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Passivity of Lotka–Volterra and quasi-polynomial systems

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Abstract
This study approaches the stability analysis and controller design of Lotka–Volterra and quasi-polynomial systems from the perspective of passivity theory. The passivity based approach requires to extend the autonomous system model with a suitable input structure. The condition of passivity for Lotka–Volterra systems is less strict than the classic asymptotic stability criterion. It is shown that each Lotka–Volterra system is feedback equivalent to a passive system and a passifying state feedback controller is proposed. The passivity based approach enables the design of novel state feedback controllers to Lotka–Volterra systems. The asymptotic stability can be achieved by applying an additional diagonal state feedback having arbitrarily small gains. This result was further explored to achieve rate disturbance attenuation in controlled Lotka–Volterra systems. By exploiting the dynamical similarities between the Lotka–Volterra and quasi-polynomial systems, it was shown that the passivity related results, developed for Lotka–Volterra systems, are also valid for a large class of quasi-polynomial systems. The methods and tools developed have been illustrated through simulation case studies.

Keywords: passivity, Lotka–Volterra system, quasi-polynomial system, disturbance attenuation, stability

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(Some figures may appear in colour only in the online journal)

1. Introduction

Lotka–Volterra systems are widely-used models to describe the dynamic behavior of interactive species or agents [1]. Their properties are continually studied by many researchers, see e.g. [2, 3]. In the paper [4] the equivalences among the different stability types of Lotka–Volterra systems were discussed.

The family of quasi-polynomial (QP) systems is regarded as a generalisation of the Lotka–Volterra form. One important property of the QP system class is that it admits a partitioning, where each class of equivalence shares the basic dynamical properties with a Lotka–Volterra model. This advantageous property is a basis of several results in the field, see e.g. [5–7].

The classical stability result developed for Lotka–Volterra systems relates the positive definiteness of a linear matrix inequality with the asymptotic stability to a positive equilibrium point of the system [8]. This is not applicable when the Lotka–Volterra system model is rank deficient, i.e. it originates from a QP system.

Passivity theory offers a possibility to overcome this difficulty, that is one of the motivations of our present work. The idea of applying passivity theory to Lotka–Volterra systems is not new. Early results related to the passivity-based control in a class of Lotka–Volterra systems can be found in [9]. Other related feedback control approaches for Lotka–Volterra systems were presented e.g. in [10–12].

Passivity is an important input–output property of many physical systems. It allows a system categorisation in terms of energy transfer between the system and its environment. Roughly speaking a system is passive if it cannot store more energy than it is supplied to it from the environment. The internal energy of the system is characterised by a non-negative, state-dependent storage function assigned to the system. In the passivity theory framework, the rate of the supplied energy is taken as the inner product of the 'power-coupled' input and output vectors of the system.

Stability analysis and control design are two important applications of the passivity theory [13]. The passivity property of input affine systems involves the stability of the autonomous part under mild conditions. The passivity can also be applied to achieve desired dynamic behavior (e.g. disturbance rejection) for the system by appropriately manipulating the system's input.

The models of many physical systems do not possess the passivity property. However, there exist such system models that can be transformed into passive systems using feedback. A system is called feedback equivalent to a passive system if it can be rendered to a passive system by performing a proper static, state-dependent, affine transformation of the original input [13].

Motivated by the above results, the contributions of this paper are as follows. Using the classical logarithmic Lyapunov function of autonomous Lotka–Volterra and QP systems we derive passivity conditions of the open version of these systems and show that each Lotka–Volterra system is feedback equivalent to a passive system. The results were extended to the class of QP systems, too. In addition, a control design method is presented to attenuate the effect of death rate or birth rate disturbances in controlled Lotka–Volterra systems.

2. Basic notions

2.1. Passive systems

In this section relevant definitions and theorems from passivity theory are reviewed, see e.g. [13, 14].

We consider a dynamic system which is modeled using a non-autonomous, input affine ordinary differential equation (ODE) in the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}), \end{aligned} \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$, $t \geqslant 0$ is state vector, $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{y}, \mathbf{u} \in \mathbb{R}^m$ are the output- and input vectors, $\mathbf{f}(\cdot), \mathbf{h}(\cdot), G(\cdot)$ are smooth mappings with appropriate dimensions, and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}(\mathbf{0}) = \mathbf{0}$ where $\mathbf{0} = (0\ldots0)^{\mathrm{T}}$ with appropriate dimension.

The condition $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$ is an equilibrium point of the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

We assign to the system (1) a continuously differentiable, nonnegative *storage function* $S(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geqslant 0}$ such that $S(\mathbf{0}) = 0$.

Definition 1. The system (1) is *passive* with respect to the storage function S if $\dot{S} \leqslant \mathbf{y}^{\mathrm{T}}\mathbf{u}, \ \forall\, \mathbf{u}, \mathbf{x}$.

Theorem 1. *The input-affine system* (1) *is passive w.r.t. S if and only if the following conditions hold:*

$$\frac{\partial S}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) \leqslant 0, \tag{2}$$

$$\frac{\partial S}{\partial \mathbf{x}}G(\mathbf{x}) = \mathbf{h}(\mathbf{x})^T. \tag{3}$$

Definition 2. The system (1) is *zero state detectable* if $\mathbf{y}(t) = \mathbf{0}$ and $\mathbf{u}(t) = \mathbf{0}$, $\forall t \geqslant 0$, imply that $\lim_{t\rightarrow\infty} \mathbf{x}(t) = \mathbf{0}$.

Theorem 2. *If the system* (1) *is zero state detectable and passive w.r.t. S, then the equilibrium state $\mathbf{x} = \mathbf{0}$ of the unforced system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is Lyapunov stable, i.e. $\forall \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that $\|\mathbf{x}_0\| \leqslant \delta(\epsilon)$ implies $\|\mathbf{x}(t)\| < \epsilon\, \forall t > 0$.*

Theorem 3. *Assume that the system* (1) *is passive w.r.t. S such that $\frac{\partial^2 S}{\partial \mathbf{x}^2}$ exists and it is continuous. If $\operatorname{rank}\{\frac{\partial \mathbf{h}}{\partial \mathbf{x}}G(\mathbf{x})\}$ is constant in a neighborhood of $\mathbf{0}$, then the system has a vector relative degree $\{1, 1, \ldots, 1\}$ at $\mathbf{x} = \mathbf{0}$.*

Theorem 4. *Consider that the system* (1) *is zero state detectable and passive w.r.t. S. Then the control law $\mathbf{u} = -K\mathbf{y}$, where $K = \operatorname{diag}(k_i) \in R^{m\times m}$, $k_i > 0$, asymptotically stabilises the equilibrium state $\mathbf{x} = \mathbf{0}$, i.e. $\lim_{t\rightarrow\infty}\|\mathbf{x}(t)\| = 0$.*

2.2. Lotka–Volterra systems

The dynamic behavior of an autonomous Lotka–Volterra system is described by the following ODE [1, 8]:

$$\dot{\mathbf{x}} = \operatorname{diag}(\mathbf{x})\,(M\mathbf{x} + \mathbf{l}), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^m_{\geqslant 0}$ is the state vector in which each entry represents a species population; the matrix $M = (m_{ij}) \in \mathbb{R}^{m\times m}$ describes the interactions among the species; $\mathbf{l} \in \mathbb{R}^m$ is the natural rate vector, $\mathbf{x}_0 \in \mathbb{R}^m_{>0}$ is the constant vector of the initial states.

Equilibrium points: a natural equilibrium point of the system (4) is $\mathbf{x}^* = \mathbf{0}$.

If $\operatorname{rank}[M\,\mathbf{l}] = \operatorname{rank} M$, the system admits other equilibrium points, which satisfy the equation

$$M\mathbf{x}^* = -\mathbf{l}. \tag{5}$$

Let us assume that the system admits a strictly positive equilibrium point $\mathbf{x}^* = (x_i^*)$, $x_i^* \in \mathbb{R}_{>0}$, $i = 1, \ldots, m$. The stability of the system (4) around the positive equilibrium can be analysed using the storage function:

$$S = \sum_{i=1}^{m} c_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*} \right), \tag{6}$$

where $c_i \in \mathbb{R}_{>0}$, $i = 1, \ldots, m$.

The time derivative of the storage function reads as [8]:

$$\dot{S} = \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^{\mathrm{T}}(MC + CM^{\mathrm{T}})(\mathbf{x} - \mathbf{x}^*). \tag{7}$$

Here $C = \operatorname{diag}(c_1 c_2 \ldots c_m)$.

Definition 3 [15]. A matrix M is Volterra–Lyapunov stable if there exist a positive definite, diagonal matrix C such that

$$MC + CM^{\mathrm{T}} < 0. \tag{8}$$

Remarks:

- The notion of Volterra–Lyapunov stability (see [15]) differs from the common Lyapunov stability definition invoked in theorem 2. The Volterra–Lyapunov matrix stability is used here to conclude on the asymptotic stability of Lotka–Volterra systems.
- The stability of Lotka–Volterra systems is independent of the offset vector $\mathbf{l}$.
- The strict inequality (8) is a sufficient asymptotic stability condition for the autonomous Lotka–Volterra system. However, if the coefficient matrix M is rank-deficient (i.e. not invertible), this condition cannot be applied.

2.3. Quasi-polynomial systems

A generalisation of the Lotka–Volterra system class is the so-called QP or generalised Lotka–Volterra model (9):

$$\dot{\mathbf{z}} = \operatorname{diag}(\mathbf{z})\,(A\mathbf{x}(\mathbf{z}) + \lambda), \quad \mathbf{z}(0) = \mathbf{z}_0. \tag{9}$$

In the model (9) $\mathbf{z} \in \mathbb{R}^m_{>0}$ denotes the state vector, $A = (a_{ij}) \in \mathbb{R}^{n\times m}$. Let $B = (b_{ij}) \in \mathbb{R}^{m\times n}$. The vector $\mathbf{x}(\mathbf{z})$ of the so-called quasi-monomials are defined as below:

$$x_j = \prod_{k=1}^{n} z_k^{B_{jk}}, \quad j = 1, \ldots, m. \tag{10}$$

It is an important property of QP systems, that the set of quasi-monomials $\mathbf{x}(\mathbf{z})$ admit a Lotka–Volterra dynamics having the following parameters [16]:

$$\begin{aligned} M &= B \cdot A \\ \mathbf{l} &= B \cdot \lambda. \end{aligned} \quad (11)$$

It is also easy to see in (9), that $B = I$ and $n = m$ yields a Lotka–Volterra model with $M = A$.

Given a QP model with its parameters (A, B, λ) the quasi-monomial transformation is defined as

$$z'_i = \prod_{k=1}^{n} z_k^{Q_{ik}}, \quad i = 1, \ldots, n, \quad (12)$$

where Q is an arbitrary invertible matrix. It is important to note, that QP models of the form (9) are form-invariant with respect to the transformation (12). The parameters of the transformed QP model are given by

$$B' = B \cdot Q, \quad A' = Q^{-1} \cdot A, \quad \lambda' = Q^{-1} \cdot \lambda. \quad (13)$$

The stability analysis of the system (9) is based on the coordinates transformation (12) and the storage (Lyapunov) function (6), [6, 17], i.e.

$$S_{\mathrm{QP}}(\mathbf{x}(\mathbf{z})) = \sum_{i=1}^{m} c_i \left(x_i(\mathbf{z}) - x_i(\mathbf{z}^*) - x_i(\mathbf{z}^*) \ln \frac{x_i(\mathbf{z})}{x_i(\mathbf{z}^*)} \right), \quad (14)$$

where $c_i \in \mathbb{R}_{>0}$, $i = 1, \ldots, m$ are the same parameters as in (6).

It also means that any QP system is dynamically similar [6] to a Lotka–Volterra model defined by the invariants (11).

Remark. In the usual case, when $m > n$, the coefficient matrix $M = BA$ of a dynamically equivalent Lotka–Volterra model of a QP model (A, B, λ) will be rank-deficient and the system dynamics evolves on a lower (n)-dimensional manifold of the m-dimensional state space. This results in difficulties in performing asymptotic stability analysis when checking the sufficient condition (8).

3. Passivity properties of Lotka–Volterra and quasi-polynomial systems

3.1. Passivity of Lotka–Volterra systems

The open Lotka–Volterra system is obtained by extending the natural rate vector with such an additive component that is set by external actions such as a feedback controller or disturbance effects:

$$\dot{\mathbf{x}} = \mathrm{diag}\,(\mathbf{x})\,(M\mathbf{x} + \mathbf{l} + \mathbf{u})\,, \quad (15)$$

where $\mathbf{u} \in \mathbb{R}^m$ denotes the input vector.

For passivity analysis define the error state:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^*, \quad (16)$$

where $\mathbf{x}^*$ is a strictly positive equilibrium point satisfying (5).

The dynamics of the error state reads as

$$\dot{\mathbf{e}} = \mathrm{diag}\,(\mathbf{e} + \mathbf{x}^*)\,(M\mathbf{e} + \mathbf{u}), \quad (17)$$

It can be seen that (17) has an input affine form $\dot{\mathbf{e}} = \mathbf{f}(\mathbf{e}) + G(\mathbf{e})\mathbf{u}$, where $\mathbf{f}(\mathbf{e}) = \text{diag}\,(\mathbf{e} + \mathbf{x}^*)M\mathbf{e}$ and $G(\mathbf{e}) = \text{diag}\,(\mathbf{e} + \mathbf{x}^*)$.

The storage function (6) as a function of $\mathbf{e}$ has the form

$$S(\mathbf{e}) = \sum_{i=1}^{m} c_i \left(e_i - x_i^* \ln \frac{e_i + x_i^*}{x_i^*} \right). \tag{18}$$

The gradient of S satisfies

$$\frac{\partial S(\mathbf{e})}{\partial \mathbf{e}} = \left(\frac{c_1 e_1}{e_1 + x_1^*} \cdots \frac{c_m e_m}{e_m + x_m^*} \right). \tag{19}$$

The Hessian of S has the form

$$\frac{\partial S^2(\mathbf{e})}{\partial \mathbf{e}^2} = \text{diag} \left(\frac{c_1 x_1^*}{(e_1 + x_1^*)^2} \cdots \frac{c_m x_m^*}{(e_m + x_m^*)^2} \right). \tag{20}$$

If $\mathbf{e} = \mathbf{0}$, then $\mathbf{f}(\mathbf{e}) = \mathbf{0}$, $S(\mathbf{0}) = 0$ and $\frac{\partial S(\mathbf{e})}{\partial \mathbf{e}}^{\mathrm{T}} = \mathbf{0}$.

By equation (7), if there exists a positive definite, diagonal C such that

$$MC + CM^{\mathrm{T}} \leqslant 0, \tag{21}$$

then the storage function (18) of the system (17) with $\mathbf{u} = \mathbf{0}$ is non-increasing.

If there exists a positive definite, diagonal C such that (21) *holds, then the open Lotka–Volterra system* (17) *possesses the following passivity-related properties*:

- By theorem 1 the open Lotka–Volterra model (17) is passive from the input $\mathbf{u}$ to the (artificial) *passive output*

$$\mathbf{y} = \text{diag}\,(\mathbf{e} + \mathbf{x}^*) \frac{\partial S(\mathbf{e})}{\partial \mathbf{e}}^{\mathrm{T}} = C\mathbf{e}. \tag{22}$$

  The condition (2) of theorem 1 directly yields from (21). The passive output (22) was computed based on equation (3).
- Since $x_i^* \in \mathbb{R}_{>0}\ \forall\, i = 1, \ldots, m$, the rank of the Hessian (20) in a neighborhood of $\mathbf{e} = \mathbf{0}$ is m. Hence, by theorem 3, the Lotka–Volterra system (17) has no internal dynamics in the neighborhood of $\mathbf{e} = \mathbf{0}$.
- The conditions $\mathbf{u} = \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$ imply that $\mathbf{e} = \mathbf{0}$. Hence (17) is zero state detectable and, by theorem 4, the control $\mathbf{u} = -K\mathbf{y}$ asymptotically stabilises the equilibrium state $\mathbf{x}^*$. The matrix K is defined in the statement of theorem 4.

The asymptotic Lotka–Volterra stability of passive Lotka–Volterra systems with diagonal output feedback ($\mathbf{u} = -KC\mathbf{e}$) directly follows from the inequality (21). The interaction matrix of the controlled system is $\mathcal{M} = M - KC$. The stability condition (8) has the form

$$\begin{aligned} &\mathcal{M}C + C\mathcal{M}^{\mathrm{T}} < 0, \\ &MC + CM^{\mathrm{T}} - 2\,KC^2 < 0. \end{aligned} \tag{23}$$

Since KC^2 is negative definite, the asymptotic stability directly yields. It was exploited that the sum of a negative definite matrix and a negative semidefinite matrix having the same dimensions is a negative definite matrix.

Note, that the full rank of matrix M is not necessary for passivity. On the other hand, the passivity of the dynamics (15) implies the stability of its autonomous part.

3.2. Special Lotka–Volterra systems

As follows, the passivity properties of two important classes of Lotka–Volterra systems are analysed.

Example 1 (Lotka–Volterra predator–prey model). Consider a two-dimensional open Lotka–Volterra system which describes the dynamics of a coexisting predator and prey population:

$$\begin{aligned}\dot{x}_1 &= x_1(l_1 - m_{12}x_2 + u_1)\\ \dot{x}_2 &= x_2(-l_2 + m_{21}x_1 + u_2),\end{aligned} \tag{24}$$

where $l_1, l_2 \in \mathbb{R}_{>0}$ represent the constant prey birth rate and predator death rate respectively. The coefficients $m_{12}, m_{21} \in \mathbb{R}_{>0}$ determine the inter-influence of the predator–prey population. The entries of $\mathbf{u} = (u_1\ u_2)^{\mathrm{T}} \in \mathbb{R}^2$ are the input rates.

The autonomous system ($\mathbf{u} = \mathbf{0}$) always admits a positive equilibrium point $\mathbf{x}^* = (l_2/m_{21}\ l_1/m_{12})^{\mathrm{T}}$.

The interconnection matrix of the system is

$$M = \begin{pmatrix} 0 & -m_{12} \\ m_{21} & 0 \end{pmatrix}. \tag{25}$$

The matrix inequality (21) with $C = \mathrm{diag}\,(c_1 \sim c_2)$ has the form

$$\begin{pmatrix} 0 & m_{21}c_2 - m_{12}c_1 \\ m_{21}c_2 - m_{12}c_1 & 0 \end{pmatrix} \leqslant 0 \tag{26}$$

The passivity of the system yields e.g. by choosing $C = \mathrm{diag}(c \sim \frac{m_{12}}{m_{21}}c)$, $c \in \mathbb{R}_{>0}$.

Example 2 (Competitive Lotka–Volterra model). In this case the two-dimensional Lotka–Volterra model reads as

$$\begin{aligned}\dot{x}_1 &= x_1\,(l_1 - m_{11}x_1 - m_{12}x_2 + u_1)\\ \dot{x}_2 &= x_2\,(l_2 - m_{21}x_1 - m_{22}x_2 + u_2),\end{aligned} \tag{27}$$

where $l_1, l_2 \in R_{>0}$ are constant linear growth rates, $m_{11}, m_{22} \in R_{>0}$ are the intra-species competition rates, $m_{12}, m_{21} \in R_{>0}$ represent the inter-species competition rates.

It is known [18] that the autonomous part of the system has a positive equilibrium point

$$\mathbf{x}^* = \begin{pmatrix} \dfrac{l_2m_{11} - l_1m_{21}}{m_{11}m_{22} - m_{12}m_{21}} & \dfrac{l_1m_{22} - l_2m_{12}}{m_{11}m_{22} - m_{12}m_{21}} \end{pmatrix}^{\mathrm{T}} \tag{28}$$

if $l_2m_{11} > l_1m_{21}$, $l_1m_{22} > l_2m_{12}$ and $m_{11}m_{22} > m_{12}m_{21}$.

In this example the passivity condition (21) with $C = \mathrm{diag}\,c_1c_2$ takes the form

$$\begin{pmatrix} 2m_{11}c_1 & m_{21}c_1 + m_{12}c_2 \\ m_{21}c_1 + m_{12}c_2 & 2m_{22}c_2 \end{pmatrix} \geqslant 0. \tag{29}$$

The system is passive if $\exists c_1, c_2 \in \mathbb{R}_{>0}$ such that $4m_{11}m_{22}c_1c_2 \geqslant (m_{21}c_1 + m_{12}c_2)^2$. If $2m_{11}m_{22} < m_{21}m_{12}$, then the inequality does not hold. If $4m_{11}m_{22} \geqslant (m_{21} + m_{12})^2$, the inequality holds for $c_1 = c_2 = 1$.

Hence the passivity condition of the competitive Lotka–Volterra system related to the error state is *model parameter dependent*. As will be shown in section 4, in such cases the passivity can be assured by state feedback control.

3.3. Passivity of quasi-polynomial systems

The generalisation of the passivity-related results for Lotka–Volterra systems to the QP case is based on the joint storage function family of the two system classes.

Suppose a QP system with an additive general input:

$$\dot{\mathbf{z}} = \mathrm{diag}\,(\mathbf{z})\left(A\mathbf{x}(\mathbf{z}) + \lambda + \mathbf{u}_I^{\mathrm{QP}}\right), \tag{30}$$

where $\mathbf{u}_I^{\mathrm{QP}} = B^{+}\mathbf{v}$, and B^{+} denotes the pseudo-inverse of matrix B. The passivity of the QP system (30) can be investigated in the monimial space, i.e. through the Lotka–Volterra variables, by analysing the Lotka–Volterra model that corresponds to (30). It is easy to see that the Lotka–Volterra model of (30) is in the form (15). The error state of the QP model is then defined using the Lotka–Volterra states [i.e. the QP quasimonomials of the QP system (30)]. This way, the passivity of the QP model can be handled using the above results on the passivity of Lotka–Volterra models and on the fact, that QP and Lotka–Voltera models share their storage functions (6) and (14).

4. Feedback equivalence of Lotka–Volterra systems to passive systems

In the case of a Lotka–Volterra system that is not passive or it is passive only for some set of its parameters, one can use the notion of feedback equivalence to ensure and investigate the passivity.

Definition 4 [13]. The system (1) is feedback equivalent to a passive system if there exists an input in the form $\mathbf{u} = \alpha(\mathbf{x}) + \beta(\mathbf{x})\mathbf{u}_p$, $\alpha(\mathbf{x}) \in \mathbb{R}^m$, $\beta(\mathbf{x}) \in \mathbb{R}^{m\times m}$ such that the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\alpha(\mathbf{x}) + G(\mathbf{x})\beta(\mathbf{x})\mathbf{u}_p, \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned} \tag{31}$$

is passive.

The notion of feedback equivalence introduces a class of input-affine systems, that are not necessarily passive, but they can be made passive by applying suitable state feedback to them. As follows, we show that an open Lotka–Volterra system is feedback equivalent to a passive system, and we construct the feedback that makes it passive.

Consider an open Lotka–Volterra system in which the state dynamics is defined by (17) and the output is given by (22).

Passivity with skew-symmetric interaction matrix: consider that the interaction matrix of the Lotka–Volterra system $M = (m_{ij})$ is skew-symmetric, i.e. $m_{ij} = -m_{ji}$, $\forall i \neq j$.

In this case the matrix inequality (21) has the form:

$$MC + CM^{\mathrm{T}} = MC - CM \leqslant 0. \tag{32}$$

The inequality (32) with skew-symmetric interaction matrix directly yields e.g. by choosing $C = I_m$, where $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix, i.e. the skew-symmetry property of the interaction matrix implies the passivity of the open Lotka–Volterra system.

In the general case, the skew-symmetry property of the interconnection matrix of the controlled system can be assured by choosing

$$\mathbf{u} = K_p\mathbf{e} + \mathbf{u}_p, K_p \in \mathbb{R}^{m\times m}. \tag{33}$$

With this input the interconnection matrix M_p of the system yields as

$$M_p = M + K_p. \tag{34}$$

A possible choice for K_p which guarantees that M_p is skew symmetric is

$$K_p(i,j) = \begin{cases} -m_{ji} - m_{ij}, & \text{if } |m_{ij}| \geqslant |m_{ji}| \text{ and } i \neq j \\ -m_{ij}, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

Accordingly, the Lotka–Volterra system is equivalent to a passive system where the input is defined in equations (33) and (35).

It is important to note, that applicability of the LMI sufficient condition (32) can be extended by means of time-reparameterization transformation [19]. This method includes extra parameters into the Lotka–Volterra coefficient matrix M in a bilinear way and turns the LMI (32) into a *bilinear matrix inequality*, which belongs to an $\mathcal{NP}$-complete problem class. Although it is computationally more demanding, in some cases the time reparameterization yields result even when the LMI (32) is infeasible. This method is not necessary in the present case.

5. Disturbance attenuation

Passivity theory opens the possibility to design powerful controllers not only for stabilising but also for disturbance rejection purposes [20]. This section proposes a physically meaningful yet realisable controller for disturbance attenuation of Lotka–Volterra systems.

5.1. The control problem

In realistic Lotka–Volterra systems the natural rate cannot be considered constant, its nominal value may change in time. The deviation of the rate vector from its nominal value can be viewed as an additive, non-constant, bounded disturbance.

It is also considered that the rate vector has a component that can be externally manipulated by a control mechanism.

Let an open Lotka–Volterra system both with control- and disturbance inputs in the form

$$\dot{\mathbf{x}} = \operatorname{diag}(\mathbf{x})\left(M\mathbf{x} + \mathbf{l} + \mathbf{u}_\mathrm{c} + \mathbf{w}\right), \tag{36}$$

where $\mathbf{u}_\mathrm{c} \in \mathbb{R}^n$ is the control input rate vector, $\mathbf{w} \in \mathbb{R}^n$ represents the unknown disturbance rate vector.

Control objective: let $\mathbf{x}_\mathrm{SP} \in \mathbb{R}^n_{>0}$ a prescribed setpoint for the system (36) and

$$\mathbf{y} = C\mathbf{e} = C(\mathbf{x} - \mathbf{x}_\mathrm{SP}), \tag{37}$$

where $C = \operatorname{diag}(c_i) \in \mathbb{R}^{m\times m}$, $c_i \in \mathbb{R}_{>0}$. Design a control input $\mathbf{u}_\mathrm{c}$, which assures that $\lim_{t\to\infty}\mathbf{y} = \mathbf{0}$ if $\mathbf{w} = \mathbf{0}$. Otherwise, ensure that

$$\int_0^t \mathbf{y}^\mathrm{T}\mathbf{y}\,\mathrm{d}\tau \leqslant \gamma \int_0^t \mathbf{w}^\mathrm{T}\mathbf{w}\,\mathrm{d}\tau + \sigma_0 \tag{38}$$

for a *prescribed* $\gamma \in \mathbb{R}_{>0}$. $\sigma_0 \in \mathbb{R}$ is an initial condition-dependent constant.

To achieve the control objective, formulate the control input as

$$\mathbf{u}_\mathrm{c} = K_p\mathbf{e} - K\mathbf{y} + \mathbf{u}_\mathrm{ff}, \tag{39}$$

where $\mathbf{y}$ is given in (37), K_p is given by (35), $K \in R^{m\times m}$ and $\mathbf{u}_\mathrm{ff} \in R^m$ is a constant feed-forward control term.

With such a feedback design, the model of the controlled Lotka–Volterra system has the form

$$\dot{\mathbf{e}} = \mathrm{diag}\,(\mathbf{e} + \mathbf{x}_\mathrm{SP})\left(M_p\mathbf{e} + \mathbf{l} + M\mathbf{x}_\mathrm{SP} - K\mathbf{y} + \mathbf{u}_\mathrm{ff} + \mathbf{w}\right), \tag{40}$$

where M_p is given by (34).

5.2. Equilibrium point shift

The feed-forward term is meant to ensure that the original positive equilibrium point of the system is shifted into the setpoint by additively modifying the natural rate vector. It is formulated as

$$\mathbf{u}_\mathrm{ff} = -\mathbf{l} - M\mathbf{x}_\mathrm{SP}. \tag{41}$$

With this choice the controlled Lotka–Volterra system takes the form

$$\dot{\mathbf{e}} = \mathrm{diag}\,(\mathbf{e} + \mathbf{x}_\mathrm{SP})\left(M_p\mathbf{e} - K\mathbf{y} + \mathbf{w}\right). \tag{42}$$

As it was presented in section 3, the control guarantees the objective $\lim_{t\to\infty}\mathbf{y} = \mathbf{0}$ provided that $\mathbf{w} = \mathbf{0}$.

5.3. Disturbance attenuation

Choose the feedback gain matrix K as a diagonal matrix with positive entries: $K = \mathrm{diag}(k_i), \quad k_i \in \mathbb{R}_{>0}$.

If disturbances are present in the control system ($\mathbf{w} \neq \mathbf{0}$), due to the passivity property, the time-derivative of the storage function (18) of the controlled system (42) satisfies

$$\dot{S} \leqslant \mathbf{y}^\mathrm{T}\left(-K\mathbf{y} + \mathbf{w}\right), \tag{43}$$

$$\int_0^t \mathbf{y}^\mathrm{T}K\mathbf{y}\mathrm{d}\tau \leqslant \int_0^t \mathbf{y}^\mathrm{T}\mathbf{w}\mathrm{d}\tau + S(0). \tag{44}$$

Let $k = \min_i\{(k_i)\}$. Since $k\int_0^t \mathbf{y}^\mathrm{T}\mathbf{y}\mathrm{d}\tau \leqslant \int_0^t \mathbf{y}^\mathrm{T}K\mathbf{y}\mathrm{d}\tau$ and $\mathbf{y}^\mathrm{T}\mathbf{w} \leqslant \frac{1}{2}\left(\mathbf{y}^\mathrm{T}\mathbf{y} + \mathbf{w}^\mathrm{T}\mathbf{w}\right)$ it follows that

$$2k\int_0^t \mathbf{y}^\mathrm{T}\mathbf{y}\mathrm{d}\tau \leqslant \int_0^t \mathbf{y}^\mathrm{T}\mathbf{y}\mathrm{d}\tau + \int_0^t \mathbf{w}^\mathrm{T}\mathbf{w}\mathrm{d}\tau + 2S(0), \tag{45}$$

$$\int_0^t \mathbf{y}^\mathrm{T}\mathbf{y}\mathrm{d}\tau \leqslant \frac{1}{2k-1}\int_0^t \mathbf{w}^\mathrm{T}\mathbf{w}\mathrm{d}\tau + \frac{2}{2k-1}S(0). \tag{46}$$

Let the prescribed attenuation gain be γ. It yields that, if the controller gain matrix is chosen such that

$$k > \frac{1}{2}\left(1 + \frac{1}{\gamma}\right), \tag{47}$$

then the disturbance attenuation control objective is achieved.

Remark. The controlled system (40) preserves the structure of the open Lotka–Volterra model (17) where $\mathbf{x}^* = \mathbf{x}_{SP}$, the interaction matrix is $M_p - KC$ and the input vector is $\mathbf{w}$.

5.4. Generalisation to quasi-polynomial systems

Due to the close connection between the two system classes, the results of this Section can be generalised to QP systems.

Suppose an open QP system of the form (48) below:

$$\dot{\mathbf{z}} = \mathrm{diag}\,(\mathbf{z})\,(A\mathbf{x}(\mathbf{z}) + \lambda + u)\,, \tag{48}$$

where

$$u = B^+ \left(K_p\mathbf{x}(\mathbf{z}) - KC\left(\mathbf{x}(\mathbf{z}) - \mathbf{x}(\mathbf{z}_{SS}) + \mathbf{u}_{\mathrm{ff}}^{\mathrm{QP}}\right)\right) + \mathbf{w} \tag{49}$$

is the input applied to the system, where $\mathbf{u}_{\mathrm{ff}}^{\mathrm{QP}} = -\lambda - (A + B^+K_p)\mathbf{x}(\mathbf{z}_{\mathrm{ss}})$. It is easy to see, that the corresponding Lotka–Volterra dynamics of the closed loop QP system (48) and (49) is in the form (40) with a rescaled disturbance $B\mathbf{w}$:

$$\dot{\mathbf{x}} = \mathrm{diag}\,(\mathbf{x})\left(M_p\mathbf{x} + \mathbf{l} - K\mathbf{y} + \mathbf{u}_{\mathrm{ff}} + B\mathbf{w}\right). \tag{50}$$

Using the feedback form (49) the proposed method of passivity based control and setpoint design can be directly applied to a subset QP systems for which the exponent matrix B is invertible.

By using B^+, i.e. the pseudo-inverse of the exponent matrix in (49) instead of the inverse, the method can also be applied to QP systems for which $m \neq n$. However, it is important to note, that this generalisation works only if $BB^+ = I$ [21].

The disturbance rejection properties of the controller extended to QP systems can be derived in the same manner as in section 5.3.

6. Case studies

The performances of the developed passivity based controller design methods are illustrated in this section using simulated case studies.

6.1. Lotka–Volterra model

A three-dimensional Lotka–Volterra predator–prey system is considered which describes the behavior of three coexisting species: two predators and one prey. The dynamics of this system reads as

$$\begin{aligned}
\dot{x}_1 &= x_1\,(-m_{12}x_2 - m_{13}x_3 + (l_1 + u_1 + w_1))\\
\dot{x}_2 &= x_2\,(m_{21}x_1 + (-l_2 + u_2 + w_2))\\
\dot{x}_3 &= x_3\,(m_{31}x_1 + (-l_3 + u_3 + w_3))\,,
\end{aligned} \tag{51}$$

where $m_{ij}, l_i \in R_{>0}, i, j = 1, 2, 3$.

The system with $\mathbf{u}_c = \mathbf{w} = \mathbf{0}$ admits non-zero equilibrium points for all the three populations only if $m_{21}/l_2 = m_{31}/l_3$, which is an eminently restrictive condition. Otherwise, one

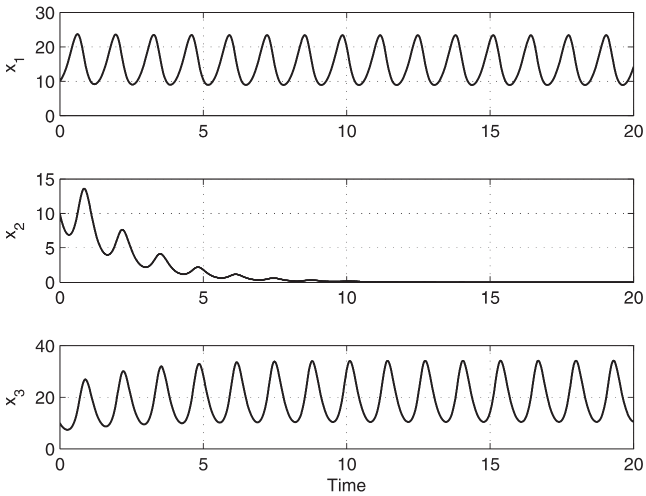

Figure 1. State trajectories of the Lotka–Volterra system without control (section 6.1).

of the predator species dies out [22] and the dynamics of the system degenerates to a two-dimensional Lotka–Volterra system, see example 1. However, using feedback control, the non-zero equilibrium for all the three species' populations can be assured.

The interaction matrix M of the system is

$$M = \begin{pmatrix} 0 & -m_{12} & -m_{13} \\ m_{21} & 0 & 0 \\ m_{31} & 0 & 0 \end{pmatrix}. \tag{52}$$

The rank of M is 2. The equilibrium states of the predators' populations always satisfy $m_{12}x_2^* + m_{13}x_3^* = l_1$.

During the simulation experiments the following parameters were chosen: $m_{12} = 0.1$, $m_{13} = 0.2$, $m_{21} = 0.3$, $m_{31} = 0.4$ and $l_1 = 4$, $l_2 = 5$, $l_3 = 6$.

Figures 1 and 2 show the dynamic response of the Lotka–Volterra system with no input, i.e. $\mathbf{u}_c = \mathbf{w} = \mathbf{0}$, and with initial values $x_1(0) = x_2(0) = x_3(0) = 10$. Without control, the extinction of one of the predator populations can be observed. The trajectories of the prey population and the surviving predator population converge to the trajectories of a typical two-dimensional predator–prey system having concordant parameters.

During the control experiments, the disturbance was modeled as increased death rate in the case of the predators and decreased death rate in the case of the prey. The entries of the disturbance vector $\mathbf{w}$ were set as: $w_1 = w_2 = w_3 = -0.5$.

The setpoint was chosen as: $\mathbf{x}_{SP} = (20\,20\,20)^T$.

The matrix K_P in the control law (33) was computed using equation (35). The control was tested for three prescribed disturbance attenuation gain values: $\gamma = 10$, $\gamma = 1$ and $\gamma = 0.1$. The diagonal control gain matrix was chosen as $K = \text{diag}\,(k_i)$, where $k_i = \gamma, i = 1, 2, 3$.

The simulation results presented in figures 3 and 4 show that the proposed disturbance attenuation approach ensures the convergence of the controlled states to the setpoint. Smaller prescribed disturbance attenuation level ensures smaller steady-state error.

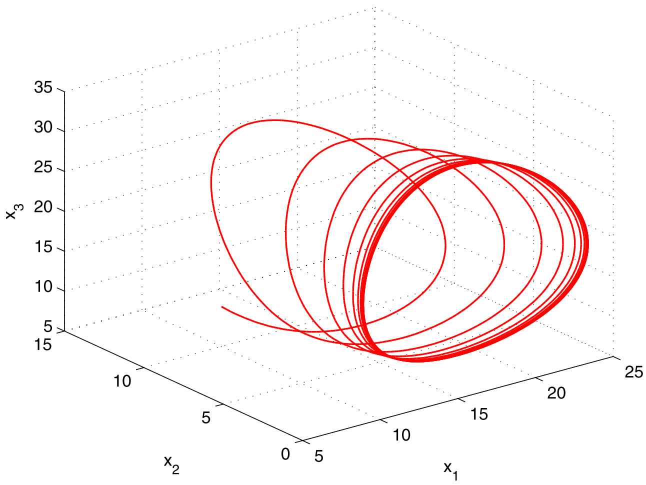

Figure 2. Phase portrait of the Lotka–Volterra system without control (section 6.1).

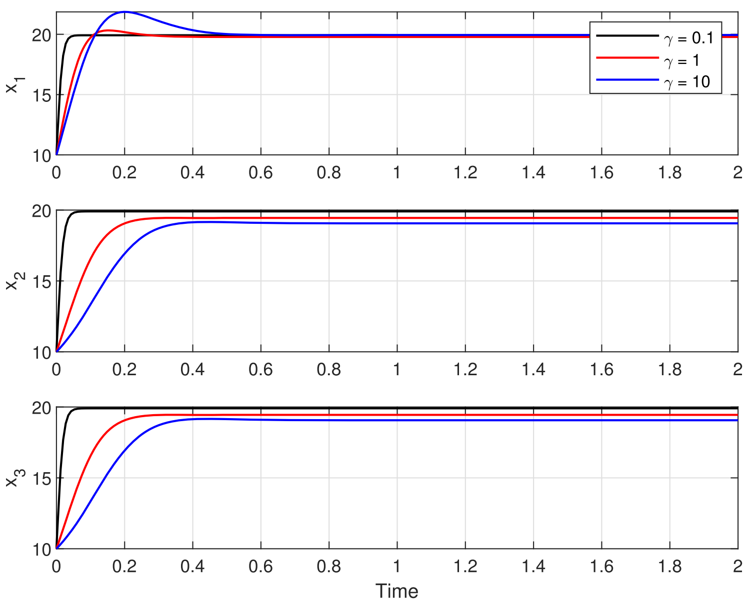

Figure 3. State trajectories of the Lotka–Volterra system with control (section 6.1).

6.2. Quasi-polynomial model with $m = n$

The examined QP system belongs to the same equivalence class as the Lotka–Volterra model (51) of the previous example. For the sake of simplicity, the numerical parameter values of section 6.1 has been used here. The ODE form of the QP system is given in (53) below.

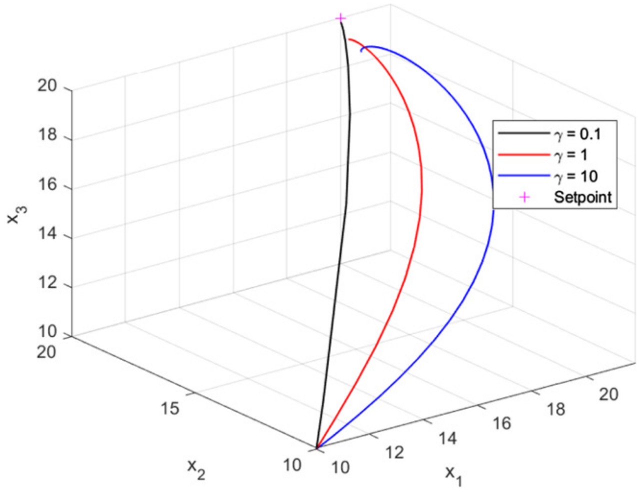

Figure 4. Phase portrait of the Lotka–Volterra system with control (section 6.1).

$$\begin{aligned}
\dot{z}_1 &= z_1 \left(-0.1090 z_1 z_2^{\frac{1}{2}} - 0.1090 z_2 z_3^{\frac{1}{4}} - 0.2180 z_1^{\frac{1}{3}} z_3 + 5.9946 \right. \\
&\quad \left. + 1.0899(u_1 + w_1) - 0.2725(u_3 + w_3) \right) \\
\dot{z}_2 &= z_2 \left(0.3545 z_1 z_2^{\frac{1}{2}} + 0.0545 z_2 z_3^{\frac{1}{4}} + 0.1090 z_1^{\frac{1}{3}} z_3 - 7.9973 \right. \\
&\quad \left. - 0.5450(u_1 + w_1) + (u_2 + w_2) + 0.1362(u_3 + w_3) \right) \\
\dot{z}_3 &= z_3 \left(0.4360 z_1 z_2^{\frac{1}{2}} + 0.0360 z_2 z_3^{\frac{1}{4}} + 0.0719 z_1^{\frac{1}{3}} z_3 - 7.9782 \right. \\
&\quad \left. - 0.3597(u_1 + w_1) + 1.0899(u_3 + w_3) \right).
\end{aligned} \tag{53}$$

It can be seen, that the state space of the QP system (53) is three dimensional, i.e. $m = n$. The QP exponent matrix B_1 that describes the monomial structure is

$$B_1 = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}, \tag{54}$$

the coefficient matrix has the value

$$A_1 = \begin{pmatrix} -0.1090 & -0.1090 & -0.2180 \\ 0.3545 & 0.0545 & 0.1090 \\ 0.4360 & 0.0360 & 0.0719 \end{pmatrix}. \tag{55}$$

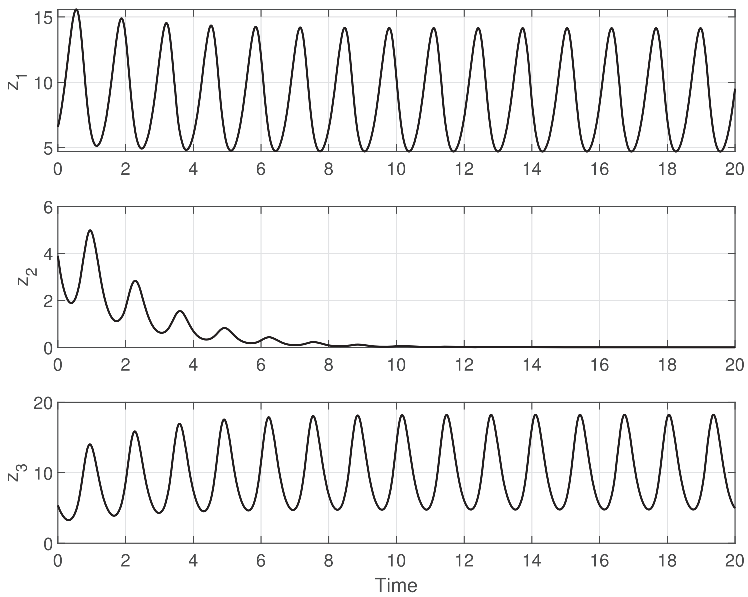

Figure 5. State trajectories of the QP system without control (section 6.2).

It can be easily checked, that $M = B_1 \cdot A_1$, i.e. the QP system (53) belongs to the same class of equivalence as the Lotka–Volterra model (51). This also means, that they are dynamically similar. The monomials of the QP system are $z_1 z_2^{\frac{1}{2}},\ z_2 z_3^{\frac{1}{4}},\ z_1^{\frac{1}{3}} z_3$, according to the rows of matrix B_1. These quasimonomials correspond to the Lotka–Volterra system (51)'s state variables x_1, x_2 and x_3, respectively. Figures 5 and 6 show the dynamic response of the QP system with no control input, i.e. $\mathbf{u}_c = \mathbf{w} = \mathbf{0}$. It is important to note, that the state trajectory (figure 6) evolves on a two dimensional manifold, just like in the Lotka–Volterra case, since the rank of the Lotka–Volterra coefficient matrix is two.

The setpoint was determined from the Lotka–Volterra case using the quasi-monomial transformation corresponding to the QP system, its value is $\mathbf{z}_{SP} = (11.5748 \quad 5.8786 \quad 8.9140)^T$. The simulation results presented in figures 7 and 8 show that the proposed disturbance attenuation approach ensures the convergence of the controlled states to the setpoint. Similarly to the Lotka–Volterra case, smaller prescribed disturbance attenuation level ensures smaller steady-state error.

6.3. Quasi-polynomial model with $m \neq n$

Usually, the Lotka–Volterra and the QP systems has state spaces of different dimensions [7]. The typical situation is when the QP system is embedded into the Lotka–Volterra form. The Lotka–Volterra embedding inflates the originally n dimensional state space to an m dimensional Lotka–Volterra dynamics. The QP system of this example also belongs to the same equivalence class as the Lotka–Volterra model (51) of the previous example, however, in this case the dimension n of QP state vector is two, i.e. $m > n$. The ODE form of the QP system is given in (56) below.

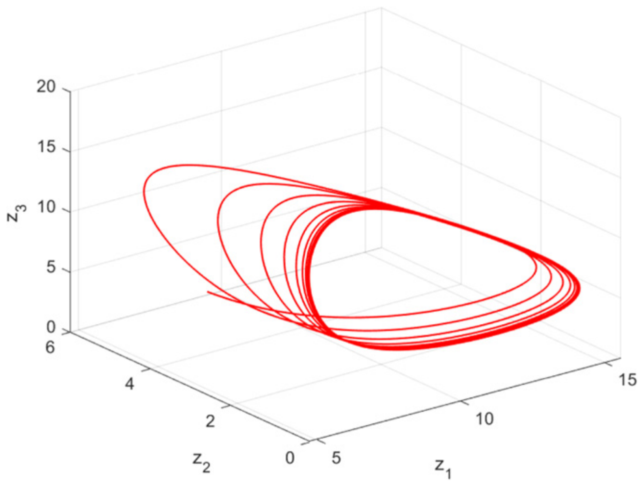

Figure 6. Phase portrait of the QP system without control (section 6.2).

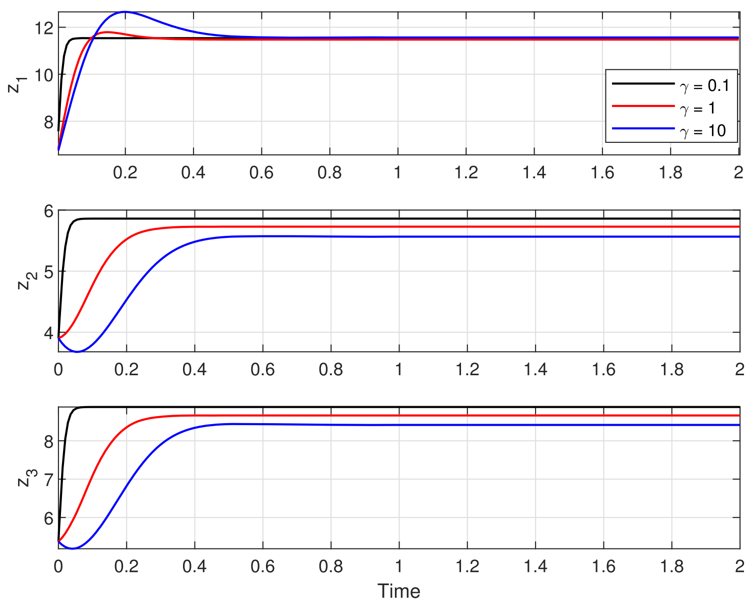

Figure 7. State trajectories of the QP system with control (section 6.2).

$$\begin{aligned}\dot{z}_1 &= z_1\,(0.5z_2 - 7.8 + 0.6(u_2 + w_2) + 0.8(u_3 + w_3))\\ \dot{z}_2 &= z_2\left(-0.1z_1^{\frac{3}{5}} - 0.2z_1^{\frac{4}{5}} + 4 + (u_1 + w_1)\right).\end{aligned} \tag{56}$$

The QP exponent matrix B_2 that describes the monomial structure is given in (57).

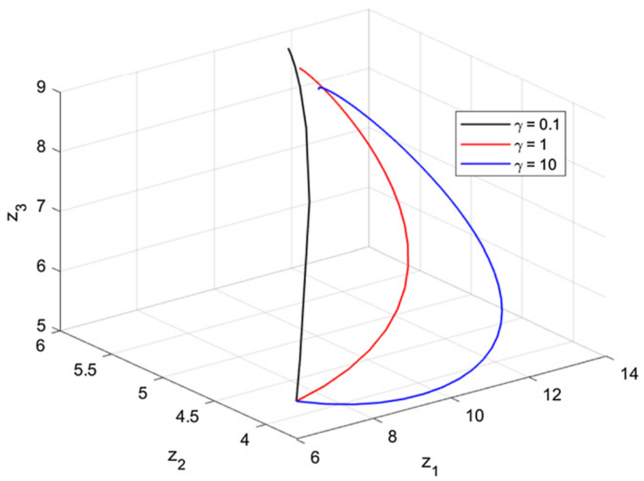

Figure 8. Phase portrait of the QP system with control (section 6.2).

$$B_2 = \begin{pmatrix} 0 & 1 \\ \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \end{pmatrix}. \tag{57}$$

The QP coefficient matrix A_2 is given in (58) below:

$$A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{2}{10} \end{pmatrix}. \tag{58}$$

The monomials of the system described by the matrices (57) and (58) are z_2, $z_1^{\frac{3}{5}}$ and $z_1^{\frac{4}{5}}$ according to the rows of matrix B_2. Figures 9 and 10 show the dynamic response of the QP system with no input, i.e. $\mathbf{u}_c = \mathbf{w} = \mathbf{0}$. The simulation results presented in figures 11 and 12 show that the proposed disturbance attenuation approach ensures the convergence of the controlled states to the setpoint which was determined from the Lotka–Volterra case using the quasi-monomial transformation corresponding to the QP system. Similarly to the Lotka–Volterra case, smaller prescribed disturbance attenuation level ensures smaller steady state error.

7. Discussion and conclusions

Using the classical logarithmic Lyapunov function of autonomous Lotka–Volterra and QP systems we derive passivity conditions of the open version of these systems. Because of the joint Lyapunov function of an equivalence class of QP systems and its Lotka–Volterra canonical form, we could extend the results to QP systems, too.

It is important to note that one should extend the classical Lotka–Volterra and QP system models with suitable input and output variables for which passivity properties hold. This means that one considers the possibility of manipulating the death/birth rate vectors of the species as inputs.

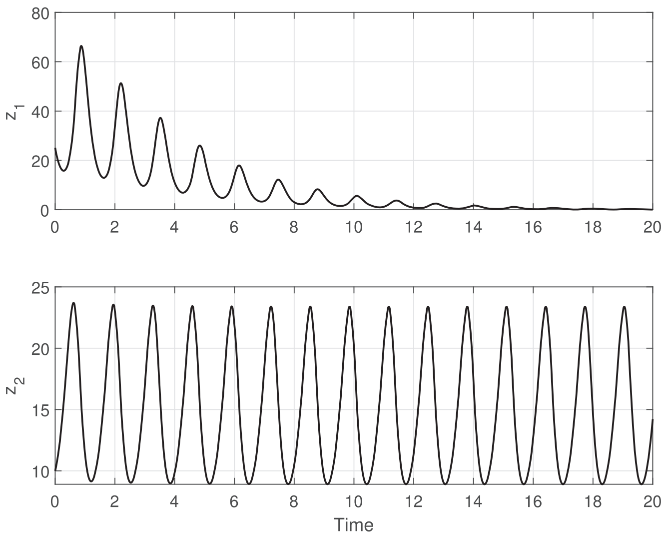

Figure 9. State trajectories of the QP system without control (section 6.3).

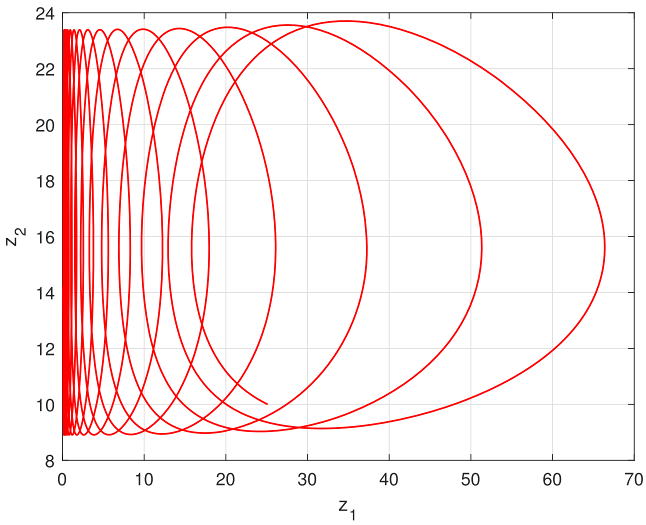

Figure 10. Phase portrait of the QP system without control (section 6.3).

The condition of passivity in (21) is less strict than the classical one by requiring only semi-definiteness. By using the input-extended model of the system, it was shown that the asymptotic stability i.e. the convergence into the equilibrium point, can be achieved by using diagonal state feedback having arbitrarily small gains.

Moreover, passivity theory provides a novel and powerful approach to design linear static feedback control laws for Lotka–Volterra systems. Based on our result that each Lotka–Volterra system is feedback equivalent to a passive system, a constructive approach was given to compute the passifying state-dependent affine input transformation. This control also

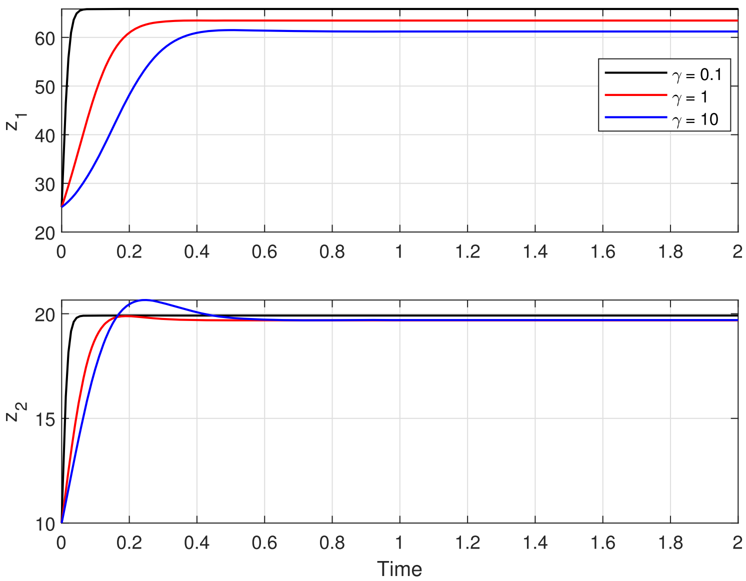

Figure 11. State trajectories of the QP system with control (section 6.3).

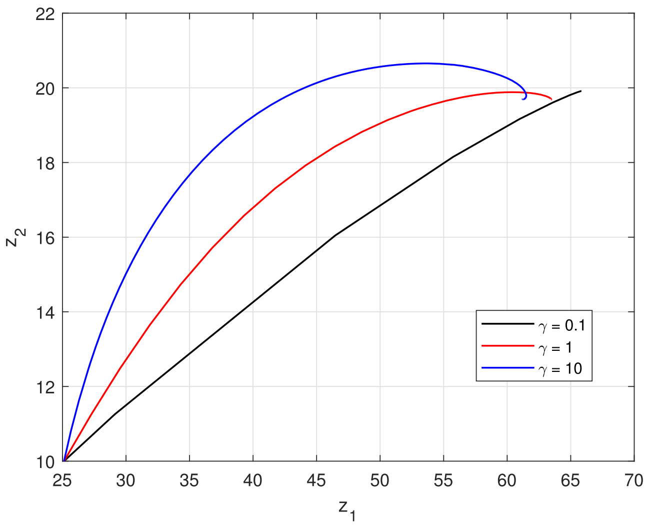

Figure 12. Phase portrait of the QP system with control (section 6.3).

assures the Lyapunov stability of Lotka–Volterra systems. Asymptotic stability of the closed-loop system, i.e. the convergence into the equilibrium point, can also be achieved by extending the previous control with diagonal small gain state feedback. Finally, a proper relation was given to design the feedback gain such to attenuate the effect of unmodelled disturbances on the asymptotic convergence.

Acknowledgments

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