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Relativistically into Finance

Vitor H. Carvalho* Raquel M. Gaspar †

April 2021

Abstract

The change of information near the light speed, advances in high-speed trading, spatial arbitrage strategies and foreseen space exploration, suggest the need to consider the effects of the theory of relativity into finance models. Time and space, under certain circumstances, are not dissociated and no longer can be interpreted as Euclidean.

This paper provides an overview of research made on this field, while formally defining the key notions of spacetime and proper time. Further progression in this field does require a common ground of concepts and an understanding of how time dilation impacts financial models.

For illustration purposes, we compute relativistic effects for option prices when viewed from the viewpoint of two distinct reference frames, based upon the classical Balck-Scholes model. We show relativistic effects are non-negligible and illustrate how they depend on option characteristics such as maturity of the contract and volatility of the underlying.

Keywords: econophysics, spacetime finance, proper time, time dilation.

JEL classification: E430, G100, G120, G130

Contents

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1 Introduction

A part of Finance focuses on the analysis of financial markets and products, modelling the way agents interact in the markets and the way products should be priced or hedged. Models are constantly adapting, though necessarily constrained by "reality". That is, they depend not only on social characteristics such as ideology, legal systems or political aspects, but also on more physical characteristics, in terms of available resources, locations, distances or communication times, among other. Thus, economics and financial constructs and behaviours are subject to physical cosmos rules.

The connection between the disciplines of physics and economics in general (finance included) is a long one. Hetherington (1983) suggests that "Adam Smith's (1723–1790) efforts to discover the general laws of economics were directly inspired and shaped by the examples of Newton's (1643–1727) success end discovering the natural laws of motion". Likewise, the economist Walras (1834-1910) was influenced by the physical sciences. "His law of general equilibrium was based on the work of the mathematician Poinsot (1777–1859)" (Paula, 2002).

At the beginning of the twentieth century, Bachelier (1900), admitted that the prices of financial assets followed a random walk. Curiously, Bachelier (1900), known as the founder of stochastic mathematical finance, anticipated the ideas from Einstein et al. (1905) in five years on the mathematical formalization of random walk (Courtault et al., 2000). Bachelier is, thus, the precursor of modern finance the efficient markets hypothesis (Samuelson, 1965; Eugene, 1970; Fama, 1991) and the well-know Black–Scholes-Merton pricing formula for options (Black and Scholes, 1973; Merton, 1973).

It was, however, much later that the *econophysics* name emerges, possibly used for the first time by Stanley et al. (1996). According to Schinkus (2010), this "new" discipline keeps arisen making important contributions to the economy, especially in the field of financial markets. For a historical overviews on econophysics see, for instance, Savoiu and Siman (2013) or Pereira et al. (2017).

The econophysics literature nowadays is extremely broad. It cover, not only, subjects such as nonlinear dynamics, chaos, stochastic and diffusive processes, (Mantegna and Stanley, 1999) but also more recent topics such as big data (Ferreira et al., 2020).

Here we look at a relatively small sub-field of econophysics which is that of the applicability of relativity theories to finance, hoping to provide a smooth, yet rigorous, read to both finance professionals and physicists.

Technical developments (as high-speed communications and trading) as well as possible future challenges (as out of Earth trade and cosmos exploration), require integration of relativistic theories into finance models. Unfortunately, the literature on the matter is still relatively scarce and sometimes inconsistent.

Time is a fundamental dimension and is key to all financial models. However, under the theory of relativity time is not absolute, instead its is intertwined with spatial dimensions. The composition of these spatial dimensions and a temporal one, allied with the speed of light, creates a reference frame, called *spacetime*. Events should, thus, be understood as situated in a spacetime reference framework.

The reference to spatial dimensions and the need to introduce them on financial models, at first, may appear odd, as they commonly do not appear finance models, at least in a straightforward way. Doubtlessly, if one looks closer and deeper it is possible to identify that space dimensions are, actually, under consideration. In fact, exchanges can be interpreted as "spatial zones", defined by a set of (not necessarily just financial) conditions, i.e defined by spatial coordinates. Moreover, information propagation times between exchanges involves space, and may even lead to spatial arbitrages.

In a spacetime framework, objects or events are not defined absolutely, instead, events are interpreted relatively to the observers motion. In other words, there are no simultaneity nor an absolute reality between different observers in different inertial reference frames. Each market participant's reality depends on its own referential frame velocity relative to the observed event's reference frame. As a result, an asset value can be different for different reference frames.

Einstein's relativistic theories can be divided in two: (i) special theory of relativity, that concerns a spacetime with no gravity (Einstein, 1905), and (ii) general theory of relativity that takes gravity into consideration (Einstein, 1916). On the present study we focus on finance applications to a gravity free spacetime structure, in the context of the special theory of relativity (STR).

In gravity free spacetimes, we are in the presence of an important type of reference frame – inertial frames – in which the relations between space dimensions are Euclidean and there exists a time dimension in which events either stay at rest, or continue to move, in straight lines, with constant speed (Rindler, 1982). Minkowski Minkowski (1908) spacetime metric is known to be the cosmos simplest space conceptualisation, under STR (Mohajan, 2013).

In this paper, we start by presenting an overview of literature that applies relativity theories to finance, in Section 2. In Section 3 we focus on STR and the Minkowski spacetime conceptualisation, and formally introduce the necessary physical concepts and presents a possible financial model setup. In Section 4 we illustrate the usage of the proposed model to identify possible option prices discrepancies, due to time dilation and non-simultaneity of communications. Section 5 concludes summarising the proposed ideas, and discussing further research challenges.

2 State of the Art

Einstein's axioms state that the laws of physics are identical in all inertial reference frames and that there exists an inertial reference frame in which light, in vacuum, always travels rectilinearly at constant speed, in all directions, independently of its source (Rindler, 1982). The relevance of Einstein's axioms resides on the universal constant value of the light speed $c = 299792458m/s$, in vacuum, and that laws of physics are identical in all inertial reference frames. However, the light speed light leads to non-simultaneity, when considering interplanetary trade.

2.1 Interplanetary Trade

Consider, for instance, the case of Earth and Mars that distance themselves between 5.57×10^{10}m and 4.01×10^{11}m (NASA, 2018)[1]. Any buy/sell order travelling between the two planets take ≈ 3.1 to ≈ 22.3 minutes to arrive. This alone creates a non-simultaneity situation. Auer (2015) argues that, due to this non-simultaneity effect, significant bid-ask spreads on interplanetary exchanges would be common and more significant than the time dilation effects.

Angel (2014) claims that the no-simultaneity would produce differences in prices for markets participants (MPs) in different reference frames. Concerning the same reference frame Krugman (2010) established two fundamental interstellar trade theorems: (i) that the interest costs should consider a common time measure to all planets reference frame (not the reference frame of any spacecraft) and (ii) that interest rates would equalize across planets.

The concern with the establishment of a common reference frame is also highlighted by Morton (2016). It mentions that in order to avoid arbitrage or misconduct, firms balance sheet should be linked to a concrete inertial reference frame. In this sense all MPs, in their own reference frames, would evaluate the firms balance sheet relative to benchmark reference frame.

Another extreme example from Morton (2016), is that a firm could be considered to perform badly by an MP and to perform well by another MP, in a different reference frame, which highlight the need to consider relativity in the definitions of asset value and risk.

Haug (2004) and Auer (2015) refer to the terms *proper interest* to correct for the non-simultaneity effect, prevent arbitrage and comply with the law of one price. Haug (2004) also refers to *proper volatility* in connection with proper time, so that MPs in different reference frames would consider the same volatility (instead of different volatility values for different reference frames).

Although full of good ideas, the above mentioned notion of *proper interest* concept, as a way to compensate the differences due to the coordinate and proper time differences, may be hard to implement. Concretely, Auer (2015) considers proper interest as a constant time dilation which hardly exists, i.e finding an interest rate process compatible with such adjustment, may be extremely difficult. The problem lies on the fact that this proper interest concept merges the Lorentz factor effect with the interest rates dynamics, instead of keeping it separate. To put it differently, even in a scenario of no (or zero) interest rate, there is still non-simultaneity in interplanetary trading. For this reason, in our option pricing application, we consider a zero interest rate setup as our base scenario, to distinguish pure proper time adjustments, from mixed (interest rate and proper time) effects when computing present or future value of assets.

Considering interplanetary financial trading may, at first, seem far fetched. It is probably not as far fetched as high-speed trading between very distance exchanges on Earth would look, some time ago, when there were no telecommunications. Space exploration is daily on the news and according to (Haug, 2004) "spacetime finance will play some role in the future". The question is not whether finance will play some role in the future space exploration, but rather a question of *when* it will happen.

[1]The distance between the planets is not always the same. Planets have elliptical orbits around the Sun. All planets have different elliptics, so distance between them is not constant.

Even with our present day technology level, delays, due to no-simultaneity, are of utmost importance. Wissner-Gross and Freer (2010) demonstrated that light propagation delays present opportunities for statistical arbitrage, at Earth scale. They identify a nodes map across Earth's surface by which the propagation of financial information can be slowed or stopped. There can be an arbitrage at a mid-point – i.e. in land, sea or space – between two exchange financial centres (Buchanan, 2015; Haug, 2018).

In fact, it is in the area of high speed trading that relativity has contributed most to finance.

2.2 High speed trading

In the works of Angel (2014), Laughlin et al. (2014), Buchanan (2015) and Haug (2018), relativistic effects on high speed trading and communications, have been referred, revealing their potential and where they can be more significant.
The race to the fastest trading speeds with investment of US $300 Million to get $2.6 \times 10^{-3}s$, between London and New York stock exchanges, or US $430 Million to get $3 \times 10^{-3}s$, between Singapore and Tokyo stock exchanges, or in hollow-core fibre cables or even neutrinos, shows how relativity is becoming ever present in finance (Laughlin et al., 2014; Buchanan, 2015). Likewise, as is clear from (Buchanan, 2015) the development of lasers or very short waves, between two points, over a geodesic, preferably in line-of-sight are a reality. Laughlin et al. (2014) reports a $3 \times 10^{-3}s$ decrease time in one-way communication between New York and Chicago due to a relativistic correct millisecond resolution tick data.
So, the light speed limit already brings challenges not only to (future) interplanetary, but also to (present, current) intraplanetary financial trading due to delays in communications, high frequency trading, non-simultaneity, spatial and speed arbitrages as highlighted by Haug (2004), Wissner-Gross and Freer (2010), Angel (2014), Laughlin et al. (2014), Auer (2015), Buchanan (2015), Morton (2016) or Haug (2018).

2.3 Other

Formal physical relativistic relationships have also been used to address other finance issues, sometimes with not so straightforward mapping considerations.

Mannix (2016) calls the attention to the revision of the **efficient markets hypothesis** concept, under a relativistic spacetime, because there is no instantaneous incorporation of all available information. Angel (2014) reports that the no simultaneity produces different best prices for market participants that are not in the same reference frames. Under relativistic quantum mechanics any measurement procedure takes some finite time, so there are no immediate values of the measured quantity (Saptsin and Soloviev, 2009). In brief, this puts into evidence the Heisenberg's uncertainty principle which combined with relativity can bring a higher uncertainty in the asset valuation and increase the no simultaneity of the incorporation of all available information. In conclusion it can reinforce an Efficient Markets Hypothesis revision. The Heisenberg's uncertainty principle affirms that the increased precision on a particle position decreases the precision in the momentum (Heisenberg, 1927).

Up to now, we have focus, relativity for Human physical scales. Although is transverse to all scales, even in the quantum reality. Literature contributions are being developed in the field of **quantum relativity** in econophysics, that adapt, use and apply quantum model processes, analogies or ideas (Jacobson and Schulman, 1984; Saptsin and Soloviev, 2009; Romero et al., 2013; Romero and Zubieta-Martínez, 2016; Trzetrzelewski, 2017).

Under relativistic quantum mechanics any measurement procedure takes some finite time, so there are no immediate values of the measured quantity (Saptsin and Soloviev, 2009). In brief, this puts into evidence the Heisenberg's uncertainty principle (Heisenberg, 1927) which combined with relativity can bring a higher uncertainty in the asset valuation and increase the no simultaneity of the incorporation of all available information.

In the works of (Romero et al., 2013), (Romero and Zubieta-Martínez, 2016) and (Trzetrzelewski, 2017) there are mapping consideration for the variables that require more theoretical and empirical support with a financial or economic interpretation. For instance (Romero and Zubieta-Martínez, 2016) considers that the physical variables mass *m* and position *x* can have their corresponding finance relations, as $m = 1/\sigma^2$ and $x = ln(S)$. Where *S* is the underlying asset price and σ the volatility. In (Trzetrzelewski, 2017) volatility has dimensions of $s^{-1/2}$. Although these models incorporate relativity and quantum ideas in finance models, empirical results are required to validate them.

In fact, the lack of, economic and or financial, direct reasoning for the variables mapping considerations applied to the quantum relativistic models, does not give proper support to these models adoption. The present study will not lean over this area of study.

Some literature contributions consider relativity independently of the physical spacetime reference frame. Trzetrzelewski (2017) considered the concept of relativity under high speed trading, where the speed of light is substituted by a frequency interpretation of orders per second. In Jacobson and Schulman (1984), Dunkel and Hänggi (2009) and Trzetrzelewski (2017), authors performed works in **relativistic Brownian motions**. Dunkel and Hänggi (2009) have developed extensive work in relativistic Brownian motions constructed under mathematical and physical considerations, with some potential to be integrated in finance models. Under a relativistic extension of the Brownian motion Kakushadze (2017) studied the volatility smile as a relativistic effect.

In these studies, relativity however is not associated with our living spacetime structure.

Relativity is a time reversal invariance theory, like all basic theories on physics. Macroscopic world is not time reversal invariance as explained by thermodynamics and entropy. Zumbach (2007) refers that time reversal invariance is only observed in stochastic volatility and regime switching processes, and that GARCH(1,1) can only explain some asymmetry. Tenreiro Machado (2014) applied relativity in financial **time series** and Pincak and Kanjamapornkul (2018) used relativity in financial time series forecast models. Pincak and Kanjamapornkul (2018) considered a special Minkowski metric where price and time can not be separated.

The heterogeneity of the above mentioned literature has one common feature: the fact that each author adapts STR differently! In fact, except for the cases of interplanetary trade and (intraplanetary) high speed trading, where some consistency (finally) seems to appear, in almost all other cases, key concept of relativity theory change, depending on the concrete application. Sometimes, even without taking into consideration the physical properties they must obey, which may lead to lost of sense resulting from the calculations.

To avoid following that "trap", in Section 3, we present a possible formal setup, focusing on properly defining the necessary physical concepts.

3 Spacetime Finance

We start by revisiting and discussing some key concepts from physics, then we go on formalizing Minkowshi Minkowski (1908) spacetime, the associated Lorentz transformations and the idea of proper time.

3.1 Concepts

- *Spacetime*

  It is a space concept where *time* and *spatial dimensions* are intertwined and undissociated, and where a reference frame is defined. Its dimensions can be interpreted as "degrees of freedom", that theoretically provides an infinite set of coordinates available to the event.

  However, spacetime dimensions are *isotropic* which means the relation between different reference frames must be deterministic. And, thus, cannot be model using stochastic processes. Furthermore, the isotropy of the time dimension does not mean that a *"back-in-time"* happening is possible, it only states that the time flow direction does not matter. Taking a finance perspective, it means we may to calculate, future values, or present values – i.e. time flow direction can be what better suits us – but, of course, there is no *"back-in-time"* possibility. These are the most common mistakes identified in the literature.

- *Market participants (and observers)*

  The term "observer" is widely used in physics and relativity literature. It intends to describes someone – e.g. researcher – that does not interfere with what is being studied, nor with the fundamental laws of physics. When taking a financial perspective is difficult conceive such person or entity, just is looking at the market without playing a role in it. Therefore, the term "market participant" (MP) seems to us a better fit for financial applications.

  A MP can have a more direct intervention in the market – e.g. issuer, broker, investor – or a lesser one, but still cannot disobey he fundamental laws of physics. We save the term "observer" to refer to and outsider person or entity that we can guarantee that it does not interferes in the market (e.g. researcher, supervision authority).

- *Relativity*

  In the present study the term relativity is used in the context of relativity that is not Euclidean and is gravity free, under STR. It affects the spacetime metric and produces market measurable effects. This implies very high velocities and an exact definition, that may depend on the concrete application.

- *Event and object*

  Object and event terms commonly have different meanings. A MP, may interpret a nickel mine as an object that is inanimate. Although another MP can interpret it as a set of material points travelling through the cosmos, at thousands of meters per second. The latter description is more frequently called as an event. The term "event" is also more suitable to refer to a deal between two MPs.[2] So, throughout, we refer only to events (E), instead of events and objects.

3.2 Minkowski spacetime

To situate an event and deal with different inertial reference frames, we need to use a free gravity space conceptualisation. The Minkowski spacetime is a suitable four-dimensional real vector space, under

[2] The term "event" has also a wider meaning – it can define an happening or an object.

STR on which a symmetric, nondegenerate metric is defined Naber (2012). It considers the Cartesian coordinates (x, y, z) , or the polar coordinates, as space coordinates, plus time t. Since space axis dimensions in Minkowski spacetime are all in meters m, and time t is often multiplied by the speed of light constant value c, to give a new spatial dimension ct.

The reasoning for considering ct, resides in the fact that, it is immediate to interpret what a w displacement in the ct axis is: it corresponds to the time taken by light to travel the same distance w (Siklos, 2011). In addition, given the common space coordinate[3], m, time t can always be extracted from the ct dimension.

In relativity, it is widely used representations like that of Figure 1, but with the equivalent time represented vertically. Here we opted to represent it in the horizontal axis, that is the typically time-related axis in finance. We hope the change of axis not to be considered a physical "heresy", and that it does help those from a financial background to visualize better the concepts. It presents a spacetime diagram where the axis z is omitted. Since events can take any direction and dimensions are isotropic, this produces a four-dimension cone called the light cone.[4]

There are two possible light cones for each event E at each moment in time, a past light cone and a future one. The light cone surface is only accessible to light, because the slope line is $45°$, between ct and x. Thus, the distance that light travels, in vacuum, in one second[5] is $299, 792, 458\ m$ is the same distance travelled in all axis. This means that a w displacement in the ct axis is the same w displacement value in the space axes. Inside the light cone resides the four-dimensional coordinates available to all real events defined at the origin. Events inside the cone are time-like events and corresponds to all set of coordinates available to the E or MP defined at the origin. Space-like events are not accessible to MP because implies speeds higher than c.

3.3 Lorentz transformations

Suppose L and L' defines, respectively, the stationary and moving inertial reference frames.[6]

Let us consider a market participant, MP_A, on the four dimensions inertial reference frame L with coordinates (ct, x, y, z). Recall all coordinates are in meters m, and time t is obtained by dividing ct by c. In addition, we have a second market participant, MP_B, on the four dimensions inertial reference frame L' with coordinates (ct', x', y', z'). Furthermore, the L' reference frame is moving away from L, according to MP_1, with velocity v. An event E coordinates transformation between the inertial reference frames L and L', is provided by the Lorentz transformations

$$ct' = \gamma(ct - \frac{v}{c}x) \ , \qquad x' = \gamma(x - vt) \ , \qquad y' = y \ , \qquad z' = z \ , \tag{1}$$

where $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$ is the so-called Lorentz factor (Rindler, 1982).

Lorentz transformations show that time and space are not invariant, but reference frame dependent (Siklos, 2011). In Equation (1) the transformed y' and z' axes coincide with the y and z axes, which although standard, is a simplification and assumes the direction of motion happens only in the x' axis (Naber, 2012)[7].

[3]It allows to create a metric tensor to perform coordinate transformations between different inertial reference frames.

[4]Only three-dimensions are represented in Figure 1.

[5]Recall $c = 299, 792, 458\ m/s$ in vacuum.

[6]The $'$ symbol should not be interpreted as a differentiation notation. Also, as opposed to Naber (2012), Siklos (2011), among other authors, who identify a reference frames by S, here we opt from the letter L, as in finance S is commonly used to identify the price of a stock.

[7]The extension of this setup to other spacetime formulations is possible. For the purpose of this paper, the simplest Minkowski spacetime definition suffices.

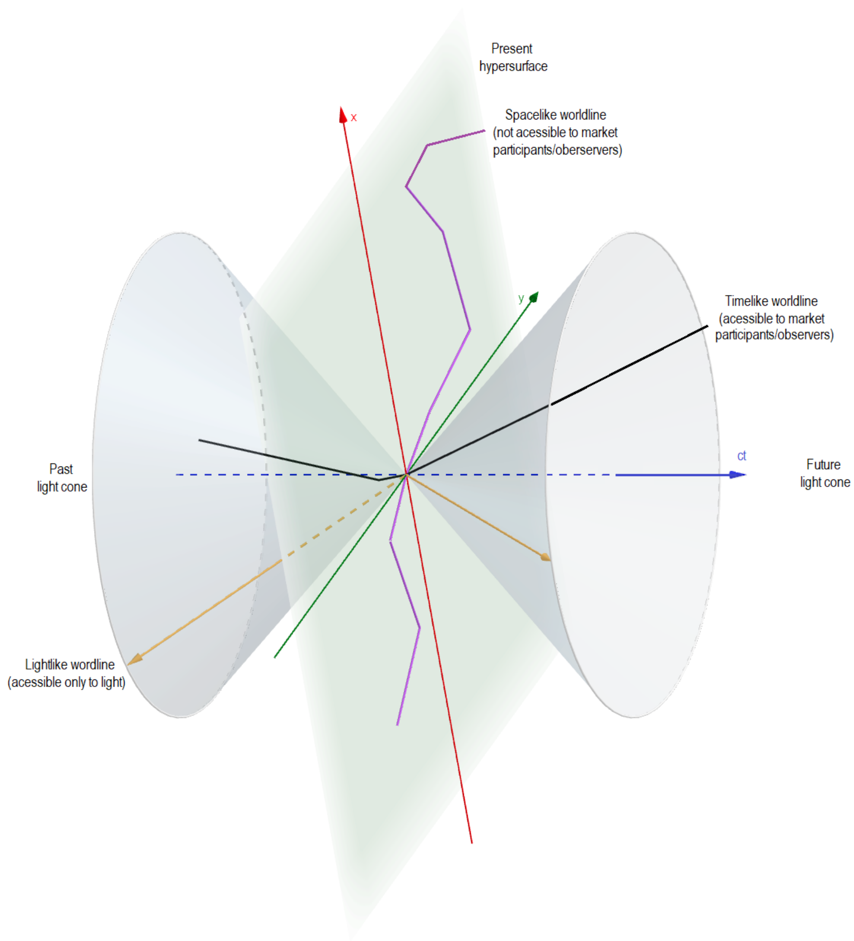

Figure 1: Spacetime diagram with past and future light cones, and timelike, lightlike and spacelike trajectories representation.

3.3.1 Space contraction and time dilation

On Figure 2, reference frames L and L' are drawn, only with ct, ct', x and x' axes, for purposes of illustration.
Green and light blue dashed lines represent simultaneity lines in L and L' reference frames, respectively. The ratio between the reference frame's relative velocity v and c, can also be defined as the $\arctan$ of the α.[8] The simultaneity line of ct_1 is constant in the L but the simultaneity line of L', represented on L, has a slope. And vice-versa, i.e. the simultaneity line ct'_1 in its own reference frame L', has no slope.

Space contraction and time dilation are implicit from the first two expressions in (1).
Let us consider the reference frame L', where an event E', starts at t'_1 and finishes at t'_2, and is stationary, so $x'_1 = x'_2 = 0$. The time interval of E' is therefore $\Delta t' = t'_2 - t'_1$. According to L reference frame, however, the event E' start and finishing moments have coordinates (ct_1, x_1) and

[8] Lorentz transformations in Equations (1) many times appear in the literature, written in hyperbolic geometric terms: $ct' = \gamma(ct - x\tanh\beta)$, $x' = \gamma(x - ct\tanh\beta)$, $y' = y$ and $z' = z$, where $\gamma = 1/\sqrt{1-\tanh^2\beta} = \cosh\beta$. The relation between α in Figure 2 and the β in these expressions is as follows: $v/c = \tan\alpha = \tanh\beta$.

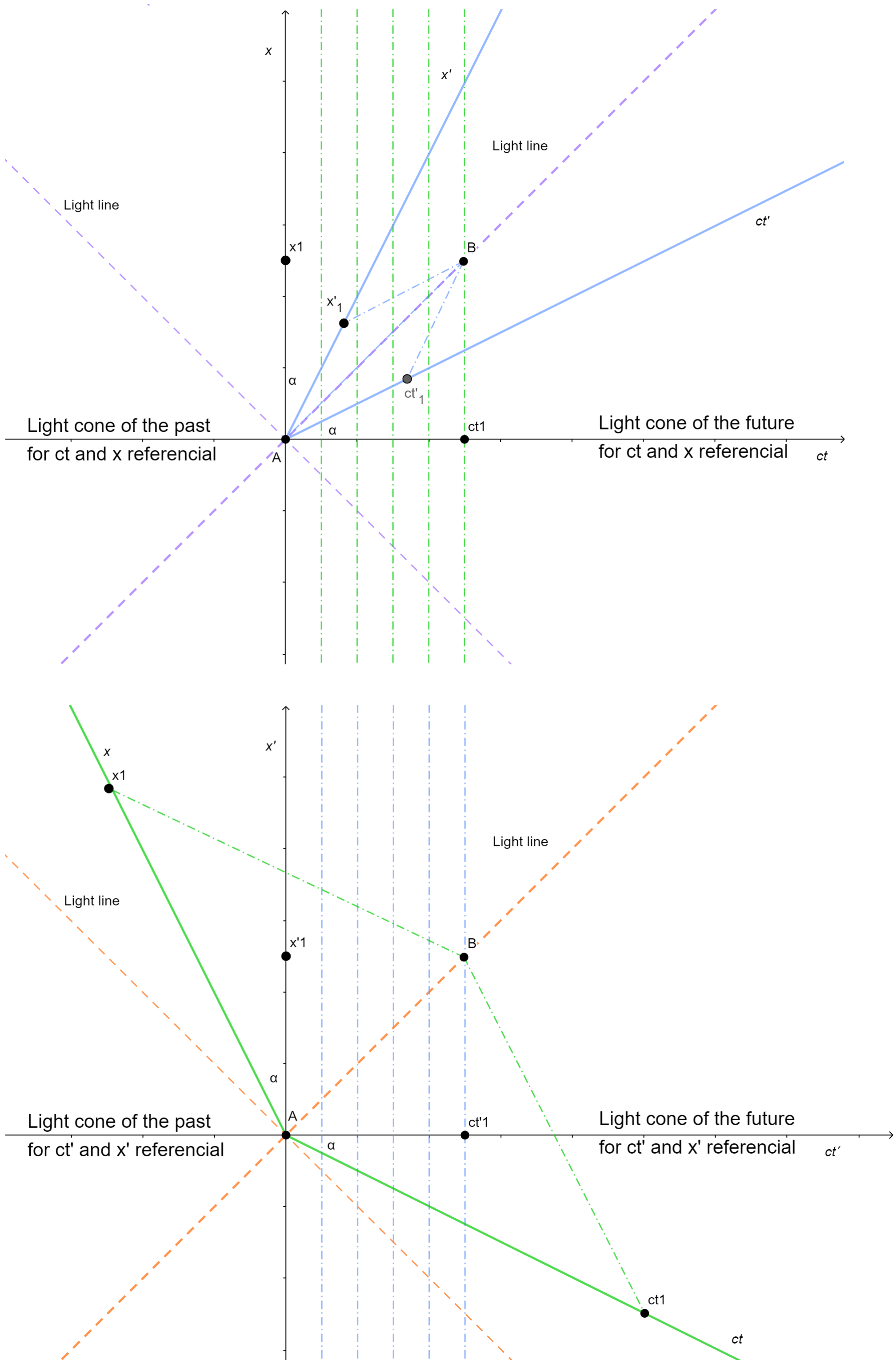

Figure 2: L (top image) and L' (bottom image) spacetime representations.

(ct_2, x_2). Since L' reference frame is moving at a constant velocity v according to L, thus the time interval in L is $\Delta t = \gamma \Delta t'$. In conclusion, the L' time interval is shorter than in L, so, time passage on L' is slower than on L so, from L' perspective, time dilates.

On the contrary, in terms of space we find a contraction. Consider now a second event $\tilde{E}'$ also taking place in L', but that it is instantaneous, i.e. $(t'_1 = t'_2 = 0)$ and has length $\Delta x' = x'_2 - x'_1$. According to L, the event $\tilde{E}'$ is, also, measured instantaneously $(t_1 = t_2 = 0)$, with its start and finishing coordinates $(0, x_1)$ and $(0, x_2)$, respectively, thus, $\Delta x = x_2 - x_1$. Since the reference frame L' is moving at a constant velocity v, according to L, we have $x_1 = x'_1/\gamma$ and $x_2 = x'_2/\gamma$. Since we have $\gamma > 1$, the length in L' is expanded, or the space in L is contracted.
Overall, in L' one experiences time dilation (time passes slowly) and a space contraction, relative to what happens in L.

Suppose, for instance, that a market participant MP$_A$ is in L and that another, MP$'_B$, is in L'. MP$_A$ at instant t_1, perceives MP$'_B$ at t'_1, that is a moment in the past of t_1. On the other hand, MP$'_B$ perceives MP$_A$ at instant t_1, already, i.e. at moment that is in the future of t_1.
So, an asset can be valued by MP$_A$ with price P_{t1} at time t_1, but since t_1 is not in the simultaneity line of L', MP$'_B$ values it differently getting $P_{t'_1}$, different from P_{t_1}. Both MP$_A$ and MP$'_B$ may be correctly pricing the asset, from the point of view of their own reference frames, which are L and L', respectively. The obtained difference in the assets price is explained by the time dilation and space contraction that MP$'_B$ really feels in is L' reference frame, relative to L. The price $P_{t'_1}$ is a past value of the asset in L. If we wish that MPs in different reference frames would trade with one another, they must agree on the "fair" asset valuations. One way to achieve this is to use what is known as proper time, instead of coordinate time.

3.3.2 Proper Time

Minkowski (1908) introduced the concepts of *proper time* that is Lorentz invariant, i.e. it is the same to all MPs, independently of their coordinates systems (Siklos, 2011).
In fact, proper time can be interpreted as temporal length (distance[9] between the event start and finishing moments), of a vector $\Delta\tau$, that measures the passage of time – e.g. lifetime, duration – of an event E, experienced by a MP.

Proper time, in L and L', respectively, are defined as

$$\Delta\tau = \sqrt{(t_f - t_i)^2 - \frac{(x_f - x_i)^2}{c^2}} \qquad \Delta\tau' = \sqrt{(t'_f - t'_i)^2 - \frac{(x'_f - x'_i)^2}{c^2}} \,, \tag{2}$$

where the subscripts i and f stand for initial and final moments of an event.

The invariant result of Lemma 3.1 follows from Equations (1). This is also visible in Figure 2 where the distance between points A and B is the same on both L and L'.

Lemma 3.1. *Given two different reference frames L and L', with associated Lorentz transformations as in Equations (1), has equal proper times. That is, for* $\Delta\tau = \sqrt{(t_f - t_i)^2 - \frac{(x_f - x_i)^2}{c^2}}$ *and* $\Delta\tau' = \sqrt{(t'_f - t'_i)^2 - \frac{(x'_f - x'_i)^2}{c^2}}$ *we have*

$$\Delta\tau = \Delta\tau' \,. \tag{3}$$

Proof. Take $\Delta t' = (t'_f - t'_i)$, $\Delta x' = (x'_f - x'_i)$. By squaring $\Delta\tau'$ in Equations (2 and multiplying by

[9] That is why in some of the literature proper time is also referred to as *proper distance* or Minkowski interval.

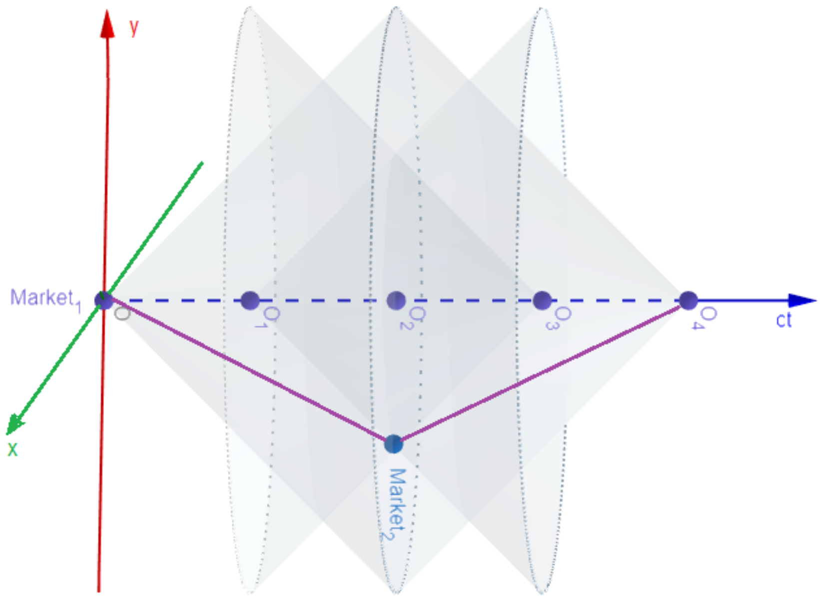

Figure 3: Two market participants case in a spacetime diagram, adapted from Siklos (2011)

c^2, we obtain the result $c^2(\Delta\tau')^2 = c^2(\Delta t')^2 - (\Delta x')^2$. From Equations (1) we get

$$\begin{aligned} c^2(\Delta\tau')^2 &= c^2\left[\gamma\left(\Delta t - v\frac{\Delta x}{c^2}\right)\right]^2 - \left[\gamma\left(\Delta x - v\Delta t\right)\right]^2 \\ &= \gamma^2\left(c^2(\Delta t)^2 - v^2(\Delta t)^2\right) + \gamma^2\left(v^2\frac{(\Delta x)^2}{c^2} - (\Delta x)^2\right) \\ &= c^2(\Delta t)^2 - (\Delta x)^2 \\ &= c^2(\Delta\tau)^2 \\ \Delta\tau' &= \Delta\tau \end{aligned}$$

■

If the vector joining events E_i and E_f is timelike, then $(\Delta\tau)^2 > 0$. These are the events accessible to us. If $(\Delta\tau)^2 = 0$ the vector is lightlike – only accessible to light speed – and when $(\Delta\tau)^2 < 0$ (implies complex numbers) the vector is spacelike – not accessible nor to us nor to light.

3.3.3 Example

Let us consider two market participants: MP$_A$ and MP$_B$ and a concrete possible trade[10].

Figure 3 illustrates the situation, from MP$_A$ and MP$_B$ perspectives. Past and future light cones for all relevant ct points are drawn. In this example, MP$_A$ is stationary in is referential frame and the time elapsed between points O and O_2 is T. The time interval between each consecutive $O_{i=1,2,3,4}$ points is T/γ.

[10] This example can be understood as an adaptation, to a financial setting, of the well-known "Twin Paradox" (Siklos, 2011).

Trade and MPs movements:

- At point $t = 0$ in Market_1, (point O), MP_A and MP_B, agree on the price of an asset[11].
- Then MP_B initiates a journey to Market_2.
- Exactly the moment when MP_B reaches Market_2, this is a simultaneous moment for MP_A. MP_A is at point O_2 and measures an elapsed time of T.
- Although from MP_B perspective, MP_A is at point O_1. So, the elapsed time measured by MP_B is $T/\gamma < T$.
- From the Market_2 perspective, the elapsed time is $T/\gamma < T$. So, when MP_B reaches Market_2, he sees the asset price for T/γ time (point O_1).
- According to MP_A, when MP_B reaches Market_2, he sees the asset price for T (point O_2).
- Now let us consider MP_B turns around and gets back to Market_1.
- In this case, from MP_B perspective, while he is turning back the reality of MP_A shifts rapidly (from point O_1 to O_3).
- Although both meet back at point O_4, in Market_1, MP_B spent $2T/\gamma$ time units, while for MP_A it took longer $2T$.
- Both MP_A and MP_B agree again on the asset price when they meet again at point O_4 (the law of one price holds)[12]. However one of them have experienced the possible gains or losses in less time than the other, which may be understood as some sort of "spacetime arbitrage".

From the above description, it follows that in the case where MPs – i.e. the buy and sell sides of a deal or regulation entities – are in different inertial reference frames. One needs to consider the spacetime structure, considering the associated Lorentz transformations and proper time.
The following axioms[13] should hold.

- **Axiom 1**: For all financial events and market participants, when different inertial reference frames are involved a settlement spacetime reference frame must be considered to serve as a benchmark.
- **Axiom 2**: When only time, incorporates the relativity effects, then, *proper time* is the time measure that makes the asset or financial instrument pricing model, invariant, to all inertial reference frames. All market participants should follow the financial event proper time – i.e. deal or asset duration – to evaluate the asset or financial instrument pricing conditions.

[11] Or other characteristic of the asset. For illustration purpose, we consider the price.
[12] For this to happen MP_A and MP_B must have different pricing models for the asset price, as they experienced different time spans between their meetings. For instance, travelling in space of MP_B, may be modelled using price jumps to account to for the time dilation experience, specially when MP_B turns back and sees MP_A passing from O_1 to O_3.
[13] Axiom 1 is a generalization of Krugman (2010) theorems to take into account different reference frames.

4 Relativistic Option Pricing

From the previous section it follows that proper time is the right concept to measure an event's lifetime, and that this quantity is invariant. So, when dealing with outer space or relativistic trading, one needs to re-define every event E – i.e. financial products, commercial deals, etc – so that all MPs, independently of their inertial reference frame, agree even if they are in different reference frames simultaneity lines.

That is, a "proper spacetime stamp" may be a required for future deals, whenever MPs need to considers different inertial frames. Haug (2004) refers to the possibility that the asset trade should register its own proper time, and that this, may be solved by implementing a spacetime stamp on each deal so that, independently of the MP times they will all agree and follow according to the assets deals spacetime stamp values.

In this section we take the case of plain vanilla at-the-money (ATM) European call options to illustrate the relativistic effects presented in the previous section.

Essentially, an European call option is a contract that confers the holder the right, but not the obligation, to purchase a certain underlying asset (e.g. a stock) for a fixed price K on a fixed expiry date T, after which the option becomes worthless.

We consider the Black and Scholes (1973) model setup, as this is one of the greatest econophysics contributions to finance, where the heat diffusion equation, widely used in physics, helped to solve the problem of finding the fair price to option contracts.

Here we focus only ATM calls, i.e. the case when at inception $t = 0$ the strike price K equals the underlying asset current price S. Without loss of generality we also take $S = K = 1$. For simplicity we also assume a zero interest rate $r = 0\%$. The fact we consider interest rates to be zero allows us to focus on time dilation effects alone (avoiding mixed times effects resulting from discounting). Under these assumptions the option price depends on two key parameters: (i) the time to maturity T and (ii) its volatility σ as it follows from Lemma 4.1.

Lemma 4.1. *Considering the Black and Scholes (1973) model on a reference frame L, with $r = 0\%$ and $S = K = 1$, the price of an at-the-money call (or put) with time to maturity T and an underlying with volatility σ is given by,*

$$Call = 2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1 \ , \tag{4}$$

where $N(\cdot)$ stands for the cumulative distribution function for the Gaussian distribution.

Proof. It follows from setting $S = K = 1$ and $r = 0\%$ in the standard Black-Scholes formula $c = SN(d_1) - Ke^{(-rT)}N(d_2)$ and realising that, under that setting we also have $d1 = -d2 = \frac{\sigma\sqrt{T}}{2}$. The result for puts follows from put-call parity when setting $S = K = 1$ and $r = 0\%$. ■

Let us consider a trade between two MPs agree on the contract/settlement reference frame, L. That is, MP_1 sells to MP_2, ATM calls for a given maturity T, at the "fair" premium in L.

Suppose, however, that after the deal is done MP_1 stays stationary in L, but MP_2 starts a journey, moving relative to MP_1. MP_2 is in a different reference frame L' and is also stationary in is L' frame.

For every day that is accounted on L – i.e. the coordinate time – less time is measured by MP_2 on L'. Recall Figure 2.

Thus, from the MP_2 perspective, the option premium paid is higher than the "fair" theoretical premium, if he had accounted for the time to maturity he/she truly experiences, $T' = T/\gamma$.

Proposition 4.2. *Under the same assumption as in Lemma 4.1, but for the perspective of the reference frame L' (as defined in Section 3), the "illusion"*[14] *price of the at-the-money call (or put) is given by,*

$$Call' = 2N\Big(\frac{\sigma}{2}\sqrt{\frac{T}{\gamma}}\Big) - 1 \tag{5}$$

where $N(\cdot)$ stands for the cumulative distribution function for the Gaussian distribution and γ is the Lorentz factor as defined in (1).

Proof. Since the settlement reference frame is L, the contracted time to maturity is T in L. However $T' = T/\gamma$ in L', as the Lorentz transformation from Equations (1) apply. The result follows from Lemma 4.1 solution, with the same assumptions, and by changing T by $T' = T/\gamma$. As before put-call-parity guarantee $c' = p'$, for $S = K = 1$ and $r = 0\%$. ■

To understand how sizable option price differences are we also define the *option price ratio*, $\frac{Call}{Call'}$, with $Call$ and $Call'$ as defined in Equations (4) and (5), respectively.

We start by analysing the option prices in L and L' and their ratio for varying maturities, assuming a constant volatility $\sigma = 15\%$.

Figure 4 shows the option prices $Call$ and $Call'$ surfaces for maturities T between 0 and 15 years, and various velocities as a percentage of the light speed constant c. On Figure 5 a surface presents the ratio. Table 1 presents concrete values for the theoretical $Call$, $Call'$ prices and the ratio $Call/Call'$ for the maturities $T = \{1/12, 3/12, 6/12, 1, 10, 15\}$ and is divided in sets of different % of c velocity $c = \{0.0\%, 12.5\%, 25.0\%, 37.5\%, 50.0\%, 62.5\%, 75.0\%, 87.5\%, 99.0\%\}$. As velocity increases so the effect of relativity in the time dilation due to the γ factor. The $Call'$ prices increase related to the settlement reference frame price $Call$. Maturities of 10 and 15 years were considered to highlight the relativistic effects.

Both from the different shape in prices surfaces in L and L', respectively on the left and right of Figure 4), and from their ratio surface (Figure 5) it is clear that the differences in prices is non-negligible. The price surface in L is insensitive to velocity changes, as its is settlement reference frame. Naturally prices of options increase with maturity. However, in terms of the reference frame L' velocity does play an important role, as expected, in particular for high maturity options.
It is clear that as velocity increases, so does the time dilation and correspondingly the ratio between the two prices on the different referential frames. It is also important to notice that the price impact is considerable, as a ratio of 1.5 means $Call$ is 50% higher than the $Call'$.

Figures 6 and 7 show time dilation effects for volatility values ranging from 1% to 30%, for a fixed $T = 1$.Table 2, concrete volatility levels $\sigma = \{1.0\%, 5.0\%, 10.0\%, 15.0\%, 20.0\%, 25.0\%, 30.0\%\}$, for the % velocities of $c = \{0.0\%, 12.5\%, 25.0\%, 37.5\%, 50.0\%, 62.5\%, 75.0\%, 87.5\%, 99.0\%\}$, presents the $Call$, $Call'$ and $Call/Call'$ ratio.

As expected time dilation effects get larger with increasing volatility. From Figure 6 it is clear from the right image that for high volatility levels (above 15%) there starts to exist significant option price differences. These effect naturally depend on the velocity at which L' departs from L, become meaningful from 25% of the speed of light c. From the left image we observe that, as expected option prices growth with volatility. The increase may seem almost linear in the image, but it is note check values in Table 2. The almost non-visible non-linearity has to do with the relative short maturity chosen, $T = 1$.

[14] Assuming only time dilation effects and not proper time.

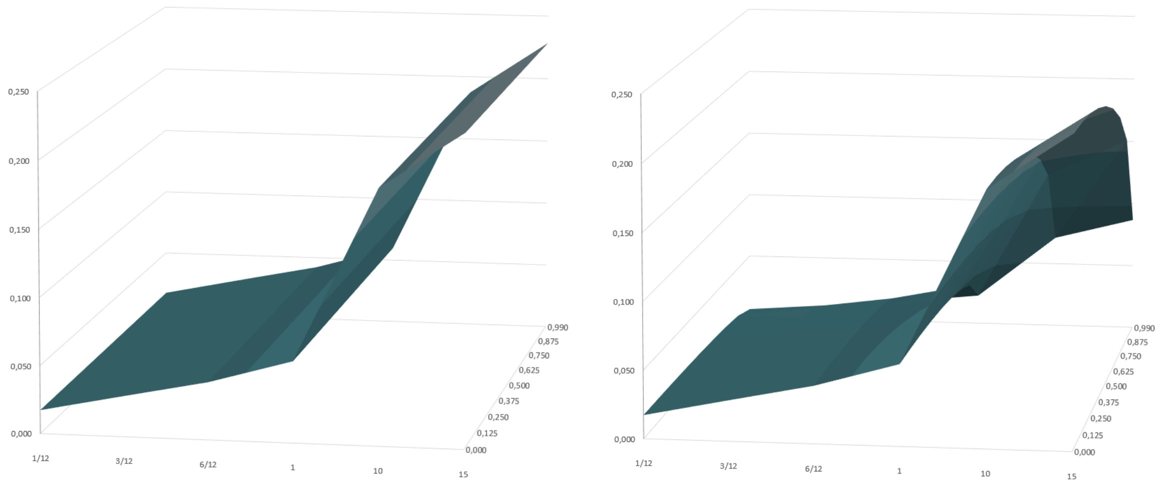

Figure 4: Surfaces of European ATM call (or put) prices (z-axis) in the reference frames L (left figure) and L' (right figure), for velocities ranging from $0.0\%c$ to $99\%c$ (y-axis), and maturities T (x-axis) of 1/12, 3/12, 6/12 and 1, 10 and 15 year. The asset volatility is fixed at $\sigma = 15\%$. For simplicity, we take $r = 0\%$ and both asset price at inception and strike equal to one $S = K = 1$.

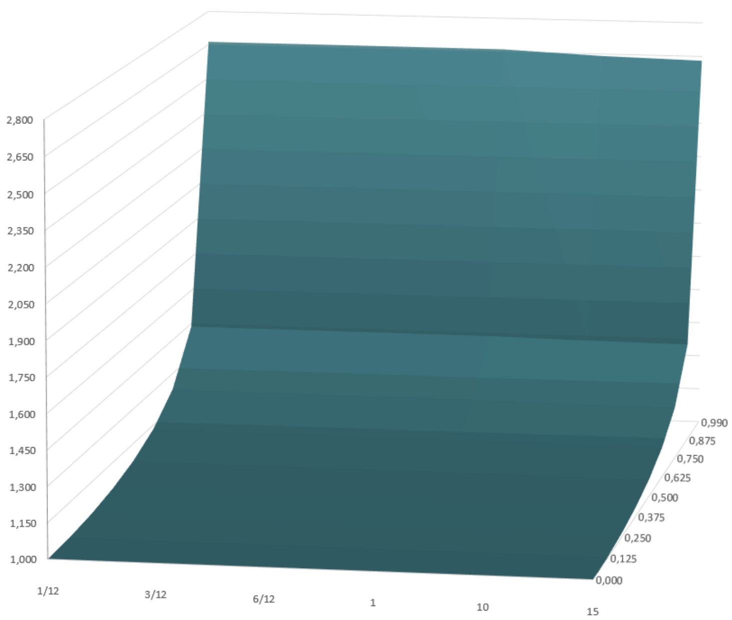

Figure 5: Surface of the ratio $Call/Call'$ (or $Put_t/Put_{t'}$), displayed on the z axis, for velocities ranging from $0.0\%c$ to $99\%c$ (y-axis), and maturities T (x-axis) of 1/12, 3/12, 6/12 and 1, 10 and 15 year. The asset volatility is fixed at $\sigma = 15\%$. For simplicity, we take $r = 0\%$ and both asset price at inception and strike equal to one $S = K = 1$.

Velocity 0.0% of c				Velocity 12.5% of c				Velocity 25.0% of c			
T	Call	Call'	Call/Call'	T	Call	Call'	Call/Call'	T	Call	Call'	Call/Call'
1/12	1,73%	1,73%	1,00000	1/12	1,73%	1,72%	1,00394	1/12	1,73%	1,70%	1,01626
3/12	2,99%	2,99%	1,00000	3/12	2,99%	2,98%	1,00394	3/12	2,99%	2,94%	1,01626
6/12	4,23%	4,23%	1,00000	6/12	4,23%	4,21%	1,00394	6/12	4,23%	4,16%	1,01625
1	5,98%	5,98%	1,00000	1	5,98%	5,96%	1,00394	1	5,98%	5,88%	1,01624
10	18,75%	18,75%	1,00000	10	18,75%	18,68%	1,00387	10	18,75%	18,45%	1,01597
15	22,85%	22,85%	1,00000	15	22,85%	22,77%	1,00384	15	22,85%	22,50%	1,01582

Velocity 37.5% of c				Velocity 50.0% of c				Velocity 62.5% of c			
T	Call	Call'	Call/Call'	T	Call	Call'	Call/Call'	T	Call	Call'	Call/Call'
1/12	1,73%	1,66%	1,03861	1/12	1,73%	1,61%	1,07456	1/12	1,73%	1,53%	1,13180
3/12	2,99%	2,88%	1,03860	3/12	2,99%	2,78%	1,07454	3/12	2,99%	2,64%	1,13177
6/12	4,23%	4,07%	1,03858	6/12	4,23%	3,94%	1,07450	6/12	4,23%	3,74%	1,13171
1	5,98%	5,76%	1,03854	1	5,98%	5,56%	1,07444	1	5,98%	5,28%	1,13159
10	18,75%	18,06%	1,03791	10	18,75%	17,47%	1,07323	10	18,75%	16,60%	1,12951
15	22,85%	22,03%	1,03756	15	22,85%	21,31%	1,07257	15	22,85%	20,25%	1,12837

Velocity 75.0% of c				Velocity 87.5% of c				Velocity 99.0% of c			
T	Call	Call'	Call/Call'	T	Call	Call'	Call/Call'	T	Call	Call'	Call/Call'
1/12	1,73%	1,40%	1,22954	1/12	1,73%	1,20%	1,43716	1/12	1,73%	0,65%	2,66230
3/12	2,99%	2,43%	1,22948	3/12	2,99%	2,08%	1,43704	3/12	2,99%	1,12%	2,66195
6/12	4,23%	3,44%	1,22938	6/12	4,23%	2,94%	1,43687	6/12	4,23%	1,59%	2,66141
1	5,98%	4,86%	1,22919	1	5,98%	4,16%	1,43652	1	5,98%	2,25%	2,66034
10	18,75%	15,30%	1,22570	10	18,75%	13,11%	1,43032	10	18,75%	7,10%	2,64122
15	22,85%	18,68%	1,22379	15	22,85%	16,02%	1,42691	15	22,85%	8,69%	2,63072

Table 1: Prices $Call$ and $Call'$, as well as the ratio $Call/Call'$ ratio, for the maturities 1/12, 3/12, 6/12, 1, 10 and 15 years and for the velocities 0.0%, 12.5%, 25.0%, 37.5%, 50.0%, 62.5%, 75.0%, 87.5% and 99.0% of c

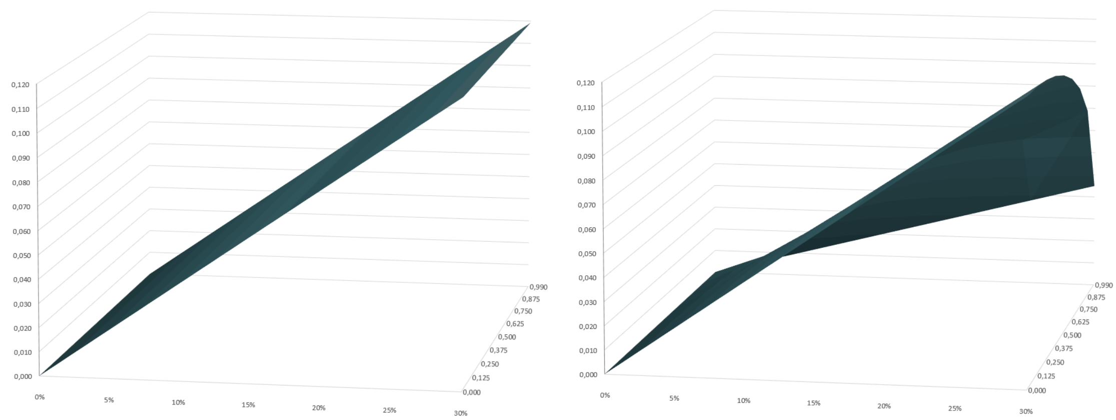

Figure 6: Surfaces of European ATM call (or put) prices (z-axis) in the reference frames L (left figure) and L' (right figure), for velocities ranging from $0.0\%c$ to $99\%c$ (y-axis), and volatility's σ (x-axis) of 1%, 5%, 10%, 15%, 20%, 25% and 30%, for maturity $T = 1$ year. For simplicity, we take $r = 0\%$ and both asset price at inception and strike equal to one $S = K = 1$.

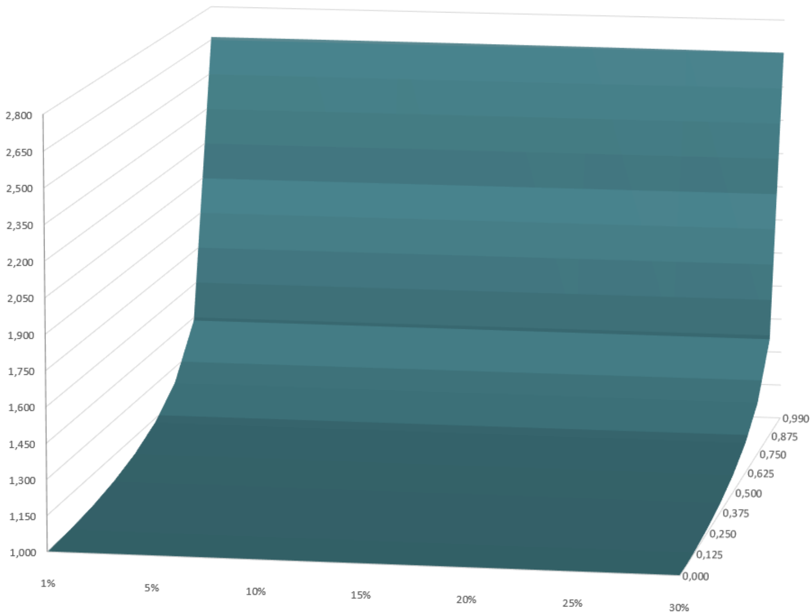

Figure 7: Surface of the ratio $Call/Call'$ (or $Put_t/Put_{t'}$), displayed on the z axis, for velocities ranging from $0.0\%c$ to $99\%c$ (y-axis), and volatility's σ (x-axis) of 1%, 5%, 10%, 15%, 20%, 25% and 30%, for maturity $T = 1$ year. For simplicity, we take $r = 0\%$ and both asset price at inception and strike equal to one $S = K = 1$.

Velocity 0.0% of c				Velocity 12.5% of c				Velocity 25.0% of c			
σ	Call	Call'	Call/Call'	σ	Call	Call'	Call/Call'	σ	Call	Call'	Call/Call'
1%	0.40%	0.40%	1.0000	1%	0.40%	0.40%	1.0039	1%	0.40%	0.39%	1.0163
5%	1.99%	1.99%	1.0000	5%	1.99%	1.99%	1.0039	5%	1.99%	1.96%	1.0163
10%	3.99%	3.99%	1.0000	10%	3.99%	3.97%	1.0039	10%	3.99%	3.92%	1.0163
15%	5.98%	5.98%	1.0000	15%	5.98%	5.96%	1.0039	15%	5.98%	5.88%	1.0162
20%	7.97%	7.97%	1.0000	20%	7.97%	7.93%	1.0039	20%	7.97%	7.84%	1.0162
25%	9.95%	9.95%	1.0000	25%	9.95%	9.91%	1.0039	25%	9.95%	9.79%	1.0162
30%	11.92%	11.92%	1.0000	30%	11.92%	11.88%	1.0039	30%	11.92%	11.73%	1.0161

Velocity 37.5% of c				Velocity 50.0% of c				Velocity 62.5% of c			
σ	Call	Call'	Call/Call'	σ	Call	Call'	Call/Call'	σ	Call	Call'	Call/Call'
1%	0.40%	0.38%	1.0386	1%	0.40%	0.37%	1.0476	1%	0.40%	0.35%	1.1318
5%	1.99%	1.92%	1.0386	5%	1.99%	1.86%	1.0746	5%	1.99%	1.76%	1.1318
10%	3.99%	3.84%	1.0386	10%	3.99%	3.71%	1.0745	10%	3.99%	3.52%	1.1317
15%	5.98%	5.76%	1.0385	15%	5.98%	5.56%	1.0744	15%	5.98%	5.28%	1.1316
20%	7.97%	7.67%	1.0385	20%	7.97%	7.41%	1.0743	20%	7.97%	7.04%	1.1314
25%	9.95%	9.58%	1.0384	25%	9.95%	9.26%	1.0742	25%	9.95%	8.79%	1.1312
30%	11.92%	11.48%	1.0383	30%	11.92%	11.10%	1.0740	30%	11.92%	10.54%	1.1309

Velocity 75.0% of c				Velocity 87.5% of c				Velocity 99.0% of c			
σ	Call	Call'	Call/Call'	σ	Call	Call'	Call/Call'	σ	Call	Call'	Call/Call'
1%	0.40%	0.28%	1.2296	1%	0.40%	0.28%	1.4372	1%	0.40%	0.15%	2.6625
5%	1.99%	1.39%	1.2295	5%	1.99%	1.39%	1.4371	5%	1.99%	0.75%	2.6622
10%	3.99%	2.78%	1.2294	10%	3.99%	2.78%	1.4369	10%	3.99%	1.50%	2.6615
15%	5.98%	4.16%	1.2292	15%	5.98%	4.16%	1.4365	15%	5.98%	2.25%	2.6603
20%	7.97%	5.55%	1.2289	20%	7.97%	5.55%	1.4360	20%	7.97%	3.00%	2.6587
25%	9.95%	6.93%	1.2285	25%	9.95%	6.93%	1.4353	25%	9.95%	3.74%	2.6565
30%	11.92%	8.31%	1.2280	30%	11.92%	8.31%	1.4344	30%	11.92%	4.49%	2.6539

Table 2: Prices $Call$ and $Call'$, as well as the ratio $Call/Call'$ ratio, for volatility's 1.0%, 5.0%, 10.0%, 15.0%, 20.0%, 25.0% and 30.0%, for maturity $T = 1$ year and for the velocities 0.0%, 12.5%, 25.0%, 37.5%, 50.0%, 62.5%, 75.0%, 87.5% and 99.0% of c

The ratio Figure 7,as expected, higher price differences the higher the velocity under consideration. Finally it matters to note that although, for each fixed velocity the ratios seem rather flat in volatility that is not the case. This is better understood by looking at the number in Table 2.

From the analysis in this section it is clear that "relativistic arbitrages" are non negligible and that whenever relativistic effect take place financial contracts should be redefined in a common time-like measure just as proper time. From our previous results, it follows.

Corollary 4.3. Under the same assumption as in Lemma 4.1, and for both the settlement reference frame L and any other reference frame L' as defined in Section 3. The fair price of an at-the-money call (or put) with time to maturity T, on the settlement reference frame L, and an underlying with volatility σ is given by,

$$Call = 2N\left(\frac{\sigma}{2}\sqrt{\Delta\tau}\right) - 1 \quad \text{where} \quad \Delta\tau = \sqrt{(\Delta t)^2 - \frac{(\Delta x)^2}{c^2}} \tag{6}$$

Although the results here presented depends upon the fact we assumed $S = K = 1$ and $r = 0\%$, both these assumptions can be easily relaxed with the appropriate straightforward to generalization of the results in Equations (4) and (5), for any S, K and r.

5 Conclusion

In conclusion, the theoretical need to incorporate relativity in finance models have been put into evidence. At the same time, there is a lack of common concepts, definitions and rules, we propose a simple market set up wit h proper time.

In fact, the introduction of relativity in finance models, without a physical time concept, to finance, has created the opportunity to develop supposedly ever better fitting models that lack economic or financial meaning, as have been the case of doubt mapping considerations.

The non-simultaneously between market participants in different inertial reference frames, due to light speed limit, brings the possibility for arbitrages opportunities and erroneous evaluations.

Proper time is the correct measure of temporal length to consider, when evaluating a financial event, in a spacetime reference frame structure.

To illustrate the above mentioned erroneous evaluations, we show time dilation effects on the prices of plain vanilla European options are significant, and particularly sizable for long maturity options on volatile underlings as velocity grows.

Finally we suggest the usage of proper time as the appropriate time measure and established the following "relativistic axioms": *(1)* For all financial events and market participants, when different inertial reference frames are involved a settlement spacetime reference frame must be considered to serve as a benchmark. *(2)* When only time, incorporates the relativity effects, then,proper time is the time measure that makes the asset or financial instrument pricing model, invariant, to all inertial reference frames. All market participants should follow the financial event proper time – i.e. deal or asset duration – to evaluate the asset or financial instrument pricing conditions.

The results here presented can be generalized to other assets, not only for inertial reference frames with accelerations and for spacetime with gravity, but also under the General Theory of Relativity, bringing the theory developments to a more real scenario.

In addition developments may be conducted in spatial arbitrages techniques, high frequency trading and performing empirical test on models, with the introduction of relativity theory.

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