

# Option pricing models without probability: A rough paths approach

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The authors would like to dedicate this paper to their late colleague Mark Davis (1945–2020). His acumen, brilliance, and determination in facing fundamental questions, his disarming laughter and good-natured common sense will be missed. Each of the authors benefited greatly from discussions with Mark over the years and did so, in particular, during the preparation of an early version of this manuscript. One aspect of the presentation below is a perspective on the so-called Fundamental Theorem of Derivative Trading. Mark often stressed the importance of this result to the understanding and effectiveness of real-world derivatives trading; indeed he included a version of it in his entry “Black-Scholes Formula” in the Encyclopedia of Quantitative Finance.

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## Abstract

We describe the pricing and hedging of financial options without the use of probability using rough paths. By encoding the volatility of assets in an enhancement of the price trajectory, we give a pathwise presentation of the replication of European options. The continuity properties of rough-paths allow us to generalize the so-called fundamental theorem of derivative trading, showing that a small misspecification of the model will yield only a small excess profit or loss of the replication strategy. Our hedging strategy is an enhanced version of classical delta hedging where we use volatility swaps to hedge the second-order terms arising in rough-path integrals, resulting in improved robustness.

## KEYWORDS

enhanced price path, fundamental theorem of derivatives trading, historical vs implied volatility, option pricing, pathwise finance, pathwise hedging, pathwise pricing, price robustness, probability-free finance, rough brackets, rough paths theory, volatility swaps

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## 1 | INTRODUCTION

The theory of rough-paths provides a framework for understanding differential systems driven by irregular input signals. An asset-price process arising from a diffusion model may be associated a rough-path. Conversely, we will find a necessary condition for a rough-path to arise from a given diffusion model, and we will call a rough-path satisfying this condition a *diffusive rough-path*. An investment strategy gives rise to a rough differential equation (RDE) describing the evolution of the profit and loss (P&L) of the strategy under a given asset price signal. Given an option with a smooth payoff function, we will show that the P&L of a modified version of the classical delta hedging strategy replicates the option payoff for any diffusive rough-path. The modification we make to achieve this replication is to augment the delta-hedging strategy with additional trades determined by a particular type of volatility swap. By assuming that the price of these swaps is well controlled we see that, in the continuous time limit, purchasing these swaps will not influence the P&L.

A core property of RDE solutions is their continuity with respect to the input rough-path. A first consequence therefore of our rough-path approach is robustness of our proposed hedging strategy: if the true asset price signal is close to a diffusive signal, our hedging strategy will still approximately replicate the option payoff. This relates to the classical Fundamental Theorem of Derivative Trading (Cont, 2010; Ellersgaard et al., 2017; El Karoui et al., 1998), which shows that if one hedges according to a given diffusion model but the actual asset price process is determined by a nearby diffusion model, the error of the delta hedging strategy will be small. Our approach goes beyond this in that it allows for asset price signals that do not arise from any diffusion model at all. Due to phenomena such as market-impact and front-running, any differential equation describing the dynamics of the P&L of an investment strategy in terms of the asset price dynamics is likely to contain some error. A perturbation of the second-order term of the asset price dynamics allows us to model such an error, and hence explain the robustness of hedging strategies in more realistic markets than those given by diffusion models.

A second consequence of our rough-path approach is that it demonstrates that a theory of hedging is possible without the need for probability theory, despite the central role of probability in the classical treatment of hedging (Harrison & Kreps, 1979; Harrison & Pliska, 1981). Our work clarifies the use of probability theory in justifying prices by identifying two steps: (i) showing that the asset price paths of a diffusion model satisfy our diffusivity condition; and (ii) deducing the uniqueness of the price of an option from the existence of a replicating strategy via a no-arbitrage argument. In a market with an arbitrage any price is possible, so there is no hope of obtaining uniqueness without invoking a no-arbitrage condition, and hence involving probability theory. We see, therefore, that the correct probability-free analogue of classical pricing is demonstrating the existence of a replicating strategy for a given initial endowment. In this way, we may interpret our theory as giving a probability-free approach to pricing.

In diffusion models, the quadratic variation is a well-defined pathwise notion which determines the price. Our definition of a diffusive rough-path identifies the exact property needed for the delta hedging strategy to work in a rough-path context. A continuous pricing signal is enhanced with a specification for its *rough bracket* to obtain a *reduced rough path* (see Friz & Hairer, 2014, Chapter 5) which we will term an *enhanced price path*. Our financial model will take the form of a specification for the properties of the rough bracket. Thus, our model specification is tantamount to a choice of enhancer, and it is this rough bracket which provides the appropriate analogue of quadratic variation for our asset pricing model. In our version of the Fundamental Theorem of Derivative Pricing, we will study the effect of a misspecification of the financial model by examining the sensitivity of our strategy to the choice of enhancer.

The purely pathwise nature of the enhancer, the price and hence the implied volatility is in marked contrast to the statistical (and therefore probabilistic) notion of historical volatility. This dichotomy between pathwise and probabilistic properties has been noted before. For example, it is exploited in (Brigo & Mercurio, 2000), which partly inspired the present work (see also Brigo, 2019), to give examples of diffusion models which are statistically indistinguishable using samples on a fixed time grid yet which have arbitrarily different option prices.

Ours is not the first non-probabilistic formulation of option pricing. (Bender et al., 2008; Bick & Willinger, 1994), and (Schied & Voloshchenko, 2016) obtained pathwise formulations of option pricing using the non-probabilistic approach to Itô calculus given in (Foellmer, 1981). In addition, (Riga, 2015) extended the pathwise framework to functional Itô calculus.

The usage of Foellmer calculus has the caveat that the continuous-time integrals depend upon the discrete-time approximating sequence, which more or less precludes obtaining robustness in their approach. To circumvent this dependence, our proposed strategy is an augmented version of delta-hedging where one also invests in volatility swaps in order to hedge the second-order part of the pricing signal. This yields a robust trading strategy, however at the expense of introducing assumptions on the price of volatility swaps to ensure our strategy is self-financing.

Moreover, the above-mentioned pathwise formulations required paths with semimartingale roughness, that is, of finite  $p$ -variation for all  $p > 2$ . Using Rough Path Theory we are able to accommodate paths of finite  $p$ -variation for  $2 < p < 3$ , hence showing how the delta hedging can be extended to a wider class of price signals.

One additional assumption that we must make in our approach is that the option payoff is differentiable. We will show that for a European call option with strike  $K$ , one can find diffusive rough-paths for which our strategy fails to replicate the option payoff. However, these rough-paths must have a stock price exactly equal to the strike at maturity. In a probabilistic theory, such paths occur with probability zero, so may be neglected. However, our interpretation is that the existence of such paths demonstrates a genuine lack of robustness of the classical delta hedging strategy. The need for a robust strategy becomes more important towards maturity as classical diffusion models break down and new phenomena occur such as the “pinning” of stock prices around exchange traded strikes (see e.g., Avellaneda & Lipkin, 2003; Avellaneda et al., 2012; Golez & Jackwerth, 2012; Jeannin et al., 2008). The failure of our strategy for certain stock paths indicates that one should switch strategy near maturity to a genuinely probabilistic strategy, such as a buy-and-hold strategy. This reflects actual trading practice, where delta hedging strategies are abandoned and quite different strategies adopted near maturity.

The article is organized as follows. In Section 2, we recall the classical theory of hedging and establish our notation. In Section 3, we describe the machinery on reduced rough path integration we will use. In Section 4, we define what is meant by a diffusive rough-path. In Section 5, we demonstrate formally how to obtain a pathwise formulation of the classical formulas of Mathematical Finance in continuous time. Section 6 shows how our continuous time trading strategy can be interpreted as a limit of discrete time trading strategies in volatility swaps. Section 7 demonstrates that our proposed strategy fails in the Black-Scholes model for call options when the stock price terminates at the strike. Section 8 presents our conclusions.

## 2 | NOTATION AND PRELIMINARIES

We will develop a rough-path version of a classical diffusion model, and will begin by describing the classical model. Let  $S_t^0$  denote the value of a riskless asset at time  $t$ , and assume that

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 = 1.$$

Let  $S_t \in \mathbb{R}^d$  denote the price vector of  $d$  non-dividend paying stocks, representing the risky asset. We let  $\tilde{S}_t$  be the discounted price of the risky asset at time  $t$ , namely  $\tilde{S}_t = S_t/S_t^0 = e^{-rt}S_t$ . We suppose that each component of the price vector  $\tilde{S}_t$  displays the following dynamics in the pricing measure

$$d\tilde{S}_t^i = \sigma_j^i(\tilde{S}_t)dB_t^j, \quad i = 1, \dots, d, \quad \tilde{S}_0 = s_0 \in \mathbb{R}^d,$$

on a stochastic base  $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t)_t, (B_t)_t)$  carrying a standard  $n$ -dimensional Brownian motion  $(B_t)_t$ . We assume  $\sigma$  in  $C_{\text{loc}}^{\alpha-\text{Hö}}(\mathbb{R}^d, \mathbb{R}^{d \times n})$ ,  $0 < \alpha < 1$ . Einstein's summation convention on double indices is employed and will be throughout all the paper.

Let  $f(S_T)$  be the payoff of a Vanilla option on the underlying  $S$ . We assume that  $f$  is a continuous and bounded function on  $\mathbb{R}^d$ . Let

$$h(x) := f(e^{rT}x) \tag{1}$$

and  $\tilde{h} := e^{-rT}h$ . The payoff is therefore equivalently written as  $h(\tilde{S}_T)$ , and its discounted value is  $\tilde{h}(\tilde{S}_T)$ .

The classical theory of (Harrison & Kreps, 1979; Harrison & Pliska, 1981) tells us that the option payoff can be replicated at time  $t$  for a price,  $\tilde{V}_t$  satisfying

$$\tilde{V}_t = P_{T-t}\tilde{h}(\tilde{S}_t),$$

where  $(P_t)_t$  is the semigroup on  $C_b(\mathbb{R}^d)$  generated by the infinitesimal operator

$$A\varphi(x) = \frac{1}{2} \left( \sum_k \sigma_k^i \sigma_k^j \right) \partial_{i,j}^2 \varphi(x), \quad \varphi \in C^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d),$$

of the dynamics of  $\tilde{S}$ . We call  $A$  the *volatility operator*. To ensure the absence of arbitrage, we must make some assumptions to ensure that the solutions to the Black–Scholes PDE are unique. In this paper will typically assume that the volatility operator is uniformly elliptic.

The stochastic process  $\tilde{V}_t$  is then a deterministic function  $w = w(t, x)$  of time and space applied after  $(t, \tilde{S}_t)$  which solves

$$\begin{cases} (\partial_t + A)w = 0 & \text{in } [0, T) \times \mathbb{R}^d \\ w(T, x) = \tilde{h}(x) & \text{on } \{T\} \times \mathbb{R}^d. \end{cases} \tag{2}$$

Equation (2) is the discounted version of the Black-Scholes partial differential equation.

We will write  $v(t, z) = e^{rt}w(t, e^{-rt}z)$  for the undiscounted value function and will use following notation for the Greeks:

$$\text{Delta}_t := \nabla_z v(t, S_t) = \nabla_x w(t, \tilde{S}_t),$$

taking values in  $\mathbb{R}^d \cong \text{Hom}(\mathbb{R}^d, \mathbb{R})$ , and

$$\text{Gamma}_t := \nabla_{zz}^2 v(t, S_t) = e^{rt} \nabla_{xx}^2 w(t, \tilde{S}_t),$$

taking values in  $\mathbb{R}^{d \times d} \cong \text{Hom}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}) \cong \text{Hom}(\mathbb{R}^d, \text{Hom}(\mathbb{R}^d, \mathbb{R}))$ .

In our setup, the pricing PDE is justified by the existence of a replicating strategy for the payoff. An investment strategy may be viewed as a pair  $(H_t^0, H_t)$  indicating the quantities to purchase at each time of the riskless and risky asset. By Itô's formula,

$$\begin{aligned} w(t, \tilde{S}_t) - w(0, \tilde{S}_0) &= \int_0^t \nabla_x w(u, \tilde{S}_u) \sigma(\tilde{S}_u) dW_u + \int_0^t (\partial_t + A)w(u, \tilde{S}_u) du \\ &= \int_0^t \nabla_x w(u, \tilde{S}_u) d\tilde{S}_u. \end{aligned} \quad (3)$$

It follows that the delta hedging strategy  $\phi_t = (H_t^0, H_t)$  given by

$$\begin{aligned} H_t &:= \nabla_x w(t, \tilde{S}_t) \\ H_t^0 &:= w(t, \tilde{S}_t) - H_t \tilde{S}_t \end{aligned} \quad (4)$$

is such that the undiscounted portfolio process

$$V_t(\phi) = H_t^0 e^{rt} + H_t S_t = e^{rt} w(t, \tilde{S}_t),$$

is self-financing, that is, it satisfies

$$V_t(\phi) = V_0(\phi) + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u, \quad (5)$$

and replicates the option payoff, that is,  $V_T(\phi) = f(S_T)$ .

### 3 | PATHWISE INTEGRALS

In this section we review the elements of rough path theory that we will need. The results are standard, or minor variations of standard results and so proofs have been omitted, but may be found in the Arxiv version of this paper.

#### 3.1 | Additivity, approximate additivity, and the Sewing lemma

Let  $B$  be a Banach space and let  $X : [0, T] \rightarrow B$  be a continuous path with trajectory in  $B$ . The increments

$$X_{s,t} := X_t - X_s, \quad 0 \leq s, t \leq T, \quad (6)$$

of such path define a two-parameter function  $X = X_{s,t}$  on the square  $[0, T] \times [0, T]$ . We employ the notation in (6) throughout this paper. Rather than considering general  $s$  and  $t$  in  $[0, T]$ , we will often restrict to the simplex  $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \subset [0, T]^2$ .  $X$  is *additive*, by which we mean that for all  $0 \leq s, u, t \leq T$

$$X_{s,t} = X_{s,u} + X_{u,t}. \quad (7)$$

If  $X$  is defined on the simplex but is additive, then it can be extended to an additive function on  $[0, T] \times [0, T]$  by setting  $X_{t,s} := -X_{s,t}$ .

Additivity characterizes those functions  $X$  on  $[0, T] \times [0, T]$  that descend from increments of paths, in the following sense.

**Proposition 3.1.** *Let  $X : [0, T] \times [0, T] \rightarrow \mathcal{B}$  be additive. Then, there exists a path  $x$  on  $\mathcal{B}$  such that*

$$X_{s,t} = x_t - x_s, \quad \forall 0 \leq s, t \leq T.$$

Moreover, if  $y$  is another path whose increments coincide with  $X$ , then  $y - x$  is constant.

Given a partition  $\pi$  of  $[0, T]$  and a time instant  $t$  in  $[0, T]$ , we adopt the following notational convention:

$$\begin{aligned} t' &:= \inf\{u \in \pi : u > t\}, & [t] &:= \sup\{u \in \pi : u \leq t\}, \\ t- &:= \sup\{u \in \pi : u' \leq t\}, & t\star &:= \begin{cases} t- & \text{if } t \in \pi \\ [t] & \text{if } t \notin \pi, \end{cases} \\ |\pi| &:= \sup\{|u' - u| : u \in \pi\}, & \pi_t &:= (\pi \cup \{t\}) \cap [0, t]. \end{aligned} \quad (8)$$

Let  $X : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow \mathcal{B}$ . We say that  $X$  is of finite  $p$ -variation for some  $p \geq 1$  if

$$\|X\|_{p\text{-var}, [0, T]}^p := \sup \left\{ \sum_{u \in \pi} |X_{u, u'}|^p : \pi \text{ partition of } [0, T] \right\} < \infty.$$

If  $X$  is additive, this notation is the usual  $p$ -variation norm of the underlying path.

For  $s \leq u \leq t$  we introduce the symbol

$$\delta X_{s,u,t} := X_{s,t} - X_{s,u} - X_{u,t}.$$

If  $X$  is additive, then  $\delta X \equiv 0$ .

**Definition 3.2** (“Control function”). A *control function*  $\omega$  is a non-negative continuous function on  $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ , null on the diagonal and such that

1.  $\omega(s_1, t_1) \leq \omega(s_2, t_2)$ , if the interval  $[s_1, t_1]$  is contained in the interval  $[s_2, t_2]$ ;
2.  $\omega(s, u) + \omega(u, t) \leq \omega(s, t)$ , for all  $s \leq u \leq t$ .

A control function generalizes the concept of the length of an interval. Common controls are  $\omega(s, t) := |t - s|$  and, for a continuous path  $x$  of finite  $p$ -variation,  $\omega(s, t) := \|x\|_{p\text{-var}, [s, t]}^p$ . From these, new controls can be defined by linear combinations  $c_1\omega_1 + c_2\omega_2$  with non-negative coefficients  $c_1, c_2 \in \mathbb{R}_{\geq 0}$ , and by products  $\omega_1^{\gamma_1}\omega_2^{\gamma_2}$  with exponents  $\gamma_1$  and  $\gamma_2$  satisfying  $\gamma_1 + \gamma_2 \geq 1$ , see (Friz & Victoir, 2010, Exercise 1.9).

Given a partition  $\pi$  of  $[s, t] \subset [0, T]$  we may use a control function to measure the mesh-size.

**Definition 3.3.** The modulus of continuity of  $\omega$  on a scale smaller or equal than the mesh-size  $|\pi|$  is given by

$$\text{osc}(\omega, |\pi|) := \sup\{\omega(s, t) : |t - s| \leq |\pi|\}.$$

**Definition 3.4** (“Approximate additivity”). A function  $\Xi : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow \mathcal{B}$  is said *approximately additive* if

1. it is null and right-continuous on the diagonal, that is,  $\Xi_{s,s} = \lim_{t \downarrow s} \Xi_{s,t} = 0$  for all  $s$  in  $[0, T]$ ;
2. there exist  $\gamma > 1$  and a control function  $\omega$  such that

$$|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}| \leq \omega^\gamma(s, t), \tag{9}$$

for all  $s \leq u \leq t$ .

Notice that Equation (9) implies that for all  $1 < \gamma' < \gamma$

$$\|\delta\Xi\|_{\omega, \gamma'} := \sup_{s \leq u \leq t} \frac{|\delta\Xi_{s,u,t}|}{\omega^{\gamma'}(s, t)} \leq \omega^{\gamma-\gamma'}(0, T).$$

Therefore, condition 2 above is equivalent to the existence of a control  $\omega$  and some  $\gamma > 1$  such that  $\|\delta\Xi\|_{\omega, \gamma} < \infty$ .

**Proposition 3.5** (“Sewing Lemma”). *Let  $\Xi : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow \mathcal{B}$  be approximately additive and let the control  $\omega$  and the exponent  $\gamma > 1$  be such that  $\|\delta\Xi\|_{\omega, \gamma} < \infty$ . Then, there exists a unique continuous path*

$$\int \Xi : [0, T] \rightarrow \mathcal{B},$$

whose increments we denote by  $\int_s^t \Xi$ , such that for all  $0 \leq s \leq t \leq T$

1.  $\int_s^t \Xi = \lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \Xi_{u,u'}$  with limit in  $\mathcal{B}$ ;
- 2.

$$\left| \int_s^t \Xi - \Xi_{s,t} \right| \leq \frac{\|\delta\Xi\|_{\omega, \gamma}}{1 - 2^{1-\gamma}} \omega^\gamma(s, t). \tag{10}$$

By Proposition 3.5, we can regard the integral as the map

$$\int : \left\{ \begin{array}{c} \text{approximately} \\ \text{additive} \\ \text{functionals} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{additive} \\ \text{functionals} \end{array} \right\}.$$

By Proposition 3.1, we can unambiguously replace the range  $\left\{ \begin{array}{c} \text{additive} \\ \text{functionals} \end{array} \right\}$  with the space of continuous paths on  $\mathcal{B}$  starting at  $0 \in \mathcal{B}$ . Let  $AA_{p\text{-var}}([0, T]; \mathcal{B})$  be the family of approximately additive functions  $\Xi : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow \mathcal{B}$  that are of finite  $p$ -variation,  $p \geq 1$ . Then, we have:

**Corollary 3.6.** *The restriction of the integral map  $\int$  to  $AA_{p\text{-var}}([0, T]; \mathcal{B})$  takes value in the space  $C_0^{p\text{-var}}([0, T]; \mathcal{B})$  of continuous paths on  $\mathcal{B}$  that start at the origin  $0 \in \mathcal{B}$  and are of finite  $p$ -variation. Moreover,*

$$\int : AA_{p\text{-var}}([0, T]; \mathcal{B}) \longrightarrow C_0^{p\text{-var}}([0, T]; \mathcal{B})$$

$$\Xi \mapsto \lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \Xi_{u, u'}$$

is continuous in  $p$ -variation norm.

### 3.2 | Young integrals

Let  $X : [0, T] \rightarrow \mathcal{B}$  be continuous and of finite  $p$ -variation. Let  $H : [0, T] \rightarrow W$  be continuous and of finite  $q$ -variation, where  $W = \text{Hom}(\mathcal{B}; V)$  and  $V$  is a Banach space. We say that  $p$  and  $q$  are *Young complementary* if  $1/p + 1/q > 1$ .

**Proposition 3.7** (“Young integral”). *Let  $p$  and  $q$  be Young complementary and set*

$$\Xi_{s,t} := H_s X_{s,t}$$

for all  $0 \leq s \leq t \leq T$ , or  $\Xi_{s,t} := H_t X_{s,t}$  for all  $0 \leq s \leq t \leq T$ . Then,  $\Xi$  is approximately additive and of finite  $p$ -variation. As a consequence, the integral

$$H.X := \lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \Xi_{u, u'} \tag{11}$$

defines a continuous path in  $V$  of finite  $p$ -variation. The integral in (11) does not depend on whether  $\Xi$  is defined according to  $\Xi_{s,t} = H_s X_{s,t}$  or to  $\Xi_{s,t} = H_t X_{s,t}$ .

The continuity of  $H$  is only used to show that the choice to evaluate  $H$  at the beginning or at the end of the partition subintervals does not affect the integral. The two choices are respectively referred to as adapted evaluation and terminal evaluation. If  $H$  is not continuous but of bounded variation, the Young integral is defined (because  $q = 1$ ), but depends on the evaluation choice. If  $\pi$  is a partition of  $[0, T]$ , we set

$${}^\pi H_t := \sum_{u \in \pi} H_u \mathbb{1}\{t \in (u, u']\},$$

which denotes the piecewise constant caglad approximation of  $H$  on the grid  $\pi$ . We let  ${}^\pi H.X$  be the Young integral of  $H$  against  $X$  with adapted evaluation, namely

$$({}^\pi H.X)_{0,t} := \sum_{u \in \pi_t} H_u X_{u, u'}$$

In this way, for  $H$  continuous and of finite  $q$ -variation,  $1/p + 1/q > 1$ , we can write

$$H.X = \lim_{|\pi| \downarrow 0} {}^\pi H.X. \tag{12}$$



### 3.3 | Compensated integrals

When the complementary regularities of integrand  $H$  and integrator  $X$  are not sufficient for Young integration, we introduce the enhancement of a path and we will define the integral using compensated Riemann sums. In particular this is the case if  $H$  and  $X$  have the same  $p$ -variation regularity for some  $p$  greater than 2.

As above, let  $X$  be a continuous path of finite  $p$ -variation with trajectory in the Banach space  $\mathcal{B}$ . Recall that  $W$  denotes  $\text{Hom}(\mathcal{B}; V)$ . We use the identification  $\text{Hom}(\mathcal{B}, W) \cong \text{Hom}(\mathcal{B} \otimes \mathcal{B}; V)$ , and we write  $\text{Hom}_{\text{sym}}(\mathcal{B} \otimes \mathcal{B}; V)$  for the subset of those  $\ell$  in  $\text{Hom}(\mathcal{B} \otimes \mathcal{B}; V)$  such that  $\ell(a \otimes b) = \ell(b \otimes a)$  for all  $a, b \in \mathcal{B}$ . Also, the symbol  $\mathcal{B} \odot \mathcal{B}$  will denote the symmetric tensor product of the Banach space  $\mathcal{B}$ , so that we can identify  $\text{Hom}_{\text{sym}}(\mathcal{B} \otimes \mathcal{B}; V) \cong \text{Hom}(\mathcal{B} \odot \mathcal{B}; V)$ . We say that a continuous path  $H : [0, T] \rightarrow W$  admits a symmetric *Gubinelli derivative*  $H'$  with respect to  $X$  if there exists a continuous path  $H' : [0, T] \rightarrow \text{Hom}_{\text{sym}}(\mathcal{B} \otimes \mathcal{B}; V)$  of finite  $q$ -variation such that

1.  $q$  and  $p/2$  are Young complementary;
2.  $R_{s,t}^H := H_{s,t} - H'_s X_{s,t}$  is of finite  $pq/(p+q)$ -variation.

In this case we say that the pair  $(H, H')$  is  $X$ -controlled of  $(p, q)$ -variation regularity. Notice that the regularities of  $R^H$  and of  $X$  imply that  $H$  is of finite  $p$ -variation.

**Definition 3.8** (“Enhancement of a path”). Let  $X$  be in  $C^{p\text{-var}}([0, T]; \mathcal{B})$  and let  $A$  be in  $C^{p/2\text{-var}}([0, T]; \mathcal{B} \odot \mathcal{B})$ . The  $A$ -*enhancement* of  $X$  is the pair  $\mathbf{X} = (X, \mathbb{X})$ , where

$$2\mathbb{X}_{s,t} = X_{s,t} \otimes X_{s,t} - A_{s,t}.$$

**Definition 3.9** (“Enhanced path”). The pair  $\mathbf{X} = (X, \mathbb{X})$  is called *enhanced path* of  $p$ -variation regularity if  $X$  is in  $C^{p\text{-var}}([0, T]; \mathcal{B})$  and  $(s, t) \mapsto X_{s,t} \otimes X_{s,t} - 2\mathbb{X}_{s,t}$  defines an additive  $\mathcal{B} \odot \mathcal{B}$ -valued function of finite  $p/2$ -variation.

Given the enhanced path  $\mathbf{X} = (X, \mathbb{X})$ , the path  $A_{s,t} := X_{s,t} \otimes X_{s,t} - 2\mathbb{X}_{s,t}$  is called the *enhancer* of  $\mathbf{X}$  and we often denote such enhancer with the symbol

$$[\mathbf{X}]_{s,t} := X_{s,t} \otimes X_{s,t} - 2\mathbb{X}_{s,t}.$$

The symbol  $[\mathbf{X}]$  will be referred to as *volatility enhancer* when the financial meaning of it is to be stressed. We say that  $\mathbf{X} = (X, \mathbb{X})$  is a *bounded-variation enhancement* of  $X$  if

$$\sup \left\{ \sum_{u \in \pi} |[\mathbf{X}]_{u,u'}| : \pi \text{ partition of } [0, T] \right\} < \infty.$$

Notice that  $\delta\mathbb{X}$  does not depend on the enhancer because  $[\mathbf{X}]$  is additive; moreover, for all  $s \leq u \leq t$  the following reduced Chen identity holds

$$\delta\mathbb{X}_{s,u,t} = X_{s,u} \odot X_{u,t}.$$

**Lemma 3.10.** *Let  $\mathbf{X} = (X, \mathbb{X})$  be an enhanced path and let  $(H, H')$  be  $X$ -controlled of  $(p, q)$ -variation regularity, with  $H'$  being symmetric. Then,*

$$\Xi_{s,t} := H_s X_{s,t} + H'_s \mathbb{X}_{s,t}$$

is approximately additive.

As a consequence of Lemma 3.10, the integral given by the compensated Riemann sum

$$(H, H').(X, \mathbb{X}) = \lim_{|\pi| \downarrow 0} \sum_{u \in \pi} [H_u X_{u,u'} + H'_u \mathbb{X}_{u \wedge t, u' \wedge t}]$$

is well-defined. Analogously to (12), we write

$$(\pi H, \pi H').(X, \mathbb{X}) = \sum_{u \in \pi} [H_u X_{u,u'} + H'_u \mathbb{X}_{u \wedge t, u' \wedge t}],$$

so that

$$(H, H').(X, \mathbb{X}) = \lim_{|\pi| \downarrow 0} (\pi H, \pi H').(X, \mathbb{X}).$$

If  $J$  is a time interval,  $n$  and  $m$  are non-negative integers and  $\alpha, \beta$  are in  $[0, 1)$ , consider the space

$$C_{\text{loc}}^{m+\beta, n+\alpha}(J \times \mathbb{R}^d; \mathbb{R}^e)$$

of  $\mathbb{R}^e$ -valued functions that are  $m$  times continuously differentiable in time with the  $m$ -th time derivative of local  $\beta$ -Hölder regularity, and  $n$  times continuously differentiable in space with all the  $n$ -th order space derivatives of local  $\alpha$ -Hölder regularity. Notice that nothing is assumed about the cross derivatives in time and space of functions in  $C_{\text{loc}}^{m+\beta, n+\alpha}$ . Let  $C_{\text{cross}}^{m+\beta, n+\alpha}([0, T] \times \mathbb{R}^d; \mathbb{R}^e)$  be the subspace of  $C_{\text{loc}}^{m+\beta, n+\alpha}([0, T] \times \mathbb{R}^d; \mathbb{R}^e)$  consisting of functions  $f$  such that

1. for every multi-index  $I$  with  $|I| = n$  and every compact  $K \Subset \mathbb{R}^d$ ,

$$\sup \left\{ \left\| \partial_x^I f(t, \cdot) \right\|_{\alpha\text{-Höl}, K} : 0 \leq t \leq T \right\} < \infty;$$

2. for every compact  $K \Subset \mathbb{R}^d$ ,

$$\sup \left\{ \left\| \partial_t^m f(\cdot, x) \right\|_{\beta\text{-Höl}, [0, T]} : x \in K \right\} < \infty.$$

Let  $C^\alpha$  be the space

$$C^\alpha := C_{\text{loc}}^{1+\alpha/2, 2+\alpha}([0, T] \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d). \tag{13}$$

**Definition 3.11** (“ $q$ -Moderation”). Let  $w$  be in  $C^\alpha$  and let  $X$  be a continuous path on  $\mathbb{R}^d$  of finite  $p$ -variation, with  $p - 2 < \alpha < 1$ . We say that the pair  $(w, X)$  is  $q$ -moderate if

1. the paths

$$\begin{aligned} H & : t \mapsto \nabla_x w(t, X_t) \\ H' & : t \mapsto \nabla_{xx}^2 w(t, X_t), \\ & \quad 0 \leq t < T, \end{aligned}$$

can be continuously extended up to  $[0, T]$ , and  $H'$  is of finite  $q$ -variation for some  $1 - 2/p < 1/q < \alpha/p$ ;

2. there exists a control function  $\omega$  such that for all  $x$  in the trace  $X[0, T]$  and all  $0 \leq s \leq t \leq T$

$$|\nabla_x w(t, x) - \nabla_x w(s, x)|^{p*} \leq \omega(s, t),$$

where  $p^* = pq/(p + q)$ ;

3.

$$\sup_{0 \leq s \leq T} \left\| \nabla_{xx}^2 w(s, \cdot) \right\|_{\alpha\text{-H\"{o}l, Conv}X[0, T]} < \infty,$$

where  $\text{Conv}X[0, T]$  is the convex hull of the trace of  $X$ .

*Remark 3.12.* Let  $0 < \alpha < 1$  and  $p, q \geq 1$  be such that  $1 - 2/p < 1/q < \alpha/p$ . Assume that  $w \in C_{\text{loc}}^{1+\alpha/2, 2+\alpha}([0, T] \times \mathbb{R}^d; \mathbb{R})$  is such that  $\nabla_x w$  is in  $C_{\text{cross}}^{1/p+1/q, 1+\alpha}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$  and  $\nabla_{xx}^2 w$  is in  $C_{\text{cross}}^{1/q, \alpha}([0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d})$ . Then for all  $X$  in  $C^{p\text{-var}}([0, T]; \mathbb{R}^d)$  the pair  $(w, X)$  is  $q$ -moderate. In particular this holds if  $w$  is twice continuously differentiable in the combined time-space variable  $(t, x)$  with second derivatives of  $\alpha$ -H\"{o}lder regularity.

**Lemma 3.13.** *Let  $w$  be in  $C^\alpha$  and let  $X$  be a continuous  $\mathbb{R}^d$ -valued path of finite  $p$ -variation, with  $p - 2 < \alpha < 1$ . Assume that the pair  $(w, X)$  is  $q$ -moderate,  $1 - 2/p < 1/q < \alpha/p$ . Then,*

$$(H, H') := (\nabla_x w(t, X_t), \nabla_{xx}^2 w(t, X_t))$$

is a Gubinelli  $X$ -controlled path of  $(p, q)$ -variation regularity.

## 4 | ENHANCED PATHS OF DIFFUSION TYPE

We now isolate the pathwise features of price trajectories that affect hedging practice.

Until further notice, we adopt the perspective of discounted prices, so that only the second-order part of  $A$  is considered, with coefficients thought of as functions of the discounted stock price.

**Definition 4.1** (“ $\alpha$ -H\"{o}lder volatility operator”). Let  $\alpha$  be in the open interval  $(0, 1)$ . An  $\alpha$ -H\"{o}lder volatility operator is a second-order elliptic differential operator of the form

$$A = \text{trace}(\mathbf{a}\nabla^2)/2 = a^{i,j}\partial_{i,j}^2/2,$$

where  $\mathbf{a} = (a^{i,j})_{1 \leq i, j \leq d}$  is symmetric and such that all coefficients  $a^{i,j} : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $1 \leq i, j \leq d$ , are  $\alpha$ -H\"{o}lder regular.

Given an  $\alpha$ -Hölder volatility operator  $A = \text{trace}(\mathbf{a}\nabla^2)/2$  and a continuous path  $X : [0, T] \rightarrow \mathbb{R}^d$  of finite  $p$ -variation, we can consider the  $\int_s^t \mathbf{a}(X_u)du$ -enhancement  $\mathbf{X} = (X, \mathbb{X})$  of  $X$  given by

$$\mathbb{X}_{s,t} = \frac{1}{2}(X_{s,t} \otimes X_{s,t}) - \int_s^t \frac{\mathbf{a}(X_u)}{2} du.$$

For brevity we will henceforth call this the  $A$ -enhancement of  $X$ . Notice that such construction yields a bounded variation enhancement. The converse construction, which starts from a bounded variation enhancement and defines a differential operator, is formalized in the following

**Definition 4.2** (“Enhanced path of  $\alpha$ -diffusion type”). Let  $\mathbf{X} = (X, \mathbb{X})$  be an enhanced path of  $p$ -variation regularity. We say that  $\mathbf{X}$  is of  $\alpha$ -diffusion type,  $p - 2 < \alpha < 1$ , if by setting

$$m^{i,j}((s, t]) := [\mathbf{X}]_{s,t}^{i,j}, \quad 0 \leq s < t \leq T, \quad 1 \leq i, j \leq d, \tag{14}$$

absolute continuous measures are defined on the interval  $[0, T]$ , and if their densities with respect to the Lebesgue measure are given by

$$\frac{dm^{i,j}}{dt} = a^{i,j}(X_t),$$

for some  $\mathbf{a} = (a^{i,j})_{1 \leq i, j \leq d}$  in  $C_{\text{loc}}^{\alpha-\text{Hö}}(\mathbb{R}^d, \mathbb{R}^{d \times d})$  satisfying the ellipticity condition

$$a^{i,j}(x)\xi_i\xi_j \geq c(x)|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d, \tag{15}$$

with some continuous strictly positive  $c : \mathbb{R}^d \rightarrow \mathbb{R}_+$ . The operator  $A^{[\mathbf{X}]} := a^{i,j}(x)\partial_{i,j}^2/2$  is called  $[\mathbf{X}]$ -volatility operator, and we say that a diffusive price with Markov generator  $L$  is  $[\mathbf{X}]$ -compatible if the second-order part of  $L$  is equal to  $A^{[\mathbf{X}]}$ .

*Remark 4.3.* The ellipticity condition in Equation (15) allows us to apply the theory from (Lorenzi & Bertoldi, 2007, Chapter 2) to the existence and uniqueness of semigroups on  $C_b(\mathbb{R}^d)$  associated with the volatility operator  $A$ . If the solution to the PDE associated with  $A$  is known to possess a unique solution, the assumed ellipticity can be removed. This is the case for example of the classical Black-Schoels partial differential equation with volatility operator  $\sigma^2 x^2 \partial_{xx}^2/2$ .

*Remark 4.4.* Definition 4.2 is reminiscent of the class of price trajectories considered in (Schied & Voloshchenko, 2016). In both cases, the idea is to define the minimal pathwise requirements that link the dynamics of the underlying to a parabolic PDE. This link hinges on a differential operator. In the theory of Markov diffusions, this differential operator is the generator of the Markov semigroup and characterizes the probability law of the diffusion. However, the class of trajectories of Markov diffusions is strictly contained in the class of trajectories considered in (Schied & Voloshchenko, 2016), which in turn is strictly contained in the class of enhanced paths of diffusion type. Indeed, in (Schied & Voloshchenko, 2016) trajectories are only required to possess a quadratic variation: for example the sum  $X = B + B^H$  of a standard one-dimensional Brownian motion  $B$  and a fractional Brownian motion  $B^H$  with Hurst exponent  $H > 1/2$  is not a Markov diffusion but it is encompassed by (Schied & Voloshchenko, 2016) and by our framework. The case of a

fractional Brownian motion  $B^H$  with  $H > 1/3$  is encompassed by our framework, but not by (Schied & Voloshchenko, 2016).

An enhanced path of  $\alpha$ -diffusion type is the minimal information that the PDE pricing approach requires from a probabilistic model. Indeed, assume that we wish to use the PDE approach to price a contingent claim  $h(X_T)$ , where  $h$  is in  $C_b(\mathbb{R}^d)$  and  $X_T$  is the terminal value of a continuous price path  $X$  of finite  $p$ -variation. Let  $\mathbf{X} = (X, \mathbb{X})$  be an enhancement of  $X$  of  $\alpha$ -diffusion type and consider the equation

$$\begin{cases} (\partial_t + A^{[\mathbf{X}]}) w = 0 & \text{in } [0, T) \times \mathbb{R}^d \\ w(T, \cdot) = \tilde{h}(\cdot) & \text{on } \{T\} \times \mathbb{R}^d. \end{cases} \tag{16}$$

Then, the Cauchy problem (16) admits a solution  $w$  in  $C^\alpha$  (Lorenzi & Bertoldi, 2007, Theorem 2.2.1) and, on any  $[\mathbf{X}]$ -compatible market model, the value  $w(t, X_t)$  is the discounted price at time  $t < T$  of the option maturing at  $T$  and yielding  $h(X_T)$ .

We are in the position to give the pathwise counterpart to Equation (3).

**Proposition 4.5.** *Let  $\mathbf{X} = (X, \mathbb{X})$  be an enhanced path of  $\alpha$ -diffusion type. Let  $w$  be the solution to (16) and assume that the pair  $(w, X)$  is  $q$ -moderate, for some  $1 - 2/p < 1/q < \alpha/p$ . Then,*

$$(H_t, H'_t) := (\nabla_x w(t, X_t), \nabla_{xx}^2 w(t, X_t))$$

is a Gubinelli  $X$ -controlled path of  $(p, q)$ -variation regularity, and it is such that

$$((H, H').(X, \mathbb{X}))_{s,t} = w(t, X_t) - w(s, X_s) \tag{17}$$

for all  $0 \leq s \leq t \leq T$ .

See Appendix 4 for the proof.

We may use higher order sensitivities and pathwise integration to estimate errors arising from time discretization of integral quantities. Consider the *cost of financing* of a hedging strategy, defined as

$$C_t(\phi) := \phi_t^0 S_t^0 + \phi_t^1 S_t - (\phi^0 \cdot S^0)_t - (\phi^1 \cdot S)_t, \tag{18}$$

where  $(\phi^0, \phi^1) \in \mathbb{R} \times \mathbb{R}^d$  is the strategy and  $S^0, S$  are respectively the riskless asset and the risky asset. The symbols  $(\phi^0 \cdot S^0)_t$  and  $(\phi^1 \cdot S)_t$  denote the time- $t$  marginals of the integral processes of  $\phi^0$  and  $\phi^1$  respectively against  $S^0$  and  $S$ . Thus, the cost of financing in Equation (18) is the difference between the value of the portfolio at time  $t$  and the cost of rebalancing the portfolio during the time window  $[0, t]$  in order to follow the hedging strategy. If continuous hedging were possible and one were able to take  $(\phi^0, \phi^1) = (H^0, H)$  as defined in (4), then this cost<sup>1</sup> would match  $V_0 = w(0, X_0)$ , the price at time  $t = 0$  of the option, on a  $P$ -full set. We remark that the probability  $P$  is the measure of the stochastic base on which in the continuous-time case the Itô integral  $(\phi^1 \cdot S)_t$  would be defined. In practice, the cost of financing has two components: the theoretical price  $V_0$  and the cost arising from time discretization, which is  $C_T(\phi) - V_0$ . For the latter, with  $\phi$  replaced by the discretization  $({}^\pi H^0, {}^\pi H)$  of (4), we now provide a pathwise estimate that relies

on integration bounds. Recall that  $X$  in Proposition 4.5 plays the role of the discounted trajectory  $\tilde{S}_t = e^{-rt}S_t$ .

**Corollary 4.6.** *Assume the setting of Proposition 4.5. Let  $\omega$  be the control function whose  $(2/p + 1/q)$ -th power asserts the approximate additivity of  $H_s X_{s,t} + H'_s \mathbb{X}_{s,t}$ . Along any partition  $\pi$  of  $[0, T]$ , the discretized strategy  $({}^\pi H^0, {}^\pi H)$  stemming from (4) with  $\tilde{S} = X$  has a cost of financing  $C({}^\pi H^0, {}^\pi H)$  that is bounded as follows:*

$$\begin{aligned} C_T({}^\pi H^0, {}^\pi H) &\leq |V_0| + e^{rT} (K\omega(0, T)\text{osc}(\omega, |\pi|)^{2/p+1/q-1} + |w_{T-,T}|) \\ &\quad + \left| \sum_{\substack{u \in \pi \\ u' < T}} e^{ru'} H'_u \mathbb{X}_{u,u'} \right|, \end{aligned} \tag{19}$$

where  $\text{osc}(\omega, |\pi|)$  is the modulus of continuity of  $\omega$  on a scale smaller or equal than the mesh-size of the partition, and  $w_{T-,T}$  is the difference between  $w(T, X_T) = \tilde{h}(X_T)$  and the discounted value  $w(T-, X_{T-})$  of the option at the second last node of the partition. The path-dependent constant  $K$  appearing in the bound is not greater than

$$\frac{1}{1 - 2^{1-(2/p+1/q)}} \left( \omega_{RH}^{1/p+1/q}(0, T) \|X\|_{p\text{-var}, [0, T]} + \|H'\|_{q\text{var}, [0, T]} \|\mathbb{X}\|_{p/2\text{-var}, [0, T]} \right),$$

where  $\omega_{RH}$  is the  $pq/(p + q)$ -variation control of  $H_{s,t} - H'_s X_{s,t}$ .

*Proof.* Let  $w_t$  be the path  $t \mapsto w(t, X_t)$ . Fix a partition  $\pi$  of  $[0, T]$  and recall the notation in (8). We preliminarily observe that

$$\begin{aligned} ({}^\pi H^0 \cdot S^0)_t + ({}^\pi H \cdot S)_t &= \sum_{u \in \pi} \left[ w_u S_{u \wedge t, u' \wedge t}^0 + H_u S_{u' \wedge t}^0 \tilde{S}_{u \wedge t, u' \wedge t} \right] \\ &= w_{t \star} S_t^0 - w_0 + \sum_{u \in \pi} S_{u' \wedge t}^0 [-w_{u \wedge t, u' \wedge t} + H_u \tilde{S}_{u \wedge t, u' \wedge t}], \end{aligned}$$

where in the second line we have used summation by parts. Then,

$$\begin{aligned} C_t({}^\pi H^0, {}^\pi H) &= \pi w_t S_t^0 - \pi H_t \tilde{S}_t S_t^0 + \pi H_t S_t - w_{t \star} S_t^0 \\ &\quad + w_0 + \sum_{u \in \pi} S_{u' \wedge t}^0 [w_{u \wedge t, u' \wedge t} - H_u \tilde{S}_{u \wedge t, u' \wedge t}] \\ &= S_t^0 H_{t \star} (\tilde{S}_t - \tilde{S}_{t \star}) + V_0 + \sum_{u \in \pi} S_{u' \wedge t}^0 [w_{u \wedge t, u' \wedge t} - H_u \tilde{S}_{u \wedge t, u' \wedge t}] \\ &= V_0 + S_t^0 w_{t \star, t} + \sum_{\substack{u \in \pi \\ u' < t}} S_{u'}^0 [w_{u, u'} - H_u \tilde{S}_{u, u'}]. \end{aligned} \tag{20}$$

By adding and subtracting the compensation, we can apply the Sewing Lemma (Proposition 3.5) to complete the proof.  $\square$

So far, we have worked with the identification  $\mathbf{X} = \tilde{\mathbf{S}}$ , i.e. the enhanced path at hand has represented the actual enhanced path of the discounted stock price. In other words, the market models have been  $[\tilde{\mathbf{S}}]$ -compatible. This amounts to considering the square  $\alpha = \sigma\sigma^\top$  of co-volatilities a *true* parameter. In Corollary 4.7 below, we no longer do so and we distinguish the *modeled* enhancer of  $\mathbf{X}$  from the *actual* enhancer of  $\tilde{\mathbf{S}}$ . The only assumption on  $\tilde{\mathbf{S}}$  is that it is an enhanced path, that is, its trace  $\tilde{S}$  is a continuous path of finite  $p$ -variation,  $2 < p < 3$ , and its second-order process

$\tilde{\mathbb{S}} = (\tilde{S} \otimes \tilde{S} - [\tilde{\mathbf{S}}])/2$  is a continuous two-parameter function of finite  $p/2$ -variation with values in  $\mathbb{R}^d \diamond \mathbb{R}^d$ ; the enhancer  $[\tilde{\mathbf{S}}]$  is not required to be of bounded variation and the integrals against it will be interpreted as Young integrals.

**Corollary 4.7.** *Let  $\tilde{\mathbf{S}} = (\tilde{S}, \tilde{\mathbb{S}})$  be an enhanced path above the  $\mathbb{R}^d$ -valued discounted price trajectory  $\tilde{S}$  of  $p$ -variation regularity. Let  $A$  be an  $\alpha$ -Hölder volatility operator, with  $\alpha > p - 2$ . Consider the  $A$ -enhancement  $\mathbf{X} = (\tilde{S}, \mathbb{X})$  of  $\tilde{S}$ . If  $h$  and  $w$  are as in Proposition 4.5, then  $(H_t, H'_t) := (\nabla_x w(t, \tilde{S}_t), \nabla_{xx}^2 w(t, \tilde{S}_t))$  is a Gubinelli  $\tilde{S}$ -controlled path of  $(p, q)$ -variation regularity and*

$$\begin{aligned} \tilde{h}(\tilde{S}_T) - V_0 &= ((H, H') \cdot (\tilde{S}, \tilde{\mathbb{S}}))_{0,T} \\ &\quad + \frac{1}{2} (H' \cdot ([\tilde{\mathbf{S}}] - [\mathbf{X}]))_{0,T}, \end{aligned} \tag{21}$$

where the second summand on the right hand side is a well-defined Young integral. As a consequence, if  ${}^\pi H$  denotes the strategy obtained by discretizing along  $\pi$  the  $A$ -delta hedging, then its cost of financing  $C_T({}^\pi H^0, {}^\pi H)$  is bounded by

$$|V_0| + \left| \sum_{\substack{u \in \pi \\ u' < T}} e^{ru'} H'_u \tilde{\mathbb{S}}_{u,u'} \right| + e^{rT} (K\omega(0, T) \text{osc}(\omega, |\pi|)^{2/p+1/q-1} + |w_{T-,T}| + K_{H'} \|[\tilde{\mathbf{S}}] - [\mathbf{X}]\|_{p/2\text{-var}, [0, T]}), \tag{22}$$

where  $\omega, K$  and  $|w_{T-,T}|$  are as in Corollary 4.6 and

$$K_{H'} = \frac{2^{-(1-2/p)^2}}{1 - 2^{1-(4/p+1/q)}} \|H'\|_{q\text{-var}, [0, T]} + 2^{-(1-2/p)^2} \|H'\|_{\infty, [0, T]}.$$

See Appendix A for the proof.

## 5 | PATHWISE FORMULATION OF FUNDAMENTAL EQUATIONS OF HEDGING

By adopting the perspective of undiscounted price paths, we recover the classical formulas of Mathematical Finance within our pathwise setting. Given a price path  $S$ , we say that a model for  $S$  has been specified when a choice for the enhancement  $\mathbf{S} = (S, \mathbb{S})$  is made. This means choosing the enhancer  $[\mathbf{S}]$ , see Section 3. We speak of an  $\alpha$ -diffusive model specification if the enhancer is given by

$$[\mathbf{S}]_{u,v}^{i,j} = \int_u^v e^{2rt} a^{i,j}(e^{-rt} S_t) dt, \quad 0 \leq u \leq v \leq T, \quad 1 \leq i, j \leq d,$$

where  $a^{i,j}, 1 \leq i, j \leq d$  are the coefficients of an  $\alpha$ -Hölder volatility operator and  $r$  is the constant interest rate. In other words, an  $\alpha$ -diffusive model specification is the undiscounted counterpart to an  $A$ -enhancement of some discounted price path, where  $A$  is an  $\alpha$ -Hölder volatility operator as defined in Definition 4.1.

**Theorem 5.1.** *Let  $f(S_T)$  be a contingent claim, where  $f$  is in  $C_b(\mathbb{R}^d)$  and  $S_T$  is the terminal value of a continuous  $d$ -dimensional price path  $S$  of finite  $p$ -variation. Let  $\mathbf{S} = (S, \mathbb{S})$  be an  $\alpha$ -diffusive model*

specification, with  $\alpha > p - 2$ , and let  $A = a^{i,j} \partial_{i,j}^2 / 2$  be the corresponding volatility operator. Then, the Black-Scholes partial differential equation

$$\begin{cases} e^{2rt} a^{i,j} (e^{-rt} z) \partial_{z^i, z^j}^2 v + r z^i \partial_{z^i} v + \partial_t v = r v & \text{in } [0, T) \times \mathbb{R}^d \\ v(T, z) = f(z) & \text{on } \{T\} \times \mathbb{R}^d \end{cases} \quad (23)$$

admits a solution  $v$  in  $C^\alpha$  and this solution is unique. Moreover, there exist a probability space  $(\Omega, \mathfrak{F}, Q, (\mathfrak{F}_t)_t)$  and a Markov diffusion process  $\tilde{S}$  defined on it, such that for all  $0 \leq t \leq T$  it holds

$$v(t, z) = E_{t,z}^Q [e^{-r(T-t)} h(\tilde{S}_T)],$$

where  $h(x) := f(e^{rT} x)$ .

*Proof of Theorem 5.1.* The change of variable  $x := e^{-rt} z$  allows us to rewrite Equation (23) as

$$\begin{cases} (\partial_t + A)w = 0 & \text{in } [0, T) \times \mathbb{R}^d \\ w(T, x) = e^{-rT} f(e^{rT} x) & \text{on } \{T\} \times \mathbb{R}^d, \end{cases}$$

where  $w(t, x) = e^{-rt} v(t, z)$ . Therefore, existence, uniqueness and regularity of the solution follow from those of Equation (16).

Let  $\tilde{S}$  be the Markov diffusion associated with  $A$ , and let  $(\Omega, \mathfrak{F}, Q)$  be the probability space where  $\tilde{S}$  is defined. On the one hand, by construction it holds

$$E_Q [e^{-r(T-t)} h(\tilde{S}_T) | \mathfrak{F}_t] = e^{(T-t)A} h(\tilde{S}_t),$$

where  $e^{tA}$  is the semigroup associated with  $A$ . On the other hand, the Itô integral  $\tilde{V}_t := \int_0^t \nabla_z v(u, S_u) d\tilde{S}_u$  is a  $Q$ -martingale and it is such that  $\tilde{V}_T = e^{-rT} h(\tilde{S}_T)$ . Therefore,

$$e^{rt} E_Q [\tilde{V}_T | \mathfrak{F}_t] = e^{rt} \tilde{V}_t, \quad (24)$$

because  $\tilde{V}$  is a martingale. Combining (24) and (24) we obtain the second claim.  $\square$

**Proposition 5.2.** *Let  $f$  and  $S$  be as in Theorem 5.1. Let  $\mathbf{S} = (S, \mathbb{S})$  be an  $\alpha$ -diffusive model specification, with  $\alpha > p - 2$ , and let  $v = v(t, z)$  solve Equation (23). If  $(v, S)$  is  $q$ -moderate, for some  $1 - 2/p < 1/q < \alpha/p$ , then*

$$(\Delta_t, \Gamma_t) := (\nabla_z v(t, S_t), \nabla_{zz}^2 v(t, S_t))$$

is a Gubinelli  $S$ -controlled path of  $(p, q)$ -variation regularity, and

$$\begin{aligned} V_t - V_0 &= ((\Delta, \Gamma).(S, \mathbb{S}))_{0,t} \\ &+ \int_0^t (V_u - \Delta_u S_u) dS_u^0, \end{aligned} \quad (25)$$

where  $V_t = v(t, S_t)$  and  $S_t^0 = \exp(rt)$ .



*Proof.* The proof is analogous to the one of Proposition 4.5. Indeed, the same Taylor expansion shows that for some  $\gamma > 1$  and some control function  $\omega$ , on the subintervals  $[u, u']$  of any partition  $\pi$ , it holds

$$\begin{aligned} v(u', S_{u'}) - v(u, S_u) &= \nabla_z v(u, S_u) S_{u,u'} + \nabla_{zz}^2 v(u, S_u) \mathbb{S}_{u,u'} \\ &\quad + \partial_t v(u, S_u)(u' - u) + \frac{1}{2} \nabla_{zz}^2 v(u, S_u) [\mathbf{S}]_{u,u'} \\ &\quad + O(\omega^\gamma(u, u')). \end{aligned}$$

By applying the operator  $\lim_{|\pi| \rightarrow 0} \sum_{u \in \pi}$  to both sides of this expansion, we obtain (25) since  $v$  solves the Black-Scholes partial differential Equation (23). □

The pathwise differential equation in (25) coincides with the classical SDE for the portfolio process in the delta hedging. In addition, the definition of the pathwise integral  $(\text{Delta}, \text{Gamma}).(S, \mathbb{S})$  explicitly expresses the dependence on the gamma sensitivity, which is not captured by the classical stochastic integral.

### 5.1 | Fundamental theorem of derivative trading

The formulas for pricing and hedging heavily depend on the diffusive model specification. In classical terms of Mathematical Finance, such specification amounts to specifying the diffusion coefficient (volatility) in Itô’s price dynamics. Volatility is not directly observable and consequently a trader is liable to misspecify volatility and to use coefficients that do not faithfully represent the true price dynamics. The *Fundamental Theorem of Derivative Trading* addresses such misspecification. It provides a formula that computes the profit&loss that a trader incurs into when hedging with the wrong volatility – see (Cont, 2010; Ellersgaard et al., 2017; Karoui et al., 1998). Proposition 5.3 contributes to the assessment of model misspecification in two ways: on the one hand, it shows the pathwise nature of the P&L formula (this aligns with the unifying theme of the section); on the other hand, it provides a generalization of the classical P&L formula. The generalization consists in removing the assumption that the “true” price evolution is governed by an Itô SDE: we capture the misspecification that arises not just between two diffusive enhancements but between a diffusive enhancement (used by the trader) and a general enhanced path (the “true” dynamics).

**Proposition 5.3** (“Fundamental Theorem Of Derivative Trading”). *Let  $f(S_T)$  be a contingent claim, where  $f$  is in  $C_b(\mathbb{R}^d)$  and  $S_T$  is the terminal value of a continuous  $d$ -dimensional price path  $S$  of finite  $p$ -variation. Let  $\mathbf{S}^{\text{true}} = (S, \mathbb{S}^{\text{true}})$  be the true enhanced path above the trace  $S$ . Let  $\mathbf{S} = (S, \mathbb{S})$  be an  $\alpha$ -diffusive model specification,  $\alpha > p - 2$ , and let  $A, v, \text{Delta}$  and  $\text{Gamma}$  be as in Proposition 5.2. Then,*

$$\text{P\&L} = \tilde{V}_T - f(S_T) = \frac{1}{2} (\text{Gamma}.([\mathbf{S}] - [\mathbf{S}^{\text{true}}]))_{0,T}, \tag{26}$$

where the integral on the right hand side is a well-defined Young integral, and  $\hat{V}_t$  is the value at time  $0 \leq t \leq T$  of the  $A$ -hedging portfolio applied to the true enhancement  $\mathbf{S}^{\text{true}}$ , defined by

$$\begin{aligned} \tilde{V}_t &:= v(0, S_0) + ((\text{Delta}, \text{Gamma}).(S, \mathbb{S}^{\text{true}}))_{0,t} \\ &\quad + \int_0^t (v(u, S_u) - \text{Delta}_u S_u) dS_u^0. \end{aligned}$$

*Remark 5.4.* If  $\mathbf{S}^{\text{true}}$  arises from a diffusion model then as compensation terms in our integrals vanish in probability, our definition of the value of the portfolio,  $\hat{V}_t$ , can be justified as a self-financing condition. We will justify this definition for general pricing signals in Section 6 below.

In order to recognize the extension of the classical Fundamental Theorem of Derivative Trading, we rewrite the Young integral in Equation (26) as

$$\frac{1}{2} \int_0^T \nabla_{zz}^2 v(t, S_t) d([\mathbf{S}]_t - [\mathbf{S}^{\text{true}}]_t).$$

In the case where  $\mathbf{S}^{\text{true}}$  is a diffusive enhancement, we have that  $[\mathbf{S}^{\text{true}}]_t = \int_0^t e^{2ru} a_{\text{true}}^{i,j}(e^{-ru} S_u) du$ , so that the integral is turned in the familiar form

$$\frac{1}{2} \int_0^T e^{2rt} \partial_{z^i, z^j}^2 v(t, S_t) \left( a^{i,j}(e^{-rt} S_t) - a_{\text{true}}^{i,j}(e^{-rt} S_t) \right) dt.$$

*Proof of Proposition 5.3.* We manipulate the Taylor expansion in the proof of Proposition 5.2 and, for  $0 \leq u \leq t \leq T$ , we write

$$\begin{aligned} v(t, S_t) - v(u, S_u) &= \nabla_z v(u, S_u) S_{u,t} + \nabla_{zz}^2 v(u, S_u) \mathbb{S}_{u,t}^{\text{true}} \\ &\quad + \partial_t v(u, S_u)(t - u) + \frac{1}{2} \nabla_{zz}^2 v(u, S_u) [\mathbf{S}]_{u,t} \\ &\quad + \frac{1}{2} \nabla_{zz}^2 v(u, S_u) ([\mathbf{S}^{\text{true}}]_{u,t} - [\mathbf{S}]_{u,t}) + O(\omega^\gamma(u, t)), \end{aligned}$$

where  $v$  is the solution to the  $d$ -dimensional Black-Scholes partial differential Equation (23),  $\omega$  is a control function and  $\gamma > 1$ . We sum over the nodes of a partition and then we let the mesh-size shrink to zero, obtaining (26). The good definition of the Young integral of Gamma against  $[\mathbf{S}^{\text{true}}]$  and  $[\mathbf{S}]$  holds as in Corollary 4.7. □

## 6 | ENLARGED HEDGING STRATEGIES

Given an enhanced price path  $\mathbf{S} = (S, \mathbb{S})$ , we interpreted the pathwise integral  $(H, H') \cdot (S, \mathbb{S})$  as the portfolio trajectory arising from the position  $H$  on the risky asset  $S$ .

In this section, we explore the possibility to modify the interpretation of  $(H, H') \cdot (S, \mathbb{S})$ . We will not only consider it as representing the values of the position  $H$  on  $S$ , but we will give a financial interpretation to the compensation  $H' \mathbb{S}$  as well. This requires to analyze the mechanics of rebalancing portfolios during hedging periods.

Given a (continuous) path  $\varphi$  in  $\mathbb{R}^m$  and a partition  $\pi$  we write  ${}^\pi \varphi$  piecewise constant caglad approximation

$${}^\pi \varphi_t = \sum_{u \in \pi} \varphi_u \mathbb{1}\{t \in (u, u']\}. \tag{27}$$

Classically, given the partition  $\pi$  and the discretized strategy  $({}^\pi H^0, {}^\pi H)$ , the cost of rebalancing the portfolio from  $(u-, u]$  to  $(u, u']$  is

$$\text{rebal}^\pi(u) = {}^\pi H_{u'}^0 S_u^0 + {}^\pi H_{u'} S_u - {}^\pi H_u^0 S_u^0 - {}^\pi H_u S_u.$$

Such discretized strategy is self-financing on the grid  $\pi$  if and only if for all  $u > 0$  in  $\pi$  it holds  $\text{rebal}^\pi(u) = 0$ , or equivalently if and only if

$$H_{u'}^0 S_{u'}^0 + H_{u'} S_{u'} - H_u^0 S_u^0 - H_u S_u = H_u^0 S_{u,u'}^0 + H_u S_{u,u'} \quad \forall u \in \pi \cap [0, T).$$

Given  $t$  in  $(0, T]$ , set  $\pi_t := (\pi \cup \{t\}) \cap [0, t]$ . By summing over  $u \in \pi_t$ ,  $u < t$ , we have

$$H_t^0 S_t^0 + H_t S_t - H_0^0 S_0^0 - H_0 S_0 = \sum_{\substack{u \in \pi_t \\ u < t}} H_u^0 S_{u,u'}^0 + \underbrace{\sum_{\substack{u \in \pi_t \\ u < t}} H_u S_{u,u'}}_{=(\pi H.S)_t}.$$

If  $S$  is a semimartingale on  $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t)_t)$ , then taking the  $P$ -limit as  $|\pi| \rightarrow 0$  justifies the axiomatic condition (5), owing in particular to

$$\sup \left\{ \limsup_{|\pi| \rightarrow 0} P \left( \left| (\pi H.S)_t - \int_0^t H dS \right| > \epsilon \right) : \epsilon > 0 \right\} = 0.$$

Here the probabilistic model comes into play to guarantee the convergence of the Riemann sums to the Itô integral  $\int_0^t H dS$  of  $H$  against the semimartingale  $S = S_t(\omega)$ , of which the actual price trajectory is thought of as a realization.

Considering an enhancement  $\mathbf{S}$  of  $S$  and incorporating the appropriate compensation within the rebalancing mechanics, we can avoid using probabilistic arguments when assessing continuously rebalanced hedging strategies.

Given a symmetric  $G_t$  in  $\mathbb{R}^{d \times d} \cong \text{Hom}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R})$  and a subinterval  $[s, t] \subset [0, T]$  we interpret the real quantity  $G_s \mathbb{S}_{s,t}$  as the sum of the payoffs at time  $t$  of the  $d(d-1)/2$  positions  $2G_s^{i,j} = 2G_s^{j,i}$ ,  $1 \leq i < j \leq d$ , on the swap contracts

$$S_{s,t}^i S_{s,t}^j - [\mathbf{S}]_{s,t}^{i,j}, \quad 1 \leq i < j \leq d,$$

and of the  $d$  positions  $G_s^{i,i}$ ,  $1 \leq i \leq d$ , on the swap contracts

$$\left( S_{s,t}^i \right)^2 - [\mathbf{S}]_{s,t}^{i,i}, \quad 1 \leq i \leq d.$$

Hence, for every continuous  $\phi_t = (\phi_t^0, \phi_t^1, \phi_t^2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_{\text{sym}}^{d \times d}$  we can interpret

$$\pi \phi_u^0 S_u^0 + \pi \phi_u^1 S_u + \pi \phi_u^2 \mathbb{S}_{u-,u}$$

as the value of our portfolio at time  $u$  if on the subinterval  $(u-, u]$  we have held  $\pi \phi_u^0 = \phi_{u-}^0$  positions in cash,  $\pi \phi_u^1 = \phi_{u-}^1$  positions in stocks and  $\pi \phi_u^2 = \phi_{u-}^2$  positions in swaps. Strategies that adopt positions in cash, stocks and swaps shall be referred to as *enlarged strategies*. For an enlarged strategy, the rebalancing cost from  $(u-, u]$  to  $(u, u']$  is

$$\text{rebal}^\pi(u) = \phi_u^0 S_u^0 + \phi_u^1 S_u + \phi_u^2 \mathfrak{p}(u, u') - \{ \phi_{u-}^0 S_u^0 + \phi_{u-}^1 S_u + \phi_{u-}^2 \mathbb{S}_{u-,u} \},$$

where, for  $0 \leq s < t \leq T$  and  $1 \leq i \leq j \leq d$ , the amount  $\mathbf{p}^{i,j}(s, t) = \mathbf{p}^{j,i}(s, t)$  denotes the (exogenously-given) price at time  $s$  of the swap  $\mathbb{S}_{s,t}^{i,j}$  with maturity  $t$ . Notice that, since swap contracts are not primitive financial instruments, in the equation above the payoff  $\mathbb{S}_{u-,u}$  at time  $u$  is disentangled from the price  $\mathbf{p}(u, u')$  required at time  $u$  to take a unit position on the next swap  $\mathbb{S}_{u,u'}$ .

We assume that the price  $\mathbf{p}(s, t)$  of the swap contracts  $\mathbb{S}_{s,t}$  defines a  $\mathbb{R}^d \odot \mathbb{R}^d$ -valued function on  $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ , null and right-continuous on the diagonal,<sup>2</sup> and such that  $\mathbf{p}(s, t)$  is of finite  $p/2$ -variation. Let  $\phi^2$  be a continuous path of finite  $q$ -variation on  $\text{Hom}(\mathbb{R}^d \odot \mathbb{R}^d; \mathbb{R})$ , where  $q$  and  $p/2$  are Young complementary. Then, the integral path

$$Y_t := (\phi^2 \cdot \mathbf{p})_{0,t}$$

exists and represents the accumulated cost in the time interval  $[0, t]$  consumed by a continuously rebalanced enlarged strategy in order to adopt the positions  $\phi^2$  on the swap contracts.

**Definition 6.1.** Let  $f(S_T)$  be a contingent claim, where  $f$  is in  $C_b(\mathbb{R}^d)$  and  $S_T$  is the terminal value of a continuous  $d$ -dimensional price path  $S$  of finite  $p$ -variation. Let  $\mathbf{S} = (S, \mathbb{S})$  be an  $\alpha$ -diffusive model specification,  $\alpha > p - 2$ , and let  $\mathbf{A}$ ,  $\mathbf{v}$ ,  $\mathbf{Delta}$  and  $\mathbf{Gamma}$  be as in Proposition 5.2. Let  $C$  be a continuous real valued function on  $[0, T]$ . Then, the  $C$ -enlarged delta hedging is the enlarged strategy defined as

$$\begin{aligned} \phi_t^0 &= C_t e^{-rt} - \mathbf{Delta}_t S_t e^{-rt} - Y_t e^{-rt}, \\ \phi_t^1 &= \mathbf{Delta}_t, \\ \phi_t^2 &= \mathbf{Gamma}_t, \end{aligned} \quad (28)$$

where  $Y_t := (\phi^2 \cdot \mathbf{p})_{0,t}$ .

A desirable property of a hedging strategy is the self-financing condition, i.e. the fact that the strategy does not require money to readjust its positions during the hedging period. The following Proposition 6.2 gives the explicit formula for  $C$  in (28) that guarantees a null rebalancing cost of the  $C$ -enlarged delta hedging.

**Proposition 6.2.** *The continuous real valued function*

$$C_t = v(t, S_t) - r \int_0^t e^{r(t-u)} Y_u du, \quad (29)$$

where  $Y_t := (\mathbf{Gamma} \cdot \mathbf{p})_{0,t}$ , is such that the  $C$ -enlarged delta hedging has zero cost of continuous rebalancing.

*Proof.* We adopt the notation in Definition 6.1. Furthermore, we set

$$y_t := -r \int_0^t e^{r(t-u)} Y_u du.$$

We can write

$$y_{0,t} - r \int_0^t (Y_u - Y_u)du = 0. \tag{30}$$

The cost of rebalancing along a partition  $\pi$  is

$$\begin{aligned} \text{rebal}^\pi(u) &= \pi\phi_{u'}^0 S_u^0 + \pi\text{Delta}_{u'} S_u + \pi\text{Gamma}_{u'} \mathbf{p}(u, u') \\ &\quad - \{ \pi\phi_u^0 S_u^0 + \pi\text{Delta}_u S_u + \pi\text{Gamma}_u S_{u-,u} \} \\ &= C_{u-,u} + \text{Gamma}_u \mathbf{p}(u, u') - Y_{u-,u} \\ &\quad - \{ \phi_{u-}^0 S_{u-,u}^0 + \text{Delta}_{u-} S_{u-,u} + \text{Gamma}_{u-} S_{u-,u} \}. \end{aligned}$$

Hence, summing over  $u \in \pi_t, u > 0$ , we have

$$\begin{aligned} \sum_{\substack{u \in \pi_t \\ u > 0}} \text{rebal}^\pi(u) &= V_{0,t} + y_{0,t} - Y_t + \sum_{u \in \pi_t} \text{Gamma}_u \mathbf{p}(u, u') - \text{Gamma}_0 \mathbf{p}(0, 0') \\ &\quad - (\pi\phi^0 \cdot S^0)_t - ((\pi\text{Delta}, \pi\text{Gamma}) \cdot (S, S))_t. \end{aligned}$$

In the limit as  $|\pi| \rightarrow 0$  we conclude

$$\begin{aligned} \lim_{|\pi| \rightarrow 0} \sum_{\substack{u \in \pi_t \\ u > 0}} \text{rebal}^\pi(u) &= V_{0,t} + y_{0,t} - r \int_0^t V_u du \\ &\quad - r \int_0^t (Y_u - Y_u)du + r \int_0^t \text{Delta}_u S_u du \\ &\quad - ((\text{Delta}, \text{Gamma}) \cdot (S, S))_{0,t} = 0, \end{aligned}$$

owing to (25) and (30). □

The classical delta hedging is such that the initial endowment  $V_0 = v(0, S_0)$  is precisely what the replicating strategy requires in order to yield the amount  $f(S_T)$  at maturity  $T$ . Therefore, the writer of an option invests  $V_0$  in the delta hedging strategy, and such strategy will cover the contingent claim at maturity. Since delta hedging has no additional costs of financing (i.e. rebalancing the portfolio does not consume money) the writer’s profit&loss is null. For the  $C$ -enlarged delta hedging in Proposition 6.2, the self-financing condition holds. Therefore, the option writer’s P&L is exclusively given by the cost of replication, namely by the difference between the due payment  $f(S_T)$  and the final value  $\phi_T^0 S_T^0 + \phi_T^1 S_T$  of the portfolio. Notice that the latter does not comprise the payoff of the swaps, because such endowments are consumed in the rebalancing process.

**Proposition 6.3.** *The profit&loss of the C-enlarged delta hedging with C given as in (29) is*

$$P\&L = Y_T + r \int_0^T e^{r(T-t)} Y_t dt,$$

where  $Y_t = (\text{Gamma} \cdot S)_{0,t}$ .

*Proof.* The profit&loss is given by the difference  $P\&L = v(T, S_T) - \phi_T^0 S_T^0 + \phi_T^1 S_T$ . Hence, the statement follows immediately from the definitions in Equation (28) with  $C$  given as in Equation (29).  $\square$

## 7 | NON-SMOOTH OPTION PAYOUTS

We now consider the case of call options in the Black–Scholes model. We will see that as a result of the non-smooth payoff function one must employ a different, and truly probabilistic, trading strategy towards maturity.

Our setting is the one presented in Section 2, and we take the dimension  $d$  equal to 1. The volatility operator  $A$  is

$$A\phi(x) = \frac{\sigma^2}{2} x^2 \partial_{xx}^2 \phi(x), \quad \phi \in C^2(\mathbb{R}), \quad (31)$$

where  $\sigma > 0$  is the volatility coefficient.

Pricing a European option with payoff  $f(S_T)$  requires solving the partial differential Equation (2) where the terminal constraint  $\tilde{h} = e^{-rT} h$  appearing in this PDE stands in relation to the payoff function  $f$  as expressed in Equation (1).

The volatility operator in Equation (31) is not locally uniformly elliptic, i.e. it does not satisfy the requirement in Equation (15). However as pointed out in Remark 4.3, we do not require ellipticity itself only the existence and uniqueness of solutions to the equation in (16). Existence and uniqueness of solutions to the Black–Scholes PDE is well-known.

In our framework, the classical Black-Scholes model is specified by the following enhancer

$$[\mathbf{S}]_{u,v} = \sigma^2 \int_u^v S_t^2 dt. \quad (32)$$

Under this specification, we now discuss the application of our pathwise framework to the case of European call options, where the payoff is

$$f(z) = (z - K)_+, \quad (33)$$

for some fixed strike  $K > 0$ .

This payoff is not bounded, so in principle it is not included in the general discussion above. However despite the fact that the semigroup associated with the PDE pricing equation was defined on the set  $C_b(\mathbb{R})$ , this semigroup extends to a wider class than  $C_b(\mathbb{R})$ , hence allowing to treat the European call option. Even if the model specification did not allow for such an extension, pricing European call options could always be reduced to pricing European put options due to put-call parity.

In order to be able to apply Proposition 5.2, it remains to discuss the assumption on the  $q$ -moderation of the pair  $(v, S)$ . Unfortunately, here we see that the non-smoothness of the payoff of the call option (or equivalently of the put option) prevents us from applying directly the results established above. We will discuss this in details now.

Recall the three conditions in Definition 3.11. Let  $H_t$  and  $H'_t$  be the delta and the gamma sensitivities namely

$$H_t = \text{Delta}_t = \partial_z v(t, S_t) = N(d_1(t, S_t)), \tag{34}$$

$$H'_t = \text{Gamma}_t = \partial_{zz}^2 v(t, S) = \frac{N'(d_1(t, S_t))}{S_t \sigma \sqrt{T-t}}, \tag{35}$$

where

$$d_1(t, S_t) = (\sigma^2(T-t))^{-\frac{1}{2}} \left[ \ln(S_t/K) + \left( r + \frac{\sigma^2}{2} \right) (T-t) \right], \quad d_2(t, S_t) = d_1(t, S_t) - \sigma \sqrt{T-t}.$$

and  $N$  denotes the distribution function of the standard normal distribution. The fulfilment of the three conditions in Definition 3.11 depends on the terminal value  $S_T$  of the price path. Depending on this terminal value we have the following asymptotics as  $t \uparrow T$ :

$$\begin{aligned} d_1(t, S_t) \sim d_2(t, S_t) &\sim (T-t)^{-\frac{1}{2}} && \text{if } S_T > K; \\ d_1(t, S_t) \sim d_2(t, S_t) &\sim -(T-t)^{-\frac{1}{2}} && \text{if } S_T < K. \end{aligned} \tag{36}$$

Instead, if  $S_T = K$ , then neither  $d_1$  nor  $d_2$  have a limit as  $t \uparrow T$ . To see this we use the law of iterated logarithm, which gives a precise statement on the small time asymptotics of the Brownian path. We have that the terminal value  $S_T$  is

$$S_T = S_t \exp \left\{ \sigma W_T - \sigma W_t + \left( r - \frac{\sigma^2}{2} \right) (T-t) \right\}.$$

If  $S_T = K$ , then by taking logarithm on both sides of this equation we have

$$\ln K - \ln S_t = \sigma W_T - \sigma W_t + \left( r - \frac{\sigma^2}{2} \right) (T-t).$$

Hence, as  $t \uparrow T$  we have

$$\frac{\ln K - \ln S_t}{\sigma \sqrt{T-t}} \sim \frac{W_T - W_t}{\sqrt{T-t}} = \underbrace{\frac{W_T - W_t}{\sqrt{2(T-t) \ln \ln(1/(T-t))}}}_{\limsup=1; \liminf=-1} \cdot \sqrt{2 \ln \ln \frac{1}{T-t}}. \tag{37}$$

The first factor on the right hand side is such that the limsup as  $t \uparrow T$  is equal to 1, and the liminf is equal to  $-1$ . Therefore, if  $S_T = K$ , then

$$\begin{aligned} \limsup_{t \uparrow T} d_1(t, S_t) &= \limsup_{t \uparrow T} d_2(t, S_t) = +\infty, \\ \liminf_{t \uparrow T} d_1(t, S_t) &= \liminf_{t \uparrow T} d_2(t, S_t) = -\infty. \end{aligned} \tag{38}$$

Because of Equation (38), conditions 1 and 2 in Definition 3.11 will not always be satisfied. Moreover, the singularity at  $T$  will also impact condition 3.

One could circumvent this issue by a smooth approximation of the option payoff that could eliminate the point of non-differentiability. Here instead, we comment on what this says about option trading in practice, and on how these singularities, exposed by our pathwise framework, could be regarded as an underpinning of the practicality of option hedging.

The unstable behavior of the sensitivities when time is close to maturity is known in practice, in particular in the case of options that are at-the-money (i.e., the underlying has a price equal or very close to the strike). Because of this, it is common to stop the delta hedging before the actual option maturity, and to continue with a simpler strategy as buy-and-hold. This is described by introducing a time horizon  $\hat{T}$  smaller than the option maturity  $T$ ; then the Black–Scholes price at  $\hat{T}$  is smooth and so our framework can be applied up to time  $\hat{T}$  subject to assuming that the option can be sold at this time at the Black–Scholes price.

After  $\hat{T}$  and in the limit as time approaches  $T$ , the sensitivity Gamma in Equation (35) no longer controls Delta of Equation (34) in the sense of Gubinelli. In the case of at-the-money options, the gamma sensitivity diverges to infinity as time approaches  $T$ . This has an impact on the profit&loss formula of Proposition 5.3, as described in the following proposition.

**Proposition 7.1.** *Assume that  $S_T = K$ . Consider the Black-Scholes model specified by the enhancer in Equation (32) and consider the rough bracket  $[\mathbf{S}^{\text{true}}]$  of the true price signal. Let  $\frac{1}{2} < \gamma < 1$ . Assume that for all  $\epsilon > 0$  there exists a partition such that  $|\pi| < \epsilon$  and*

$$\inf\{[\mathbf{S}^{\text{true}}]_{u,u'} - [\mathbf{S}]_{u,u'} : u \in \pi\} > \epsilon^{1-\gamma}. \quad (39)$$

*Then, there always exists an arbitrary fine trading grid such that the profit&loss of the delta hedging on this trading grid diverges to  $-\infty$  as time approaches the option maturity.*

*Remark 7.2.* Proposition 7.1 says that, in the case of at-the-money options, if the misspecification of the Black-Scholes model is such that the volatility is underestimated, then there exist trading times when following the delta hedging will make the trader incur in unbounded losses. Instead, in the cases of in-the-money and out-the-money options ( $S_T > K$  and  $S_T < K$  respectively), the gamma sensitivity has a limit as time approaches maturity and this limit is zero. Therefore, in these two cases, the Young integral describing profit&loss can be bounded relying on the integration bounds of Section 3.

*Proof of Proposition 7.1.* Let  $\pi$  be a trading grid up to the option maturity. Consider the approximation of the Young integral in Equation (26) on this trading grid, namely

$$\sum_{u \in \pi} \text{Gamma}_u([\mathbf{S}]_{u,u'} - [\mathbf{S}^{\text{true}}]_{u,u'}). \quad (40)$$

The condition in Equation (39) says that for every  $\epsilon > 0$  there exists  $\pi = \pi(\epsilon)$  such that for all  $u$  in  $\pi$  it holds

$$[\mathbf{S}]_{u,u'} - [\mathbf{S}^{\text{true}}]_{u,u'} \leq -\epsilon^{1-\gamma}.$$

Hence, if the sum in Equation (40) is performed on this partition, then such a sum is upper bounded by

$$-\epsilon^{1-\gamma} \Gamma_{T-},$$



where  $T-$  denotes the partition point immediately before the option maturity. Using Equation (37), we have

$$\frac{N'(d_1(t, S_t))}{\sqrt{T-t}} \sim \frac{(2\pi)^{-\frac{1}{2}} \exp \left\{ - \left( \frac{W_T - W_t}{\sqrt{2(T-t) \ln \ln(1/(T-t))}} \right)^2 \cdot 2 \ln \ln \frac{1}{T-t} \right\}}{\sqrt{T-t}}. \quad (41)$$

We see that as  $\epsilon \downarrow 0$  the quantity  $-\epsilon^{1-\gamma} \Gamma_{T-}$  goes to  $-\infty$ .  $\square$

## 8 | CONCLUSIONS

In this work, we have shown that European options may be replicated in a framework that does not use probability. We instead study enhanced price paths defined in the spirit of Rough Path Theory. On the one hand, their enhancements are essential for pathwise integration, as discussed in Section 3. On the other hand, they encapsulate the specification of a model for the valuation of derivatives, carrying the information needed for the hedging (Section 4). Moreover, these enhancements allow to assess model misspecification: a P&L formula for the hedging under “wrong” volatility was proved, generalizing the so-called fundamental theorem of derivative trading (Section 5).

We stated the precise assumptions that allow for the application of Gubinelli integrals in the description of hedging strategies. These assumptions are satisfied in the standard Black-Scholes case of European call and put options only up to a time  $\hat{T}$  that strictly precedes the option maturity  $T$ . On the one hand, this opens the question about suitable approximations for the limiting case as  $\hat{T}$  converges to  $T$  (without using probability); on the other hand, it provides a mathematical underpinning to some hedging practises linked to unstable option sensitivities, in particular in the at-the-money case.

The fact that our enhanced paths extend to trajectories other than semimartingales would make the no-arbitrage arguments suitable for models with transaction costs and other market imperfections. Indeed, in these cases price trajectories are usually less regular than semimartingales. Moreover, we would like to point out that the classical arguments for no-arbitrage under transaction costs is based on consistent price systems, see (Guasoni, 2006; Guasoni et al., 2008). This means that the absence of arbitrage is ultimately based on support theorems, hence presenting the opportunity to apply Rough Path Theory, whose application in support-type arguments has proved to be fruitful (see (Friz & Victoir, 2010, Chapter 19)). In this direction, a recent MSc Thesis at Imperial College London moved the first step (Pei, 2019).

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## DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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## ENDNOTES

<sup>1</sup> In the continuous-time abstraction, the term  $(\phi^1.S)_t$  is to be read as the Itô integral of the continuous adapted process  $\phi^1$  against the continuous semimartingale  $S$ ; the term  $(\phi^0.S^0)_t$  would instead refer to the Lebesgue integral  $r \int_0^t \phi_u^0 e^{ru} du$ .

<sup>2</sup> By this we mean:  $\mathfrak{p}(s, s) = \lim_{t \downarrow s} \mathfrak{p}(s, t) = 0$  for all  $0 \leq s \leq T$ .

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## APPENDIX A: PROOFS FOR SECTION 4

*Proof of Proposition 4.5.* The fact that  $(H, H')$  is  $X$ -controlled follows from Lemma 3.13. We can expand the increments of  $w_t := w(t, X_t)$  as

$$\begin{aligned} w(t, X_t) - w(s, X_s) &= (t-s) \int_0^1 [\partial_t w(s+y(t-s), X_t) - \partial_t w(s, X_t)] dy \\ &\quad - \frac{t-s}{2} \left[ a^{i,j}(X_t) \partial_{i,j}^2 w(s, X_t) - a^{i,j}(X_s) \partial_{i,j}^2 w(s, X_s) \right] + \partial_t w(s, X_s)(t-s) \\ &\quad + \left( \int_0^1 \int_0^1 [\nabla_{xx}^2 w(s, X_s + y_1 y_2 X_{s,t}) - \nabla_{xx}^2 w(s, X_s) y_1] dy_2 dy_1 \right) (X_{s,t} \otimes X_{s,t}) \\ &\quad + \nabla_x w(s, X_s) X_{s,t} + \frac{1}{2} \nabla_{xx}^2 w(s, X_s) (X_{s,t} \otimes X_{s,t}). \end{aligned}$$

We have used (16) on the second line to re-express time derivatives as spatial ones. The assumed  $q$ -moderation allows to control the three increment-type summands in the expansion. Let  $K := \sup_{0 \leq s \leq T} \|\nabla_{xx}^2 w(s, \cdot)\|_{\alpha\text{-HöL}, \text{Conv}X[0,T]}$ . Then,

$$\left| \int_0^1 [\partial_t w(s+y(t-s), X_t) - \partial_t w(s, X_t)] dy \right| \leq \|\mathbf{a}\|_{\infty, X[0,T]} \left[ K \omega_X^{\alpha/p} + \omega_{H'}^{1/q} \right](s, t);$$

and

$$\begin{aligned} &\left| a^{i,j}(X_t) \partial_{i,j}^2 w(s, X_t) - a^{i,j}(X_s) \partial_{i,j}^2 w(s, X_s) \right| \\ &\leq \|\mathbf{a}\|_{\infty, X[0,T]} K \omega_X^{\alpha/p}(s, t) \\ &\quad + \|\mathbf{H}'\|_{\infty, [0,T]} \|\mathbf{a}\|_{\alpha\text{-HöL}, \text{Conv}X[0,T]} \omega_X^{\alpha/p}(s, t); \end{aligned}$$

and

$$\begin{aligned} &\left| \left( \int_0^1 \int_0^1 [\nabla_{xx}^2 w(s, X_s + y_1 y_2 X_{s,t}) - \nabla_{xx}^2 w(s, X_s) y_1] dy_2 dy_1 \right) (X_{s,t} \otimes X_{s,t}) \right| \\ &\leq \frac{K}{(1+\alpha)(2+\alpha)} \omega_X^{(2+\alpha)/p}(s, t). \end{aligned}$$

Recall that, in particular,  $\frac{2+\alpha}{p} > 1$  by the choice of  $\alpha$  in the definition of enhanced path of  $\alpha$ -diffusion type. Then, the three estimations above say that, for the expansion of the increments  $w_{s,t}$ , the following holds: there exists a control  $\omega$  and an exponent  $\gamma > 1$  such that

$$\begin{aligned} &\left| w_{s,t} - \nabla_x w(s, X_s) X_{s,t} - \nabla_{xx}^2 w(s, X_s) \mathbb{X}_{s,t} - \partial_t w(s, X_s) - \frac{1}{2} \nabla_{xx}^2 w(s, X_s) [\mathbf{X}]_{s,t} \right| \\ &= \left| w_{s,t} - \partial_t w(s, X_s) - \nabla_x w(s, X_s) X_{s,t} - \nabla_{xx}^2 w(s, X_s) (X_{s,t} \otimes X_{s,t}) \right| \leq \omega^\gamma(s, t). \end{aligned}$$

Hence,

$$\begin{aligned} w_{s,t} &= \lim_{|\pi| \rightarrow 0} \sum_{u \in \pi \cap [s,t]} \left[ \partial_t w(u, X_u) (u' - u) + \frac{1}{2} \partial_{i,j}^2 w(u, X_u) [\mathbf{X}]_{u,u'}^{i,j} \right] \\ &\quad + \underbrace{\lim_{|\pi| \rightarrow 0} \sum_{u \in \pi \cap [s,t]} [H_u X_{u \wedge t, u' \wedge t} + H'_u \mathbb{X}_{u \wedge t, u' \wedge t}]}_{=:(\pi H, \pi H')(X, \mathbb{X})_{s,t}} \\ &= \underbrace{\quad}_{=:(H, H')(X, \mathbb{X})_{s,t}} \end{aligned}$$

The possibility to split the limit descends from the already-known convergence of  $(({}^\pi H, {}^\pi H').(X, \mathbb{X}))$  as  $|\pi| \rightarrow 0$ . For any  $i, j$  the discrete sum  $\sum_{u \in \pi} \partial_{i,j}^2 w(u, X_u) [\mathbf{X}]_{u,u'}$  approximates the Stieltjes integral of the continuous function  $u \mapsto \partial_{i,j}^2 w(u, X_u)$  against the measure  $m^{i,j}$  of (14). Hence, in the limit as  $|\pi| \rightarrow 0$  it converges to  $\int_0^t \partial_{i,j}^2 w(u, X_u) a^{i,j}(X_u) du$ . The cancellation guaranteed by (16) then implies (17).  $\square$

*Proof of Corollary 4.7.* The fact that  $(\nabla_x w(t, \tilde{S}_t), \nabla_{xx}^2 w(t, \tilde{S}_t))$  is  $\tilde{S}$ -controlled of  $(p, q)$ -variation regularity is already contained in Proposition 4.5, because it does not involve the second-order component of  $\tilde{\mathbf{S}}$ . Also, the Taylor expansion of Proposition 4.5 yields a control function  $\omega$  and an exponent  $\gamma > 1$  such that however chosen a subinterval  $[s, t]$  of  $[0, T]$ , it holds

$$\begin{aligned} w(t, \tilde{S}_t) - w(s, \tilde{S}_s) &= \nabla_x w(s, \tilde{S}_s) \tilde{S}_{s,t} + \partial_t w(s, \tilde{S}_s)(t-s) \\ &\quad + \frac{1}{2} \nabla_{xx}^2 w(s, \tilde{S}_s) \tilde{S}_{s,t} \otimes \tilde{S}_{s,t} + O(\omega^\gamma(s, t)) \\ &= \nabla_x w(s, \tilde{S}_s) \tilde{S}_{s,t} + \nabla_{xx}^2 w(s, \tilde{S}_s) \tilde{S}_{s,t} \\ &\quad + \partial_t w(s, \tilde{S}_s)(t-s) + \frac{1}{2} \nabla_{xx}^2 w(s, \tilde{S}_s) [\mathbf{X}]_{s,t} \\ &\quad + \frac{1}{2} \nabla_{xx}^2 w(s, \tilde{S}_s) ([\tilde{\mathbf{S}}]_{s,t} - [\mathbf{X}]_{s,t}) + O(\omega^\gamma(s, t)). \end{aligned}$$

Therefore, by considering the subintervals  $[u, u']$  of a partition  $\pi$  of  $[s, t]$ , summing over these, and letting  $|\pi| \rightarrow 0$ , we obtain

$$w(t, \tilde{S}_t) - w(s, \tilde{S}_s) = ((H, H').(\tilde{S}, \tilde{\mathbf{S}}))_{s,t} + \frac{1}{2} (H'.([\tilde{\mathbf{S}}] - [\mathbf{X}]))_{s,t}, \quad (\text{A1})$$

and in particular (21). The second summand on the right hand side is a well-defined Young integral because  $t \mapsto \nabla_{xx}^2 w(t, \tilde{S}_t)$  is of bounded  $q$ -variation,  $q < p/\alpha$ , and  $\alpha > p - 2$  by assumption.

Write  $w_{s,t}$  for the increments  $w(t, \tilde{S}_t) - w(s, \tilde{S}_s)$ ,  $0 \leq s \leq t \leq T$ . Owing to (A1), for every subinterval  $[u, u']$  of a partition  $\pi$  we can write

$$\begin{aligned} w_{u,u'} - H_u \tilde{S}_{u,u'} &= ((H, H').(\tilde{S}, \tilde{\mathbf{S}}))_{u,u'} - H_u \tilde{S}_{u,u'} - H'_u \tilde{\mathbf{S}}_{u,u'} \\ &\quad + H'_u \tilde{\mathbf{S}}_{u,u'} + \frac{1}{2} (H'.([\tilde{\mathbf{S}}] - [\mathbf{X}]))_{u,u'}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \sum_{\substack{u \in \pi \\ u' < t}} S_{u'}^0 [w_{u,u'} - H_u \tilde{S}_{u,u'}] \right| &\leq e^{rT} K \omega(0, T) \text{osc}(\omega, |\pi|)^{2/p+1/q-1} + \left| \sum_{\substack{u \in \pi \\ u' < t}} e^{ru'} H'_u \tilde{\mathbf{S}}_{u,u'} \right| \\ &\quad + \frac{1}{2} e^{rT} \|H'.([\tilde{\mathbf{S}}] - [\mathbf{X}])\|_{p/2\text{-var}, [0, T]}, \end{aligned}$$

where, by applying the bounds in (Friz & Victoir, 2010, Theorem 6.8) we see

$$\begin{aligned} &\|H'.([\tilde{\mathbf{S}}] - [\mathbf{X}])\|_{p/2\text{-var}, [0, T]} \\ &\leq 2 \left(1 - \frac{2}{p}\right)^{\frac{2}{p}} \|[\tilde{\mathbf{S}}] - [\mathbf{X}]\|_{p/2\text{-var}, [0, T]} \\ &\left( \frac{1}{1 - 2^{1-(4/p+1/q)}} \|H'\|_{q\text{-var}, [0, T]} + \|H'\|_{\infty, [0, T]} \right). \end{aligned}$$

By plugging in (20), we obtain the desired results.  $\square$