n-1} \\ &= \left[\frac{\delta + \Phi(p)\frac{\sigma^2}{2} - \frac{p}{\Phi(p)}}{\Phi(p)\sigma^2 + \delta}\right]\left[\frac{\psi'(\Phi(p))}{\Phi(p)\sigma^2 + \delta}\right]\left[\frac{\Phi(p)\sigma^2 + \delta - \psi'(\Phi(p))}{\Phi(p)\sigma^2 + \delta}\right]^{n-1}, \qquad n \geq 1.\end{aligned}$$

In any case, we have that for any $x \in \mathbb{R}$ and $\alpha \geq 0$,

$$\mathbb{E}_x(e^{-\alpha L_{e_p}}) = 1 - e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)\frac{\alpha\sigma^2}{\alpha\sigma^2 + \psi'(\Phi(p))}.$$

Note that the result above agreess with the one derived in Li and Zhou (2020) (see Corollary 3.4) up to a multiplicative constant.

Now we proceed to study the Laplace transform of i-th time that the process X makes a downcrossing by jump below the level zero from the interval $[0, \infty)$, i.e. we study the distribution of the random time

$$\kappa_i := \inf\{t \geq 0 : J_t \geq i\},$$

for $i \in \{0, 1, 2, \ldots\}$, where as usual $\inf \emptyset = \infty$.

Recall $t \mapsto J_t$ is non negative and non decreasing. We then have that $\kappa_0 = 0$ and $\kappa_i \leq \kappa_{i+1}$ for all $i \geq 0$. Moreover, it is easy to show that for each $t \geq 0$, J_t is $\mathcal{F}_t$ measurable and a right-continuous process. These facts imply that for each $i \geq 0$, the random variable κ_i is a stopping time with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$.

We calculate joint Laplace transform of the random vector (κ_i, X_{κ_i}). The method is mainly based on an exponential change of measure technique and the result derived in Theorem 5.2.1.

Corollary 5.2.3. *Let X be any spectrally negative Lévy process. Then for any $p \geq 0$ and $\beta \geq 0$,*

$$\mathbb{E}_x(e^{-p\kappa_1 + \beta X_{\kappa_1}} \mathbb{I}_{\{\kappa_1 < \infty\}}) = \begin{cases} \theta^{(p)}(\beta) e^{\Phi(p)x} & x \leq 0 \\ e^{\beta x} \mathcal{I}^{(p,\beta)}(x) - \mathcal{C}^{(p)}(x)[1 - \theta^{(p)}(\beta)] & x > 0 \end{cases}. \tag{5.5}$$

Moreover, for $i \geq 2$, we have that for any $p \geq 0$ and $\beta \geq 0$,

$$\begin{aligned}
&\mathbb{E}_x(e^{-p\kappa_i + \beta X_{\kappa_i}} \mathbb{I}_{\{\kappa_i < \infty\}}) \\
&\quad = \theta^{(p)}(\beta) \theta^{(p)}(\Phi(p))^{i-2} \mathbb{E}_x(e^{-p\kappa_1 + \Phi(p) X_{\kappa_1}} \mathbb{I}_{\{\kappa_1 < \infty\}}) \\
&\quad = \begin{cases} \theta^{(p)}(\beta) \theta^{(p)}(\Phi(p))^{i-1} e^{\Phi(p)x} & x \leq 0 \\ \theta^{(p)}(\beta) \theta^{(p)}(\Phi(p))^{i-2} \left[e^{\Phi(p)x} \mathcal{I}^{(p,\Phi(p))}(x) - \mathcal{C}^{(p)}(x)[1 - \theta^{(p)}(\Phi(p))] \right] & x > 0 \end{cases},
\end{aligned}$$

where the functions $\mathcal{I}$, $\mathcal{C}$ and θ are given in (1.12), (1.15) and (5.1) respectively.

Remark 5.2.4. *Note that when X has jumps of infinite variation, we have that $\delta = \infty$ and then $\theta^{(p)}(\beta) = 1$ for all $p, \beta \geq 0$. We then have that for all $p \geq 0$ and $\beta \geq 0$*

$$\mathbb{E}_x(e^{-p\kappa_1 + \beta X_{\kappa_1}} \mathbb{I}_{\{\kappa_1 < \infty\}}) = \begin{cases} e^{\Phi(p)x} & x \leq 0 \\ e^{\beta x} \mathcal{I}^{(p,\beta)}(x) & x > 0 \end{cases}$$

and

$$\mathbb{E}_x(e^{-p\kappa_i+\beta X_{\kappa_i}}\mathbb{I}_{\{\kappa_i<\infty\}}) = \mathbb{E}_x(e^{-p\kappa_1+\Phi(p)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}).$$

for all $i \geq 2$*. This agrees with the fact that when* X *has jumps of infinite variation,* $\tau_0^- = 0$ $\mathbb{P}$*-a.s. even when there is no Brownian motion component and then* $\kappa_i \stackrel{d}{=} \tau_1$ *for all* $i \geq 2$*, where* $\tau_1 = \inf\{t \geq \kappa_1 : X_t > 0\}$*.*

Proof. Let $p \geq 0$ and $\mathbf{e}_p$ be an independent exponential random variable with parameter p. Note that the event $\{\kappa_1 < \mathbf{e}_p\}$ is equivalent to $\{J_{\mathbf{e}_p} \geq 1\}$. Then for all $x \in \mathbb{R}$, we have that

$$\begin{aligned}\mathbb{E}_x(e^{-p\kappa_1}\mathbb{I}_{\{\kappa_1<\infty\}}) &= \mathbb{P}_x(\kappa_1 < \mathbf{e}_p)\\ &= 1 - \mathbb{P}_x(J_{\mathbf{e}_p} = 0).\end{aligned}$$

Taking $\alpha = 0$ and $\beta \to \infty$ on equation (5.3), we obtain that

$$\mathbb{E}_x(e^{-p\kappa_1}\mathbb{I}_{\{\kappa_1<\infty\}}) = \begin{cases} e^{\Phi(p)x}\theta^{(p)}(0) & x \leq 0 \\ \mathcal{I}^{(p,0)}(x) - \mathcal{C}^{(p)}(x)[1-\theta^{(p)}(0)] & x > 0 \end{cases}. \tag{5.6}$$

Take $\beta \geq 0$, using an exponential change of measure, we get

$$\begin{aligned}\mathbb{E}_x(e^{-p\kappa_1+\beta X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}) &= \mathbb{E}_x(e^{-\psi(\beta)\kappa_1+\beta X_{\kappa_1}}e^{-(p-\psi(\beta))\kappa_1}\mathbb{I}_{\{\kappa_1<\infty\}})\\ &= e^{\beta x}\mathbb{E}_x^{\beta}(e^{-(p-\psi(\beta))\kappa_1}\mathbb{I}_{\{\kappa_1<\infty\}}),\end{aligned}$$

where $\mathbb{P}^{\beta}$ is defined in (1.22). Recall that under $\mathbb{P}^{\beta}$, X has Lévy triplet

$$\left(\mu - \sigma^2\beta - \int_{(-1,0)} y(e^{\beta y}-1)\Pi(\mathrm{d}y), \sigma^2, e^{\beta y}\Pi(\mathrm{d}y)\right).$$

and then $\Phi_\beta(q) = \Phi(q+\psi(\beta)) - \beta$ for all $q \geq -\psi(\beta)$. Moreover, it can be shown that $\theta_\beta^{(q)}(\lambda) = \theta^{(q+\psi(\beta))}(\lambda+\beta)$ for all $q \geq -\psi(\beta)$ and $\lambda \geq 0$. Hence, using (5.6) under $\mathbb{P}_x^{\beta}$, we have that for $x \leq 0$,

$$\begin{aligned}
\mathbb{E}_x(e^{-p\kappa_1+\beta X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}) &= e^{\beta x}\mathbb{E}_x^\beta(e^{-(p-\psi(\beta))\kappa_1}\mathbb{I}_{\{\kappa_1<\infty\}}) \\
&= e^{\beta x}e^{\Phi_\beta(p-\psi(\beta))x}\theta_\beta^{(p-\psi(\beta))}(0) \\
&= e^{\Phi(p)x}\theta^{(p)}(\beta).
\end{aligned}$$

Similarly, for $x > 0$, we obtain from equation (5.6) under $\mathbb{P}_x^\beta$ and the definition of $\mathcal{I}$ and $\mathcal{C}$ (see equations (1.12) and (1.15)) that

$$\begin{aligned}
&\mathbb{E}_x(e^{-p\kappa_1+\beta X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}) \\
&\qquad = e^{\beta x}\mathbb{E}_x^\beta(e^{-(p-\psi(\beta))\kappa_1}\mathbb{I}_{\{\kappa_1<\infty\}}) \\
&\qquad = e^{\beta x}Z_\beta^{(p-\psi(\beta))}(x) - e^{\beta x}\frac{p-\psi(\beta)}{\Phi(p)-\beta}W_\beta^{(p-\psi(\beta))}(x) \\
&\qquad\quad - e^{\beta x}\frac{\sigma^2}{2}\{W_\beta^{(p-\psi(\beta))\prime}(x) - (\Phi(p)-\beta)W_\beta^{(p-\psi(\beta))}(x)\}[1-\theta_\beta^{(p-\psi(\beta))}(0)],
\end{aligned}$$

where $W_\beta^{(q)}$ and $Z_\beta^{(q)}$ are the scale functions of X under the measure $\mathbb{P}^\beta$. Computing the Laplace transform of $W_\beta^{(p-\psi(\beta))}$, we can easily show that $W_\beta^{(p-\psi(\beta))}(x) = e^{-\beta x}W^{(p)}(x)$ for all $x \in \mathbb{R}$. Thus, for all $\beta, p \geq 0$ and $x > 0$,

$$\begin{aligned}
\mathbb{E}_x(e^{-p\kappa_1+\beta X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}) &= e^{\beta x} + (p-\psi(\beta))\int_0^x e^{\beta(x-y)}W^{(p)}(y)\mathrm{d}y - \frac{p-\psi(\beta)}{\Phi(p)-\beta}W^{(p)}(x) \\
&\quad + \mathcal{C}^{(p)}(x)[\theta^{(p)}(\beta)-1] \\
&= e^{\beta x}\mathcal{I}^{(p,\beta)}(x) - \mathcal{C}^{(p)}(x)[1-\theta^{(p)}(\beta)].
\end{aligned}$$

Now we assume that $i \geq 2$. Conditioning with respect to the filtration at time κ_{i-1}, we get that for any $p \geq 0$ and $\beta \geq 0$

$$\begin{aligned}
\mathbb{E}_x(e^{-p\kappa_i+\beta X_{\kappa_i}}\mathbb{I}_{\{\kappa_i<\infty\}}) &= \mathbb{E}_x(e^{-p\kappa_{i-1}}\mathbb{I}_{\{\kappa_{i-1}<\infty\}}\mathbb{E}_x(e^{-p(\kappa_i-\kappa_{i-1})+\beta X_{\kappa_i}}\mathbb{I}_{\{\kappa_i<\infty\}}|\mathcal{F}_{\kappa_{i-1}})) \\
&= \mathbb{E}_x(e^{-p\kappa_{i-1}}\mathbb{I}_{\{\kappa_{i-1}<\infty\}}\mathbb{E}_{X_{\kappa_{i-1}}}(e^{-p\kappa_1+\beta X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}))
\end{aligned}$$

where we used the strong Markov property for Lévy processes in the last equality. Using the fact that $X_{\kappa_j} \leq 0$ for all $j \geq 1$, we obtain from formula (5.5) that

$$\mathbb{E}_x(e^{-p\kappa_i+\beta X_{\kappa_i}}\mathbb{I}_{\{\kappa_i<\infty\}}) = \theta^{(p)}(\beta)\mathbb{E}_x(e^{-p\kappa_{i-1}+\Phi(p)X_{\kappa_{i-1}}}\mathbb{I}_{\{\kappa_{i-1}<\infty\}}).$$

Thus, by an induction argument, we conclude that for $i \geq 2$,

$$\mathbb{E}_x(e^{-p\kappa_i+\beta X_{\kappa_i}}\mathbb{I}_{\{\kappa_i<\infty\}}) = \theta^{(p)}(\beta)\theta^{(p)}(\Phi(p))^{i-2}\mathbb{E}_x(e^{-p\kappa_1+\Phi(p)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}).$$

The proof is now complete. □

Remark 5.2.5. *From the fact that for all $i \geq 1$, $J_{\boldsymbol{e}_p} \geq i$ if and only if $\kappa_i < \boldsymbol{e}_p$, we can easily deduce the probability function of $J_{\boldsymbol{e}_p}$ in terms of the Laplace transform the variables κ_i. Indeed, for all $p \geq 0$ and $x \in \mathbb{R}$,*

$$\mathbb{P}_x(J_{\boldsymbol{e}_p} \geq i) = \mathbb{P}_x(\kappa_i < \boldsymbol{e}_p) = \mathbb{E}_x(e^{-p\kappa_i}\mathbb{I}_{\{\kappa_i<\infty\}}).$$

5.3 Applications

In this section, we consider that X is a Lévy risk process so that the capital of an insurance company is modelled by X with initial capital $x \in \mathbb{R}$. We further assume that X is a spectrally negative Lévy process with jumps of finite variation. In the classical risk theory, we consider the moment of ruin as the first time the risk process crosses below zero. If we assume that there is a Brownian motion component ($\sigma > 0$), the moment of ruin can occur either by creeping or by a jump below the level zero. Note that if the time of ruin is made by creeping, with probability one, the process is above the level zero instantly. Thus, it is also of interest to consider the time of ruin as the first downcrossing by jump below the negative half line. Inspired by the Gerber–Shiu function (see Gerber and Shiu (1997) and Gerber and Shiu (1998)), we define the expected present value of the economic cost of the insurer of the first downcrossing by jump below the level zero as

$$\mathbb{E}_x(e^{-r\kappa_1}\omega(X_{\kappa_1}, X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}),$$

where $r > 0$ is the force of interest and $\omega : (-\infty, 0) \times (0, \infty) \mapsto \mathbb{R}$ is a measurable function representing the cost of the ruin by jump as a function of the capital before and after such moment of ruin. The value above can be calculated as

$$\mathbb{E}_x(e^{-r\kappa_1}\omega(X_{\kappa_1}, X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}) = \int_{(-\infty,0)}\int_{(0,\infty)} \omega(y,z) f_x^{(r)}(\mathrm{d}y, \mathrm{d}z),$$

where for any $y<0$ and $z>0$, the measure $f^{(r)}(\mathrm{d}y, \mathrm{d}z)$ is such that for any set $A \subset (-\infty, 0)$ and $B \subset [0,\infty)$,

$$\mathbb{E}_x\left(e^{-r\kappa_1}\mathbb{I}_{\{\kappa_1<\infty, X_{\kappa_1}\in A, X_{\kappa_1}\in B\}}\right) = \int_A\int_B f_x^{(r)}(\mathrm{d}y, \mathrm{d}z)$$

It turns out that the measure $f^{(r)}$ is absolutely continuous with respect to the Lebesgue measure. As a direct consequence of Corollary 5.2.3, we derive a formula for the density in terms of the scale functions.

Corollary 5.3.1. *Let X be a spectrally negative Lévy process with jumps of finite variation. For all $r>0$, the function $f^{(r)}$ satisfies*

$$\begin{aligned} &f_x^{(r)}(\mathrm{d}y, \mathrm{d}z) \\ &\quad = \Pi(\mathrm{d}y - z)\left(\Phi'(r)e^{-\Phi(r)z}\left[e^{\Phi(r)x} - \mathbb{E}_x\left(\mathbb{I}_{\{\kappa_1<\infty\}}e^{-r\kappa_1+\Phi(r)X_{\kappa_1}}\right)\right] - W^{(r)}(x-z)\right)\mathrm{d}z \end{aligned}$$

for all $x\in\mathbb{R}$, $y<0$ and $z>0$.

Proof. Let $A\subset(-\infty,0)$ and $B\subset[0,\infty)$. Since $J_t = 0$ if and only if $\kappa_1 > t$ and the compensation formula for Poisson random measures (see (1.25)), we have that

$$\begin{aligned} &\mathbb{E}_x(e^{-r\kappa_1}\mathbb{I}_{\{\kappa_1<\infty, X_{\kappa_1}\in A, X_{\kappa_1-}\in B\}}) \\ &\quad = \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)} e^{-rt}\mathbb{I}_{\{X_{t-}+y\in A, X_{t-}\in B\}}\mathbb{I}_{\{J_{t-}=0\}}\mathbb{I}_{\{X_{t-}>0\}}\mathbb{I}_{\{X_{t-}+y<0\}}N(\mathrm{d}t, \mathrm{d}y)\right) \\ &\quad = \mathbb{E}_x\left(\int_0^\infty\int_{(-\infty,0)} e^{-rt}\mathbb{I}_{\{X_t+y\in A, X_t\in B\}}\mathbb{I}_{\{J_t=0\}}\mathbb{I}_{\{X_t>0\}}\mathbb{I}_{\{X_t+y<0\}}\Pi(\mathrm{d}y)\mathrm{d}t\right) \\ &\quad = \mathbb{E}_x\left(\int_0^\infty e^{-rt}\Pi(A - X_t)\mathbb{I}_{\{X_t\in B\}}\mathbb{I}_{\{J_t=0\}}\mathrm{d}t\right) \\ &\quad = \int_A\int_B \Pi(\mathrm{d}y - z)\int_0^\infty e^{-rt}\mathbb{P}_x(X_t\in\mathrm{d}z, \kappa_1>t)\mathrm{d}t, \end{aligned}$$

where the last equality follows from Fubini's theorem. Conditioning with respect to the filtration at time κ_1 and from the strong Markov property, we obtain that

$$
\begin{aligned}
\int_0^\infty & e^{-rt}\mathbb{P}_x(X_t \in \mathrm{d}z, \kappa_1 > t)\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\mathbb{P}_x(X_t \in \mathrm{d}z)\mathrm{d}t - \int_0^\infty e^{-rt}\mathbb{P}_x(X_t \in \mathrm{d}z, \kappa_1 < t)\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\mathbb{P}_x(X_t \in \mathrm{d}z)\mathrm{d}t - \mathbb{E}_x\left(\mathbb{I}_{\{\kappa_1<\infty\}}e^{-r\kappa_1}\mathbb{E}_{X_{\kappa_1}}\left(\int_0^\infty e^{-rt}\mathbb{I}_{\{X_t\in \mathrm{d}z\}}\mathrm{d}t\right)\right) \\
&= \left[\Phi'(r)e^{-\Phi(r)(z-x)} - W^{(r)}(x-z)\right]\mathrm{d}z \\
&\quad - \mathbb{E}_x\left(\mathbb{I}_{\{\kappa_1<\infty\}}e^{-r\kappa_1}\left[\Phi'(r)e^{-\Phi(r)(z-X_{\kappa_1})} - W^{(r)}(X_{\kappa_1}-z)\right]\right)\mathrm{d}z \\
&= \left[\Phi'(r)e^{-\Phi(r)(z-x)} - W^{(r)}(x-z)\right]\mathrm{d}z \\
&\quad - \Phi'(r)e^{-\Phi(r)z}\mathbb{E}_x\left(e^{-r\kappa_1+\Phi(r)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}\right)\mathrm{d}z,
\end{aligned}
$$

where the third equality follows from equation (1.21) and the last is due to $X_{\kappa_1} < 0$ and $z > 0$. We can then conclude that

$$
\begin{aligned}
&\mathbb{E}_x(e^{-r\kappa_1}\mathbb{I}_{\{\kappa_1<\infty, X_{\kappa_i}\in A, X_{\kappa_i-}\in B\}}) \\
&= \int_A\int_B \Pi(\mathrm{d}y - z)\left[\Phi'(r)e^{-\Phi(r)z}\left[e^{\Phi(r)x} - \mathbb{E}_x\left(e^{-r\kappa_1+\Phi(r)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}}\right)\right] - W^{(r)}(x-z)\right]\mathrm{d}z.
\end{aligned}
$$

□

Assume that the insurance company can endure a negative capital for a while so it can go back to having positive capital. An important quantity to consider in this setting is the expected present value of the total economic costs of all the downcrossing by jumps below the level zero before an exponential time, that is,

$$
R(p, r, x) = \mathbb{E}_x\left(\sum_{i=1}^{J_{\mathbf{e}_p}} e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\right)
$$

for $r, p > 0$ and $x \in \mathbb{R}$. Using Corollary 5.2.3, we give a formula to calculate the value of R.

Corollary 5.3.2. *Let X be a spectrally negative Lévy process with finite variation jumps. For any $r, p > 0$ and $x \in \mathbb{R}$, we have that*

$$R(p,r,x) = \mathbb{E}_x\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1}, X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right) + \frac{\mathbb{E}_x(e^{-(r+p)\kappa_1+\Phi(r+p)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}})\mathbb{E}\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1}, X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right)}{1-\theta^{(r+p)}(\Phi(r+p))},$$

where the function θ *is given in* (5.1).

Proof. Let $p > 0$ and $r > 0$. Since $\mathbf{e}_p$ is independent of X, we have that

$$\begin{aligned}
\mathbb{E}_x\left(\sum_{i=1}^{J_{\mathbf{e}_p}} e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\right) &= \sum_{n=1}^{\infty}\mathbb{E}_x\left(\sum_{i=1}^{n} e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\mathbb{I}_{\{J_{\mathbf{e}_p}=n\}}\right)\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathbb{E}_x\left(e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\mathbb{I}_{\{\kappa_n<\mathbf{e}_p<\kappa_{n+1}\}}\right)\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathbb{E}_x\left(e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})[e^{-p\kappa_n} - e^{-p\kappa_{n+1}}]\right).
\end{aligned}$$

Using Fubini's theorem, we obtain that

$$\begin{aligned}
\mathbb{E}_x\left(\sum_{i=1}^{J_{\mathbf{e}_p}} e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\right) &= \sum_{i=1}^{\infty}\mathbb{E}_x\left(e^{-r\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\sum_{n=i}^{\infty}[e^{-p\kappa_n} - e^{-p\kappa_{n+1}}]\right)\\
&= \sum_{i=1}^{\infty}\mathbb{E}_x\left(e^{-(r+p)\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\mathbb{I}_{\{\kappa_i<\infty\}}\right).
\end{aligned}$$

For each $i \geq 1$, define the stopping time

$$\tau_i = \inf\{t \geq \kappa_i : X_t \geq 0\}.$$

Then, for every $i \geq 2$, conditioning with respect to the filtration at time τ_{i-1} and the strong Markov property, we obtain that

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-(r+p)\kappa_i}\omega(X_{\kappa_i}, X_{\kappa_i-})\mathbb{I}_{\{\kappa_i<\infty\}}\right)\\
&\quad= \mathbb{E}_x\left(\mathbb{E}_x\left(e^{-(r+p)\kappa_i}\omega(X_{\kappa_n}, X_{\kappa_i-})\mathbb{I}_{\{\kappa_i<\infty\}}|F_{\tau_{i-1}}\right)\right)\\
&\quad= \mathbb{E}_x\left(e^{-(r+p)\tau_{i-1}}\mathbb{I}_{\{\tau_{i-1}<\infty\}}\right)\mathbb{E}\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1}, X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right),
\end{aligned}$$

where in the last equality we used the fact that X creeps upwards. Conditioning the first expectation above with respect to the filtration at time κ_{i-1}, we obtain that

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-(r+p)\kappa_i}\omega(X_{\kappa_i},X_{\kappa_i-})\mathbb{I}_{\{\kappa_i<\infty\}}\right)\\
&=\mathbb{E}_x\left(e^{-(r+p)\kappa_{i-1}}\mathbb{I}_{\{\kappa_{i-1}<\infty\}}\mathbb{E}_{X_{\kappa_{i-1}}}\left(e^{-(r+p)\tau_0^+}\mathbb{I}_{\{\tau_0^+<\infty\}}\right)\right)\mathbb{E}\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1},X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right)\\
&=\mathbb{E}_x\left(e^{-(r+p)\kappa_{i-1}}\mathbb{I}_{\{\kappa_{i-1}<\infty\}}e^{\Phi(r+p)X_{\kappa_{i-1}}}\right)\mathbb{E}\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1},X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right)\\
&=\theta^{(r+p)}(\Phi(r+p))^{i-2}\mathbb{E}_x(e^{-(r+p)\kappa_1+\Phi(r+p)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}})\mathbb{E}\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1},X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right),
\end{aligned}$$

where the second equality follows from equation (1.3) and the last by Corollary 5.2.3. Hence, using the geometric series formula, we obtain that

$$\begin{aligned}
&\mathbb{E}_x\left(\sum_{i=1}^{J_{\mathbf{e}_p}}e^{-r\kappa_i}\omega(X_{\kappa_i},X_{\kappa_i-})\right)\\
&\qquad=\mathbb{E}_x\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1},X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right)\\
&\qquad\quad+\frac{\mathbb{E}_x(e^{-(r+p)\kappa_1+\Phi(r+p)X_{\kappa_1}}\mathbb{I}_{\{\kappa_1<\infty\}})\mathbb{E}\left(e^{-(r+p)\kappa_1}\omega(X_{\kappa_1},X_{\kappa_1-})\mathbb{I}_{\{\kappa_1<\infty\}}\right)}{1-\theta^{(r+p)}(\Phi(r+p))}.
\end{aligned}$$

The proof is now complete. □

5.4 Proof of Theorem 5.2.1

Suppose that X is a spectrally negative Lévy process of finite variation. Starting from zero, it takes a positive amount of time to enter the set $(-\infty,0)$, that is, $\tau_0^->0$ $\mathbb{P}$-a.s. Hence, stopping at the sequence of times in which X enters the set $(-\infty,0)$ after visiting the level zero, using the strong Markov property for Lévy process and the lack of memory property of the exponential distribution, we can find directly the distribution of $J_{\mathbf{e}_p}$. However, in the case where X is of infinite variation, it is well known that the closed zero set of X is perfect and nowhere dense, rendering the latter approach unhelpful (since we have that $\tau_0^-=0$ a.s). Therefore, in order to exploit the idea applicable for finite variation processes to help us prove Theorem 5.2.1, we make use of the perturbation method used in Section 3.3.1. This method is mainly based on the works of Dassios and Wu (2011) and Revuz and Yor (1999) (see Theorem VI.1.10) which consist of the construction of a new "perturbed" process $X^{(\varepsilon)}$ (for ε sufficiently small) that approximates X with the property that $X^{(\varepsilon)}$ visits the level zero a finite number of times before any time $t \geq 0$. We then approximate J_t by the number of downcrossing by jumps of $X^{(\varepsilon)}$.

Recall that for $\varepsilon > 0$,

$$X_t^{(\varepsilon)} = \begin{cases} X_t - \varepsilon & \text{if} \quad \sigma_{k,\varepsilon}^- \leq t < \sigma_{k,\varepsilon}^+ \\ X_t & \text{if} \quad \sigma_{k,\varepsilon}^+ \leq t < \sigma_{k+1,\varepsilon}^-, \end{cases}$$

where

$$\sigma_{k,\varepsilon}^+ = \inf\{t > \sigma_{k,\varepsilon}^- : X_t \geq \varepsilon\}$$
$$\sigma_{k+1,\varepsilon}^- = \inf\{t > \sigma_{k,\varepsilon}^+ : X_t < 0\}.$$

For $\varepsilon > 0$, we defined in Section 3.3.1 the random variable $M_t^{(\varepsilon)}$ as the number of times the process $X_t^{(\varepsilon)}$ is below the level zero at time $t \geq 0$ (see (3.12) for its definition). Moreover, in Lemma 3.3.3 the distribution of $M^{(\varepsilon)}$ at an independent exponential time is found. We have the following remark about its distribution.

Remark 5.4.1. *From the proof of Lemma 3.3.3, we can give a probabilistic interpretation to the functions $\mathcal{I}^{(p,0)}$ and $\mathcal{I}^{(p,\Phi(p))}$. The function $\mathcal{I}^{(p,0)}(x)$ corresponds to the probability of visiting the interval $(-\infty, 0)$, starting from the level x, before an exponential time of parameter p. The function $\mathcal{I}^{(p,\Phi(p))}(x)$ corresponds to the probability that, given X starts from x, there is a visit to the interval $(-\infty, 0)$ and then again a visit to the point x, before an exponential time. Whilst for all $x < \varepsilon$, the term $e^{-\Phi(p)(\varepsilon - x)}$ corresponds to the visit of the point ε, starting from x before an exponential time of parameter p.*

Hence, formula (3.13) has the following interpretation: given that $X^{(\varepsilon)}$ starts from $x < \varepsilon$, to have $n-1$ visits to the interval $(-\infty, 0]$ before an exponential time, we need to have a first visit to the point ε (with probability $e^{\Phi(p)(x-\varepsilon)}$) then $n-2$ excursions before $\boldsymbol{e}_p$ (with probability $\mathcal{I}^{(p,\Phi(p))}(\varepsilon)^{n-2}$) and a last visit to the interval $(-\infty, 0)$ (with probability $\mathcal{I}^{(p,0)}(\varepsilon)$) with no excursions afterwards (with probability $1 - \mathcal{I}^{(p,\Phi(p))}(\varepsilon)$).

From the definition of $X_t^{(\varepsilon)}$ and from the fact that X creeps upwards we have that between any two times the process $X^{(\varepsilon)}$ is below zero there is an intermediate time in which the process X has to be at exactly at level ε. This fact together with the Strong Markov property of Lévy processes let us prove that conditioned to $\{M^{(\varepsilon)}_{\mathbf{e}_p}=n\}$ the random variables $\{X_{\sigma^-_{k,\varepsilon}}, 2\le k\le n\}$ are independent.

Lemma 5.4.2. *Suppose that X is a spectrally negative Lévy process. Let $\varepsilon>0$ and $p\ge 0$, then for any $x\in\mathbb{R}$, $n\ge 3$ and $\beta_2,\ldots,\beta_n\ge 0$, we have that*

$$\mathbb{E}_x\left(\prod_{k=2}^n e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=n\right)=\prod_{k=2}^n \mathbb{E}_x(e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=n).$$

Proof. Let $n\ge 3$. We calculate for $x\in\mathbb{R}$ and $\beta_2,\ldots,\beta_n\ge 0$,

$$\mathbb{E}_x\left(\prod_{k=2}^n e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=n\right)=\frac{\mathbb{E}_x\left(\prod_{k=2}^n e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\mathbf{e}_p\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}>\mathbf{e}_p\}}\right)}{\mathbb{P}_x(\sigma^-_{n,\varepsilon}<\mathbf{e}_p,\sigma^-_{n+1,\varepsilon}>\mathbf{e}_p)}.$$

Conditioning with respect to $\mathcal{F}_{\sigma^+_{n-1,\varepsilon}}$, we have that from the Markov property for Lévy process and the fact that X can only creep upwards, i.e. $X_{\sigma^+_{n-1,\varepsilon}}=\varepsilon$,

$$\begin{aligned}
&\mathbb{E}_x\left(\prod_{k=2}^n e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=n\right)\\
&=\frac{\mathbb{E}_x\left(\prod_{k=2}^{n-1} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}\mathbb{E}_x(e^{-\beta_n X_{\sigma^-_{n,\varepsilon}}}\mathbb{I}_{\{\sigma^-_{n,\varepsilon}<\mathbf{e}_p\}}\mathbb{I}_{\{\sigma^-_{n+1,\varepsilon}>\mathbf{e}_p\}}|\mathcal{F}_{\sigma^+_{n-1,\varepsilon}})\right)}{\mathbb{E}_x(\mathbb{P}_x(\sigma^-_{n,\varepsilon}<\mathbf{e}_p,\sigma^-_{n+1,\varepsilon}>\mathbf{e}_p|\mathcal{F}_{\sigma^+_{n-1,\varepsilon}}))}\\
&=\frac{\mathbb{E}_x\left(\prod_{k=2}^{n-1} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\right)\mathbb{E}_\varepsilon(e^{-\beta_n X_{\sigma^-_{2,\varepsilon}}}\mathbb{I}_{\{\sigma^-_{2,\varepsilon}<\mathbf{e}_p\}}\mathbb{I}_{\{\sigma^-_{3,\varepsilon}>\mathbf{e}_p\}})}{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}})\mathbb{P}_\varepsilon(\sigma^-_{2,\varepsilon}<\mathbf{e}_p,\sigma^-_{3,\varepsilon}>\mathbf{e}_p)}\\
&=\frac{\mathbb{E}_x\left(\prod_{k=2}^{n-1} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}}e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\right)}{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}})}\mathbb{E}_\varepsilon(e^{-\beta_n X_{\sigma^-_{2,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=2),
\end{aligned}$$

where in the second equality we used the loss of memory property of the exponential distribution. Taking $\beta_k=0$ for all $k\in\{2,\ldots,n-1\}$ in the above calculation, we get that for all $n\ge 3$ and $x\in\mathbb{R}$,

$$\mathbb{E}_x\left(e^{-\beta_n X_{\sigma^-_{n,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=n\right)=\mathbb{E}_\varepsilon(e^{-\beta_n X_{\sigma^-_{2,\varepsilon}}}|M^{(\varepsilon)}_{\mathbf{e}_p}=2).$$

Hence, we conclude that

$$\mathbb{E}_x\left(\prod_{k=2}^{n} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n\right)$$
$$= \frac{\mathbb{E}_x\left(\prod_{k=2}^{n-1} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} e^{-p\sigma^+_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\right)}{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}})} \mathbb{E}_x\left(e^{-\beta_n X_{\sigma^-_{n,\varepsilon}}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n\right). \tag{5.7}$$

Moreover, for every $k \in \{2, \ldots, n-1\}$, setting $\beta_i = 0$ for all $i \neq k$ in (5.7), we obtain that

$$\mathbb{E}_x\left(e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n\right) = \frac{\mathbb{E}_x\left(e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} e^{-p\sigma^+_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\right)}{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}})}. \tag{5.8}$$

Similarly, conditioning with respect to the filtration at time $\sigma^+_{n-2,\varepsilon}$ and by an induction argument, we have that for all $n \geq 3$ and $x \in \mathbb{R}$,

$$\frac{\mathbb{E}_x\left(\prod_{k=2}^{n-1} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} e^{-p\sigma^+_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\right)}{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}} \mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}})}$$
$$= \frac{\mathbb{E}_x(e^{-\beta_2 X_{\sigma^-_{2,\varepsilon}}} e^{-p\sigma^+_{2,\varepsilon}} \mathbb{I}_{\{\sigma^+_{2,\varepsilon}<\infty\}})}{\mathbb{E}_x(e^{-p\sigma^+_{2,\varepsilon}} \mathbb{I}_{\{\sigma^+_{2,\varepsilon}<\infty\}})} \prod_{k=3}^{n-1} \frac{\mathbb{E}_\varepsilon(e^{-\beta_k X_{\sigma^-_{2,\varepsilon}}} e^{-p\sigma^+_{2,\varepsilon}} \mathbb{I}_{\{\sigma^+_{2,\varepsilon}<\infty\}})}{\mathbb{E}_\varepsilon(e^{-p\sigma^+_{2,\varepsilon}} \mathbb{I}_{\{\sigma^+_{2,\varepsilon}<\infty\}})},$$

where we understand that $\prod_{k=3}^{n-1} = 1$ when $n-1 < 3$. Hence, for any $k \in \{2, \ldots, n-1\}$, taking $\beta_i = 0$ for all $i \neq k$ in the equation above and using the result given in equation (5.8), we obtain that

$$\mathbb{E}_x(e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n) = \frac{\mathbb{E}_\varepsilon(e^{-\beta_k X_{\sigma^-_{2,\varepsilon}}} e^{-p\sigma^+_{2,\varepsilon}} \mathbb{I}_{\{\sigma^+_{2,\varepsilon}<\infty\}})}{\mathbb{E}_\varepsilon(e^{-p\sigma^+_{2,\varepsilon}} \mathbb{I}_{\{\sigma^+_{2,\varepsilon}<\infty\}})}.$$

Therefore from above and equation (5.7), we have just proved that for all $n \geq 3$, $x \in \mathbb{R}$ and $\beta_1, \ldots, \beta_n \geq 0$,

$$\mathbb{E}_x\left(\prod_{k=2}^{n} e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n\right) = \prod_{k=2}^{n} \mathbb{E}_x(e^{-\beta_k X_{\sigma^-_{k,\varepsilon}}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n).$$

The proof is complete. $\square$

Note that the event $\{M_{\mathbf{e}_p}^{(\varepsilon)} = n\}$ means that the process enters the interval $(-\infty, 0)$ a total of $n-1$ times either by creeping or by a jump. Hence, in the event $\{M_{\mathbf{e}_p} = n\}$, we find the probability that the k-th visit to the interval $(-\infty, 0)$ is made by creeping. For any $p \geq 0$ and $x \in \mathbb{R}$, we define the auxiliary functions

$$\rho(x,p) := 1 - \frac{\sigma^2/2\{W^{(p)\prime}(x) - \Phi(p)W^{(p)}(x)\}}{e^{\Phi(p)x}(1-\psi'(\Phi(p))e^{-\Phi(p)x}W^{(p)}(x))} = 1 - \frac{\mathcal{C}^{(p)}(x)}{e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)}, \tag{5.9}$$

$$\varrho_\varepsilon(x,p) := 1 - \frac{[1-e^{-\Phi(p)\varepsilon}\mathcal{I}^{(p,0)}(\varepsilon)]\mathcal{C}^{(p)}(x)}{\mathcal{I}^{(p,0)}(x) - \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon-x)}\mathcal{I}^{(p,\Phi(p))}(x)}, \tag{5.10}$$

where we understand that $\rho(x,p) = \varrho_\varepsilon(x,p) = 1$ for all $x \in \mathbb{R}$ when $\sigma = 0$.

Lemma 5.4.3. *Let $\varepsilon > 0$ and $p \geq 0$. We have for all $n \geq 2$ and $x < \varepsilon$ that*

$$\mathbb{P}_x(X_{\sigma_{n,\varepsilon}^-} < 0 | M_{\mathbf{e}_p}^{(\varepsilon)} = n) = \varrho_\varepsilon(\varepsilon, p)$$

and for any $2 \leq k < n$,

$$\mathbb{P}_x(X_{\sigma_{k,\varepsilon}^-} < 0 | M_{\mathbf{e}_p}^{(\varepsilon)} = n) = \rho(\varepsilon, p).$$

Proof. For $2 \leq k < n$ and $x < \varepsilon$, we have

$$\begin{aligned}
\mathbb{P}_x(X_{\sigma_{k,\varepsilon}^-} = 0 | M_{\mathbf{e}_p}^{(\varepsilon)} = n) &= \frac{\mathbb{P}_x(X_{\sigma_{k,\varepsilon}^-} = 0, \sigma_{n,\varepsilon}^- < \mathbf{e}_p, \sigma_{n+1,\varepsilon}^- > \mathbf{e}_p)}{\mathbb{P}_x(\sigma_{n,\varepsilon}^- < \mathbf{e}_p, \sigma_{n+1,\varepsilon}^- > \mathbf{e}_p)} \\
&= \frac{\mathbb{P}_x(X_{\sigma_{k,\varepsilon}^-} = 0, \sigma_{k,\varepsilon}^+ < \mathbf{e}_p, \sigma_{n,\varepsilon}^- < \mathbf{e}_p, \sigma_{n+1,\varepsilon}^- > \mathbf{e}_p)}{\mathbb{P}_x(\sigma_{k,\varepsilon}^+ < \mathbf{e}_p, \sigma_{n,\varepsilon}^- < \mathbf{e}_p, \sigma_{n+1,\varepsilon}^- > \mathbf{e}_p)},
\end{aligned}$$

where the last equality follows since $\sigma_{k,\varepsilon}^+ < \sigma_{n,\varepsilon}^-$. Conditioning with respect to the filtration at time $\sigma_{k,\varepsilon}^+$ for both the numerator and the denominator and from the lack of memory property of the exponential distribution, we get

$$\mathbb{P}_x(X_{\sigma_{k,\varepsilon}^-} = 0 | M_{\mathbf{e}_p}^{(\varepsilon)} = n) = \frac{\mathbb{E}_x(\mathbb{I}_{\{X_{\sigma_{k,\varepsilon}^-} = 0\}} e^{-p\sigma_{k,\varepsilon}^+} \mathbb{I}_{\{\sigma_{k,\varepsilon}^+ < \infty\}})}{\mathbb{E}_x(e^{-p\sigma_{k,\varepsilon}^+} \mathbb{I}_{\{\sigma_{k,\varepsilon}^+ < \infty\}})}.$$

Conditioning the above with respect to the filtration at time $\sigma_{k,\varepsilon}^-$ and using the strong Markov property, we obtain that

$$\mathbb{P}_x(X_{\sigma^-_{k,\varepsilon}} = 0 | M^{(\varepsilon)}_{\mathbf{e}_p} = n) = \frac{\mathbb{E}_x(\mathbb{I}_{\{X_{\sigma^-_{k,\varepsilon}}=0\}}\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}}e^{-p\sigma^-_{k,\varepsilon}}\mathbb{E}_{X_{\sigma^-_{k,\varepsilon}}}(e^{-p\tau^+_\varepsilon}\mathbb{I}_{\{\tau^+_\varepsilon<\infty\}}))}{\mathbb{E}_x(e^{-p\sigma^-_{k,\varepsilon}}\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}}\mathbb{E}_{X_{\sigma^-_{k,\varepsilon}}}(e^{-p\tau^+_\varepsilon}\mathbb{I}_{\{\tau^+_\varepsilon<\infty\}})}$$
$$= \frac{\mathbb{E}_x(\mathbb{I}_{\{X_{\sigma^-_{k,\varepsilon}}=0\}}\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}}e^{-p\sigma^-_{k,\varepsilon}}e^{-\Phi(p)\varepsilon})}{\mathbb{E}_x(e^{-p\sigma^-_{k,\varepsilon}+\Phi(p)(X_{\sigma^-_{k,\varepsilon}}-\varepsilon)}\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}})},$$

where in the last equality we used the formula for the Laplace transform of τ^+_0 given in (1.3) and the event of $\{X_{\sigma^-_{k,\varepsilon}} = 0\}$ in the numerator. Then, conditioning with respect to the filtration at time σ^+_{k-1}, using the strong Markov property and the fact that X creeps upwards, we get

$$\mathbb{P}_x(X_{\sigma^-_{k,\varepsilon}} = 0 | M^{(\varepsilon)}_{\mathbf{e}_p} = n) = \frac{\mathbb{E}_x(\mathbb{I}_{\{X_{\sigma^-_{k,\varepsilon}}=0\}}\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}}e^{-p\sigma^-_{k,\varepsilon}}e^{-\Phi(p)\varepsilon})}{\mathbb{E}_x(e^{-p\sigma^-_{k,\varepsilon}+\Phi(p)(X_{\sigma^-_{k,\varepsilon}}-\varepsilon)}\mathbb{I}_{\{\sigma^-_{k,\varepsilon}<\infty\}})}$$
$$= \frac{\mathbb{E}_x(e^{-\sigma^+_{k,\varepsilon}}\mathbb{I}_{\{\sigma^+_{k,\varepsilon}<\infty\}})\mathbb{E}_\varepsilon(e^{-p\tau^-_0}\mathbb{I}_{\{X_{\tau^-_0}=0\}}\mathbb{I}_{\{\tau^-_0<\infty\}})}{\mathbb{E}_x(e^{-\sigma^+_{k,\varepsilon}}\mathbb{I}_{\{\sigma^+_{k,\varepsilon}<\infty\}})\mathbb{E}_\varepsilon(e^{-p\tau^-_0+\Phi(p)X_{\tau^-_0}}\mathbb{I}_{\{\tau^-_0<\infty\}})}$$
$$= 1 - \rho(\varepsilon, p),$$

where the last equation follows from and (1.11) and (1.14) and the definition of ρ given in (5.9). On the other hand, take $n \geq 2$ and $x < \varepsilon$. Conditioning with respect to the filtration at time $\sigma^+_{n-1,\varepsilon}$ and from the lack of memory property of the exponential distribution, we get that

$$\mathbb{P}_x(X_{\sigma^-_{n,\varepsilon}} = 0 | M^{(\varepsilon)}_{\mathbf{e}_p} = n) = \frac{\mathbb{P}_x(X_{\sigma^-_{n,\varepsilon}} = 0, \sigma^-_{n,\varepsilon} < \mathbf{e}_p, \sigma^-_{n+1,\varepsilon} > \mathbf{e}_p)}{\mathbb{P}_x(\sigma^-_{n,\varepsilon} < \mathbf{e}_p, \sigma^-_{n+1,\varepsilon} > \mathbf{e}_p)}$$
$$= \frac{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\mathbb{P}_\varepsilon(X_{\sigma^-_{2,\varepsilon}} = 0, \sigma^-_{2,\varepsilon} < \mathbf{e}_p, \sigma^-_{3,\varepsilon} > \mathbf{e}_p))}{\mathbb{E}_x(e^{-p\sigma^+_{n-1,\varepsilon}}\mathbb{I}_{\{\sigma^+_{n-1,\varepsilon}<\infty\}}\mathbb{P}_\varepsilon(\sigma^-_{2,\varepsilon} < \mathbf{e}_p, \sigma^-_{3,\varepsilon} > \mathbf{e}_p))}$$
$$= \frac{\mathbb{P}_\varepsilon(X_{\sigma^-_{2,\varepsilon}} = 0, \sigma^-_{2,\varepsilon} < \mathbf{e}_p, \sigma^-_{3,\varepsilon} > \mathbf{e}_p)}{\mathbb{P}_\varepsilon(\sigma^-_{2,\varepsilon} < \mathbf{e}_p, \sigma^-_{3,\varepsilon} > \mathbf{e}_p)},$$

where in the second equality we used the fact that $X_{\sigma^+_{n-1}} = \varepsilon$. In a similar way, conditioning with respect to the time $\sigma^-_{2,\varepsilon}$ and using the lack of memory property of the exponential distribution, we get

$$\mathbb{P}_x(X_{\sigma^-_{n,\varepsilon}} = 0 | M^{(\varepsilon)}_{\mathbf{e}_p} = n) = \frac{\mathbb{E}_\varepsilon(e^{-p\sigma^-_{2,\varepsilon}}\mathbb{I}_{\{X_{\sigma^-_{2,\varepsilon}}=0\}}\mathbb{P}_{X_{\sigma^-_{2,\varepsilon}}}(M_{\mathbf{e}_p}=1))}{\mathbb{E}_\varepsilon(e^{-p\sigma^-_{2,\varepsilon}}\mathbb{I}_{\{\sigma^-_{2,\varepsilon}<\infty\}}\mathbb{P}_{X_{\sigma^-_{2,\varepsilon}}}(M_{\mathbf{e}_p}=1))}$$
$$= \frac{\mathbb{E}_\varepsilon(e^{-p\sigma^-_{2,\varepsilon}}\mathbb{I}_{\{X_{\sigma^-_{2,\varepsilon}}=0\}}[1-\mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)\varepsilon}])}{\mathbb{E}_\varepsilon(e^{-p\sigma^-_{2,\varepsilon}}\mathbb{I}_{\{\sigma^-_{2,\varepsilon}<\infty\}}[1-\mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)(\varepsilon-X_{\sigma^-_{2,\varepsilon}})}])},$$

where the last equality follows from formula (3.13). We then obtain that for $x \geq \varepsilon$,

$$\mathbb{P}_x(X_{\sigma^-_{2,\varepsilon}} = 0 | M^{(\varepsilon)}_{\mathbf{e}_p} = 2) = \frac{[1-e^{-\Phi(p)\varepsilon}\mathcal{I}^{(p,0)}(\varepsilon)]\mathbb{E}_\varepsilon(e^{-p\tau_0^-}\mathbb{I}_{\{X_{\tau_0^-}=0\}}\mathbb{I}_{\{\tau_0^-<\infty\}})}{\mathbb{E}_\varepsilon(e^{-p\tau_0^-}\mathbb{I}_{\{\tau_0^-<\infty\}}) - \mathcal{I}^{(p,0)}(\varepsilon)e^{-\Phi(p)\varepsilon}\mathbb{E}_\varepsilon(e^{-p\tau_0^-+\Phi(p)X_{\tau_0^-}}\mathbb{I}_{\{\tau_0^-<\infty\}})}$$
$$= 1 - \varrho_\varepsilon(\varepsilon, p),$$

where the last equality follows from equation (1.11) and the definition of ϱ_ε (see equation (5.9)). The proof is now complete. □

For all $t \geq 0$ and $\varepsilon > 0$, we define the random variable

$$J^{(\varepsilon)}_t = \sum_{k=2}^{\infty} \mathbb{I}_{\{\sigma^-_{k,\varepsilon}<t\}}\mathbb{I}_{\{X^{(\varepsilon)}_{\sigma^-_{k,\varepsilon}}<-\varepsilon\}} = \int_{[0,t]}\int_{(-\infty,0)} \mathbb{I}_{\{X^{(\varepsilon)}_{s-}>0\}}\mathbb{I}_{\{X^{(\varepsilon)}_{s-}+y<-\varepsilon\}}N(\mathrm{d}s\times\mathrm{d}y).$$

We prove in the following Lemma that the random variable $J^{(\varepsilon)}_t$ converges to J_t for all $t \geq 0$ when $\varepsilon \downarrow 0$.

Lemma 5.4.4. *For each $t \geq 0$, $J^{(\varepsilon)}_t \uparrow J_t$ when $\varepsilon \downarrow 0$, where J_t is defined by* (5.2).

Proof. For a fixed $t \geq 0$, we first prove that the mapping $\varepsilon \mapsto J^{(\varepsilon)}_t$ increases when $\varepsilon \downarrow 0$. Indeed, from Lemma 3.3.1, we know that for all $s \geq 0$, $X^{(\varepsilon)}_s \uparrow X_s$ when $\varepsilon \downarrow 0$. Hence, for a fixed $s \geq 0$ and for $0 < \varepsilon_1 \leq \varepsilon_2$, we have that $X^{(\varepsilon_2)}_s \leq X^{(\varepsilon_1)}_s \leq X_s$ which we can easily take limits to conclude that $X^{(\varepsilon_2)}_{s-} \leq X^{(\varepsilon_1)}_{s-} \leq X_{s-}$. This fact implies that $\{X^{(\varepsilon_2)}_{s-} > 0\} \subset \{X^{(\varepsilon_1)}_{s-} > 0\} \subset \{X_{s-} > 0\}$. Moreover, from Lemma 3.3.1, we have that if $s \geq 0$ is such that $X^{(\varepsilon)}_s > 0$, then there exists a value $k \geq 0$ such that $s \in [\sigma^+_{k,\varepsilon}, \sigma^-_{k+1,\varepsilon})$ and then $X^{(\varepsilon)}_s = X_s$. Thus, we have for all $y \in (-\infty, 0)$ that

$$\begin{aligned}\{X^{(\varepsilon_2)}_{s-} > 0\} \cap \{X^{(\varepsilon_2)}_{s-} + y < -\varepsilon_2\} &= \{X^{(\varepsilon_2)}_{s-} > 0\} \cap \{X_{s-} + y < -\varepsilon_2\} \\ &\subset \{X^{(\varepsilon_1)}_{s-} > 0\} \cap \{X_{s-} + y < -\varepsilon_1\} \\ &= \{X^{(\varepsilon_1)}_{s-} > 0\} \cap \{X^{(\varepsilon_1)}_{s-} + y < -\varepsilon_1\}.\end{aligned}$$

Similarly, we have that

$$\{X^{(\varepsilon_2)}_{s-} > 0\} \cap \{X^{(\varepsilon_2)}_{s-} + y < -\varepsilon_2\} \subset \{X_{s-} > 0\} \cap \{X_{s-} + y < 0\}.$$

Hence, we have that for all $t \geq 0$,

$$J^{(\varepsilon_2)}_t \leq J^{(\varepsilon_1)}_t \leq J_t.$$

We have just proved that the sequence $\{J^{(1/n)}_t, n \geq 1\}$ is a positive increasing sequence bounded by J_t. Thus, the limit exist and $\lim_{n\to\infty} J^{(1/n)}_t \leq J_t$. On the other hand, using Fatou's Lemma, and the fact that $X^{(1/n)}_{s-} = X_{s-}$ in the event $\{X^{(1/n)}_{s-} > 0\}$, we obtain that

$$\begin{aligned}\liminf_{n\to\infty} J^{(1/n)}_t &= \liminf_{n\to\infty} \int_{[0,t]} \int_{(-\infty,0)} \mathbb{I}_{\{X^{(1/n)}_{s-}>0\}} \mathbb{I}_{\{X^{(1/n)}_{s-}+y<-1/n\}} N(\mathrm{d}s \times \mathrm{d}y) \\ &\geq \int_{[0,t]} \int_{(-\infty,0)} \liminf_{n\to\infty} \mathbb{I}_{\{X^{(1/n)}_{s-}>0\}} \mathbb{I}_{\{X_{s-}+y+1/n<\}} N(\mathrm{d}s \times \mathrm{d}y) \\ &= \int_{[0,t]} \int_{(-\infty,0)} \mathbb{I}_{\{X_{s-}>0\}} \mathbb{I}_{\{X_{s-}+y<0\}} N(\mathrm{d}s \times \mathrm{d}y) \\ &= J_t,\end{aligned}$$

where the second last equality follows since $X^{(1/n)}_{s-} \uparrow X_{s-}$ when $n \to \infty$, the function $x \mapsto \mathbb{I}_{\{x>0\}}$ is left-continuous and $x \mapsto \mathbb{I}_{\{x<0\}}$ is right-continuous. Therefore, we have that for all $t \geq 0$, $J^{(\varepsilon)}_t \uparrow J_t$ when $\varepsilon \downarrow 0$. □

Similar to Revuz and Yor (1999) (Chapter VI, Theorem 1.10), it turns out that $M^{(\varepsilon)}_t$ are approximations to L_t in some sense. Now we are ready to prove Theorem 5.2.1. For a fixed $\varepsilon > 0$, with the help of Lemmas 5.4.2 and 5.4.3, we calculate the joint Laplace transform of $(2\varepsilon M^{(\varepsilon)}_{\mathbf{e}_p}, J^{(\varepsilon)}_{\mathbf{e}_p})$. Then, taking $\varepsilon \downarrow 0$, from Lemmas 3.3.2 and 5.4.4 and using the dominated convergence theorem, we find the Laplace transform of $(L_{\mathbf{e}_p}, J_{\mathbf{e}_p})$.

Proof of Theorem 5.2.1. First, consider the case when $x \le 0$. From Lemmas 5.4.2 and 5.4.3, we have that given $\{M^{(\varepsilon)}_{\mathbf{e}_p} = n\}$, with $n \ge 2$, the random variable $J^{(\varepsilon)}_{\mathbf{e}_p}$ can be seen as a sum of a binomial random variable with parameters $(n-2, \rho(\varepsilon, p))$ and an independent Bernoulli random variable with parameter $\varrho_\varepsilon(\varepsilon, p)$. Hence, conditioning with respect to the number of downcrossings of $X^{(\varepsilon)}$, we have that for all $\beta \ge 0$ and $x \le 0$,

$$
\begin{aligned}
&\mathbb{E}_x(e^{-2\alpha\varepsilon M^{(\varepsilon)}_{\mathbf{e}_p} - \beta J^{(\varepsilon)}_{\mathbf{e}_p}}) \\
&\quad = \sum_{n=1}^{\infty} e^{-2\alpha\varepsilon n} \mathbb{E}_x(e^{-\beta J^{(\varepsilon)}_{\mathbf{e}_p}} | M^{(\varepsilon)}_{\mathbf{e}_p} = n) \mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} = n) \\
&\quad = e^{-2\alpha\varepsilon} \mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} = 1) \\
&\qquad + \sum_{n=2}^{\infty} e^{-2\alpha\varepsilon n} [e^{-\beta}\rho(\varepsilon, p) + 1 - \rho(\varepsilon, p)]^{n-2} [e^{-\beta}\varrho_\varepsilon(\varepsilon, p) + 1 - \varrho_\varepsilon(\varepsilon, p)] \mathbb{P}_x(M^{(\varepsilon)}_{\mathbf{e}_p} = n).
\end{aligned}
$$

Then, using the formulas for the distribution of $M_{\mathbf{e}_p}$ derived in equation (3.13), we have that

$$
\begin{aligned}
&\mathbb{E}_x(e^{-2\alpha\varepsilon M^{(\varepsilon)}_{\mathbf{e}_p} - \beta J^{(\varepsilon)}_{\mathbf{e}_p}}) \\
&= e^{-2\alpha\varepsilon}[1 - \mathcal{I}^{(p,0)}(\varepsilon) e^{-\Phi(p)(\varepsilon - x)}] \\
&\quad + e^{-4\alpha\varepsilon}[e^{-\beta}\varrho_\varepsilon(\varepsilon, p) + 1 - \varrho_\varepsilon(\varepsilon, p)] \mathcal{I}^{(p,0)}(\varepsilon) e^{-\Phi(p)(\varepsilon - x)} [1 - \mathcal{I}^{(p,\Phi(p))}(\varepsilon)] \\
&\qquad \times \sum_{n=2}^{\infty} e^{-2\alpha\varepsilon(n-2)} [e^{-\beta}\rho(\varepsilon, p) + 1 - \rho(\varepsilon, p)]^{n-2} [\mathcal{I}^{(p,\Phi(p))}(\varepsilon)]^{n-2} \\
&= e^{-2\alpha\varepsilon}[1 - \mathcal{I}^{(p,0)}(\varepsilon) e^{-\Phi(p)(\varepsilon - x)}] \\
&\quad + \frac{e^{-4\alpha\varepsilon}[e^{-\beta}\varrho_\varepsilon(\varepsilon, p) + 1 - \varrho_\varepsilon(\varepsilon, p)] \mathcal{I}^{(p,0)}(\varepsilon) e^{-\Phi(p)(\varepsilon - x)} [1 - \mathcal{I}^{(p,\Phi(p))}(\varepsilon)]}{1 - e^{-2\alpha\varepsilon}[e^{-\beta}\rho(\varepsilon, p) + 1 - \rho(\varepsilon, p)][\mathcal{I}^{(p,\Phi(p))}(\varepsilon)]} \\
&= e^{-2\alpha\varepsilon} + e^{-2\alpha\varepsilon} e^{-\Phi(p)(\varepsilon - x)} \mathcal{I}^{(p,0)}(\varepsilon) \\
&\quad \times \frac{e^{-2\alpha\varepsilon}(e^{-\beta} - 1)\varrho_\varepsilon(\varepsilon, p)[1 - \mathcal{I}^{(p,\Phi(p))}(\varepsilon)] + e^{-2\alpha\varepsilon}(e^{-\beta} - 1)\rho(\varepsilon, p)[\mathcal{I}^{(p,\Phi(p))}(\varepsilon)] + e^{-2\alpha\varepsilon} - 1}{1 - e^{-2\alpha\varepsilon}\mathcal{I}^{(p,\Phi(p))}(\varepsilon) - e^{-2\alpha\varepsilon}(e^{-\beta} - 1)\rho(\varepsilon, p)\mathcal{I}^{(p,\Phi(p))}(\varepsilon)},
\end{aligned}
$$

where the second equality follows from the geometric sum. From the definition of ρ and ϱ_ε (see equations (5.9) and (5.10)), we have that

$$
\begin{aligned}
&\mathbb{E}_x(e^{-2\alpha\varepsilon M^{(\varepsilon)}_{\mathbf{e}_p} - \beta J^{(\varepsilon)}_{\mathbf{e}_p}}) \\
&\quad = e^{-2\alpha\varepsilon} + \frac{e^{-2\alpha\varepsilon} e^{-\Phi(p)(\varepsilon - x)} \left[e^{-2\alpha\varepsilon}(e^{-\beta} - 1)\left(\mathcal{I}^{(p,0)}(\varepsilon) - \mathcal{C}^{(p)}(\varepsilon)\right) + e^{-2\alpha\varepsilon} - 1\right]}{1 - e^{-2\alpha\varepsilon}\mathcal{I}^{(p,\Phi(p))}(\varepsilon) - e^{-2\alpha\varepsilon}(e^{-\beta} - 1)[\mathcal{I}^{(p,\Phi(p))}(\varepsilon) - \mathcal{C}^{(p)}(\varepsilon) e^{-\Phi(p)\varepsilon}]}.
\end{aligned} \tag{5.11}
$$

Suppose that X is a process of finite variation. Then $\sigma^2 = 0$, $\mathcal{C}^{(p)}(x) = 0$ for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}^{(p,0)}(\varepsilon) = 1 - \frac{p}{\Phi(p)\delta} \qquad \text{and} \qquad \lim_{\varepsilon \downarrow 0} \mathcal{I}^{(p,\Phi(p))}(\varepsilon) = 1 - \frac{\psi'(\Phi(p))}{\delta}.$$

Thus, by the dominated convergence theorem and Lemmas 5.4.4 and 3.3.2, we have that for all $x \le 0$,

$$\begin{aligned}\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p} - \beta J_{\mathbf{e}_p}}) &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x(e^{-2\alpha\varepsilon M^{(\varepsilon)}_{\mathbf{e}_p} - \beta J^{(\varepsilon)}_{\mathbf{e}_p}}) \\ &= 1 + e^{\Phi(p)x} \frac{(e^{-\beta} - 1)\left(\delta - \frac{p}{\Phi(p)}\right)}{\delta - e^{-\beta}[\delta - \psi'(\Phi(p))]}.\end{aligned}$$

Using the fact that $\sigma^2 = 0$, we can see that the equation above corresponds to (5.3).

For the case in which X is of infinite variation with no Gaussian component, i.e. $\sigma^2 = 0$ and $\int_{(-1,0)} y\Pi(\mathrm{d}y) = -\infty$, we have also that $\mathcal{C}^{(p)}(x) = 0$ for all $x \in \mathbb{R}$ and

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}^{(p,0)}(\varepsilon) = 1 \qquad \text{and} \qquad \lim_{\varepsilon \downarrow 0} \mathcal{I}^{(p,\Phi(p))}(\varepsilon) = 1.$$

Hence, taking $\varepsilon \downarrow 0$ in (5.11), we obtain that

$$\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p} - \beta J_{\mathbf{e}_p}}) = 1 - e^{\Phi(p)x}.$$

From the fact that $\delta = \infty$, it is easy to see that (5.3) also holds in this case. Lastly, we assume that $\sigma^2 > 0$. For this case, we have that for all $p \ge 0$,

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}^{(p,0)}(\varepsilon) = \lim_{\varepsilon \downarrow 0} \mathcal{I}^{(p,\Phi(p))}(\varepsilon) = \lim_{\varepsilon \downarrow 0} \mathcal{C}^{(p)}(\varepsilon) = 1.$$

Then, the joint Laplace transform of $(L_{\mathbf{e}_p}, J_{\mathbf{e}_p})$ under the measure $\mathbb{P}_x$, for $x \le 0$, is given by

$$\begin{aligned}&\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p} - \beta J_{\mathbf{e}_p}}) \\ &\quad = \lim_{\varepsilon \downarrow 0} \mathbb{E}_x(e^{-\alpha\varepsilon M^{(\varepsilon)}_{\mathbf{e}_p} - \beta J^{(\varepsilon)}_{\mathbf{e}_p}}) \\ &\quad = 1 + e^{\Phi(p)x} \lim_{\varepsilon \downarrow 0} \frac{e^{-2\alpha\varepsilon}(e^{-\beta} - 1)\left(\mathcal{I}^{(p,0)}(\varepsilon) - \mathcal{C}^{(p)}(\varepsilon)\right) + e^{-2\alpha\varepsilon} - 1}{1 - e^{-2\alpha\varepsilon}\mathcal{I}^{(p,\Phi(p))}(\varepsilon) - e^{-2\alpha\varepsilon}(e^{-\beta} - 1)[\mathcal{I}^{(p,\Phi(p))}(\varepsilon) - \mathcal{C}^{(p)}(\varepsilon)e^{-\Phi(p)\varepsilon}]}.\end{aligned}$$

Using L'Hopital's rule, we obtain that

$$\begin{aligned}\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}) &= 1+e^{\Phi(p)x}\frac{(e^{-\beta}-1)\left(\delta\frac{2}{\sigma^2}+\Phi(p)-\frac{p}{\Phi(p)}\frac{2}{\sigma^2}\right)-2\alpha}{2\alpha+\frac{2}{\sigma^2}\psi'(\Phi(p))+(e^{-\beta}-1)\left[\frac{2}{\sigma^2}\psi'(\Phi(p))-2\Phi(p)-\delta\frac{2}{\sigma^2}\right]}\\ &= 1+e^{\Phi(p)x}\frac{(e^{-\beta}-1)\left(\delta+\Phi(p)\frac{\sigma^2}{2}-\frac{p}{\Phi(p)}\right)-\alpha\sigma^2}{\alpha\sigma^2+\psi'(\Phi(p))+(e^{-\beta}-1)\left[\psi'(\Phi(p))-\Phi(p)\sigma^2-\delta\right]},\end{aligned}$$

where we used equations (1.5) and (1.6) so that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial x}\mathcal{I}^{(p,0)}(\varepsilon) &= -\frac{p}{\Phi(p)}\frac{2}{\sigma^2}\\ \lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial x}\mathcal{I}^{(p,\Phi(p))}(\varepsilon) &= -\frac{2}{\sigma^2}\psi'(\Phi(p))\\ \lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial x}\mathcal{C}^{(p)}(\varepsilon) &= -\delta\frac{2}{\sigma^2}-\Phi(p).\end{aligned}$$

Note that when X has jumps of infinite variation, we understand $\delta=\infty$ in the limiting sense and then, in this case, we have that

$$\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}) = 1-e^{\Phi(p)x}.$$

Now we consider the case where $x>0$. Define the stopping time

$$T_0=\inf\{t>0: X_t=0\}.$$

Then, we have that

$$\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}) = \mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}\mathbb{I}_{\{T_0>\mathbf{e}_p\}})+\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}\mathbb{I}_{\{T_0<\mathbf{e}_p\}}).$$

Note that since X starts from $x>0$, it can reach the level 0 either by creeping downwards or upwards. In view of the negative jumps, the second case happens if and only if there is a jump below the level zero and the process creeps upwards to zero after that. Using the fact that $L_{\mathbf{e}_p}=0$ on $\{T_0>\mathbf{e}_p\}$ and that $J_{\mathbf{e}_p}=0$ on the event $\{\tau_0^->\mathbf{e}_p\}$ and $J_{\mathbf{e}_p}=1$ on

$\{\tau_0^- < \mathbf{e}_p, T_0 > \mathbf{e}_p\}$, we have that the first term on the equation above becomes

$$\begin{aligned}
\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}\mathbb{I}_{\{T_0>\mathbf{e}_p\}}) &= \mathbb{E}_x(e^{-\beta J_{\mathbf{e}_p}}\mathbb{I}_{\{\tau_0^-<\mathbf{e}_p\}}\mathbb{I}_{\{T_0>\mathbf{e}_p\}})) + \mathbb{E}_x(e^{-\beta J_{\mathbf{e}_p}}\mathbb{I}_{\{\tau_0^->\mathbf{e}_p\}}) \\
&= e^{-\beta}\mathbb{E}_x\left(e^{-p\tau_0^-}\mathbb{I}_{\{X_{\tau_0^-}<0\}}\mathbb{P}_{X_{\tau_0^-}}(\tau_0^+ > \mathbf{e}_p)\right) + \mathbb{P}_x(\tau_0^- > \mathbf{e}_p) \\
&= e^{-\beta}\mathbb{E}_x\left(e^{-p\tau_0^-}\mathbb{I}_{\{X_{\tau_0^-}<0\}}\right) - e^{-\beta}\mathbb{E}_x\left(e^{-p\tau_0^-+\Phi(p)X_{\tau_0^-}}\mathbb{I}_{\{X_{\tau_0^-}<0\}}\right) \\
&\quad + 1 - \mathbb{E}_x(e^{-p\tau_0^-}\mathbb{I}_{\{\tau_0^-<\infty\}}) \\
&= e^{-\beta}[\mathcal{I}^{(p,0)}(x) - e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)] + 1 - \mathcal{I}^{(p,0)}(x),
\end{aligned}$$

where in the second equality we used the strong Markov property applied at the filtration at time τ_0^- and the lack of memory property of the exponential random variable, the third follows from the Laplace transform of τ_0^+ given in(1.3) and the last from equations (1.11) and (1.14). Similarly, conditioning with respect to the filtration at time T_0, we get that

$$\begin{aligned}
\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}\mathbb{I}_{\{T_0<\mathbf{e}_p\}}) &= \mathbb{E}_x(e^{-\beta J_{T_0}}\mathbb{E}_x(e^{-\alpha(L_{\mathbf{e}_p}-L_{T_0})-\beta(J_{\mathbf{e}_p}-J_{T_0})}\mathbb{I}_{\{T_0<\mathbf{e}_p\}}|\mathcal{F}_{T_0})) \\
&= \mathbb{E}_x(e^{-pT_0-\beta J_{T_0}}\mathbb{I}_{\{T_0<\infty\}})\mathbb{E}(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}),
\end{aligned}$$

where we used the fact that $(L_{T_0+t} - L_{T_0}, X_{t+T_0})$ is independent of $\mathcal{F}_{T_0}$ and has the same law as (L_t, X_t) for all $t \geq 0$, $L_{T_0} = 0$ by continuity of L and the lack of memory property of the exponential distribution. Conditioning with respect to $\mathcal{F}_{\tau_0^-}$, using the strong Markov property and the fact that since $J_{\tau_0^-} = 0$ if $X_{\tau_0^-} = 0$ and $J_{\tau_0^-} = 1$ otherwise, we have that

$$\begin{aligned}
\mathbb{E}_x(e^{-pT_0-\beta J_{T_0}}\mathbb{I}_{\{T_0<\infty\}}) &= \mathbb{E}_x(e^{-\beta J_{\tau_0^-}}\mathbb{I}_{\{\tau_0^-<\infty\}}\mathbb{E}_x(e^{-pT_0}\mathbb{I}_{\{T_0<\infty\}}|\mathcal{F}_{\tau_0^-})) \\
&= \mathbb{E}_x(e^{-\beta J_{\tau_0^-}}e^{-p\tau_0^-+\Phi(p)X_{\tau_0^-}}\mathbb{I}_{\{\tau_0^-<\infty\}}) \\
&= e^{-\beta}[e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x) - \mathcal{C}^{(p)}(x)] + \mathcal{C}^{(p)}(x).
\end{aligned}$$

Therefore for all $x > 0$, we have that

$$\begin{aligned}
\mathbb{E}_x(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}) &= 1 - \mathcal{I}^{(p,0)}(x) + e^{-\beta}[\mathcal{I}^{(p,0)}(x) - e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x)] \\
&\quad + \left(e^{-\beta}[e^{\Phi(p)x}\mathcal{I}^{(p,\Phi(p))}(x) - \mathcal{C}^{(p)}(x)] + \mathcal{C}^{(p)}(x)\right)\mathbb{E}(e^{-\alpha L_{\mathbf{e}_p}-\beta J_{\mathbf{e}_p}}).
\end{aligned}$$

The proof is now complete. □

5.5 Conclusions

Using the same perturbation method as presented in Chapter 2, we have studied the joint Laplace transform of the local time at zero and the number of crosses below the level zero as a consequence of a jump before an exponential time. The main difficulty of the study of this random variable is that in the infinite variation case, (starting from zero) the process enters the set $(-\infty, 0)$ immediately, making it a difficult task to count the number of downcrossings by a jump. In the proof of Theorem 5.2.1, we have used the fact that the perturbed Lévy process, $X^{(\varepsilon)}$, has a finite number of downcrossings. We have first shown that for $X^{(\varepsilon)}$, conditional on the total number of downcrosings before an exponential time, the total number of downcrossing by jumps follows a Binomial random variable plus a Bernoulli random variable (that represents the last downcrossing) which depends on the probability of crossing the level zero by creeping, whereas the local time is approximated using the weighted number of downcrossings of $X^{(\varepsilon)}$. The final result then follows by a limit argument.

From the Laplace transform of $J_{\mathbf{e}_p}$ derived in Theorem 5.2.1, we are able to draw some conclusions about its distribution (see Remark 5.2.2). When X has jumps of infinite variation, the random variable $J_{\mathbf{e}_p}$ is degenerate where it takes only the value of infinity (under the measure $\mathbb{P}$). This agrees with the fact that X visits the level zero an infinite number of times (regardles of the value of σ) and hence by the lack of memory property of the exponential distribution, we have an infinite number of (small) downcrossings by jump. When the jumps of X are of finite variation, the distribution of X coincides with the distribution of a geometric distribution multiplied by a Bernoulli random variable whose parameters depend on the function $\theta^{(p)}$.

A key point in the derivation of the results presented in this Chapter is that spectrally negative Lévy processes have no positive jumps and therefore they can only creep upwards. This property allows the process to start afresh or regenerate whenever a visit to level zero happens, making all the calculations tractable. For instance, the derivation of the Laplace transform of the i-th downcrossing by jump uses the fact that there is always a visit to the level zero made by creeping in between two consecutive times $\{\kappa_i, i \geq 1\}$. This fact allows us to write all the formulas as derived in Section 5.3 in terms of the time of the first downcrossing by jump and its overshoot, (κ_1, X_{κ_1}).

Appendix A

Variational inequalities for spectrally negative Lévy processes

Let X be a spectrally negative Lévy process with the following representation:

$$X_t = -\mu t + \sigma B_t + \int_0^t \int_{(-\infty,-1)} xN(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{(-1,0)} x(N(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\Pi(\mathrm{d}x)),$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ $\{B_t, t \geq 0\}$ is a standard Brownian motion, N is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}\{0\}$ with intensity $\mathrm{d}t \times \Pi(\mathrm{d}y)$ where Π is a Lévy measure, i.e., Π satisfies

$$\int_{\mathbb{R}} (1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty.$$

Fix $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, the set of all bounded $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ functions with bounded derivatives. By applying Itô formula, we obtain the following decomposition

$$f(t, X_t) = f(0, X_0) + M_t + \int_0^t \mathcal{A}_{(t,X)}(f)(s, X_s)\mathrm{d}s,$$

where M is a martingale starting at zero and $\mathcal{A}_{(t,X)}(f)$ is the infinitesimal generator of (t, X) applied to f given by

$$\mathcal{A}_{(t,X)}(f)(t, x) = \frac{\partial}{\partial t} f(t, x) - \mu \frac{\partial}{\partial x} f(t, x) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f(t, x) + B_X(f)(t, x),$$

where

$$B_X(f)(t,x) = \int_{(-\infty,0)} \left(f(t,x+y) - f(t,x) - y\mathbb{I}_{\{y>-1\}} \frac{\partial}{\partial x} f(t,x) \right) \Pi(\mathrm{d}y).$$

Note that in order to the derivatives in the operator $\mathcal{A}_{(t,X)}$ to be defined it is only needed that $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. In the next Lemma we show that B_X can be defined in a subset $B \subset \mathbb{R}_+ \times \mathbb{R}$ provided that some conditions are met in the set B.

Lemma A.1. *Let $B \subset \mathbb{R}_+ \times \mathbb{R}$ an open set. Assume that f is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ function and that*

$$\int_{(-\infty,-1)} |f(t,x+y)| \Pi(\mathrm{d}y) < \infty$$

for all $(x,y) \in B$. Then we have that $|B_X(f)(t,x)| < \infty$ for all $(t,x) \in B$. Moreover if f, its derivatives and $(t,x) \mapsto \int_{(-\infty,-1)} |f(t,x+y)| \Pi(\mathrm{d}y)$ are bounded functions in B we have that $B_X(f)$ is bounded in $B \cap [0,T] \times \mathbb{R}$ for any $T > 0$ and continuous in B.

Proof. Take $(t,x) \in B$. By Taylor's theorem we know that for each $y \in (-1,0)$ there exists $c_y \in [x+y,x] \subset [x-1,x]$ such that

$$f(t,x+y) - f(t,x) - y\frac{\partial}{\partial x} f(t,x) = y^2 \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t,c_y)$$

Hence, we have that for any $(t,x) \in B$ that

$$\begin{aligned}
|B_X(f)(t,x)| &= \left| \int_{(-1,0)} \left(f(t,x+y) - f(t,x) - y\frac{\partial}{\partial x} f(t,x) \right) \Pi(\mathrm{d}y) \right. \\
&\qquad \left. + \int_{(-\infty,-1]} \left(f(t,x+y) - f(t,x) \right) \Pi(\mathrm{d}y) \right| \\
&\leq \int_{(-1,0)} y^2 \frac{1}{2} \left| \frac{\partial^2}{\partial x^2} f(t,c_y) \right| \Pi(\mathrm{d}y) + \int_{(-\infty,-1]} |f(t,x+y) - f(t,x)| \, \Pi(\mathrm{d}y) \\
&\leq \sup_{z \in [x-1,x]} \frac{1}{2} \left| \frac{\partial^2}{\partial x^2} f(t,z) \right| \int_{(-1,0)} y^2 \Pi(\mathrm{d}y) + \int_{(-\infty,-1]} |f(t,x+y)| \, \Pi(\mathrm{d}y) \\
&\qquad + |f(t,x)| \Pi(-\infty,-1] \\
&< \infty,
\end{aligned}$$

where we used that Π is a Lévy measure and is finite on any set away from zero and that the derivatives of f are continuous on B and then bounded on compact sets. The second assertion follows by the fact that the second derivative is continuous and bounded on the compact set containing the set $\tilde{B} = \{(t, x-1) \in [0,T] \times \mathbb{R} : (t,x) \in B \text{ and } (t, x-1) \notin B\}$ and since f and $(t,x) \mapsto \int_{(-\infty,-1)} |f(t, x+y)|\Pi(\mathrm{d}y)$ are bounded in B. The continuity of $B_X(f)$ in B follows from the fact that f is continuous and the dominated convergence theorem. □

Consider the stopping time τ_B as the first time the process (t, X) is outside the open set B, i.e.,

$$\tau_B^{(s,x)} = \inf\{t \geq 0 : (s+t, X_t + x) \notin B\}.$$

Lemma A.2. *Let $B \subset \mathbb{R}_+ \times \mathbb{R}$ an open set. Assume that f is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ function such that f, its derivatives and $(t,x) \mapsto \int_{(-\infty,-1)} |f(t,x+y)|\Pi(\mathrm{d}y)$ are bounded in B. Then we have the following decomposition*

$$f(u + t \wedge \tau_B^{(u.x)}, X_{t\wedge\tau_B^{(u.x)}} + x) = f(u,x) + M_t + \int_0^{t\wedge\tau_B^{(u.x)}} \mathcal{A}_{(t,X)}(u+s, X_s + x)\mathrm{d}s, \qquad \text{(A.1)}$$

where $\{M_t, t \geq 0\}$ is a $\mathbb{P}$-martingale.

Proof. Let $(s,x) \in B$. Since f is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ function we have by Itô formula that

$$\begin{aligned}
&f(u+t, X_t + x) - f(u,x) \\
&= \int_0^t \frac{\partial f}{\partial t}(u+s, X_s + x)\mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(u+s, X_{s-} + x)\mathrm{d}X_s + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 f}{\partial x^2}(u+s, X_s + x)\mathrm{d}s \\
&+ \int_0^t \int_{(-\infty,0)} \left[f(u+s, X_{s-} + x + y) - f(u+s, X_{s-}+x) - y\frac{\partial f}{\partial x}(u+s, X_{s-}+x)\right] N(\mathrm{d}s, \mathrm{d}y) \\
&= \int_0^t \frac{\partial f}{\partial t}(u+s, X_s + x)\mathrm{d}s - \mu \int_0^t \frac{\partial f}{\partial x}(u+s, X_{s-} + x)\mathrm{d}t + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 f}{\partial x^2}(u+s, X_s + x)\mathrm{d}s \\
&+ \sigma \int_0^t \frac{\partial f}{\partial x}(u+s, X_{s-}+x)\mathrm{d}B_s + \int_0^t \int_{(-\infty,-1)} y\frac{\partial f}{\partial x}(u+s, X_{s-}+x)N(\mathrm{d}s, \mathrm{d}x) \\
&+ \int_0^t \int_{(-1,0)} y\frac{\partial f}{\partial x}(u+s, X_{s-}+x)\widetilde{N}(\mathrm{d}s, \mathrm{d}x) \\
&+ \int_0^t \int_{(-\infty,0)} \left[f(u+s, X_{s-} + x + y) - f(u+s, X_{s-}+x) - y\frac{\partial f}{\partial x}(u+s, X_{s-}+x)\right] N(\mathrm{d}s, \mathrm{d}y) \\
&= M_t^{(1)} + M_t^{(2)} + \int_0^t \mathcal{A}_{(t,X)}(u+s, X_s + x)\mathrm{d}s,
\end{aligned}$$

where

$$M_t^{(1)} = \sigma \int_0^t \frac{\partial f}{\partial x}(u+s, X_{s-}+x)\mathrm{d}B_s + \int_0^t \int_{(-1,0)} y \frac{\partial f}{\partial x}(u+s, X_{s-}+x)\widetilde{N}(\mathrm{d}s, \mathrm{d}x)$$

and

$$M_t^{(2)} = \int_0^t \int_{(-\infty,0)} \bigg[f(u+s, X_{s-}+x+y) - f(u+s, X_{s-}+x) \\ - y\mathbb{I}_{\{y>-1\}} \frac{\partial f}{\partial x}(u+s, X_{s-}+x) \bigg] \widetilde{N}(\mathrm{d}s, \mathrm{d}y).$$

Note that for any $s < \tau_B^{(u,x)}$ we have that $(u+s, X_s+x) \in B$. Hence, since $\frac{\partial f}{\partial x}$ is bounded in the set B have that the stopped process $\{M^{(1)}_{t\wedge\tau_B^{(u,x)}}, t \geq 0\}$ is a martingale. Moreover, from Lemma A.1 we have that $B_X(f)$ is a bounded function on $[0,t] \times \mathbb{R} \cap B$ so we have that

$$\mathbb{E}\left(\int_0^{t\wedge\tau_B^{(u,x)}} B_X(u+s, X_s+x)\mathrm{d}s \right) < \infty$$

for all $t \geq 0$. Then the process $\{M^{(2)}_{t\wedge\tau_B^{(u,x)}}, t \geq 0\}$ is also a martingale. □

Let G be a right-continuous function. Define the process $Z^{(s,x)} = \{Z_t^{(s,x)}, t \geq 0\}$, where

$$Z_t^{(s,x)} = f(s+t, X_t+x) + \int_0^t G(r+s, X_r+x)\mathrm{d}r, \qquad t \geq 0.$$

We have the following proposition.

Proposition A.3. *Let $B \subset \mathbb{R}_+ \times \mathbb{R}$ an open set. Assume that f is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ function such that f, its derivatives and $(t,x) \mapsto \int_{(-\infty,-1)} |f(t,x+y)|\Pi(\mathrm{d}y)$ are bounded in B and $G : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ a continuous function. Then for all $(s,x) \in B$ the process $\{Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}}, t \geq 0\}$ is a submartingale if and only if for all $(s,x) \in B$, $\mathcal{A}_{(t,X)}(f)(s,x) + G(s,x) \geq 0$.*

Proof. Suppose that $\{Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}}, t \geq 0\}$ is a submartingale for all $(s,x) \in B$. We prove that $\mathcal{A}_{(t,X)}(f)(s,x) + G(s,x) \geq 0$ for all $(s,x) \in B$. Fix $(s,x) \in B$, since $\{Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}}, t \geq 0\}$ is a submartingale we have, for every $t \geq 0$, that

$$\mathbb{E}\left[\frac{1}{t}(Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}} - Z_0^{(s,x)}) \right] \geq 0$$

which implies that

$$\mathbb{E}\left[\frac{1}{t}[f(s+t\wedge\tau_B^{(s,x)}, X_{t\wedge\tau_B^{(s,x)}}+x)-f(s,x)]\right]+\mathbb{E}\left[\frac{1}{t}\int_0^{t\wedge\tau_B^{(s,x)}} G(s+r,X_r+x)\mathrm{d}r\right]\geq 0$$

So, by the decomposition (A.1) we get

$$\mathbb{E}\left[\frac{1}{t}\int_0^{t\wedge\tau_B^{(s,x)}} \mathcal{A}_{(t,X)}(f)(s+r,X_r+x)\mathrm{d}s\right]+\mathbb{E}\left[\frac{1}{t}\int_0^{t\wedge\tau_B^{(s,x)}} G(s+r,X_r+x)\mathrm{d}r\right]\geq 0.$$

Note that, due to the right continuity of (t,X) and since B is open, we have $\tau_B^{(s,x)}>0$ almost surely. Therefore, tending t to zero in the above inequality, by Fubini's theorem and fundamental theorem of calculus (since $r\mapsto X_r$ is right continuous and G is continuous) we deduce that

$$\mathcal{A}_{(t,X)}(f)(s,x)+G(s,x)\geq 0.$$

Now we prove the "only if" statement. Suppose that for all $(s,x)\in B$, $\mathcal{A}_{(t,X)}(f)(s,x)+G(s,x)\geq 0$. We show that the process $\{Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}}, t\geq 0\}$ is a submartingale. By the semi-martingale decomposition (A.1) and since $\mathcal{A}_{(t,X)}$ is bounded in B (see Lemma 4.4.16) we have that $\mathbb{E}(|Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}}|)<\infty$ for all $t\geq 0$. Moreover, we have, for any $(s,x)\in B$ and $0\leq r\leq t$,

$$\begin{aligned}
\mathbb{E}(Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}}|\mathcal{F}_r) &= \mathbb{E}\left[f(s,x)+M_{t\wedge\tau_B^{(s,x)}}+\int_0^{t\wedge\tau_B^{(s,x)}}\mathcal{A}_{(t,X)}(f)(v+s,X_v+x)\mathrm{d}v\middle|\mathcal{F}_r\right]\\
&\quad+\mathbb{E}\left[\int_0^{t\wedge\tau_B^{(s,x)}} G(v+s,X_v+x)\mathrm{d}v\middle|\mathcal{F}_r\right]\\
&= Z^{(s,x)}_{r\wedge\tau_B^{(s,x)}}+\mathbb{E}\left[\int_{r\wedge\tau_B^{(s,x)}}^{t\wedge\tau_B^{(s,x)}}\mathcal{A}_{(t,X)}(f)(v+s,X_v+x)\mathrm{d}v\middle|\mathcal{F}_r\right]\\
&\quad+\mathbb{E}\left[\int_{r\wedge\tau_B^{(s,x)}}^{t\wedge\tau_B^{(s,x)}} G(v+s,X_v+x)\mathrm{d}v\middle|\mathcal{F}_r\right]\\
&\geq Z^{(s,x)}_{s\wedge\tau_B^{(s,x)}},
\end{aligned}$$

where the last inequality follows from the fact that $(v+s,X_v+x)\in B$ for all $v\in(r\wedge$

$\tau_B^{(s,x)}, t \wedge \tau_B^{(s,x)})$ and that $\mathcal{A}_{(t,X)}(f)(s,x) + G(s,x) \geq 0$ in B. Therefore the process $Z^{(s,x)}_{t\wedge\tau_B}$ is a submartingale. □

It turns out that the above proposition can be extended to a more general class of functions, provided that the inequality $\mathcal{A}_{(t,X)}(f) + G \geq 0$ is taken in the sense of distributions. For this recall some facts and notation from the theory of distributions (see e.g. Friedlander et al. (1998) for further details). We introduce the multi-index notation. A multi-index is a n-tuplet $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers with order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we set the notation

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, we denote by $\mathcal{D}(\mathcal{O})$ the set of all C^∞ functions with compact support in $\mathcal{O}$ and by $\mathcal{D}'(\mathcal{O})$ the space of distributions on $\mathcal{O}$. That is, $\mathcal{D}'(\mathcal{O})$ is the space of linear forms u in $\mathcal{D}(\mathcal{O})$ such that for every compact set $K \subset O$, there is a real number $C \geq 0$ and a nonnegative integer N such that

$$|\langle u, \psi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \psi|$$

for all $\psi \in \mathcal{D}(\mathcal{O})$, where $\langle u, \varphi \rangle$ denotes the evaluation on the test function φ of the distribution u. Inspired by the integration by parts formula, the derivative of the distribution u is defined by

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

If u is a locally integrable function on $\mathcal{O}$ (u is a measurable function and $\int_K |u(x)| \mathrm{d}x < \infty$ for any compact set $K \subset \mathcal{O}$) we can define the distribution

$$\langle u, \varphi \rangle = \int u(x)\varphi(x)\mathrm{d}x, \qquad \varphi \in \mathcal{D}(\mathcal{O}).$$

Which is usually identified only with the function u. Hence, if g is a locally integrable function on $(0, \infty) \times \mathbb{R}$, the differential operator, $\mathcal{A}^0_{(t,X)}(g)$, given by

$$\mathcal{A}^0_{(t,X)}(g)(u,x) := \frac{\partial}{\partial t}g(t,x) - \mu\frac{\partial}{\partial x}g(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}g(t,x)$$

can be defined in the sense of distributions. For any test function φ with compact support in $O \subset \mathbb{R}_+ \times \mathbb{R}$, we define

$$\langle \mathcal{A}^0_{(t,X)}(g), \varphi\rangle := \int_{\mathbb{R}_+}\int_{\mathbb{R}} g(t,x)\left[-\frac{\partial}{\partial t}\varphi(t,x) + \mu\frac{\partial}{\partial x}\varphi(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\varphi(t,x)\right]\mathrm{d}x\mathrm{d}t. \quad \text{(A.2)}$$

Moreover, Lamberton and Mikou (2008) showed (see Proposition 2.1) that the operator defined by

$$B_X(g)(t,x) := \int_{(-\infty,0)}(g(t,x+y) - g(t,x) - y\frac{\partial}{\partial x}g(t,x)\mathbb{I}_{\{y>-1\}})\Pi(\mathrm{d}y)$$

can be also defined in the sense of distributions when g is a bounded Borel measurable function. For $\varphi \in \mathcal{D}((0,\infty)\times\mathbb{R})$, consider the operator B^*_X given by

$$B^*_X(\varphi)(t,x) = \int_{(-\infty,0)}[\varphi(t,x-y) - \varphi(t,x) + y\frac{\partial}{\partial x}\varphi(t,x)\mathbb{I}_{\{y>-1\}}]\Pi(\mathrm{d}y), \quad \text{(A.3)}$$

for any $(t,x) \in (0,\infty)\times\mathbb{R}$. Then from Proposition 2.1 in Lamberton and Mikou (2008) we know that $B^*_X(\varphi)$ is continuous and integrable on $(0,\infty)\times\mathbb{R}$ and the operator

$$\langle B_X(g), \varphi\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}} g(u,x)B^*_X(\varphi)(u,x)\mathrm{d}x\mathrm{d}u, \quad \text{(A.4)}$$

defines a distribution. In the next lemma we show that the boundedness condition of g can be relaxed.

Lemma A.4. *Let g be a locally integrable function in $\mathbb{R}_+\times\mathbb{R}$ such that*

$$(u,x) \mapsto \int_{(-\infty,-1)}|g(u,x+y)|\Pi(\mathrm{d}y) \quad \text{(A.5)}$$

is locally integrable. The linear operator $B_X(g)$ defined in (A.4) defines a distribution on any open set $O \subset \mathbb{R}_+\times\mathbb{R}$. Moreover, if in addition we suppose that g is a $C^{1,2}(\mathbb{R}_+\times\mathbb{R})$ function *we have that*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} B_X(g)(t,x)\varphi(t,x)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}_+}\int_{\mathbb{R}} g(t,x)B_X^*(\varphi)(t,x)\mathrm{d}x\mathrm{d}t$$

for any $\varphi \in \mathbb{R}_+ \times \mathbb{R}$.

Proof. It is clear that the operator defined in (A.4) is linear. Take a test function φ with support in a compact set $H \times K \subset \mathbb{R}_+ \times \mathbb{R}$. We have

$$\begin{aligned}
|\langle B_X(g),\varphi\rangle| &\le \int_{\mathbb{R}_+}\int_{\mathbb{R}} |g(u,x)||B_X^*(\varphi)(u,x)|\mathrm{d}x\mathrm{d}u \\
&\le \int_H\int_{\mathbb{R}} |g(u,x)| \int_{(-1,0)} |\varphi(u,x-y)-\varphi(u,x)+y\frac{\partial}{\partial x}\varphi(u,x)|\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u \\
&\quad + \int_H\int_{\mathbb{R}} |g(u,x)| \int_{(-\infty,-1)} |\varphi(u,x-y)-\varphi(u,x)|\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u.
\end{aligned}$$

Note that if $x \notin K+(-1,0]$ we have that $x \notin K$ (if we assume that $x \in K$ then $x = x+0 \in K+(-1,0]$ which is a contradiction) and $x-y \notin K$ for all $y \in (-1,0)$ (if $z = x-y \in K$ then $x = z+y \in K+(-1,0) \subset K+(-1,0]$ then we have got a contradiction), then $\varphi(u,x-y)-\varphi(u,x)+y\frac{\partial}{\partial x}\varphi(u,x) = 0$. Denote $k_* = \inf K$, since $x \mapsto \varphi(u,x)$ has support in K and using Taylor's formula we obtain

$$\begin{aligned}
|\langle B_X(g),\varphi\rangle| &\le \int_H\int_{K+(-1,0]} |g(u,x)| \int_{(-1,0)} |\varphi(u,x-y)-\varphi(u,x)+y\frac{\partial}{\partial x}\varphi(u,x)|\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u \\
&\quad + \int_H\int_K |g(u,x)| \int_{(-\infty,-1)} |\varphi(u,x-y)-\varphi(u,x)|\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u \\
&\quad + \int_H\int_{-\infty}^{k_*} |g(u,x)| \int_{(-\infty,-1)} |\varphi(u,x-y)|\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u \\
&\le \frac{1}{2}\sup|\frac{\partial^2}{\partial x^2}\varphi| \int_{(-1,0)} y^2\Pi(\mathrm{d}y) \int_H\int_{K+(-1,0]} |g(u,x)|\mathrm{d}x\mathrm{d}u \\
&\quad + 2\sup|\varphi|\Pi((-\infty,-1)) \int_H\int_K |g(u,x)|\mathrm{d}x\mathrm{d}u \\
&\quad + \sup|\varphi| \int_H\int_K\int_{(-\infty,-1)} |g(u,x+y)|\Pi(\mathrm{d}y)\mathrm{d}x\mathrm{d}u,
\end{aligned}$$

which proves the assertion since Π is a Lévy measure and $(u,x) \mapsto \int_{(-\infty,-1)} |g(u,x+y)|\Pi(\mathrm{d}y)$ is locally integrable by assumption. The last assertion follows by the same argument as in Lamberton and Mikou (2008) (see Proposition 2.1) so the proof is omitted. □

Therefore, if g is a locally integrable function in $\mathbb{R}_+ \times \mathbb{R}$ such that the function defined in (A.5) is locally integrable, we can define the distribution $\mathcal{A}_{(t,X)}(g) = \mathcal{A}^0_{(t,X)}(g) + B_X(g)$ in the set $\mathbb{R}_+ \times \mathbb{R}$.

Let u a distribution and $\theta \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$. Then the function

$$(\theta * u)(t,x) = \langle u(s,y), \theta(t-s, x-y)\rangle$$

is a member of $C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and defines a distribution given by

$$\langle \theta * u, \phi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \langle u(s,y), \theta(t-s, x-y)\rangle \phi(t,x) \mathrm{d}x\mathrm{d}t$$

for any $\phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$.

It turns out that Proposition 2.3 in Lamberton and Mikou (2008) can also be extended to this case. The proof remains is very similar but it is included for completeness.

Proposition A.5. *Let g be a Borel and locally integrable function in $\mathbb{R}_+ \times \mathbb{R}$ such that the function $\int_{(-\infty,-1)} |g(u, x+y)|\Pi(\mathrm{d}y)$ is locally integrable. We have that for every θ and φ in $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$,*

$$\langle \mathcal{A}_{(t,X)}(g * \theta), \varphi \rangle = \langle \mathcal{A}_{(t,X)}(g), \varphi * \check{\theta} \rangle = \langle \mathcal{A}_{(t,X)}(g) * \theta, \varphi \rangle,$$

where $\check{\theta}(u,x) = \theta(-u,-x)$ for any $(u,x) \in \mathbb{R}_+ \times \mathbb{R}$.

Proof. Take a $\varphi, \theta \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$. Then by the definition of convolution we have that

$$\begin{aligned} \langle B_X(g*\theta), \varphi \rangle &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} (g*\theta)(u,x) B_X^*(\varphi)(u,x) \mathrm{d}x\mathrm{d}u \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(u-v, x-y)\theta(v,y) B_X^*(\varphi)(u,x) \mathrm{d}v\mathrm{d}y\mathrm{d}x\mathrm{d}u \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(u,x) \int_{\mathbb{R}_+} \int_{\mathbb{R}} \theta(v,y) B_X^*(\varphi)(u+v, x+y) \mathrm{d}v\mathrm{d}y\mathrm{d}x\mathrm{d}u. \end{aligned}$$

Now, by Fubini's theorem and the definition of $B_X(\varphi)$ we have that

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}}\theta(v,y)B_X^*(\varphi)(u+v,x+y)\mathrm{d}v\mathrm{d}y\\
&=\int_{(-\infty,0)}\Pi(\mathrm{d}z)\int_{\mathbb{R}_+}\int_{\mathbb{R}}\theta(v,y)\\
&\qquad\qquad\times[\varphi(u+v,x+y-z)-\varphi(u+v,x+y)+z\mathbb{I}_{\{z>-1\}}\frac{\partial}{\partial x}\varphi(u+v,x+y)]\mathrm{d}v\mathrm{d}y\\
&=\int_{(-\infty,0)}\Pi(\mathrm{d}z)[(\varphi*\check{\theta})(u,x-z)-(\varphi*\check{\theta})(u,x)+z\mathbb{I}_{\{z>-1\}}\frac{\partial}{\partial x}(\varphi*\check{\theta})(u,x)]\mathrm{d}v\mathrm{d}y\\
&=B_X^*(\varphi*\check{\theta})(u,x),
\end{aligned}
$$

where $\check{\theta}(u,x)=\theta(-u,-x)$. Hence,

$$
\langle B_X(g*\theta),\varphi\rangle=\int_{\mathbb{R}_+}\int_{\mathbb{R}}g(u,x)B_X^*(\varphi*\check{\theta})(u,x)\mathrm{d}x\mathrm{d}u=\langle B_X(g),\varphi*\check{\theta}\rangle.
$$

On the other hand, using Fubini's theorem and a change of variable we obtain that

$$
\begin{aligned}
&\langle\mathcal{A}^0_{(t,X)}(g*\theta),\varphi\rangle\\
&\quad=\int_{\mathbb{R}_+}\int_{\mathbb{R}}(g*\theta)(u,x)\left[-\frac{\partial}{\partial u}\varphi(u,x)+\mu\frac{\partial}{\partial x}\varphi(u,x)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\varphi(u,x)\right]\mathrm{d}x\mathrm{d}u\\
&\quad=\int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}}g(u-v,x-y)\theta(v,y)dvdy\\
&\qquad\qquad\times\left[-\frac{\partial}{\partial u}\varphi(u,x)+\mu\frac{\partial}{\partial x}\varphi(u,x)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\varphi(u,x)\right]\mathrm{d}x\mathrm{d}u\\
&\quad=\int_{\mathbb{R}_+}\int_{\mathbb{R}}g(u,x)\int_{\mathbb{R}_+}\int_{\mathbb{R}}\theta(v,y)dvdy\\
&\qquad\times\left[-\frac{\partial}{\partial u}\varphi(u+v,x+y)+\mu\frac{\partial}{\partial x}\varphi(u+v,x+y)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\varphi(u+v,x+y)\right]\mathrm{d}x\mathrm{d}u\\
&\quad=\int_{\mathbb{R}_+}\int_{\mathbb{R}}g(u,x)\left[-\frac{\partial}{\partial u}(\varphi*\check{\theta})(u,x)+\mu\frac{\partial}{\partial x}(\varphi*\check{\theta})(u,x)+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}(\varphi*\check{\theta})(u,x)\right]\mathrm{d}x\mathrm{d}u\\
&\quad=\langle\mathcal{A}^0_{(t,X)}(g),\varphi*\check{\theta}\rangle
\end{aligned}
$$

Therefore we have that

$$\begin{aligned}\langle \mathcal{A}^0_{(t,X)}(g*\theta),\varphi\rangle &= \langle \mathcal{A}^0_{(t,X)}(g),\varphi*\check{\theta}\rangle\\ &= \left\langle \mathcal{A}^0_{(t,X)}(g)(s,y), \int_{\mathbb{R}_+}\int_{\mathbb{R}}\theta(t-s,x-y)\varphi(t,x)\mathrm{d}x\mathrm{d}t\right\rangle\\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}}\langle \mathcal{A}^0_{(t,X)}(g)(s,y),\theta(t-s,x-y)\rangle\varphi(t,x)\mathrm{d}x\mathrm{d}t\\ &= \langle \mathcal{A}^0_{(t,X)}(g)*\theta,\varphi\rangle.\end{aligned}$$

The proof is now complete. □

Let u a distribution in $\mathcal{O}$, we say that u is non-negative if for any non-negative test function $\varphi\in\mathcal{D}(\mathcal{O})$,

$$\langle u,\varphi\rangle\geq 0.$$

The next result is an extension of Proposition 2.5 in Lamberton and Mikou (2008). The proof is essentially the same but we include it for completeness.

Proposition A.6. *Let B be an open set in $\mathbb{R}_+\times\mathbb{R}$. Suppose that $f:\mathbb{R}_+\times\mathbb{R}\mapsto\mathbb{R}$ is such that f and $(u,x)\mapsto\int_{(-\infty,-1)}|f(u,x+y)|\Pi(\mathrm{d}y)$ are locally integrable functions in $\mathbb{R}_+\times\mathbb{R}$ and bounded in B and that $G:\mathbb{R}_+\times\mathbb{R}\mapsto\mathbb{R}$ is a locally integrable function. Assume that the process $\{Z^{(s,x)}_{t\wedge\tau_B^{(s,x)}},t\geq 0\}$ is a submartingale for every $(s,x)\in B$, where $Z^{(s,x)}_t = f(s+t,X_t+x)+\int_0^t G(s+r,X_r+x)\mathrm{d}r$ and $\tau_B^{(s,x)}=\inf\{t\geq 0:(t+s,X_t+x)\notin B\}$. Then $\mathcal{A}_{(t,X)}(f)+G$ is a non-negative distribution on B.*

Proof. Take $z_0=(u_0,x_0)\in B$ and choose $a>0$ such that $\mathbf{B}(z_0,2a)\subset B$, where $\mathbf{B}(z_0,2a)$ is the open ball with center z_0 and radius $2a$. We define the stopping time

$$\tau_B=\inf\{t\geq 0: \text{ there exists } z\in\mathbf{B}(z_0,a) \text{ such that } z+(t,X_t)\notin B\}.$$

Note that for every $(u,x)\in\mathbf{B}(z_0,a/2)$ and $(v,y)\in\mathbf{B}(0,a/2)$ we have that $(u-v,x-y)\in\mathbf{B}(z_0,a)\subset B$ and then $\tau_B\leq\tau_B^{(u-v,x-y)}$. Hence, the process $\{Z^{(u-v,x-y)}_{t\wedge\tau_B},t\geq 0\}$ is a submartingale and then

$$\mathbb{E}\left(f(u-v+t\wedge\tau_B, X_{t\wedge\tau_B}+x-y)+\int_0^{t\wedge\tau_B} G(u-v+r, X_r+x-y)\mathrm{d}r\right)\geq f(u-v,x-y).$$

Next, we consider a sequence of even nonnegative functions $\{\rho_n, n\geq 1\}$ in C^∞ such that for each $n\geq 1$, the support of ρ_n is in $\mathbf{B}(0,a/(2n))$ and $\int_{\mathbb{R}^2}\rho_n(v,y)\mathrm{d}v\mathrm{d}y=1$. Then by integrating the equation above with respect to ρ_n and due tu Fubini's theorem we get that

$$\mathbb{E}\left((f*\rho_n)(u+t\wedge\tau_B, X_{t\wedge\tau_B}+x)\right)+\mathbb{E}\left(\int_0^{t\wedge\tau_B}(G*\rho_n)(u+r,X_r+x)\mathrm{d}r\right)\geq (f*\rho_n)(u,x) \tag{A.6}$$

Fix $(u,x)\in\mathbf{B}(z_0,a/2)$. Note that since f is bounded we have that for all $n\geq 1$, the function $(s,w)\mapsto f*\rho_n(u+s,w+x)$ is $C^\infty(\mathbb{R}_+\times\mathbb{R})$ and has bounded derivatives in the open set $\widetilde{B}=\{(s,w)\in\mathbb{R}_+\times\mathbb{R}: z+(s,w)\in B \text{ for all } z\in\mathbf{B}(z_0,a)\}$. Indeed, for any $(s,w)\in\widetilde{B}$ and any $(v,y)\in\mathbf{B}(0,a/(2n))$ we have that $(u+s-v,w+x-y)\in B$ and then

$$\begin{aligned}
&\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}(f*\rho_n)(u+s,w+x)\right|\\
&\quad\leq\int\int_{\mathbf{B}(0,a/(2n))}|f(u+s-v,w+x-y)|\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}\rho_n(v,y)\right|\mathrm{d}v\mathrm{d}y\\
&\quad\leq\sup_{(u',x')\in B}|f(u',x')|\int\int_{\mathbf{B}(0,a/(2n))}\left|\frac{\partial^{i+j}}{\partial u^i\partial x^j}\rho_n(v,y)\right|\mathrm{d}v\mathrm{d}y
\end{aligned}$$

for any $i,j\in\{0,1,2,\ldots\}$. Moreover, by Fubini's theorem we have that

$$\begin{aligned}
&\left|\int_{(-\infty,-1)}(f*\rho_n)(u+s,w+x+y)\Pi(\mathrm{d}y)\right|\\
&\quad\leq\int\int_{\mathbf{B}(0,a/(2n))}\left|\int_{(-\infty,-1)}f(u+s-v,w+x-v+y)\Pi(\mathrm{d}y)\right|\rho_n(v,y)\mathrm{d}v\mathrm{d}y\\
&\quad\leq\sup_{(u',x')\in B}\left|\int_{(-\infty,-1)}f(u',x'+y)\Pi(\mathrm{d}y)\right|.
\end{aligned}$$

Hence, the function $(s,w)\mapsto\int_{(-\infty,-1)}(f*\rho_n)(u+s,w+x+y)\Pi(\mathrm{d}y)$ is bounded in $\widetilde{B}$. Thus, since τ_B is the first exit time of (s, X_s) from the set $\widetilde{B}$, we get from Lemma A.2 that

$$(f * \rho_n)(u + t\wedge\tau_B, X_{t\wedge\tau_B} + x) \\ = (f * \rho_n)(u, x) + M_t^{(u,x)} + \int_0^{t\wedge\tau_B} \mathcal{A}_{(t,X)}(f * \rho_n)(u + s, X_s + x)\mathrm{d}s,$$

where $\{M_t^{(u,x)}, t \geq 0\}$ is a martingale. Therefore equation (A.6) reads

$$\mathbb{E}\left(\int_0^{t\wedge\tau_B} \left[\mathcal{A}_{(t,X)}(f * \rho_n)(u + r, X_r + x) + (G * \rho_n)(u + r, X_r + x)\right] \mathrm{d}r\right) \geq 0.$$

Note that $\tau_B > 0$ a.s. (since $\mathbf{B}(0, a) \subset \widetilde{B}$) so the dividing by $t > 0$ the equation above and taking $t \downarrow 0$ we obtain that $\mathcal{A}_{(t,X)}(f * \rho_n)(u, x) + (G * \rho_n)(u, x) \geq 0$ for all $n \geq 1$ and $(u, x) \in \mathbf{B}(z_0, a/2)$. That implies that for any test function ψ in $\mathbf{B}(z_0, a/2)$

$$\langle \mathcal{A}_{(t,x)}(f * \rho_n) + G * \rho_n, \psi\rangle \geq 0.$$

Then from Proposition A.5 we conclude that $\mathcal{A}_{(t,X)}(f) * \rho_n + G * \rho_n \geq 0$ in the sense of distributions on $\mathbf{B}(Z_0, a/2)$. By letting n go to infinity, we conclude that $\mathcal{A}_{(t,X)}(f) + G \geq 0$ on $\mathbf{B}(z_0, a/2)$. Since z_0 is any arbitrary point in B, using a partition of unity argument, we conclude that $\mathcal{A}_{(t,X)}(f) + G \geq 0$ in the sense of distributions on B.

□

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