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Renewable energy biodiesel: A mathematical approach from ecology to production

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Abstract

Biodiesel is one of promising renewable energy source and used as an alternative of conventional hydrocarbon fuels. *Jatropha curcas* plant oil (JCPO) is the most cost effective sources of biodiesel. The plant can be cultivated in wastelands and grows on almost any type of territory, even on sandy and saline soils. Judicious agricultural practices and effective crop management of *Jatropha curcas* is preliminary requisite to get maximum yield of oil. Production of biodiesel by transesterification of Jatropha oil significantly depends on four reaction parameters viz., reaction time, temperature, oil to alcohol molar ratio and stirrer speed. In this work, we have formulated a mathematical model of *Jatropha curcas* plant, which is affected by many type of pest with the aim to control the pest through *Nuclear Polyhedrosis Virus* (NPV). Here we have also concentrated on insecticide spraying as controlling measure to reduce the pest, to get maximum yield of Jatropha seeds, which gives Jatropha oil. We have also shown the effect of different variants on mass transfer in biodiesel production from JC oil and how the control theoretic approach flags the maximum production of biodiesel under the mathematical paradigm. Our analytical results provide an idea of the cost effective faster rate of biodiesel production, which satisfies our numerical conclusions.

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Mathematics Subject Classification (2010). 34A08, 34K37, 37N25, 49J15.

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1. Introduction

In today's world, petroleum is clearly the most important energy source, providing more than half of the world's power, as well as being a basic material used in the manufacture of fertilizer, synthetic fibers, plastics and synthetic rubber. Demand is ever increasing worldwide, yet petroleum resources are finite and non-renewable. Concerns about diminishing supplies, rising cost and environmental problems of hydrocarbon fuels have been motivating researchers globally to seek more extensively alternative, renewable energy sources to harness future energy demands. Jatropha oil is environmentally safe, cost effective renewable source of non-conventional energy and a promising substitute for diesel. Biodiesel can be derived from Jatropha oil and it is clean-burning, biodegradable, natural and it can help in reducing mineral oil dependence also. It is one of the promising and profitable agro forestry crop grown for its bio-diesel production

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[1, 2]. The cultivation of Jatropha plant has been considered as one of the most promising solutions to the problems created by climate change and energy insecurity in developing countries for several years. Also, Jatropha is a multipurpose shrub of significant economic importance because of its several potential industrial and medicinal uses. Jatropha is depicted as miracle tree that could simultaneously produce biodiesel, reclaim waste land and enhance rural development without compromising food production or ecosystem services [3, 4, 5].

Jatropha curcas seed oil is proven to be toxic to many microorganisms, insects and animals. Despite its toxicity, Jatropha plant is not pest and disease resistant. The major pests and diseases affecting Jatropha are: 1) the leaf miner *Stomphastis thraustica*, 2) the leaf and stem miner *Pempelia morosalis*, and 3) the shield-backed bug *Calidea panaethiopica*, which can cause flower and fruit abortion. Damage from these pests particularly during the second year after the plantations and before later receding is dangerous [6].

To get sufficient oil from this plant, protection from different diseases is essential. In this respect study of disease dynamics of Jatropha plant can be done with the help of an eco-epidemiological mathematical model for controlling and/or minimizing diseases. There are very few information available on the eco-epidemiological mathematical model for the study of pest and disease dynamics to save *Jatropha curcas* plant from different pests. Mathematical models for pest control have been proposed and studied by several researchers [7, 8]. They have studied the effect of environmental fluctuations on a sterile insect release method. Bhattacharya and Bhattacharya [9] have designed and analysed a mathematical model under sterile insect population and pesticide. Pest management models through control strategies have been designed and analysed by many researchers mathematically oftentimes[10, 11, 12].

Vegetable oil from Jatropha plant is the most viable source for the production of biodiesel. It is significant to point out that the biodiesel from Jatropha plant has the requisite potential of providing a promising and commercially viable alternative to diesel oil since it has desirable physicochemical and performance characteristics comparable to diesel. It is utmost important to protect the crop from pest in order to obtain uninterupted supply of Jatropha fruits which is the key raw material for Jatropha seeds.

Oil which is extracted from the seeds of *Jatropha curcas* plant is converted to biodiesel by transesterification reaction with the help of chemical or biological catalyst. This transesterification reaction depends on some reaction conditions such as temperature, catalyst concentration, speed of stirrer, molar ratios of oil and alcohol. To get maximum conversion, certain major physical and chemical property of the reaction system should be maintained properly [13], [14], [15]. Mathematical modeling can be helpful and its implementation in reality would be beneficial in this regard. It was found that in the production of biodiesel from Jatropha oil by transesterification, the high FFA in the Jatropha oil can react with the catalyst (KOH) to produce soap [4], [16]. This reaction causes lower yield of biodiesel, washing difficulties and purity of biodiesel.

Thus, the main objectives of this article are two fold. Initially we formulate an Eco-epidemiological mathematical model for studying the *Jatropha curcas* plant pest management for controlling and/or minimizing damages of the crop and successively to formulate a mathematical model to optimize the production of biodiesel from Jatropha oil with chemical catalysts by controlling reaction parameters. We determine the optimal control condition with the help of simple mathematical technique and discuss the relevant analysis with respect to the specified time interval, by varying control factors to maximize the biodiesel production.

2. Formulation of The Mathematical Model for Jatropha Pest Control

The model, we analyze in this paper, has four populations, viz.

(i) The plant population, J,

(ii) The susceptible pest population, S,

(iii) The infected pest, I and

(iv) The virus, V.

The following assumptions are taken to formulate the mathematical model:

(A1) In the absence of the pest, plant population grows according to a logistic curve with carrying capacity $K_J(K_J > 0)$ and with an intrinsic growth rate constant R_J. Thus the rate equation for Jatropha plant is:

$$\frac{dJ}{dT} = R_J J(1 - \frac{J}{K_J}). \tag{1}$$

Now, Jatropha plant gets infected with pest thereby causing considerable crop reduction. Let λ be the per capita effective contact rate with pest. Hence, the rate of change in plant population is given by the following equation:

$$\frac{dJ}{dT} = R_J J(1 - \frac{J}{K_J}) - \lambda SV. \tag{2}$$

(A2) The infected individuals do not reproduce due to resource limitations. However, it contributes with S to population growth with carrying capacity K_S. In absence of virus (NPV), the intrinsic growth rate of the susceptible pest population cab be described in logistic fashion [17]. Let, R_J be the intrinsic birth rate and λ be the effective per capita contact rate of pest with viruses. Also pest consumes Jatropha plant, so the reproduction rate of pest is enhanced. Let, β_1 be the rate increase of reproduction of susceptible pest. Hence the differential equation of the susceptible pest is given by:

$$\frac{dS}{dT} = R_S S(1 - \frac{S+I}{K_S}) + \beta_1 JS - \lambda SV.$$

(A3) The infected class of pest is removed by lysis before having the possibility of reproducing. Let ξ_1 be the rate of mortality of infected pest and hence the growth rate of infected pest is given by the following equation:

$$\frac{dI}{dT} = \lambda SV - \xi_1 I.$$

(A4) Let κ be the rate of production of virus per pest from lysis which called virus replication parameter. Now, μ_V be the rate of mortality of virus. Also there are free virus in this environment which are reproduced constantly and we consider π_v be the constant rate of reproduction of free virus. Then the rate of change of virus is given by the following differential equation:

$$\frac{dV}{dT} = \pi_v + \kappa\xi_1 I - \mu_V V.$$

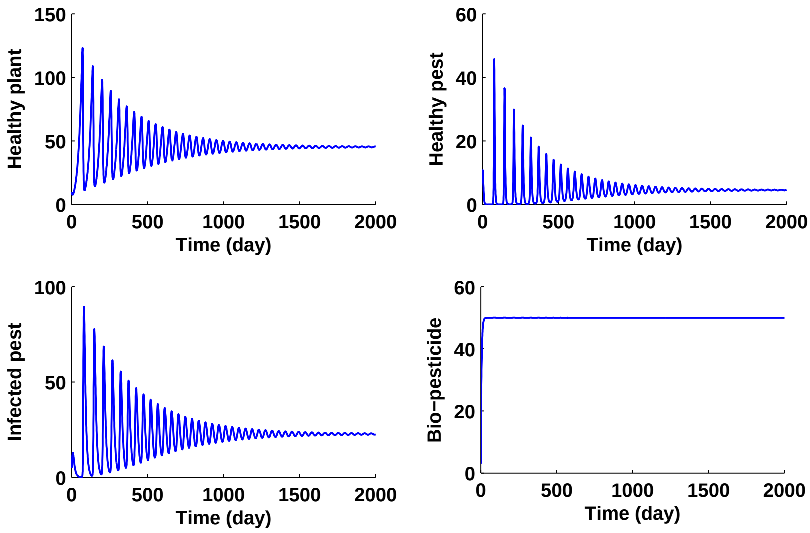

FIGURE 1. Trajectory portrait of model system (3) with parameter value as in Table 1 and $\kappa = 0.01$.

noindent Based on the above assumptions (A1) - (A4), we can further formulate the following mathematical model:

$$\begin{aligned}
\frac{dJ}{dT} &= R_J J(1 - \frac{J}{K_J}) - \beta_1 JS, \\
\frac{dS}{dT} &= R_S S(1 - \frac{S+I}{K_S}) + \beta_1 JS - \lambda SV, \\
\frac{dI}{dT} &= \lambda SV - \xi_1 I, \\
\frac{dV}{dT} &= \pi_v + \kappa \xi_1 I - \mu_V V, \qquad (3)
\end{aligned}$$

with initial conditions as $J(0) = J_0$, $S(0) = S_0$, $I(0) = I_0$ and $V(0) = V_0$.

3. The Mathematical Model for Biodiesel Production from Jatropha Oil

Biodiesel can be produced by the transesterification of triglycerides and methanol in the presence of an alkaline catalyst such as potassium hydroxide (KOH). The reaction consists of three steps and reversible reactions, where triglycerides (TG) is converted to diglycerides (DG), diglycerides (DG) to monoglycerides (MG) and finally monoglycerides (MG) to glycerol. The reaction steps are given below by schematic diagram:

$$\begin{aligned}
TG + M &\underset{k_2}{\overset{k_1}{\rightleftharpoons}} DG + E, \\
DG + M &\underset{k_4}{\overset{k_3}{\rightleftharpoons}} MG + E, \\
MG + M &\underset{k_6}{\overset{k_5}{\rightleftharpoons}} G + E.
\end{aligned} \qquad (4)$$

At each reaction step, one molecule of methyl ester is produced for each molecule of methanol consumed [18], [19].

It is shown above that the alcoholysis kinetic reaction scheme consists of three reversible reactions. Beside, there are also FFA and BD saponification reactions. It is shown below that the saponification kinetic reaction scheme consists of mainly three irreversible reactions [20] as:

$$FFA + OH \xrightarrow{k_7} A + W,$$
$$E + OH \xrightarrow{k_8} A + M. \tag{5}$$

We denote the concentrations of triglycerides, diglycerides, monoglycerides, biodiesel (methyl ester), methanol(alcohol) and glycerol by x_T, x_D, x_M, x_E, x_A and x_G respectively. Also, concentration of catalyst, fatty acid, soap and water are denoted by x_H, x_F, x_P and x_W respectively. Using the above assumption and considering the law of mass action, we get the following differential equations to characterize the stepwise reactions,

$$\begin{aligned}
\frac{dx_T}{dt} &= -k_1 x_T x_A + k_2 x_D x_E,\\
\frac{dx_D}{dt} &= k_1 x_T x_A - k_2 x_D x_E - k_3 x_D x_A + k_4 x_M x_E,\\
\frac{dx_M}{dt} &= -k_5 x_M x_A + k_6 x_G x_E + k_3 x_D x_A - k_4 x_M x_E,\\
\frac{dx_A}{dt} &= -k_1 x_T x_A + k_2 x_D x_E - k_3 x_D x_A + k_4 x_M x_E - k_5 x_M x_A + k_6 x_G x_E + k_8 x_E x_H,\\
\frac{dx_E}{dt} &= k_1 x_T x_A - k_2 x_D x_E + k_3 x_D x_A - k_4 x_M x_E + k_5 x_M x_A - k_6 x_G x_E - k_8 x_E x_H,\\
\frac{dx_G}{dt} &= k_5 x_M x_A - k_6 x_G x_E,\\
\frac{dx_F}{dt} &= -k_7 x_F x_H,\\
\frac{dx_H}{dt} &= -k_7 x_F x_H - k_8 x_E x_H,\\
\frac{dx_P}{dt} &= k_7 x_F x_H + k_8 x_E x_H,\\
\frac{dx_W}{dt} &= k_7 x_F x_H, \qquad (6)
\end{aligned}$$

with the initial conditions:

$$\begin{aligned}
x_T(0) &= x_{T_0},\ x_D(0) = 0,\ x_M(0) = 0, x_A(0) = x_{A_0},\ x_E(0) = 0,\\
x_G(0) &= 0, x_F(0) = 0,\ x_H(0) = x_{H_0}, x_P(0) = 0 \text{and } x_W(0) = 0. \qquad (7)
\end{aligned}$$

Here k_i, (i=1, 2,...,8) are reaction rate constants. Again, the reaction constant, k_i, is expressed by the following equation:

$$k_i = a_i e^{\frac{-b_i}{T}}.$$

T is the reaction temperature in Kelvin scale (K), a_i is the frequency factor, and

$$b_i = \frac{Ea_i}{R}.$$

in which Ea_i is the activation energy for each component and R is the universal gas constant. The values of a_i and b_i are given in Table 2.

4. Mathematical Analysis of Model System (3)

4.1. Dimensionless form of the model (3)

To reduce the number of parameters and for the simplicity of analytical calculations we take the following dimensionless form of the system (3). Taking dimensionless time t=$\frac{T}{\lambda K_S}$ and using the transformations $j = \frac{J}{K_J}$, $s = \frac{S}{K_S}$, $i = \frac{I}{K_S}$ and $v = \frac{V}{K_S}$, we have the dimensionless form of the model given by

$$\begin{aligned} \frac{dj}{dt} &= aj(1-j) - \beta js, \\ \frac{ds}{dt} &= bs(1-(s+i)) + \alpha js - sv, \\ \frac{di}{dt} &= sv - \xi i, \\ \frac{dv}{dt} &= v_0 + \kappa\xi i - \mu v, \end{aligned} \tag{8}$$

Where $a = \frac{R_J}{\lambda K_S}$, $\beta = \frac{\beta_1}{\lambda}$, $b = \frac{R_S}{\lambda K_S}$, $\alpha = \frac{\beta_1 K_J}{\lambda K_S}$, $\xi = \frac{\xi_1}{\lambda K_S}$, $v_0 = \frac{\pi_v}{\lambda K_S^2}$, $\mu = \frac{\mu_V}{\lambda K_S}$.

4.2. Positivity and Boundedness of the system

We make an obvious assumption that all parameter used in the model are positive. The initial condition are given by,

$$\begin{aligned} &j(0) = j_0,\ s(0) = s_0,\ i(0) = i_0,\ v(0) = v_0, \\ &\text{with } j_0 > 0,\ s_0 > 0,\ i_0 > 0 \text{ and } v_0 > 0. \end{aligned} \tag{9}$$

Now we prove the positivity and boundedness of the system by following theorem and lemma.

Theorem 4.1. *Each component of the system (8) with initial conditions (9) remains positive for all $t > 0$.*

Proof. Let $(j(t), s(t), i(t), v(t))$ be any solution of the system (8). Now, it is clear that $j(t) > 0$ for all $t > 0$.
and

$$\frac{ds}{dt} = bs(1-(s+i)) + \alpha js - sv \geq -sv.$$

Hence,

$$s \geq s_0 e^{\int_0^t (v(t))dt} > 0, \text{ since } \int_0^t (v(t))dt < \infty \text{ for all } t > 0. \tag{10}$$

Similarly, it can prove that

$$i \geq i_0 e^{-\xi t} > 0, \text{for all } t > 0 \text{ } and \text{ } v \geq v_0 e^{-\mu t} > 0 \text{ for all } t > 0. \tag{11}$$

■

Lemma 4.2. *Define the function $H_1(t) = \alpha j(t) + \beta s(t);\ t \in [0, \infty)$. Then for all $t > 0$, $H_1(t) \leq M_1$. where, $M_1 = \frac{\alpha(a+\alpha)^2}{4a} + \frac{\beta(b+\beta)^2}{4a} + H(j(0), s(0))e^{-t}$. Hence, j(t) and s(t) are bounded.*

Proof. Let $(j(t), s(t), i(t), v(t))$ be any solution of the system (8). Here $H_1(t) = \alpha j(t) + \beta s(t)$.

Therefore,

$$\begin{aligned}\frac{dH_1}{dt} &= \alpha\frac{dj}{dt}+\beta\frac{ds}{dt}\\ &= \alpha aj(1-j)-\alpha\beta js+\beta bs(1-(s+i))+\alpha\beta js-\beta sv\\ &\le \alpha aj(1-j)+\beta bs(1-s)\\ &\le \alpha aj(1+\frac{\alpha}{a}-j)+\beta bs(1+\frac{\beta}{b}-s)-H_1.\end{aligned}$$

Hence,

$$\frac{dH_1}{dt}+H_1\le \alpha aj(1+\frac{\alpha}{a}-j)+\beta bs(1+\frac{\beta}{b}-s),$$

which implies that

$$H_1(j(t),s(t))\le\frac{\alpha(a+\alpha)^2}{4a}+\frac{\beta(b+\beta)^2}{4a}+H(j(0),s(0))e^{-t}=M_1.$$

■

Lemma 4.3. *Define the function $H_2(t)=s(t)+i(t), t\in[0,\infty)$. There exist two real number M_2 and M_3 such that $s(t)\le M_2$ and $j(t)\le M_3$ and we have, for all $t>0$, $H_2(t)\le M_4$. Where $M_4=\frac{b+\alpha M_3}{b}+H_2(s(0),i(0))e^{-bM_2t}$. Then $i(t)$ bounded.*

Proof. Let $(j(t),s(t),i(t),v(t))$ be any solution of the system (8). Since $j(t)$ and $s(t)$ are bounded, there exist two real number M_2 and M_3 such that $s(t)\le M_2$ and $j(t)\le M_3$ Since, $H_2(t)=s(t)+i(t)$. We have,

$$\begin{aligned}\frac{dH_2}{dt} &= \frac{ds}{dt}+\frac{di}{dt}\\ &= bs(1-(s+i))+\alpha js-sv+sv-\xi i\\ &\le bM_2(1-(s+i))+\alpha M_2M_3\\ &\le bM_2-bM_2H_2+\alpha M_2M_3,\end{aligned}$$

which implies,

$$\frac{dH_2}{dt}+bM_2H_2\le bM_2+\alpha M_2M_3.$$

Hence,

$$H_2(s(t),i(t))\le\frac{b+\alpha M_3}{b}+H_2(s(0),i(0))e^{-bM_2t}=M_4.$$

■

Lemma 4.4. *Let $H_3(t)=v(t), t\in[0,\infty)$. there exists a real number M_5, s.t $i(t)\le M_5$. Then for all $t>0$, $H_3(t)\le M_6$, where $M_6=\frac{v_0+\kappa\xi M_5}{\mu}+H_3(v_0)e^{-\mu t}$. Hence $v(t)$ is bounded.*

Proof. Let $(j(t),s(t),i(t),v(t))$ be any solution of the system (8). Since $i(t)$ is bounded, there exists a real number M_5, such that $i(t)\le M_5$, since $H_3(t)=v(t)$. We have,

$$\begin{aligned}\frac{dH_3}{dt} &= \frac{dv}{dt}\\ &= v_0+\kappa\xi i-\mu v\le v_0+\kappa\xi M_5-\mu H_3,\end{aligned}$$

which implies $\frac{dH_3}{dt}+\mu H_3\le v_0+\kappa\xi M_5$. Hence,

$$H_3(v(t))\le\frac{v_0+\kappa\xi M_5}{\mu}+H_3(v_0)e^{-\mu t}=M_6.$$

■

Theorem 4.5. *: All solution of the system (8) that start in R^4_+ are uniformly bounded.*

The proof follows directly from Lemma 1, Lemma 2 and Lemma 3.

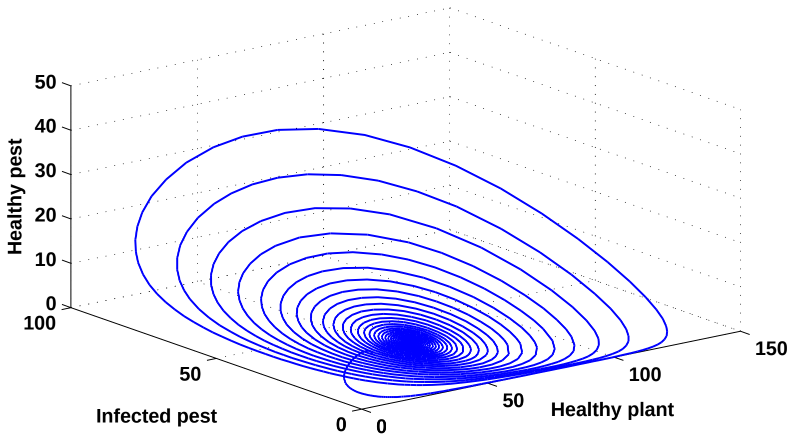

FIGURE 2. Phase portrait of Healthy plant (J(t)), Healthy pest (S(t)) and Virus (V(t)) with parameter value as in Table 1 and $\kappa = 0.01$.

4.3. Equilibria and Stability

In this sub-section we find the different equilibrium point of the dimensionless system (8) and analyze the stability of this system around these points. The above system (equation 8) has four equilibrium points, viz.

(a) The axial equilibrium equilibrium point $E_0(0, 0, 0, \frac{v_0}{\mu})$,

(b) Pest free equilibrium point $E_1(1, 0, 0, \frac{v_0}{\mu})$,

(c) Virus free equilibrium point $E_2(1 - \frac{\beta}{a}, 1, 0, 0)$ and

(d) The interior equilibrium point $E^*(j^*, s^*, i^*, v^*)$.

where,

$j^* = 1 - \frac{\beta}{a}\frac{\mu\xi i^*}{\kappa\xi i^* + v_0}$,

$s^* = \frac{\mu\xi i^*}{\kappa\xi i^* + v_0}$,

$v^* = \frac{\kappa\xi i^* + v_0}{\mu}$ and i^* is the positive root of the following equation:

$$\begin{aligned} q_1 {i^*}^2 + q_2 i^* + q_3 &= 0, \text{ where,} \\ q_1 &= \mu ab\kappa\xi + a\kappa^2\xi^2 \\ q_2 &= \mu^2\xi(ab + \alpha\beta) + v_0 ab\mu - \mu a\kappa\xi(b + \alpha) + a\kappa^2\xi^2 \\ q_3 &= av_0^2 - \mu\alpha a v_0 - \mu ab v_0. \end{aligned} \tag{12}$$

For positivity of j^* implies that $i^* < \frac{av_0}{\beta\mu\xi - a\kappa\xi}$. Also for positivity of E_2 we must have $a > \beta$.

Now, the following cases may arise.

Case(I): If $\mu = \frac{v_0}{\alpha+b}$, then the coefficient q_3 vanishes and other coefficients are positive, which implies that $i^* = 0$ and consequently $s^* = 1$, $v^* = 0$ and $j^* = 1 - \frac{\beta}{a}$ i.e susceptible pest population is maximum and growth of Jatropha plant will be minimum.

Case(II): If $\mu > \frac{v_0}{\alpha+b}$, then the coefficient q_3 is negative. Also, the coefficient q_1 is always positive then for any real value of q_2, by Descartes' rule of sign, it can be said that equation (12) has exactly one positive solution i.e unique interior equilibrium E^* exists.

Case(III): Finally if, $\mu < \frac{v_0}{\alpha+b}$, then the coefficient q_3 is positive. Also, the coefficient q_1 is always positive. Now, $q_2 > 0$ if

$$(\mu - p_1)(\mu - p_2) > 0 \tag{13}$$

where,

$$\begin{aligned} p_1 &= \frac{a\kappa\xi(b+\alpha) - abv_0 + [(a\kappa\xi(b+\alpha) - abv_0)^2 - 8\kappa\xi^2 v_0 a(ab+\alpha\beta)]^{\frac{1}{2}}}{2\xi(ab+\alpha\beta)}, \\ p_2 &= \frac{a\kappa\xi(b+\alpha) - abv_0 - [(a\kappa\xi(b+\alpha) - abv_0)^2 - 8\kappa\xi^2 v_0 a(ab+\alpha\beta)]^{\frac{1}{2}}}{2\xi(ab+\alpha\beta)}. \end{aligned} \tag{14}$$

For real and positive value of μ we must have

$$\begin{aligned} & \kappa\xi(b+\alpha) - abv_0 > [(a\kappa\xi(b+\alpha) - abv_0)^2 - 8\kappa\xi^2 v_0 a(ab+\alpha\beta)]^{\frac{1}{2}}, \\ \text{and} \quad & (\kappa - p_3)(\kappa - p_4) > 0. \end{aligned} \tag{15}$$

Here,

$$\begin{aligned} p_3 &= \frac{8\xi^2 v_0 a(ab+\alpha\beta) + 2a^2 b\xi(bv_0+\alpha) + [A^2 - 4a^4 b^2 v_0^2 \xi^2 (b+\alpha)^2]^{\frac{1}{2}}}{2a^2\xi^2(b+\alpha)^2}, \\ p_4 &= \frac{8\xi^2 v_0 a(ab+\alpha\beta) + 2a^2 b\xi(bv_0+\alpha) - [A^2 - 4a^4 b^2 v_0^2 \xi^2 (b+\alpha)^2]^{\frac{1}{2}}}{2a^2\xi^2(b+\alpha)^2}, \end{aligned} \tag{16}$$

where, $A = (8\xi^2 v_0 a(ab+\alpha\beta) + 2a^2 b\xi(bv_0+\alpha))$,
For real and positive value of κ we must have the following conditions,

$$\begin{aligned} A^2 &> 4a^4 b^2 v_0^2 \xi^2 (b+\alpha)^2 \quad \text{and} \\ A &> [A^2 - 4a^4 b^2 v_0^2 \xi^2 (b+\alpha)^2]^{\frac{1}{2}}. \end{aligned}$$

Also, condition (13) and (15) hold if $\mu > p_1$ and $\kappa > p_3$. In this case E^* dose not exist. But when $p_1 < \frac{v_0}{\alpha+b}$ i.e when $\kappa > \frac{v_0\xi(ab+\alpha\beta)+2abv_0(\alpha+b)}{2a\xi(\alpha+b)(2-(\alpha+b))}$, the flowing subcases may arise,

Subcase(I): When $p_1 < \mu < \frac{v_0}{\alpha+b}$, then all coefficient $q_i (i = 1, 2, 3)$ of the equation 12 are positive. This implies that E^* does not exit.

Subcase(II: When $p_2 < \mu < p_1$, then the coefficient q_2 is negative and other coefficient q_1 and q_2 are positive. So by Descartes' rule of sign the equation 12 has at least one positive root. Thus at least one positive equilibrium E^* exists.

Subcase(III): For $0 < \mu < p_2$, then all coefficient $q_i (i = 1, 2, 3)$ of the equation 12 are positive. Here E^* does not exit in that case.

Proposition 4.6. *The vanishing equilibrium point E_0 is always an unstable equilibrium point. The system is is locally asymptotically stable around the pest free equilibrium point E_1 if, $\mu < \frac{v_0}{\alpha+b}$ and the system is unstable around this point if, $\mu > \frac{v_0}{\alpha+b}$. Finally the point critically stable if, $\mu = \frac{v_0}{\alpha+b}$.*

Proof. The Jacobian matrix of the system at vanishing equilibrium point $E_0(0,0,0,\frac{v_0}{\mu})$ is given by:

$$J(0,0,0,\frac{v_0}{\mu}) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-\frac{v_0}{\mu} & 0 & 0 \\ 0 & \frac{v_0}{\mu} & -\xi & 0 \\ 0 & 0 & \kappa\xi & -\mu \end{bmatrix}.$$

Note that the above Jacobian matrix has at least on positive eigenvalue. Hence vanishing equilibrium point is unstable. The Jacobian matrix of the system at pest-free equilibrium point $E_1(1,0,0,\frac{v_0}{\mu})$ is given by:

$$J(0,0,0,\frac{v_0}{\mu}) = \begin{bmatrix} -a & -\beta & 0 & 0 \\ 0 & b+\alpha-\frac{v_0}{\mu} & 0 & 0 \\ 0 & \frac{v_0}{\mu} & -\xi & 0 \\ 0 & 0 & \kappa\xi & -\mu \end{bmatrix}.$$

Which gives the following characteristic equation:

$$(\lambda+a)(\lambda+\frac{v_0}{\mu}-(b+\alpha))(\lambda+\xi)(\lambda+\mu)=0. \tag{17}$$

Here, three eigenvalues are always real and negative and other eigenvalue are given by,

$$\lambda+\frac{v_0}{\mu}-(b+\alpha)=0. \tag{18}$$

Three cases arise here,
Case(I): When $\mu < \frac{v_0}{\alpha+b}$ then the eigenvalue is negative thus E_1 is stable.
Case(II): When $\mu > \frac{v_0}{\alpha+b}$ then the eigenvalue is positive thus E_1 is unstable.
Case(III): When $\mu = \frac{v_0}{\alpha+b}$ then the eigenvalue is zero. In this case the system is critically stable at E_1.

■

Proposition 4.7. *The virus free equilibrium point E_2 is locally asymptotically stable under $\kappa<\mu$. Whenever $\kappa=\mu$, the system enter into saddle-node bifurcation around this equilibrium point. Lastly, for $\kappa>\mu$, virus free equilibrium point E_2 is unstable.*

Proof. The Jacobian matrix of the system at virus free equilibrium point $E_2(1-\frac{\beta}{a},1,0,0)$ is given by:

$$J(1-\frac{\beta}{a},1,0,0) = \begin{bmatrix} \beta-a & \frac{beta^2}{a}-\beta & 0 & 0 \\ \alpha & \alpha(1-\frac{\beta}{a})-b & -b & -1 \\ 0 & 0 & -\xi & 1 \\ 0 & 0 & \kappa\xi & -\mu \end{bmatrix}.$$

The characteristic equation is given by:

$$(\lambda^2+d_1\lambda+d_2)(\lambda^2+d_3\lambda+d_4) = 0, \tag{19}$$

where,

$$\begin{aligned} d_1 &= \mu+\xi,\ d_2=\xi(\mu-\kappa) \\ d_3 &= a+b+\frac{\alpha\beta}{a}-(\alpha+\beta) \\ d_4 &= (a-\beta)(b+\frac{2\alpha\beta}{a}-\alpha). \end{aligned}$$

Now, under the conditions: $a+b+\frac{\alpha\beta}{a}>\alpha+\beta$, $a>\beta$ and $b+2\frac{\alpha\beta}{a}>\alpha$ the following three cases are arise:
Case(I): For $\kappa<\mu$ all eigenvalue are negative, hence the virus free equilibrium point is stable.

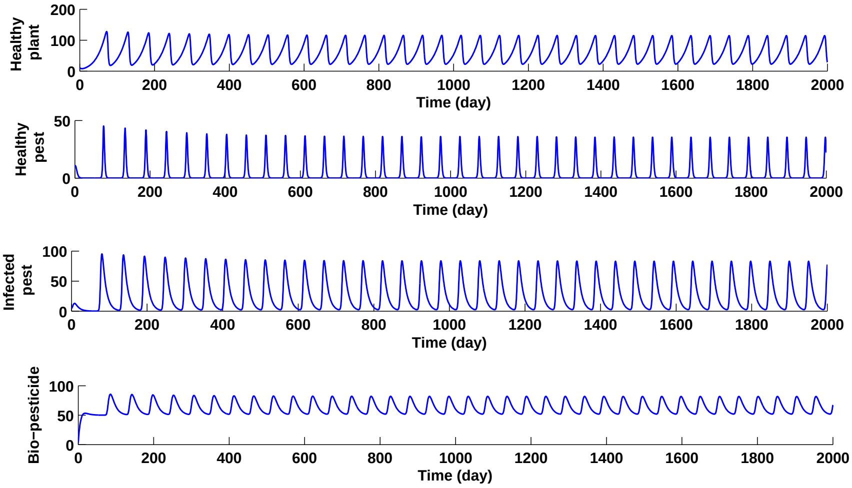

FIGURE 3. Trajectory portrait of model system (3) with $\kappa = 1$ and other parameters value as given in Table 1.

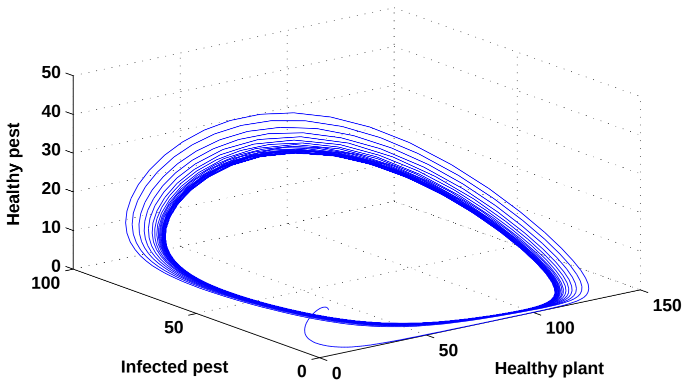

FIGURE 4. Phase portrait of Healthy plant (J(t)), Healthy pest (S(t)) and Virus (V(t)) with parameter value as in Table 1 and $\kappa = 1$.

Case(II): For $\kappa = \mu$ then one eigenvalue is zero and other three eigenvalue are negative hence the system enter into saddle-node bifurcation at E_2.
Case(III): For $\kappa > \mu$ then at least one eigenvalue is positive. Hence, the virus free equilibrium point E_2 is unstable in this case.

■

4.4. Stability of Interior Equilibrium

The Jacobian matrix of the system at interior equilibrium point $E^*(j^*, s^*, i^*, v^*)$ is given by:

$$J(j^*, s^*, i^*, v^*) = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix},$$

where,

$$\begin{aligned} s_{11} &= a - 2aj^* - \beta s^*, s_{12} = -\beta j^*, s_{21} = \alpha s^*, \\ s_{22} &= b - 2bs^* + \alpha j^* - v^*, s_{23} = -bv^*, \\ s_{24} &= -s^*, s_{32} = v^*, s_{33} = -\xi, s_{34} = s^*, \\ s_{43} &= \kappa\xi, s_{44} = -\mu. \end{aligned} \tag{20}$$

The characteristic equation of J_{E^*} is

$$\rho^4 + \sigma_1\rho^3 + \sigma_2\rho^2 + \sigma_3\rho + \sigma_4 = 0, \tag{21}$$

where,

$$\begin{aligned} \sigma_1 &= -(s_{11} + s_{22} + s_{33} + s_{44}), \\ \sigma_2 &= s_{11}s_{22} + s_{11}s_{33} + s_{11}s_{44} + s_{22}s_{33} + s_{22}s_{44} + \\ &\quad s_{33}s_{44} - s_{34}s_{43} - s_{23}s_{32} - s_{12}s_{21}, \\ \sigma_3 &= -(s_{11}s_{22}s_{33} + s_{11}s_{22}s_{44} + s_{11}s_{33}s_{44} \\ &\quad + s_{22}s_{33}s_{44}) + s_{34}s_{43}(s_{11} + s_{22}) \\ &\quad + s_{23}s_{32}(s_{11} + s_{44}) + s_{12}s_{21}(s_{33} + s_{44}) - s_{24}s_{32}s_{43}, \\ \sigma_4 &= s_{12}s_{21}s_{34}s_{43}. \end{aligned} \tag{22}$$

Then by Routh-Hurwitz criterion it follows that the interior equilibrium point locally asymptotically stable if

(i) $\sigma_1 > 0, \sigma_4 > 0$,

(ii) $\sigma_1\sigma_2 - \sigma_3 > 0$ and

(iii) $\sigma_3(\sigma_1\sigma_2 - \sigma_3) - \sigma_1^2\sigma_4 > 0$.

5. Mathematical Study of the System (6)

5.1. Boundedness of the System

In this section we show that the solution of the system is bounded using the following theorem.

Theorem 5.1. *All solution of the system that start in R_+^{10} are uniformly bounded.*

Proof. We define the function $W(t)$ as follows:

$$W(t) = x_T(t) + x_D(t) + x_M(t) + x_A(t) + x_E(t) + x_G(t) + x_F(t) + x_H(t) + x_P(t) + x_W(t)$$

Therefore,

$$\frac{dW}{dt} = \frac{dx_T}{dt} + \frac{dx_D}{dt} + \frac{dx_M}{dt} + \frac{dx_A}{dt} + \frac{dx_E}{dt} + \frac{dx_G}{dt} + \frac{dx_F}{dt} + \frac{dx_H}{dt} + \frac{dx_P}{dt} + \frac{dx_W}{dt}.$$

From system (2) we have, $\frac{dW}{dt} = 0$.

Hence, $W(t) = k$, where k is a positive constant (since $x_T(0), x_A(0), x_H(0)$ are positive and others components are zero.)

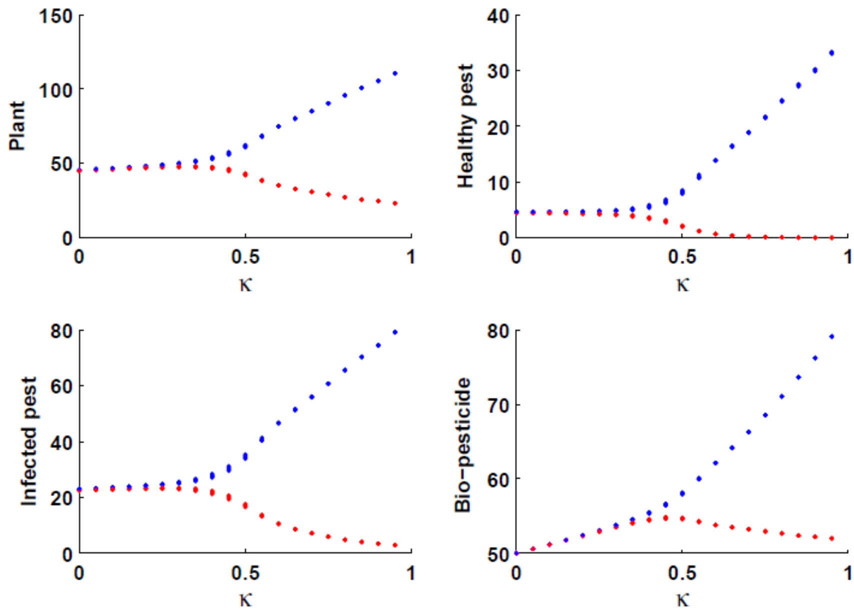

FIGURE 5. System dynamics for varying κ and parameter are as given in Table 1.

From the above analysis we have $W(t) < c$ for some $c > k$. Thus solution of the system (6) is bounded.

■

5.2. The Optimal Control Problem for Biodiesel Production

Here we have introduced three control parameters $u_1(t)$, $u_2(t)$ to reduce soap formation and catalyst loss. They are introduced in the first stage, second stage of saponification reaction respectively. The corresponding reaction mechanism is given by the following schematic diagram

$$FFA + OH \overset{k_7, u_1}{\rightarrow} S + W,$$
$$BD + OH \overset{k_8, u_2}{\rightarrow} S + AL.$$

We apply the control inputs $u_1(t)$ and $u_2(t)$ to reduce the side reaction of biodiesel production to get more biodiesel in each step of transesterification reaction as quick as possible. Here, u_1, u_2 satisfy $0 \le u_i(t) \le 1$ [21]. Also $u_i(t) = 1$ represents the maximal use of control and $u_i(t) = 0$, which signifies no control. Thus, introducing control input parameters, the system (6) becomes,

$$
\begin{aligned}
\frac{dx_T}{dt} &= -k_1x_Tx_A + k_2x_Dx_E,\\
\frac{dx_D}{dt} &= k_1x_Tx_A - k_2x_Dx_E - k_3x_Dx_A + k_4x_Mx_E,\\
\frac{dx_M}{dt} &= -k_5x_Mx_A + k_6x_Gx_E + k_3x_Dx_A - k_4x_Mx_E,\\
\frac{dx_A}{dt} &= -k_1x_Tx_A + k_2x_Dx_E - k_3x_Dx_A + k_4x_Mx_E - k_5x_Mx_A + k_6x_Gx_E + u_2k_8x_Ex_H,\\
\frac{dx_E}{dt} &= k_1x_Tx_A - k_2x_Dx_E + k_3x_Dx_A - k_4x_Mx_E + k_5x_Mx_A - k_6x_Gx_E - u_2k_8x_Ex_H,\\
\frac{dx_G}{dt} &= k_5x_Mx_A - k_6x_Gx_E,\\
\frac{dx_F}{dt} &= -u_1k_7x_Fx_H,\\
\frac{dx_H}{dt} &= -u_1k_7x_Fx_H - u_2k_8x_Ex_H,\\
\frac{dx_P}{dt} &= u_1k_7x_Fx_H + u_2k_8x_Ex_H,\\
\frac{dx_W}{dt} &= u_1k_7x_Fx_H, \qquad (23)
\end{aligned}
$$

with the initial conditions:

$$
\begin{aligned}
x_T(0) &= x_{T_0},\ x_D(0)=0,\ x_M(0)=0, x_A(0)=x_{A_0},\ x_E(0)=0,\\
x_G(0) &= 0, x_F(0)=x_{F_0},\ x_H(0)=0, x_P(0)=0 \text{and } x_W(0)=0. \qquad (24)
\end{aligned}
$$

The above system can be written as:

$$\frac{dx_i}{dt} = f_i(x_i, k_i, u_1, u_2, t), i = 1, 2, ...10. \qquad (25)$$

5.2.1. The Optimal system. Here we want to maximize bio-diesel x_E and minimize soap x_P, so that we define the objective cost function for the minimization problem as,

$$J(u_1,u_2) = \int_{t_i}^{t_f} [Pu_1^2(t) + Qu_2^2(t) - Rx_E^2(t) + Tx_P^2(t)]dt, \qquad (26)$$

The parameters P, Q, R and T are the positive weight constants on the benefit of the cost of production. The benefit is based on the minimization of cost together with maximization of biodiesel concentration. Our aim is to find out the optimal control pair $u^*=(u_1^*, u_2^*)$ such that

$$J(u_1^*, u_2^*) = min\ (J(u_1,u_2) : (u_1,u_2) \in U),$$

where,

$U = U_1 \times U_2$ and
$U_1 = \{u_1(t) :\ u_1 \text{ is measurable and } 0 \le u_1 \le 1,\ t \in [t_i, t_f]\}$
$U_2 = \{u_2(t) :\ u_2 \text{ is measurable and } 0 \le u_2 \le 1,\ t \in [t_i, t_f]\}$.

Here we use "Pontryagin Minimum Principle" [22] to find $u^*(t)$.
We define the Hamiltonian as follows:

$$H = [Pu_1^2(t) + Qu_2^2(t) - Rx_B^2(t) + Tx_S^2] + \sum \xi_i f_i, i = 1, 2, ..., 10, \qquad (27)$$

where, $\xi_1, \xi_2, ..., \xi_{10}$ are adjoint variables. According to Pontryagin, adjoint variables satisfy the following equations,

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, ..., 10, \qquad (28)$$

where $x_1 = x_T, x_2 = x_D, \cdots, x_{10} = x_W$.
This gives,

$$\begin{aligned}
\frac{d\xi_1}{dt} &= \xi_1 k_1 x_A - \xi_2 k_1 x_A + \xi_4 k_1 x_A - \xi_5 k_1 x_A \\
\frac{d\xi_2}{dt} &= -\xi_1 k_2 x_E + \xi_2 k_2 x_E + \xi_2 k_3 x_A - \xi_3 k_3 x_A + \xi_4 k_3 x_A + \xi_5 k_2 x_E - \xi_5 k_3 x_A \\
\frac{d\xi_3}{dt} &= -\xi_2 k_4 x_E + \xi_3 k_5 x_A + \xi_3 k_4 x_E - \xi_4 k_4 x_E + \xi_4 k_5 x_A + \xi_4 k_4 x_E - \xi_4 k_5 x_A \\
&\quad -\xi_6 k_5 x_A, \\
\frac{d\xi_4}{dt} &= \xi_1 k_1 x_T - \xi_2 k_1 x_T + \xi_3 k_5 x_M - \xi_3 k_3 x_D + \xi_4 k_1 x_T + \xi_4 k_3 x_D + \xi_4 k_5 x_M \\
&\quad -\xi_5 k_1 x_T - \xi_5 k_3 x_D - \xi_5 k_5 x_M - \xi_6 k_5 x_M, \\
\frac{d\xi_5}{dt} &= 2R x_E - \xi_1 k_2 x_D + \xi_2 k_2 x_D - \xi_2 k_4 x_M - \xi_3 k_6 x_G + \xi_3 k_4 x_M - \xi_4 k_2 x_D - \xi_4 k_4 x_M \\
&\quad -\xi_4 k_6 x_G - u_2 \xi_4 k_8 x_H + \xi_5 k_2 x_D + \xi_5 k_4 x_M + \xi_5 k_6 x_G + u_2 \xi_5 k_8 x_H + \xi_6 k_6 x_G \\
&\quad +u_2 \xi_5 k_8 x_H - u_2 \xi_9 k_8 x_H \\
\frac{d\xi_6}{dt} &= -\xi_3 k_6 x_E - \xi_4 k_6 x_E + \xi_5 k_6 x_E + \xi_6 k_6 x_E \\
\frac{d\xi_7}{dt} &= u_1 \xi_7 k_7 x_H + u_1 \xi_8 k_7 x_H - u_1 \xi_9 k_7 x_H - u_1 \xi_{10} k_7 x_H \\
\frac{d\xi_8}{dt} &= u_1 \xi_7 k_7 x_F + u_1 \xi_8 k_7 x_F - u_1 \xi_9 k_7 x_F - u_1 \xi_{10} k_7 x_F - u_2 \xi_8 k_8 x_E - u_2 \xi_9 k_8 x_E \\
\frac{d\xi_9}{dt} &= -2T x_P, \\
\frac{d\xi_{10}}{dt} &= 0, \qquad (29)
\end{aligned}$$

along with the transversality conditions $\xi_i(t_f) = 0$ for $i = 1, 2, ..., 10$.
According to Pontryagin Minimum Principle [22], the unconstrained optimal control variables u_1^* and u_2^* satisfy,

$$\frac{\partial H}{\partial u_i^*} = 0, i = 1, 2. \qquad (30)$$

Thus from (27) and (30), we have

$$u_1^*(t) = \frac{k_7 x_F x_H (\xi_7 - \xi_9 + \xi_8 - \xi_{10})}{2P}, \qquad (31)$$

$$u_2^*(t) = \frac{k_8 x_E x_H (\xi_5 - \xi_4 + \xi_8 - \xi_9)}{2Q}. \qquad (32)$$

Due to the boundedness of the standard control we get,

$$u_1^*(t) = max(0,\ min(1,\ \frac{k_7 x_F x_H (\xi_7 - \xi_9 + \xi_8 - \xi_{10})}{2P})), \qquad (33)$$

In a similar way, we have the compact form of $u_2^*(t)$ as:

$$u_2^*(t) = max(0,\ min(1,\ \frac{k_8 x_E x_H (\xi_5 - \xi_4 + \xi_8 - \xi_9)}{2Q})). \qquad (34)$$

Thus, we have the following theorem.

Theorem 5.2. *If the objective cost function $J(u_1^*, u_2^*)$ over U is minimum for the optimal control u^* corresponding to the interior equilibrium (x_i^*, i=1, 2, ...,10), then there exist adjoint variables $\xi_1, \xi_2, ..., \xi_{10}$ which satisfy the the system of equations (29).*

Table 1. Values of parameters used in numerical calculation for system (3) [23, 24].

Parameters	Definition	Values (Unit)
R_J	The maximum rate of plantation	0.05 Day^{-1}
K_J	The plant density	500 M^{-2}
λ	The infection rate	0.01 Vector^{-1} Day^{-1}
β	The rete of interaction	0.05 Day^{-1}
ξ	The rate of plant loss	0.1 Day^{-1}
R_S	The growth rate of pest	0.05 Day^{-1}
β_V	The acquisition rate	0.01 Plant^{-1} Day^{-1}
K_S	The pest carrying capacity	10000 Day^{-1}

Table 2. Values of parameters used in numerical calculation for model system (6) [19].

Parameters	Value	Parameters	Value
a_1	3.92e7	b_1	6614.83
a_2	5.77e5	b_2	4997.98
a_3	5.88e12	b_3	9993.96
a_4	0.098e10	b_4	7366.64
a_5	5.35e3	b_5	3231.18
a_6	2.15e4	b_6	4824.87
a_7	3.92e7	b_7	6614.83
a_8	5.77e5	b_8	4997.98

6. Numerical Simulation

6.1. Numerical Simulation for Model System (3)

The dynamics of the model system are analyzed using numerical methods in MATLAB. The numerical results of the system (3) are obtained to verify the analytical predictions obtained in the previous sections.

Figure 1 shows that the model variables J(t), S(t), I(t) and V(t) oscillate initially before the system moves toward its stable region as time increases. Length of oscillation is decreasing as the time goes on. Figure 2 is the phase portrait of healthy plant, healthy pest and infected pest which shows the asymptotic stability of the system around the interior equilibrium E^* for $\kappa = 0.01$ and other parameters value as given in Table 1.

The changes in the behavior of the dynamics of the system for different values of parameter β is seen in Figure 3. Numerically, we have shown that system (3) is stable asymptotically for $\kappa < 0.3$ (approx) and when κ cross the value (0.3), the system becomes unstable and for κ =1 gives a limit cycle in (J-S-I) plane which is indicated by Figure 4.

Figure 5 depicts the bifurcation diagram of four populations to observe a perfect behavior of the system for variation of the parameter κ (the rate of interaction between Healthy Plant and Pest). We observe that when κ passes through the value 0.35 (approx.), the interior equilibrium E^* bifurcates towards a periodic solution.

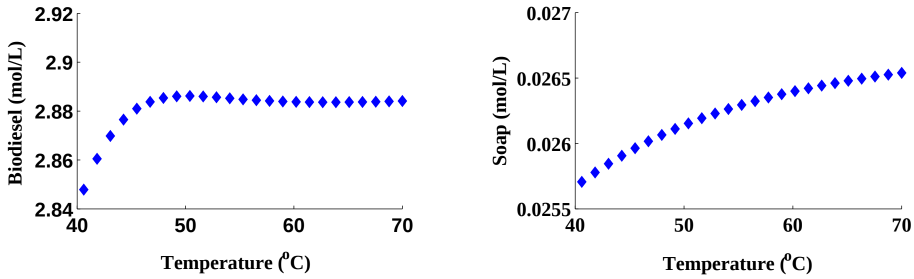

Figure 6. Effect of temperature on BD production and soap formation taking parameter as in Table 2 (without control).

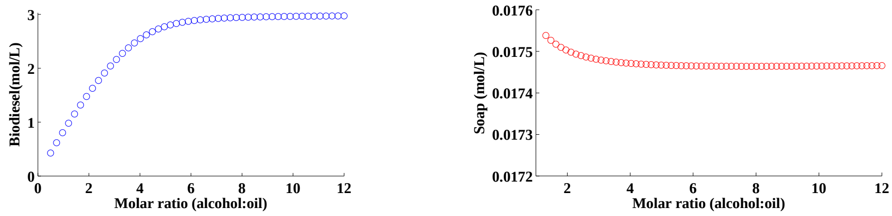

Figure 7. Effect of molar ratio on biodiesel production (left panel) and formation of soap (right panel) (without control).

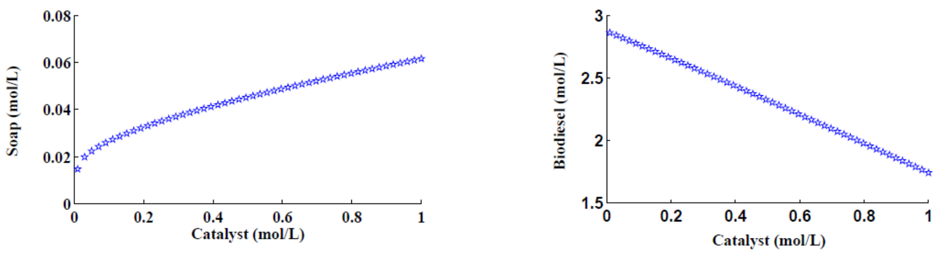

Figure 8. Effect of catalyst on BD production (right panel) and formation of soap (left panel) without control.

6.2. Numerical Simulation for Model System (6)

We solved the model equations (6) numerically to understand the behaviour of the transesterification reaction. The kinetics of the system has been analyzed using numerical methods in the presence and absence of the control parameters. Here, temperature, molar ratio, catalyst concentration are employed for understanding their effects in biodiesel production process.

Figure 6 shows the effect of the temperature on conversion of *Jatropha curcas* oil using 6:1 methanol to oil molar ratio with KOH concentration 1%wt of oil. It is also cleared that with

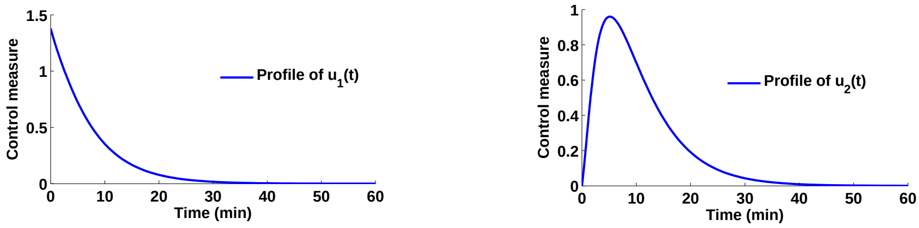

FIGURE 9. Control profiles are plotted as a function of time for biodiesel production process.

an increasing reaction temperature at or above 50^oC the initial reaction conversion from oil to biodiesel taking 1 hour reaction time, the ultimate biodiesel conversion appears to be lower compared to that of using a lower temperature. The probable reason for this is that the saponification of biodiesel and FFA by the alkali catalysts is much faster than the transesterification reaction at temperature above 50^oC. In other words, higher temperature accelerates the side saponification reaction. Thus operating the reactor under the conditions of the experiments at temperatures higher than 50^oC was not economical.

Molar ratio between alcohol and oil influences the yield of biodiesel production. Biodiesel concentration increases with the increasing ethanol/oil molar ratio. Alcohol to oil molar ratio is varied from 6:1 to 12:1 and the results are shown in figure 7. It is seen that with the increasing in the methanol to oil molar ratio, the percentage molar conversion increased rapidly with time but after 1 hour the ultimate percentage mole conversion is lower compared to that of using lower methanol to oil molar ratio.

Catalyst concentration has a significant effect on alkali-catalyzed methanolysis. We vary catalyst concentration to see the effects of catalyst concentration on the methyl ester (x_E) and soap (x_P) which is shown in Figure 8. Biodiesel yield is decreased and soap is increased as the catalyst concentrations increases. Although, the catalyst concentration of 1.5% w/w of oil provided a lower biodiesel yield than that of 1% w/w of oil for all reaction times, such a concentration should be avoided. Moreover, the methyl ester layer obtained from using this catalyst concentration has to be washed with hot distilled water several times in the water washing step. So there is a possibility of losing some biodiesel product to emulsion formation. For these reasons, 1% w/w of oil is considered to be the optimum catalyst concentration.

Figure 9 illustrates how the control policies act on the system with respect to time. From this figure we see that the control approach on each step of saponification reaction we obtain soap and if we apply control on it, then the rate of reaction monotonically decreases and soap formation reduces. We also see from this figure that the concentration of biodiesel is also enhanced. Thus, initially higher control is needed for maximum biodiesel production but after 20 minutes of the reaction, comparatively less control is needed.

In figure 10, concentration profiles of biodiesel is plotted in two different avenues which are mentioned as without control and at the optimal control level. From this figure, it is observed that how the control induced system give more production of biodiesel with respect to time. Our result shows that at 60 minute of reaction time, the concentration of biodiesel at optimal level reaches its maximum value as 2.98 mol/L, while the maximum concentration is 2.795 mol/L if

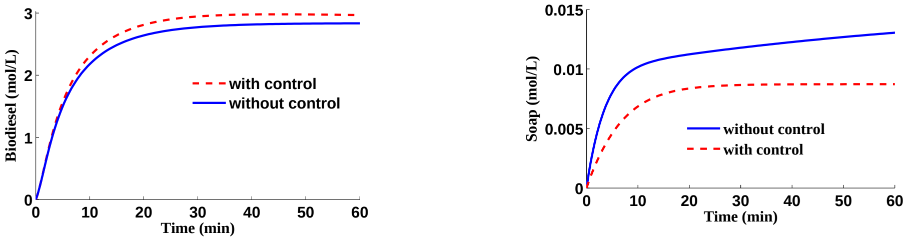

FIGURE 10. Left panel: Comparison between two cases: (1) BD concentration with optimal control, and (2) BD concentration without control; right panel:Comparison between two cases: (1) soap formation with optimal control, and (2) soap formation without control.

there is no control policy (like change of temperature, molar ratio, catalyst etc.) present on the reaction. In this way we can get 6.62% more biodiesel after 60 min of reaction time.

7. Discussion

In this article, we desire to observe the effects of biopesticide for controlling the pest of *Jatropha curcas* plant . We have explored the local stability of the positive interior equilibrium point E^* and also local Hopf-bifurcation. we have shown how the dynamics changes with the increase in the value of the parameter κ of the system. The dynamical behavior of the system is investigated from the point of view of stability and persistence. The model shows that infection can be sustained only above a threshold force of infection resulted by virus replication parameter κ. On increasing the value of κ, the endemic equilibrium bifurcates towards a periodic solution. A numerical simulation is then presented under a different choice of virus replication parameter' i.e. κ.

We have also presented another mathematical model of transesterification reaction for bio-diesel production. We have shown that the reaction depends significantly on molar ratio, catalyst concentration and temperature. Applying mathematical control approach we see that control on saponification reaction is essential in the transesterification reaction to increase bio-diesel production. By implying control approach on each step of saponification reaction, we have seen that administering the control parameters (such as temperature, catalyst concentration etc.) saponification can be reduced as well as production of bio-diesel can be increased.

8. Conclusion

The effectiveness of viral infection as a biological pest control in Jatropha plantation is discussed by mathematical modelling. It is certainly possible to eradicate pest population through release of viral pesticide. Finally, our analytical and numerical analysis indicate an important role of viral infection in pest control in *Jatropha curcas* plantation management. Thus, Application of biological pesticide for the integrated pest management policy with release of viral pesticide as biological control is the most favorable one [25]. In case of biodiesel production, fatty acid level in *Jatropha curcas* oil should be reduced as less as possible before catalytic transesterification of oil to avoid saponification problem. Also, catalyst concentration should be maintained properly. Numerical simulation offers a better understanding of optimal control for the maximum production of biodiesel. Thus, this article provides an idea to protect Jatropha plant from different pests and gives the conditions for getting maximum biodiesel production from Jatropha oil.

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