5-14-2021

Semiotics and Symbiosis – “Gap-Closing”: How Signs, Symbols, and Structure Impact the Teaching and Learning of Mathematics for Middle School African American Students

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ACCEPTANCE

This dissertation, SEMIOTICS AND SYMBIOSIS— "GAP-CLOSING": HOW SIGNS, SYMBOLS, AND STRUCTURE IMPACT THE TEACHING AND LEARNING OF MATHEMATICS FOR MIDDLE SCHOOL AFRICAN AMERICAN STUDENTS, by ELIJAH PORTER, II, was prepared under the direction of the candidate's Dissertation Advisory Committee. It is accepted by the committee members in partial fulfillment of the requirements for the degree, Doctor of Philosophy, in the College of Education & Human Development, Georgia State University.

The Dissertation Advisory Committee and the student's Department Chairperson, as representatives of the faculty, certify that this dissertation has met all standards of excellence and scholarship as determined by the faculty.

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SEMIOTICS AND SYMBIOSIS— “GAP-CLOSING”:
HOW SIGNS, SYMBOLS, AND STRUCTURE IMPACT THE TEACHING AND LEARNING OF MATHEMATICS FOR MIDDLE SCHOOL AFRICAN AMERICAN STUDENTS

by

ELIJAH PORTER, II

Under the Direction of David W. Stinson, Ph.D.

ABSTRACT

The purpose of this theory-building project was to generate a scientific platform through which society might stop using the data from standardized mathematics assessments to evaluate and scrutinize students and instead evaluate, scrutinize, and improve the processes and activities through which students engage in mathematics learning (Hilliard, 1994: Kozol, 2005; Ladson-Billings, 1997; Martin, 2000; Steele, 1992). In particular, this project focused on the syntax, semantics, and pragmatics (Peirce, 1902; Saussure, 1908/1998) of mathematics situations and the activities through which students might leverage these tools to construct their own mathematics knowledge in an effort to achieve mathematics proficiency (Kilpatrick, Swafford, & Findell, 2001). The participants of this project were self-identified African American male and female middle school students located in the southeastern region of the Unite States.

This theory-building project used a re-engineered teaching experiment methodology (Steffe, 1991) located within a sociocultural and radical constructivist ideological frame (von Glasersfeld, 1983; Vygotsky, 1930/1978). More specifically, the students were mentored through a cycle of exploration, introduction, application, and inquiry when given mathematical situations. Data from observations and Socratic inquiries were collected and analyzed using cultural-historical activity theory (CHAT; Vygotsky, 1930/1978) and a newly developed coding protocol in order to seek aspects of metacognition, cognition, and mathematics proficiency (Saldaña, 2016).

The reporting and analysis of the data revealed that the students could demonstrate progressive acts in their pursuit of mathematics proficiency. How the students were able to make such achievements were to be found, in part, in how they understood the semiotic aspects of any given mathematical situation—its syntax, semantics, and problem-solving elements. In addition, the students gave deeper and intentional attention to the metacognitive knowledge and metacognitive skills necessary to emphasize these semiotic aspects (Veenman & Spaan, 2005). Consistently, the responses from and the observations of each student were unique representations of their experiential selves. In the end, the aim of this theory-building project was to capture these unique representations and determine the specifics that might serve as components of a preliminary mathematics learning model.

INDEX WORDS: Activity Theory, Cognition, Discourse, Linguistics, Logic, Metacognition, Radical Constructivism, Semantic Domain, Semiotics, Teaching Experiment, Mathematics Education

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A Dissertation

Presented in Partial Fulfillment of Requirements for the

Degree of

Doctor of Philosophy

in

Teaching and Learning with a Concentration in Mathematics Education

in

the Department of Middle and Secondary Education

in

the College of Education and Human Development

Georgia State University

Atlanta, GA
2021

DEDICATION

This dissertation is dedicated with honor and obedience to The Grand Architect, for whom I serve as a humble instrument. I love and respect my parents, Dr. Elijah Porter, Sr. and LaVerne Porter, who taught me the ways of humanity, and the brilliance that can be achieved through education. To all of the many students who gave me a chance to learn you…I salute you all!

ACKNOWLEDGMENTS

I would like to thank my doctoral committee for being patient and allowing me to find my way along this doctoral journey. To my academic advisor, Dr. David Stinson, thank you for enduring my engineering analyses at each and every step. You gave me the map and the tools to grow as a social scientist. To Dr. Peggy Albers, thank you for opening my eyes to the creative genius that exists in all of us. You taught me how to connect so that I may know that there is infinite knowledge in the smallest of messages. To Dr. Viviana Cortes, thank you for showing me the underlying structure that exists in all things. You taught me to seek the underlying structure whenever I needed guidance. To Dr. Jennifer Esposito, thank you for officially bringing mixed methods methodology into my purview. Your example of stretching into new areas became the model that fueled my stretching into new areas. I bring honor to Dr. Pier Junor Clarke and Dr. Gertrude Tinker Sachs, two leaders in culture and academia who granted me a snapshot of the world through their eyes and worldly experience. I also acknowledge the rest of the faculty in the Department of Middle and Secondary Education and across the Georgia State University campus for embracing a Mathematics Education student who simply wanted to learn more about what knowledge has to offer. To my writing group, we made it through the bumps and bruises. High applause goes to the bedrock of the group, the mathematician from Morehouse College, who journeyed through every line that I wrote. As for my family and friends, I thank you for simply being…and for demonstrating openness, honesty, and transparency. Last, I acknowledge the constant adjustments necessary for this product.

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FOREWORD

This dissertation is made up of three broad sections. The first section focuses on the idea of attributes. I provide attributes and characteristics of my own identity as a mathematics educator. I also detail the attributes and tendencies that seem to characterize the literature that is relevant to the mathematics experiences of African American students and that is relevant to the connections between mathematics and semiotics. The second section focuses on the constituency of this theory-building project. I first discuss the theoretical elements that constitute the conceptual framework used to structure this work. Then, I provide details on the set of micro-experiences, which generated the macro-experience that I used to find the optimal methodological tool. This methodological tool was then adapted (i.e., re-engineered) to meet the needs of this project and to move or evolve me from a classroom practitioner to a mathematics education theorist, in particular, and then into a social scientist, in general. The third section focuses on my abilities as an engineer. First, I used my engineering skills to transform the traditional teaching experiment into a re-engineered teaching experiment where the foci are the student as well as the activity. Second, I used my engineering skills and archived data to simulate a scientific study using my re-engineered teaching experiment. Last, I conclude this dissertation with a discussion about my findings, implications, and the answers to my research questions. In what follows, I provide greater details into each chapter.

In Chapter 1, I provide a timeline of sorts that captures my transformation as a mathematics teacher. This account is not an autoethnographic treatise. But rather, this first chapter is my effort to orient the reader to my demographic, historic, and epistemic profile. I hope that this profile allows the reader to better understand the decisions and actions that are discussed throughout. I begin by discussing the teacher certification process that I completed. I continue by providing many of the reflections that I had as a new teacher. One of my many reflections concerned the concept of hegemony, and I direct the reader's attention to a connection between hegemony and the phenomenon called the "achievement gap." Then, I provide a brief commentary on the achievement gap and its forever-present existence in the literature. Toward the end of the chapter, I offer a counter-narrative to the achievement gap. This counter-narrative leads to offering a suggestion to address the differentials in the mathematics performance of middle school students. I conclude the chapter by declaring my academic niche and stating the dissertation problem statement and research questions.

In Chapter 2, I provide the reader with a synthesis, within the context of mathematics education, of several of the key areas of my work: semiotics, semantics, and metacognition. I begin by giving attention to the historical development of semiotics within mathematics education. Such attention then requires that I provide a commentary on the funds of knowledge and the importance of personal experience. This discourse on knowledge construction through experience, then, provides me the opportunity to further draw attention to the critical nature of semantics and metacognition for mathematics learning. I conclude the chapter with a few summative closing remarks.

In Chapter 3, I provide a synthesis of three theoretical traditions, located in the understand paradigm of inquiry, that provide the theoretical foundations for this theory-building project: cultural-historical activity theory (CHAT), semiotics, and radical constructivism. I begin by providing a historical trajectory of CHAT. I continue by detailing semiotics and one of its manifestations in society, the semiosphere. Next, the meaning-making nature of semiotics requires that I give commentary on epistemology and symbolic interactionism. I conclude the chapter with a detailed analysis of radical constructivism and its relevance to the learning of mathematics.

In Chapter 4, I begin by discussing my high school teaching experiences and how I utilized my knowledge of and experiences with the engineering design process (EDP) to address some of the challenges that I have faced as a mathematics teacher. Next, I provide details of my middle school teaching experiences and the tandem of pilot studies that I was able to conduct in my classrooms. These pilot studies allowed me to investigate several hypotheses that I had developed about my middle school students. In large part, I followed the scientific method to conduct these investigations. My use of the EDP and the scientific method led me to searching for a methodological tool that I could use to not only test new hypotheses but also to address my practical classroom challenges. The methodology that I found which best met these two criteria is the teaching experiment; the basic principles of the teaching experience are then outlined. My findings from many informal implementations of the teaching experiment inspired me to consider making adaptations, extensions, or, maybe better yet, re-engineer the teaching experiment in an effort to broaden its investigative lens, its investigative power. In that, I desired to investigate the mathematics activity within an assignment, specifically its context, its constituent parts, and the role creation that it imparts onto the student with the same level of scrutiny that I investigated the mathematics learning of a student. As I provide insight into this re-engineering process, I conclude the chapter by detailing my own evolution from a sole practitioner, teacher, into a neophyte academic, theorist. This evolution was essential given that this dissertation project led me into the field of theory-building so to ensure that my re-engineering of the teaching experiment would be received as theoretically sound, robust, and critical.

In Chapter 5, I provide my thoughts and processes for my unique re-engineering of the teaching experiment. I begin by offering my perspective on the concept of generalizability; given that, the concept has such a large impact on theory-building. Next, I bring attention to a phenomenon that I call activity *dis*-aggregation, the dissection of an activity into its constituent parts. Activity disaggregation was a finding from my second pilot study; it is an integral part of my dissertation work. A comprehensive understanding of activity disaggregation requires a clear understanding of the semiosphere; therefore, for reference, I provide a brief review of the semiosphere and its analysis. My re-engineered teaching experiment compelled me to contend with certain ideological tensions; therefore, I provide an explanation of how I resolved the tensions between radical constructivism and sociocultural theory so to extend the teaching experiment into broader investigative areas. I then offer details into my actual adjustments to the teaching experiment. My first adjustment was the creation of a set of data collection templates. The second adjustment was the design of a new procedure for the teaching experiment that brings attention to both the actions of the student and the actions of me, the researcher. Next, I transition the discussion to the data analysis portion of the re-engineered teaching experiment by first presenting my perspective on coding and then detailing the creation of a set of data analysis templates. I conclude this chapter with a summary of the entire process.

In Chapter 6, I provide a simulation of the data analysis process that is possible from my re-engineered teaching experiment. Before I begin the simulation, however, I provide insight on how the COVID-19 pandemic affected this work. Due to the integration of so many ideas and concepts within this work, I present a review of the main points from each chapter. Next, I offer my stance on why this work is positioned within qualitative science. I then present examples of two archived datasets as exemplars of what is possible when using the data collection charts presented in Chapter 4 created to align with the re-engineered teaching experiment. I continue by providing a more comprehensive overview of thematic coding before I detail the first-cycle and second-cycle coding techniques that I used to analyze the archived data. I conclude the data analysis of the datasets from my tutoring students by offering the resulting theoretical paradigms for consideration. I repeat this process of first-cycle and second-cycle coding for my three-member classroom groups, and provide the resulting theoretical paradigms from the three-member classroom groups. After completing the thematic coding section of this chapter, I discuss the metacognition analysis process that I used. Next, I provide samples of this analysis as I conducted it on the archived data from my tutoring students and on my three-member classroom groups. I conclude this chapter by unifying the thematic coding paradigms and the metacognition paradigms and by providing the resulting over-arching theoretical and methodological perspective.

In Chapter 7, I re-present the main points of this theory-building project. I begin by discussing the need for combining CHAT with the teaching experiment. I then discuss the importance of activity dis-aggregation and its impact on this work. Next, I detail how involvement in mathematics discourse leads to the development of mathematics proficiency. I continue by discussing the idea of mathematics proficiency and show how it can extend to elements beyond the solution to a mathematics problem. I continue by emphasizing the need of semiotics within the mathematics classroom and acknowledge the presence of one ever-present phenomenon within this work—metacognition. I then present several conceptual models that serve as visual representations of this theory-building project. These visual aids helped me to refine my thinking throughout the project. Before I conclude, I provide explicit answers to my guiding research questions. I also thought it helpful to present a traditional lesson plan for the current classroom teacher who may wonder how the structure and findings of this project might be implemented in a mathematics classroom.

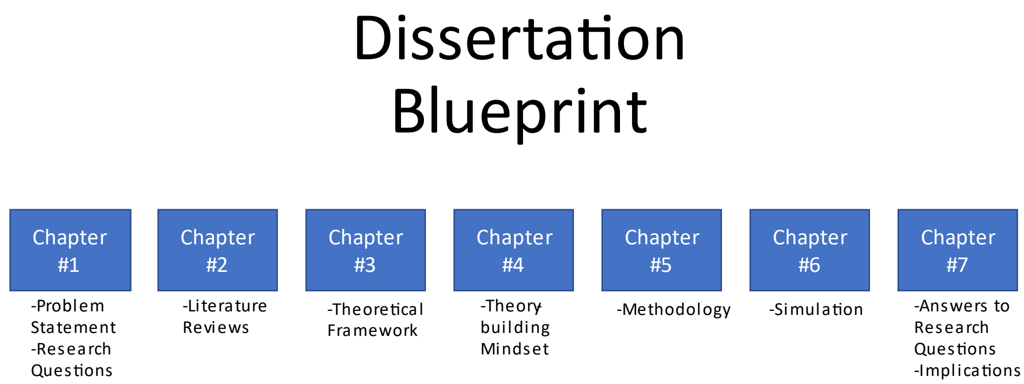

Figure F.1. Dissertation blueprint.

Figure F.1 above provides a visual map of the blueprint for this dissertation. In sum, I attempted to provide the reader with a small snapshot into my lived experiences as an expert mathematics classroom teacher as well as insight into my thinking as a novice mathematics education researcher. At most, the reader will understand how I have increased the mathematics proficiency of my students; and at minimum, the reader will understand how I have propelled forward the discussion about the evolution of mathematics education in the United States through the use of CHAT and the Teaching Experiment.

CHAPTER 1

INTRODUCTION

In this chapter, I provide a timeline of sorts that captures my transformation as a mathematics teacher. This account is not an autoethnographic treatise. But rather, this first chapter is my effort to orient the reader toward my demographic, historic, and epistemic profile. I hope that this profile allows the reader to better understand the decisions and actions that are discussed throughout. I begin by discussing the teacher certification process that I completed. I continue by providing many of the reflections that I had as a new teacher. One of my many reflections concerned the concept of hegemony, and I direct the reader's attention to a connection between hegemony and the phenomenon called the "achievement gap." Then, I provide a brief commentary on the achievement gap and its ongoing existence in the literature. Toward the end of the chapter, I offer a counter-narrative for the achievement gap. This counter-narrative leads to offering a suggestion to address the differentials in the mathematics performance of middle school students. I conclude the chapter by declaring my academic niche and stating the dissertation problem statement and research questions.

A Personal Journey: Making Sense of the Science of Teaching Mathematics

To be respectful to the process of writing a dissertation, it is only fair and necessary that I also present my subjectivities. To get to this point, it has been nearly ten years of intense historical, discursive, and epistemological analyses and ongoing philosophical and theoretical critiques. At times, these analyses and critiques have been conscious, and at other times unconscious. The type, degree, and timespan of the analyses and critiques are important to note because of my identity—projected and perceived. I am an African American man born in 1970 in Detroit, Michigan, educated and employed as an engineer, educated at both a historically Black university (Prairie View A&M University) and a predominantly White university (University of Michigan). Currently, I work as a mathematics teacher in the K–12 environment. I already possess a bachelor's and a master's degree, and this dissertation fulfills the remaining requirements for my doctorate. I am six feet, six inches tall; I wear glasses and I have excellent command of both Standard English and the African American Vernacular.

I am cognizant that I represent all these labels as I study how to guide the learning of mathematics more effectively for African American students. I am also cognizant that I generate specific perceptions in others—some threatening and some non-threatening. Within many, based on the context that I have established, I am perceived as the "angry Black man—a weapon of destruction" due to the systemic injustices and grand (master) narratives that exist throughout society. To others, I am perceived as the "intellectual Black man—a tool for change" due to the generations of hope, the organized networks of support, and the demand for social justice that systematically challenge the foundations of society.

I am familiar with the statement, "Perception is reality." This mindset is simplistic, reductionistic, and represents the very opposite of what genuine philosophical inquiry is all about. I cannot control perception; I can only control that which I endeavor to project. I endeavor to project compassion and not competition, healing and not hurting, construction and not destruction, reparation through remediation and not resentment, an evolution and not an evacuation. In what follows is a junior scholar's attempt to not only explore, describe, and explain the U.S. approach to educating the African American, but also to provide suggestions to move the country beyond this sub-standard standard. U.S. society is crumbling under its own weight of hegemony and social injustice; it is time that it evolved. Hegemony pertains to an individual's subscription to ideas that manage his or her behavior for the benefit of an oppressive system or institution (Adamson, 2014).

I spent 15 years working as an engineer. However, an opportunity presented itself that allowed me to have greater significance in the community and not simply in the corporate industry. The transition from a corporate engineer into the field of education was bumpy. I was fortunate enough to be hired provisionally as a high school mathematics teacher based on my 15 years of experience as an electrical engineer. The hire was provisional because I had to complete the state of Georgia teacher certification process, which included a lengthy list of courses and two challenging mathematics examinations.[1] The state of Georgia granted me 3 years to complete the teacher certification process. It was a particularly arduous challenge to successfully learn the amount of information that needed to be learned in both the area of mathematics pedagogy and the area of mathematics content in the time that was granted. And given that I was also teaching mathematics in a predominantly African American high school within the Atlanta metropolitan area took this challenge to an even higher height. Fortunately, my years of experience as an engineer allowed me to plan my work and work my plan; I successfully completed the Georgia teacher certification process during the last possible month of July 2008.

I offer this story of insight to provide a frame for what is to come. For many, the doctoral process begins the day that the candidate makes the decision to apply to the graduate school of a particular university. Also, for many, the purpose of the doctoral degree is to equip the person for a specific future career. These were not the cases for me. For me, the doctoral process began during the first day of school in 2005 when I was given an Algebra I class to teach to African

[1] Passing the state of Georgia Mathematics Proficiency Exams, the GACE Math I exam and the GACE Math II exam certify the test-taker as a "highly-qualified" teacher of mathematics. Challenging, because both tests required demonstrated proficiencies in mathematics content that I had not seen since my high school and early undergraduate years.

American teenagers. Most of my students had limited intrinsic motivation to learn the art of manipulating mathematical sentences. I did not know what I was doing because I had no prior experience in this area; my inability to perform and lack of experience did not seem to concern the system because no one came to my aid on that first day. It was on that day that I realized that knowing mathematics and utilizing it for 15 years did not transfer well into the skills needed to guide the mathematics learning of disinterested[2] teenagers. It was that realization which forced me to acknowledge that I had to teach myself to become a teacher of mathematics. I therefore continued within the Georgia teacher certification process with an altered motivation. I was not simply trying to complete the program. I was trying to make myself better than I was; I endeavored to evolve myself. It was that day that I began my doctoral program. In other words, I began 6 years before I formally entered the PhD in Teaching and Learning with a concentration in Mathematics Education degree program at Georgia State University.

For many at the university, it came as a surprise that I knew exactly what I wanted to study during my doctoral program. I was told by the committee that interviewed me that many aspirants have a general idea of their area of interest but require the reflection made possible through the graduate courses to truly focus and decide upon their actual topic. What this committee of university professors did not know was that I had spent the last 6 years engaged in the reflection of experiences that was granted to me by the combination of the Georgia teacher certification program, the 900 days of mathematics classroom experience, and the countless student interactions. My engineering background took hold during my 6 years of reflection. During those initial 6 years, I recall repeatedly thinking to myself and saying aloud, "The

[2] My use of the term "dis-interested" refers to the response that my teaching approach created within my students. The term in no way suggests the perception that my students had toward education.

fundamentals of mathematics have not changed in thousands of years. It should not be this difficult to teach something that has not changed to individuals who do not know it." It took me 5 of those 6 years of reflection to determine that I had a one-sided relationship with mathematics. During my own high school and college years, I was only aware of the computational aspect of mathematics where the goal is to calculate an answer. In fact, this particular aspect of mathematics was reinforced during my 15 years of engineering experience. I did not embrace the descriptive nature of mathematics until after 5 years of reflecting on that one particular question: "The fundamentals of mathematics have not changed in thousands of years, so why is it so difficult to teach something that has not changed to individuals who do not know it?"

It has been suggested that if a person commits to a goal for at least 10,000 hours then that person becomes an expert for that particular goal (Gladwell, 2008). For someone who commits 40 hours each week for 50 weeks each year to a particular goal, then after approximately five years that person becomes an expert for that particular goal. This commitment of hours is exactly what I achieved teaching high school mathematics. Although to be considered an expert is an intriguing thought, due to the limited proficiency[3] of my students, I did not feel as if I were an expert mathematics teacher. What I will affirm is that after 5 years of reflecting on that one particular question and trying many different strategies and tactics in the classroom with a variety of students during those 5 years, I resolved that there is a major flaw, I believe, in how mathematics is approached in the United States. Consequently, how mathematics is taught, particularly to African America students. After 5 years of what I consider failure as a mathematics teacher and the reflection that accompanied it, several focal points surfaced. I had been teaching mathematics from a flawed vantage point, as if I had captured mathematics, placed

[3] My use of the term "limited proficiency" reflects my position that proficiency is a nurtured phenomenon and not a native phenomenon. Wherever one finds limited proficiency, one can also find limited nurturing.

it in a jar, and now had to distribute it to my students. One clear problem with this approach was that I placed myself as omnipotent and omniscient as I presumed that mathematics was a tangible entity to be captured and that I in fact had the ability to capture it. A second problem with this approach was that I positioned myself as superior to my students. A third problem was that I had converted what should have been an educational experience for my students into a commercial experience for them as I presumed what I knew to be of value to my students. Realizing these three major failings in my approach to the teaching and learning of mathematics was the first sign of growth for me. The second sign of growth was the subsequent 3 months that it took for me to transform myself and my approach.

Now that I had determined that my approach contained major flaws, I thought about what allowed for my personal success in mathematics, specifically learning it. I quickly realized that I could not privilege my learning of mathematics during my elementary, middle school, high school, or college years because the teaching approaches that I had personally experienced were exactly the same flawed teaching approaches that I was utilizing with my own students. The only other learning experience that I had in mathematics was the informal process that I developed for myself during my teacher certification program. As previously mentioned, the state of Georgia granted me only 3 years to complete its teacher certification program. The state offered many courses to address the pedagogical dynamics of mathematics, but there were not many courses to address the mathematics content. But the two standardized mathematics certification examinations assessed arithmetic, algebra, geometry, trigonometry, probability, statistics, pre-calculus, and calculus concepts. I will not speak for any other person who had to or must endure such a battery, but I admit that I had to reteach myself all eight content areas in the 3 years that the state of Georgia granted me.

The traditional and formal approach to teaching these eight content areas is a minimum of 1 year per content area. I did not have the necessary 8 years; in fact, I had less than half the traditional time. So, I took a chance and developed a completely different approach to learning mathematics. The approach had to be comprehensive enough to embrace eight different focal points of mathematics and robust enough to endure the 3-year time constraint. The informal approach that I developed privileged two points. The first point was the fact that the foundations of mathematics had not changed in thousands of years. The second point was that each of the eight content areas was actually nothing more than eight different views of the same entity. This new perspective that these eight different courses of mathematics or eight different content areas of mathematics were actually describing the same phenomenon meant that there had to be an underlying commonality and consistency within each of the eight areas. So, instead of attempting to reteach myself eight different content areas of mathematics, I endeavored to discover the underlying commonality and consistency that exists within all of mathematics. What I realized is that mathematics is a language that humanity has created to describe, analyze, and predict behavior that occurs in nature. The level of the description, analysis, or prediction is revealed through the mathematics proficiency of the observer. Admittedly naïve, for me, this discovery was life changing.

My informal approach to learning those eight content areas required that I acutely attend to the language of each area and the language that was commonly used in questions typical of each area. Over time, I realized that the words used within texts that focused on one content area often referred to ideas and words that I found in other content areas. So, again, it was the language that caught my attention. I also became aware of the semiotic process by which a word not only has its own meaning but also how it can refer to a different word that has its own meaning. My ultimate insight was this iterative referential process that is embedded within mathematics. It is this constant referencing that allows a calculus problem to be analyzed and understood as an arithmetic problem, and all the other perspectives between. The semiotics of mathematics was my major awakening. During the 2010 school year, I taught my students to engage and interpret mathematics as a language and to attend to its constant referencing to other simpler mathematical concepts.

Hegemony and the "Achievement Gap" Hegemony

Experiencing such an awakening provided me with a tremendous sensation. Nonetheless, lest my ego take hold, I was convinced that this referential connection within and throughout mathematics had to be known by other mathematicians, but simply was not shared and presented within the classroom. This belief led me back to that particular question, "The fundamentals of mathematics have not changed in thousands of years; so, why is it so difficult to teach something that has not changed to individuals who do not know it?" This question provided the energy, perspective, and focus that motivated me to apply for the doctoral program in 2010, beginning the program in 2011. During my initial coursework, the awareness of such major flaws in such a fundamental system as the public school system, rather a fundamental institution of the cultural fabric of the United States, caused me to pause. I then endeavored to identify flaws in other U.S. systems and institutions beyond the public education system. Upon further inquiry, reflection, and discursive exchanges at the university, one fundamental theme continued to surface: flawed systems are allowed to exist specifically to establish or maintain some particular situation or structure. This contrived structure must serve a purpose. I discovered that the purpose of such a contrived structure replete with its inherent flaws is to maintain a hierarchy of racism, sexism, and classism through a network of elaborate forms of oppression. As an African American, I understood this implicitly; however, as a junior scholar, the awareness of the degree of facelessness, and degree of intricate systemic oppression became explicit. During the middle stages of my dissertation studies, specifically, just prior to my comprehensive examinations, I was introduced to the concept of hegemony.

Hegemony pertains to an individual's subscription to ideas that manage his or her behavior for the benefit of an oppressive system or institution (Adamson, 2014). An insightful perspective is that hegemony is the voluntary replacement by the individual of cultural practices the purpose of which is the survival, sustenance, and evolution of societal structures and institutions. Slightly more inspective than Adamson's definition, I define hegemony as emphasizing and supplying an external want to generate a situation or circumstance of inadequacy for specific individuals so as to create and reify an oppressive system or institution in the society. I would propose that hegemony is the oil that keeps unjust social structures active and efficient. I now had greater insight into why if the fundamentals of mathematics have not changed in thousands of years, why it is so difficult to teach mathematics to individuals who do not know it? My 15 years of teaching experience have convinced me that intentional efforts have been made to make the effective teaching of mathematics difficult and consequently the proficient learning of mathematics improbable to reify an oppressive social system. Again, acknowledging my naiveté, this was my second major awakening.

The Achievement Gap in Literature Databases

In North America, a common label that is used to refer to what I view as hegemonic forces within public education—the intersection of race and academic performance—is the term *achievement gap* (Gutierrez & Dixon-Rom, 2011). My interest lay at a deeper level than the general idea of the achievement gap, which, in many cases, incorporates the ideas of reading comprehension and writing of the English language. Due to my interpretation of mathematics as its own language, I wanted to focus not only on the reading, the writing, and the execution of mathematics beyond its representation in a standardized test environment, but also on its representation within the classroom learning environment. This focus led me to conduct a review of the literature to better understand this intersection of race and academic performance. This literature search contained three steps. The first step was to search only for the key words "achievement gap" in the title of the article. The second step was to conduct a search that required the key words "achievement gap" and "mathematics" in the title of the article. The third step was to conduct a search that required the key words "achievement gap," "mathematics," and "African American" in the title of the article. All searches were conducted for academic literature published within the years 2011 and 2016 using the Georgia State University literature search engine. This search engine accesses the EBSCOhost and the Galileo database systems. The EBSCOhost system has access to nearly 375 different literary databases, while the Galileo system has access to over 100 different literary databases. A total of almost 500 databases exists in both EBSCOhost and Galileo.

The Intersection of Hegemony and the Achievement Gap in the Literature

For the sake of my work, as mentioned earlier, I define hegemony as a denial of a genuine need while emphasizing and supplying an external want to generate a situation or circumstance of inadequacy for the purpose of creating and reifying an oppressive system or institution. The findings of a research study conducted by the Study Committee on Mathematics Learning and commissioned by the National Research Council was released in 2001 (Kilpatrick, Swafford, & Findell, 2001). The report entitled *Adding It Up: Helping Children Learn Mathematics* is a document that investigated the necessary elements that represent effective mathematics teaching (pre-K through eighth grade).[4] The document is quite explicit in revealing the most impactful characteristics of a learning environment that cultivates proficient mathematics learning. The report specifies that the aim of every mathematics teacher and every mathematics classroom should be the attainment of mathematics proficiency by the students. According to the study, mathematics proficiency is evident when the student displays the following five intertwined attributes: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (see Figure 1.1).

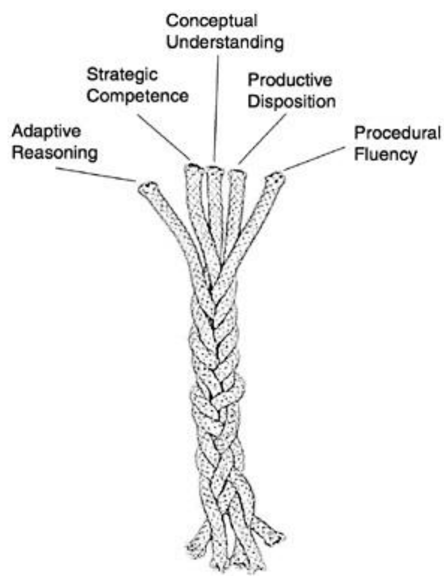

Figure 1.1. Intertwined strands of mathematics proficiency (Kilpatrick et al., 2001, p. 5).

Conceptual understanding focuses on students' "comprehension of mathematical concepts, operations, and relationships"; procedural fluency on students' "skills in carrying out procedures flexibly, accurately, efficiently, and appropriately"; strategic competence on students'

[4] At nearly 500 pages, *Adding it Up* is a book-length report approved by the Governing Board of the National Research Council—whose members are drawn from the councils of the National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine—that provides a comprehensive view of what it means to be successful in mathematics. This comprehensive view is based on syntheses of over a half century of research. The report has made significant contributions to mathematics education since its 2001 publication and has been cited more than 7,000 times. Its contributions are also indicated by its "Best Seller" status on the following platforms: STEM Education; Science and Technology Teaching Materials, and Math Teaching Materials, among others.

“ability to formulate, represent, and solve mathematical problems”; adaptive reasoning on students’ “capacity for logical thought, reflection, explanation, and justification”; and productive disposition on students’ “inclination to see mathematics as sensible, useful, and worthwhile” and on their mathematics agency (Kilpatrick et al., 2001, p. 5). It seems reasonable to presume that due to the extent of the study and nearly 500 pages of findings published in 2001 that any genuine effort to study U.S. mathematics education in the years after 2001 would frame the research around the five critical strands detailed in the report. Unfortunately, none of the literature that I found published in the years 2011 to 2016 investigating the achievement gap focused on any of the five strands. Instead, based on my evaluation of the published material, a hegemonic posture and a deficit-thinking perspective of the performance of African American students were consistent throughout the literature.

The prominent themes of the published work[5] during this timeframe include teacher quality,[6] family investment,[7] meritocracy,[8] psychology of the student,[9] multi-culturalism,[10] empirical data,[11] America's commitment,[12] open communication,[13] and education as a public service.[14] Despite the prevalence of these nine themes within the mathematics education

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[5] During the doctoral process, I found myself reading about topics and concepts that I initially believed would be integral to my proposed work. Over the years, however, I realized that knowledge construction can be integral in the development of my cognitive framework, or integral in the development of a particular research study. For the sake of the reader, I have made two decisions. First, I have made a conscious effort to locate the readings which have been integral to my cognitive framework in the footnotes. Second, I have made conscious effort to locate the readings which have been integral to this project in the body of the document.

[6] Torff (2014), Peterson, Rubie-Davies, Osborne, and Sibley (2016), Turner, Rubie-Davies, and Webber (2015), and Frye (2015) suggested that despite completion of a teacher preparation program, aspiring teachers who develop a low performance expectancy level for their students also develop a resistance to change this perception and do not provide their students with the appropriate rigor needed in the classroom.

[7] Shoraka, Arnold, Kim, Salinitri, and Kromrey (2015) directed the ideas of Rowley and Wright (2011) to mathematics and the parents' educational attainment and employment type. Specifically, their work highlights a correlation between college-educated parents who are employed in one of the STEM fields and their student's mathematics achievement.

[8] Craft and Slate (2012) suggested that disparities caused by individualism, ecology, and social capital within racial and ethnic groups lead to adverse effects in academic performance.

[9] Stereotype threat theory states that an individual will be at risk of confirming a negative stereotype about one's identity group (Steele & Aronson, 1995). In this case, the negative stereotype is that African American students perform poorly on standardized assessments (Ochoa, 2013).

[10] McKown (2013) explored the validity and reliability of social equity theory to describe the inequities in the social structure that lead to disparities in academic performance among racial groups. Hartney and Flavin (2013) identified that the academic performance of marginalized students is de-prioritized as a matter of local, state, or national government concern.

[11] Condron, Tope, Steidl, and Freeman (2012) explored the idea that academic performance is contingent upon non-school factors, within-school factors, and between-school factors. Each of these factors requires a qualitative analysis instead of simply a quantitative analysis.

[12] Society has made perfunctory attempts at addressing the differences in academic performance across racial groups (Vijil, Slate, & Combs 2012; Westerman, 2015). Specifically, Rojas-LeBouef (2012) explored the effect of the vast resources that institutions have committed to addressing the achievement gap and how prominent the achievement gap remains.

[13] A study conducted by Hays (2013) focused on the communication and collaboration between school administrators and their superiors, district administrators, state administrators, and elected officials. The effort revealed the benefits to students that could be realized by providing more autonomy to school administrators.

[14] Gutierrez and Dixon-Roman (2011) provided additional insight into the concept "pedagogy of poverty" developed by Haberman (2010). This concept describes an approach to mathematics teaching and learning that is focused on

literature, each theme, I believe, is a divergence from the five attributes needed to achieve mathematics proficiency as detailed in the report *Adding It Up* (Kilpatrick et al., 2001). Any effort to criticize mathematics performance before mathematics proficiency is attained is an intentional effort to distract, disrupt, and obfuscate the necessary focus. The revelation of this stark representation of hegemony that had penetrated the discourse of mathematics performance was disheartening but did not dissuade me. It provided me with a clearer line of focus, and the ability to generate a counter-narrative.

The Achievement Gap Counter-narrative

The term achievement gap is a misnomer. The approach should not be to place the blame on the student for a performance that places her or him in a quartile that is less than the optimal quartile. The effort should be to hold accountable the process that was used to prepare the student. The task at hand is to acknowledge the situation whereby the student is the product of an educational process, and a misdirected critique of the student should be redirected and become a fully enthralled critique of the educational process (Hilliard, 1994; Kozol, 2005; Ladson-Billings, 1997; Martin, 2000; Steele 1992).

Predominantly existing within journals and through the discursive practices of the academy, the impact of this misdirection can be seen in the world of standardized testing and the evaluation of the academic ability of demographic groups (Steele & Aronson, 1995). African American students are the target of discursive practices in the effort to place blame on them for their performance on standardized mathematics assessments. The effort at establishing a counter-narrative has been undertaken (e.g., Ladson-Billings, 1994). Nonetheless, more work needs to be done to redirect the momentum of hundreds of years of deficit-thinking about African

student preparation for a standardized exam. The cognitive development of the student for the student's sake is a peripheral idea. Gutierrez was effective in connecting such a pedagogical agenda to hegemony.

Americans. The project reported here has the humble goal of providing the reader with a cultural-historical activity theory (CHAT) and linguistic analysis of the meaning-making experiences of middle school African American mathematics students as they work to develop mathematics proficiency. A focus on the cultural-historical components of activities and on language during the meaning-making processes of mathematics learning, I believe, can provide additional insight into the cultivation of proficient mathematical knowledge for students.

Investigating the meaning-making process of students ultimately means that the learning experiences of the individual student must be studied. These learning experiences represent the conscious and unconscious processes by which relationships and associations are sought by an individual. Attention to this process is referred to as knowledge construction (von Glasersfeld, 1989b). Another word for knowledge construction and its investigation is epistemology. Historically, there has been much attention given to epistemology and its two major paths: positivism and constructivism. The epistemological battle between positivism and constructivism has been waged for many years (Comte & Martineau, 1853/2009; Piaget, 1952).

The mis-guided interpretations and assertions of those loyal to the ideas of positivism have created a culture and environment of deficit-thinking as it relates to the academic capacity of the African American student. Such assertions are fixed within the data analysis that is conducted on standardized test performance. Because such critics embrace the idea of an absolute truth and time-independence in human development, the test performance of an individual at a particular point in time is viewed as a representation of the full capacity of the individual. For the constructivist, such an opinion is filled with fallacies. The perspective of the constructivist is that the capacity of humanity is boundless and that a snapshot of a person’s existence at one time interval is not a definitive indicator of her or his potentiality. In other words, both the knowledge-construction processes and the knowledge-construction environments affect the learning experiences of the learner. As such, results from standardized assessments should be interpreted as evaluations of the quality of the process and of the environment within which the student exists. In short, mathematics standardized assessments should serve as summative measurements of how well the mathematics teacher and the mathematics classroom cultivate mathematics proficiency within each of the students.

A Suggestion: Semiotics and Meaning-Making Processes

I self-identify as a constructivist. Beyond my prior reference to the historical description of constructivism, my lived experience in my own mathematics learning and my own increasing mathematics proficiency serve as artifacts of my own cognitive evolution. As a constructivist, I read the data from standardized assessments as indicators of the deficiencies that exist within the learning processes and learning environments of many U.S. students. If the results show that a deficit exists and I locate the deficit within the processes or within the environments instead of within the student, then the meaning-making processes and its constructs should be scrutinized. Specifically, scrutiny should be given to the use, or lack thereof, of semiotics within the mathematics classroom. Semiotics, the art and science of meaning-making, establishes the optimal use of meaning-making constructs (Chandler, 2007; Eco, 1978). Deliberate and informed use of semiotics can address, I believe, the deficiencies within the learning processes and the learning environments.

I believe that the utility of semiotics can lead first to cultivating environments for the student. Second, it can lead to the development of mathematics proficiency by the student. Third, it can result in an optimal performing student. In the effort to evaluate the characteristics and viability of explicitly and intentionally integrating semiotics within a mathematics education context, I conducted a comprehensive search of the journal data base. As displayed in Table 1.1, the search revealed some dis-heartening insight about the number of publications regarding semiotics and its associated words within mathematics education, specifically, between the years 2013 and 2018. The point that I want to present here is that the larger theme of problem-solving occurred over nine-times more often than the more constituent themes of linguistics, semiotics, and modeling in the publications of the 2013–2018 years. My challenge with this publication frequency and rate is that it conflates the constituent elements of meaning-making into an indistinct nebula as represented by this larger theme. Arguably, for mathematics, it is the constituent elements and their interrelationships that provide the foundation for proficient mathematical thinking and performance (Kilpatrick et al., 2001). It is unfortunate that the published research in mathematics education does not display this awareness, and more importantly, one effect of such a lapse is to obfuscate the analysis of a literary search for the foundational elements of mathematics learning when such a search is conducted by a non-suspecting reader.

Table 1.1
Findings of Publications Regarding Semiotics and its Associated Words 2013–2018

Publications	*Listings*
Problem-solving and mathematics education	2,856
Linguistics and mathematics education	378
Semiotics and mathematics education	375
Critical thinking and mathematics education	262
Modeling and mathematics education	240
Semantics and mathematics education	22
Syntax and mathematics education	5
Pragmatics and mathematics education	5
Computational fluency and mathematics education	3

Mathematics proficiency and mathematics education	2
Multiple representations and mathematics education	2
Deductive reasoning and mathematics education	2
Inductive reasoning and mathematics education	1
Abductive reasoning and mathematics education	1
Procedural fluency and mathematics education	1
Conceptual fluency and mathematics education	0
Multi-modality and mathematics education	0
Multi-modal literacy and mathematics education	0

A Discovery: An Academic Niche

The point of a dissertation is to find a niche and position oneself within it. Based on the dominant themes of recent years, I have identified my niche and this research project represents my initial efforts to position myself within it. In my view, it is unacceptable that syntax, semantics, and pragmatics, the foundational elements of semiotics (Saussure 1998; Peirce 1870/1984; Mead 1934), are the subjects of so few publications in recent years. I use the current discourse of the achievement gap as my launch pad, and I leverage the dearth of published work on the impact of semiotics, syntax, semantics, and pragmatics on mathematics education and their role in the development of mathematics proficiency as the guideposts for my trajectory.

Problem Statement and Research Questions

I have resolved that somehow language and my inability to address it properly were lapses in my pedagogy. Coincidently, based on the findings of my literature analysis of the term achievement gap, my inadequacies within the mathematics classroom have been duplicated across the nation and across a considerable time span. Teaching mathematics successfully is a procedural and systemic problem; it is not a student-contextualized problem. I draw solace from the potentiality that can be experienced by explicitly and intentionally integrating semiotics within the mathematics classroom.

With a focus on semiotics and its constituent tools of syntax, semantics, and pragmatics, rich meaning-making is available with any mathematics situation. A problem that I have noticed in the literature is that not enough attention is placed on these aspects when students are learning mathematics. I propose a re-direction. My project specifically focused on African American middle school students. It also focused on the syntax, semantics, and pragmatics of mathematics situations and the activities through which students can leverage these tools to construct their own mathematics knowledge in the effort to achieve mathematics proficiency. My research is one small part in my evolution from using the data from standardized assessments to evaluate and scrutinize the student to instead evaluate, scrutinize, *and* improve the processes and activities through which students construct their mathematics knowledge.

A new question that draws my attention is what insight can an analysis of mathematics as a language provide for improving my pedagogy? Although mathematics proficiency is the end-goal, mathematics proficiency cannot be hoped for if the student does not first learn and become confident in constructing his or her own mathematics knowledge. It is anticipated that the effort to study the nuances, patterns, and manifestations of language in the mathematics register could chart a different trajectory within mathematics education. Using the constructive process of the teaching experiment (Steffe, 1991) within a cultural-historical activity theory (CHAT) framework (Engstrom, 2008; Leont'ev, 1981; Vygotsky, 1930/1978), this research inquiry generated a model with the intention of addressing the following questions:

1. How does teaching mathematics as a language system affect the construction of mathematical knowledge (learning of mathematics) by African American students?

2. What can be understood about the construction of mathematical knowledge (learning of mathematics) by African American students when different language systems beyond numbers and operations (visual imagery, movement, written/oral language, for example) are integrated into the mathematics curriculum? (In short, how can semiotics assist with the interpretation and learning of mathematics by African American students?)
3. What are the dispositions of African American students toward mathematics when different language systems (visual imagery, movement, written/oral language, etc.) are integrated into their learning? (In short, how can semiotics impact the disposition of African American students toward mathematics?)

As detailed throughout, hegemonic conventions promote that many African American students produce lower than average test scores on standardized mathematics exams for a variety of reasons. Therefore, my foundational question is: What insight might an analysis of mathematics as a language provide in the formulation of a counter-narrative to this hegemonic propaganda? It is hoped that the efforts demonstrated within this project will chart a course for pivotal transformation of mathematics teaching and learning for all. It is my hope that my dissertation work will be effective as one insightful and instructive example for releasing mathematics education from the repressive tentacles of hegemonic forces.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I provide the reader with a synthesis, within the context of mathematics education, of several of the key areas of my work: semiotics, semantics, and metacognition. I begin by giving attention to the historical development of semiotics within mathematics education. Such attention then requires that I provide a commentary on the funds of knowledge and the importance of personal experience. This discourse on knowledge construction through experience, then, provides me the opportunity to further draw attention to the critical nature of semantics and metacognition for mathematics learning. I conclude the chapter with a couple of closing remarks.

Evaluation of the Educational System

It is due to my engineering background and experience in quality control that I can confidently say that over the years the educational system in the United States, I believe, has become dysfunctional for a large number of students. A sizable percentage of these underserved students are neither from the ruling class nor the ruling culture. Consequently, due to the fact that the rituals and routines of the public school system are aligned with the rituals and routines of the ruling class and the ruling culture, without formal training in these rituals and routines, many students do not have as productive an educational experience as is possible (Durkheim, 1922/1956; Quantz, 2011). I have a unique perspective in making this statement because I teach mathematics to marginalized [15]students. Ever since I stepped into the classroom, I have been amazed at how many moving parts there are to the U.S. educational system, and how both

[15] I vacillate between the words “marginalized” and “non-dominant” to refer to a group of people who are intentionally mistreated. Due to the political and power implications that marginalized generates in the United States, I want the reader to understand that my work is an affront to not only the word that references the African American, but also an affront to the politics and the power that accompanies this reference.

informal and formal training in the nuances of the rituals and routines of the school and classroom environments is a necessity. Due to this interdependence, it is unreasonable, I believe, to evaluate the U.S. education process by focusing on individual variables. No part works in isolation. Many years ago, Bronfenbrenner (1977) commented on the massive task of studying human social systems. According to Bronfenbrenner[16], a robust and productive evaluation of a human social system requires an analysis of human development that goes—

> beyond the direct observation of behavior on the part of one or two persons in the same place; it requires examination of multiperson systems of interaction not limited to a single setting and must take into account aspects of the environment beyond the immediate situation containing the subject. (p. 514)

The resilience of rituals and routines across ever-changing social structures is achieved through discourse[17] (Halliday, 1978). Within the educational system, published social theories and cognitive theories are effective not only in highlighting advances in these areas but also in maintaining existing rituals and routines (Vygotsky, 1930/1978). Work in social theories and cognitive theories are most prominent in the university academic setting. In academia, one constant focus is to conduct research and publish journal articles. As a doctoral student, I have had my own experience with the use of published work to reinforce existing rituals and routines. As an aspiring researcher, I have had several university professors suggest that I evaluate the

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[16] Bronfenbrenner (1977) continued by detailing the four structures that constitute human social systems. The Microsystem is the system of relations between the subject under inquiry and his or her immediate environment. The Mesosystem is the system of interrelations of various settings containing the subject under inquiry at a particular point in time. The Exosystem is the system that contains formal and informal social spaces which do not interact with the subject under inquiry but do impact the subject under inquiry. The Macrosystem is the system that contains the institutional elements of culture under whose power and influence the subject under inquiry is immersed.

[17] I use *discourse* here in the same sense as Halliday (1978), which is to say that discourse is a phenomenon which includes the total communicative event and activity of the speaker or writer, the function of the speech or word, and the style of the speech or word.

leading journals in my field of interest and investigate the leading themes of their recent publications. I believe the point of this advice was to align my research interests with the interests of the editing team of the leading journals who could facilitate my path to becoming a published researcher. It would not be too presumptuous to think that other senior researchers are sharing this same advice with other aspiring scientists and junior scientists. The point here is that in many cases, the alignment of ideas with the ideology of an editing team, leads to situations where, in many cases, the dominant topics stay dominant and establish a density and momentum that the non-dominant topics simply do not or cannot achieve. The consequence of such rigidity in the dominance of particular topics is the reification of the rituals and routines of the ruling class and the ruling culture (Durkheim, 1922/1956; Quantz, 2011).

Historical Reviews

As previously displayed in Table 1.1 (see Chapter 1), according to my preliminary review of the literature for the use of semiotics within mathematics education, just fewer than four-hundred articles were published on that topic from 2013–2018. As important as I believe that semiotics is to mathematics education, I view this as a dismal number of publications and so few publications definitely makes this particular topic a non-dominant topic within academia.[18] I was also curious about the more prevalent themes within the topic of semiotics and mathematics education and I have attempted to catalogue these themes below. The purpose for this course of action was to facilitate my identification of the most potent niche in the literature that would allow me to expound upon my own particular ideas.

[18] I made every effort to focus my literature analyses on the years 2013–2018, specifically because my aim is to offer the field of mathematics education an academic ecological alternative and my immersion in the literature of this more recent timeframe best positions me to locate any gaps in the literature and identify my niche. There are a few references however that are before 2013 which have important relevance to my work in which I made the decision to include here.

Semiotics and Mathematics Education

I grouped the published work from 2013 until 2018 into the themes detailed in Chapter 1, which are the five strands that constitute mathematics proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick et al., 2001). The following analysis of the literature is not meant to be exhaustive, but rather I wanted to review the work of several of the more prominent researchers in the field of mathematics education and determine how they aligned (or not) semiotics and mathematics education with the strands necessary for mathematics proficiency.

Conceptual Understanding

The articles that I classified as the conceptual understanding strand focus on the clarity of the students' thinking. Three groups of researchers offered work that focused on cognitive clarity: Arzarello and Sabena (2011); Sandefur, Mason, Stylianides, and Watson (2013); and Pino-Fan, Guzman, Font, and Duval (2017). Within the three teams of researchers, the attributes of reasoning were highlighted. Although explicitly detailed in only one of the articles, each article described features that reference deductive reasoning, inductive reasoning, and abductive reasoning. According to Arzarello and Sabena (2011), deductive reasoning starts with a general rule, evaluates a specific situation, and establishes a conclusion. Inductive reasoning starts with a specific situation, evaluates many different trials of that situation, and establishes a general rule. Abductive reasoning starts with a general rule, evaluates a conclusion, and articulates the possibilities for the specific situation.

Additionally, the articles describe the supplementary attributes of semiotic control, diagrammatic reasoning, and theoretic control. Semiotic control refers to the selection and treatment of representative forms (Arzarello & Sabena, 2011). Diagrammatic reasoning is the use of visual modeling to represent and expound upon a given situation (Arzarello & Sabena, 2011; Radford, 2014). Theoretic control refers to the utility of mathematics concepts. The intermixing of semiotic and theoretic control produces a situation that the researchers refer to as implicit insight. Implicit insight refers to the awareness that is gained through the modeling of interrelationships which exist between phenomena but are not explicitly stated within a given description (Arzarello & Sabena, 2011). In my view, it is this idea of implicit insight that distinguishes one person's proficiency with problem-solving tasks from another person's struggles.

Strategic Competence

The articles that I have placed in the strategic competence strand focus on the various forms by which a student interacts with mathematics. As detailed previously, strategic competence focuses on the ability to represent and solve mathematical situations (Kilpatrick et al., 2001). The key to this definition is the representation of mathematical situations, which is synonymous to the idea of conceptual representation presented as by Ajose (1999). According to Ajose, an insightful definition for conceptual representation was established by Professor Arcavi who referred to conceptual representation as visualization:

> the ability, the process, and the product of creation, interpretation, and use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. xx)

Arcavi (see Ajose, 1999) went further to offer three functions of visualization: (a) support and illustration of essential symbolic results, (b) a possible way of resolving conflict between correct symbolic solutions and incorrect intuitions, and (c) a way to help the engagement with and recovery of conceptual underpinnings that may be easily bypassed by formal solutions. In short, Arcavi discussed the significance of visualization on the process of meaning-making.

Duval (1996) presented the idea of the semiotic register to refer to a system of representations in the form of a numerical representation, a graphical representation, a tabular representation, a pictorial representation, or a linguistic representation. These representations allow for three cognitive activities: the production of a concept, the evaluation of the concept, and the conversion of the concept between different semiotic registers (Duval). This idea of conversion between different semiotic registers has come to be called semiotic bundling (Arzarello, Paola, Robutti, & Sebena, 2009).

Meaning-making through the semiotic registers does require some degree of mental acuity. de Almeida and da Silva (2018) provided a clear interpretation of the triadic relationship that exists with conceptual representations which is aligned with the work of Peirce (1870/1984). First, there is the intention or the intended concept, as determined by the primary interlocutor, which generates thought activity. Second, there is the semiotic register, as detailed by Duval (1996) that serves as the representation of the intended concept and is decided upon by the primary interlocutor. Finally, there is the interpretation of the semiotic register by the primary interlocutor or a potential secondary interlocutor. The primary aim is to have the intention and the actual concept align as closely as possible. The secondary aim is to have the intention and the interpretation be aligned as closely as possible.

Another form of semiotic symbolism is the gesture. There are three perspectives for the utility of gesturing to convey meaning. One perspective is that gesturing is not a supplement to speech but is an extension of speech in that gestures contain meaning which is not presented through the speech effort (Johansson, 2014). Another perspective is that gestures are the precursors to the representation of abstract ideas in a speech act (Sfard, 2009). The third perspective is that gestures serve to coordinate and construct thought, and not simply as a representation of a reflection of thought (Radford, 2003).

Gestures can be considered as pictorial representations and are physical movements that exist due to the person's imagination. Gestures are framed as multi-modal representations which support the idea that "thinking does not occur solely in the head but also *in* and *through* a sophisticated semiotic coordination of speech, body, gestures, symbols and tools" (L. Radford as cited in Ferrara, 2014, p. 111). An interesting assertion established by Ferrara is that gesturing is a manifestation of the creativity and imagination of the student due to her or his "emotional, immersive, and animated experiences of the mathematics that the students could live" (p. x). This assertion echoes the point mentioned earlier that students must have a connection with the mathematics. Gesturing is one example of such a connection.

The Learning Environment

The articles classified for the learning environment group focus on the semiotic elements of the structured places that promote the construction of mathematical knowledge. Unfortunately, the concept of a learning environment is not explicitly established as one of the strands of mathematics proficiency; however, the authors of the report *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick et al., 2001) exerted great effort to make the implication clear that they were referring to the cultivation of mathematics proficiency within a learning environment—the classroom. Therefore, I am comfortable with using learning environment as a category for the synthesis of this literature review.

A comprehensive investigation of the mathematics learning environment and its constituent parts was detailed in Kuzniak, Tanguay, and Elias (2016); this research team came to refer to a mathematics learning environment as a mathematical working space (MWS). In establishing the description of the MWS, Kuzniak and colleagues suggested that such a learning environment contains a task with a mathematical goal. The task is a contextualized situation that contains a set of instructions. Continuing, they submitted that an activity is the work that is done to accomplish the instructions within the confines of the given task. Much of the work outlined in the article is aligned with my own theoretical and practical concerns and proposed resolutions. Although there are many similarities between our two approaches, I am most interested in the experience of the individual student, whereas the MWS paradigm is most interested in the collective experiences of students in the classroom.

An aspect beyond the activity of the student to be considered in the mathematics learning environment is the activity of the instructor. Arzarello, Ascari, and Sabena (2011) contended that the role of the teacher is to serve as the expert in a cognitive apprenticeship relationship with the student and prompt the student through her or his meaning-making processes. One artifact highlighted in their research centers on the creation of examples by the students so that the students have a supply of reflective experiences on which to refer when addressing future mathematical situations. According to Mason (2008), “it seems patently clear that what teachers can do for learners, indeed perhaps the only thing they can actually do for learners is to direct learners’ attention” (Mason, 2008. p. 31). The role of the expert in this relationship is to direct the attention of the apprentice (i.e., the student) to situations that require effort to identify differences in examples of the same phenomena and to identify features of potential examples which eliminate them as acceptable models. The aim of the creation of such an example space is for students to activate their deductive reasoning, inductive reasoning, and abductive reasoning. Arguably, these are challenging aspects of thought for many students to activate, but such activation establishes a distinct and deliberate connection to meaning-making. This process is guided by the teacher and (hopefully) internalized by the student. This level of knowledge construction by the student can serve as an indicator or predictor of the student's level of mathematics proficiency.

Maracci and Mariotti (2013) offered another perspective on the connection between teaching, learning, and a cognitive apprenticeship relationship:

> Teaching–learning is a semiotic process: it involves both teaching–learning to do and teaching–learning to mean and originates from an intricate interplay of signs. One of the objectives of teaching–learning is the development of the individual's higher mental functions, which entails in particular the formation of "scientific concepts" related to specific disciplines. The term "scientific concepts" was used by Vygotsky to designate abstract general concepts organized in conceptual systems as distinct from "everyday concepts." While everyday concepts are based on direct experience, the formation of scientific concepts needs conscious awareness and volitional control. Individuals have to be involved in semiotic processes leading to the explicit formulation of the meanings they have developed in relation to an activity, in order to become conscious of such meanings. (p. 21)

In addition, the perspective of Maracci and Mariotti (2013) is that the goal of a student that is in the learning mode is to accumulate information and experience through a teacher-created activity. After such accumulation, the goal of a student who is now in the processing mode should use his or her skills in deductive reasoning, inductive reasoning, and abductive reasoning skills to filter through the collection of information and experience and locate the commonalities or dominant themes. It is the elevation of these themes into concepts and principles that is the over-riding aim of the complete activity. It is the teacher who guides the student through this process.

Adaptive Reasoning

The articles in the adaptive reasoning strand focus on mathematics discourse in the classroom. An important element to classroom discourse is the idea of a social collective or collaborative environment whereby groups of students in the classroom work together to produce a comprehensible object (Sfard, 2008). This object could be a diagram, an equation, a picture, or any other semiotic representation. This type of effort facilitates learning and actually extends it (Schreiber, 2012). Through oral exchange, Kaartinen and Latomaa (2012) suggested the importance of students using their own social language at the onset to discuss mathematical objects and having the instructor begin to transform their social language into the appropriate mathematical vocabulary and discourse. In this way, the instructor serves as a bridge from the language and experience with which the students are most familiar to language and experience that is new (Radford & Roth, 2010; Vygotsky, 1930/1978). This idea of social language or social culture as distinct from academic language or academic culture is accentuated in the work by Wake and Williams (2004). These researchers demonstrate how metaphor, analogy, and mathematical modeling assist in bridging the gap between social experience and academic experience, specifically, as it relates to understanding mathematical concepts and their relationships in much the same way that Maracci and Mariotti (2012) detailed in the teaching-learning semiotic process previously mentioned.

Productive Disposition

The articles in the productive disposition strand focus on the societal responsibilities of mathematics and its broader implications. The work of Straehler-Pohl, Fernandez, Gellert, and

Figueiras (2014) showed how human perception affects teacher expectation in the classroom. They concluded that social stratification in society permeates into the classroom through classroom discourse as made manifest in teacher–student interactions. In many cases, the students are unaware of this intellectual inequity and how it results in their social stifling. The idea of social stratification is also studied in the formally constructed mathematics standards and in standardized mathematics assessments (Morgan & Sfard, 2016). Morgan and Sfard studied the language used within the standards as well as various assessments and highlighted how discursive manipulation does indeed exist, changes over time, impacts the classroom dynamic, and reifies the societal structure.

The topics of teacher preparation and the importance of the teacher as an integral bridge between social culture and academic culture in the mathematics classroom are investigated by de Freitas and Zolkower (2011) as well as by Iori (2018). These researchers found that mathematics teacher preparation programs focus more on the existence of formal mathematical principles than on the transference of these principles across multiple semiotic modalities or on the transference of these principles into the social culture of students. It is this transference of formal mathematical principles into the lived experience of students that establishes a direct line between mathematics and social justice (de Freitas & Zolkower, 2009).

Commentary on Identified Themes

Respectful work has been conducted in the area of semiotics and its integration into mathematics education; however, based upon my review, there is a litany of ideas that emerge which may obfuscate the efforts of one who attempts to identify pivotal themes. In my effort to develop a student-focused process for teaching and learning mathematics that can serve as an alternative to the teacher-focused process, I use the many ideas presented in the literature for semiotics and its integration into mathematics education as important elements. From these elements, I find the various forms of reasoning, activity, and cognitive apprenticeship to be the most substantive.

Although the themes as discussed present noble efforts by researchers in highlighting the necessity of the student to develop conceptual understanding before expecting to display strategic competence or adaptive reasoning in the pursuit to establish a productive disposition with mathematics, I think the prior knowledge and prior experience of the student should serve as the catalysts from which all efforts should commence (Resnick, 1985). It is extremely challenging for a teacher to guide a student to mathematics proficiency if an accurate start point for the student has not been identified. A student's prior mathematical knowledge and prior mathematical experience serve as that student's start point. Unfortunately, in the literature that I reviewed, not much attention seemed to have been given to the student's prior knowledge and prior experience. In what follows, I attempt to provide a broader discussion on these important items.

Funds of Knowledge[19] *and Personal Experience*

Although I applaud the effort extended by many of the researchers, I do find opportunities for improvement. I think that the most robust knowledge is that which is constructed through one's own experience. This experience could be a completely new experience or an elaborate connection of new experiences with old experiences. In either case,

[19] The totality of this project is focused on student learning as made possible through metacognition. Historically, "funds of knowledge" has been privileged as a tool for an instructor's teaching (see, e.g., Gonzalez, Moss, & Amanti, 2005); however, in my work, I am privileging funds of knowledge as a tool for metacognition to facilitate and cultivate student learning.

the foundation of a person's knowledge construction is firmest when the foundation is the person's own experience and not the presented experience of another. Connecting new experiences with old experiences is a reference to a person's funds of knowledge[20] (Gonzalez, Moss & Amanti, 2005). As Gonzalez, Moss, and Amanti (2005) detail—

> The point of the funds of knowledge project was indeed to immerse the teachers into a family's living environment and have the teacher uncover the social, cultural, historical, labor-oriented, ecological, economic, and political knowledge that the family constructed through social networking for its survival. The transformative education and discourse that occurred as a result was in direct opposition to the grand narratives. (p. 287)

A line of inquiry that was stimulated within me as I read the work of Gonzalez, Moss, and Amanti (2005) is the similarity that seems to exist between prior knowledge and the concept of funds of knowledge. The two seem to have more similarities than differences; and the only resolution that I could establish for their distinction is that funds of knowledge may refer to a person's prior knowledge that is based upon his or her personal experience.

Semantics

As I continued to work to understand the distinction between funds of knowledge and prior knowledge, I conducted another literature search. This particular search led me to the ideas of meaning-making and semantics within mathematics education. Amongst the 22 articles that have been published between 2013 and 2018 detailing various connections between semantics

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[20] The great distinction, I believe, between the ideas of *Funds of Knowledge* and *Prior Knowledge* is that the former centers on the general view that a student gains knowledge through his or her family, social, or cultural base through informal means. A reductionist view limits the informal means to food, dance, clothing, music, holidays, and dialect. The latter centers on a more comprehensive prospect whereby specific topic knowledge that the student constructs is based upon experiences with his or her peers, teachers, family, social, or cultural base through informal and/or formal means. The specific effort was to provide a counter-narrative to the idea that culture is homogenous and static and can only be represented through food, dance, and holidays, and not extrapolated to include actions, activities, tools, and interdependence (Vygotsky, 1930/1978).

and mathematics education, a few have relevance to the ideas of personalized experience and prior knowledge. McGowen and Tall (2010) posit that new information which is connected to the person's prior knowledge leads to new knowledge construction through appropriate relationships and this new knowledge has greater robustness and greater retention. Bruya and Ardelt (2018) provided a distinction between object and concept. According to Bruya and Ardelt, an object is a noun or verb that is disconnected from the person's prior knowledge or personal experience and is reduced to memorization and regurgitation.

Conversely, a concept is a noun or verb that must be interwoven into the person's prior knowledge or connected to a personal experience for comprehension to occur. Students are ineffective in their comprehension and application if they attempt to only memorize or regurgitate objects. Cheng (2003) offered a unique insight on the idea of semantics by providing a new word, semantic transparency, as the center of his work. According to Cheng, semantic transparency is the making of the conceptual structure of knowledge readily apparent in the structure of a representational system. Due to Cheng's position that information-of-interest already exists within some type of a representational system, I extend his idea of semantic transparency to the phenomenon of the semantic domain as first presented by Trier (1931), as well as to the idea of prior knowledge. According to Trier, a semantic domain is an interconnectivity of phenomena belonging to the same field of interest. Within prior knowledge, the pre-existing relationships of information or knowledge seemed to seamlessly reference the ideas of a semantic domain. To achieve semantic transparency, Cheng states that there are four essential elements of the semantic domain: (a) primary meaningful distinctions, (b) categories of things, (c) universal invariants, and (d) overarching relations.

Ideas consistent with semantic transparency appear in Ott, Brunken, Vogel, and Malone (2018). Ott and colleagues presented two ideas that can lead to greater robustness of a semantic domain. As highlighted by Ott and colleagues, but established by Seufert (2003), intra-representational coherence is the understanding of a phenomenon within a specific representational style by establishing relationships and connections within its constituent elements. Seufert also established the idea of inter-representational coherence which is the integration of relationships across different representational styles. O'Halloran (2015) provided insight into the various representational styles that exist, specifically within mathematics. According to O'Halloran, there are three representational styles, which she referred to as grammar systems: linguistic grammar system, visual imagery grammar system, and the symbolic grammar system. The linguistic grammar system focuses on the natural language that is spoken between individuals. The visual imagery grammar system focuses on the visual representations of concepts and phenomena. The symbolic grammar system focuses on the efficient use of signs and symbols to encode the connections and relationships amongst various phenomena.

Last, Pinto (2018) presented the idea of semantic density as an important feature of a semantic domain. Pinto defines semantic density as the degree of compaction of meaning in a particular area. It can be argued that utilizing the ideas of intra-representational coherence and inter-representational coherence can lead to a student developing a high degree of semantic density (Seufert, 2003). Based on work conducted in integrating semiotics and semantics with mathematics education, the most robust learning for students occurs when mathematical ideas are connected to the students' personal experiences so that they may generate their own individualized semantic domain and then integrate this new semantic domain into their prior knowledge.

Metacognition

The general sequence that I present as a pathway to robust learning requires an extensive amount of thinking and reflection on the part of the learner. This thinking is particular and quite specific because the learner must think about his or her own thinking, whether it be in the form of personal experience or prior knowledge. A term that is used to refer to thinking about one's thinking is the word metacognition (Flavell & Wellman, 1977). So, I can restate my previous statement and say that the most robust learning for students occurs when students use metacognition to not only generate their own individualized semantic domain but also to integrate the new semantic domain into their own prior knowledge.

Before I provide my own understanding of metacognition, I think it wise that I bring attention to the scholars that have helped frame my understanding. According to my non-exhaustive search of the literature, it seems that the first published documents on metacognition occurred during the 1970s. Flavell and Wellman (1977) are the often-cited researchers on the topic of metacognition. Flavell (1976) defined metacognition as referring—

> to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data. … Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (p. 232)

Flavell also posited a list of three variable types for metacognition and offered their descriptions: person variables, task variables, and strategy variables. Person variables consist of what one believes about oneself and others as cognitive beings**.** Task variables refer to the scope and requirements of tasks as well as knowledge about the factors and conditions that make some tasks more difficult than others. Strategy variables consist of knowledge of general and specific cognitive strategies along with an awareness of their potential usefulness for approaching and carrying out certain tasks.

During the 1980s, Flavell's (1976) work inspired the work of Garofalo and Lester (1985). Garofalo and Lester extended Flavell's research into the field of mathematics problem-solving as they created the cognitive-metacognitive framework for mathematics tasks. This framework contains four elements: orientation, organization, execution, and verification. Orientation focuses on the strategic behavior necessary to assess and understand a problem. Organization focuses on the planning of behavior and choice of actions. Execution focuses on the regulation of behavior to conform to plans. Verification focuses on the evaluation of decisions made and the evaluation of the outcomes of executed plans**.** More important than the creation of this framework is the mindset that Garofalo and Lester held that allowed it to happen. Garofalo and Lester viewed cognition as knowledge-as-a-product, and they viewed metacognition as knowledge-as-a-process. As such, their cognition-metacognition framework centers on the product and process natures of mathematical knowledge. Garofalo and Lester also stressed the person variable aspect of metacognition by stressing the importance of mathematics agency, motivation, and knowledge transfer.

Additionally, in the 1980s, Resnick (1985) made great contributions in the area of metacognition in her own effort to explicitly establish a relationship between prior knowledge and semantic domains. According to Resnick, knowledge is the process of connecting new experience with old experiences. Continuing, she conjectured that meaning is the relationship and connections that exist between experiences that are explicit or inferred within a semantic domain. As such, according to Resnick, it is not only important for a learner to connect new and old experiences but also it is perhaps more important that the learner seek to understand and articulate the relationships between them. The relationship of experience, semantic domain, prior knowledge, current knowledge, and meaning that is outlined by Resnick led to a paradigm shift for me. It was through her work that I was able to understand the superordinate nature of experience and its accumulation.

In the 1990s and 2000s, the theoretical work in metacognition from the 70s and 80s transformed into the practical application of metacognition, specifically in the classroom. Santos (1995) focused on the features of a learning environment and its activities that cultivate metacognition. His research centered on using student work as an indication or measurement of the metacognitive ability of the student. Stillman and Galbraith (1998) published their research that centered on the work of both Flavell (1976) and Garofalo and Lester (1985). Their work was valuable because it outlined an attempt to chart the metacognitive trajectory of students' thinking as they attempted to solve a "real-world" math problem. Adibnia and Putt (1998) provided a new perspective on metacognition because it highlighted the many roles held by the classroom teacher to facilitate the student's awareness and use of metacognition. An evolution in the perception of metacognition occurred through the years and the result was highlighted by Veenman and Spaan (2005). They outlined the constituent parts of metacognition: metacognitive knowledge and metacognitive skills. Metacognitive knowledge is the declarative knowledge one has about the interplay between personal characteristics, task characteristics, and available strategies in a learning situation. Metacognitive skills are concerned with the procedural knowledge that is required for the actual regulation of and control over one's learning activities. An additional contribution by Veenman and Spaan (2005) is that metacognitive knowledge develops in children between the ages of 4–10; while metacognitive skills develop within children between the ages of 11–12.

Now that I have presented short summaries on the researchers and theorists who have most influenced my thinking on metacognition, I offer my perspective. I find that metacognition is best defined generally as the generation of, attention to, and utility of experience. More specifically, metacognitive knowledge is the awareness and monitoring of the entire process of experience-generation and one's sense of self-efficacy[21] within this process. Metacognitive skills, first, are the skills necessary in the design, implementation, evaluation, and optimization of the awareness of one's experiences. Second, metacognitive skills are the skills necessary in the awareness of the existence of relationships amongst one's experiences. Third, metacognitive skills are the skills necessary in the determination of the nature of the relationships and the utilization of these relationships and experiences to solve problems.

Some Concluding Words on the Literature Review

As I conclude this chapter that summarizes the literature that has informed my thinking, I take pause to reflect on the various topics that have been pertinent to my evolution. Although I began with the ideas of the achievement gap and semiotics, I was led to the more pressing ideas of prior knowledge, semantic domain, and metacognition in my effort to understand the plight of mathematics education in this country. So, it would seem that these more pressing ideas could inform the development of a process that would be more effective in the learning of mathematics by students who have not been adequately served by current processes and approaches. New

[21] My use of "self-efficacy" here is not to direct the reader to the volumes of literature on self-efficacy. But rather, my use of the term is to direct the attention of the reader to the productive disposition attribute of mathematical proficiency (Kilpatrick et al., 2001).

insight could indeed be gained if mathematics knowledge was viewed by both teacher and student as a process and not reduced to a product (Garofalo & Lester, 1985).

CHAPTER 3

A THEORETICAL FRAMING OF THEORY-BUILDING

In this chapter, I provide a synthesis of three theoretical traditions, located in the understand paradigm of inquiry (see Stinson & Walshaw, 2017, p. 1407), that provide the theoretical (and methodological) foundations for this theory-building project: cultural-historical activity theory (CHAT), semiotics, and radical constructivism. I begin by providing a historical trajectory of CHAT. I continue by detailing semiotics and one of its manifestations in society, the semiosphere. Next, the meaning-making nature of semiotics requires that I give commentary on epistemology and symbolic interactionism. I conclude the chapter with a detailed analysis of radical constructivism and its relevance to the learning of mathematics.

Cultural-Historical Activity Theory (CHAT)

Cognitive theory is the theoretical lens that is used to understand how sensory input is transformed, reduced, elaborated, stored, recovered, and used (Neisser, 1967). Within cognitive theory, the social psychological paradigm focuses on the phenomenon of knowledge construction and how humans engage and interact in the knowledge construction process (Lourenco, 2012). Noted leaders of this social psychological paradigm of cognition were Jean Piaget and Lev Vygotsky (Lourenco, 2012; Tryphon & Voneche, 2013). Although each believed that the individual, social interaction, and activity play a role in knowledge construction, each held his own perspective. Piaget (1936/1952) believed that the individual has the more dominant role in his or her knowledge construction. Conversely, Vygotsky (1930/1978) believed that human activity is the causative agent for knowledge construction. The perspective held by Vygotsky, referred to as social cultural theory, resonates the most with me because I too believe that the activity with which one engages has the dominant influence on the person's knowledge construction. This focus on activity led Vygotsky (and others, e.g., Leont'ev and Luria) to the further development of cultural-historical activity theory (CHAT), which established that activity is culturally, socially, and historically defined. A Vygotskian perspective on knowledge construction and the CHAT framework are theoretical traditions through which the mathematics learning experiences of students in this theory-building project are framed.

Vygotsky was greatly influenced by the work of Karl Marx (1818–1883), who is given credit for saying—

> the production of ideas, of conceptions, of consciousness is irreducibly tied up with material activity and material interactions of people, language of real life. … The people are the producers of their conceptions, ideas, and so on, but the real, acting people, as they are determined by the determinate development of productive forces and the relations that correspond to them up to the most advanced formations thereof. Consciousness never can be anything other than conscious being, and the being of humans is the real process of life. (as cited in Roth & Lee, 2010, p. 1)

A quote by Roth and Lee (2010) is effective in highlighting the message in Marx's words in Vygotsky's CHAT theory:

> Put in another way, cultural-historical activity theory understands consciousness generally and knowing and learning more specifically as a function of the activity, where 'activity' is understood in terms of systems that satisfy the continuation of human society. These systems constitute the minimal unit of analysis, fulfilling Vygotsky's (1986) call for doing unit analysis rather than element analysis. (p. 3)

Vygotsky's (1930/1978) initial thought identified the relationship amongst the individual, the object, and the outcome. In addition, Vygotsky emphasized the cognitive structures and thoughts that are generated through the use of mediating tools by the individual on the object. According to Vygotsky, these mediating tools could be tangible items like stones, sticks, pencils, computers, and so on, as well as abstract items like signs, symbols, and language. The use of the mediating tool on the object is referred to as the activity, and the specificity of the objects of interest was determined by the culture of the individual. As Vygotsky was committed to understanding the cognitive development of the individual, he privileged signs and semiotic mediation in the form of speech and word meaning, specifically, the tools of the activity as the unit of analysis.

As his theories developed, students of Vygotsky, Leont'ev (1981) and Luria (1976), were drawn to investigate the impact that a group of individuals can have on the cognitive development of each individual as well as the impact that the group can have on the activity itself. Such a feedback system emphasized the affect that social interaction has on cognition as well as the affect that the advancement of an individual action toward a collective activity also has on cognition. This type of investigation would be similar in nature to deconstructing a larger entity into its constituent parts. It can be argued that while Vygotsky, Leont'ev, and Luria each studied cognitive development, Vygotsky focused on cognitive development where the individual and the activity are co-agents, while Leont'ev and Luria focused on cognitive development where the collective and the activity are co-agents. Ultimately, Leont'ev and Luria used social, cultural, and historical dimensions of practical activity, specifically, the context of the activity as the unit of analysis. It was during Leont'ev's influence that the nomenclature cultural historical activity theory (CHAT) became popular.

Over the last 30 years, CHAT has undergone another evolution. Through the efforts of Michael Cole and Yrjo Engestrom (1993), CHAT became a global phenomenon. The multiplicities of social ecological systems, cultural systems, language systems, collective activity systems, and epistemologies led to the investigation of networks of interacting activity systems. Here again, the unit of analysis is the activity, but now it has acquired the consideration of the influence of multiple-layers of activities on the activities themselves, on the collective group and on the isolated individual. This latest evolution highlights these diverse activity systems as the result of a continuous historical process of progression.

Despite its evolutions, the general precepts of CHAT are the same. Within CHAT, there is an ordering which is evident. First, there is the activity, which is motive-driven and is based upon collective needs. Second, there is action which is goal-oriented and is a constituent of activity. According to Roth and Lee (2010), conscious operations and unconscious operations exist. Operations that are consciously executed become actions. There is an iterative aspect to CHAT, in that an action in one system can become an activity in another system. The opposite can also occur, whereby an activity in one system can become an action in another. Another important precept of CHAT is that consciousness and its expression through thought and language is understood according to its dependence on the cultural-historical and societal processes of human beings. Human consciousness cannot be understood without context. It is this focus on context that identifies CHAT as a valuable tool to not only understand the world but also to transform it.

For Vygotsky (1930/1978), as it is for other social-cultural theorists, human learning and development are rooted in social interactions and as a result they are impacted by the context defined by said social interactions (Roth & Lee, 2007). As Russian dolls are nested one within the other, human interaction is nested within larger socially organized activities. CHAT centers on the idea that learning is linguistically mediated where signs and symbols are used to make meaning. There are three levels of language use. First, the operation level is the unconscious choice of words and grammar. Second, the action level is the explicit process of reflection and representation. The third level is the activity level which is the level whereby language is used for theorizing. Specifically, learning is rooted within culturally and historically organized activities whereby a child learns about the use of language according to his or her metacognition and learns through the use of language according to his or her cognition. Such human activities have the following properties: all activity is goal-oriented, all activity is mediated by culturally constructed tools and artifacts, and all activity is historical.

Beyond these three levels of language use, language has a specific influence on activity. According to Rolf and Lee (2007), there are three levels of the cultural evolution of language that impact the degree and quality of activity:

- Social relations evolve as a function of the societal infrastructures,
- The communication and the verbal exchanges evolve within the framework of the social relations, and
- The form of speech acts evolves from verbal interaction.

The distinct lines of inquiry offered by CHAT motivated the various evolutions. Each line of inquiry, each evolution, however, makes an effort to study the same phenomenon—the experience generated by activity. Vygotsky studied the experience of the individual caused by the activity and influenced by the mediating tools of speech or word meaning. Luria (1976) and Leont’ev (1981) advocated for the study of the experience of the individual caused by the activity and influenced by the social collective. And most recently, Engestrom (2008) advocates for the study of the experience of the individual caused by the activity and influenced by other nested activities. It would seem that Vygotsky suggested the deepest investigation into the experience of the individual by studying the individual's selected speech and word meanings.

Activity, learning, and the experience of learning occur in all types of spaces. As such, there is a constant ebb and flow of metacognition and cognition within the learner (recall the discussion of metacognition in Chapter 2). In addition, there are a collection of goals, and culturally constructed tools and artifacts that the learner must navigate. Because activity, learning, and the experience of knowledge construction are not space-restricted, a challenge exists with the three properties of language that impact human activity (previously detailed). The challenge is the intersection of traditional classroom knowledge and the out-of-school knowledge. More specifically, for the student and teacher, tension exists due to the funds of knowledge possessed by the students and the academic knowledge possessed by the teacher (Gonzalez, Moss, & Amanti, 2005; see Chapter 2). It is not uncommon for the teacher to have an academic intention for an assignment or activity and the students to have a non-academic interpretation based on their historical and cultural experiences as represented by their funds of knowledge. This tension has been called the "third space" by Razfar and Rumenapp (2013). These opposing entities inform the researcher of another important element of CHAT, which is dialectics. Dialectics is the study of mutually exclusive categorical pairs that exist within an activity system. One element of the pair pre-supposes the existence of the other. The function of one element cannot be understood decoupled from the function of the other, or decoupled from the function of the integrated whole. A few other examples are the individual versus the collective, the body versus the mind, the subject versus the object, agency versus subservience, and concrete versus abstract.

The Semiosphere

A major component of CHAT is the mediating tools on which Vygotsky focused much of his effort. As previously mentioned, Vygotsky considered language, signs, and symbols as mediating tools. It is interesting to note that another researcher, Halliday (1978), held similar beliefs about the importance of signs, symbols, and language in influencing the involvement of people with activity and subsequently the cognitive development of the participants. According to Halliday, the semiosphere is composed of a semantic domain and discursive practices. It has both concrete elements and abstract elements. Halliday continued by segmenting the semiosphere further into three attributes: field, tenor, and mode. For example, within the context of a classroom, field is the setting, the subject matter, and the activities that exist therein (e.g., what is the classroom discussion about). Tenor is the style of the communication, the relationships between individuals and the emotions of the individuals (e.g., what are the connections and power relations amongst the individuals involved in the discussion). Mode is the form of language that is used, its role and its dialect (e.g., how is the discussion scaffolded to facilitate learning by the individuals). These three attributes are the fundamental elements of Halliday's (2014) discourse analysis tool systemic functional linguistics (SFL). He further separated the semiosphere into its cultural components and its contextual components because every person has a cultural aspect and a contextual aspect, and these aspects influence how the semiosphere is perceived. These two further partitions are represented by Halliday in his reference to register. For Halliday, register is what is talked about and how it is expressed. It is the configuration of cultural and semantic resources as determined by the situation that the individual perceives within the semiosphere. Specifically, according to Halliday, register is the meaning potential that is accessible to the individual in a given social context. In sum, the mediating tools as studied by Vygotsky are a sub-set of the semiosphere as studied by Halliday, and both embraced the significance of signs, symbols, and language on the development of cognition.

It is necessary to explicate the utility and potential of the semiosphere because it is the semiosphere that facilitates the teacher's effort in cultivating students who are proficient in all the requisite mathematical language forms. The entire planet is a semiosphere and the classroom is a small but specific piece of it. The semiosphere is available to be used for activity and learning; it does not discriminate. The distinctions and diversity that exist within and amongst the learning by individuals are due to the distinctions and diversity in the social and cultural identities represented by the individuals. As such, the classroom should also be ripe for activity and learning, with all its distinctions and diversity; it too should not discriminate.

As previously mentioned, the classroom can be viewed as a small sample of the world, at least in terms of its semiotic potential. Therefore, its meaning potential is without bound. The infusion of a student into a classroom produces the following situation. First, the classroom possesses an amount of meaning potential. Second, according to the student, the classroom has a cultural aspect and it has a contextual aspect due to the cultural and contextual attributes of the student. Third, according to the funds of knowledge possessed by the student, he or she will have his or her own unique experience within the classroom according to what elements within the classroom have significance or meaning for the student (Gonzalez, Moss, & Amanti, 2005). Fourth, according to the student's unique register as cultivated by his or her social code, within the meaning potential, specific meaning possibilities are constructed by the student. Fifth, the student selects a particular meaning from the litany of possibilities that are available to him or her.

Halliday (1978) referred to this selected meaning as text and suggests that this text can be represented by the student in spoken form or written form. It is the responsibility of the teacher to understand these stages of engagement by the student with the semiosphere as they occur within the classroom. In addition, despite the multitude of convergent and divergent meaning possibilities represented by each student, the teacher must also attempt to coordinate these meaning possibilities so that they are relevant to the topic at hand. This coordination by the teacher is why the concept of semantic domain is so vitally important. An awareness of the semantic domain allows for the refinement of the meaning possibilities. The teacher must serve as a guide and assist the students in aligning and fitting their meaning selections with each other. Last, the teacher must also guide these meaning selections so that the students understand the alignment and fit of their discourse with the discourse of the larger society. Please note that the primary goal of student discourse should not necessarily be to match or be consistent with the larger social discourse; the primary goal of student discourse should be the attainment of meaning. The secondary goal can be the comparison of this meaning to the meaning within the larger social discourse. A tertiary goal can be to determine the consonance of the meaning with the larger social discourse. As complicated as this process as posited by Halliday may be, analyses of these sort are facilitated by CHAT. The most comprehensive inquiries, however, are performed by researchers who are also competent in the field of semiotics—the study of signs and symbols.

Semiotics

What is the symbol? The study of signs and symbols is a classical one in academics. Signs and symbols are important because they mediate thoughts and thinking (Chandler, 2007; Mead, 1934). The field of signs and symbols is called semiotics, but has no statically agreed upon definition. Chandler (2007) compiled a litany of functional definitions. A few include the study of signs; the meaning-making ability of signs and symbols; the study of the different representations of meaning; and the study of drawings, paintings, photographs, words, sounds, and body language.

Here, I define semiotics as "the study of the semiosphere—the collection of signs and symbols used as tools for information exchange within a particular context. Signs and symbols exist in two groups: analog (visual images, gestures, textures, tastes and smells) and digital (words and whole numbers) (Tomaselli, 1996). Analogical signs and symbols assist with contextual descriptions and can reveal mood and intention. Digital signs and symbols, on the other hand, tend to be decontextualized, but assist in the refinement of abstract thought. Semiotics has many practitioners, but only two fundamental poles—Ferdinand de Saussure (1857–1913) and Charles Peirce (1839–1914) (Chandler, 2007; Eco, 1978). Saussure was a Swiss linguist and Peirce was an American philosopher. Both realized that meaning-making is a human construct based upon social, cultural, and historical dynamics. This realization led to a key distinction between these two theorists. In the case of Saussure, he was troubled by the apparent arbitrariness of the meaning-making process. Consequently, in his work, he avoided the meaning-making aspect of signs and symbols, choosing instead to focus on the structure within a language by which words are organized. In present day terms, this structure is called the syntax of the language (Chandler, 2007; Eco, 1978). Peirce, on the other hand, appreciated this arbitrariness and found that it expanded the meaning-making ability of humans. As such, much of Peirce's work focused on what is presently called semantics, which is the meaning of signs and symbols (Chandler, 2007; Eco, 1978).

Within semiotics, there is an interesting inquiry that persists. The critical question: "whether the system precedes and determines usage or whether usage precedes and determines the system?" (Chandler, 2007, p. 13) Here, the word *system* could be interpreted as language, but it also can refer to the semiosphere, which is the natural environment, as coined by Lotman (2001). This inquiry stems from the epistemology that an individual espouses. The awareness of the relationship between knowing and unknowing, as a pragmatist might internalize it, or between the known and the unknown, as a foundationalist[22] may view it, is requisite here as one engages in this inquiry. Knowing is subordinate to the unknowing; consequently, this theory-building project emerges firmly from the position that the world exists as a semiosphere, a place to make meaning from and with, and humans have the opportunity through language to construct and exchange such meaning. Chandler (2007) amassed a collection of thoughts on this point:

> Distinctively, we make meanings through our creation and interpretation of 'signs'. … Signs take the form of words, images, sounds, odours, flavours, acts or objects, but such things have no intrinsic meaning and become signs only when we invest them with meaning. (p. 17)
>
> Anything can be a sign as long as someone interprets it as 'signifying' something—referring to or standing for something other than itself. We interpret things as signs largely unconsciously by relating them to familiar systems of conventions. (p. 17)
>
> No sign makes sense on its own, but only in relation to other signs. (p. 22)

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[22] For further epistemological investigation in this matter, see Alston (1976).

> The sign is more than the sum of its parts. While signification—what is signified—clearly depends on the relationship between the two parts of the sign, the value of a sign is determined by the relationships between the sign and other signs within the system as a whole. (p. 24)

Moving beyond this idea of environments saturated with signs and symbols, I provide a summary of the work of Saussure and Peirce below.

The Saussurean Model

The Saussurean model of semiology contains three extremely important concepts: arbitrariness, structure, and dyadicism (Saussure, 1908/1998). Saussure was not convinced that there was any pattern or process by which signs were connected to that which they represented. In fact, it troubled him that he could not isolate a unique and consistent meaning, the semantics, for any particular sign. This arbitrariness of both the sign and its meaning remained an unresolved area within his approach. Instead, Saussure focused on the syntax of a language system—the sign and its characteristics. For Saussure, language was the combination of both syntax and semantics. As such, language structure was a formal system of meaning-making and meaning-exchange. His evaluation of language as a comprehensive whole led to his realization that specificity and consistency existed with the orientation of the elements of a language. For example, there are specific and consistent locations for a noun in the English language. There are also specific and consistent locations for verbs and adjectives with respect to the location of the noun, within the English language. It was to these syntactic factors that Saussure attended to the most in his semiology approach.

Within his approach, Saussure established a dyadic characteristic for signs (Saussure 1908/1998). He preferred spoken language, as he was not as inspired by written language. In his dyadic approach, a sign had a signifier-component and a signified-component. The signifier carried the structure or syntax of the element, while the signified carried its meaning. Saussure realized that this pairing of syntax and semantics could be quite random because there were no set rules as to which signifier was attached to a particular signified. This pairing occurred as a result of contexts and social dynamics.

Saussure (1908/1998) also made a distinction between intention and interpretation. Due to the arbitrariness of signifier and signified, Saussure did not envision signifieds as representing real items given that the shape or image of real items is not random. Instead, he envisioned signifieds as representing the mental representations of items experienced in nature. Despite his commitment to the arbitrariness of signs, Saussure did acknowledge that after a sign and its constituent signifier and signified parts are established, they are not arbitrarily changed. Once the history of a sign is established within the community of users, the sign and its constituent parts, signifier and signified, are not easily altered.

The Peircean Model

Peirce, on the other hand, had a slightly different approach to semiotics (Chandler, 2007). In the effort to capture his important facets, the following three terms are used: three-pronged, impact, and generative. Peirce established a triadic model for his study of signs and symbols; the components of which are representamen, interpretant, and object. Representamen is the form that the sign takes. Interpretant is the sense made of the sign. The object is the entity to which the sign refers. Peirce's perspective on semiotics:

> A sign is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the interpretant

> of the first sign. The sign stands for something, its object. It stands for the object not in all respects, but only in reference to a sort of idea, which I have sometimes called the ground of the representamen. (as cited in Chandler, 2007, p. 32)

This "creation in the mind" is what is referred to as a sign's impact. This interpretation was extremely important to Peirce because the impact of a sign could be different according to the person involved. The iterative nature of Peirce's concept of creation in the mind could go on indefinitely as its only constraints would be imagination or impact on the individual and the context of the situation. It is this iterative process that is generative.

Peirce espoused a three-tiered relationship: the symbol, the icon, and the index (Chandler, 2007). These are the three classifications by which the before-mentioned relationship from Saussure regarding the signifier and signified pair can be understood. The symbol classification is used when there is an arbitrary relationship between the signifier and the signified. Due to its arbitrariness, symbols must be learned. It would seem that symbols are what frustrated Saussure the most. The icon classification is used when there is a salient connection to the signified based upon the perception of the user. The icon displays limited arbitrariness. The index classification is used when there is a direct connection between signifier and the signified. Such indices direct the attention of the user to the object through pure compulsion. Peirce showed great interest in the symbol. He was firm in his position that the symbol was the most general tool and as a result it would assist the most in the reasoning ability of the user. Peirce's focus on meaning-making allows for the internal reflection of the representamen by the originator which is aligned with Vygotsky's concept of internal thinking (Chandler, 2007).

Classroom Relevance of Saussure and Peirce

The relevance of the works by Saussure and Peirce in the classroom, specifically a mathematics classroom is immediate. The signs and symbols that are used within mathematics are quite arbitrary. As Saussure posited, however, once pairing between signifier and signified is established, it becomes fixed. This fixed paring explains why the symbols used within a particular culture to represent mathematics endure the test of time. It can be argued that the subsequent meaning-making represented by such arbitrary and general signs and symbols is what makes mathematics so profound but also so frustrating for many of its learners. When students ask, “Why?”, they are trying to locate themselves on this continuum of understanding as they attempt to not only understand the identity and relationships of the symbols used, Saussure’s syntax, but also the impact that the symbols have on them, Peirce’s semantics.

Epistemology and Fallibilism

Is the meaning of the symbol known and can it be articulated? What does the symbol mean? What meaning is represented by the symbol? Each of these is an important question.

Epistemology is the study of knowledge. Unfortunately, knowledge is an abstract entity; therefore, it is troublesome for people to capture its meaning. Ultimately, however, I think that people would be forced to agree that however one defines knowledge, it represents an essence for some, or an ability to be possessed, for others. Otherwise, how would the following phrases be sensible “I know,” “He is knowledgeable,” and “She knows a lot.” This essence, if you will, is the epistemological nature of knowledge. According to Peirce, as detailed in his collected works (see Kloesel & Houser, 1992), thoughts and thinking are what produce knowledge. Peirce has elaborated on this connection between thoughts and knowledge, specifically, as it relates to the essence that knowledge possesses. According to Peirce, there are four elements to thought and thinking:

- Thought is a representation of a relationship,
- A single disconnected thought is un-intelligible; it becomes intelligible only by the mind relating ideas and concepts together,
- Thinking requires signs, and
- Signs provide the material essence which gives thought its quality.

It can be argued that because Peirce was a semiotician, that his perspective in many areas, including knowledge, was clouded by his immersion in the prevalence of signs and symbols. Admittedly, I am also an advocate of the predominance of signs and symbols on the human psyche, and how such affects lead to the regress or progress of humanity. Peirce continued by suggesting that thoughts are connected or related to one another due to their sign or symbolic nature. In fact, due to this connection, thoughts suggest other thoughts, which in turn suggest still other thoughts ad infinitum. Peirce provided a bit of detail on signs by suggesting that they are not generated by introspection or by intuition, but through language and use. He also suggested that signs can only represent an object or point, within “some respect.” Signs cannot be the true and complete essence of an object or point. This incompleteness is the fallibilist nature of knowledge—the ability to be error-prone.

A pertinent question develops regarding the origin of that first thought or sign. According to Peirce, as detailed in his collected works (see Kloesel & Houser, 1992), thoughts are made manifest by the environment or external world to which one is exposed. A social constructionist aspect of his work is revealed when he maintained that thoughts depend in large part upon the community for their derivation of meaning and correct use. Depending upon the situation, his use of community refers to either a group of people or one's natural environment.

Peirce, as detailed in his collected works (see Kloesel & Houser, 1992), has also spoken to the contextual nature of knowledge.[23] He has suggested that knowledge proceeds within a historical, social, cultural, linguistic, and scientific context. Specifically, he presented that trends, beliefs, and practices influence how knowledge is developed and gained, but that these trends, beliefs, and practices change over time. These changes determine if knowledge regresses or progresses. Peirce viewed these changes as having the capacity to disprove trends, beliefs, and practices of the past. This changeability is further evidence of the fallibilist nature of knowledge.

Symbolic Interactionism

What does the symbol want me to do? What action is represented by the symbol? Symbolic Interactionism was created by George Herbert Mead (1863–1931) and extended through the work of Herbert Blumer (1900–1987). Where semiotics centers on the existence of signs and symbols; symbolic interactionism centers on the human utility of these signs and symbols. It can be argued that it is through symbolic interactionism that the signs and symbols become a language. The following statement may be helpful to better relate the work of Saussure, Peirce, and Mead. The fusion of these three theorists is invaluable as an educator who studies mathematics through language, where mathematics can also be viewed independently as a language. Mathematics, like spoken languages, allows relationships to be stratified. This stratification results in explicit and implicit relationships. Unlike spoken languages, however, the

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[23] Beyond Peirce, there are other popular perspectives on knowledge. A few of the highly critiqued perspectives are Descartes's approach, which advocated that knowledge could be deduced from a set of foundational knowledge claims; Kant's approach, which advocated for an *a priori* approach to knowledge; and Hume's approach, which advocated for a sensory-intuitive approach to knowledge.

allusions within the syntax and semantics of mathematics require greater attention by the user because in the classroom, mathematics requires that action be taken, not just that comprehension be gained. The effective transference of mathematical interpretation to mathematical activity is a key indicator of semiotic fluency and is integral to mathematics proficiency and mathematics agency.

This transference can occur in several ways. Two approaches will be highlighted. Herbel-Eisenmann, Wagner, and Cortes (2010) discussed the distinction between formal mathematics language and the language discussed within the mathematics classroom. This difference is the center of the two approaches. First, the formal mathematics language is oftentimes represented by the traditional lecture format utilized by the teacher. The formal mathematics language is also presented in the linguistic material found in the text. This discourse includes the “correct” mathematical vocabulary, along with its syntax and semantics. This utterance is the socially consensually constructed discourse. Second, the informal mathematics classroom discourse is represented by the mathematics conversation often had amongst students while the teacher is lecturing or while the students are engaged in collaborative problem-solving projects. This utterance is the experientially constructed discourse and includes the funds of knowledge registers of the individual students.

Oftentimes, the syntax and semantics of the traditional mathematics languages are filtered through the culture of the students as well as their socialization process so that meaning can be constructed and action can be performed. Whether it is viewed positively or negatively, the reality is that the mathematics language is transformed from standard into a more culturally and socially contemporary form as the students mature in their mathematical learning. To demonstrate mathematics proficiency, however, the reality is that the language and knowledge constructed from a cultural and social mathematics experience must be transformed into standard form. The individual mathematics conversations between the teacher and student can serve as a bridge for the student to mature from informal mathematics classroom discourse to the formal mathematics discourse. It is this maturing experience that is well theorized by symbolic interactionism as mathematical action is connected to mathematical discourse.

Now that the theoretical foundations for what follows have been provided, a fusion of the principles of mathematical philosophy, epistemology, semiotics, and human experience into a comprehensive theoretical approach is presented.

Fusion of Mathematics Proficiency, CHAT, and Semiotics

Historically, in the United States, mathematics proficiency has been measured quantitatively (Hilliard, 1994). This type of measurement assumes that there is a common standard to which all students are to be compared. But it seems invalid to make definitive conclusions about someone's cognition as compared to another if everyone is uniquely different. In addition, it seems unreliable to assert a permanent classification on someone's cognition if knowledge is never fixed, but instead is ever evolving. Traditional use of CHAT does not seem to address such issues of unreliability and invalidity. In what follows, I am more explicit in my concern.

CHAT can have a significant impact in the realm of education, in this theory-building project, and specifically, on mathematics educators. In the effort to gain a deeper and more holistic understanding of CHAT, some of the work of several of the current-day stalwarts of the CHAT framework have been reviewed. Presently, often-cited works in the field include Yamagata-Lynch (2007), Razfar and Rumenapp (2013), Roth and Lee (2010), Engestrom (2008), Foot (2014), Radford (2000), and Anderson and Stillman (2013). After reading the works of these CHAT theorists, a deeper and more holistic understanding of CHAT was gained, but I was left unfilled in the fundamental area of interest-knowledge construction and agency through language and activity. The components for such work are evident within CHAT, but current work seems to highlight snapshots of human performance and interaction as consequences of activity systems and only alludes to the impact that activity systems have on knowledge construction and agency. Instead, I endeavor to use CHAT to analyze activity systems, within a time-continuum, to highlight the knowledge construction and agency that are consequences of the analyzed activity. In other words, attention must be drawn most to the perspective held by Vygotsky as it relates to CHAT, understanding the cognitive development of the individual and embrace the extensions as offered by its subsequent evolutions.

A focus on semiotics within the activity, the discourse, and the knowledge-construction processes seems to allow for an effective and evolving link amongst the three. This link allows for the construction of the underlying propositional knowledge for the activity, as well as the underlying procedural knowledge for the activity. In addition, a focus on the semiotics of the activity improves the deduction, induction, abduction, retention, and processing by the cognitively evolving individual.

As detailed previously, traditional use of CHAT seems to correlate the output and outcome of an activity predominantly to the quality of the activity. Undoubtedly, the activity and its quality are major contributors; however, the more immediate effect of the activity and its quality is the language-use that they generate within the participants and the knowledge-construction which results. The output and outcome of the activity are then consequences of the participant’s language proficiency and knowledge-construction, perhaps more so than simply as a consequence of the activity and its quality. Unfortunately, traditional CHAT researchers have not focused on this perspective and perhaps it is because CHAT, in its utility, does not have a mechanism to evaluate or investigate discourse or knowledge-construction. There is an investigative tool that does provide insight into discourse and knowledge construction, specifically for mathematics activities. The tool is the "teaching experiment" and it comes from the epistemological stance of radical constructivism. (See Chapter 4, Methodology, for details about the teaching experiment.) In addition, the explicit focus on semiotics also implies a deeper focus on meaning-making. This investigation and focus directly necessitates a more fertile relationship with epistemology and the attributes of knowledge. In sum, this theory-building project, borrowing from multiple theoretical traditions, requires the re-evaluation of the relevance of epistemology within the classroom.

Constructivism and Radical Constructivism

An epistemological assessment within the classroom is necessary. The teacher and the students must embrace an epistemology that rejects the idea of a fixed knowledge capacity or the idea that events frozen in time endure forever. It is important for an educator to seek an epistemological stance that is consistent with his or her own experiences and passion for teaching, where teaching is identified as either knowledge transference or knowledge construction. Constructivism, for me, is consistent with the demands of my perspectives of teaching, specifically radical constructivism. The battle between knowledge transference and knowledge construction seems to have a long history (Freire, 2000).[24]

Radical constructivism establishes that knowledge is the result of a learner's activity rather than the passive reception of information or instruction (Steffe & Thompson, 2000). It also establishes that knowledge is experientially constructed but extends beyond this point and

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[24]Freire (1970/2000) critiqued this battle and specifically the knowledge transference aspect of the "banking system of education."

submits that knowledge is also contextually orchestrated (Steffe & Thompson, 2000). Knowledge is not some objective element of a world that is separate from our experience. Unfortunately, knowledge will be socially constructed for a person who is too immature to attune to his or her own lived experience. Nevertheless, when the person matures and achieves awareness of self then he or she attunes to his or her own lived experience. For such a self-aware person, his or her world is experientially constructed and his or her knowledge is experientially constructed (Steffe & Thompson, 2000). To consider more fully the idea of constructivism, Steffe and Thompson (2000) suggested that the radical constructivist is a person who believes that knowledge is the result of a learner's activity and that knowledge is a representation of and insight into a perceptible world that exists external to the person. The key difference between a constructivist and a radical constructivist is the attribute of the external world. For the constructivist, the external world has objectiveness and a "real truth" to it. For the radical constructivist, the external world does not possess objectiveness and it has no foundational truth. For the radical constructivist, the world is as it is experienced; it does not have an external reality. In addition, for the radical constructivist, the learner's interpretation of his or her experiences is influenced by the social, cultural, and historical aspects of his or her existence. For the radical constructivist, knowledge is adaptive insight that allows a person to cope within his or her experientially constructed world. There are no multiple realities; there are multiple experiences. Knowledge is adaptive according to the individual's attenuated experiences; it is examined and evaluated according to its impact on the individual. Consequently, knowledge is subjective, but not such that it leads to relativism, more on that point later. It is important to note that for radical constructivism, the person will choose the construction that is most viable for him or her, where this viability measurement depends upon the type of experience.

The authority on radical constructivism is Ernst von Glasersfeld (1917–2010). von Glasersfeld lived a life filled with a variety of experiences. The set of experiences that are most intriguing were his linguistic experiences. von Glasersfeld was first a linguist before he became an epistemologist. It was during his time as a linguist that von Glasersfeld realized that the language with which a person communicates and organizes his or her thoughts has a significant impact on the person's perception, interpretation, and understanding. This fundamental realization spawned his work in epistemology and his development of radical constructivism. Some of the critiques that radical constructivism has received are due to von Glasersfeld's resistance to identifying knowledge-construction as focused on being and not just knowing. Specifically, von Glasersfeld refused to embrace the concept of "the self," a decontextualized and externalized entity, as the center of subjective awareness (Lewin, 2000). This idea of the "center of subjective awareness" is the interpretation of one's experience and instead must be viewed as contingent upon the person's location and integration within the human ecological system where time, culture, social factors, and ethics are indices to which the person must attune himself or herself. The person's individual ethics are the truest indications within the human ecological system of a person's identity—one's "self."

Fundamental Principles of Radical Constructivism

A digital search for published material on radical constructivism for the years starting in 1950 and ending in 1979 was performed and no published material was found. It was not until the 1980s that Ernst von Glasersfeld brought forth his ideology. Important philosophers that von Glasersfeld studied before he organized his own ideology around the concept of knowledge include Vico, Locke, Descartes, Berkeley, Hume, and Kant. In his studies, von Glasersfeld realized that humanity had two unresolved issues with respect to knowledge, as captured by these two questions:

- “What is knowledge?”
- “How does humanity attain its knowledge?”

For von Glasersfeld, the concept of “fit/viability” is an essential attribute of radical constructivism. von Glasersfeld defined these terms in accordance with how well an interpretation allows predictions of future events/activities (Glasersfeld, 1983). This concept of fit/viability” is von Glasersfeld’s defense for the critiques that radical constructivism results in relativism. Relativism suggests that each possible interpretation is subjective and is equally valid (Baghramian, 2004). Radical constructivism, however, suggests that, although each possible interpretation is indeed subjective, the only valid perspectives are those that consistently and correctly predict future events/activities.

In 1984, von Glasersfeld oriented the reader around several key points of his ideology. According to von Glasersfeld (1984), there is a parallel between the theory of evolution and radical constructivism. The theory of evolution focuses on physical survival despite environmental constraints, and radical constructivism focuses on cognitive survival despite ideological constraints. The ultimate constraint inhibiting humanity from “knowing” reality is that humanity cannot “know” anything without using its senses or its own thoughts. Both sources establish the human as the central entity which is called anthropocentrism. As the central instrument, being human is humanity’s limitation on its own “knowing.” Although in agreement that humanity’s own senses and thoughts frame its cognitive boundaries, von Glasersfeld’s perspective is slightly different on the meaning of knowledge in that he established that radical constructivism is deemed radical because it explicitly admits that it is an epistemology that asserts that knowledge can only identify what works, not what is.

Among these early steps there is, of course, the relationship between knowledge and reality, and that is precisely the point where radical constructivism steps out of the traditional scenario of epistemology. Once knowing is no longer understood as the search for an iconic representation of ontological reality but, instead, as a search for *fitting* ways of behaving and thinking, the traditional problem disappears. Knowledge can now be seen as something which the organism builds up in the attempt to order the as such amorphous flow of experiences by establishing repeatable experiences and relatively reliable relations between them. The possibilities of constructing such an order are determined and perpetually constrained by the preceding steps in the construction (von Glasersfeld, 1984). This construction means that the “real” world manifests itself exclusively at the point where our constructions break down and are no longer valid, reliable, or accurate. We can only describe and explain these breakdowns using the very concepts, ideas, and language that we have used to build the failing structures; as a result, this process can never yield a picture of a world that extends beyond our conceptual awareness, thought patterns, or linguistic skill. In fact, from this perspective, the very idea and existence of human failure corroborate the existence of phenomena beyond our conceptual awareness, thought patterns, and linguistic skill (von Glasersfeld, 1984). Once this clarification on the meaning of knowledge has been fully understood, it becomes apparent that radical constructivism itself must not be interpreted as a picture or description of any absolute reality but as a possible model of knowing and the construction of knowledge in cognitive organisms that are capable of constructing for themselves, on the basis of their own experience, a more or less reliable world. It should be explicitly stated that for the radical constructivist, as a person increases his or her conceptual awareness, thought patterns, and linguistic skills, the person will have a greater experience with the phenomenon of interest and subsequently construct a deeper knowing of it. As humanity evolves, so to do the descriptions of the phenomena that humanity experiences; consequently, there can be no absolute reality of or for phenomena, where absolute refers to a definitive end.

von Glasersfeld (1989a) provided the following points as The Fundamental Principles of Radical Constructivism:

- Knowledge is not passively received either through the senses or by way of communication,
- Knowledge is actively built up by the cognizing subject,
- The function of cognition is adaptive, in the biological sense of the term, tending toward fit or viability, and
- Cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality.

Next, I provide details into these principles through brief discussions of epistemology, ontology, relativism, and learning environments within the context of radical constructivism.

Epistemology. Radical constructivism does not allow for the idea of an absolute truth. Human experience includes human sensing, human acting, and human thinking. Human experience is the limiting factor in humanity knowing the world because we can only know what we experience; and what we experience is specifically unique to the person experiencing it. As such, there can be no absolute truth. von Glasersfeld (1983) suggested that it was Plato who proposed that humans are born with the knowledge of the world and universe already within them, but it is unclear, scrambled, and half-forgotten. Consequently, humans spend their existence trying to “re-member/put-back-together” this knowledge into a cohesive, coherent, and cogent whole. Radical Constructivism can be viewed as humanity’s effort, through experience, to re-acquaint itself with this insight.

Due to the dependence of humanity’s consciousness and existence on space and time, it is impossible for humans to know anything before it is framed within humanity’s conceptual tools of space and time. As such, von Glasersfeld (1989b) argued that no person can ever know anything until he or she has experienced it according to his or her framing within space and time. von Glasersfeld posited that if reality existed, as an absolutist internalizes it, then it would be beyond the confines of space and time. If reality is beyond the confines of space and time, then it does not have structure, because structure implies a specific relationship with space and time. If reality does not have structure, then reality does not exist because existence demands structure. Existence demands a specific relationship with space and time. As such, humanity cannot know an absolute reality, because an absolute reality does not exist. Another consideration regarding the existence of an absolute reality offered by von Glasersfeld is how could one confirm one’s knowledge of reality when one could not gather the true example of the phenomenon for comparison for which his or her knowledge is a signifier?

von Glasersfeld (1989b) acknowledged a key connection to the Chicago School of Pragmatism in that constructivism is a form of pragmatism and is consistent with its attitude toward knowledge and truth. A distinction between constructivism and pragmatism as depicted by von Glasersfeld is in how knowledge is attained. von Glasersfeld presented a definition of knowledge whereby knowledge is the conceptual structures that epistemic agents, given the range of present experience within their tradition of thought and language, would consider viable. According to von Glasersfeld, the mistranslation of German philosophy, specifically, Kant's *Critique of Pure Reason*, has led to the unfortunate mis-use of the English word "representation" and its role in knowledge construction. For von Glasersfeld, "perception" was based upon a present-time activity or event. However, "representation" is based upon a past-time activity or event. A re-presentation is a replay of one's lived experiences. Conversely, perception is an accumulation of current, at the moment, sensory inputs. Notice the addendum, "re-presentation" can also mean the English word prediction, which is concerned with forecasts based upon one's prior experiences. Note the aspect of time that was privileged by von Glasersfeld. His linguistic abilities allowed him to read Kant's *Critique of Pure Reason* differently than others. It was his particular reading of that text that led him to develop an epistemology which captured the significance of time and context. For von Glasersfeld, the value of knowledge, the interpretation of one's experiences, is two-fold:

- The ability to re-present what has happened in the past, and
- The ability to understand what one can do and what one likes, as well as what one cannot do and what one does not like.

von Glasersfeld's definition can be understood to be the conceptual means to make sense of one's experience. His definition was not centered on the external world; consequently, his definition was not an attempt to represent a phenomenon that lies beyond one's experience. von Glasersfeld advocated humanity moving itself beyond the idea of affirming causation and instead into the idea of affirming correlation. Incidents are related with one another, but humanity should not assume that it understands all the underlying dynamics of a situation or existence. I find knowledge to be the awareness of ideas, the interrelationships of those ideas and the reasons for those interrelationships as constructed by one's interpretation of one's experiences. Humanity is limited to experiencing only projections of an entity's essence; humanity can never experience the definitive essence of an entity because existence has no singular form.

Ontology. It is important to emphasize that although radical constructivism establishes that one constructs one's interpretation of one's experiences, it does not, however, establish that a person can construct any reality that he or she desires. In this case, reality refers to the existence of a world external to the person (von Glasersfeld, 1989b). Interestingly, the person interpreting his or her perception of his or her experiences would consider reality as the perception itself. So, unfortunately, we are using the same word, reality to refer to two different concepts. In the first case, it would be the world external to one's experiences. In the second case, it would be the interpretations of the person's perceptions of the world. von Glasersfeld (1989c) offered commentary on this point. First, it is due to this duality of the word reality that his radical constructivism fails under the microscope of ontology because he was in fact not trying to frame radical constructivism as an ontology. He was not using the term reality in the first case. He was using the term reality in the second case; he was framing it as an epistemology.

Second, he posited that representations of reality can be compared with other representations of reality, ad infinitum. Representations of reality, however, cannot be compared with actual reality, because actual reality is outside and beyond the experiential world(s) of humanity. Recall that for von Glasersfeld, neither of his two points regarding knowledge had anything to do with an absolute reality. This perspective is because he knows and understands that humanity is constrained and limited from ever truly knowing a definitive reality. This constraint would be for two reasons. Humanity cannot have the infinite number of experiences necessary to achieve reality. Also, phenomena are never able to reveal their true essence, their reality; because phenomena demonstrate and reveal themselves based upon their interactions and relationships with other entities, and no phenomena can have the infinite number of interactions and relationships necessary to demonstrate and reveal all of what it is.

Although von Glasersfeld (1983) was firm that an absolute reality does not exist, he was clear that in fact an ontological reality, the concept of reality does exist. His straightforward evidence for an ontological reality was that one is not and cannot be 100% successful in the pursuit of one's goals and the obstacles and frictions which deny humanity its desires is ontological reality. He established that ontological reality is how existence actually is, and not just our limited experiences with it. His interesting description of ontological reality included the essence that impedes, resists, and denies the pursuits of humanity (von Glasersfeld, 1983). The presence of an ontological reality can only be assured when it is establishing the failures in humanity's actions or in humanity's thinking. For von Glasersfeld, reality actually refers to the existence of humanity's perceptual and conceptual structures that exist due to our interpretations of our experiences. It is our experiential reality, not the objective or absolute reality.

The inquiry regarding how stability and durability are possible if humanity is not capturing objective reality is acknowledged by von Glasersfeld (1989c) in his explicating the importance of repetition in one's experiences. As one experiences the repetition of experiences or aspects thereof, it not only establishes/reinforces the fit or viability of the person's knowledge and interpretation but also it establishes the stability and durability of the person's experiential reality. Here, von Glasersfeld used the word real to mean repetitive, stable, and durable. All of which are still about the person's experiential reality and not an objective reality.

Relativism. If radical constructivism is based upon one's unique experiences, how does radical constructivism differ from solipsism or other forms of relativism? von Glasersfeld's (1983) "abstraction of regularities" posits that an entity can have a multitude of experiences, but that within this multitude, there is some amount of consistency. This consistency is the description of the entity or phenomenon (von Glasersfeld, 1983). von Glasersfeld's ideology establishes that the individual must develop and establish his or her own knowledge first. In so doing, the "self" is constructed. According to von Glasersfeld (1989a), the self is the awareness of what one is doing or of what one is experiencing. von Glasersfeld's perspective is that while the individual establishes her or his experiential world, the individual is also constructing his or her own identity. Then, the individual's perspective, knowledge, and identity can be refined, honed, and adapted to fit and align with future experiences and tested against social consensus through social interaction. It stands to reason that if the individual's experience did not fit with that of the society, then von Glasersfeld would have advocated that the individual stay consistent with his or her own experiential reality. von Glasersfeld viewed the idea of society itself as yet another human construction. This view suggests two forms of experiential fit or alignment. The term "coherent fit" refers to knowledge that aligns with one's future experiences. The term "cohesive fit" refers to an individual's knowledge that aligns with socially constructed and socially consensual knowledge as established between individuals of a particular social group. Cohesive fit does not mean that the individuals' experiences or their interpretations are the same. It means that the individuals' experiences and interpretations are compatible.

There is an interesting corollary that exists when the phenomenon-under-study is another human or sentient being. Let us take the interaction between a teacher and a student, for example. The knowing that the teacher has of the student is limited by both the experience that the teacher has with the student, as well as by the conceptual awareness, thought patterns, or linguistic skill of the teacher. This same situation exists for the student. The amount of knowing that the student has of the teacher is limited by both the experience that the student has with the teacher, as well as by the conceptual awareness, thought patterns, or linguistic skill of the student. There is yet a third phenomenon with which both the teacher and the student have interaction. This third phenomenon is the actual learning environment, the semiosphere. Again, the amount of knowing that both the teacher and the student construct about the learning environment is limited by their experiences with the learning environment, as well as by their conceptual awareness, thought patterns, or linguistic skill. As complicated and complex as the learning experience is for both the teacher and the student, any researcher who proposes definitive and absolute findings from a study of an educational scenario has drastically under-estimated the complexities of the phenomenon and drastically over-estimated his or her own conceptual awareness, thought patterns, and linguistic skill. In an effort to bring this broad picture into focus, Peirce (e.g., Peirce, 1902) focused on the potentiality of meaning-making within the individual, while Mead (e.g., Mead, 1910) focused on the actuality of meaning-making between and amongst individuals, and Saussure (e.g., 1916/2011) focused on the aspects of language and its structure in facilitating meaning-making between and amongst individuals. Radical constructivism accentuates this knowledge-construction process by focusing on the experience of the meaning-making in terms of its extent, boundaries, contextualization, and quality.

It would seem that for von Glasersfeld (1983), cognitive dissonance was a key aspect in establishing the viability of one’s thoughts and interpretations of one’s experiences. Such cognitive dissonance provides the greatest opportunity for accommodation, assimilation, and cognitive equilibrium (Piaget, 1936/1952). According to von Glasersfeld, the greatest opportunity for cognitive dissonance was through social interaction, where communicative fit between individuals is only possible through iterative social interaction. Iterative social interaction is established through ideological orientation and adaptation.

Radical constructivism recognizes social order through its process of fit or alignment with social consensus, which I referred to as cohesive fit. Radical constructivism, I believe, is appropriate in its acknowledgement and is correct in refusing to allow knowledge to be confined by social consensus. The power of radical constructivism is its provision that the individual should privilege his or her own experiential constructions of knowledge once coherent fit is achieved and viability is attained, even if the individual's knowledge construction is contrary to the social consensus. In sum, radical constructivism embraces both social order and social dis-order specifically because both possibilities are nothing more than another set of human constructs.

Learning environments. A learning environment that embraces social interaction amongst students for the purpose of cognitive refinement should be the expectation in every classroom. Assuredly, as mathematical knowledge is constructed and refined, a student's mathematical productive dispositions toward greater mathematical proficiency is also impacted. The immediate challenge with this approach is the competency of the teacher. It requires, I believe, a teacher with his or her own elevated level of mathematical proficiency and a high level of semiotic fluency in order to guide the converging and diverging mathematics discourse generated by 20 or 30 students.

According to radical constructivism, learning is viewed as cognitive change. According to Piaget (1936/1952, 1977), this cognitive change is achieved through a combination of assimilation and accommodation. More specifically, Piaget (1977) stated:

> Thus, when a subject takes cognizance of or relates to an object, there is a pair of processes going on. It is not just straight association. There is a bipolarity, in which the subject is assimilating the object into his schemes and at the same time accommodating his schemes to the special characteristics of the object. (p .3)

von Glasersfeld (1989a) suggested that a teacher should encourage and orient a student's interpretative process and not give the student externally organized abstractions. von Glasersfeld's distinction is intriguing. He said that *teaching* centers on the construction and refinement of a person's knowledge, while *training* centers on the adjustment of one's activity, performance, or behavior (von Glasersfeld). This difference connects with the perspective that meaning is in the mind, while performance is in the hands. These points establish that no one but the individual can produce the individual's experience. And no one but the individual can interpret the experience and produce the resulting cognitive structures and relations amongst them, the knowledge. In sum, the role of the teacher should be to assist the student in ascertaining the fit or viability of his or her knowledge construction. It should not be the role of the teacher to somehow give his or her knowledge to the student through some process of cognitive transfer.

Some Concluding Words on Theoretical Framework

What has been detailed here represents a two-fold arduous process: (a) theoretically framing a theory-building project; and (b) developing a model to represent mathematical learning. CHAT provides the overarching ideological approach for the analysis of the student's learning environment and constituent activities. CHAT however has not established itself in the literature as having a firm grasp on the meaning-making and knowledge-construction of the individual. To address this particular concern, radical constructivism and its focus on language-use is a functional tool for effective linguistic and semiotic analyses. It is the linguistic analysis that draws greatest attention because it allows for a greater understanding of the cognitive components which facilitate a greater understanding of the experience had by the individual.

A Historical Fist of Cuffs: Radical Constructivism v. Sociocultural Theory

Now, to be clear, although I state that a combination of CHAT with radical constructivism allows for a theoretical framework that addresses my needs for this theory-building project, such a combination is not without its tensions. CHAT provides the investigative tools and perspective for analyzing an activity; and radical constructivism provides the investigative tools and perspective, through the teaching experiment (to be detailed in Chapter 4), for analyzing the individual's language and actions within an activity. At this point of combining these two theoretical frameworks, a duality exists. For me, the junior theorist, this match should have been problematic because of the ideological differences which are purported to exist between sociocultural theory, the foundation of CHAT, and radical constructivism, the foundation of the teaching experiment. For me, the seasoned practitioner, this match was clear and evident because it addressed a practical dilemma. As a classroom teacher, I was convinced that this combination would be beneficial for my students; however, it was not until recently that I learned of the ideological fist of cuffs. The issue seems to be about the generative source of knowledge (Jaworski, 2016). For the sociocultural researcher, community and its culture are the generative sources of knowledge (Lerman, 1996). For the radical constructivist, the individual's experience is the generative source of knowledge (von Glasersfeld, 1984). Steffe and Thompson (2000) presented a cogent argument that there is an overlay of the two ideologies when people engage in discourse because the influences of one's community and culture are captured within one's discourse. In other words, the individual's experience, as presented within the discourse, is enshrined in the community and the culture of his or her immersion.

In the same fashion as Steffe and Thompson (2000), I also believe that there is evidence of a hand-shaking of sorts between the two ideologies. I wish to provide the idea of apprenticeship as another element which establishes that there is indeed an overlay of the two ideologies. I present apprenticeship as my evidence because a metacognitive apprenticeship is exactly what I practice in my tutoring sessions and in my mathematics classroom. I expound on the topic of apprenticeship in Chapter 5.

On this subject, I close by first saying that I appreciate the intellectual exercise that was necessary for me, myself, to establish my own position on this issue because the foundation of this theoretical project and its underlying methodology hinged on my ability to bring together two ideologies that some had believed to be mutually exclusive. Secondly, I state that for the African American student in a mathematics learning environment, the traditional use of CHAT seems to reify an already existing grand (master) narrative: African Americans are deficient in mathematics. But if this combined theoretical framework were employed, one that highlighted effective linguistic and semiotic analyses for the purpose of understanding the cognitive activity and knowledge-construction within the individual, then perhaps a counter-narrative could be derived.

CHAPTER 4

THE PRACTICING AND EVOLVING "TEACHER–RESEARCHER"

In this chapter, I begin by discussing my high school teaching experiences and how I utilized my knowledge of and experiences with the engineering design process (EDP) (e.g., Berland, Steingut, & Ko, 2014) to address some of the challenges that I have faced as a mathematics teacher. Next, I provide details of my middle school teaching experiences and the tandem of pilot studies that I was able to conduct in my classrooms. The pilot studies allowed me to investigate several hypotheses that I had developed about my middle school students. In large part, I followed the scientific method (e.g., Haig, 2019) to conduct these investigations. My use of the EDP and the scientific method led me to searching for a methodological tool that I could use to not only test new hypotheses but also to address my practical classroom challenges. The methodology that I found which best met these two criteria is the teaching experiment (e.g., Cobb & Steffe, 1983; Steffe & Thompson, 2000); the basic principles of the teaching experience are outlined here. My findings from many informal implementations of the teaching experiment inspired me to consider making adaptations, extensions, or, maybe better yet, re-engineer the teaching experiment in an effort to broaden its investigative lens, its investigative power. In that, I desired to investigate the mathematics activity within an assignment, specifically its context, its constituent parts, and the role creation that it imparts onto the student with the same level of scrutiny that I investigated the mathematics learning of a student. I felt that the teaching experiment might be re-engineered to accommodate my need. As I provide insight into this re-engineering process, I conclude the chapter by detailing my own evolution from a sole practitioner, teacher, into a neophyte academic, theorist. This evolution was essential given that this dissertation project led me into the field of theory-building so to ensure that my re-engineering of the teaching experiment would be received as theoretically sound, robust, and critical.

My High School Teaching Experiences – The Engineering Design Process

The engineering design process (EDP) is a problem-solving approach that I have been trained in and used for almost twenty years. As an engineering major during my undergraduate and graduate education, I had 7 years of formal education in the EDP before I spent 10 years applying it in the corporate world. My point is that it is quite a normal response for me to engage the EDP whenever I am confronted with a problem. As I discussed in Chapter 1, teaching high school mathematics was definitely a problem that I had confronted. If you recall, it was within Chapter 1 that I stated that I was faced with the challenge of teaching mathematics to individuals who did not seem to be interested. I have since learned that a lack of interest was not the problem. The problem was the approach. To be more specific, the problem was the approach for the teaching—not the learning. I offer a bit more insight in what follows.

The EDP contains the following general stages: (a) definition of the problem, (b) conduct background research, (c) specify conditions and requirements, (d) generation of action options, (e) selection of choice, (f) modeling of prototype, (g) development and implementation of the prototype, (h) review and revise the prototype; and (i) communicate the results (Berland, Steingut, & Ko, 2014; Cash, Hicks, & Culley, 2015; Daugherty & Mentzer, 2008; David, 2013; Gagnon, Leduc, & Savard, 2012; Mesutoglu & Baran, 2020; Pieper & Mentzer, 2013). My educational training and work experience provided me with a few more specifics, and I utilized the general framework as well as these specifics to face the challenges in my high school classroom. Allow me to demonstrate the process.

First, I had to define the problem, as I experienced it. The perceived problem was that my high school math students could not solve fundamental mathematics problems. When I did some background research on my high school students, I discovered that many of my students did not read at grade level. This knowledge brought me a bit of clarity because my own data analyses revealed that the more deficient was the student's reading proficiency, then the more deficient was the student's mathematics proficiency. Upon understanding this connection, I began the next stage of the EDP which was to specify the conditions and requirements. One major condition was that my students did not believe that reading and mathematics were complementary acts. For my students, mathematics was understood to be the steps taken to solve a numerical problem. The idea of first establishing an understanding of the numerical problem was not aligned with their lived experience. A second condition was that I did not have the benefit of receiving desired resources or increasing my allotted instructional time with the students. As such, I had to develop a process that immersed the students in reading comprehension mathematics scenarios while at the same time staying aligned with the official curriculum pacing schedule—all without additional resources or additional time.

Second, I had to do some brainstorming. During my brainstorming of possible actions to take, one particular approach resonated the most with me. The approach was to emphasize the multiple representation of each and every mathematics topic. Such a multiple representation strategy actually required the students to engage in modeling, which first required the students to gain an understanding of the given mathematical situation. So, I was able to tie the classroom activity of creating multiple representations of a mathematics concept back to my goal of incorporating reading comprehension into my mathematics classroom.

Working within the guidelines of the EDP, the next step was for me to develop a prototype approach for having my students generate the multiple representations. I determined that the best way to achieve this goal was for me to change the method by which I introduced a mathematical concept and instead, introduce each concept in each of the five modes: (a) pictorially, (b) graphically, (c) tabularly, (d) numerically, and (e) conceptually. Now, granted, not every mathematical concept can be represented in each of these five modes, but to do so was the goal. Despite complaints from my students, I became more consistent with this approach. As I became more consistent, my students also became more consistent in doing the same.

Eventually, my students had become so comfortable with this approach that I was able to assess the depth of their knowledge of a mathematical concept by how effectively and efficiently they were able to generate each of the five modes for the respective mathematical concept. Such effectiveness and efficiency required the students to *read* and *understand* the mathematics to ensure that the same information was correctly represented across all of the modes. Indeed, this was the ultimate goal.

Beyond my frequent classroom formative evaluations of my students, I did have opportunities to formally evaluate the impact of my approach. One year, during the time when the High School Graduation Test was still in effect, my twelfth-grade students had a writing assessment to complete. After the results had been analyzed, it was determined that my group of twelfth graders out-performed all the others in the school on the writing assessment. Although the writing assessment was purely an essay writing task, I believe that my effort in focusing their attention to reading and reading comprehension within my mathematics classroom equipped my twelfth graders to have a high performance on their writing assessment. This outcome was the first indication that my approach of not only presenting mathematical concepts in all five modes but also expecting my students to do the same may have had a positive effect.

A second opportunity at a formal evaluation of my approach came during the spring semester of that same school year when my twelfth-grade students had to take their High School Graduation Test. I was tasked to develop a test preparation program for the school's twelfth graders, specifically a program for the mathematics preparation. It was an extremely arduous task to introduce my mathematics approach to all of the twelfth graders in the building over a span of a mere 2 months. Nevertheless, after all of the results for my district had been analyzed, the twelfth graders at my school had performed well enough to be rewarded for outstanding achievement by the district.[25] Based upon the performance of my students, I was convinced that the presentation of mathematics concepts across the various five modalities of picture, graph, table, equation, and words was beneficial for my students, and it allowed me to focus on my students' reading comprehension in a new and creative way.

Now, with regards to the larger impact that this development had on my dissertation work, I must say that the realization that having my students comprehend a given mathematical concept or situation in multiple representations was the seed that propelled me toward the study of semiotics, in general. But more specifically, it propelled me toward the realization that having my students comprehend a given mathematical concept or situation in multiple representations should be the first action taken when given a mathematical concept or situation, irrespective of the accompanying question or instruction. This new insight compelled me to examine and consider activities in a different way years before I was introduced to CHAT.

My Middle School Teaching Experiences – The Scientific Method

After my work with the twelfth-grade students at my high school, I was presented an opportunity to become the Director of Mathematics at a K–8 public charter school. I took the

[25]An internal district report showed that the pass rate for our twelfth-graders at first administration was 78%. This was significantly higher than the pass-rate at first administration for our twelfth graders of previous years.

opportunity for two reasons. The first reason was that I would be able to lead my own team of mathematics teachers. The second reason was that I would have the opportunity to experience firsthand the mathematics learning experience of middle school students. Based upon my high school mathematics teaching experiences, I had a developing and somewhat verified idea of which techniques worked best for my high school students. Now, I had been given a chance to present these same techniques to middle school students with the hope that their future high school mathematics experiences would be easier and more enjoyable for them. I was also curious to know what new and as yet unrealized techniques I might learn by working in a middle school that I could use to propel the students even further in their mathematics learning.

By the time of this promotion, I had just completed my first year of graduate school, and I was being exposed to new and interesting ideas from my doctoral readings every week from several different areas: mathematics education, linguistics, literacy, and research methods. With respect to my new employment position, as it turned out, I was also exposed to a new and interesting idea in the form of an opportunity to teach a 3-year long accelerated mathematics program at the public charter school. This opportunity was fantastic for me because I was able to not only present my techniques to students as young as eleven years of age but also to watch these young students construct and grow into their own mathematical identities.

During my first year at the public charter middle school, despite my best efforts to present and emphasize the value of multiple mathematical representations to the students, there was initial resistance. The students were never disrespectful. But rather it seemed as though the students were confused as to why I would emphasize producing so many different representations of the same situation when one was sufficient to get the answer. They viewed the entire process as extra and unnecessary work. It also seemed as if the students wanted to get the answers as quickly as possible at the risk of missing some of the more subtle mathematical ideas. What quickly became apparent was that in my students' rush to get to the answer, they were getting many problems wrong.

During this entire time, I was taking three or four graduate courses from a variety of departments. There were two courses which have had a lasting impact on my pedagogical development. In both courses, I was given projects to complete with the option of conducting the projects within my classroom. I took advantage of these opportunities with the hope that I could gain insight into why my middle school math students were getting so many problems wrong, beyond the simplistic explanation they were rushing through the mathematics. I never believed that it was because the students were too young to learn the mathematics. I actually thought that it was because of the literacy level of the mathematics problems.

By this time, I had developed two different hypotheses, which interestingly enough, were much aligned with the scientific method (Haig, 2019; Kosso, 2009; McGuire, 2007; Tang, Coffey, Elby, & Levin, 2009; Woodcock, 2014). The first hypothesis was that my students had a cursory understanding of mathematical situations and were driven to *act* first and *understand* second. This first–second explanation was my hypothesis because I thought that the sentence structure and sentence type of the mathematics problems were too complex and far exceeded the standard sentence structure of the middle school grades curriculum. In other words, I believed that the sentence structure, type, and literacy levels were so high that my students did not attempt to decode the sentences in the math word problems for understanding. But rather, they were product focused and immediately attempted to reference classroom examples to solve the word problems. But I was actually attempting to develop, within the students, a comprehension first, and solution second mindset, which is a process-oriented approach. The second hypothesis was that irrespective of the sentence structure and type, the nature of the connections and relationships of the math word problems were too abstract for the middle school grades curriculum and therefore my students had no conceptual reference to the word problems.

An opportunity developed that allowed me to conduct two small pilot studies to test both of these hypotheses. Although not intentional, these pilot study opportunities also allowed me to continue along the path of the scientific method. The first pilot study was generated within one of my Applied Linguistics courses, and the second pilot study was generated within one of my Language and Literacy courses. These two pilot studies were valuable to my work because they allowed me to do preliminary and informal investigations of the manner and depth by which language influenced the mathematics learning in my classroom. The findings of both of these pilot studies led to the specific nature and structure of this dissertation project. The first pilot study focused on language syntax and was coordinated under the guidance of Dr. Viviana Cortes, Associate Professor of Applied Linguistics. The second pilot study also focused on language, but not its structure, but rather the second pilot study investigated the meaning constructed from language by the students, the conceptual reference which the students may have had, and the resulting discourse within the classroom. This second pilot study was coordinated under the guidance of Dr. Peggy Albers, Professor of Language and Literacy Education. Below, I provide a description of the pilot studies.

Pilot Study #1

The purpose of the first of two pilot studies was aligned with the Saussurean perspective of semiotics. Recall that Saussure focused on the structure and position of linguistic elements, the syntax, to provide information (Chandler, 2007; Eco, 1978). Specifically, this first pilot study used syntax to search for and identify characteristic patterns in the language used in algebraic mathematic word problems. I hypothesized that the domain-specific structure of the language used within algebraic mathematics word problems was a major contributing cause for the low performance of my students and resulted in the low performance of my students on standardized mathematics exams.

Pilot Study #1 used a variety of syntax-oriented tools to examine underlying linguistic patterns in mathematics word problems. The use of such tools has come to be called corpus linguistics and requires a corpus of linguistic elements—a large supply of words, phrases, or sentences (Biber, Douglas, Conrad, & Reppen, 1998). Corpus linguistics is the process of studying or evaluating a linguistic situation using computer-based tools (Biber, Douglas, Conrad, & Reppen, 1998). Some non-supporters of corpus linguistics refer to this approach as simply "word counting." Leaders in the field, however, know the power of this approach and understand its ability to challenge the intuition of language speakers by seeking patterns in language use. As such, this approach was ideal for my study because I was looking for patterns in the language used in algebraic mathematics word problems.

In this pilot study, it was necessary to collect a corpus of algebraic mathematics word problems. This necessity was accomplished by purchasing a software tool that allowed for the generation of a multitude of mathematical problem types. The software tool is called Kuta Software (kutasoftware.com website) and allows the user to select the problem type, its difficulty level, and various other mathematical attributes. I used this tool to generate a minimum of 60 algebraic word problems in each of the following three categories: Pre-Algebra, Algebra 1, and Algebra 2. Each of the categories contained several of the following sub-categories: arithmetic, rates, distance problems, systems of equations, and age problems.

This corpus contained nearly two hundred mathematics word problems. The word problems were analyzed for their linguistic structure and patterns. According to the analyses conducted, there were three points that were quite prominent throughout the available corpus of mathematics word problems:

- the sentence structure and type of the mathematics word problems was of the simple sentence type, which is the most basic of the sentence structures,
- the word choice of the word problems was not academically rigorous, and
- over 70% of the words used in the word problems were from the list of the first 2000 words in the General Service List (GSL), which is a list of 2000 of the most frequent words in the English language (Gilner, 2011).

At the completion of this pilot study, I had insight into both the linguistic complexity and syntax of the word problems and their readability levels. The overall theme of the analyses was that the linguistic structure and patterns of the word problems were not representative of linguistically complex examples of the English language. In fact, the structures and patterns were the most basic and within the parameters of middle school literacy. This finding convinced me that my middle school students should have been able to understand the situations described by the mathematics problems that they were given. The cause for difficulty of so many of my math students was not the linguistic structure and patterns found in the word problems. It was something else. Therefore, after completing the first pilot study, I decided to conduct a second pilot study and attend to the conceptual complexities of mathematics problems—their semantics. Details of this second pilot study are below.

Pilot Study #2

The purpose of the second pilot study was aligned with the Peircian perspective of semiotics. Recall that Peirce focused on what is presently called semantics, which is the meaning of signs and symbols (Chandler, 2007; Eco, 1978). Specifically, this second pilot study used semantics to search for and identify meaning in the signs and symbols used by me and the students in the classroom, as well as in mathematics problems and in texts. I hypothesized that the relationships and representations of the mathematical concepts used within the field of mathematics were too obscure for my middle school students; therefore, my students had no conceptual reference. Ultimately, this lack of a conceptual reference was a major contributing cause for the low performance of my students on standardized mathematics exams. In this pilot study, it was necessary for my students to build their own conceptual reference and engage in their own mathematical discourse. This necessity was achieved by having my students investigate mathematical situations in a variety of representations and to various depths in accordance with their own meaning-making needs. In short, the purpose of the second pilot study was to study the effect that guiding the students through a process of constructing their own conceptual mathematical reference could have on my students.

The participants were my sixth-, seventh-, and eighth-grade classroom students from the public charter school where I worked. I chose two students from each class—one male student and one female student. All students were either African American or Latinx and met the criteria for moderate-to-low socioeconomic status. For this second pilot study, I gave the students a math problem and then had the students create their own inquiry mathematics question. The inquiry question could be of any form and purpose, but had to be created by the individual student. Next, the student accessed the internet with their inquiry question in mind and, from the multitude of resources listed on the internet, selected at least three websites, at least two videos, and at least one text, for a minimum of six sources, which would be helpful in addressing their particular inquiry question. After addressing their particular inquiry question, the students would continue in their investigative effort to further understand the given problem, and then, to solve the problem. Although each student conducted what he or she viewed as an investigation necessary to solve the problem, there was no guarantee that the student's solution was the correct solution. It was only after engaging in discourse with other students that each student gained a realistic assessment of what had been learned.

During this second pilot study, I provided the students the time and opportunity to work independently, according to their own learning rate in order to identify, gather, and understand explicit details provided within the problem. In addition, during this process, the students gained implicit insight on the problem that may have escaped their awareness at first examination. This implicit insight was gained through the intertextuality of internet links, the multiple modal representations offered by internet websites, and the variety of teaching styles offered by instructional videos which were available on the internet.

After engaging in discourse with each other, the students collectively converted their joint knowledge into visual models, in the effort to produce a tangible representation of the results of their discourse. These models were different than the original information given in the math problem and represented the combined synthesis of the resulting discourse amongst the students. My access to various student data in the form of video, audio, and student work required that I chose an analytic tool that allowed me to examine the semiotic essence of each data form—the act and its perceived meaning. I chose Multi-Modal Interaction Analysis which is an approach that allows the researcher to penetrate beyond an activity and its output, which were foci of Vygotsky, and investigate the semiotic acts within the activity and their potential meaning, which were foci of Halliday (Halliday, 1978; Norris, 2014; Vygotsky, 1930/1978). According to Norris (2014), Multi-modal Interaction Analysis extends beyond mediated discourse analysis, interactional sociolinguistics, and social semiotics. Norris (2014) presents Multi-Modal Interaction Analysis as follows:

> With the mediated action as the unit of analysis, multimodal (inter)action analysis focuses each study on *what* social actors do (the action that is performed), and *how* the action is performed (the mediational means/cultural tool used to perform the action). This focus on social actors as they perform an action highlights three interconnected elements: the social actor, the action itself, and the tools that are being used. (p. 71)

This analytic tool was valuable not only because of my use of video, audio, and the various forms of student work which all served as data within this second pilot study but also was valuable due to the ability of this analytic tool to, as Norris (2014) stated, "guide the researcher in how to investigate the micro, the intermediate, and the macro of the (inter)action" (p. 71). I viewed the micro as that level of analysis that focused only on the body language exhibited during the interview, the non-verbal communication. I viewed the intermediate as that level of analysis that focused on the intermixing of the verbal response and the non-verbal accompaniment. Last, I viewed the macro as the analysis that focused on the overarching message that the student was attempting to communicate, by my interpretation. In other words, Multi-Modal Interaction Analysis allowed me to shift my focus from a student's answer to a completed mathematics activity to the student and his or her discursive practices during the mathematics activity. Although the purpose of this second pilot study was not to establish a correlation between quality of discursive practice during a mathematics activity and the accuracy of the final answer, I did indeed find a general relationship that students who engaged in such discourse performed better on assessments.

Based upon my analyses, I concluded the following four points. First, it seemed that the classroom that allowed for a predominantly collaborative and discursive culture resulted in a more positive, pleasant, and successful educational environment for the students. Second, I also noticed that the quality and content of the collaboration and discourse was higher according to the degree of conceptual reference development and communicative competency which existed within the group. Third, it was also clear that the students found value in expressing to each other how they were thinking while they were solving their mathematics problems. Fourth, language skill became equally as important in my mathematics classroom as was mathematics skill. For example, challenges with the linguistic and critical skills of decoding, encoding, and modeling adversely affected the comprehension, learning, discourse, and performance of my students. This affect was particularly present when the mathematics was presented in written standard English and the students had to engage in explanatory discussions, as represented in the following example problem: "Two times three less than a number is equal to the quotient of five more than a number and seven. What is the value of the number?" In other words, although the literacy level of the mathematics problems was appropriate for middle school students, if the students did not have the experience in decoding and encoding, or the appropriate conceptual reference, the students would not be able to engage in the necessary discursive practices with themselves or others to solve the given problem.

Overall Findings from Both Pilot Studies

After the completion of the second pilot study, I converted the format of all of my future classes to align with the findings that I made from these two pilot studies. In addition, due to the findings from the second pilot study that the discourse between students could be an invaluable aid to my teaching and to the learning of students, I focused on investigating this student-with-student discourse and became curious as to how I could influence its content and quality. As a mathematics teacher, I foresaw that a student-with-student discourse that was rich with mathematics content could be a pathway to greater mathematics proficiency for my students especially given that the findings from the first pilot study revealed that most mathematics word problems are formed using simple to moderate sentence structure and vocabulary. The fundamental conclusions that I made at the conclusion of both of these pilot studies were:

- mathematics word problems consist of simple to moderate sentence structures and contain simple to moderate vocabulary, and
- the mathematics learning environment is more constructive and productive for the students when the students are engaged in activities that emphasize the development of their own conceptual reference, promote their own unique forms of mathematics discourse, and free them from prescribed discussions or isolated mathematical calculations.

With these conclusions in mind, I needed to transform the mathematics experience of my students from a solitary calculational experience into a collaborative discursive experience where the topics of discussion were mathematical relationships and their many forms. I now know this type of environment to be called the semiosphere, and it was discussed in Chapter 3, and I would need to transform my practices so that opportunities to engage and interpret this semiosphere were prominent in my classroom. Influencing such experiences and the resulting discourse is what led me to this dissertation study, and to the teaching experiment.

My Informal Teaching Experiment

As I completed the second pilot study, it became clearer and clearer to me that a focus on a student's development of an individualized conceptual reference and on a student's unique discourse were ultimately a focus on the student's mathematical experience and the student's expression of that mathematical experience. Attending to a student's mathematical experience and its expression as precursors to the student's performance allowed me to focus on the qualitative aspects of a mathematical situation and not simply its quantitative aspects. In addition, such a qualitative perspective had me promoting the fundamental principles of radical constructivism and its methodological arm, the teaching experiment, from their practical benefits in my classroom before I learned of their theoretical values in my doctoral program.

As I detailed in Chapter 3, for the radical constructivist, the world is as it is experienced; it does not have an external reality. In addition, for the radical constructivist, the learner's interpretation of his or her experiences is influenced by the social, cultural, and historical aspects of his or her existence. Knowledge then for the radical constructivist is adaptive insight that allows a person to cope within his or her experientially constructed world. There are no multiple realities; there are multiple experiences. Knowledge is adaptive according to the individual's attenuated experiences; and is examined and evaluated according to its impact on the individual. It is important to note that for radical constructivism, the person will choose the construction that is most viable for him or her, where this viability measurement depends upon the type of experience. The authority on radical constructivism, Ernst von Glasersfeld (1917–2010), realized that the language with which a person communicates and organizes his or her thoughts has a significant impact on the person's perception, interpretation, and understanding. Each of these points aligns well with and describes what I concluded from my second pilot study. Having such an experience in my classroom before I received formal exposure to radical constructivism and the teaching experiment is what convinced me that both radical constructivism and the teaching experiment would have to be major components of my dissertation study.

As I reflected upon experiences from the two pilot studies, two points were clear. First, the language structure and literacy levels of the mathematics word problems were not beyond what was expected at the middle school level. Second, each person in a mathematics learning environment has a uniquely different mathematical experience in accordance with his or her conceptual reference and his or her discursive practices—his or her language and action.

This awareness directed my path toward investigating and examining the form and characteristics of some of these unique mathematical experiences. This awareness is indeed what led me to the teaching experiment, because as previously detailed, such an investigation and examination is the explicit purpose of the teaching experiment. Because I was the classroom teacher in this situation, I functioned as a teacher/researcher when I made the effort to implement an informal teaching experiment. I refer to it as an informal teaching experiment because I did not seek IRB approval, I did not make any video or audio recordings, I did not have a consistent external observer in my classroom, and I specifically focused on the question-posing aspect of the teaching experiment. For my purposes, the infusion of experiential questioning into my daily classroom culture was most important; the deviations from the formality of the teaching experiment were inconsequential to my needs. Some of the questions that I asked my students about their experiences and found to be the most informative were:

- Are you able to correctly read the given situation mathematically?
- Are you able to understand the mathematics within what you have read about the given situation?

- What can you do/create based upon your understanding of the given situation?
- From all that you have done, understood, and created mathematically, what are you able to write down in clear and complete sentences?
- From all that you have done, understood, and created mathematically, what connections and relationships are you able to explain to me?
- What can you do/create for me that reveals what you have learned from the given situation?
- What can you write for me that reveals what you have learned from the given situation?
- What can you discuss with me that reveals what you have learned from the given situation?

Within my classroom, I had already begun the practice of allowing my students to work in groups as I observed their work practices. So, the inclusion of these questions into my practice was not at all difficult, whether I wanted to pose these questions to an individual student or to a small group of students.

Based on my decision to fortify my work with the radical constructivist epistemology, as I detailed in Chapter 3, I needed a methodology that not only was aligned with the ideology of radical constructivism but also a methodology that had a history of effective investigation of student mathematics learning. The methodology that met both of these requirements is the teaching experiment (Steffe, 1984). In addition, due to the fact that I also position my work as an effective theory-building effort of student mathematics learning, I had to also ensure that the chosen methodology had a rich history of generating mathematical theory. The teaching experiment has such a history. In what follows, I provide a brief history and detail some of the descriptive features of the teaching experiment.

Purpose and Characteristics of the Teaching Experiment

Cobb and Steffe (1983) explained that the teaching experiment was an approach utilized by researchers in the Soviet Union to study the mathematical thinking of students. Specifically, the teaching experiment allows for the real-time inspection of correlations amongst the cognitive activity, cognitive development, pedagogy, and psychological traits of the student during their dynamic states of transitions. Menchinskaya (1969) wrote that since the 1920s, three forms of the teaching experiment had been developed in the Soviet Union. The first type focused on the cognitive activity of the student and studied the student's performance within one activity with children of different ages. As developments in research occurred, the teaching experiment evolved and the focus shifted to the cognitive development of the student and instead studied the student's abilities within scaffolded activities with children of the same age. This second form allowed for greater inspection of the correlation amongst cognitive activity, cognitive development, and instruction. The third form is the implementation of the teaching experiment to include full classes of students conducted over a period of several years with changes not only in the pedagogy but also in the curriculum.

Menchinskaya (1969) identified that throughout the ongoing development of the teaching experiment, there have been two centers of attention. One center of attention has been the activity, which includes the learning environment. Two forms of this approach were developed. One form has students of the same age exposed to different stages within the activity. The second form has students of various ages engaged with one stage of the activity. These particular forms of the teaching experiment, when the focus of attention is the activity and the learning environment, are referred to as the macroschemes of the teaching experiment. The second center of attention has been the student. In this form, one particular student is studied through his or her process of ability and performance across a variety of activities, specifically the student's process of cognitive activity, cognitive development, analysis, synthesis, and achievement. This particular form of the teaching experiment, when the focus of attention is the student, is referred to as the microscheme of the teaching experiment. According to my research interests, cultural-historical activity theory (CHAT) would be useful in the evaluation of the macroschemes of the teaching experiment.

To gain the benefits of either focus of the teaching experiment, Steffe (1991) presents four roles of the researcher:

- The researcher must also be a practitioner and serve in the role of teacher.
- The researcher must also be an instructional analyst and evaluate pedagogy.
- The researcher must also be a linguist and analyze the meanings of discourse.
- The researcher must also be a theorist and generate theories and models which represent his or her best perception of the knowledge construction of the student.

Steffe also identified ten goals around which the teaching experiment is conducted:

- Communicate mathematically with the student.
- Engage students in goal-directed activity.
- Embrace the mathematics that the students already possess.
- Create a variety of mathematical environments.
- Interpret the mathematical experience of the student.
- Make mathematics relevant for the student.
- Engage in reflection and abstraction.

- Motivate students to communicate mathematically with each other.
- Motivate persistent student learning.
- Engage with other mathematics educators.

History of the Teaching Experiment

Due to the work by researchers in the Soviet Union, the teaching experiment and its ideology have had a long history. Beginning in the 1920s, instructional psychologists from the Soviet Union began to study the connection between instruction and cognitive development (Menchinskaya, 1969). Exploring this connection was the beginning of the cultural-historical analysis of cognition. It was also during this time when studies were conducted to correlate memory and cognition. According to Menchinskaya, Soviet researchers also evaluated connections between a person's learning process and his or her personality traits.

During the 1930s, the teaching experiment was used to study variations in cognitive development according to particular subject matter (Menchinskaya, 1969). It was also during this time when unique differences in the separate research methodologies between Soviet researchers and American researchers became evident. The Soviet researchers entered classrooms and utilized qualitative approaches in their work, while American researchers preferred quantitative analyses conducted in laboratories. According to Wirszup and Kilpatrick (1978), American researchers committed themselves to the correlations among student age, educational background, and cognitive development. Soviet researchers, however, committed themselves to studying the connections among pedagogy, subject matter, and cognitive development. It can be argued that the focus of Soviet research was content mastery, while American research focused on content attainment. It would also seem evident that American researchers perceived cognitive development to be a linear process dependent upon time alone, while Soviet researchers perceived cognitive development to be a non-linear process dependent upon many factors of which time is only one.

From the 1940s to the 1970s, Soviet research had full momentum and Soviet researchers selected the areas of mathematics, grammar, and reading for their studies in content mastery (Menchinskaya, 1969). Scientifically, great strides were made in distinguishing concrete operations from abstract operations. It was also during this time frame that American researchers learned of the work of the Soviet researchers and learned about the teaching experiment, although the political climate between the two countries was rather volatile.

During the 1960s, American researchers took hold of the teaching experiment and began their own work. For example, Steffe and Parr (1968) used the teaching experiment to study the use of manipulatives in mathematical problem-solving. Steffe and Carey (1972) found the teaching experiment quite helpful in their study of the conceptualization of measurement by elementary school students. The use of manipulatives by elementary school children was also studied by Behr (1976a; 1976b). Also, in the late 1970s, Denmark, Barco, and Voran (1976) used the teaching experiment to study the conceptual understanding of the equal sign held by elementary school students. De Corte and Somers (1981) had great success with the methodology in studying the problem-solving techniques of sixth graders. Behr, Wachsmuth, Post, and Lesh (1984) continued their use of the methodology and made great gains in research focused on equivalence and comparison of rational numbers. Although many American researchers utilized the methodology, it is important to note that Steffe had been one of the initial American researchers to deeply understand the benefits offered by the teaching experiment. This depth of understanding is evident in a publication by Cobb and Steffe in which the ideology and methods of the teaching experiment are provided (Cobb & Steffe, 1983).

During the 1990s, non-Soviet researchers and non-American researchers utilized the teaching experiment to evaluate abstract operations within elementary and middle school students. For example, Verschaffel and De Corte (1997) conducted a study on mathematical modeling with fifth graders. Viiri (1996) expanded the teaching experiment into the field of engineering by studying the cognitive development of first-year engineering students as they learned the physics and mathematics behind natural forces.

By the 2000s, results from the teaching experiment had become substantial. Researchers around the world were accustomed to utilizing this tool and sharing their results through publications. De Corte (2004) published an article detailing how the teaching experiment could be used to create a learning environment that would be optimal for the cognitive development of fifth-grade math students. van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004) used the teaching experiment to generate experimental lesson plans and instructional guides to facilitate the learning of ratios and proportional reasoning by eighth graders. Lamberg and Middleton (2009) offered a great example of extending the teaching experiment beyond the individual student and utilized the tool for whole-class instruction. Norton and McCloskey (2008) took the teaching experiment into a new area by demonstrating its use in the professional development of two elementary school teachers.

Influential Scholars of the Teaching Experiment

Now that I have presented an overview of the teaching experiment and offered a limited set of examples of the use of this methodology, I think it important to highlight the scholars who have had the greatest individual impact on my learning and understanding of the teaching experiment. In what follows, I bring attention to Les Steffe, Paul Cobb, and Patrick Thompson, their unique insights on the teaching experiment, and what I have gained from each of them.

Les Steffe, professor at the University of Georgia, Athens is the prominent American authority on the teaching experiment. He was a part of the initial team that studied the teaching experiment as it was implemented in the Soviet Union. After his own early use of the teaching experiment, he led the effort to incorporate several customizations that established distinctions between the American form of the teaching experiment and the Soviet Union form (Cobb & Steffe, 1983). Two of these customizations were the inclusion of the clinical interview as practiced by Piaget and the fusion of the etic and emic perspectives of the situation under study, which allowed for the inclusion of a composite and multi-dimensional viewpoint of the situation under study. I have chosen two publications, Cobb and Steffe (1983) and Steffe and Thompson (2000) that I believe best represents the orientation of the teaching experiment here in the United States as well as Steffe's academic introspection regarding it.

In Cobb and Steffe (1983), Steffe partnered with Paul Cobb, a prominent researcher himself in mathematics education, to co-author the *Journal for Research in Mathematics Education* article "*The Constructivist Researcher as Teacher and Model Builder*." At the time of the publication, Paul Cobb was a doctoral student of Steffe's at the University of Georgia. In the article, the two bring attention to several points regarding the teaching experiment:

- The researcher must also serve as teacher,
- The assertion of the constructivist view of teaching,
- The use of qualitative data: observations and clinical interviews,
- The long-term interaction between the researcher and the students,
- The trajectory of a study is rhizomatic and not determined by prescribed conceptualizations of the researcher, and

- The goal is the development of a model, as a plausible explanation, of the meaning-making and knowledge-construction by the students in the study.

In the following sections, I provide details of each point. On the emphasis that the researcher also functions as a teacher, Cobb and Steffe (1983) offered two reasons. First, they posit that a theoretical perspective without experiential support results in an incomplete perspective. Second, they posit that the meaning-making and knowledge-construction experiences are influenced by social interaction. According to Cobb and Steffe, one way that the mathematics experience of the student is influenced is through the mathematics discourse which occurs with a mathematically competent researcher. The second way that the mathematics experience of the students is influenced is through the ability of the mathematically competent researcher to draw the student's attention to particularities in the contexts of various mathematical situations. This cooperative collaboration between researcher and students whereby mathematical experiences, of both the researcher and the student's, is generated, experienced, nurtured, and altered is formally referred to as the teaching episode (Steffe, 1991). It is during the teaching episode that meaning-making and knowledge-construction experiences are influenced by social and mathematical interaction.

The second point presented by Cobb and Steffe (1983) is that the teaching experiment should be based upon a constructivist epistemology. A constructivist epistemology states that a cognizing entity constructs meaning through experience (von Glasersfeld, 1984). Based upon the prior section, within a teaching experiment, the researcher functions both as an investigator and as a teacher. This dual role means that there are at least three cognizing entities participating in the teaching experiment, the researcher, the teacher, and the student. Because the constructivist epistemology posits that each cognizing entity makes meaning through his or her own experience, then there are, in fact, three separate experiences which must receive attention in the teaching experiment: the researcher's experience, and the teacher's experience, the student's experience. In addition, due to the qualitative nature of the constructivist epistemology, the student's experience is privileged. So, in short, a teaching experiment that is based upon a constructivist epistemology requires the researcher to focus his or her efforts on interpreting all actions—the researcher's, the teacher's, and the student's—from the student's point of view. Undoubtedly, this is a challenging task for the researcher; however, the benefit of such an effort is that the researcher reduces the critique that his or her findings are limited to any a priori ideologies. For the teaching experiment, such forms of data include observations, field notes, video recordings, audio recordings, narratives, discursive sessions, and reflective journaling.

The third point presented by Cobb and Steffe (1983) is that the teaching experiment uses qualitative data for its data analysis. Qualitative data is information that is in a form that is non-numeric (Denzin & Lincoln, 2000). Qualitative data is necessary during a teaching experiment because it is imperative that the researcher develop and evaluate learning hypotheses according to the experiences of the actual students, as detailed in the previous section.

The fourth point presented by Cobb and Steffe (1983) is that the teaching experiment encompasses a long-term interaction between the researcher and the students. A study which lasts for a minimum of 6 weeks would qualify as a long-term interaction. A specific time period is a prerequisite for a teaching experiment because the researcher is making a deliberate attempt to understand the mathematical language and the mathematical skill that the student embodies. Such awareness requires focused attention, rigorous reflection, and constant verification by the researcher. As Cobb and Steffe suggested, "The processes of a dynamic passage from one state of knowledge to another are studied. What students do is of concern, but of greater concern is how they do it" (p. 87). None of these efforts can be rushed.

The fifth point presented by Cobb and Steffe (1983) is that the teaching experiment is dynamic and organic and is not framed within any prescribed conceptualizations of the researcher. It is indeed not a genuine research study of any kind if the researcher begins the study having already established the findings. The credibility of the study and of the researcher would not endure the scrutiny of the peer-review process. The capacity of the researcher to foretell the future in such a manner is limited, as indicated in this statement by Cobb and Steffe (1983):

> We, too, believe that adults can help children as they attempt to learn mathematics. However, it is not the adult's interventions per se that influence children's constructions, but the children's experiences of these interventions as interpreted in terms of their own conceptual structures. In other words, the adult cannot cause the child to have experience *qua* experience. Further, as the construction of knowledge is based on experience, the adult cannot cause the child to construct knowledge. (p. 88)

The sixth point presented by Cobb and Steffe (1983) is that the objective of a teaching experiment, based upon the context of the actual teaching experiment, is to generate a theory of the mathematics learning of students. Despite the specifics of the setting of the teaching experiment, or of the particular students involved in the teaching experiment, or of the identity of the researcher involved in the teaching experiment, the model of mathematics learning that is generated by the teaching experiment should be rigorous and robust enough that it displays specificity and generality of the unique mathematical phenomenon that has been modeled. The term generality here means that the theory is a viable explanation of the mathematical construct that would be constructed by other students of similar age, language, and skill from some other setting (Steffe, 1991). The stance by Cobb and Steffe on this point is given below:

> The reader might well have inferred that our sole objective is to account for the mathematical progress made by the small number of students who participate in a teaching experiment. However, we strive to build models that are general as well as specific. On the one hand, the model should be general enough to account for other student's mathematical progress. On the other hand, it should be specific enough to account for a particular student's progress in a particular instructional setting. We attempt to attain these seemingly contradictory objectives by ensuring that there is a dialectical interaction between the theoretical and empirical aspects of our work. (p. 91)

In summary, Cobb and Steffe (1983) asserted that a rigorous and credible model results from a teaching experiment whereby the researcher takes the necessary time to engage with the students in quality mathematics discourse in order to establish and influence the mathematics language and mathematics skills of the students, individually and collectively.

I now provide a similar evaluation of the Steffe and Thompson (2000) publication. Steffe partnered with Patrick Thompson (also a former doctoral student), a well-published researcher in mathematics education, to coauthor the publication entitled "Teaching Experiment Methodology: Underlying Principles and Essential Elements." At the time of the publication, Patrick Thompson was a professor of mathematics education at Vanderbilt University. In this article, the two reflected upon their experiences as members on various research teams that conducted many teaching experiments at the University of Georgia. In addition to restating the previously mentioned six points, they bring attention to several additional points regarding the teaching experiment:

- The methodology is not a standardized protocol,
- The researcher should engage in exploratory teaching,
- Research hypotheses should be created and tested according to students' language and actions,
- Critique of suppositions requires discourse with fellow researchers,
- Compare the boundaries within which the students are collectively successful and individually successful,
- A historical assessment of the qualitative data is essential in order to track student learning and development, and
- The replication of a teaching experiment is to extend a theory through generalizability, not simply to prove a theory.

I provide details on each these points in the sections that follow.

The first point presented by Steffe and Thompson (2000) is that the teaching experiment does not present the researcher with a standardized protocol. The fact, simply stated, is that no two students are exactly the same; each student is uniquely different. Consequently, each experience will be uniquely different; therefore, each teaching experiment should, in fact, be procedurally different to accommodate the uniqueness of each experience that is being studied. Two comments by Steffe and Thompson speak to this point:

> Rather, the teaching experiment is a conceptual tool that researchers use in the organization of their activities. It is primarily an exploratory tool, derived from Piaget's clinical interview and aimed at exploring students' mathematics. (p. 273)

> It is a dynamic way of operating, serving a functional role in the lives of researchers as they strive to organize their activity to achieve their purposes and goals. In this, it is a living methodology designed initially for the exploration and explanation of students' mathematical activity. (p. 273)

The second point presented by Steffe and Thompson (2000) is that the researcher should engage in exploratory teaching. According to Steffe and Thompson, exploratory teaching is a traditional classroom teaching experience during which time the researcher engages with a class of students in order to better understand the traditional classroom environment from the teacher's perspective, from the group of students' perspectives, and from the perspective of individual students. It is during this time where the researcher, in the role of the traditional classroom teacher, can best understand the multiple dimensions which exist in the learning experience of a student. The expectation is that with this greater awareness, the researcher can lead a more successful teaching experiment. Steffe and Thompson stated:

> It is important that one becomes thoroughly acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest. In understanding this, one must adopt a certain attitude if substantial progress is to be made toward learning a mathematics of students. The teacher-researcher must attempt to put aside his or her own concepts and operations, and not insist that the students learn what he or she knows. (p. 274)

The third point presented by Steffe and Thompson (2000) is that within the teaching experiment, research hypotheses are to be created according to the communication and activity of the students. This point is a reformulation of a prior point in that the researcher must neither frame any hypothesis within the boundaries of preconceived notions nor consider a hypothesis verified simply because the researcher believes it to be so. Without confirmatory experiential evidence from the student, either during a discussion or through action, a hypothesis remains as a conjecture of the researcher. The importance of this need to subordinate the researcher's hypotheses and other forms of thinking is made evident in the following statements from Steffe and Thompson:

> The research hypotheses one formulates prior to a teaching experiment guide the initial selection of the students and the researchers' overall general intentions. However, the researchers do their best to 'forget' these hypotheses during the course of the teaching episodes, in favor of adapting to the constraints they experience in interacting with the students. The researchers' intention is to remain aware of the students' contributions to the trajectory of teaching interactions and for the students to test the researcher hypotheses seriously. (p. 275)

> From the side of the researchers, a teaching experiment includes the generation and testing of hypotheses to see whether or not the experiential world that the students' language and actions comprise allows the current interpretation that the developing model proposes. (p. 298)

The fourth point presented by Steffe and Thompson (2000) is that evaluations of credibility of a teaching experiment, its analysis, or its model-building require the scrutiny of fellow researchers. The value of the involvement of additional researchers is made evident in these following statements:

> However, the teacher-researcher should expect to encounter students operating in unanticipated and apparently novel ways as well as their making unexpected mistakes

> and becoming unable to operate. In these cases, it is often helpful to be able to appeal to an observer of the teaching episode for an alternative interpretation of events. (p. 283)

The fifth point presented by Steffe and Thompson (2000) is that the teaching experiment is an effective exploratory and explanatory tool to use with both individual students and with groups of students. Due to the non-standardized nature of the teaching experiment, it would not be a mistake to consider that a credible teaching experiment can be conducted by simply having any number of students. In fact, as has been emphasized many times, the key element of a teaching experiment is the experience of the student or students. Now, it can be said that a teaching experiment with a group of students can offer the researcher more data, in the form of observations and discussion, for analysis. However, the number of participants is not as important as the need for having a cognizing agent. According to Steffe and Thompson,

> As indicated earlier, a primary goal of the teacher in a teaching experiment is to establish living models of students' mathematics, as exemplified in the protocols. The goal of establishing living models is sensible only when the idea of teaching is predicated on an understanding of human beings as self-organizing and self-regulating. If students were not self-regulating and self-organizing, a researcher would find that they would make no independent contributions. (p. 284)

The sixth point presented by Steffe and Thompson (2000) is that a chronological analysis of the data is necessary in order to produce a credible model of a student's conceptual construction of his or her mathematical reality. Conducting evaluations and re-evaluations of a student's progress over time, specifically their cognitive constructions, is an important element of the teaching experiment. In many cases, it is the review of previously collected data as a precursor to more recently collected data that facilitates the making of hypotheses, as well as the testing of hypotheses (Steffe, 1991). Such a benefit is made clear when the researcher reviews previously collected data and registers something that escaped his or her prior awareness.

The final point presented by Steffe and Thompson (2000) is that the purpose for replicating a teaching experiment is to advance the model which was originally developed. If the primary goals of a teaching experiment are to build models and establish theories of a student's cognitive constructions, then the secondary goal is to advance such models and theories. The mindset of the researcher who conducts teaching experiments should be focused on developing models that explain the learning and development of students by building upon previously established models (Steffe, 1991). From one perspective, for the researcher conducting teaching experiments, the ideas of replication and generalizability are naturally embedded within the teaching experiment methodology because one cannot advance a model or theory until one first re-asserts the model or theory. Consequently, as long as the researcher studies the teaching experiments of other researchers, and then decides to evaluate a similar cognitive construct, then replication and generalizability are consequences of the subsequent teaching experiment due to the mere fact that different but similarly functioning students are involved in the subsequent teaching experiment. A statement by Steffe and Thompson (2000) may make this point clearer:

> Further, if we can reorganize our previous ways of thinking in a new teaching experiment, that is, if we can learn, aspects of the old model become involved in new relations in the new model and, thus, become generalized conceptually. When it is possible to communicate with other researchers doing teaching experiments independently of us, this also serves as a vital confirmation of our way of thinking and perhaps as a site for each of us to construct a superseding model. The element of generalizability that is involved is strengthened if that other researcher launched his or

her teaching experiment for the purpose of constructing a superseding model of our current model of students' mathematics. (p. 300)

In summary, Steffe and Thompson (2000) provided more details and examples that extend beyond the necessary elements of a productive and viable teaching experiment, as outlined in Cobb and Steffe (1983). To their credit, Steffe and Thompson seemed to focus their discussion on the execution of a teaching experiment, and the building of a quality theory or model. They suggested that a rigorous and credible model results from a teaching experiment when the researcher understands that there is no pre-ordained path to engaging in academic discussions with students, and then developing and testing hypotheses. In addition, the hypotheses and resulting model can endure the scrutiny and replication of fellow theorists when the researcher considers the distinctions that exist between individual students, groups of students, and their cognitive changes over time.

Similar Research

There have been several researchers whose work has been similar to my specific research interests and whose efforts have resonated with me. Some of these researchers are Paul Cobb, Janet Bowers, Kay McClain, Gail Fitzsimons, Mitchell Nathan, Martha Alibali, Kate Masarik, Ana Stephens, Kenneth Koedinger, and Sunae Kim. Each of these researchers has conducted research within the context of the education environment that included CHAT or the teaching experiment or some combination of the two Below, I provide short summaries that contain points of divergence between our respective research interests.

Paul Cobb (1986) conducted a research study that archived the mathematics performance of a first-grade female student as she attempted to solve problems involving two basic arithmetic operations—addition and subtraction. I noticed that Cobb focused on his "observations" instead of the discourse and articulation of the participant. One difference in the approach that Cobb took is that there does not seem to be an effort to seek the overall experience of the participant from the participant's point of view. However, there is an effort made by Cobb to have the participant perform a "think-aloud" as she solved the math problem. Also, Cobb did not seem to seek to know about her construction of knowledge as a result of the various activities and problems. Cobb's interests seemed to be entirely centered on her ability to calculate the answer to a numerical arithmetic problem.

Paul Cobb (1995) conducted another research study that focused on the use of a counting manipulative by four second graders to perform addition and subtraction problems of numbers less than 100. In this study, there are several points where my work converges with Cobb's. First, it seems that his attention here was drawn to situation-specific occurrences where the specific context activates the student's thinking. Second, it would seem that his reference to image-independence would suggest evidence of conceptual awareness by the students where the conceptual understanding activates the students' knowing. Unfortunately, my anticipation to read of evidence of metacognition within his article, however, was premature. Although such phrases as "could reflect on," "monitor her activity," "objects of reflection," "unable to step back and monitor what she was doing," and "significant in making this reflection possible" are evident throughout his work, Cobb used them to identify what the student was not able to do as the student completed the assignments. Specifically, Cobb stated that the students had trouble distancing themselves from the specific situation at hand to understand the broader concept that was represented by the specific situation. This inability to see or register the broader concept led to the students' being unable to step back and monitor what she or he was doing. It was this inability that led me to my conclusion that the students were not explicitly taught to develop their metacognition and to establish its importance in the hierarchy of not only solving problems but also seeking to solve the problems most efficiently. Within this study, Cobb did focus on the knowledge construction of the student. In so doing, his pursuit centered on the students' ability to:

- understand the task that was given,
- understand the objects involved in the task,
- understand the actions represented in the task,
- select appropriate techniques and tools for use during the task,
- evaluate the correctness of a solution,
- explain his or her approach, and
- justify his or her approach.

Although these are all noble and noteworthy points of interest, each of these points are condition dependent, or more specifically, these points are object, action, and task dependent. A valuable addition to Cobb's work would have been to extend the study beyond the knowledge construction of the students for the given task and investigate their ability to extrapolate their knowledge construction to the concept or abstract level.

Bowers, Cobb, and McClain (1999) provided the findings of a case study that was based upon a teaching experiment conducted in a third-grade mathematics class. The focus of the teaching experiment was to facilitate the learning of place value. Here, Bowers and colleagues were interested in the mathematics understanding of the individual learner as a result of the learner's interaction within a community of other learners in the classroom. A point emphasized by Bowers and colleagues was that actions taken on abstract symbols have the same significance for the learner as performing the action on concrete objects. This point is consistent with semiotic bundling discussed previously (Arzarello, Paola, Robutti, & Sebena, 2009). Unlike Cobb (1995), there does not seem to be a focus on the particular activity or its constituent parts. According to my reading, the data from the student interviews that normally would follow the investigation-segment of a teaching experiment were not included. This absence of the student responses limits the ability of the reader to gain a broader understanding of the study and to gain insight into the student's experience.

In summary, the explicit nurturing of metacognition within a CHAT framework is what separates my work from the work of the researchers detailed here as they explicitly investigated cognition using the teaching experiment methodology. In addition, my particular focus is not limited to object and action. My work involves the students' ability to discern objects and their relationships. The elicited action is of secondary interest for me. It is the objects and their relationships that are paramount, and this awareness is what allows a student to see beyond the concrete nature of a problem and discern the abstract or conceptual nature. In identifying the concept that the task represents, the student can better activate his or her metacognition about the concept. Then, the specific conditions of the task become tools to be used within the student's metacognitive scheme. Developing such a proficiency in abstract thinking is the reason why teaching mathematics as a language system that is anchored by the cultivation of metacognition and semantic domains is the center of my work.

My Student–Teacher Conferences

I structured my class so that my students routinely worked together in groups, and I spent the majority of my time moving through the classroom observing and listening to the groups work as well as randomly observing and listening to the individual students within a group. In addition, I created a schedule by which I would have a student–teacher conference each day where I would pose my experience-gathering questions to the student. Within one class period, I was able to conduct four to five such conferences that allowed me to successfully confer with each of my students each week. This conferencing was possible because my classes were no larger than 20 students in any one semester. I repeatedly followed this format for 4 years.

What I learned is that my students were not familiar with the student–teacher conference aspect of a mathematics class. Their thinking did not seem to naturally accommodate a question that inquired why they chose a particular strategy or problem-solving path for a given problem. I had to make it clear that they could not respond to my questions with the more comfortable answer of "because that is what we did in the example." Consequently, the students needed time to coordinate their thoughts, language, and actions in such a way that they could effectively articulate themselves to me.

I also learned that each question seemed to have forced the student to think in a different manner or to reference a different portion of the problem-solving process. As this referencing became more and more prevalent with my students, I decided to facilitate their efforts to articulate themselves to me by demonstrating and modeling some of these various portions of the problem-solving process in the class. I discovered that as I modeled specific portions of the process to my students, they began to recognize a pattern or a sequence of acts that were more optimal for problem-solving. Such a recognition allowed my students to give greater attention to their own problem-solving actions and to discern the various stages of their own approaches. This pre-emptive discernment enhanced the student responses during the student–teacher conferences. Undoubtedly, having more and more practice with answering my questions during the conferences also led to the enhancement of the student responses.

A key finding from these student–teacher conferences was that my students performed better at a task when they were allowed to repeat the task, and this included the student–teacher conferences. So, if I did not assign a grade to the student–teacher conferences, and had the conference seem more like a casual discussion about a mathematics problem during which time the student could explain his or her decision-making and problem-solving approaches, then the student appeared to be more comfortable and the responses were more authentic. Greater comfort and more authentic responses led to an overwhelming improvement in student performance in my class, as well as on standardized mathematics assessments. In addition, a classroom discourse developed around the various portions of the problem-solving process, which allowed for a collective alignment of thought and language. This collective alignment was an unexpected development because the class chose its own labeling of the various portions of the problem-solving process and an appropriate sequence. I think that once such a collective alignment occurred, the students realized that problem-solving does not have to be a random set of actions, and instead could be an intentional and replicable course of action.

Transitioning to a Theory-Building Mindset

Due to the fact that I am also positioning my work as an effective theory-building effort for not only student mathematics learning but also for the investigation of student mathematics activity, I had to understand what exactly is the building of theory. Admittedly, I have already discussed the theory-building capacity of the teaching experiment. But now, my work has pushed me to move beyond the current boundaries of the traditional form of the teaching experiment as it has been conducted in the United States. Recall, the microscheme, the focus on the individual, has been the focus of the teaching experiment as it has been conducted in the United States. My work has pushed me to now incorporate the macroscheme, the focus on the learning environment and the activity. In order to do this with fidelity, I must first develop a firm understanding of the phenomenon called theory-building, and then develop a firm understanding of the process by which theory-building is achieved. Only after establishing such a bipolar foundation can I hope to re-engineer the traditional teaching experiment in a way that allows it to still produce rigorous and robust theory. To be clear, the effort here is to re-engineer the teaching experiment into a new ideology and new methodology that empowers those who employ it to develop theory not only for student mathematics learning but also theory for the mathematics learning environment and for the mathematics activity.

By the time in my journey when I arrived at my actual dissertation study, I had already completed an engineering design process, two pilot studies, and an informal teaching experiment. All of these allowed me to privilege my role as a professional mathematics educator. Now, with the actual dissertation study, I had to privilege my role as a neophyte theorist and researcher. Although this could have been uncomfortable for some doctoral students, I was quite comfortable. I think that my comfort level was heightened due to the fact that I had already spent 14 years, 7 years at the high school level and 7 years at the middle school level, targeting that one specific question: “The fundamentals of mathematics have not changed in thousands of years, so why is it so difficult to teach something that has not changed to individuals who do not know it?” After those 14 years, I had developed a level of confidence that convinced me that I should move my efforts beyond the current academic and pedagogic boundaries.

As I detailed in Chapter 3, I found CHAT to be quite descriptive of what occurred within my classroom; but it was missing something. I also explained that I conducted informal teaching experiments in my middle school classrooms for more than four years, but I realized that it was missing something as well. Upon further inspection of both CHAT and the teaching experiment, I determined that what I felt was missing in CHAT was actually the emphasis of the teaching experiment, and what I felt was missing in the teaching experiment was actually the emphasis of CHAT. In short, CHAT needed the student experience and the teaching experiment needed the activity dissection. Although I had not found much evidence of such a partnership in the literature, I knew the value of such a partnership from my own classroom experiences.

This clarity of vision and heightened sense of confidence placed me in unknown territory. My dissertation would no longer be tenable as an empirical study that led to new paradigms on student mathematics learning without including an analysis of the mathematics activity. My dissertation could only be tenable if I could determine how it could be transformed into a theory-building study that could generate paradigms for not only student mathematics learning and how student mathematics learning could be investigated but also generate new paradigms for the general phenomenon of mathematics activity. Such a transformation required that I first understood how experienced theory-builders achieved such a goal of producing new conceptual frameworks and methodologies. Fortunately, I discovered that there is an entire body of literature on exactly this topic. In what follows is the core of what I learned.

What is Theory?

In the effort to build theory, it is first necessary to grasp a meaning of the word theory. After a diligent effort to gain such an insight, I discovered that the academy has not yet agreed upon a singular definition for the word theory. Upon further analysis, I determined that there are various perspectives on the word theory. One perspective is that theory is a way to get something done in practice. Scholars who represent this mindset include Swanson, Lynham, Ruona, and Torraco (2000). A second perspective is that theory is a way to describe a complex situation or phenomenon. Scholars who represent this mindset include Dubin (1976) and Swanson, Lynham, Ruona, and Torraco (2000). A third perspective is that theory is a way to understand any type of situation or phenomenon. Scholars who represent this mindset include Gioia and Pitre (1990), Pool and Van de Ven (1989), Swanson (2001), and Torraco (2005). A fourth perspective is the intersection of the first and third perspectives. Specifically, such scholars suggest that from all of the varied perspectives, a comprehensive definition for theory could be that theory is a way to understand a situation or phenomenon for the purpose of getting something done. Scholars who represent this mindset include Simon (1967), Lynham (2002), Van de Ven and Johnson (2006), Swanson and Chermak (2013). A consistent theme throughout all of the perspectives is that a theory is a way to describe a complex situation or phenomenon and to get something done with the situation or phenomenon in practice. An actual definition that I believe each scholar would accept is offered by Lynham (2002):

> By virtue of its application nature, good theory is of value precisely because it fulfills one primary purpose. That purpose is to explain the meaning, nature, and challenges of a phenomenon, often experienced but unexplained in the world in which we live, so that we may use that knowledge and understanding to act in more informed and effective ways. (p. 222)

Although it would be possible for me to simply accept these varied perspectives and definitions of the word theory, I prefer to position myself within the commonality that I find exists across each of these variants. As such, the definition for theory on which this work is based is the following: *Theory* – a representation of underlying relationships as contextualized within a situation or phenomenon that not only allows for greater understanding of such a situation and phenomenon but also allows for influence over such a situation and phenomenon.

Importance of Theory-building

Now that I have established my own definition for the word theory, and located it within the various perspectives and definitions for theory as presented within the academy, it is now necessary for me to establish my own understanding of theory-building. In preparation for doing so, I investigated the literature again for any available guidance that I might gain from experienced scholars. What I found was consistent with the approach taken within the academy in embracing multiple definitions for the word theory, the academy has also embraced multiple perspectives on the concept of theory-building. Upon further analysis, I determined that there are at least four perspectives on the concept of theory-building.

One perspective is that theory-building is an ongoing process (e.g., Lynham, 2000; Steffe, 1991; Swanson, 1999). A second perspective is that theory-building is a scholarly process (e.g., Dubin, 1976; Lynham, 2000; Marsick 1990; Swanson, 2001; Van de Ven, 1989). A third perspective is that theory-building corrects misconceptions and errors (e.g., Swanson & Chermak, 2013). A fourth perspective is that theory-building enhances understanding (e.g., Gioia & Pitre, 1990; Lynham, 2002; Torraco, 2005). As with the available position made possible through the various perspectives and definitions for theory, it would also be possible for me to simply accept these various perspectives on the concept of theory-building. Again, however, I prefer to position myself within the commonality that I find exists across each of these variants. As such, the perspective on theory-building on which this work is based is the following: *Theory-building* – an iterative process through which refined understanding of a situation or phenomenon is sought and rigor and relevance are established.

Process for Quality Theory-building

Thus far, I have provided a definition for theory and a perspective on theory-building that orient this work. I must now bring attention to the perspectives held by the academy on the approach used to effectively build theory. As with the definition of theory and the significance of theory-building, several processes for theory-building can be found within the academy. I was able to identify four unique perspectives. One perspective is that a quality process is established and developed through its continued use (e.g., Cohen, 1991; Dubin, 1983; Swanson, 1999). The second perspective is that a quality process is achieved by following a given sequence of specific steps. A broad overview of such a sequence includes the following steps: (a) research design, (b) data collection, (c) data analysis, and (d) grounded theory articulation (Gioia, Corley, & Hamilton, 2012; Gioia & Pitre, 1990; Reynolds, 1971). The third perspective is that a quality process is established by aligning one's efforts with a prescribed rubric or template (e.g., Dubin, 1978; Lynham, 2002; Patterson, 1983; Snow, 1973; Torraco, 1997; Wilson, 1998). The fourth perspective is that a quality process is established through rigor, where following a prescribed sequence of steps or adhering to some prescribed guidelines may indeed prove necessary, but it is not at all sufficient. A remark by Lynham (2002) supports the position that mechanically following a sequence of steps or a guideline does not in and of itself establish a rigorous process:

> It is further evident that theory-building research methods are of a duo deductive-inductive nature. Although some theory-building research methods may begin with deduction, at some point they become informed by induction. With other theory-building methods, the relationships between deduction and induction may be the other way around. What is important in theory-building inquiry, whether one starts with theory and then moves to research and/or application, or vice versa is that the choice of specific

> theory-building research methods should be based on the nature of the phenomenon, issue, or problem that is the focus of the theory-building endeavor. And not by the theorist's preferred specific method of theory-building. (p. 237)

Although each of these perspectives is credible, I am of the mindset that a quality process for building theory requires the aspects that each of these four perspectives can offer. I posit that a rigorous theory-building protocol contains evidence of practical application, adherence to a prescribed rubric or template, and inclusion of a prescribed sequence of steps. Based upon my position, I chose the five-element protocol as outlined in Lynham (2002): "The General Method of Theory-Building Research in Applied Disciplines." Below I list the five elements and describe each element in the sections that follow:

1. Conceptual Development
2. Operationalization
3. Application
4. Confirmation or Disconfirmation
5. Continuous Refinement and Development

Theory-building of Student Mathematics Learning

Up to this point, I have provided my definition for theory, I have provided my perspective on theory-building, and I have presented the rigorous theory-building protocol that guided my theory-building process. Now, it is important to explain why an additional effort in building a theory for scientific investigation of mathematics activity is warranted.

As I had done for the prior three topic areas—theory, theory-building, and process—I also conducted a search of the literature in order to develop an informed stance as to why an additional effort in building a theory for scientific investigation of mathematics activity would be warranted. My search of the literature revealed five perspectives that have been used to address this point. There are three perspectives that provide insight on the benefits of continuous theory-building in a familiar field, in general. One perspective is that building new theories challenges false theories (Swanson & Chermak, 2013). Swanson and Chermak posit that building theory in fields of practice and application can lead to not only the refutation of false theories but also can lead to positive impact in the forms of high integrity and effectiveness by the people and the systems involved. A second perspective is that building additional theory in a familiar area can extend prior knowledge (Gioia, Corley, & Hamilton, 2012; Torraco, 2005). A third perspective is that when investigating a phenomenon that consists of both social and learning elements, then a social-constructionist perspective can be insightful. Social-constructionism is an epistemology that orients meaning-making and knowledge construction around social interaction and individualized experience (Bauersfeld, 1988; Lourenco, 2012; Steffe, 1991; Steffe & Thompson, 2000; Torraco, 2005; Tryphon & Voneche, 2013; von Foerster, 1984; von Glasersfeld, 1990).

In addition, there are two perspectives that refer to ongoing theory-building in mathematics education, specifically. The first perspective that buttresses continued theoretical work within mathematics education posits that theory is more authentic and more comprehensive when it privileges the student’s perspective and experience (Hackenberg, 2010; Piaget, 1977/2001; Steffe, 1991; Cobb & Steffe, 1983; Steffe & Wiegel, 1992; Thompson, 1979; von Glasersfeld, 1995). The final perspective that resonated with me was that it is imperative that theory within mathematics education acknowledges and includes the teacher–student contrast. The similarities, differences, and commonalities that exist between teacher and student within a mathematics learning environment and activity can enhance the clarity of a robust and rigorous theory (McKay, 1969; Cobb & Steffe, 1983; Steffe & Wiegel, 1992; von Glasersfeld, 1978).

Although each of these five perspectives offer substantial contributions to theory-building within mathematics education, generally, and for scientific investigations of mathematics activity, specifically, I find that one particular perspective is most aligned with my theoretical frame and the purpose of my work. This one particular perspective is that when investigating a phenomenon that consists of both social and learning elements, then a social-constructionist perspective, in my case, specifically, a radical-*social* constructivist perspective is necessary. As I detailed in Chapter 3, radical constructivism is the epistemological platform on which all of my work rests. It is my opinion that basing my work on the radical constructivist epistemological platform grants me the benefits of each of the other detailed perspectives in this section. I contend that building theory of student mathematics learning and of the scientific investigation of mathematics activity from a radical constructivist lens does in fact lead to theory that (a) challenges false theories, (b) achieves high integrity and effectiveness, (c) extends prior knowledge, (d) privileges the student perspective and student experience, and (e) acknowledges the teacher–student dichotomy. I close this section with the following quote from Steffe (1991):

> As mathematics educators, we have a choice between using mathematics of children or conventional school mathematics as the basis on which to teach mathematics. Choosing the former is a fundamental requirement of constructivism for mathematics education. (p. 181)

Necessary Characteristics of Theory-building Methodology

When conducting academic research, for both the researcher and the work-product to endure the scrutiny that results from the peer-review process, a key requirement is that the research methodology must align with the theoretical framework (Denzin & Lincoln, 2000). In my case, I have taken on another element of burden: my choice of research methodology must also possess the characteristics of theory-building methodology. To ensure that I met this burden, I had to first learn what would be those necessary characteristics. Learning these attributes required that I again study the literature. Analyzing the literature led me to the following conclusion: there are six fundamental characteristics and one aspect that a research methodology must possess for it to also be an effective theory-building methodology. I have characterized these six characteristics into two groups: multi-dimensional and organic. Multi-dimensional refers to the ability to integrate more than one perspective. Organic refers to the ability to be responsive to real-time contexts.

Before I present these two groups of characteristics, I begin by offering what I feel is the most critical aspect—the purpose of the theory-building methodology is to actually produce theory. If the fundamental premise of the methodology is not for the researcher to generate a theory, but instead, is for the researcher to collect and present data, then the methodology has been crafted to only allow the researcher to be descriptive, at the risk of being overly etic. However, for a methodology that has been crafted to guide the researcher to generate theory, then the researcher transcends the descriptive mindset and enters into an inferential mindset (Saldaña, 2016). It is within the inferential mindset where the researcher can embrace not only the etic and the emic but also their intersectionalities. It is also within the inferential mindset that the researcher can think beyond the given context and extend toward the more general and abstract contexts (Saldaña, 2016; Thorsten, 2017). Taken together, theory-building methodologies that are not oriented to generate theory as the final work-product are at an increased risk of succumbing to the intense scrutiny of not only other researchers and scholars but also, most specifically, to the intense scrutiny of fellow theory-builders (Cobb & Steffe, 1983).

Now that I have presented what I have found to be the most important aspect, I present the three characteristics that I have placed within the multi-dimensional group. One characteristic is that theory-building methodology must blend theory with practice. This blending means that the theorist must also function as a practitioner in the area within which the theory is to be applied. A benefit of this particular characteristic is that the methodology identifies what to do and why to do it, so that other researchers may understand the details of the study and so that future studies may serve as replicates of prior studies (Corley and Gioia, 2011; Dreyfus and Dreyfus, 1998; Dubin, 1978; Lynham, 2002; Cobb & Steffe, 1983; Swanson & Chermak, 2013; Van de Ven & Johnson, 2006; Weick, 1995).

The second characteristic is that a theory-building methodology must use a variety of tools and techniques. Such a diverse set of methods increases the opportunities for data collection and increases the likelihood for data analysis triangulation (Levi-Strauss, 1966; Thorsten, 2017; Torraco, 2005). In fact, Levi-Strauss (1966) is given credit for coining the phrase bricolage as a reference to this particular mindset. The third characteristic is that a theory-building methodology be able to investigate both the individual dynamics and any group dynamics. As I discussed within the chapter which detailed my theoretical framework (Chapter 3), the social and cultural aspects of an experience may change depending upon if the person is experiencing a situation or activity alone or experiencing them within a group (Vygotsky, 1930/1978; von Glasersfeld, 1984). A methodology that is used to build theory regarding a social context must be effective in allowing the researcher to investigate both scenarios (Thompson, 1979).

Now that I have presented three of the six characteristics, I present the remaining three characteristics that I have placed within the organic group. The first characteristic is that a theory-building methodology must not be fixed upon any a priori paradigm or expectations. Such openness by the researcher allows for the data to reveal the underlying relationships, instead of the researcher attempting to force the data to fit any number of preconceived notions (Harrison, 1997; Mitroff & Linstone, 1993; Van de Ven & Johnson, 2006). Harrison (1997) offered the term intellectual arbitrage to refer to this ability of the researcher to triangulate on problems through the evolution of theses, antitheses, and varying perspectives. The second characteristic is that a theory-building methodology must privilege the experiences of the participants. I interpret this characteristic to mean that the researcher must resist any urge to have the etic perspective dominate the work. When the endeavor is to build theory within a social context, the experiences of the participants, the emic perspective, must be allowed to lead the trajectory of the work (Confrey & Lachance, 2000; Gioia & Pitre, 1990; Hackenberg, 2010; MacKay, 1966 and 1969; Cobb & Steffe, 1983; Thompson, 1979 and 2010; von Glasersfeld, 1974 and 1995). In fact, I argue that the best theory-building would occur when and where the etic and the emic perspectives coincide with one another. The last characteristic is that a theory-building methodology must not be standardized. This characteristic means that the methodology should serve as a guide and not as some prescriptive sequenced ritual. I interpret this guiding to mean that in order to produce a viable and practical theory, the methodology must be organic and dynamic to allow for the situations and the data to lead the theory-building process (Gioia, Corley, & Hamilton, 2012).

Each of these six characteristics resonates well with my own effort to build a theory regarding students' mathematics learning and mathematics activity. However, if taken from a broader vantage point, each of the six characteristics can well be captured within the union of the strategy coined bricolage by Levi-Strauss (1966) and the strategy labeled intellectual arbitrage by

Harrison (1997). Each of these two strategies demands a scientific approach that is both organic and dynamic to the extent that the resulting methodology embraces the idea of producing a cogent theme from a variety of and sometimes even divergent sets of elements, which is what the six characteristics seem to represent.

The General Method of Theory-building

Now that effort has been made to provide details for theory, theory-building, and a theory-building methodology, a similar effort will be made to provide details for the well-respected theory-building approach as crafted by Lynham (2002). The General Method of Theory-Building Research in Applied Discipline is an approach that has been well received and accepted as a viable approach to building rigorous and robust theory. As previously noted, this approach contains five elements: conceptual development, operationalization, application, confirmation or disconfirmation, and continuous refinement and development. Details of each element are outlined below.

The conceptual development element serves as a first-hand and informal experience with the phenomenon under study. The purpose is to make certain that the researcher has some preliminary awareness of the event or situation, and its key attributes. According to Lynham (2002):

> The process of conceptual development varies according to the theory-building method employed by the theorist. However, at a minimum this process will include the development of the key elements of the theory, an initial explanation of their interdependence, and the general limitations and conditions under which the theoretical framework can be expected to operate. (p. 232)

The result of the conceptual development stage is a preliminary model or metaphor that is based upon the theorist's constructed knowledge and experience with the phenomenon under study (Lynham, 2002).

The operationalization element serves to have the theorist generate hypotheses from the current form of the theory, and then to test those hypotheses. The manner by which this is done is for the theorist to transform the theory from its written, model, or metaphoric form into an observable or analytical form so that this transformed form can be applied and empirically evaluated in the real-world setting where the phenomenon, issue, or problem exists (Lynham, 2002). According to Lynham:

> A primary output of the theorizing component of theory-building research in applied disciplines is therefore an *operationalized theoretical framework* (italics in original), that is, an informed theoretical framework that has been converted into components or elements that can be further inquired into and confirmed through rigorous research and relevant application. (p. 233)

The application element serves to focus the efforts of the theorist on establishing the relevance and effectiveness of the developing theory. Lynham (2002) posits that it is not sufficient to confirm that the theory is viable in the contextual world in which it exists—the operationalization of the theory. The theorist must still commit the theory to genuine practice situations so that its findings and conclusions can be evaluated and scrutinized by fellow theorists and practitioners. Insight on this matter from Lynham is as follows:

> An important outcome of this application phase of theory building is therefore that it enables the theorist to use the experience and learning from the real-world application of the theory to further inform, develop, and refine the theory. It is in the application of a

> theory that practice gets to judge and inform the usefulness and relevance of the theory for improved action and problem solving. And it is through this application that the practical world becomes an essential source of knowledge and experience for ongoing development of applied theory. (p. 233)

For me, a key aspect of this element that distinguishes it from the operationalization element is that the theory and its findings are evaluated and scrutinized by fellow theorists, scholars, and practitioners across a litany of settings and conditions. Indeed, the true robustness of the theory is investigated.

The next element to discuss is the confirmation or disconfirmation aspect of the approach. This particular element allows for a determination of the viability and credibility of the theory. To make such a determination, the theorist must plan, design, and implement a rigorous investigative protocol of the theory and then evaluate the results. The goal is for the theorist to purposefully seek out deficiencies in his or her theory while it is still in the preliminary stages of development. According to Lynham (2002), “When adequately addressed, this aspect results in a confirmed and trustworthy theory that can then be used with some confidence to inform better action and practice” (p. 233).

The remaining element of this approach is for the continuous refinement and development of the theory. This particular element demands that the theorist accept the lack of absolute certainty as it relates to the omniscience of the theory. The theorist must present a constant effort at evolving the theory and insuring its utility in practice. Such an effort helps to establish, investigate, and maintain the relevance of the theory throughout time. According to Lynham (2002):

> The intentional outcome of this phase is thus to ensure that the theory is kept current and relevant and that it continues to work and have utility in the practical world. It also ensures that when the theory is no longer useful, or is found to be ‘false’, that it is shown to be as such and adapted or discarded accordingly. (p. 234)

In conclusion, Lynham (2002) has taken much effort to synthesize the theory-building process down to these fundamental five elements regardless of the framework of the theorist—qualitative or quantitative. In addition, she has concentrated these five elements down further to a constituent pair of components, and their intersectionality—the theorizing component and the practice component. The theorizing component consists of the conceptual development element. The practice component consists of the confirmation or disconfirmation element, and the application element. Lynham also acknowledges the importance of the intersectionality of these two components. She states that the operationalization element and the continuous refinement and development element are located within this intersectionality. What is also important to note from Lynham, and probably what has allowed her approach to be so well accepted, is that she embraced both the qualitative framework as well as the quantitative framework and made sure that both frameworks could produce rigorous, robust, and relevant theory when using her approach.

Re-engineering The Teaching Experiment

Based on my decision to fortify my work with the radical constructivist epistemology, as I detailed in Chapter 3, I needed a methodology that not only was aligned with the ideology of radical constructivism but also has a history of effective investigation of student mathematics learning through mathematics activity. The methodology that meets both of these requirements is the teaching experiment (Steffe, 1984). But because I am also positioning my work as an effective theory-building effort of the scientific investigation of student mathematics activity, I had to also ensure that the chosen methodology contained the characteristics previously discussed: blends theory with practice, sustains scientific review, uses various tools and techniques, permits multiple perspectives, privileges participants' experiences, refutes standardization, investigates individual and group dynamics, and commits to formulating a theoretical model. The teaching experiment is inclusive of these characteristics as well.

New Insights

Over these last several sections, I have attempted to provide my deeper understanding of the decades-long work of Les Steffe and his colleagues in the teaching experiment. Steffe has been instrumental in establishing the necessary foundation of the teaching experiment here in the United States. Next, I provide a brief review of two other researchers, Hackenberg (2005, 2010) and the research team of Engelhardt, Corpuz, Ozimek, and Robello (2004), who had a direct impact on my work and whose foci were only made possible because of the foundational work of Steffe and his many colleagues. It was these foci that served as guideposts for my own re-engineering of the teaching experiment. I was confident that a re-engineered teaching experiment would allow me to investigate not only student mathematics learning but to also investigate the mathematics learning environment and the mathematics activity, specifically their contexts, their constituent parts, and the role creation that they impart onto the student.

Hackenberg. In her adaptation of the teaching experiment, Hackenberg (2005, 2010) focused her attention on the effect of care toward students in the teaching experiment. For me and my study, this inclusion of care toward the students in the teaching experiment elevates the teaching experiment from a potentially mechanical teacher–student association that is focused on mathematics learning to an intentional sentimental community relationship that is focused on mentoring the student in cultivating a productive disposition toward mathematics (Kilpatrick, Swafford, & Findell, 2001). Hackenberg calls this form of caring mathematical caring relationships (MCRs). To better appreciate her perspective on how MCRs elevate the teaching experiment to new heights, I offer Hackenberg's (2010) own words:

> In my adaptation of this methodology, I posed tasks aimed not only at constructing a working model of the students' mental activity, but also at generating a model of their energetic responses to our mathematical interaction. In other words, I was invested not exclusively in constructing the students as thinkers, but also in constructing them as affective beings. This aim meant that I was sensitive to their energy levels and might curtail our activity if depletion seemed too great, or prolong our activity in some way (often by extending problems or posing what I thought were greater challenges) if the students appeared to be experiencing a balance between depletion and enhancement. (p. 243)

In her work, Hackenberg (2010) was able to assert the existence of a correlation with not only MCRs and the cultivation of a productive disposition by the student but also with a positive self-image:

> Establishing MCRs appears to influence teachers' personal teaching efficacy and students' construction of mathematical self-concepts. Linking what one knows to one's construction of oneself is not uncommon. This link is of considerable importance in mathematics education, because persistent feelings of inferiority in mathematical activity can lead to judgements about one's ability to do mathematics, which may in turn engender broader, and often relatively permanent, judgements about whether one is, or is not, mathematical. (p. 269)

This effort by Hackenberg to infuse the teaching experiment with MCRs emphasizes the humane aspects of the teaching experiment and shows how the teaching experiment aligns quite well with all of the five pillars of mathematics proficiency as outlined in *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001).

Engelhardt and colleagues. The research team Engelhardt, Corpuz, Ozimek, and Robello (2004) focused their attention on the importance of student activity in the teaching experiment. Specifically, Engelhardt and colleagues brought attention to the constituent parts of the teaching episode. Recall my earlier reference to the teaching episode as the social and mathematical interaction between the researcher and the students whereby meaning-making and knowledge-construction experiences are influenced (Steffe, 1991). I offer the following two quotes to represent the perspective of Engelhardt and colleagues:

> The teaching experiment embraces both the learning cycle and Socratic teaching in its tenets. The structure of the interview resembles a Socratic dialog. Students are repeatedly asked probing questions to try and elicit as much of their reasoning and thought process as possible. The questions tend to be focused around the activities or tasks that the students are asked to think about and explain. (p.2)

> The teaching experiment is also related to the learning cycle. A typical learning cycle consists of three stages, an exploration phase, conceptual introduction phase and concept application phase. In the exploration phase, students explore the concept under investigation through hands-on activities. In the conceptual introduction phase, an explanation of the observations that were performed in the exploratory phase is given a

name and further refined. In the concept application phase, students apply the concept that they explored and later named to new situations. (p. 2)

The synthesis by Engelhardt and colleagues of the teaching episode provided me with the insight to no longer consider the teaching episode as an informal exchange of mathematics discourse, but rather as a set of coordinated social-cultural-historical-linguistic activities. This evolution in my understanding allowed me to explicitly make a connection between the teaching experiment and CHAT. Recall details given in Chapter 3 about CHAT and Vygotsky's two major motivations: (a) transitioning activities from informal to formal so that greater gains in learning could occur; and (b) revealing the social, cultural, historical, and linguistic attributes of these formal activities (Vygotsky, 1930/1978). Remaining committed to the mandate that the teaching experiment is not to be a standardized protocol (Steffe & Thompson, 2000), I now had a non-standardized set of activities, all housed within the domain of the teaching episode, that could greatly assist with the analyses of student learning and activity and the building of a more rigorous and robust theory.

In summary, both Hackenberg (2005, 2010) and the research team of Engelhardt, Corpuz, Ozimek, and Robello (2004) advanced my thinking as to the insights that the teaching experiment could offer to better understand student learning and activity and the building of theory that aims to explicate student learning. Hackenberg brought to my attention aspects of the teaching experiment, specifically the teaching episode, which promote a correlation between MCRs, the cultivation of a productive disposition by the student, and the development of a positive self-image by the student. In short, the benefit of an apprenticeship. Engelhardt and colleagues brought to my attention a non-standardized set of activities, all housed within the domain of the teaching episode, which allowed me to explicitly make a connection between the teaching experiment and CHAT. In short, the dis-aggregation of a mathematics activity.

Teaching Experiment v. General Method of Theory-building

Now that I have provided a coherent synopsis of the teaching experiment, which is the methodology for my work, and I have provided a coherent synopsis of The General Method of Theory-Building Research in Applied Disciplines, which is a well-respected theory-building approach, I now make an effort to highlight the intersections of these two approaches to address any outstanding debate that the teaching experiment is the appropriate methodological foundation for my work. In short, the teaching experiment is the best core methodology for extending mathematics education theory in the convergent point of theory, practice, activity, and experience because it coincides with the rigorous demands of the primary evaluation system, The General Method of Theory-Building Research in Applied Disciplines.

Based upon the necessary characteristics of the teaching experiment as brought forth by Steffe and his many colleagues, there is indeed an overlap between the teaching experiment and The General Method of Theory-Building Research in Applied Disciplines (Lynham, 2002). As detailed previously, Lynham (2002) gave great attention to both the theorizing component of her theory-building approach and to the practice component. In addition, she gave attention to the overlap between these two components. In the theorizing component, one can find the conceptual development phase as detailed by Lynham and four of the characteristics of the teaching experiment: (a) the researcher must serve as teacher, (b) the constructivist view of teaching must be used, (c) qualitative data must be emphasized, and (d) a long-term interaction between the researcher and students must be designed. In the practicing component, one can find the application and the confirmation or disconfirmation phases as detailed by Lynham and five of the characteristics of the teaching experiment: (a) the trajectory of the study is rhizomatic, (b) the researcher must perform exploratory teaching, (c) the researcher must recognize individual learning and collective learning, (d) the researcher must conduct a historical analysis of the qualitative data, and (e) the teaching experiment is not standardized. Finally, in the intersection of the theorizing and practice components, one can find the operationalization and the continuous refinement and development phases as detailed in Lynham and the four remaining characteristics of the teaching experiment: (a) research hypotheses should be created and tested according to the students language and actions, (b) critique of suppositions requires discourse with fellow researchers, (c) theory-building of mathematics learning is the goal, and (d) replication should lead to theory-evolution, not theory-proving.

To this point, I have included important nuances and details about the teaching experiment and its theory-building capacity as set forth by Steffe (e.g., Steffe, 1991) and corroborated by Lynham (2002). I have also included some of the innovations as suggested by Hackenberg (2005, 2010) and the research team of Engelhardt, Corpuz, Ozimek, and Robello (2004). My effort, in this study, was to unite the microscheme with the macroscheme of the teaching experiment in order to extend the teaching experiment into new spaces of inquiry within mathematics education. Recall that a macroscheme is the type of teaching experiment when the foci of attention are the activity and the learning environment, and the microscheme is the type of teaching experiment when the focus of attention is the cognitive development of the student. These two centers of the teaching experiment address exactly the areas of interest of my research study—the development of a “real” learning environment context for seeking to understand how students perceive their experiences and construct mathematics knowledge from their experiences.

Summative Remarks

The goal of this chapter was to explain the mindset necessary to expand a tool, the teaching experiment, from its current use into a much broader use without diminishing its rigor or robustness. To achieve such a goal, it is was necessary for me to first understand the phenomenon of theory-building and second, to understand the process of theory-building so that I could ensure that my re-engineering of the teaching experiment reflected a deep understanding of the two. I began by explaining what is theory. I continued by offering insight into the process of theory-building. Next, I discussed why I felt that building a theory of student's mathematics learning and activity is important. Then, I provided the characteristics of a research methodology that allow for an optimized theory-building process. I included discussion on two areas that I felt warrant my extensions of the teaching experiment: apprenticeship and activity disaggregation. I also provided the tenets in the form of a rubric, the General Method of Theory-building, which would allow me to focus my efforts and successfully extend the teaching experiment into these two areas and ensure that its rigor, robustness, and credibility would not be diminished.

In conclusion, the trajectory of my social science experience started in the high school classroom and then took me into the middle school classroom. I learned much in both places, and I experienced the difference in mathematics learning at the two distinct levels. Fortunately, I was able to detect a commonality in mathematics growth at both levels—semiotics. Nevertheless, I was not yet able to articulate a competent response. Although determining this commonality was a significant finding, I do not believe that it was the most poignant point of this chapter. I suggest that the most potent point of this chapter is that the experiences detailed in this chapter serve as the map that shows my own evolution from a teacher-researcher into a researcher-teacher. It was through the researcher-teacher role that I gained the awareness to understand that I needed to pursue a theory-building dissertation and developed the confidence to develop its design.

CHAPTER 5

THE BIRTH OF A "RESEARCHER–TEACHER"

In this chapter, I provide my thoughts and processes for my unique re-engineering of the teaching experiment. I begin by offering my perspective on the idea of generalizability; given that, the concept generalizability has such a large impact on theory-building. Next, I bring attention to a phenomenon that I call activity *dis*-aggregation, the dissection of an activity into its constituent parts. Activity disaggregation was a finding from my second pilot study; it is an integral part of my dissertation work. A comprehensive understanding of activity disaggregation requires a clear understanding of the semiosphere; therefore, for reference, I provide a brief review of the semiosphere and its analysis. My re-engineered teaching experiment compelled me to contend with certain ideological tensions; therefore, I provide an explanation of how I resolved the tensions between radical constructivism and sociocultural theory so to extend the teaching experiment into broader investigative areas. I then offer details into my actual adjustments to the teaching experiment. My first adjustment was the creation of a set of data collection templates. The second adjustment was the design of a new procedure for the teaching experiment that brings attention to both the actions of the student and the actions of me, the researcher. Next, I transition the discussion to the data analysis portion of the re-engineered teaching experiment by first presenting my perspective on coding and then detailing the creation of a set of data analysis templates. I conclude this chapter with a summary of the entire process.

My Thoughts on Generalizability

Throughout this investigation I aim to make sense of two abstract phenomena—a cognitive environment and a cognitive experience, both within a mathematical context (Steffe & Thompson, 2000). I, however, do not aim to make predictions about a larger population based upon the actions of individuals. Specifically, I am not trying to predict how a population of people will behave in a certain environment based upon the behavior of a collection of individuals. Nor am I trying to predict the experiences that will be had by a population of people based upon the experiences of a collection of individuals. I do, however, aim to discover the characteristics of a mathematics learning environment that might stimulate and cultivate the optimal learning of a person. I am also trying to discover the characteristic traits of mathematics learning experiences that might optimize the learning of a person. Such characteristics then, I believe, can be crafted into a model that can be used to shed light on the construction of mathematical knowledge within a specific population of people; and based upon those mathematical constructions shed light on the language and actions of a specific population of people.

In the scientific community, there seem to be two agreed upon scenarios for making predictions and building theory. The scenario for making predictions is to randomly select a representative subset of the population of interest and then conduct rigorous statistically sound experiments on the representative subset (Steffe & Thompson, 2000). This is a quantitative exercise (Teddlie & Tashakkori, 2009). In contrast, the scenario for theory-building is to examine numerous cases or occurrences, as similar and as dissimilar as possible, of the phenomenon of interest and seek the intersectionalities (Steffe & Thompson, 2000). A weaving of these intersectionalities may lead the researcher to the beginnings of a theory. It is important to note the distinctions between these two scenarios. In the first scenario, the prediction of population behavior is based upon a statistically limited number of investigations of the performance or behavior of randomly selected representative subsets of the population of interest and then project the findings onto the population of interest (Teddlie & Tashakkori, 2009). In the second scenario, the building of theory is based upon the ongoing investigation of the existence of the phenomenon of interest in as many different representative forms of the phenomenon (not people) as possible and then refine the theory according to new insights (Steffe & Thompson, 2000). In my case, the relevance is that my theory-building effort makes an initial declaration about the identity of a phenomenon of interest that requires the continuous examination of students in order to seek more of the nuances and intricacies of said phenomenon (Steffe & Thompson, 2000). A statistical generalization, however, makes declarations about a group of people and then projects these declarations onto the non-examined population of interest (Teddlie & Tashakkori, 2009). A statistical generalization is not at all the effort of this dissertation project.

Although both efforts require the same term *generalizability*, one effort uses generalizability to study an assortment of ever-changing people who have had an experience with the phenomenon of interest so that the researcher can seek and accumulate identifying characteristics of the phenomenon. I view this as *a multitude of experiential instances or as a multitude of experiential case studies*. The other effort uses generalizability to study a specific representative set of people who have had an experience with the phenomenon of interest so that the researcher can seek and project not only the experience onto the population of interest but also project the resulting impact of the phenomenon onto the population of interest. I view this as *a representative sample*. The goals being sought are different. The generation of a viable theory studies people to better understand the identity and intricacies of a phenomenon, while a prediction of the behavior of a population of people studies individuals to better predict the behavior of a population of people. In addition, the term generalizable has different meanings for the two scenarios. When seeking to accumulate identifying characteristics, generalizable requires the researcher to examine as many different representative forms of the phenomenon as possible; however, when seeking to predict and project the behavior of a larger population of people, generalizable requires the researcher to examine a statistically reduced number of representative randomly selected subsets of the population of interest. More specifically, any representation of the phenomenon is valuable when trying to amass identifying characteristics because each representation could actually reveal a different attribute of the phenomenon. For the other scenario, however, for a randomly selected subset to be representative, the subset must contain all of the attributes of the whole population; it must be a scaled down version of the whole population, a dilation, if you will. So, succinctly, there are two distinctions in the two scenarios: (a) the difference in the goals of generalizable, and (b) the difference in the meaning of generalizable.

Indubitably, a benefit of a long-standing viable theory is its ability to predict and project. So, I endeavor that my re-engineering of the teaching experiment, in accordance with a multitude of experiential instances, might begin the long enduring process of seeking a living model for optimal mathematical experiences and the requisite mathematical environment that can evolve into a substantive theory which can then endure the inspection and scrutiny of other researchers and become a long-standing and viable theory. Then, and only then, could anyone think to use such an asset-emphasizing theory as a source for predictions of and projections from a representative sample, and then emphatically serve as a replacement for the deficit-minded statistical sophistry that is currently in use to make deceptive predictions and projections.

Activity ***Dis*-aggregation**

As I previously mentioned, an unexpected result of the student–teacher conferences was that the students developed a particular discourse, which developed around the various portions of the problem-solving process. This discourse allowed for a collective alignment of thought and language through the labeling of the various portions of the problem-solving process and the appropriate sequence. Once such a collective alignment occurred, the students realized that problem-solving could be an intentional and duplicatable course of action. It was this dissection or disaggregation of mathematics activity into an intentional course of action that I wanted to investigate using the teaching experiment. For ease of explanation, I represent the portions of the problem-solving process, as determined by my students, with the following labels: (a) researcher, (b) analyzer, (c) designer, (d) executor, and (e) critic. It is important to note that each portion is assigned a nominative label instead of an action designation. I did this intentionally because a nominative label best describes the vocational depth and quality that my students assigned to each portion instead of a lack of depth and quality that can often be assigned to actions which students consider random and un-important.

My interest in using the teaching experiment to investigate this disaggregation of a mathematics activity pushed the teaching experiment beyond its traditional borders. For example, instead of using the teaching experiment to investigate what and how a student learned from the overall mathematics activity, now I use the teaching experiment to investigate how a student performed during each of the constituent portions of the problem-solving process and what did the student learn during each of these portions. In short, my students helped me to see that the aggregate idea of mathematics learning could be disaggregated into smaller mathematics learning segments precisely because they found it easier to understand and complete a mathematics activity by dissecting it into smaller actionable items. Therefore, I hypothesized that by using CHAT with the teaching experiment, I could produce a richer and more robust understanding of my students' overall mathematics learning if I first examined their thoughts, language, and actions at each of the constituent vocational segments of the mathematics activity.

In the preparation and reconfiguration of the teaching experiment for this study, I had to describe each of the vocational segments. I used my findings and reflections from my second pilot study and produced the following descriptions. The researcher was the role that a student enacted when gathering information about a particular topic, concept, or vocabulary word. The analyzer was the role that a student enacted when attempting to interpret or decode the semiosphere of a given mathematics problem or situation. The designer was the role that a student enacted when brainstorming and developing a plan for solving a given mathematics problem or situation. The executor was the role that a student enacted when executing the developed plan in hopes of arriving at a solution. The critic was the role that a student enacted when examining and evaluating the overall problem-solving process, as well as evaluating the accuracy of the resulting solution.

Of these five vocational segments, the analyzer required the most investigative effort because, as discussed in Chapter 3, the semiosphere has great complexity and its interpretation is unique to the particular analyzer. Therefore, to embrace this complexity and the uniqueness of the interpretation, the students needed to understand the importance of stratifying the semiosphere into its syntactic and semantic elements. Having the students target the structure, vocabulary, and explicit connections of the given mathematics problem allowed them to subsequently realize the implicit relationships that were helpful in developing a plan for solving the given mathematics problem. The phenomenon of the semiosphere proved to be more important to this project than I originally realized. Once I viewed my students' mathematics learning as a composite of smaller mathematics learning segments, then I realized that the semiosphere was a vital component within each of these smaller mathematics learning segments. Consequently, having the students target the structure, vocabulary, and explicit connections within each of the smaller learning segments facilitated their ability to complete each vocational task and transition to another vocational role. It became clear to me that not only would I have to re-engineer the teaching experiment to accommodate the disaggregated learning segments, but I would also have to re-engineer the teaching experiment to accommodate syntactic analysis and semantic analysis of the semiosphere (i.e., the specific tasks of the analyzer).

Before I discuss how I implemented the teaching experiment and its four partitions of the teaching episode in this dissertation study, I briefly review the details of the semiosphere. As I discussed in Chapter 3, the semiosphere is composed of a semantic domain and discursive practices (Halliday, 1978). It has concrete elements, abstract elements, cultural elements, and contextual elements. All of these elements influence how the semiosphere is perceived. It is the configuration of cultural and semantic resources as determined by the situation that the individual perceives within the semiosphere. Specifically, according to Halliday, register is the meaning potential that is accessible to the individual in a given social context. In sum, the mediating tools as studied by Vygotsky (1930/1978) are a sub-set of the semiosphere as studied by Halliday (1978), and both embraced the significance of signs, symbols, and language on the development of cognition.

Due to the vastness of the meaning potential of the semiosphere, it is important to distinguish the role of the researcher from the role of the teacher, when working with a learner. First, it is the semiosphere that facilitates the teacher’s effort in cultivating students who are proficient in all the requisite mathematical language forms. As previously mentioned, the classroom can be viewed as a small sample of the world, at least in terms of its semiotic potential. Therefore, meaning potential of the classroom is without bound. Second, despite the multitude of convergent and divergent meaning possibilities represented by each student, the teacher must attempt to coordinate these meaning possibilities so that they are relevant to the topic at hand. This coordination by the teacher is why the concept of semantic domain is so vitally important. An awareness of the semantic domain allows for the refinement of the meaning possibilities. Third, the teacher must serve as a guide and assist the students in aligning and fitting their meaning selections with each other. Last, the teacher must also guide these meaning selections so that the students understand the alignment and fit of their discourse with the discourse of the larger society. Now, with regards to the researcher, the researcher should also be aware of the vast meaning potential of the semiosphere; however, the researcher should refrain from any attempt to coordinate the student's meaning-making. Also, the researcher should refrain from functioning as a guide or from assisting the students in aligning their meaning selections with each other. To be clear, the role of the teacher should be to apprentice the student in making meaning from the semiosphere, while the role of the researcher should be to observe the student in his or her meaning-making process. If taken to one further level of scrutiny, it can be said that the role of the researcher is to analyze the apprenticeship of the student by the teacher in making meaning from the semiosphere. This perspective offers a clearer understanding of the position held by Vygotsky (1930/1978) that certain levels of intellect and meaning-making required a formal process.

Re-engineering the Teaching Experiment

Now, before I specify the re-engineering that I performed on the teaching experiment to accommodate the disaggregated learning segments, the syntactic analysis and the semantic analysis embedded within each learning segment, I need to acknowledge an ideological tension that exists. The tension is between the foundational ideologies of CHAT and of the teaching experiment.

A Historical Fist-of-Cuffs: Radical Constructivism v. Sociocultural Theory

The specific problem is the coupling of CHAT with the teaching experiment. For me, the mathematics classroom teacher, this coupling was an obvious match. As I detailed in Chapter 3, CHAT provides the investigative tools and perspective for analyzing an activity; and the teaching experiment provides the investigative tools and perspective for analyzing the individual's language and actions within an activity. However, for me, the junior theorist, this match should have been problematic because of the ideological differences that are purported to exist between sociocultural theory, the foundation of CHAT, and radical constructivism, the foundation of the teaching experiment. It was not until recently that I learned of this ideological fist-of-cuffs. The issue seems to be about the generative source of knowledge (Jaworski, 2015). For the sociocultural researcher, community and its culture are the generative sources of knowledge (Lerman, 1996). For the radical constructivist, the individual's experience is the generative source of knowledge (von Glasersfeld, 1984). Steffe and Thompson (2000) presented a cogent argument that there is an overlay of the two ideologies when people engage in discourse because the influences of one's community and culture are captured within one's discourse. I incorporated this perspective within my methodology by linking together the microscheme and the macroscheme of the teaching experiment.

To review my position as stated in Chapter 3, in the same fashion as Steffe and Thompson (2000), I also believe that there is evidence of a hand-shaking of sorts between the two ideologies. I wish to provide the idea of apprenticeship as another example which establishes that there is indeed an overlay of the two ideologies. I present apprenticeship as my evidence because a metacognitive apprenticeship is exactly what I practice in my mathematics classroom and in my tutoring sessions.

Apprenticeship, from the academic perspective, first appeared as a term in the literature in the mid-to-late 1980s. Collins, Brown, and Newman (1987) defined the academic form of apprenticeship as modeling, coaching, and fading. Modeling, in this context, is represented by the teacher demonstrating the respective action. Coaching is represented by the teacher guiding the student as he or she attempts to execute the same actions. Fading is represented by the teacher limiting the amount of guidance and feedback that the student receives. My use of apprenticeship as evidence that there is indeed an overlay between sociocultural theory and radical constructivism is direct. During the first stage, modeling, the community and culture demonstrate and represent how a particular action is performed. During the second stage, coaching, the community and culture guide the individual through his or her attempts at executing the same actions. During the third stage, fading, the community and culture provide limited guidance and feedback to the individual as he or she performs the respective task. Eventually, the involvement of the community and the culture approaches a level of seemingly non-existence as the individual is independently performing the respective task, with his or her own particular idiosyncrasies. It is during this sequence of stages that the overlay between the two ideologies is made clear. At the beginning when the individual has limited or no awareness as to how to perform a task, the community or culture has a dominant role in the construction of the individual's knowledge. As the knowledge constructed by the individual becomes more coherent, the dominant role that the community or culture played begins to recede. Eventually, the individual has constructed enough knowledge and attained enough experience that he or she assumes the dominant role in his or her future knowledge and experiential constructions. It is through such a knowledge-construction and experience-attainment process that sociocultural theory and radical constructivism are tied together. I agree that when this process is viewed in discrete parts, a linkage between the two is untenable. When viewed in the aggregate, however, knowledge-construction and experience-attainment are an apprenticeship process that embodies both sociocultural theory and radical constructivism. On this subject, I close by first saying that I appreciate the intellectual exercise that was necessary for me, myself, to establish my own position on this issue because the foundation of this theoretical project and its underlying methodology hinged on my ability to bring together two ideologies that some had believed to be mutually exclusive. Secondly, I state that the richness of the teaching experiment is that it is a methodology that asserts that each member in the study, the teacher, the researcher, and the student serve in the role of expert and in the role of student, depending upon the question and situation under investigation during the study.

Creation of Data Collection Charts and Templates

Now that I have provided my perspective on this ideological tension, I move on to the data collection and data analysis segments of my work. I think it best to begin by discussing the tools that I created for data collection and data analysis. I then detail the procedural adaptations that I made to the teaching experiment.

I first created a grammar analysis chart to facilitate the examination of the mathematics word problems which were used in this study. I call this chart a Syntactic Analysis Chart. There are several layers in this chart. The first layer allows each sentence within the word problem to be separated into its subject and predicate parts. The subject of the sentence is the collection of words that precede the active verb (Biber, Conrad & Leech, 2002; Kolln & Funk, 2012; Leech & Svartvik, 1975; Van Gelderen, 2002). The predicate of the sentence is the collection of words that are not a part of the subject and includes the active verb (Biber, Conrad & Leech, 2002; Kolln & Funk, 2012; Leech & Svartvik, 1975; Van Gelderen, 2002). The chart then allows the predicate to be separated into its constituent parts: (a) verb, (b) object, (c) subject complements, and (d) verb modifiers. Last, the chart allows for the sentence type to be identified: (a) simple, (b) compound, (c) complex, and (d) compound-complex. For some mathematics educators, the use of such a chart to examine a mathematics word problem may be considered as superfluous, but I have found that the use of this chart made it easier for my students to understand how and why certain relationships existed between words and how certain inferences became evident based upon those relationships. I present below (see Figure 5.1) an example of the Syntactic Analysis Chart applied to an eighth-grade mathematics word problem:

> The state fair is a popular field trip destination. This year, the senior class at High School "A" and the senior class at High School "B" both planned trips there. The senior class at High School "A" rented and filled 8 vans and 8 buses with 240 students. High School "B" rented and filled 4 vans and 1 bus with 54 students. Every van had the same number of students in it. Every bus had the same number of students in it. Find the number of students in each van and in each bus.

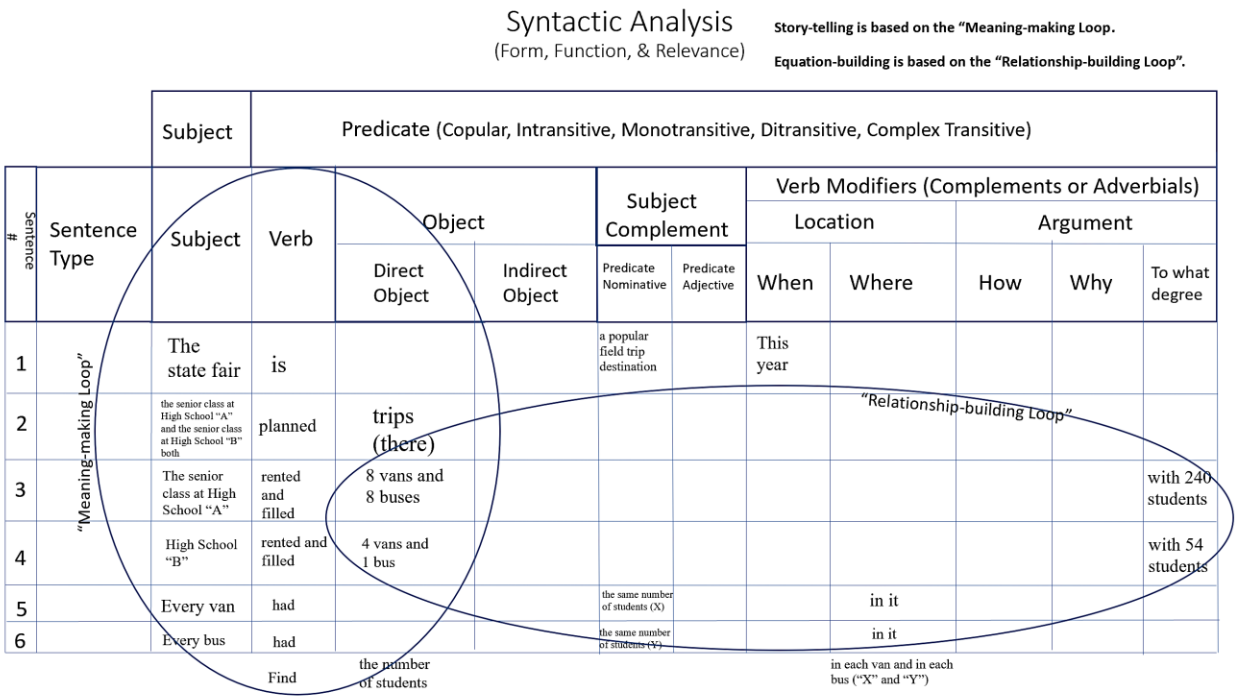

Figure 5.1. Syntactic Analysis Chart of eighth-grade word problem.

Next, I created an Analytic Memo Overview diagram. I created this diagram for two reasons. First, Saldaña (2016) strongly urges the researcher to use analytic memos:

> Analytic memo writing documents reflections on: your coding processes and code choices; how the process of inquiry is taking shape; and the emergent patterns, categories, sub-categories, themes, and concepts in your data—all possibly leading toward theory. (p. 44)

In addition, Steffe (1991) highlights that one of the aspects of the teaching experiment is that it allows for the researcher to analyze three possible interactions that exist for the student: (a) the researcher's perspective of the interaction between the student and the mathematics concept, (b) the student's perspective of his or her own experiences, and (c) the researcher's perspective of the interaction between himself or herself and the student. Due to time constraints, my work focused on my interpretation of the student's interaction with the particular mathematics concepts, and on the student's understanding of his or her own experiences with the particular mathematics concepts. Examining the interaction between the students and me, from my perspective, would have provided great insight into the caring relationship aspect of this work, but I did not have the time to investigate this particular aspect (cf. Hackenberg, 2005, 2010). This Analytic Memo Overview diagram can be seen in Figure 5.2. The demand of the teaching experiment is for the researcher to grasp insight on each of these interactions. However, it is not possible for the researcher's etic perspectives to match the student's (i.e., participant's) emic perspectives of any of the interactions. When the researcher is attempting to understand the student's experience, the teaching experiment demands that the researcher make such an effort within the bounds of the student's language and actions, not the researcher's language and actions (Steffe, 1991).

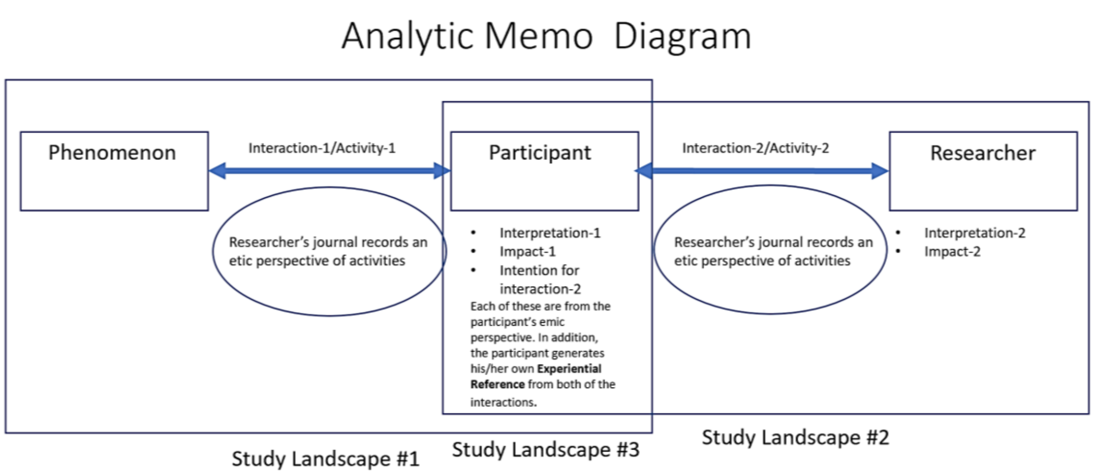

Figure 5.2. Analytic Memo Diagram.

The third item that I created is based on the elements of Cultural Historical Activity Theory (CHAT). Due to the fact that CHAT served as the theoretical framework for this study, I used its elements as the section titles within my data collection code chart. I also included several additional aspects that I thought would allow for greater inspection of the student's experience. I included syntax, semantics, and pragmatics sections in the chart. The syntax section allowed me to record my thoughts about the student's examination of the grammar represented within the mathematics word problem. The semantics section allowed me to record my thoughts about the student's meaning-making from the given mathematics word problem. The pragmatics section allowed me to record my thoughts about the student's problem-solving efforts. I also included an evaluation section that allowed me to record my thoughts on the quality of the student's work and accuracy of the answer. This CHAT Dissection chart can be seen in Figure 5.3.

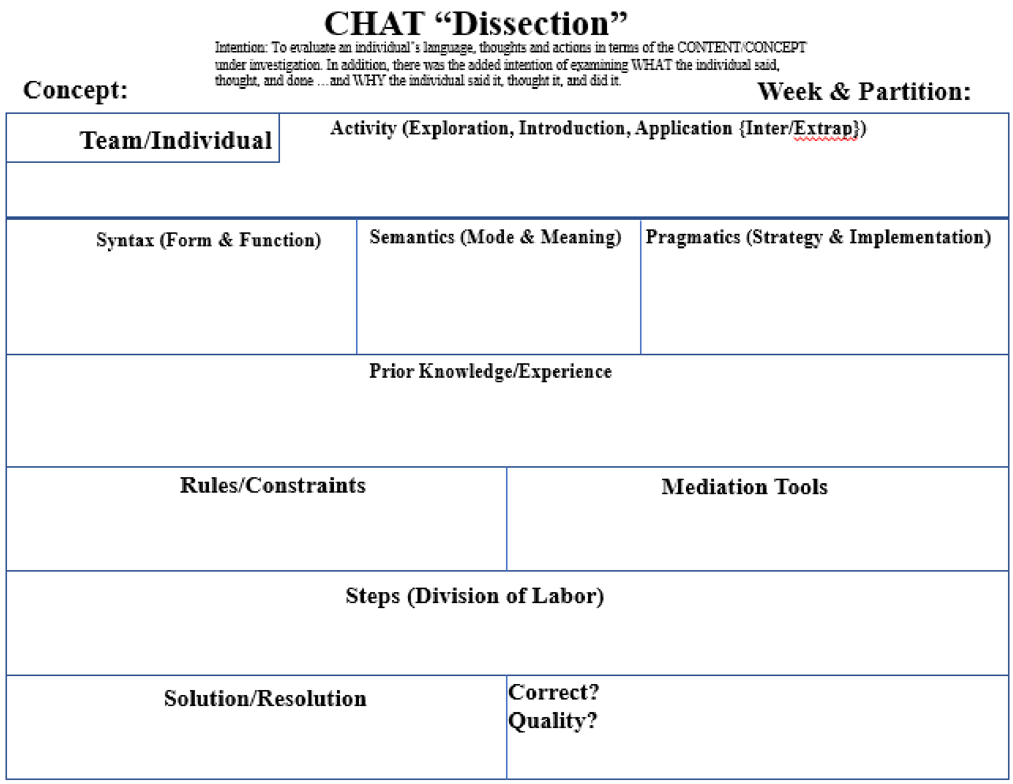

Figure 5.3: CHAT Dissection Chart

The fourth item that I created is the Socratic Inquiry Chart. This chart allowed me to guide the discourse and organize the student's responses to the various Socratic questions that I created based upon the work done by Overholser (1993a, 1993b, 1994, 1995, 1996, 1999). It included the eight areas as detailed by Overholser, and then I cross-referenced those eight areas with the four partitions of the teaching episode as detailed in the work of Steffe and Thompson (2000) and by Engelhardt, Corpuz, Ozimek, and Robello (2004): (a) exploratory teaching, (b) concept exploration, (c) concept introduction, and (d) concept application. This Socratic Inquiry Chart can be seen in Figures 5.4a, 5.4b, 5.4c, and 5.4d. Due to time constraints, I did not include an investigation into my exploratory teaching as provided for in this chart because the exploratory teaching partition was not the focus of this work. If used in total, there are thirty-two areas for investigation that can be pursued using this chart. For my work, twenty-four of these areas were relevant and I used them in accordance with the progress made by the student.

	Self-Improvement	Disavowal Knowledge	Universal Definitions	Inductive Reasoning	Questioning Elements	Systematic Questioning	Attitude/ Disposition	Problem-Solving(pg.89)
Exploratory Teaching	What can be done to improve my thinking and my teaching?	What mistakes have been made as we discuss and solve this problem? What other approach could we have taken to solve this problem?	What does the class think are the key words in the given situation/problem? What do these key words mean? Do any of the key words remind the class of any other words?	What patterns or relationships does the class notice may exist within this topic/concept? Does anything about this new topic/concept remind you of another topic/concept that the class has experienced?	What does the class understand about the given topic/concept?	Has the class ever dealt with a situation like this one before? What comes to mind as the class thinks about the given situation?	What convinces the class that the class can be successful in understanding this concept/topic? What convinces the class that we will not quit when the concept/topic becomes difficult? How do we insure that everyone in the class is doing his/her part? How do we make sure that everyone in the class is learning?	What representative form s are better for this topic? What are the key points? What are the best steps to take when working with this concept? What do you understand about the situation? How could you produce a solution for this situation? What are other ways? Based upon your options, which option will you choose? Why? What are you thinking as you solve this problem? Why do you feel that this approach has produced a correct solution?

Figure 5.4a: Socratic Inquiry Chart - Exploratory Teaching

Concept Exploration	What do I already understand about the given situation/problem? What can be done to improve my thinking and my discussion about this current situation/problem? What can be done to improve my ability to solve this current situation/problem (ex: faster with less mistakes)? What can I do when I do not understand the problem or situation? What can I do when I get an answer wrong? How do I stay focused on a challenging problem?	What mistakes have been made as you think about and solve this problem? What other approach could you have taken to solve this problem?	What do you think are the key words in the given situation/problem? What do these key words mean? Do any of the key words remind you of any other words?	What patterns or relationships do you notice may exist within this topic/concept? Does anything about this new topic/concept remind you of another topic/concept that the class has experienced? Based upon the given formula/equation, what types of values are not allowed?; what conditions are necessary?	What do you understand about the given topic/concept? What words stand-out that are difficult for you to understand?	Have you ever dealt with a situation like this one before? What comes to mind as you think about the given situation?	What convinces you that you can be successful in solving this problem? What convinces you that you will not quit when the problem becomes difficult? What needs to be done so that everyone on your team is doing his/her part? How do you make sure that everyone on the team is understanding?	What representative form s are bet for this topic? What are the key points? What are the best steps to take when working with this concept? What do you understand about the situation? How could you produce a solution for this situation? What are other ways? Based upon your options, which option will you choose? Why? What are you thinking as you solve this problem? Why do you feel that this approach has produced a correct solution?

Figure 5.4b: Socratic Inquiry Chart – Concept Exploration

	Self-Improvement humility	Disavowal Knowledge humility	Universal Definitions Syntax	Inductive Reasoning Semantics	Questioning Elements Self-Evaluation	Systematic Questioning Modeling & Strategizing	Attitude/ Disposition Self-esteem	Problem-Solving(pg.89) Problem-solving
Concept Introduction	What do I already understand about the given situation/problem? What can be done to improve my thinking and my discussion about this current situation/problem? What can be done to improve my ability to solve this current situation/problem (ex: faster with less mistakes)? What can I do when I do not understand the problem or situation? What can I do when I get an answer wrong? How do I stay focused on a challenging problem?	What mistakes have been made as we think about and solve this problem? What other approach could you have taken to solve this problem?	What do you think are the key words in the given situation/problem? What do these key words mean? Do any of the key words remind you of any other words?	What patterns or relationships do you notice may exist within this topic/concept? Does anything about this new topic/concept remind you of another topic/concept that the class has experienced? Based upon the given formula/equation, what types of values are not allowed?; what conditions are necessary?	Why do your think that your understanding is accurate? Why do you think that your approach is correct?	How would you represent what you understand about the given situation? What other way could you represent what is being described?	What convinces the class that we can be successful in solving this problem? What convinces the class that we will not quit when the problem becomes difficult? What needs to be done so that everyone in our class is doing his/her part? How do we make sure that everyone in the class is learning?	What representative form s are best for this topic? What are the key points? What are the best steps to take when working with this concept? What do you understand about the situation? How could you produce a solution for this situation? What are other ways? Based upon your options, which option will you choose? Why? What are you thinking as you solve this problem? Why do you feel that this approach has produced a correct solution?

Figure 5.4c: Socratic Inquiry Chart – Concept Introduction

Concept Application	What do I already understand about the given situation/problem? What can be done to improve my thinking and my discussion about this current situation/problem? What can be done to improve my ability to solve this current situation/problem (ex: faster with less mistakes)? What can I do when I do not understand the problem or situation? What can I do when I get an answer wrong? How do I stay focused on a challenging problem?	What mistakes have been made as you think about and solve this problem? What other approach could you have taken to solve this problem?	What do you think are the key words in the given situation/problem? What do these key words mean? Do any of the key words remind you of any other words?	As you read the given situation/problem, what main math concept do you think is being described?	Think about what you are saying and doing, and describe what you would write about in your math journal	What is being described by the given situation? What relationships are important to understand when working to solve the given problem? What ideas come to mind as you think about your approach to solving this problem? What steps are necessary for you to solve this problem? Although this may be a new type of problem for you, what ideas do you have to solve this problem? What steps have you taken to check the quality and accuracy of your work?	What convinces you that you can be successful in solving this problem? What convinces you that you will not quit when the problem becomes difficult? What needs to be done so that everyone on your team is doing his/her part? How do you make sure that everyone on the team is understanding?	What representative form s are bet for this topic? What are the key points? What are the best steps to take when working with this concept? What do you understand about the situation? How could you produce a solution for this situation? What are other ways? Based upon your options, which option will you choose? Why? What are you thinking as you solve this problem? Why do you feel that this approach has produced a correct solution?

Figure 5.4d: Socratic Inquiry Chart – Concept Application

These four items greatly facilitated my efforts to collect and record data from my observations, journal notes, analytic memos, and student conferences. Due to the COVID-19 pandemic, a more formal and optimal approach to this work was not possible. Instead of my 14 years of teacher notes, reflective journaling, and informal tutoring sessions guiding the framework and guidelines of a formally coordinated and time-specified study, data sets from the last 3 of those 14 years became the source material for my actual dissertation work. I could have been placed in disarray by the COVID-19 pandemic had it not been that I had already gathered the more impactful material and prepared it for my Prospectus and the IRB. So, over the past 14 years, although the original data was not organized in an optimal manner when I initially archived my experiences with my classroom students and tutoring students, it was just a matter of collecting data that propelled my evolution the furthest and placing that information within these items. In a traditional research study, the researcher collects the data from the participants in the study; however, for this theoretical project, I collected the data from my archives. Now, in what follows, I provide a general description of my changes to the teaching experiment.

General Overview of Reconfigured Procedure

Now that I have detailed the items that I created to collect the data, I now provide a general overview of the format that I used when I engaged the student, and the general actions of the student. First, I provided the student a mathematics problem. I chose to give the student a word problem because word problems provided me the best opportunity to investigate the connection between language and mathematics. Second, I provided the student time to independently solve the problem, and I observed the student's actions. This amount of time varied, but typically lasted between 15 minutes and 30 minutes. Third, I engaged with the student and provided important information on three topics, and I took notes on the information that I provided. The first topic was information on the relevant mathematics concept on which the problem was focused. The second topic was information on English grammar and its relevance to mathematics. The third topic was information on model-making and its connection to equation-building. With my tutoring students, I spoke on one topic each week, requiring a total of 3 weeks. With my classroom students, during a 5-day school week, I spoke on the first and second topics for 2 days, with 1 day remaining for the third topic. Next, after providing the student an opportunity to engage in reflective abstraction, I provided the student additional time to return to solving the given problem independently, and I observed the student's actions (Piaget, 1936/1952). Last, I interspersed Socratic Inquiry sessions with the student throughout this entire process, and I took notes on what the student said. Over the next several pages, I provide the following: (a) a more detailed description of the format used, (b) visual models of the format, and (c) a representative example of this data collection process with one student, and the data collected.

In Chapter 4, I referred to the suggestion by Engelhardt, Corpuz, Ozimek, and Robello (2004) that the teaching experiment be partitioned into the following four parts: (a) concept exploration, (b) concept introduction, (c) concept application, and (d) Socratic inquiry. This structure allows for descriptive and insightful data collection as each partition focuses on a different aspect of the student's learning and experience. In addition, I decided to broaden the content of the concept introduction partition to include information beyond the mathematics concept. I included syntactic skills, semantic skills, and pragmatic skills. I use the word *pragmatic* to refer to motives in context—problem-solving (Rowland, 2000). I made the decision to emphasize to the student that in language and in communication there can be multiple layers of information (Halliday, 1978). The syntactic layer refers to grammar (Chandler, 2007; Eco, 1978; Halliday, 1978). The semantic layer refers to meaning-making (Chandler, 2007; Eco, 1978; Halliday, 1978). The pragmatic layer refers to problem-solving (Halliday, 1978; Rowland, 2000). With such awareness, the aim was for the student to begin to value the different type of information that is available at each layer.

I have included a visual representation of the data collection that was conducted at each partition (see Figure 5.5). Notice that a Socratic Inquiry was conducted between the observations of each consecutive partition. This inquiring was done to gain the student's perception of the episodic experience because the observations only reflect the researcher's perspective and interpretation, in this case, mine.

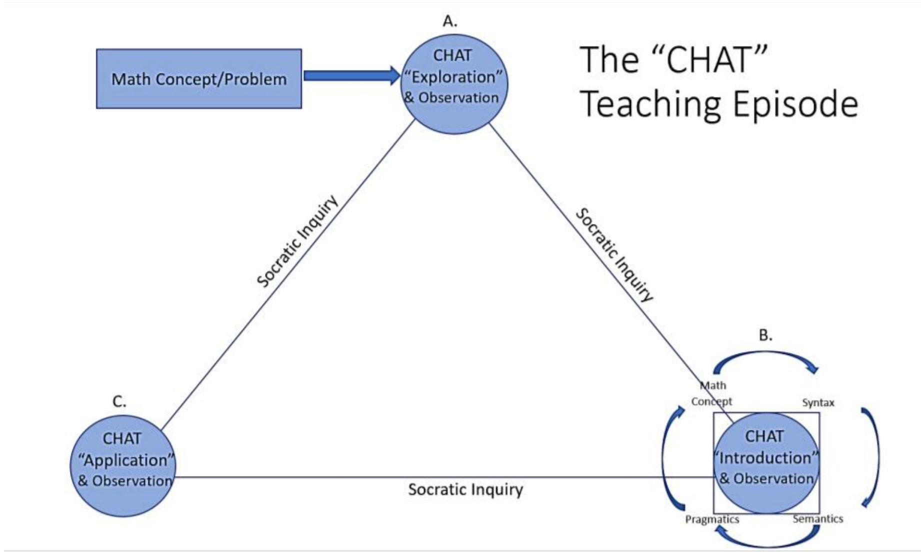

Figure 5.5. CHAT Teaching Episode.

At position "A" in the figure, I wrote my observations of the student attempting to solve the given math problem with whatever approach that he or she chose. After my observations, I engaged the student with Socratic inquiry in the effort to have the student explain to me his or her line of reasoning (Overholser, 1994). Then, the student entered into the concept introduction partition during which time I presented to the student important information regarding the current mathematics concept. I took notes of my activities and efforts during this time, using the same CHAT chart format that I had used for the other observations. After I had completed my presentation of the mathematics concept, I engaged the student with another Socratic inquiry in the effort to have the student discuss with me his or her perspective on the given mathematics problem within the context of the newly shared information on the mathematics concept. Then, the student was given another opportunity to solve the given mathematics problem; however, the student not only had the benefit of his or her own prior knowledge and experience, but now had the addition of the potentially new information that I had just provided. Last, I engaged the student with one more Socratic Inquiry in the effort to have the student discuss with me his or her most recent attempt at solving the mathematics problem (Overholser, 1994). By this time, the student would have been exposed to the mathematics concept three separate times, twice in the form of the given problem, and one time in the form of a presentation from me. These three exposures allowed for a multi-dimensional investigation of the student's learning experience. For my tutoring students, this sequence was performed two more times, during two more weeks so that I could also present syntax and semantic information to the student during the concept introduction partition. I was not able to present the pragmatics information to all students, so it is not discussed here. In total, each tutoring student completed this three-pronged protocol for three consecutive weeks, which allowed each student six separate chances to solve the same mathematics problem, one opportunity during each of the three concept exploration partitions, and one more opportunity during each of the three concept application partitions. Although it was not the initial intent of my research to evaluate the impact of time-dependence on cognitive development, this protocol makes it possible to do so in future work.

Deeper Evaluation of Reconfigured Procedure

Now that I have provided an overview of the student's actions during my research procedure, I now provide greater detail into my own specific actions during my re-engineered teaching experiment. As previously mentioned, my re-engineering of the teaching experiment all center around the four partitions of the teaching episode—concept exploration, concept introduction, concept application, and Socratic inquiry. See the Teaching Episode Macroscheme/Microscheme Diagram (Figure 5.6) for a visual display of these partitions.

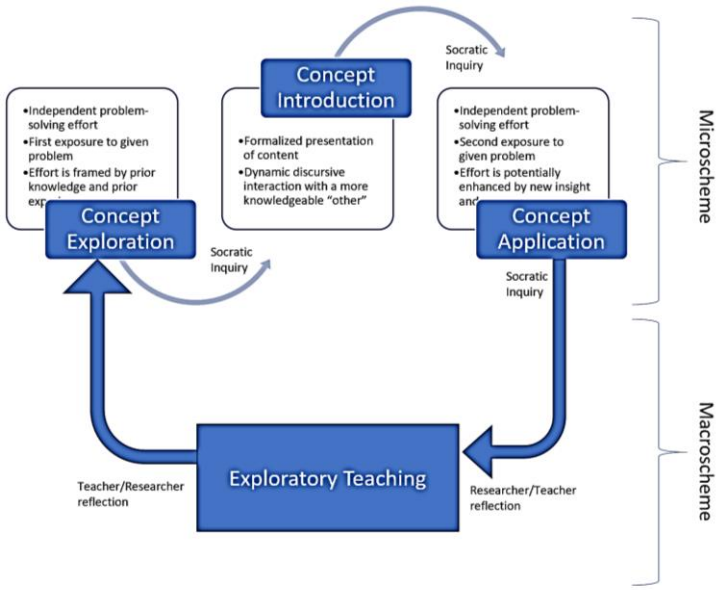

Figure 5.6. Teaching Episode Macroscheme/Microscheme Diagram.

I began my investigation during the concept exploration partition. Recall that it is during this partition that the student can attempt to solve the given math problem in any way desired. As I observed the student exploring the given problem, there were two phenomena of interest for me—the given mathematical situation and the student. My goal was to develop a descriptive profile of both phenomena. As the researcher, my goal was to describe the given mathematical situation according to its syntactic and semantic characteristics. Within the syntactic investigation, the effort was to describe the representative structure of the given situation as either conceptual, pictorial, graphical, tabular, numerical, or some composite. Also, within the syntactic investigation, the effort was to identify the part of speech and purpose of any linguistic elements. In linguistic terms, this process is referred to as form and function (Biber, Johansson, Leech, Conrad, & Finegan, 1999). Within the semantic investigation, the effort was to focus on nouns and verbs and compile a list of the vocabulary, definitions, characteristics, concepts, relationships, patterns, representations, and requisite mathematical procedures. The Syntactic

Analysis Chart (see Figure 5.7) and the CHAT "Dissection" Chart (Figure 5.3) were used to record all such data.

Syntactic Analysis

(Form, Function, & Relevance)

		Subject	Predicate (Copular, Intransitive, Monotransitive, Ditransitive, Complex Transitive)									
Sentence #	Sentence Type	Subject	Verb	Object		Subject Complement		Verb Modifiers (Complement or Adverbials)				
								Location		Argument		
				Direct Object	Indirect Object	Predicate Nominative	Predicate Adjective	When	Where	How	Why	To what degree

Figure 5.7. Syntactic Analysis Chart.

Upon completing the profile for the given mathematical situation, my next goal was to develop a descriptive profile of the student as the student made his or her first attempt at solving the given mathematical situation. As the researcher, one aim was to describe the student's use of language while attempting to solve the given mathematical situation. With regards to the student's discourse, I made effort to discern if the student engaged in colloquial speech or in mathematical speech, or some composite, and then to identify its content. A second aim was to describe the student's actions as he or she attempted to solve the given mathematical situation. With regards to his or her actions, I made effort to discern if the student attempted to investigate the linguistic features of the given mathematical situation, its syntax and semantics, as I had done. I also made the effort to determine if the student engaged in transferring his or her understanding of the given mathematical situation by generating a model of his or her understanding, or some new representation of the given situation. If the student did not engage in model generation or changing the representation, then I needed to determine what exactly did the student do. This focus on the language used and the actions performed by the student was consistent with the work of Steffe and other advocates of the teaching experiment (Steffe, 1991). However, it can be argued that my inclusion of an investigation of the syntactic and semantic aspects of the given mathematical situation, itself, is an explicit effort that is beyond what Steffe and other advocates of the teaching experiment have traditionally espoused.

To develop both of these profiles within the concept exploration partition, I engaged in reflection and observation. Specifically, I reflected upon my own mathematical experiences (i.e., my prior knowledge) in order to profile the given mathematical situation; and I observed and listened to the student in order to profile his or her discourse and actions. The reason why I engaged in such profile construction was due to the amount of information and insight that was gained on both the given mathematical situation and the student, and the influence that the information and insight had on informing my actions during the remaining partitions of the teaching episode. In short, these two profiles allowed for a more refined and customized teaching episode and insured that each teaching episode would have its own unique features with respect to the language used with the student and the tasks given to the student.

After the concept exploration partition, the concept introduction partition followed. During this partition of the teaching episode, the student was formally exposed or introduced to the mathematically important vocabulary and concepts that were contextualized within the given mathematical situation. This exposure and introduction were two separate events. The exposure was actually a student-led investigation of the vocabulary and underlying mathematical concepts, as was detailed in Pilot Study 2. The introduction was actually a teacher-led discussion, performed by me, which consisted of a discussion of the vocabulary and underlying mathematical concepts. Unlike the concept exploration partition, during this partition, there was only one phenomenon of interest, the student; but there were two actions. One action was to develop a descriptive profile of the student, different from the previous profile, and unique to this partition. The second action was to provide the student with a discussion of the formal mathematical vocabulary and underlying mathematical concepts that existed within the given mathematical situation. This first action, the development of a second descriptive profile, consisted of observations, field notes, and analytic memos whereby I kept an account of the student's use of language and actions as the student investigated the vocabulary and underlying mathematical concepts contained within the given mathematical situation. As with the first profile, in the development of this second descriptive profile, I attempted to discern if the student engaged in colloquial speech or in mathematical speech, or some composite. Also, within the development of this second descriptive profile, I attempted to keep an account of the actions taken by the students while they conducted their academic investigations. As previously discussed in Pilot Study 2, the academic investigation required the student to: (a) create his or her own research questions that represented the cognitive disconnects or disruptions caused by the given mathematical situation, (b) access the internet with the inquiry questions in mind, and (c) select at least three websites, at least two videos, and at least one text from the multitude of resources available on the internet which could be helpful. During this process, the student gained insight on details that were not evident in the initial presentation of the mathematical situation. This insight was gained through the intertextuality of internet links, the multiple modalities offered by internet websites, and the variety of teaching styles offered by the instructional videos. This insight was valuable to the student because it addressed questions that the student had which were unique to the student, according to his or her learning rate, but were not actual questions posed by the given problem.

After I completed the development of this second profile, I began the second action. The second action was to provide the students with a discussion of the formal mathematical vocabulary and underlying mathematical concepts that existed within the given mathematical situation. For some teachers, this second action is referred to as a lecture; however, I refer to it as a mathematical discussion because I did privilege the interactive communication with the students over a self-gratifying monologue (Steffe, 1991). During this mathematical discussion, some of the topics included a focus on the relevant vocabulary; a focus on relevant instructions, directions, or procedures; a focus on modeling and using multiple representations; a focus on demonstrating a problem-solving approach; and a focus on misconceptions or misinterpretations that occurred during the concept exploration partition as well as other misconceptions or misinterpretations held by the student. During these discussions, as previously detailed in Pilot Study 2, note-taking was an invaluable tool for the students because it allowed the student to compare and contrast the information that he or she gained during his or her academic investigations with the information that they gained during these mathematical discussions.

After the concept introduction partition, the student entered the concept application partition. During this partition of the teaching episode, the student returned to working on the original mathematical situation that was given as well as a collection of other similar and slightly advanced problems, if time and opportunity allowed. Some of the additional problems were similar to the original problem in terms of the amount of information provided, the complexity of the language, or the representative modality. However, some of the additional problems were slightly more advanced than the original problem in terms of the excess or reduction in the amount of information provided, the advanced complexity of the language, the representative modality, or in terms of the variety of the questions posed to the student. During this concept application partition, there were two phenomena of interest—the problems and the student. This partition was similar to the concept exploration partition in terms of the data that I collected. I developed a descriptive profile of both phenomena. As in the earlier partition, my goal was to describe the given mathematical problems according to their syntactic and semantic characteristics if they were different from the original problem. As before, within the syntactic investigation, the effort was to describe the representative structure of the problems as either conceptual, pictorial, graphical, tabular, numerical, or some composite. Also, again, within the syntactic investigation, the effort was to identify the parts of speech and purpose of any linguistic elements. Within the semantic investigation, as was performed before, if the problem was different from the original, the effort was to focus on nouns and verbs and compile a list of the vocabulary, definitions, characteristics, concepts, relationships, patterns, representations, and requisite mathematical procedures.

Upon completing the profile for the problems, the next goal was to develop a descriptive profile of the student as the student worked alone or in a small group at solving the given problems. As detailed earlier, one of my aims was to describe the student's use of language while attempting to solve the problems. With respect to the student's discourse, I made effort to discern if the student, when alone or with a group, engaged in colloquial speech or in mathematical speech, or some composite and then to determine its content. A second aim was to describe the student's actions as he or she attempted to solve the problems. With respect to his or her actions, I made effort to discern if the students attempted to investigate the linguistic features of the given problems, their syntax and semantics, as I had done. I also made an effort to determine if the student engaged in transferring or crystalizing his or her understanding of the given problems by generating a model of their understanding, or by generating some new representation of the given problems. If the students did not engage in model generation or changing the representation, then I needed to determine what exactly did the students do. Note this process was the same as the process performed during the concept exploration partition, but I repeated it here to identify if there were any alterations or adaptations in the student's language or actions based upon the events of the prior partitions.

Up to this point, I have described in detail what my actions were during the first three partitions of the teaching episode. The amount of effort required to attend to the language and actions of the students during these three partitions is not to be under-estimated. As both language and action can be organic, dynamic, and simultaneous, I had to be equally as fluid in my data collection efforts. In addition, due to the fact that this was a teaching experiment methodology, I had to be committed to the additional charge of interpreting all of the language and action from the perspective of the students. Without question, rigor existed throughout this process. Before engaging in the final stage, Socratic inquiry, I reviewed the various profiles, field notes, and memos; I then engaged in deep reflection in order to determine the line of questioning to be pursued during the Socratic inquiry partition. It was the Socratic inquiry that allowed me to gain insight of the interpretation of the language and actions from the student's perspective.

As described in great length earlier, the Socratic inquiry method is a complex, dynamic, and investigative discursive technique for stimulating the student to reveal the depth and breadth of his or her knowledge. The best approach that I found to systematically manage the plethora of responses generated by the student and the trajectory for each of those responses was to infuse a problem-solving mechanism within the Socratic inquiry method. Specifically, D'Zurilla and

Goldfried (1971) generated such a mechanism and Overholser (1993) provided examples as to how such a problem-solving mechanism could be integrated into the Socratic inquiry method. The D'Zurilla and Goldfried mechanism contains five stages. The first stage is the problem definition stage. The purpose of this stage is for the student to reveal what he or she understands about the given mathematical situation or the teaching episode partition. The second stage is the generation of alternative approaches stage. The purpose of this stage is to motivate the student to create and discuss a variety of directions by which the given mathematical situation or the teaching episode partition could be addressed before actually attempting any of the options. The third stage is the decision-making stage. The purpose of this stage is to help the student reduce the quantity of possible solution paths that were created during the prior stage. This reduction and fine-tuning are achieved by having the student explain the advantages and disadvantages of each option. The fourth stage is the implementation stage. The purpose of this stage is for the student to describe the actions and the thinking that exists while the student is actually executing the solution approach which was selected and justified in the prior stage. The fifth stage is the evaluation of work stage. The purpose of this final stage is for the student to appraise the success or failure of the chosen approach. Such an appraisal is valuable to the student because it allows the student to determine and record, for future references, the approaches that are best suited for particular mathematical situations.

I infused this problem-solving mechanism within the leading question element of the Socratic inquiry method. I decided to do so because, as previously stated, the purpose of the leading question element is to center on "an implied assumption, often serving as a spotlight to focus the student's attention onto a specific area" (Overholser, 1993, p. 71). In my case, the underlying assumption was the particular stage of the problem-solving mechanism; and the specific areas were the language used and actions performed by the student during that particular problem-solving stage. In sum, this particular approach to the Socratic inquiry method within the teaching experiment allowed for each student to have a different inquiry experience that was more aligned with his or her unique language, actions, and with his or her experiences within the teaching episode partitions and the problem-solving mechanism. In addition, I was better able to investigate the student's experiences in accordance with his or her depth of comprehension, cognitive constraints, and errors (Steffe, 1991).

It is important to note that it is the decision of the researcher to determine where and when to implement the Socratic inquiry method. Although I present it here as the last partition of the teaching episode, this ordering by no means suggests that another researcher could not re-position the Socratic inquiry method at another place within the teaching episode, or even position it multiple times throughout the teaching episode. Such mobility emphasizes the true value of the teaching experiment—the ability to have interactive mathematics discussions at any time and at any place within the student's cognitive construction process. This flexibility adds yet another degree of uniqueness to the student's mathematics learning experience.

Now that each of the partitions of the teaching episode have been detailed, and I have explained that the Socratic inquiry method is not restricted to any one location or one implementation within the teaching episode, allow me to state the obvious—there is a lot of data to be collected when attempting to understand and chart the cognitive construction of a student. Yes, it was overwhelming for me, but the ability to endure and endeavor through the process is what establishes the rigor and robustness of the analysis. Due to the fact that the researcher and teacher have an ethical obligation to provide the nurturing learning environment and relationship that each unique child needs in order to develop a positive disposition toward mathematics, there was great opportunity to collect comprehensive and descriptive data (Kilpatrick, Swafford, & Findell, 2001). In addition, the situation was quite practical in that I benefited from having a data management approach as I collected, analyzed, and interpreted such a large assortment of data for this theory-building effort. As I prefer the hands-on approach to data collection and analysis, the CHAT heuristic, analytic memoing, and the numerous charts and diagrams that I generated were effective in guiding my efforts. To be true, I felt more connected to the data using such a hands-on approach than I believe would have been possible using one of the numerous computer-assisted qualitative data analysis software programs (CAQDAS) (Saldaña, 2016).

Thoughts on Coding

In general terms, a code is a word or phrase that represents the researcher's interpretation of data (Saldaña, 2016). There are generally, at most, two cycles of coding. The first cycle leads the researcher through the process of interpreting sets of language-oriented or visual data and assigning codes to them. Then, the researcher collects the codes and categorizes them according to similarities. At the end of the first cycle, the researcher generates themes or concepts that represent the aggregates of the categories of codes (Saldaña, 2016). The second cycle leads the researcher through the process of recognizing abstractions that connect the themes or concepts from the first cycle of coding. Then, at the end of the second cycle, the researcher generates a theory that represents the aggregate of the abstractions (Saldaña, 2016).

My troubles with coding are best captured by Saldaña (2016) when he states, "Thorough—even cursory—descriptions about the researcher's code development and coding processes rarely make it into the methods section of a final report" (p. 39). Saldaña further explains, "The majority of readers would most likely find the discussion tedious or irrelevant compared to the more important features such as the major categories and findings" (p. 39).

Finally, he suggests that the time and energy one puts into coding and memo writing should be acknowledged as "private affairs between you and your data" (p. 40). Without a clear description and discussion of the code development and the coding process, the coding does indeed exist as a private affair between the researcher and the data, and makes it quite challenging for the researcher to detail his or her inferencing technique to himself or herself or another at some later date. My engineering background is the root of my discomfort because successful engineers are always preparing themselves to address the "Why" and "How" questions. For an engineer, simply stating "What" he or she did is rarely sufficient. So, in my discomfort with the data collection and data analysis processes, I have left for myself and provided to the reader a trail as to why and how I decided upon my actions and arrived at the inferences that I made. Wanting to be sensitive to the second point above from Saldaña (2016), I created charts that best represent my efforts. Saldaña (2016) was a valuable resource in revealing and emphasizing to me the benefits of code charting. My personal extension of this code charting technique was to not only use it to present to the reader the data and its analysis but also to first present to the reader the code charts that I created as templates for the forthcoming data analysis stages of this project.

Creation of Data Analysis Templates

After transferring the information from my notes into the previously described data collection charts, I realized that I also needed to create items that would facilitate the analysis of all of the information that I had placed into these items. To facilitate the analysis, I created two different analysis charts for the individual student data; I describe them here. The first analysis chart that I created was the Grade Level First-Cycle Coding Chart. This template contains the student's identification information, which I refer to as the Student Profile. It also contains locations where I record the codes that I generated from the information that I wrote in the

CHAT Dissection Chart. Recall that the CHAT Dissection Chart allowed me to archive my analytic memos from observing the student solve a mathematics word problem. The unique aspect about these analytic memos is that they each were focused on a particular element of CHAT or the relevant syntax, semantic, and pragmatics of the situation. I used In Vivo coding (Saldaña, 2016) to fill in this portion of the chart. The Grade Level First-Cycle Coding Chart also contains locations for my reflective journal entries about the student's responses to a variety of questions as outlined by the Socratic inquiry method (Overholser, 1993a, 1993b, 1994, 1995, 1996, 1999). I used process coding (Saldaña) to produce the entries for these locations. In addition, due to the fact that I had data across a 3-week period of time, I was able to separate each set of data according to particular weeks. I partitioned each classroom or tutoring session according to the teaching episodes, concept exploration, concept introduction, and concept application. As previously discussed, in a 3-week tutoring rotation, I had the opportunity to change the topic that was discussed in the concept introduction partition each week. The first week was committed to the specific mathematics concept of interest. During the second week, I focused on grammar analysis of word problems. In the last week, I focused on relationship-seeking and model-making from word problems. The student worked on the same problem each week, and each teaching or tutoring session lasted approximately sixty minutes. To accommodate this wealth of data, I used a Grade Level First-Cycle Coding Chart for each week for each student.

Saldaña (2016) emphasizes the importance of making the effort to make the connections between the data and the resulting relationships more and more abstract. To achieve this move to greater abstraction, I included an aggregate row and aggregate column in the Grade Level First-Cycle Coding Chart. This inclusion allowed me to use concept coding, as described by Saldaña

(2016), to identify an over-arching concept that most resonated with me that related the codes which I had developed using the In Vivo and Process Coding techniques. A point of interest with performing concept coding in both the row and column directions is that I was able to find over-arching concepts for the CHAT Dissection Chart, the Socratic Inquiry Chart, the concept exploration partition, and the concept application partition. This process was indeed challenging work because it required me to reflect upon the data to the extent that I needed to seek relationships beyond the specific student and find relationships that were much more generalized and more abstract. As the image of the chart shows, another demand of having an aggregate row and aggregate column is that they produce an intersection location that must also be filled. I used the technique of theming the data (Saldaña, 2016) that requires the researcher to produce a sentence or phrase that encompasses and represents the level of abstraction achieved. This Grade Level First-Cycle Coding Chart can be seen in Figures 5.8, 5.9, and 5.10. Figure 5.8 shows the math concept discussed in week 1. Figure 5.9 shows the data gained during the syntax topic of week 2. Finally, Figure 5.10 shows the data collected from the semantics topic of week 3.

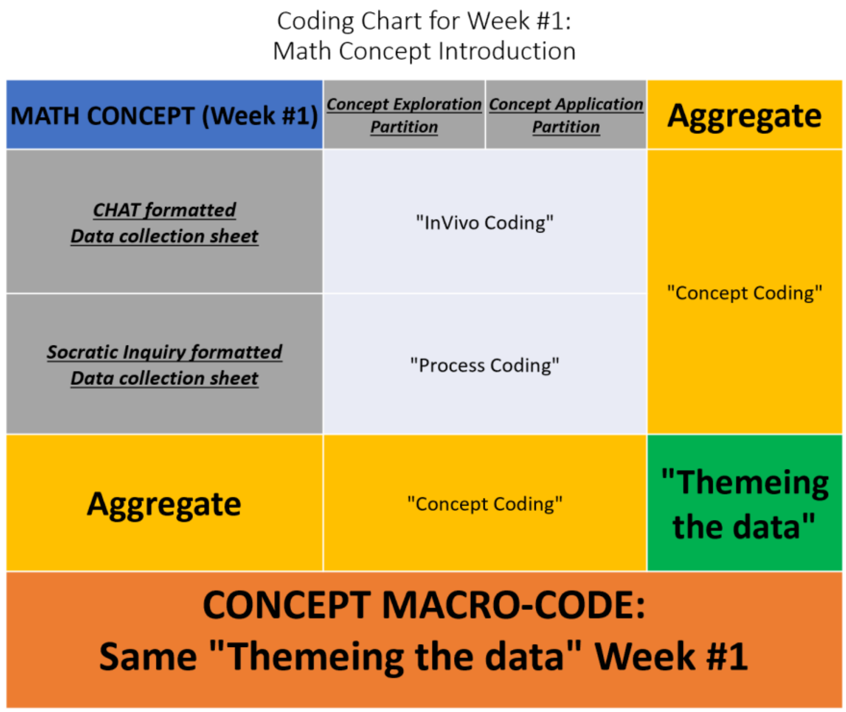

Figure 5.8. Math concept coding chart for week 1.

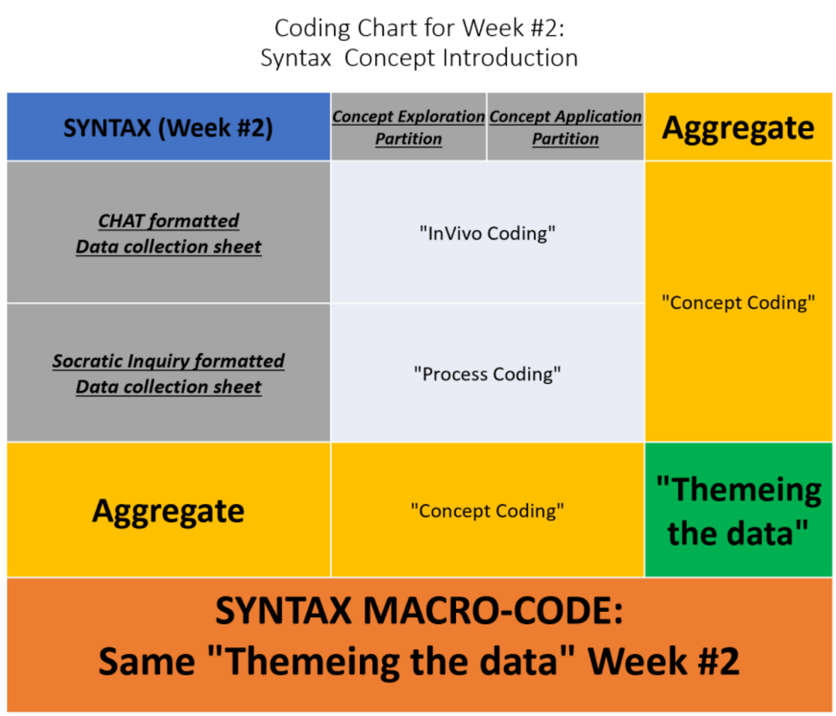

Figure 5.9. Syntax concept coding chart for week 2.

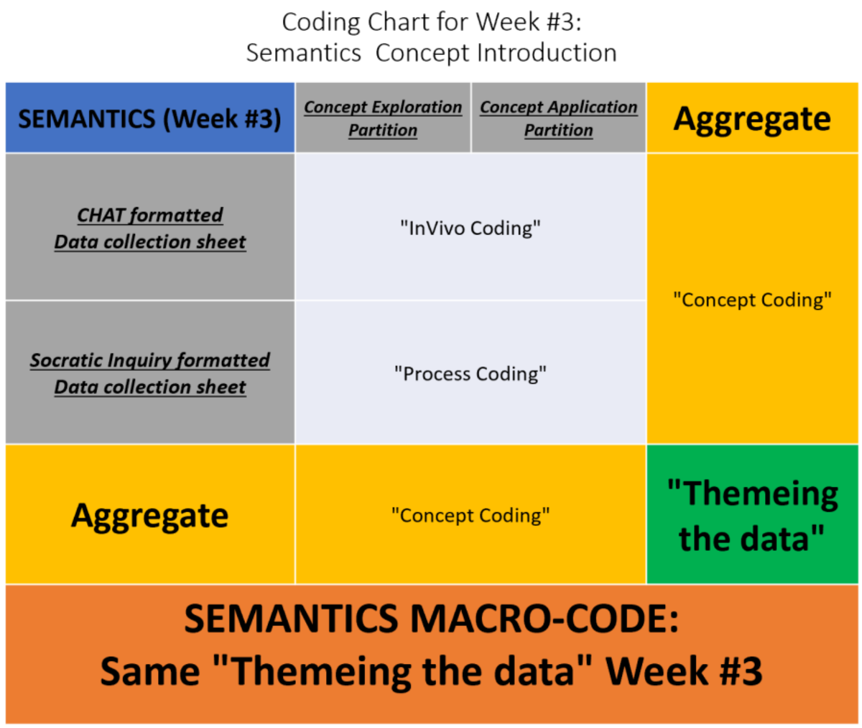

Figure 5.10. Semantics concept coding chart for week 3.

The second analysis chart that I created was the Theorizing Chart. The purpose of the Theorizing Chart was to compile all of the theming the data phrases from each student and from each grade level onto one chart. I then created aggregate rows and aggregate columns for this chart as well, and I used theoretical coding (Saldaña, 2016) to fill in each of these locations. According to Saldaña (2016), theoretical coding is a second-cycle coding technique whose objective is to assist the researcher in producing a resulting theory. As the point of my entire work was to produce a theory for the construction of mathematics knowledge, a second-cycle coding technique was necessary, and I found theoretical coding to be most effective. Again, due to the fact that I had an aggregate row and aggregate column in this chart, there was an intersection location that needed to be filled, and I used that location to arrive at the theory. The development of the theory was yet another effort at abstraction with the codes that I developed using the theoretical coding technique. This entire process was quite tedious and arduous but it allowed me to assert a degree of triangulation through the aggregate rows and aggregate columns that convinced me that I was connecting with the data and was not restricting myself to my own lived experiences. It also allowed me to use the elements of CHAT and the Socratic inquiry method to achieve the desired degree of triangulation. The Theorizing Chart can be seen in Figure 5.11.

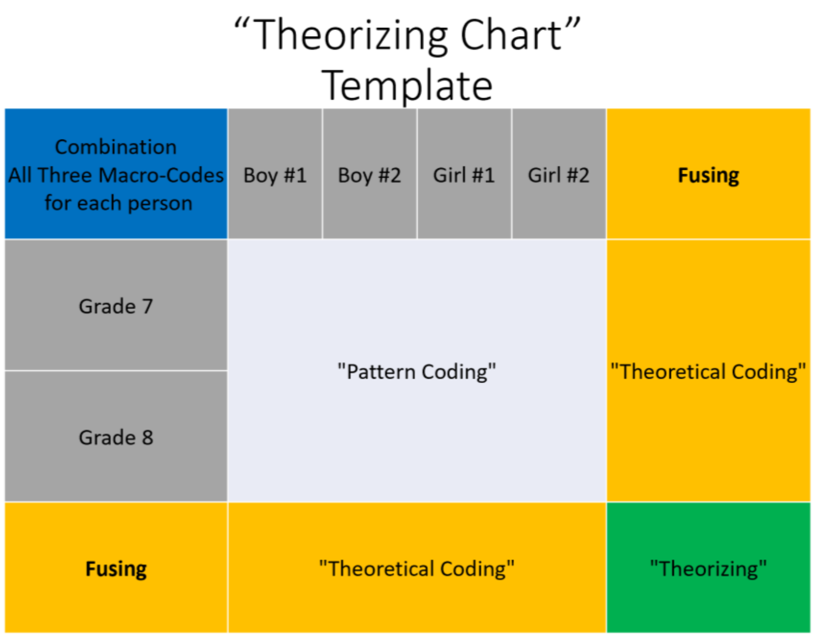

Figure 5.11. Theorizing Chart.

I had to also follow a tedious and arduous process for the group data that I had collected from my actual classrooms. To facilitate the analysis, I created two different analysis charts for the three-member group data; I describe them here. The three-member group data was collected from my students within my classroom, as such, the data was collected within the 5-day school week. The first analysis chart that I created was the Grade Group First-Cycle Coding Analysis Chart. This chart helped me to penetrate through the surface interactions of the group work and reach the depth of the underlying interdependence of the group members. It contains locations where I record the codes that I generated from the information that I wrote in the Grade Group

Data Chart. The Grade Group Data Chart allowed me to archive my analytic memos from my observations of each member of the three-member group as the group solved a mathematics word problem. As with the analytic memos for my individual tutoring students, these analytic memos also focused on a particular element of CHAT or the relevant syntax, semantic, and pragmatics of the situation. I used descriptive coding (Saldaña, 2016) to complete the interior of the chart.

I found great value in including aggregate rows and aggregate columns in the analysis charts for the individual student data analysis, so I did the same for the analysis charts for the group data. Again, I used concept coding to identify over-arching concepts that best represented the codes that I had developed using the descriptive coding technique. A similar challenge occurred with the group data that existed with the individual data: concept codes had to be generated in both the row and column directions in order to complete the analysis. It was this particular effort that allowed me to penetrate through the surface interactions of the group work and reach the depth of the underlying interdependence of the group members. I also had to complete the intersection between the aggregate row and aggregate column spaces. I used the same technique of theming the data that I used with the individual data to achieve this necessary level of abstraction. The Grade Group First-Cycle Coding Chart can be seen in Figure 5.12.

The second analysis chart that I created was the Grade Group Second-Cycle Coding Chart. This chart had the same purpose for the group data as did the Theorizing Chart for the individual data: to compile all of the theming the data phrases from each three-member group and from each grade level onto one chart. I then created aggregate rows and aggregate columns for this chart as well, and I used theoretical coding to fill in each of these locations.

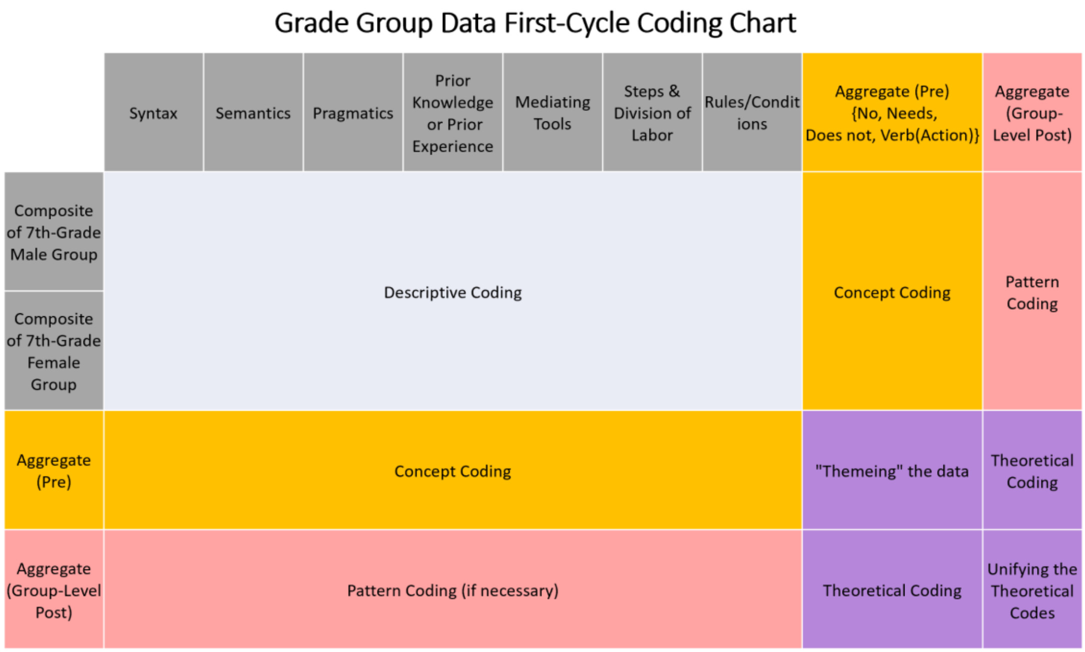

Figure 5.12. Grade group data first-cycle coding chart.

Again, due to the fact that I had an aggregate row and aggregate column in this chart, there was an intersection location that needed to be filled, and I used that location to arrive at the theory. Once I actually began the second-cycle of coding, I found it more effective to append the elements of the Grade Group Second-Cycle Coding Chart to the boundary of the Grade Group First-Cycle Coding Chart. I seemed to have connected better with the data once I made this adjustment. The separated Grade Group Second-Cycle Coding Chart can be seen in Figure 5.13; these same elements have been included at the boundary of the Grade Group First-Cycle Coding Chart (see Figure 5.12).

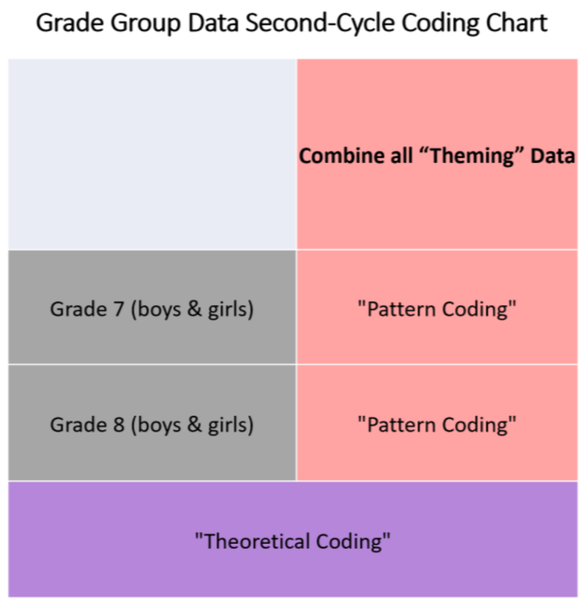

Figure 5.13. Grade group data second-cycle coding chart.

Summative Remarks

In summary, this chapter contains a lot of detail regarding the methodology and the methods of my re-engineered teaching experiment. Undoubtedly, a major achievement was the provision of a resolution to the ideological tension between radical constructivism and sociocultural theory. Without a resolution, the goal of combining CHAT with the teaching experiment would not have been tenable. Subsequently, another major achievement was the development of a new methodology for the teaching experiment that emphasized the power of both the teaching experiment and of CHAT, along with the creation of the commensurate data collection and data analysis methods. However, I posit that the potent point of this chapter was the discussion of the impact that activity disaggregation had on my work. If my middle school students had not developed their own meaning-making and knowledge construction processes, and revealed their processes to me, then my curiosity would not have been piqued to consider what could be possible in a theory-building dissertation based upon the integration of CHAT and the teaching experiment.

CHAPTER 6

SIMULATION OF A RE-ENGINEERED TEACHING EXPERIMENT

In this chapter, I provide a simulation of the data analysis process that is possible from my re-engineered teaching experiment. Before I begin the simulation, however, I provide insight on how the COVID-19 pandemic affected this work. Due to the integration of so many ideas and concepts within this work, I present a review of the main points from each chapter. Next, I offer my stance on why this work is best positioned as a qualitative project and not a quantitative project. Then, I present examples of two archived datasets as exemplars of what is possible when using the data collection charts presented in Chapter 4 created to align with the re-engineered teaching experiment. I continue by providing a more comprehensive overview of thematic coding before I detail the first-cycle and second-cycle coding techniques that I used to analyze the archived data. I conclude the data analysis of the datasets from my tutoring students by offering the resulting theoretical paradigms for consideration. I repeat this process of first-cycle and second-cycle coding for my three-member classroom groups, and provide the resulting theoretical paradigms from the three-member classroom groups. After completing the thematic coding section of this chapter, I discuss the metacognition analysis process that I used. Next, I provide samples of this analysis as I conducted it on the archived data from my tutoring students and on my three-member classroom groups. I conclude this chapter by unifying the thematic coding paradigms and the metacognition paradigms and by providing the resulting over-arching theoretical and methodological perspective.

The COVID-19 Global Pandemic—Who Knew?

In what follows is the research process that I submitted to the Georgia State University's IRB. I completed the protocol as if the COVID-19 pandemic had been resolved and that the nation-wide, social-distancing mandate no longer existed. In many instances, the IRB evaluator wanted the same level of detail that I am including here; in other instances, such detail was not required. In an effort to maintain the integrity of the submitted research protocol, I provide all of the meticulous details for each step in my process (see the Appendix for IRB correspondence and details). It is my hope that providing such specificity illustrates the degree of rigor that my research protocol contains in the data collection phase as well as the amount of triangulation that is possible in the data analysis phase.

Unfortunately, the COVID-19 pandemic has not been resolved (as of spring 2021), and neither has the nation-wide, social-distancing mandate been lifted in time for me to execute the IRB. Consequently, the impact of the COVID-19 pandemic on my dissertation research cannot be understated. In short, it was impossible for me to conduct my study with research participants. I had to rethink the entire research protocol submitted to IRB. The consensus of my dissertation committee was to unbox, so to speak, the reflective journaling, analytic memos, manual notes of student conferences, and the conceptual models that resulted from my mathematics tutoring and teaching experiences with middle school African American students and construct mathematical profiles and composites from this source data. The constructed mathematical profiles and composites of my many male and female students were to be as authentic as possible and needed to represent my experiences as best as I could recollect. Clearly, an empirical study based on singular participants and real-time data provides a greater level of authenticity and substance than a scientific project based on reflective journaling and profiles. However, both an empirical study and a theory-building scientific project provide the researcher with great opportunities for rigor and robustness.

Fortunately, it was agreed that I would use my teacher reflections and notes from the last 3 years which made this monumental effort more tenable and more authentic. Nonetheless, there is no replacement for having the real-time data from actual research participants. It would be improper to presume that I think that my note-taking ability provides greater insight into a person than that person's own experiences or that person's own perspectives. What I have to offer in what follows is the implementation of my research protocol that I submitted to IRB conducted on the profiles of students that I have tutored or taught over the last 3 years. Despite the construction of the theoretical paradigms of mathematical learning that are detailed in this chapter, I emphasize that the actual goal of this chapter is to use archived data to illustrate a simulation of my re-engineered teaching experiment and the utility of the data collection and data analysis charts created. But before I detail the simulation, I briefly review key discussions from each of the previous chapters.

Chapter Recapitulations

The purpose of my study was to construct a theory for the learning of mathematics by middle school African American students. The goal of such a theory is the establishment of a mathematics ecosystem that results in a cogent experience for mathematical proficiency which endures internal critique and external scrutiny. I have attempted to narrate my journey as I have worked toward this purpose. What has been discussed thus far suggests an amalgamation of a multitude of variables. In Chapter 1, the five strands of mathematics proficiency were emphasized: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. In Chapter 2, various forms of reasoning, personal experience, semantics, and metacognition were emphasized. In Chapter 3, CHAT, the semiosphere, semiotics, epistemology, symbolic interactionism, and radical constructivism were emphasized. In Chapter 4, I offered my definitions and perspectives of theory and theory-building, outlined the processes of theory-building, and detailed characteristics of a research methodology (i.e., the teaching experiment) that can facilitate the building of robust and rigorous theory. In Chapter 5, I discuss the fusion of CHAT with the teaching experiment. The result of such a fusion is a re-engineered teaching experiment that investigates not only the mathematics experience of the student but also investigates the characteristics of mathematics activity within an assignment, specifically its context, its constituent parts, and the role creation that it imparts onto the student. Such a fusion requires a plethora of experiential data instead of numerical data, and so the choice between a qualitative study or a quantitative study seemed clear to me.

Qualitative v. Quantitative

According to Merriam (2009), qualitative research is about experiences. It is about how the respondent constructs his or her experiences, how the respondent interprets his or her experiences, and what meaning he or she obtains from experiences. This type of information is important to gather. It would be a disservice to a student for an educator to simply know that the student performed poorly on an assessment. If an educator is committed to the charge of cultivating students who are knowledgeable of and able to navigate within the U.S. educational system, then the educator must obtain the insight that addresses the “how” and the “why” of the student’s learning experiences. These how’s and why’s are generated by research participants in the form of narratives (Teddlie & Tashakkori, 2009). In addition, these narratives are analyzed and interpreted most often using a constructivist perspective (Teddlie & Tashakkori, 2009). All in all, the key focus of qualitative research is privileging the individualization of the experience while emphasizing the context (Patton, 2002).

An alternative research approach would be a quantitative study where the foci are the "what" of the learning phenomenon and the "what" of the students' performances. These what's are gathered and represented in the form of numerical data (Patton, 2002; Teddlie & Tashakkori, 2009). In addition, the numerical data are analyzed and interpreted using most often positivist or post-positivist perspectives (Teddlie & Tashakkori, 2009). All in all, the key focus of quantitative research is privileging the generalization of the numerical data while de-emphasizing the context (Patton, 2002). At worst, a quantitative analysis informs the researcher that not all respondents respond the same on a survey, or that not all students learn the same within the same learning environment, or that not all students perform the same on the same standardized assessments. At best, a quantitative analysis provides the researcher insight into which respondents or students should be considered for a deeper and richer experiential and contextual analysis—a qualitative analysis.

There were two reasons that made it clear that the appropriate choice for this work was for it to be a qualitative project. The first reason was that my research questions centered on the experience of the student, from varying vantage points. The second reason was that the purpose of this study was to construct a theory that explicates the establishment of a mathematics learning environment that cultivates a cogent experience for mathematical proficiency. Patton (2002) describes qualitative research designs as naturalistic, taking place in "real-world" settings. Such designs allow the phenomenon under study to unfold "naturally in that it has no predetermined course established by and for the researcher such as would occur in a laboratory or other controlled setting" (p. 39). Naturalistic inquiry, according to Patton, "contrasts with controlled experimental designs where, ideally, the investigator controls study conditions by manipulating, changing, or holding constant external influences and where a limited set of outcome variables is measured" (pp. 39–40).

Although Patton (2002) distinguishes between a naturalistic study and a laboratory study, he also acknowledges that research study designs exist along a continuum where one end might be viewed as the perfect naturalistic study and the opposite end might be viewed as the perfect laboratory study. The reality of academic research studies is that the conditions, constraints, and research questions determine where on the continuum the actual study resides. In the case of my particular research study, for example, I introduced the participants to an instructional investigation, similar to what Patton described, and then I observed how they performed within the context of a learning environment. Due to the existence of an investigation, this study would not qualify as purely naturalistic. Likewise, due to the observations occurring within a natural learning environment, this study would not qualify as a pure laboratory experiment. Consequently, this study, and the teaching experiment in particular, I believe, lies along the continuum.

Presentation of Datasets

As I have mentioned previously, a large amount of data was used to complete this work. I was fortunate to be able to include my notes and reflections from my work with my classroom students, as well as my notes and reflections from my work with my tutoring students. I focused on my seventh- and eighth-grade students, and divided my data sets according to the binary labels of male and female for categorization purposes. The students self-identified the assigned label. I worked with my students individually, as well as in three-member groups, and my data sets reflect this fact. I provide a table below (see Table 6.1) that displays the various database categories that my archived data helped me to construct.

Table 6.1
Database Categories

	7th Grade		8th Grade	
Sex	*Individual*	*Group*	*Individual*	*Group*
Male	Exploratory CHAT Application CHAT Socratic Inquiry	Exploratory CHAT Application CHAT	Exploratory CHAT Application CHAT Socratic Inquiry	Exploratory CHAT Application CHAT
Female	Exploratory CHAT Application CHAT Socratic Inquiry	Exploratory CHAT Application CHAT	Exploratory CHAT Application CHAT Socratic Inquiry	Exploratory CHAT Application CHAT

For each grade level (seventh and eighth), composite profiles for two male students and two female students were constructed, resulting in composite profiles for a total of eight students who I tutored. Additionally, for each grade level, the composite profiles for one male three-member group and one female three-member group were constructed, resulting in composite profiles for a total of four three-member groups that I taught in my classroom. Due to space limitations, I do not provide details of all of these composite profiles here. I do, however, provide details of selected student–tutor and group–teacher happenings that are composite representations of my experiences and allow for an informative simulation of my re-engineered teaching experiment.

Eighth-Grade Female Student

In what follows, I provide the data for a female tutoring student. This dataset includes the CHAT dissection chart and the Socratic inquiry chart for each of the three partitions: concept exploration, concept introduction, and concept application. I changed the display format from a chart based to a text base in hopes that it eases the reading of the information. I also include my evaluative coding summary.

Week #1 – Data Collection Discussion Outline

CHAT Observations (Concept Exploration)

Mathematics Word Problem: The state fair is a popular field trip destination. This year, the senior class at High School “A” and the senior class at High School “B” both planned trips there. The senior class at High School “A” rented and filled 8 vans and 8 buses with 240 students. High School “B” rented and filled 4 vans and 1 bus with 54 students. Every van had the same number of students in it. Every bus had the same number of students in it. Find the number of students in each van and in each bus.

Student (pseudonym): Narnia (eighth-grade female student)

Syntax. Narnia quickly reads the problem and starts to write some things on her paper. Narnia does not gather any notes, so whatever she is writing she is able to do without referring to external notes. There is no indication that she is paying attention to the grammatical details of the sentences or the position of the words within the word problem.

Semantics. Narnia writes several of the numbers from the word problem on her paper. There is no clear indication of how she might use these numbers. She also includes on her paper what the numbers represent. This action is early in her problem-solving process, so I cannot discern her approach, yet.

Pragmatics. Narnia seems to be trying to randomly select numbers that would qualify for the requirements as detailed in the problem. I think Narnia will soon realize that once she figures out the appropriate numbers for one set of criteria (High School “A”), those same numbers must also qualify for the second set of criteria (High School “B”). I have to wait and see how this realization might affect her work.

Prior Knowledge/Prior Experience. At no time has Narnia referenced any notes. She, however, has been diligently working and writing on her paper. These actions suggest that Narnia has some command over her knowledge base to recognize which prior knowledge and prior experiences are relevant, and how they can most effectively be used. I have to wait and see how far in her problem-solving approach Narnia can progress before she needs to begins to reference notes of some form.

Rules/Constraints. Narnia's problem-solving approach of guessing at numbers that qualify for one set of criteria should be effective for her. I am, however, concerned that she may not have considered the secondary criteria that those same numbers have to simultaneously meet. As Narnia progresses through her selection of numbers, I will see if I can determine if she has had this insight, yet.

Mediating Tools. I see no evidence of any external tools that Narnia is using as she works through this problem. She has not yet accessed any notes.

Steps (Division of Labor). Thus far, the only steps that I can see from Narnia's problem-solving approach is to choose random numbers and determine which numbers satisfy the requirements as they are given in the word problem. If Narnia guesses at one number, she can solve an equation to determine the other number. I have to see how long she continues with this approach.

Solution/Quality. Narnia has not yet produced a solution to the problem that I can evaluate; however, her problem-solving approach is quite credible. Narnia may find that there are other approaches that are more efficient. But the approach that she has started with to solve this problem is a quality approach.

Socratic Inquiry (Concept Exploration)

Question Set #1 (Self-Improvement): What do I already understand about the given situation/problem?

Narnia: I know that this problem requires that I figure out which pairs of numbers satisfy the situation that is described in the given word problem. It's not a difficult problem to do at all, you just have to pick one number and then use an equation to calculate the second number.

Question Set #2 (Inductive Reasoning): What patterns or relationships do you notice may exist within topic/concept? Based on the given formula/equation, what types of values are not allowed?

What conditions are necessary?

Narnia: This entire problem is about working with equations. We've been working with equations for a couple of years now, so, this problem should be pretty straightforward for us. The word problem describes the relationship that has to exist between the numbers that you pick, so you just have to make sure that the pairs of numbers that you come up with satisfy what is described in the problem. For example, it wouldn't make sense to have one of the numbers be equal to zero, just based on what is described in the problem. Also, you can't have your numbers be fractions or decimals because that also wouldn't make sense based on what is described in the word problem.

Question Set #3 (Problem-Solving): What are the key points? What are the best steps to take when working this concept? What are you thinking as you solve this problem? Why do you feel that this approach has produced a correct solution?

Narnia: An important point in this problem is the connection between all of the numbers that they give you. If you can't see the connection, you probably won't be able to do this problem. Based on the problem-solving approach that I'm using, I just have to pick one number and then place that number in the equation in order to calculate the second number. Then, the two numbers make a pair of numbers that are answers to the problem. Then, I start again, and pick another number to start with. So far, with this approach, I've gotten many different pairs of numbers that are answers. I'll probably get a few more and then I'll be done. The problem doesn't say how many pairs I should find.

Evaluative Summary (Evaluation Coding)

Narnia is quite confident in her ability to solve this problem. She has developed a strategy that is effective for her. Narnia has a very positive attitude and is progressing through the problem without any issues. With her perspective, Narnia believes that she has already provided a numerical solution to this problem, and she is working to compile a list of acceptable number pairs. Narnia has not referenced any notes, but is working efficiently with her strategy. It is not clear, if she values notetaking or not.

Week #2 – Data Collection Discussion Outline

CHAT Observations (Concept Introduction)

Mathematics Word Problem: The state fair is a popular field trip destination. This year, the senior class at High School "A" and the senior class at High School "B" both planned trips there.

The senior class at High School “A” rented and filled 8 vans and 8 buses with 240 students. High School “B” rented and filled 4 vans and 1 bus with 54 students. Every van had the same number of students in it. Every bus had the same number of students in it. Find the number of students in each van and in each bus.

Student: Narnia (eighth-grade female student)

Syntax. I provide Narnia insight on the five various forms/representations of a system of equations: conceptually, pictorially, graphically, tabularly, and numerically. Since the equation of interest is a system of linear equations, I also discuss some of the more important characteristics of linear equation: slope, vertical intercept, horizontal intercept, slope-intercept form, and standard form.

Semantics. I give the meaning of “system” as it relates to equations. I also define equations just to ensure that Narnia has a firm understanding. After I show her the five various forms of a linear equation and a system of linear equations, I also show her how each of the characteristics of a linear equation manifests in each of the various forms.

Pragmatics. I demonstrate the three approaches to solving a system of linear equations problem: graphical, pictorial (substitution-method), and numerical (elimination-method). After I demonstrate each approach, I provide the details of each technique.

Prior Knowledge/Prior Experience. As I explain the structure, the meaning, the representations, and the techniques for system of equations, I make deliberate effort to connect any new information to Narnia’s prior knowledge and/or prior experience. I either ask her directly about such prior information or I presume her prior knowledge based on the math curriculum of prior years.

Rules/Constraints. I describe to Narnia the conditions under which a function or equation is linear. This insight should help her to identify linear equations in the future.

Mediating Tools. Each of the solution techniques should be considered as tools for Narnia when trying to solve a system of linear equations: graphically, pictorially (substitution-method), and numerically (elimination-method).

Steps (Division of Labor). I detail each the solution techniques multiple times and have Narnia take notes.

Solution/Quality. This iteration is an "information-session" for Narnia. As such, there are no additional practice problems given and no solutions to math problems to evaluate.

Socratic Inquiry (Concept Introduction)

Question Set #1 (Self-Improvement): What do I already understand about the given situation/problem?

What can be done to improve my thinking and my discussion about this situation/problem?

What can be done to improve my ability to solve this current situation/problem (e.g., faster with less mistakes)?

Narnia: Now that I've had a chance to get some notes on linear equations and solving linear equations, I see that I only understood one part of this problem. I didn't pay attention to the second relationship that is described in the problem. I also now know what the word "simultaneous" means, which is when two different relationships have to be satisfied at the same time. I wasn't doing this before. Being open to hear and see different approaches is really important to getting new ideas and making sure that you understand a whole situation.

Question Set #2 (Inductive Reasoning): What patterns or relationships do you notice may exist within this topic/concept? Based on the given formula/equation, what types of values are not allowed?; what conditions are necessary?

Narnia: If you really read the word problem, the two relationships are stated quite clearly. I just missed the second one when I was first doing the problem. I've never solved a system of equations before, so these approaches are new to me. But I still think that the final answer will be numbers that don't equal zero, and they won't be fractions or decimals. The numbers have to be positive whole numbers.

Question Set #3 (Problem-Solving): Based on your options, which option will you choose?

Why?

Narnia: Now that I understand this problem better, I think that I will choose to solve it using the numerical method (elimination method) because I like to work with equations. Although there may be a few more steps with the numerical approach, it is definitely less work than doing it the way that I had started to do it. Plus, I was doing it wrong when I started, so at least now, I have a way that I know will solve this problem.

Evaluative Summary (Evaluation Coding)

Narnia is displaying a positive attitude after determining that she was not solving the problem correctly and showing an appreciation for notetaking. In addition to learning a new concept, Narnia also learned several new techniques. Narnia is not resistant to learning new information and gaining new skills. These are valuable attributes to have when working with mathematics problems.

Week #3 – Data Collection Discussion Outline

CHAT Observations (Concept Application)

Mathematics Word Problem: The state fair is a popular field trip destination. This year, the senior class at High School “A” and the senior class at High School “B” both planned trips there. The senior class at High School “A” rented and filled 8 vans and 8 buses with 240 students. High School “B” rented and filled 4 vans and 1 bus with 54 students. Every van had the same number of students in it. Every bus had the same number of students in it. Find the number of students in each van and in each bus.

Student: Narnia (eighth-grade female student)

Syntax. Narnia appears to be paying slightly more attention to the structure of the word problem. It is clear that she understands what the problem describes based on her earlier work; however, now she seems to be investigating the details of the word problem as if she missed something the first time.

Semantics. Narnia starts to write equations on her paper. One of the equations is similar to the one that she had on her paper previously. Now, however, there is a second equation on her paper that I do not recall seeing before.

Pragmatics. Narnia is doing a lot of writing on her paper. It is not clear what problem-solving approach she is using to work through this problem. What is clear is that she is not solving this problem with the same approach that she first started with. Before she was working with only one equation; now she is working with two equations.

Prior Knowledge/Prior Experience. There is a clear distinction in the use of notes by Narnia currently, versus her lack of notes before. During her initial attempt at solving this problem, the approach that she was using seemed well crafted in her mind. This new approach that she is using does not seem to be organized in her mind, and she requires the use of notes to guide her through the process. This use suggests that this approach is still new to her.

Rules/Constraints. The rules and conditions necessary to execute this new approach may be located in Narnia's notes only; this would explain her absolute dependence on her notes. I think she is trying to avoid making any mistakes.

Mediating Tools. At present, all that I can see that Narnia is using is her notes. So for this problem, her notes should be classified as a tool.

Steps (Division of Labor). It is not clear what steps Narnia must follow after producing the two equations on her paper for this problem. Because I do not see a Cartesian coordinate graph on her paper, the use of the two equations will lead her to one of the other two approaches. I must watch Narnia carefully to determine her choice.

Solution/Quality. Although Narnia has not finished the problem, she has been using her notes with immense attention. I think whichever approach she selects will be the one most supported by her notes.

Socratic Inquiry (Concept Application)

Question Set #1 (Self-Improvement): What do I already understand about the given situation/problem?

What can be done to improve my thinking and my discussion about this current situation/problem? What can be done to improve my ability to solve this current situation/problem (e.g., faster with less mistakes)?

Narnia: I now understand that this problem is about Simultaneous Equations, and not just about a single equation. So, the problem-solving approach is slightly different because you need to be concerned about two different relationships at one time. I can improve myself by writing good notes and studying them. I have to keep an open mind so that I can constantly be learning.

Question Set #2 (Inductive Reasoning): As you read the given situation/problem, what main math concept do you think is being described?

Narnia: The main concept in this word problem is Simultaneous Equations. I think that it can also be called System of Equations. This problem only has two different relationships or equations that need to be worked through, but I think that you can have a lot more. It just depends on the problem that you're given.

Question Set #3 (Problem-Solving): What representative forms are best for this topic? What are you thinking as you solve this problem?

Narnia: If I had to pick a "best form," I would probably say that the graphical approach would probably be the best approach to take when solving a system of equations because this approach shows the actual graph of each equation. With graphs, you can really see what is going on with the equations. However, you can't get the exact answers to the problem using the graphical approach. You can only get an estimate. I'm kind of excited to be using the numerical approach (elimination method) so that I can see what happens with these equations, and how the actual answer is produced.

Evaluative Summary (Evaluation Coding)

Narnia is very disciplined in her movements. She studies her notes and then she makes the appropriate actions with the given problem and with the work that she has written on her paper. Although Narnia has discovered that her initial efforts were incorrect, she shows no signs of frustration or disappointment. It would seem that once the error in her thinking was identified and then the necessary corrective measures were brought to her attention, she did not internalize her mistake. Her mistake was an opportunity for her to learn. It is now clear that she values notetaking and the use of notes when solving problems. During her first attempt at solving the problem, she did not need to reference any notes because she was familiar with the problem-solving approach that she had decided to use. Conversely, when she is not familiar with the problem-solving approach that she has decided to use, she shows no resistance to depending on her notes.

Seventh-Grade, Three-member Male Classroom Group

The prior information focused on the microscheme because it involved only one student and focused on that student's learning (Menchinskaya, 1969). Because this study investigated attributes of both the individual student, groups of students, and of the learning environment, I also performed this data collection process with small groups of students in the classroom. The small classroom groups consisted of a collection of three-member male student and three-member female student groups. There were male and female groups for both the seventh and eighth grades for a total of four small classroom groups in all. Although over the years, I had compiled a wide assortment of group data from a variety of grade levels, I chose one male group and one female group for both the seventh and eighth grades here because I had data for my tutoring students who were also in the seventh and eighth grades.

The process that I used with the small groups was similar to the process that I used with my individual tutoring students except that I only conducted the observations, I did not conduct the Socratic inquiry sessions with the small classroom groups because it would not have been possible for me to hand-record notes on what each group member said. To facilitate identifying each group member, I gave each member a label in accordance with his or her verbal engagement in the problem-solving activity. The labels that I used were “most-vocal,” “moderately vocal,” and “least vocal.” This labeling system allowed me to effectively connect my observations of the group’s problem-solving efforts with the appropriate group member. To better store my analytic memos, I created a different chart that would be aligned to the elements of CHAT. This chart (see Figure 6.1) is like the other chart that I had created, but it has space for my analytic memos regarding each member of the group. I wrote down my observations of each student according to four of the elements from CHAT, and according to their syntactic work, their semantic work, and their pragmatic work. Completing the twenty-one locations in the chart was demanding work, but it made me pay closer attention to the intricacies and idiosyncrasies within each group, and not limit myself to only determining *what* the group did. I had to also evaluate *how* the group functioned across the seven categories. See Figure 6.2 for a representative example of this data collection process with the 3-member seventh-grade male group.

Group Data Chart
TEMPLATE

Concept: Week & Partition:

Who	(Syntax)	(Semantics)	(Pragmatics)	(Prior Knowledge & Prior Experience)	(Mediating Tools)	(Steps: Division of Labor)	(Rules/ Conditions)
Most Vocal							
Moderately Vocal							
Least Vocal							

Figure 6.1. Group data chart.

7th-Grade Male Group Data Chart
Part #1

Who	(Syntax)	(Semantics)	(Pragmatics)
Most Vocal	There's no indication that student (MV+) is paying any particular attention to the grammar structure of the given word problem. It doesn't seem that student (MV+) is consciously aware of the significance of the word placement within the word problem. Despite what seems as a lack of intentional focus of the functioning of the words, student (MV+) seems to be comprehending meaning from the words because student (MV+) is summarizing the situation described in the word problem.	Student (MV+) has begun writing the numbers that are given in the problem on her paper. It is not clear to me the meaning that student (MV+) has gained from these numbers because she has only written the numbers on her paper. There is no particular order to the numbers, nor is there an image or a model ascribed to the numbers. As I listen to her self-talk, it seems that the student (MV+) has a clear cursory understanding of the situation described by the word problem, but not necessarily the embedded relationships of the actions detailed in the word problem.	Student (MV+) seems to be convinced that using a proportion to solve this problem is the best approach and she has made her position known to the other members of the group. Although no leader or captain has been assigned, student (MV+) seems to have proclaimed herself as the captain based upon her tone with the other members of the group.
Moderately Vocal	I have no observable indicators that student (MV) is examining the word structure of the given word problem. This does not seem to inhibit her understanding of the given situation because she is engaged in an enriching discussion with student (MV+) about what is described in the word problem.	The conversation between student (MV) and student (MV+) seems to be balanced in terms of the contribution that each is making to the discussion. Student (MV) seems to have a similar and consistent understanding of the cursory situation described in the word problem. It is an interesting exchange between student (MV) and student (MV+), as student (MV) seems to be trying to explain the situation faster than student (MV+). Unfortunately, what the two members have not realized is that they have not detailed the underlying relationships that will allow them to actually solve the problem, and not only describe the situation.	Although it is not clear to me if student (MV) has determined her approach to this problem, she is assertively challenging the suggestion and the posture of student (MV+). Student (MV) has continuously asked student (MV+) to justify the decision to use a proportion to solve this problem. Student (MV+) seems to be dismissing the challenge from student (MV) by repeatedly saying, "Because I think so." Student (MV) is resisting this response and asking student (MV+) to show some sort of justification in her notes.
Least Vocal	I cannot make a determination if student (MV-) is investigating the word placement within the given word problem because student (MV-) is not speaking much. She seems to be listening to the back-and-forth between student (MV+) and student (MV)	The presence and contribution of student (MV-) seems to have gotten lost in the exchanges between the other two members of the group. Student (MV-) does not seem to be dis-heartened by those tense exchanges between the other two members of the group. I can only suspect that student (MV-) has not determined how to solve this problem because she is not writing anything on her paper. It also is not clear to me what student (MV-) understands about this problem because she is neither contributing to the discussion nor is she talking out loud about her meaning-making.	This student (MV-) is quietly listening to the disagreement between the other two members of the group. I have no evidence that student (MV-) had decided for herself an approach to use to solve this problem. Student (MV-) seems more attentive to the discussion between the other two members of the group.

Figure 6.2. Seventh-grade male group data chart.

Thematic Coding of Tutoring Students and Three-member Classroom Groups

In what follows, I detail my approach to coding the large quantity and assortment of data that accumulated from my approach in using the re-engineered teaching experiment. It is important to note that depending on the student, the responses and narratives that I collected from each student were different dependent on that particular student's progression through the teaching episode. As such, it simply was not possible for me to analyze each aspect of each student's experience from each particular stage of the teaching episode. Presented here, however, is the general approach that I took, regardless of where within the teaching episode the data was collected. Nevertheless, to be clear, the location from where the data was collected did frame the context of my findings and conclusions. I detail my coding approach below.

I borrow from Saldaña (2016) to present my interpretation of the activity of coding. Saldaña states, "When we reflect on a passage of data to decipher its core meaning, we are *de*coding [sic] when we determine its appropriate code and label it, we are *en*coding [sic]" (p. 5). Saldaña continues by explicating the use of the word code, by saying "coding will be the sole term. Simply understand that coding is the transitional process between data collection and more extensive data analysis" (p. 5).

In general, an approach to take when coding is to search for references to rituals, routines, roles, rules, and relationships within the narrative data (Saldaña, 2016). The approach that I took when I coded the narrative passages from the students resulted in my conducting a four-layer analysis. Layer one focused on the generation of codes, where the codes were action-oriented. Layer two focused on using the codes to generate categories, where the categories were concept-oriented. Layer three focused on using the categories to generate themes, where the themes were relationship-oriented. Last, layer four required that I analyze the themes and produce a representative theory that described and explained the students' overall experiences. With respect to the perspective of Saldaña, he would reduce my four-layer approach to two layers and refer to it as a two-cycle coding.

In addition to using data from the student to generate the categories and themes that resulted from the coding activity, analytic memos were also used. Analytic memos represent the brainstorming that I did during data collection and data analysis (Saldaña, 2016). Analytic memos represented my time-captured thoughts during fieldwork and coding. My analytic memos were helpful in guiding my attention as I completed the two-cycle coding approach as detailed by Saldaña. The inclusion of my analytic memos as source data provided me with a different vantage point to analyze and consider in addition to the work products received from the student work and my own observations.

Data Analysis Thematic Coding Procedure – Tutoring Students

With respect to data analysis, because of my view on the utility of generalizability in theory-building, I needed to analyze the data in as many different ways with as many different techniques as I could. This multiplicity required that I organize and coordinate the analyses to not just be simultaneous with the data collection but also to be sequential and generative for each other (Charmaz, 2014; Saldaña, 2016). According to the coding techniques as described by Saldaña (2016), I used the following techniques to analyze the data: attribute coding, descriptive coding, in vivo coding, process coding, concept coding, pattern coding, and theoretical coding. Before I began the data analysis process, I had no clear idea as to the path that the data would take me. Once I began collecting and evaluating the data, I realized the magnitude of the analysis task and what seemed like the endless directions that I could travel. When I inspected and reflected on the data as best I could and I cross-referenced what I did with what Saldaña describes in his text, I discovered that I had used an eclectic composite of the seven above-mentioned coding techniques, as untenable as that may sound. To help specify when and how I used each technique, I had to create the various analysis charts that I described previously (see Chapter 5).

During the analyses, I found the CHAT Code chart to also be effective in serving as a repository for the summary for the various analytic memos as advocated by Saldaña (2016). My first analytic step was to summarize my analytic memos from each of the nine areas of the CHAT Code Chart that I had amassed from my tutoring students. For each week, I compiled what I viewed were my most impactful and most general analytic memos from both my male and female tutoring students, and recorded them in a separate CHAT Code Chart. Due to the fact that I chose statements from within my analytic memos that I felt represented the most impactful and most general perspectives, in vivo coding was effective during this first analytic step (Saldaña, 2016). I have included the Week #1 seventh grade CHAT Code Summary Chart in Figure 6.3 as an indication of the type of information that is contained in the other five CHAT Code Summary Charts.

Male Contributions;
Female Contributions

7th-grade CHAT Code Summary Chart Week #1

Concept: Ratios & Proportions **Week: #1 – Investigations & Introduction**

Team/Individual	**Activity (Exploration, Introduction, Application {Inter/Extrap})**
	"Cedric and Doug each had an equal amount of money. After Cedric spent $35 and Doug spent $28, the ratio of Cedric's money to Doug's money was 2:3. How much money did each boy have at first?"

Syntax (Form & Function)	**Semantics (Mode & Meaning)**	**Pragmatics (Strategy & Implementation)**
No emphasis on the initial representation; *very reflective; intentional in her actions and not impulsive; reading and re-reading the problem; retrieves what look like notes from her binder*	Various aspects of the problem are confusing to the student; No model or equation created on the paper; *Contextualizing the notes*	Arithmetic Calculator; using some form of a "Guess-and-Check" approach; *seems quite organized in her approach*

Prior Knowledge/Experience
Experience Introvert; Resorted to his own "more familiar" strategies; *triggers some memory or understanding; model seems to serve as a record of what she already knows, understands,*

Rules/Constraints	**Mediation Tools**
only showing his calculations on his paper; *reflecting over her notes*	Still has not written out any equations or drawn any model of the problem; knows his basic arithmetic; *no indication that there is an awareness of a problem-solving process*

Steps (Division of Labor)
Impulse-over-Planning; Insistent on solving the problem in his head; It seems that he is brainstorming; *trying to search her brain for any experience that is vaguely similar to the given problem; student's Plan-of-Action*

Solution/Resolution	**Correct? Quality?**
Gotten frustrated and has guessed at the answer	No "Math-Path; only calculations on his paper

Figure 6.3. CHAT Code Summary Chart.

The normal font represents responses from my male tutoring students, and the italicized font represents statements from my female tutoring students.

I also placed these same summary points into the Grade Level First-Cycle Coding Chart for each individual student for each of the three weeks, for the concept exploration and the concept application partitions. I only used the data from the concept exploration and the concept application partitions because the student was actively working during these two partitions. I also used process coding of the feedback that I received from the students during the Socratic inquiry sessions (Saldaña, 2016). These process codes were also included in the Grade Level First-Cycle Coding Chart for each individual student for each of the concept exploration and the concept application partitions. I then used concept coding to attain a higher level of abstraction for each student's CHAT observations, Socratic inquiry sessions, concept exploration partition, and the concept application partition. I think that such aggregate abstractions from so many different perspectives of the student's language and actions allowed me to understand some of the foundational relationships which existed for the particular student. Last, I themed the data which meant that I had to produce a phrase or sentence that represented these foundational relationships (Saldaña, 2016). I have included the math concept, syntax, and semantics Grade Level First-Cycle Coding Charts for one of my seventh-grade female tutoring students in Figure 6.4, Figure 6.5, and Figure 6.6 as an indication of the type of information that is contained in the other seven Grade Level First-Cycle Coding Chart Charts.

7th-Grade female First-Cycle Coding Chart
"Math Concept Chart"

MATH CONCEPT	Exploration	Application	Aggregate
CHAT form (Etic perspective)	very reflective; intentional in her actions and not impulsive; reading and re-reading the problem; trying to search her brain for any experience that is vaguely similar to the given problem;	seems quite organized in her approach; triggers some memory or understanding; model seems to serve as a record of what she already knows, understands,	Reflective, Analytical; Model-building is record-keeping
Socratic Inquiry Form (Emic perspective)	None of the words are difficult to understand; review of some old information that I already knew and an introduction to some new information that I did not know; figure out how to mix my new information with my old information; drawing a picture of what is going on will show me some things	figure out why it's hard, then I can figure out what I need to do; sometimes the math concept is not as important as the actions in the problem; have an approach that allows me to show both	Mixing old and new information; Problem-solving concepts and actions
Aggregate	Transformative knowedge-base	Triggers model-drawing	Knowledge-evolving problem-solving representations
INVESTIGATION MACRO-CODE:	Knowledge-evolving problem-solving representations		

Figure 6.4. 7th-Grade Female First-Cycle Coding Chart – Math Concept Chart.

7th-Grade female First-Cycle Coding Chart
"Syntax Chart"

SYNTAX	Exploration	Application	Aggregate
CHAT form (Etic perspective)	going back and forth referencing the word problem	relationship between particular word types and their placement inside the model	Reflective model correspondence
Socratic Inquiry Form (Emic perspective)	trying this problem several times and thinking about it; problem-solving approach is just as important as the details in the problem; calculation comes at the very end of the entire process; analyze the words that are given in a word problem. They can actually help by describing what your model needs to show; "read" my model	Having a specific way of doing something helps to keep the person on track	Guided modeling represents comprehending and problem-solving
Aggregate	Critical decoding and encoding	Guided correspondence	Inferential modeling
SYNTAX MACRO-CODE:	Inferential modeling		

Figure 6.5. 7th-Grade Female First-Cycle Coding Chart – Syntax Chart.

7th-Grade female First-Cycle Coding Chart
"Semantics Chart"

SEMANTICS	Exploration	Application	Aggregate
CHAT form (Etic perspective)	has interwoven the knowledge and experience that she has generated on her own with the knowledge and experience that she generated through the guidance of another person; become proficient in their use based upon the number of times that she has practiced and failed with them; has a clear plan of action; constantly cross-references between the given problem and her model	a good model also can reveal what the next steps are for solving the problem; guided more by her own self-directed experience than by any presentation or guidance by someone; the calculations are the very last step	Evolving knowledge-base; Evolving proficiency; Evolving model; Self-directed problem-solving
Socratic Inquiry Form (Emic perspective)	figuring out what exactly makes it hard is kind of like a puzzle or a riddle; there is so much more to a math problem, then just doing a calculation and getting an answer; knowing the big picture tells me what to do next. So, I just need to follow the system; checking and double-checking my model with what the problem describes.	you've got to constantly be thinking about it and you have to be open to getting ideas from other people; . I just have to practice; I can see what to do next because the model is showing it to me.	Systematic big view with an attention to details; Reflection and guidance
Aggregate	Evolving awareness	Modeling is self-directed problem-solving	Evolving self-direction
SEMANTICS MACRO-CODE:	Evolving self-direction		

Figure 6.6. 7th-Grade Female First-Cycle Coding Chart – Semantics Chart.

Because the objective of this work was the development of a theory for the mathematical constructions of students, I followed Saldaña's (2016) prescription of performing a second cycle of coding. I used the same thematic statements that I generated at the end of the first-cycle of coding for each of the students in the two grades and performed pattern coding. Pattern coding allows for the attainment of a much higher degree of abstraction as the goal is to locate patterns in the thematic statements (Saldaña, 2016). After I generated the pattern codes for each student in each grade level, I then performed the last analytic step, theoretical coding. According to Saldaña, this last analytic step achieves the optimal level of abstraction necessary to generate a theory that not only represents each of the subordinate levels of abstraction but also represents the discrete elements of data that each individual student contributed. To achieve an optimal level of rigor, I extended Saldaña's suggestion in two ways. First, I performed theoretical coding on all of the patterns that resulted from each grade level, as well as on all of the patterns that resulted from each student. Then, I compiled each of those theoretical codes into a unifying statement. This unifying statement became my theory, if you will, from the data that I had compiled from my tutoring students. The resulting composite coding chart that contains all of the resultant thematic statements for the seventh-grade students is in Figure. 6.7.

7th-Grade Composite of Codes
"Individual Data Sets"

COMBINE ALL MACRO-CODES	Boy #1	Boy #2	Themeing (Boys)	Girl #1	Girl #2	Themeing (Girls)
Investigation & Concept	Arithmetic Juggler	Impulse	Impulsive Arithmetic	Knowledge-evolving problem-solving representations	Guidance through storytelling	Problem-solving story-telling
Syntax	Mathematics "shedding"	Accepting unknown trajectory	Shedding known mathematics and entering unknown mathematics	Inferential modeling	Meaning-making and action-taking process	Meaning-making and action-taking inferences
Semantics	Thinker	Under-values problem-solving through inferencing	Thinking through inferencing	Evolving self-direction	Notes allow for interpretive and innovative modeling	Self-directed interpretation and modeling
Themeing	Shedding calculations for comprehension	Evolving valuation of inferencing	From impulse to inferencing	Self-directed problem-solving inferences	Meaning-making and action-taking storytelling	Self-directed problem-solving storytelling

COMPOSITE 7th-grade STATEMENT:
From impulse to self-directed problem-solving storytelling

Figure 6.7. 7th-Grade Composite of Codes.

This unifying statement for the seventh-grade tutoring students was:

> from impulsive to self-directed problem-solving storytelling.

The resultant composite coding chart that contains all of the resultant thematic statements for the eighth-grade tutoring students is in Figure 6.8.

8th-Grade Composite Codes
"Individual Data"

COMBINE ALL MACRO-CODES	Boy #1	Boy #2	Themeing (Boys)	Girl #1	Girl #2	Themeing (Girls)
Investigation & Concept	Needs math triggers to envision replication route	Seeking relationships through contexts, concepts, and experiences	Seeking math triggers and replication routes through contexts, concepts, and experiences	Replication through triggers, comprehension, and notes	Embedded relationships are revealed through credible notes and credible understanding and lead to credible problem-solving	Replication and Problem-solving are revealed through relationships triggers, credible notes and credible understanding
Syntax	Case-specific notes do not cultivate deep understanding of mathematics	Note-taking, meaning-making and relationship-representing	Case-specific notes do not cultivate meaning-making and relationship-representing	Meaning-making and relationship-stating notes nurture a Mathematics state-of-mind and Plans-of-Action	Equation encoding and decoding; Multiple levels of understanding leading to credible Plans-of-Action	Meaning-making, and equation decoding and encoding nurture multiple levels of mathematical understanding and lead to credible Plans-of-Action
Semantics	Needs a note "taking and accessing" system that explains and emphasizes the relationships in mathematics not the actions of mathematics	Linking ideas and details through organized inferencing and mapping	Needs a note "taking and accessing" system that explains and emphasizes inferencing and mapping in mathematics	Limits in inferencing limit organized Plans-of-Action	Depth of knowledge registers situations and relationships and leads to credible Plans-of-Action	Limits in relationship-recognition limit credible Plans-of-Actions
Themeing	Case-specific notes result in math triggers and replication instead of a note "taking and accessing" system that explains and emphasizes the relationships in mathematics	Seeking meaning-making and relationships through note-taking, contexts, experiences, and modeling	Case-specific notes lead to math triggers and replication routes and do not cultivate a meaning-making, relationship-emphasizing, and mapping note "taking and accessing" system	Meaning-making and relationship-stating notes nurture Plans-of-Action and not replication	Multiple levels of understanding through credible notes and credible understanding lead to equation encoding and decoding and result in credible problem-solving	Limits in multiple levels of mathematical understanding diminish meaning-making, relationship-recognition, and equation decoding and encoding and result in replication instead of credible problem-solving

Composite 8th-grade Statement:
Case-specific notes lead to limits in multiple levels of mathematical understanding and do not cultivate a meaning-making, relationship-emphasizing, and equation-mapping note-taking system that allows for credible problem-solving

Figure 6.8. 8th-Grade Composite of Codes.

This unifying statement for the eighth-grade students was:

> Case-specific notes lead to limits in multiple levels of mathematical understanding and do not cultivate a meaning-making, relationship-emphasizing, and equation-mapping notetaking system that allows for credible problem-solving.

The final theoretical statement from all of the tutoring students' datasets was:

> A meaning-making, relationship-emphasizing, and equation-mapping note-taking system can guide a student from an impulsive student to a credible self-directed problem-solving storyteller.

Data Analysis Thematic Coding Procedure – Three-Member Classroom Groups

I used a similar analysis protocol for the data that I collected from my classroom three-member groups. Due to the fact that there were three students in a group, I found it too difficult to try to conduct the Socratic inquiry interviews and write down the various responses that came from each member. I found it to be more viable to give the group a mathematics problem to solve and then observe the group members working together to solve it. It was much more seamless for me to write down my observations of their collaborative efforts and my interpretations of their collaborative efforts. These were the analytic memos that I used in the data analysis of the male and female three-member groups. Recall, the chart for the Grade Group Data is in Figure 6.1. The analyzed data for the male and female groups from seventh-grade is located in Figure 6.9.

7th-Grade Group Data
Code Chart

	Syntax	Semantics	Pragmatics	Prior Knowledge or Prior Experience	Mediating Tools	Steps & Division of Labor	Rules/Conditions
Composite of 7th-Grade Male Group	There's no indication of an examination of the grammar structure of the given word problem	understands that this problem is about ratios; I don't see any models or images being drawn on his paper (student MV+); written the ratio on his paper: 2/3 (student MV); written the ratio as 2:3 on his paper (student MV-); seems to be reviewing his notes (students MV+ and MV-)	Three separate and different approaches taken in trying to solve the given problem.	There is no indication that students MV+ and MV are using anything other than their memory; seems to be committed to reviewing his notes (student MV-)	has constructed as a specific analytic tool (student MV+), but is unable to use this tool effectively; student (MV) has decided upon a different analytic tool, but he also seems to be having difficulty; does not seem to have started his attempt at solving this problem (student MV-).	These three students are not working together as a unit; They each seem to be working in their own way	does not seem to have the appropriate depth of understanding of the various approaches or of the notes to understand the appropriate conditions and constraints
Composite of 7th-Grade Female Group	There's no indication of any particular attention to the grammar structure of the given word problem.	begun writing the numbers that are given in the problem on her paper; there is no particular order to the numbers, nor is there an image or a model; has a cursory understanding of the situation, but not the underlying relationships that will not only allow for a description on the situtation, but also allow them to actually solve the problem; show some sort of justification in her notes; is listening to the disagreement between the other two members of the group.	no leader or captain has been assigned, but one seems to have proclaimed herself as the captain; a member is challenging the suggestion and the posturing	does not seem to be referencing her notes as she makes her decisions; only using her memory	There is no evidence of any tool being used	no evidence that this student has determined how her group members can assist her efforts to solve this problem; the members have not found agreement in the approach to use to solve this problem; a member does not engage much in conversation with them	a clear approach to solving this problem is not evident. Therefore, the appropriate pre-requisite conditions and guidelines will remain unaddressed

Figure 6.9. 7th-Grade Male and Female Groups Data Code Chart.

Again, I used a two-cycle coding approach because the objective was to build a theory from this group data. The first-cycle coding required the use of descriptive coding and concept coding. I used descriptive coding to compile the data of the various forms of language, behavior, and actions that I observed (Saldaña, 2016). I then used concept coding to establish another level of abstraction from the descriptive codes. Due to the charts that I created, I was able to gather all of the codes for the male group and the female group, for both grade levels, according to seven of the categories on the CHAT Code Chart. I did not use the last two categories, solution and quality, because I was most interested in the work environment of the group members, not the quality or accuracy of their answer. As detailed previously, a benefit of the chart that I created is that I was able to produce concept codes for each group type, male or female, as well as for each of the seven categories of the CHAT Code Chart. This multi-dimensional aspect facilitated triangulation when the next level of abstraction was performed. To achieve this next level of abstraction, I used the same theming the data (Saldaña, 2016) technique that I used for the coding of the data from my tutoring students. As discussed previously, theming the data allowed me to express an aggregate grade-level theme that I felt captured each of the concept codes that were generated by the data from both of the groups. The theme that resulted was the completion of the first-cycle coding.

After I completed the first-cycle coding protocol, I began the second-cycle coding protocol. I used the two separate sets of themes that resulted from the theming the data technique, one for both seventh-grade groups and one for both eighth-grade groups, and applied the pattern coding technique to produce yet another level of abstraction (Saldaña, 2016). This technique required that I look for any underlying patterns within the sets of themes that I had developed. The last step was for me to use the theoretical coding technique (Saldaña, 2016) to achieve the level of abstraction necessary to formulate a theory. The theoretical coding technique required that I capture the underlying relationships that I felt were dominant and comprehensive enough to describe not only the pattern codes but also the observations that I compiled from each group at each grade level (see Table 6.2).

Table 6.2
Group-level Second Cycle Coding

Grade Level (Boys & Girls)	Code
Seventh	1. Need a "problem-solving" heuristic in order to generalize problem-solving efforts 2. Need to establish a problem-solving culture of "system minded-ness," to achieve: (a) grammar analysis; (b) descriptive and inferential analysis of math concepts and their underlying relationships, conditions, constraints, and math problem-solving techniques; (c) representative "model-making"; (d) unanimous decision-making; (e) effective notetaking and note-referencing; (f) tool selection; (g) task delegation; and; (h) the establishment and demonstration of group cohesion (common language, productive discussions, general problem-solving approach, specific problem-solving plan, organized/coordinated actions) 3. Need a "solution-deriving" template
Eighth	1. Effective notetaking and note-referencing that reduces the dependency on memory and that highlights conditions and constraints of concepts and problem-solving tools/techniques 2. Grammar analysis 3. Effective situation-assessments that lead to representative model-making and equation-formation 4. Converting descriptive meaning-making and relationship-recognition into storytelling, personal experience narratives, and analogies 5. Effective decision-making 6. Effective group discussions 7. Effective identification and selection of mathematically appropriate and optimal problem-solving techniques/tools 8. Effective role-assignments 9. Effective task-delegation 10. Need a culture that cultivates: (a) a general problem-solving heuristic; (b) specific problem-solving protocol that emphasizes the reduction of big ideas into smaller "action-steps" and the accurate execution of the corresponding techniques and tool implementation (i.e., solution template); (c) preference of mathematical appropriateness and mathematical justification over memory and personal preference; (d) proficient "group-engagement" or "individual-effort"

The final theoretical statement, which resulted from the Group datasets was:

> A culture is needed that nurtures and cultivates a research environment that embodies inferencing, storytelling, model-making, synergy, and problem-solving.

Data Analysis for Metacognition of Tutoring Students and Three-member Classroom Groups

In an effort to be even more comprehensive and take further advantage of the large amount of data that this research protocol accumulated, I now provide a different type of an analysis that highlights the metacognitive influence that syntax and semantics can have on the cognitive development of the student. This effort was an activity in bricolage and intellectual arbitrage in order to assign yet another lens to the data in order to investigate if such a lens could evolve my initial theory even further (Harrison, 1997; Levis-Strauss, 1966). If such a theoretical evolution was found to be possible, then this research methodology and the described data collection and data analysis techniques would all advance further in demonstrating their theory-building potentiality.

Recall, I ended the literature review in Chapter 2 with a brief historical synopsis of metacognition. In that synopsis, I brought attention to Flavell and Wellman (1977) who are the often-cited authors on the topic of metacognition. Flavell (1976) is given credit for distinguishing the three different variable types for metacognition: (a) person variables, (b) task variables, and (c) strategy variables. Next, I highlighted the work of Garofalo and Lester (1985) who extended the work of Flavell into the field of mathematics problem-solving. Garofalo and Lester produced an analytic framework that includes four elements: orientation, organization, execution, and verification. Then, I brought attention to the relationships that exist amongst experience, semantic domain, prior knowledge, current knowledge, and meaning that were outlined by Resnick (1985). I ended the synopsis by presenting the work of Veenman and Spaan (2005) who outlined the constituent parts of metacognition: metacognitive knowledge and metacognitive skills. According to Veenman and Spaan (2005), metacognitive knowledge is the declarative knowledge one has about the interplay between personal characteristics, task characteristics, and available strategies in a learning situation. Metacognitive skills are concerned with the procedural knowledge that is required for the actual regulation of and control over one's learning activities. Based on my work in this study, it seemed a natural extension to consider metacognitive knowledge not only with regards to the declarative knowledge one has in general, but the declarative knowledge that one has about the previously discussed vocational roles of the researcher, analyzer, designer, executor, and critic. In like manner, it also seemed a natural extension to consider metacognitive skills not only with regards to the procedural knowledge one has in general but also the procedural knowledge that is required for the completion of the activities of the researcher, analyzer, designer, executor, and critic. In fact, it would seem that the more metacognitive knowledge and metacognitive skills one constructs for these vocational roles, then the more proficient the person becomes in mathematics, in general.

Metacognition Analysis Procedure – Tutoring Students

I first focused on the metacognition analysis procedure for my tutoring students; I then present the analysis for my three-member classroom groups. I have chosen one male tutoring student and one female tutoring student from each grade level to demonstrate the metacognitive analysis. The eighth-grade female student that I present here is the same eighth-grade female student, Narnia, that I evaluated earlier in this chapter. I decided to present the data that focused on the concept application partition of the teaching experiment for all three weeks, and the concept exploration partition of the first week so that I could investigate any cognitive development that may have occurred across the three weeks. I included the concept exploration partition in Week #1 so that I could investigate the student's initial cognition before any instruction from me. Recall, that during Week #1, the focus was the general math concept; Week #2 focused on the syntax of the word problem; and Week #3 focused on the semantics of the word problem. As I stated earlier, although my extensions and investigations into the syntax and the semantic layers of a math concept may only invite the interest of a small group of people, I feel that these perspectives can provide valuable insights into the mathematics learning of students.

> **Seventh-grade mathematics word problem:** Cedric and Doug each had an equal amount of money. After Cedric spent $35 and Doug spent $28, the ratio of Cedric’s money to Doug’s money was 2:3. How much money did each boy have at first?

Alvin (seventh-grade male tutoring student). Alvin self-identified as a male. Based on a conversation with his parents, at the time of our tutoring sessions, Alvin had a score of 85 in his mathematics and language arts classes. When asked about his feelings toward school, Alvin stated that he did not have much interest in school. I begin by discussing the person variable as described by Flavell (1976). I found that the person variable could be used to refer to the syntax, semantics, pragmatics elements of the CHAT chart, as well as the responses to the Socratic *i*nquiry questions. Based on the analytic memos of my observations during Week #1, Alvin did not give any observable attention to the syntactic elements of the word problem. Due to his use of the paper for the purpose of performing arithmetic calculations, I did not have insight into his problem-solving strategy or “how” he arrived at the calculations that he was conducting on his paper. Based on his responses to the Socratic *i*nquiry questions, Alvin was rather committed to the one and only way that he knew to solve ratio problems, although that approach proved to be unsuccessful. His further responses, combined with his lack of using the paper for anything other than calculations and him not “thinking aloud” to provide me access to his thinking, convinced me that initially, Alvin did not have prior knowledge, prior experience, or the recollection thereof necessary for working with problems of this type (Resnick, 1985). Even after my instruction of the mathematics concept, Alvin still struggled with this problem and returned to his prior approach. It was clear that this ratio problem required that he extend his understanding beyond the calculational aspect of ratios and into the conceptual aspect of ratios. In sum, during his initial efforts with the given word problem, Alvin's metacognitive knowledge was limited in its ability to achieve the necessary orientation of the problem and this limitation restricted his available task variables and strategy variables. Consequently, his organization of these variables was also adversely impacted which led to the truncation of his metacognitive skills, leading to his eventual inability to generate and execute a plan to successfully solve the problem (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

During Week #2, I provided information to Alvin that focused on syntax and its potential benefits to better understanding word problems. Based on the analytic memos of my observations during Week #2, Alvin did give observable attention to the syntactic elements of the word problem. In fact, he successfully identified the various parts of speech that existed within the problem. His successful parsing of the word problem led Alvin to use a modeling technique to represent the situation that was described by the word problem. This modeling technique was presented to Alvin during the session in Week #1, and he seemed to have recalled it during Week #2, and became more effective with it after the instruction on syntax. Now, due to his successful grammar analysis, Alvin used the paper to draw a model that showed his new understanding of the given situation. His model gave me insight into his thinking of the situation and insight into his potential problem-solving approach. Based on his responses to the Socratic inquiry questions, Alvin was no longer committed to the one and only way that he knew to solve ratio problems; he now had a tool that he realized could provide him greater insight into the given problem. His further responses indicated that although Alvin did not have prior knowledge or prior experience with grammar analysis, he was able to integrate this new knowledge into his schema (Resnick, 1985). Such an integration combined with the model that he had drawn on his paper convinced me that understanding the syntactic nature of the word problem extended his understanding of ratios beyond the calculational aspect and into the conceptual aspect. In sum, during his Week #2 efforts with the given word problem, Alvin's metacognitive knowledge was expanded in its ability to achieve the necessary orientation of the problem and this expansion increased his available task variables and strategy variables. Consequently, his organization of these variables was also positively impacted which led to the enhancement of his metacognitive skills. Although Alvin did not solve the problem in the time that I gave to him, he seemed pleased with what he was able to execute and achieve (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

Lastly, during Week #3, I provided information to Alvin that focused on semantics and its potential benefits to better understanding word problems. Based on the analytic memos of my observations during Week #3, Alvin again gave deliberate and observable attention to the syntactic elements of the word problem. Although this attempt was only his second time working with this problem after having been introduced to syntactic analysis, Alvin quickly identified the various parts of speech that existed within the problem. Again, his successful parsing of the word problem led to his successful drawing of a representative model. Now, Alvin began to make the cognitive transitions from "what is being described" to "what meaning is being made" and to "what is being asked." Based on his responses to the Socratic inquiry questions, Alvin was comfortable using the grammar analysis approach that he had integrated into his schema. In addition, Alvin now had a positive experience with using the information that he gained from the grammar analysis to create a representative model of the given situation. During Week #3, Alvin's cognitive ability had evolved to the extent that he was reading his model to find any underlying relationships that could help him to solve the problem. Based on his work during

Week #3, Alvin now had two cognitive tools that he could use to better understand word problem.

In sum, during his Week #3 efforts with the given word problem, Alvin's metacognitive knowledge was expanded even further in its ability to achieve the necessary orientation of the problem and this further expansion increased his available task variables and strategy variables. Consequently, his organization of these variables was also more positively impacted which led to the further enhancement of his metacognitive skills. Unfortunately, Alvin did not solve the problem in the allotted time, but he was able to develop and execute an actual problem-solving approach that propelled him closer to arriving at a solution. It is clear from this 3-week analysis that Alvin's metacognitive knowledge and metacognitive skills both increased substantially after integrating the semiotic sub-elements of syntax and semantics into his mathematics learning (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

Jayla (seventh-grade female student). Jayla self-identified as a female. Based on a conversation with her parents, at the time of our tutoring sessions, Jayla had a score of 87 in her mathematics and language arts classes. When asked about her feelings toward school, Jayla stated that she enjoyed school and had a high interest in school. I begin my analysis by discussing the person variable as described by Flavell (1976). Based on the analytic memos of my observations during Week #1, Jayla did not give any observable attention to the syntactic elements of the word problem. She did, however, seem to be reading and re-reading the word problem. Initially, although Jayla seemed quite reflective over the word problem, she only wrote the fractional number "2/3" on her paper. Consequently, I did not have much insight into her problem-solving strategy, but the writing of the fractional number on her paper did reveal to me that she had prior knowledge and prior experience of representing ratios as fractions (Resnick, 1985). In the given word problem, the ratio was listed as “2:3”, which is not its fractional form. Based on her responses to the Socratic inquiry questions, Jayla was intent on understanding the problem. She recognized key features within the problem, and also recognized that she had not had prior experience with ratio problems that contained such special features. In this case, her further responses combined with her only use for the paper was to write down the fractional number 2/3, and her not “thinking aloud” to provide me access to her thinking convinced me that Jayla did not have prior knowledge, prior experience, or the recollection thereof of working with problems that contained features of this type (Resnick, 1985). Although initially, neither Alvin nor Jayla showed much work on their papers, it is clear from their responses to the Socratic inquiry questions that their minds were working in different fashions. Alvin was focused on trying to fit the information in the problem into his prescribed strategy, while Jayla was focused on trying to understand the problem in order to develop a strategy. It was clear that this ratio problem extended Jayla’s conceptual understanding beyond the conceptual idea of the general features of a ratio problem into the conceptual aspect of ratio problems containing special features. After my instruction on the mathematics concept, Jayla did make efforts to draw a representative model on her paper. Unfortunately, she was unsuccessful in her many attempts. In sum, during her initial efforts with the given word problem, Jayla’s metacognitive knowledge was limited in its ability to achieve the necessary orientation of the problem and this limitation restricted her available task variables and strategy variables. Consequently, her organization of these variables was also adversely impacted which led to the truncation of her metacognitive skills. New insight however did lead to her eventual ability to generate a new plan, but she was unable to execute the plan to successfully solve the problem (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

During Week #2, I provided information to Jayla that focused on syntax and its potential benefits to better understanding word problems. Based on the analytic memos of my observations during Week #2, Jayla did give observable attention to the syntactic elements of the word problem. In fact, she also successfully identified the various parts of speech that existed within the problem. Her successful parsing of the word problem led Jayla to use a modeling technique to represent the situation that was described by the word problem. This modeling technique was also presented to Jayla during her session in Week #1, and she also seemed to have recalled it during Week #2. Now, due to her successful grammar analysis, Jayla used the paper to draw a model that showed her deeper understanding of the given situation. In fact, Jayla drew two models. During the session in Week #1, Jayla realized that there was an aspect of time that the problem described. There was a "before" aspect, then an event occurred, and then there was an "after" aspect to the problem. During the Week #2 session, Jayla was able to combine her awareness of the time aspect in the problem with her new focus on the grammar of the sentences to guide her effort to draw a representative model. Her attempts at model-drawing gave me insight into her thinking of the situation and insight into her potential problem-solving approach. Based on her responses to the Socratic inquiry questions, Jayla was more focused on developing a strategy for solving the problem then actually solving the problem. She seemed to understand that although there are an infinite number of mathematics problems, there does not need to be a unique way to solve each of them, and that there is not one calculational approach that solves each of them. Jayla seemed to have a much broader awareness that first having a strategy to understand the situation that the problem describes is more important than immediately trying to solve the problem. Jayla seemed to have found value in using the syntax of the word problem to enhance her understanding of the word problem. Her further responses indicate that although

Jayla did have prior knowledge and prior experience with grammar analysis, she was not aware of its utility in mathematics. She was now able to integrate this new knowledge into her schema (Piaget, 1936/1952; Resnick, 1985). Such an integration combined with the two models that she had drawn on her paper convinced me that understanding the syntactic nature of the word problem enhanced her understanding of the conceptual nature and features of ratios. In sum, during her Week #2 efforts with the given word problem, Jayla's metacognitive knowledge was expanded in its ability to achieve the necessary orientation of the problem and this expansion increased her available task variables and strategy variables. Consequently, her organization of these variables was also positively impacted which led to the enhancement of her metacognitive skills. Although Jayla did not solve the problem in the time that I gave to her, she seemed comfortable and confident with what she was able to execute and achieve (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

Lastly, during Week #3, I provided information to Jayla that focused on semantics and its potential benefits to better understanding word problems. Based on the analytic memos of my observations during Week #3, Jayla again gave deliberate and observable attention to the syntactic elements of the word problem. As she had worked multiple times with this problem, Jayla quickly identified the various parts of speech that existed within the problem. Her efficient parsing of the word problem convinced me that both her metacognitive knowledge and metacognitive skills with respect to syntax had increased. Additionally, the correct parsing of the word problem also led to Jayla successfully drawing a model that represented the two separate time aspects of the situation, as well as the event which occurred in between the two time periods. Jayla seemed to have constructed a level of cognition that exceeded that of Alvin. Not only was Jayla successfully able to make the cognitive transition from "what is being described" to "what meaning is being made" and "what is being asked" but also able to make the cognitive transition to "how do the existing relationships guide me to solve the problem." Based on her responses to the Socratic inquiry questions, Jayla was comfortable using the modeling approach that she had integrated into her schema. In addition, Jayla learned so much from making several attempts at drawing different models of the problem that she was ultimately able to draw one unifying model that contained all of the key features described within the problem. During Week #3, Jayla's cognitive ability had evolved to the extent that she was using the relationships detailed in her model to help her write the equation necessary for her to solve the problem. This evolution was a level of development further than what Alvin was able to achieve. Based on her work during Week #3, Jayla now had the cognitive tools that she felt she need to solve the word problem.

In sum, during her Week #3 efforts with the given word problem, Jayla's metacognitive knowledge was expanded even further in its ability to perform the necessary orientation of the problem and the question at hand; this further expansion increased her task variables and strategy variables. Consequently, her organization of these variables was positively impacted which led to the further enhancement of her metacognitive skills. Unfortunately, Jayla also did not solve the problem in the allotted time, but she was able to develop and execute an actual problem-solving approach that emphasized the value of a representative model, and this insight propelled her closer to arriving at a solution. After integrating the semiotic sub-elements of syntax and semantics into her mathematics learning, it is clear from this 3-week long analysis that Jayla's metacognitive knowledge and metacognitive skills both increased substantially, even beyond Alvin's development (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

> **Eighth-grade mathematics word problem:** The state fair is a popular field trip destination. This year, the senior class at High School "A" and the senior class at

> High School “B” both planned trips there. The senior class at High School “A” rented and filled 8 vans and 8 buses with 240 students. High School “B” rented and filled 4 vans and 1 bus with 54 students. Every van had the same number of students in it. Every bus had the same number of students in it. Find the number of students in each van and in each bus.

Lemuel (eighth-grade male student). Lemuel self-identified as a male. Based on a conversation with his parents, at the time of our tutoring sessions, Lemuel had a score of 65 in his mathematics class, and a score of 75 in his language arts class. When asked about his feelings toward school, Lemuel stated that he had a moderate interest in school. As I did for the prior two tutoring students, I begin by discussing the person variable as described by Flavell (1976). Based on the analytic memos of my observations during Week #1, Lemuel did not give any observable attention to the syntactic elements of the word problem. In fact, it was impossible for me to determine what aspects of the problem drew his attention because he did not write anything on paper and I did not hear him say anything. From my perspective as an observer, Lemuel looked to be just sitting and looking at the problem. Based on his responses to the Socratic inquiry questions, Lemuel was rather overwhelmed with the fact that he had no understanding of the problem and no recollection of ever having any experiences with such a problem. This sense of being overwhelmed suggested to me that Lemuel had a preconceived notion that the math problems that he would be assigned to solve would be math problems that he had experienced in the past. The idea that mathematics includes seeking meaning through relationships far exceeded his perspective that mathematics is a container of previously-solved problems that he must be given access. His further responses combined with his lack of using the paper and him not “thinking aloud” to provide me access to his thinking convinced me that Lemuel did not have prior knowledge, prior experience, or the recollection thereof of working with problems of this type (Resnick, 1985). Even after my instruction on the mathematics concept, Lemuel’s perspective on mathematics did not move too far beyond his initial stance because he was trying to duplicate the sequence of steps that I had demonstrated without adapting the approach to his specific problem. It was clear that this ratio problem was beyond his current level of mathematics understanding. In sum, during his initial efforts with the given word problem, Lemuel's metacognitive knowledge was unable to provide him with any guidance for achieving an orientation of the problem and this restriction removed the task variables and strategy variables from his use. Consequently, the elimination of these variables led to the disengagement of his metacognitive skills. Lemuel seemed convinced that replication was a viable plan although he would not achieve the cognition and metacognition necessary to become a self-guided learner (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

During Week #2, I provided information to Lemuel that focused on syntax and its potential benefits to better understanding word problems. Based on the analytic memos of my observations during Week #2, Lemuel did give observable attention to the syntactic elements of the word problem. He was able to successfully identify the various parts of speech that existed within the problem. His successful parsing of the word problem led Lemuel to then attempt to create representative equations for the situation described in the problem. This equation forming technique was presented to Lemuel during the session in Week #1, and he seems to have recalled it during Week #2. Now, due to his successful grammar analysis, Lemuel used the paper to write out the equations that showed his new understanding of the given situation. His equations gave me insight into his thinking of the situation and insight into his potential problem-solving approach. Based on his responses to the Socratic inquiry questions, Lemuel seemed to have thought that the information that he received from his teachers was exhaustive of what was available to know about a particular topic. In addition, Lemuel seemed to have thought that if he simply used the information that I gave him without alteration that he would be able to solve the given problem. Both of these perspectives seemed to suggest that Lemuel did not seek to understand the concepts behind the information that he gained, and that Lemuel did not seek to integrate any new insight into his schema. This lack of integration combined with the further struggles with what he had written on his paper convinced me that his prior understanding had not been extended into a more conceptual awareness of either syntax or the mathematics concept of System of Equations. In sum, during his Week #2 efforts with the given word problem, Lemuel's metacognitive knowledge did not seem to expand in any observable way. This static state of his metacognitive knowledge did not change his ability to achieve the necessary orientation of the problem. The static state of his metacognitive knowledge also did not change his available task variables and strategy variables by any discernible degree. Consequently, his organization of these variables was impeded which did not lead to any observable change in his metacognitive skills. Lemuel did not solve the problem in the time that I gave to him, and he seemed frustrated with what he was able to execute and achieve (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

Lastly, during Week #3, I provided information to Lemuel that focused on semantics and its potential benefits to better understanding word problems, and implementing a problem-solving approach. Based on the analytic memos of my observations during Week #3, Lemuel again gave deliberate and observable attention to the syntactic elements of the word problem. As this attempt was his second time working with this problem after having been introduced to syntactic analysis, Lemuel quickly identified the various parts of speech that existed within the problem. His successful parsing of the word problem led to the successful creation of representative equations. It was actually during Week #2 that Lemuel began to understand that knowing “what” to do is not the same as knowing “how” to do. Fortunately for Lemuel, some of the material that I shared with him during Week #3 was a review of some of the material that I shared with him during Week #1. A positive cognitive transition for Lemuel was that he could now perform a different valuation of the review material, as he now had a line of inquiry, according to his prior knowledge and prior experience, that made the material more relevant to him (Resnick, 1985). Based on his responses to the Socratic inquiry questions, Lemuel had realized that he needed to begin to view his notes as a useful and necessary tool in his learning. In addition, Lemuel now understood that he should not depend on his memory to guide him through a math problem. During Week #3, Lemuel’s cognitive ability had evolved to the extent that he was able to acknowledge that his study habits and notetaking skills needed to improve, and he could articulate how both needed to improve. Based on his work during Week #3, although Lemuel gained new cognitive tools that he could use to better understand and solve word problems, I actually feel that Lemuel gained more insight about his own self-efficacy and self-management processes.

In sum, during his entire 3 weeks of working with the given word problem, Lemuel’s metacognitive knowledge was activated to the point that he became conscious of his cognitive strengths and weaknesses. He then became aware of the need to orient the details provided within a problem. Such an orientation could then increase his available task variables and strategy variables. Subsequently, his organization of these variables would then allow for a positive impact on his metacognitive skills. Although Lemuel did not solve the problem in the allotted time, he was able to learn aspects of his learning, in general, and aspects of his mathematical learning, in particular, which were more enlightening and beneficial to him. Lemuel also came to understand that the creation of a plan and the execution of the plan are not the same occurrence. Although in a much different way, it is clear from this 3-week analysis that Lemuel's metacognitive knowledge and metacognitive skills both increased substantially after integrating the semiotic sub-elements of syntax and semantics into his mathematics learning. In Lemuel's case, this increase in metacognition was in his knowledge of self (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

Narnia (eighth-grade female student). Narnia self-identified as a female. Based on a conversation with her parents, at the time of our tutoring sessions, Narnia had a score of 88 in her mathematics and language arts classes. When asked about her feelings toward school, Narnia stated that she had a high interest in school. I begin, as I have for the other three tutoring students, by discussing the person variable as described by Flavell (1976). Based on the analytic memos of my observations during Week #1, Narnia did not give any observable attention to the syntactic elements of the word problem. Due to her immediate use of the paper for the purpose of solving mathematical equations that she generated, I had immediate and clear insight into her problem-solving strategy. Narnia mis-understood the depth of the problem and did not recognize all of the criteria that had to be met. Consequently, her approach was overly simplified and required an inordinate amount of time to complete. Narnia, however, was convinced that she was using an effective and efficient approach. In crafting this approach, she did not reference any notes. This lack of referencing suggests that she has a good awareness of her metacognitive knowledge and metacognitive skills through her prior knowledge and prior experience (Resnick, 1985). Based on her responses to the Socratic inquiry questions, Narnia was convinced that she knew how to solve the given problem. Her further responses combined with her extensive use of the paper to solve self-generated equations convinced me that Narnia had prior knowledge, prior experience, and the recollection thereof of working with problems of this type (Resnick, 1985).

Unfortunately, this system of equations problem was viewed by Narnia to be equivalent to solving a single equation, an acceptable initial thought considering that she had not yet been introduced to equations that needed to be solved simultaneously. After my instruction on the mathematics concept, Narnia made two adjustments. She now created two equations to represent the problem, and no longer attempted to solve the problem with only one equation. She also paid great attention to her notes. In sum, during her initial efforts with the given word problem, Narnia's metacognitive knowledge was active in guiding her in performing an orientation of the problem. This active guidance also guided her use of task variables and strategy variables. Narnia's organization of these variables facilitated the use of her metacognitive skills and led to her eventual generation and execution of a plan that she thought would solve the problem. Unfortunately, her approach was incorrect; however, she did demonstrate an ability to adapt based on new insight (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

During Week #2, I provided information to Narnia that focused on syntax and its potential benefits to better understanding word problems. Based on the analytic memos of my observations during Week #2, Narnia did give observable attention to the syntactic elements of the word problem. In fact, she successfully identified the various parts of speech that existed within the problem. After successfully parsing the word problem, Narnia began closely examining her parsing and comparing it to the equations that she had previously created. I interpreted this close examination as an effort by her to relate the position of a word in the word problem with the creation of the respective equation. Based on her responses to the Socratic inquiry questions, Narnia became aware of the different levels of comprehension that are available with a math problem. In her view, the more levels of comprehension that one has, the more ways that exist to solve the problem. With the introduction to syntax and the new level of comprehension for a word problem that it offers, Narnia now had a tool that she realized could provide her greater insight into such problems. Narnia’s further responses indicated that although she did not have prior knowledge or prior experience with grammar analysis, she was able to integrate this new knowledge into her schema (Piaget, 1936/1952; Resnick, 1985). Such an integration combined with the equations that she had written on her paper convinced me that understanding the syntactic nature of the word problem extended her understanding of a system of equations beyond the mathematical aspect and into the linguistic aspect.

In sum, during her Week #2 efforts with the given word problem, Narnia’s metacognitive knowledge was expanded in its ability to achieve the necessary orientation of the problem and this expansion increased her available task variables and strategy variables. Consequently, her organization of these variables was also positively impacted which led to the enhancement of her metacognitive skills. Although Narnia did not solve the problem in the time that I gave to her, she seemed pleased with what she was able to execute and achieve (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

Lastly, during Week #3, I provided information to Narnia that focused on semantics and the benefits that it offers to constructing meaning from word problems. Narnia also learned that having a clear understanding of the problem and a problem-solving technique allowed for efficient progress toward a solution. Based on the analytic memos of my observations during Week #3, Narnia again gave deliberate and observable attention to the syntactic elements of the word problem. As this was her second time working with this problem after having been introduced to syntactic analysis, Narnia quickly identified the various parts of speech that existed within the problem. She was also able to identify the relationships between word position in the word problem and the development of her equations. This identification allowed Narnia to successfully make the cognitive transitions from "what is being described" to "what meaning is being made" and to "what is being asked." Based on her responses to the Socratic[26] inquiry questions, Narnia was comfortable using the grammar analysis approach and the insight that it offered her with regards to the relationships amongst the words. In addition, Narnia had the experience of building mathematical equations using the insight gained from the syntax of the word problem. During Week #3, Narnia's cognitive ability had evolved to the extent that she was interpreting both the syntax and her equations to establish the underlying relationships that could help her to solve the problem. Based on her work during Week #3, Narnia was able to enhance her notetaking skills and incorporate all of these cognitive tools into her schema.

In sum, during her Week #3 efforts with the given word problem, Narnia's metacognitive knowledge was expanded even further in its ability to achieve the necessary orientation of the problem and this further expansion increased her available task variables and strategy variables. Consequently, her organization of these variables was also more positively impacted which led to the further enhancement of her metacognitive skills. Narnia was quite happy that she successfully completed the problem and produced the correct answer. Her focus, however, did not stay fixed on her completed problem; she soon began studying her notes so that she could effectively use the same problem-solving technique in the future without the same level of dependence on the notes that was necessary to solve this problem. In solving this problem, Narnia was able to develop and execute an actual problem-solving approach that propelled her to a solution. It is clear from this 3-week analysis that Narnia's metacognitive knowledge and metacognitive skills both increased substantially after integrating the semiotic sub-elements of

[26] It should be noted that although the use of Socratic inquiry as detailed in this project was an effective investigative tool, its use can be enhanced in future projects by incorporating lines of inquiry that investigate not only the cognitive aspects of the activity, but also the cultural and historical aspects of the individual.

syntax and semantics into her mathematics learning (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005).

My overall conclusion from these four tutoring students is the following. In a general sense, these four students present data that suggest that the incorporation of the semiotic sub-elements of syntax and semantics into mathematics problem-solving enhances metacognitive knowledge and metacognitive skills. What I found most enlightening is that my intentional effort to separate the mathematics problem into its constituent parts of situation, question, and strategy, allowed me to gain greater insight into how metacognitive knowledge and metacognitive skill can impact a student's problem-solving efforts. More specifically, these students' actions allowed me to better understand the roles that metacognitive knowledge and metacognitive skill play in guiding the student through all three of the constituent parts. In fact, the specifics of metacognitive knowledge and metacognitive skill seem to be different within each of the constituent parts. This difference may be due to the fact that the person variables, task variables, and strategy variables are themselves different within each of the constituent parts (Flavell, 1976, Garofalo & Lester, 1985; Veenman & Spaan, 2005). This finding is consistent with the before-mentioned development from my teacher–student conferences that when the given mathematics activity was dissected into the constituent vocational tasks of researcher, analyzer, designer, executor, and critic, then the progression and performance of my students increased. So, from this perspective, it would not be a coincidence that the person variables, task variables, and strategy variables of metacognition would therefore be different for each vocational task given that each task is assigned a different identifying label with its own unique objective. To be clear, what I am describing is a three-way division of a given mathematics activity into its situation, question, and strategy pieces; and a five-way division of a given student's actions into the researcher, analyzer, designer, executor, and critic roles; and then making the explicit statement that each role has a function within each piece of the given mathematics activity. In addition, the utility of the semiotic sub-elements of syntax and semantics provides greater clarity and focus for the objectives and discursive practices of each of these roles within each piece of the given mathematics activity. Therefore, my resulting theoretical statement is as follows:

> Semiotics extends one's depth of experience and facilitates the development of metacognitive knowledge and metacognitive skills.

Metacognition Analysis Procedure – Three-member Classroom Group

Now that the metacognition analysis of my observations, my interpretations and of the sentiments of my tutoring students was complete, I needed to perform this same metacognition analysis on my three-member classroom groups. Performing such an analysis for a group of students is a considerably different task because metacognition is regarded as a phenomenon which exists within one individual person, not a group of people (Flavell, 1976). In the effort to address this particular challenge, I determined that I may have success at evaluating both metacognitive knowledge and metacognitive skill from a group perspective if I attempted to examine the activities which make up or represent metacognitive knowledge and metacognitive skill. If I could, in fact, specify such activities, then I could use CHAT to guide my investigation whereby I would place my focus on the experience of community or group in the activity instead of on the experience of the subject. In short, my effort was not to investigate metacognitive knowledge or metacognitive skill per se for my classroom groups. But rather, to investigate the interactions of the members of my classroom groups as they attempted the activities that have been found to represent metacognitive knowledge and metacognitive skill. The goal here is that this analytical path would provide me with an acceptable alternative to the idea of metacognition for a group of people.

My first decision was to take advantage of my intentional effort to separate the mathematics problem into its constituent parts of situation, question, and problem-solving strategy. This decision allowed me to gain greater insight into how metacognitive knowledge and metacognitive skill impact the student's problem-solving efforts. I hoped that this separation could also be insightful when examining the activities of metacognitive knowledge and metacognitive skill performed by a group of students. If I considered that both the situation and the question contain syntactic and semantic elements, then I could establish that seeking these elements are individual activities within themselves. The third constituent part, choosing the problem-solving strategy, would also be considered as its own activity. So now within the original situation, a given mathematical problem, there are twelve constituent parts and sub-situations: (a) the described situation, (b) the question, (c) the selection of a problem-solving strategy for syntax evaluation of the described situation, (d) the selection of a problem-solving strategy for the semantic evaluation of the described situation, (e) completing a syntax evaluation of the described situation, (f) completing a semantic evaluation of the described situation, (g) the selection of a problem-solving strategy for the syntax evaluation of the question, (h) the selection of a problem-solving strategy for the semantic evaluation of the question, (i) completing a syntax evaluation of the question, (j) completing a semantic evaluation of the question, (k) the selection of a problem-solving strategy for the given mathematical problem, and (l) the generation of an answer. Each of these twelve constituent parts and sub-situations represents a vocational task to be completed by one of the designated roles. For example, items c, d, g, h, and k are tasks for the designer role. Items e, f, i, j, and l are tasks for the executor. Despite the nuance and specificity of the tasks, the roles are consistent with what was revealed during the teacher–student conferences.

Before I provide this analysis, I restate the descriptions of syntax and semantics. Recall that Saussure, the Swiss linguist, focused on syntax, the information that is provided through the structure and organization of the meaning-making situation. Peirce, the American philosopher, focused on semantics, the information that is provided through the signs and symbols used within the meaning-making situation (Chandler, 2007; Eco, 1978). So, this metacognition analysis is performed using the data from the classroom group as they were engaged in investigating the information that was provided by the structure and organization of the situation and by the question, as well as investigating the information that was provided by the signs and symbols used within the situation and within the question. I use the data from the eighth-grade girls group to begin this analysis. Due to the fact that I did not have available data across multiple weeks for the classroom groups as I did for my individual tutoring students, it was not possible to do a 3-week metacognition analysis for the classroom group as I conducted for my individual tutoring students. As such, the following metacognition analysis is only for the equivalent of one week, and it is specifically for the setting that occurred after I gave the class instruction on systems of equations. Another distinction with the forthcoming metacognition analysis is that I have isolated each activity accordingly, as previously detailed.

Three-member, eighth-grade female classroom group. Each member of the eighth-grade girl[27] group self-identified as female. Based on conversations with them, they each had a high interest in school. In what follows, I provide details for the following five sub-activities: (a) a syntax evaluation of the described situation, (b) a semantic evaluation of the described situation,

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[27] My use of the terms ‘girl’ and ‘boy’ throughout this work serves to emphasize to the reader that the participants for this project were indeed children younger than 14-years of age.

(c) a syntax evaluation of the question, (d) a semantic evaluation of the question, and (e) a problem-solving strategy for the given mathematical problem.

A syntax evaluation of the described situation

Based on my analytic memos, there was no evidence by the group of an examination of the underlying structure and organization of the words in the given word problem. Although there was evidence by all three of the students of reviewing their notes, their refencing of notes was in the effort to solve the problem, but not in trying to understand the foundational grammar of the word problem. This lack of attention to the grammatical structure of the word problem suggests that the girls did not orient or organize their initial efforts in such a way as to perform such an examination (Garofalo & Lester, 1985). This lack of attention may have been due to the fact that the value of such an examination had not yet been presented to them. Without such prior knowledge or prior experience, from a metacognitive perspective, these young girls did not have the person variables, task variables, or strategy variables necessary for such an exercise. In short, as these are the fundamental elements of metacognition, these girls did not have either the metacognitive knowledge or the metacognitive skills needed to perform a syntax evaluation of the given word problem (Flavell, 1976; Veenman & Spaan, 2005)

A semantic evaluation of the described situation

With regards to attaining a sound understanding of the given word problem, each member seemed to have had a solid understanding of the situation that was described. In addition, each member sought to produce a representative equation for the given situation. This consistency was an interesting occurrence given that there are other approaches to demonstrate comprehension of the given situation. The fact that each resorted to producing an equation signifies that the meaning-making that each member achieved was done in her head with no other observable evidence for me to investigate. To be clear, for many students in many circumstances, this mental exercise in meaning-making is a viable approach, but it does reduce the likelihood that subsequent error checking and justification can be achieved. Based on these observations, each member had the person variables, task variables, and strategy variables necessary to achieve comprehension of the given word problem (Flavell, 1976). This collection of metacognitive variables allowed them to orient and organize their individual and collective efforts toward producing a representative equation for the given situation. This particular exercise seemed easy for the group and I suspect that it was easy because each chose to produce an equation as the tangible form of evidence of comprehension. It would have been interesting to observe this group if one of the members chose to represent the given situation with a table or a picture, because then there would have been a particular need for each group member to explain her reasoning. Without evidence of a problem-solving strategy for achieving comprehension, I suspect that the group members would have had a difficult time explaining the rationale to another that each achieved in her own mind. The fact that none of the girls seemed confused in her own process of creating a representative equation suggested that each had prior knowledge and prior experience with such an activity. Nevertheless, none of the group members demonstrated a tool, evidence of conditions, or a sequence of steps that led her to the resulting equations. Their equations just appeared on their paper. My concern here is that without the use of a meaning-making strategy or tool, each girl would have to depend on her memory or re-create the meaning-making process each time she had to explain or justify her actions. The use of a strategy or tool would make such subsequent explanations and justifications easier. The absence of evidence of such a strategy or tool suggested to me that none of the girls had the person variable, task variable, or strategy variable necessary to demonstrate such a strategy or tool use (Flavell, 1976). A consequence of not having this necessary metacognitive knowledge or metacognitive skill is the inability to orient oneself or organize one's actions toward demonstrating the utility of an effective meaning-making approach or tool. In the end, the resulting representative equation will be evident to the creator, but the cognitive path will not be evident to an observer.

A syntax evaluation and semantic evaluation of the question

There was no evidence that any of the girls examined the structure of the question. From the discussion of the group members, however, there was evidence that each member understood the question. None of the girls seemed to have been troubled by this exercise. If an observer had the opinion that meaning-making is privileged over linguistic structure and organization, then the fact that the group members demonstrated a clear understanding of the question that was asked could serve as an explanation as to why there was no evidence of such a syntactic examination. Simply stated, it was not needed. In this case, I take this opinion, therefore, I cannot confirm or deny that the girls had the personal variables, task variables, or strategy variables necessary to support the metacognitive knowledge and metacognitive skill needed to perform such an evaluation. Nonetheless, the question emerges as to how one who may not have understood the question could be assisted in gaining such an understanding. Without explicit attention to the structural and organizational aspects of the linguistic elements within the question, I would suspect that each of these group members would be challenged to provide such assistance.

A problem-solving strategy for the given mathematical problem

What was clear from my analytic memo notes was that each group member referenced her notes constantly throughout the problem-solving process. Not only did each reference her notes but also each contributed to the discussion based on information that she found in her notes. This sharing suggested that each also understood her notes. Although the complete problem-solving approach was not made clear, what was clear was that good notetaking and good note-referencing were essential parts of their problem-solving strategy. From their actions and discussion, there was evidence that each girl had the person variables, task variables, and strategy variables needed to engage in this problem-solving activity. In addition, the existence of these variables led to the construction of the metacognitive knowledge and metacognitive skill needed for each group member to participate in the activity (Flavell, 1976, Veenman & Spaan, 2005). Each girl—first independently, but then collectively—oriented her thinking and organized her actions around the steps necessary to solve the given problem. Their alignment and connection were strong enough to allow them to separate the entire problem-solving process into smaller more manageable steps which they subsequently worked through together. To offer another point of interest, each group member was present during the same notetaking opportunity; consequently, they each had similar notes. Having a common source of information to reference which contained a common language to use may have allowed for such a strong alignment and connection to exist amongst them and greatly facilitated not only the individual thought and activity but also the collective thought and activity.

In conclusion, although this group of three eighth-grade, female students did not demonstrate metacognitive knowledge and metacognitive skills in each of the identified areas, they were able to effectively work together to solve the problem. This finding suggests that not all metacognitive areas need to be satisfied for effective problem-solving to occur. Based on the data, this group did not show evidence of the necessary variables to manifest metacognitive knowledge or metacognitive skill in either of the syntax areas. However, there was consistent evidence of the necessary variables in the semantic and problem-solving areas. One hypothesis, based on this limited data set, is that the same variables and activities necessary to substantiate metacognitive knowledge and metacognitive skill in an individual can exist within a group of individuals when the group is engaged in semantic and problem-solving activities, but not necessarily engaged in syntax-oriented activities. In addition, the group members must exhibit the following traits: (a) reference similar source material, (b) have a similar proficiency with the linguistic elements in the source material, and (c) have a similar level of competency from the source material. As insightful as this may seem, it must be noted that each of these three girls held a positive perspective about education and demonstrated a high degree of agency in their cognitive development.

Three-member, eighth-grade male classroom group. I conducted this same metacognition analysis on the data that I had compiled from a three-member group of seventh-grade boys. The members of the seventh-grade male group each self-identified as male. Based on conversations with them, they each had a low interest in school. In what follows, as I did with the eighth-grade female group, I provide details for the following five sub-activities: (a) a syntax evaluation of the described situation, (b) a semantic evaluation of the described situation, (c) a syntax evaluation of the question, (d) a semantic evaluation of the question, and (e) a problem-solving strategy for the given mathematical problem.

A syntax and semantic evaluation of the described situation

Based on my analytic memos, there was no evidence by the group of an examination of the underlying structure and organization of the words in the given word problem. This lack of attention to the grammatical structure of the word problem suggested that the boys did not orient or organize their initial efforts in such a way as to perform such an examination (Garofalo & Lester, 1985). As in the case with the girl group, this omission may have been due to the fact that the value of such an examination had not yet been presented to the group. Without any evidence of such prior knowledge or prior experience, from a metacognitive perspective, these boys did not show evidence of the person variables, task variables, or strategy variables necessary for such an exercise (Flavell, 1976; Veenman & Spaan, 2005).

With respect to attaining a sound understanding of the given word problem, each boy seemed to have had a solid understanding of the situation that was described, and the mathematics concept involved. Unfortunately, each member travelled his own path to solving the problem. I do not oppose the idea of pursuing one's own cognitive path because we are all individuals; however, when working in a group there does need to be some form of organization so that cooperative collaboration can be achieved. Unfortunately, during my observations of this male group, I did not see evidence of cooperative collaboration.

Based on my observations, each member had the person variables, task variables, and strategy variables necessary to achieve comprehension of the given word problem (Flavell, 1976). This collection of metacognitive variables allowed them to orient and organize their individual efforts, but not their collective efforts toward producing a representative model for the given situation. One student decided to produce a chart to represent the situation described in the given word problem, while another student decided to produce an equation to represent the situation described in the given word problem. The third student had not written anything on his paper to indicate his particular problem-solving approach. Despite their best efforts, each member struggled to produce a form, a chart, or an equation, that represented what was described in the given word problem. Although none of the boys gave any indication of not understanding the described situation, the observable difficulty with producing a representative form suggested to me that none of the boys had a good understanding of the underlying relationships that were embedded in the given word problem. I am convinced that each member could tell me what the words meant, but I am not convinced that any of the members could explain the embedded relationships. A realization of underlying relationships is a key component of Peirce's semantics (Chandler, 2007; Eco, 1978). Without evidence of a problem-solving strategy for achieving comprehension, I suspect that the group members would have a difficult time explaining the rationale to another that each achieved in his own mind. The fact that all of the boys seemed confused in his own process of creating a representative form suggested that each had not acquired proficient prior knowledge nor proficient prior experience with such an activity. My concern here, as it was with the eighth-grade female group, is that without the use of a meaning-making strategy or tool, each boy would have to depend on his memory or re-create the meaning-making process each time he had to explain or justify his actions. Granted, one group member was referencing his notes, but he had not yet started to produce a representative form of the given situation. Also, one member was attempting to create a representative chart but he was not successful in doing so. The comprehension and use of a strategy or tool would make such subsequent explanations and justifications easier. The absence of comprehension of such a strategy or tool suggested to me that none of the group members had the person variable, task variable, or strategy variable necessary to demonstrate such a strategy or tool use (Flavell, 1976). A consequence of not having the necessary metacognitive knowledge or metacognitive skill is the inability to orient oneself or organize one's actions toward demonstrating the utility of an effective meaning-making approach or tool. This finding is a main reason why I wanted to separate the act of comprehending a given situation from the act of solving the given situation; such a separation provides the observer the opportunity to discern which stage of the problem-solving process the student is engaged. In short, although these boys did not demonstrate either the metacognitive knowledge or the metacognitive skills needed to perform a syntax evaluation of the given word problem, they did show a clear, but limited understanding of the problem. Again, this emphasizes the distinction between the syntax and the semantics of a given situation, as well as indicates that the metacognitive knowledge and the metacognitive skills needed for the two evaluations are different.

A syntax evaluation and semantic evaluation of the question

There was no evidence that any of the group members examined the structure of the question that is asked in the problem. Nevertheless, through group discussion, there was evidence that each member understood the question. Interestingly, in both instances, for the situation and the questions, success in the semantics evaluation was achieved without evidence of even an attempt at the syntactic evaluation. One hypothesis regarding the value of a syntactic evaluation would be that the importance of such an evaluation increases with an increase in the quantity of unfamiliar words provided in the given situation or in the question. In this case, none of the members complained about the presence of unfamiliar words in either the described situation or in the question.

A problem-solving strategy for the given mathematical problem

What was clear from my analytic memo notes was that only one group member referenced his notes constantly throughout the problem-solving process. Although each group member was present during the same notetaking opportunity, only one person took and kept notes. This observation suggested that two of the group members had assigned a low value to the use of notes. Although the two group members who did not use notes did in fact have a suggestion for the problem-solving approach, neither could successfully complete the implementation of his particular approach. From their actions and discussion, there was evidence that none of the members had the person variables, task variables, and strategy variables needed to effectively engage in this problem-solving activity. In addition, the absence of these variables made it impossible for them to display the metacognitive knowledge and metacognitive skill needed for them to be successful. None of the members could orient his thinking and organize his actions around the steps necessary to solve the given problem. Consequently, because none of the members had established the necessary metacognitive knowledge or metacognitive skill to work independently, then it is understandable that they were not able to function as a collective unit. For this group, having a common source of information to reference that contained a common language to use did not allow for a strong alignment and connection to exist amongst them and did not facilitate individual thought and activity. Ultimately, I believe that two of the boys confused conceptual understanding with metacognitive knowledge and with metacognitive skill, which could explain their low valuation of notes.

In conclusion, this group of seventh-grade male students did not demonstrate metacognitive knowledge and metacognitive skills in any of the identified areas, Consequently, they were unable to effectively work independently or together to solve the given problem. I think the unsuccessful efforts of this group emphasize two points. First, metacognitive knowledge is more than simply understanding a situation; metacognitive knowledge is concerned with orienting oneself and organizing oneself to act based on the understanding that one has of the situation (Garofalo & Lester, 1985). Second, metacognitive skill is more than simply convincing oneself that one can duplicate the process or procedure that is demonstrated by another. Metacognitive skill is concerned with having the experiential evidence of self-directing the successful execution of the same process or procedure, independently.

In sum, the following points seem to emerge from this limited group analysis. First, a metacognitive-like state exists within the group if each member has access to the same source material, has a similar proficiency with the linguistic elements, has a similar level of meaning-making from the source material, and some amount of metacognitive knowledge and metacognitive skill. Last, in the constituent parts of the situation, question, and solution, as long as each group member can achieve the level of understanding and meaning-making necessary to propel him or her to the next step in the problem-solving activity, then a deliberate and explicit focus by the group on the structural and organizational elements of the linguistic elements in either the given situation or the question does not seem to be necessary. My resulting theoretical statement is as follows:

> A metacognitive-like state exists within a group of people if first the individuals within the group have achieved coincident metacognitive knowledge and coincident metacognitive skills.

This statement in no way diminishes the importance of the tacit focus that each group member assigned to syntactic evaluations, because ultimately the explicit meaning that is made is indeed contingent on the person having some implicit, tacit, or unconscious awareness of the form and function of each part-of-speech (Biber, Douglas, Conrad, & Reppen, 1998).

"Interpreting" the Findings

At this point, I have built two separate theories from my thematic coding process—one according to the data from my tutoring students, and one according to the data from my classroom groups. Recall that my tutoring students provided insight on the microscheme for this theoretical project, and my classroom groups provided insight on the macroscheme. As I discussed in Chapter 4, I endeavored to extend the work of Steffe beyond investigating only one of these phenomena, I wanted to explore both. So, my next step was to examine both of the two separate theories and determine if a unifying theory could be developed. I discuss the results of this particular effort next.

The two separate theories have already been stated, but can be found more succinctly in Figure 6.10. A much broader theory resulted from the data collected from my classroom groups. I surmise that such a broader theory was possible because I was examining a group of students instead of a single student and the amount of interaction and discourse within a group of students allowed for a much more diverse set of categories to be observed. I used theoretical coding on these two statements, as I decided to view my situation as a second-cycle coding scenario (Saldaña, 2016). The key ideas that most resonated with me from each data can be found in Figure 6.10. I then offer my final formulation of a representative theory at the bottom of the figure.

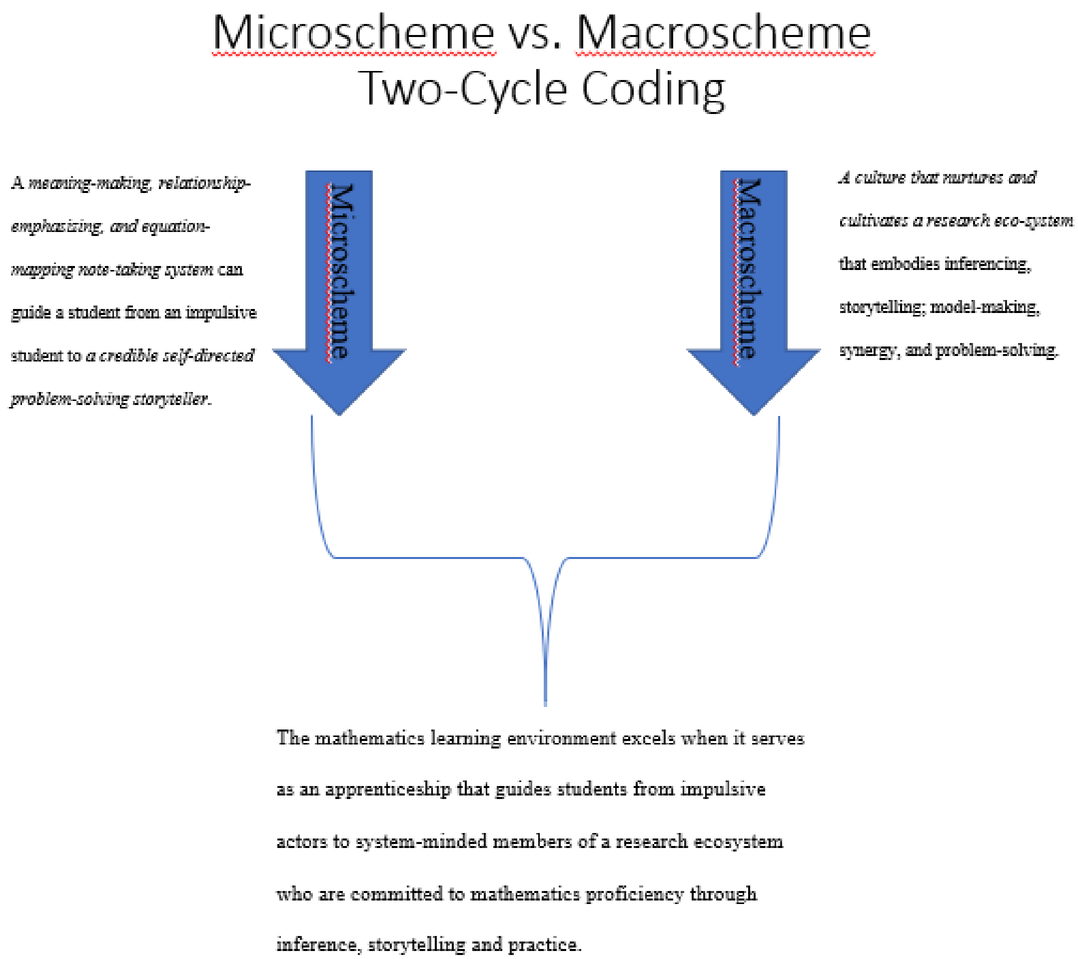


Figure 6.10. Microscheme vs. Macroscheme Two-Cycle Code Chart.

In the final formulation of this representative theory, I made every effort to not be pleonastic in its construction. As such, word choice became significant as well as word order. The final formulation is provided here:

> The mathematics learning environment excels when it serves as an apprenticeship that guides students from impulsive actors to system-minded members of a research ecosystem who are committed to mathematics proficiency through inferencing, storytelling, and practicing.

Each word is rich with meaning, and represents well the constituent themes, concepts, codes, and source data. Now that such a formulation is complete, I hope that the fact that I combined the microscheme with the macroscheme to build such a theory establishes that the theory is adequately robust to stimulate hypotheses of single-student settings as well as of student-group settings. As a classroom teacher, I must admit that there are a few elements which are highlighted in this theory that I have not incorporated with either my own tutoring students or my own classroom students. I am already envisioning the transformations that are necessary in my mathematics interactions with students.

Unifying the Metacognition Theories

In an effort to provide an encompassing summary from a metacognitive perspective, I offer the following. First, a prior metacognitive knowledge and a prior metacognitive skill experience seem to be the guidelines for an individual when determining the necessity for deliberate instruction, guidance, or success in a given situation. Second, the incorporation of the semiotic sub-elements of syntax and semantics into mathematics problem-solving enhances metacognitive knowledge and metacognitive skills. Third, an explicit focus on the syntax sub-element may not be necessary if the individual can gain the necessary semantic understanding of a given situation. Last, if the members of a group reference the same source material, have a similar proficiency with the linguistic elements in the source material, and have a similar level of competency from the source material, then a metacognitive-like state can be achieved by the group. My resultant theoretical statement from the metacognitive analysis is:

> Prior metacognitive knowledge and metacognitive skill experience with the semiotic sub-elements of syntax and semantics can exist within a group and can guide the group and its members in determining the necessity for deliberate instruction or guidance and in determining the quality of collaboration.

The effort to integrate the findings from each of these theories into a cogent whole provided me the rigor necessary to build a robust theory. Such an integration has led to the following macro-theory which I think is effective in representing the contents of this chapter:

> The mathematics learning ecosystem excels when it serves as a semiotics apprenticeship that guides students through metacognitive knowledge and metacognitive skills toward proficiency in inferencing, storytelling, and practicing.

Summative Remarks

As I conclude this chapter, I reflect on several items. First, the methodology that resulted from the fusion of CHAT with the teaching experiment provided a multitude of investigative directions. Second, the generation of the data collection charts for both the characteristics of the activity and for the experience of the student facilitated my ability to travel along several of these investigative directions. Third, the generation of data analysis templates that centered on triangulation was pivotal. The rigor of triangulation required that I move beyond my own opinion and experience and engage the data as a phenomenon with its own identity. This perspective of generalization is what I discussed in Chapter 5. This simulation was hard work, but I am not attempting to replace the quality and rigor of an actual empirical study with the execution of a simulation. What I am attempting to do is open the minds of our mathematical communities to consider the implications of this simulation. Although the product of this simulation was several theoretical paradigms, ultimately, these paradigms serve as seeds for new sets of hypotheses to be tested within future empirical studies (Steffe & Thompson, 2000). The potent point of this chapter is not the theoretical paradigms in and of themselves. The potent point of this chapter is the rigor that is represented by this newly designed methodology, the newly created data collection charts, and the newly created data analysis. In sum, the rigor of the process reveals so much more of the mathematics learning experience of the student.

CHAPTER 7

IMPLICATIONS

> *Theory-building* is the ongoing process of producing, confirming, applying and adapting theory. In a way, to live life successfully we are all obliged to engage in theory building, that is, in processes by which we observe, experience, think about, and understand and act in our worlds, and we do so continuously. However, these theories-in-practice are not always explicit and often occur in the form of implicit, unconscious knowledge on the part of the theorist.
>
> –Lynham, 2002, p. 222–223

In this chapter, I re-present the main points of this theory-building project. I begin by discussing the need for combining CHAT with the teaching experiment. I then discuss the importance of activity dis-aggregation and its impact on this work. Next, I detail how involvement in mathematics discourse leads to the development of mathematics proficiency. I continue by discussing the idea of mathematics proficiency and show how it can extend to elements beyond the solution to a mathematics problem. I continue by emphasizing the need of semiotics within the mathematics classroom and acknowledge the presence of one ever-present phenomenon within this work—metacognition. I then present several conceptual models that serve as visual representations of this theory-building project. These visual aids helped me to refine my thinking throughout the project. Before I conclude, I provide explicit answers to my guiding research questions. I also thought it helpful to present a traditional lesson plan for the current classroom teacher who may wonder how the structure and findings of this project could be implemented in a traditional mathematics classroom.

Initial Thoughts

Having spent 14 years reflecting on the same question, and of those 14 years, having 9 of those years guided by a formal doctoral program, my entire perspective and approach to mathematics education has evolved. I no longer view mathematics education as a hierarchical relationship between teacher and student. I now embrace the reciprocity and the apprenticeship nature of the symbiosis. The student requires the socio-cultural-emotional-psychological investment of the teacher to develop into a proficient mathematics student. In like manner, the teacher requires the socio-cultural-emotional-psychological investment of the student to develop into a proficient mathematics teacher. The two need each other in order to exist. Attendance of the two simply for the sake of attendance is not sufficient. In addition, this mutual investment is the nourishment that is needed to cultivate the learning environment required for the necessary harmony and cultivation of the two. All must work in concert.

With this in mind, I offer a short review of what I feel are the high points of my dissertation work. First, my students, both tutoring and classroom, revealed to me the need to combine CHAT with the teaching experiment. In so doing, I realized that achieving such a fusion required a three-tier understanding of what I call the CHAT–teaching experiment heuristic. The first tier is having an over-arching understanding of what a CHAT–teaching experiment heuristic offers the researcher, teacher, and the student. The second tier is achieving a clear resolution for the ideological tension that exists between the foundational principles of CHAT and of the teaching experiment. The final tier is establishing a rigorous and robust scientific protocol for implementing the CHAT–teaching experiment heuristic in practice. It is one point of clarity to suggest how an idea should function; it is an entirely different and deeper point of clarity to demonstrate how the idea functions in practice. I present my dissertation as one example of this different and deeper point of clarity that may inspire other researchers into further research in mathematics education.

A second high point that my students revealed to me is that an activity is so much more than what and how it is presented. A given activity is the composite whole of so many other smaller constituent activities that must be embraced and addressed. Overlooking these smaller constituent activities does not seem to facilitate a more successful completion of the given composite activity. A third high point is that mathematics proficiency is directly related to mathematics discourse. The demands of mathematics curriculum and the real-world application of mathematics have exceeded the preparation that is offered by a calculation-only mathematics classroom. Mathematics proficiency is best established when a student engages in discursive practices that reveal conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and a productive disposition (Kilpatrick et al., 2001).

This third point brings me to my most important finding. In this work, the various smaller constituent activities were best described by a set of five roles that my students developed. During the course of this dissertation study, I determined that proficiency in each of these sub-activities is what can allow for proficiency in mathematics overall. In other words, my dissertation study revealed to me that mathematics proficiency, as outlined in the *Adding It Up: Helping Children Learn Mathematics* report (Kilpatrick et al., 2001), is the result of first achieving proficiency in the five constituent activities of the researcher, analyzer, designer, executor, and critic. These five roles seem to capture, in a discursive form, the various nuances and specificities that exist within any mathematics activity. My effort to divide a mathematics activity into its situation, question, and problem-solving strategy pieces combined with these five roles allowed for even greater descriptive discourse from the students. It seems that the development of a student into a proficient researcher, analyzer, designer, executor, and critic facilitates the further development of the student into a proficient mathematics student. My view is that proficiency in mathematics is the result or convergence of these five super-ordinate proficiencies. Pursuing mathematics proficiency without first establishing these five superordinate proficiencies is difficult. I now understand that pursuing the development of my students into proficient mathematics students without first apprenticing them to become proficient in these five constituent identities is and has been my mistake. I posit that this mistake is common throughout much of mathematics education in the United States.

Semiotics

Semiotics and its three constituent parts—syntax, semantics, and pragmatics—allow the student to make meaning within the context of the given situation. (The reader can see the Afterword for further discussion on semiotics and the graphophonic cueing system.) What is interesting here is that semiotics allows the student to evaluate each part according to its syntactic, semantic, and pragmatic elements. Note here that I am using the term pragmatics to mean the practical approach to completing a task. In this case, the tasks include each of the four parts: (a) decode the context, (b) decode the instruction or inquiry statement, (c) encode the appropriate mathematical principles and procedures, and (d) execute the appropriate mathematical procedures. Although this approach is extensive meaning-making for the student, due to the thoroughness of the process, it provides the student a contextualized depth of knowledge that would not be easy to achieve in any other way. My approach requires that the student, the teacher, and the researcher attend to all four parts of this cognitive sequence, and value the depth of knowledge that each part offers. Admittedly, this is arduous work. According to the *Adding It Up: Helping Children Learn Mathematics* report (Kilpatrick et al., 2001), however, depth of knowledge allows for conceptual understanding which is one of the strands for mathematics proficiency. So, my effort was to first divide a mathematical situation into its decoding, encoding, and execution parts, as represented within the five embodiments of the researcher, analyzer, designer, executor, and critic. Second, I used semiotics to examine the language and actions embodied by students as they analyzed the semiosphere that is captured within each of these parts. These detailed investigations are the reasons why I had to adapt both CHAT and the teaching experiment. Separately, neither could give me the investigative power that I needed.

Metacognition Indoctrination

As I sit and reflect upon the journey that is represented within this theoretical project and tie those reflections with other reflections that I have had about my 14-year mathematics education career, I recently paused and took notice of what I was doing. My focus was not so much on what I was physically doing as I reflected, nor was my focus on what I was specifically contemplating. No, my focus was on the mere awareness that I was thinking about the thinking in which I was engaged during this theoretical project; and I was thinking about the thinking that I had accumulated over my 14-year mathematics education career. I was thinking about my thinking. This awareness is the last implication of this theoretical project that I address. My seamless and unconscious immersion into a metacognitive state to evaluate my work in this theoretical project and my actions over my 14-year mathematics education career is a telling factor about my identity, as a person and as a mathematician.

When I took a broad view of my approach in this theoretical project, its structure and its trajectory, I realized that it aligned with my own metacognitive tendencies. I now see that I was attempting to manifest indicators of metacognition within my students through my actions within this theoretical project. Over the last 14 years, when I would assign a problem set to my tutoring students or my classroom students, I had always been more interested in their process than in their final answer. I realize now that I was seeking an indication of their own metacognitive features (Flavell, 1976; Garofalo & Lester, 1985; Veenman & Spaan, 2005). Based on my determination of each individual student's metacognitive features, then and only then was I able to craft an individualized plan for their mathematics learning. This process has been my approach to mathematics education from the very beginning, I was just never conscious of it before now.

Therefore, before I ever knew the academic meaning of metacognition or about the long history of work that has gone into the study of metacognition, I embodied it and I sought it in others. I now realize that the individualized mathematics learning plans that I have crafted over the years have been rooted in nurturing and cultivating metacognition within my students. The details that I have presented in this theoretical project is rooted in metacognition. The increase in productive disposition that I mentioned was experienced by the students in this theoretical project despite the fact that they did not successfully solve the given problem was rooted in the student's heightened awareness of his or her own metacognition. In short, before I knew what was metacognition, I was seeking it and cultivating it within my students. Before I realized that the development of metacognition was what this theoretical project was truly about, it was embedded in every decision, action, and detail of this project. Seeking and developing metacognition within my students is what this theoretical project has been about.

Allow me to provide an example of how entrenched the idea of metacognition has been in my work. When the students divided their actions into the five roles of researcher, analyzer, designer, executor, and critic and I divided the mathematics activity into the three dis-aggregate pieces of situation, question, and strategy, it became easier for my students to engage in mathematics discourse about a given mathematics problem because they were equipped with a multi-dimensional locator of sorts. A student could specify the piece of the activity in which he or she was working; and the student could specify his or her role within the dis-aggregate of the activity. More importantly, not only could a student speak with this specificity but also listeners could understand and engage in productive discourse. The connection to metacognition occurs when one considers the various perspectives on metacognition offered by Flavell (1976) and offered by Veenman and Spaan (2005). Flavell is given credit for distinguishing the three different variable types for metacognition: (a) person variables, (b) task variables, and (c) strategy variables. Veenman and Spaan outlined the constituent parts of metacognition: metacognitive knowledge and metacognitive skills. When I consider that the different variable types posited by Flavell could actually refer to the five roles as distinguished by my students, and that the metacognitive knowledge and the metacognitive skills posited by Veenman and Spaan could refer to increased awareness and insight into these five roles, then these five roles become an explicit tangible representation of the more implicit and abstract ideas of metacognition. When considered in this manner, it offers an explanation as to why my students performed better when they engaged in this multi-dimensional discourse of role and activity dis-aggregate; they were talking about their metacognition. My students had moved beyond thinking about their thinking; they elevated to talking about the thinking of their thinking.

Re-thinking Mathematics Proficiency

During the data collection aspect of this theory-building project, there seemed to be a trend that I noticed across all of the students and the student groups—none of the students or the student groups could initially solve the problem that they received. In addition, there was never a circumstance where one of the students or one of the student groups made it known to me that there was a problem with comprehending the given problem. Although the concepts for the two math problems, ratios for the seventh graders and linear equations for the eighth graders, were familiar to the students, none could solve the respective problem. In most of the instances, it was not until the third week or session of instruction, specifically, the week of the semantics instruction, that the students began to show significant progress toward solving the respective problem. This finding is not to imply that I offered the best instruction that the students had ever experienced; on the contrary, I intentionally kept the instruction at a foundational and basic level. No, I think something else occurred. I think that after the first exposure to the respective problem, during the concept exploration episode when the student could solve the problem using whatever approach he or she desired, the student realized for himself or herself that he or she did not have enough awareness and knowledge to solve the problem. In short, each student, except for the eighth-grade female student (Narnia), had exhausted his or her knowledge base, and he or she was aware that his or her knowledge base was emptied and no longer useful as it related to the respective problem. This finding is an important point because when the student realized that he or she had a knowledge or skill void that needed to be addressed, then he or she was more attentive during the instruction because the student was now seeking specific information, which was relevant to him or her.

For ease of discussion, I use the term cognitive vacuum to refer to the state when a knowledge or skill void exists that needs to be addressed. According to Piaget (1976), accommodation is one approach to address this cognitive vacuum. Accommodation occurs when the person’s current cognitive expanse, or schema, is transformed to incorporate new experience and new knowledge because the new experience and new knowledge could not be integrated into the existing structure and organization of the existing schema (Piaget). I differentiate a cognitive disequilibrium from a cognitive vacuum by stating that a cognitive disequilibrium emphasizes an inconsistency between experience and current reference within the schema (Piaget). However, a cognitive vacuum emphasizes not just an inconsistency, but a lack of connection, a void, between experience and schema possibly due to the fact that there is no reference within the schema. The idea of making information available to someone before the person has determined the relevance of the information, and expecting the person to be attentive and engaged is a lot to ask in a society filled with a large amount of sensory input. Relevance becomes an important aspect to facilitate the person's discernment of information from the noise.

During this theory-building project, providing the students with the opportunity to exhaust their cognitive toolboxes as they attempted to solve the respective problem helped the students to determine for themselves the relevance of the subsequent instruction, because now there was a context to which the information could be applied. For these students, the context was not a real-world problem or application. No, for these students, the context was a void in their own understanding, which resulted in their inability to solve the respective problem. It was this context that made the subsequent information as relevant for the student as it needed to be and led to a heightened degree of attentiveness by the student during the period of instruction. The combination of a heightened degree of attentiveness and a designation of relevance by the student is what allowed the students to experience their own relative amount of mathematics proficiency during this project.

Recall the National Research Council's report *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick et al., 2001) that I detailed in Chapter 1 and referenced throughout. Based on the syntheses of research in that report, the student must first be attentive according to his or her own sense of a self-determined contextualized relevance before the student can experience an authentic increase in his or her mathematics proficiency. The five interwoven strands of mathematics proficiency (most likely) will not be achieved if the student does not first invest himself or herself in the mathematics experience. In other words, *conceptual* *understanding* is achieved by the student when the concept fills a cognitive void that the student experienced from the respective problem. In similar fashion, *procedural fluency* is achieved by the student when the procedure fills a cognitive or psychomotor void that the student experienced from the respective problem. *Adaptive reasoning* is demonstrated by the student when the self-reflection and reflective abstraction fill a cognitive void that the student experienced from the respective problem. And lastly, *strategic competence* is demonstrated by the student when such competence fills a cognitive void that the student experienced from the respective problem.

Arguably, the strand productive disposition is experienced by the student not only because he or she could successfully solve a mathematics problem, but also, and perhaps more importantly, productive disposition is experienced by a student when he or she is able to successfully fill a cognitive void that he or she has identified. I make this statement because each of the students in this theory-building project experienced a heightened productive disposition by the end of their participation, but only one of the students actually solved the respective problem. Therefore, I do not believe that mathematics proficiency can be singularly designated as successfully solving a math problem. On the contrary, I believe that mathematics proficiency is also achieved and demonstrated when the student resolves a cognitive disequilibrium or perturbation (Piaget, 1936/1952; Steffe, 1991). For the purpose of theory-building, this perspective on productive disposition is more insightful because it allows the focus of the effort to remain on the internal process of the construction of mathematical knowledge by the student and not simply the completion of an external event. (The reader can see the Afterword for further discussion on cognition and its higher-order abstract form). The journey in cognitive growth becomes an important aspect for each of the strands of mathematics proficiency, and not just the successful solving of a mathematics problem. To investigate this point further, I present my perspective on how I divide a mathematical situation into its primary and secondary concepts, and how this division connects to the five strands of mathematics proficiency.

As I detail these two concepts, I want to emphasize the transformation and growth of my own schemata that has occurred through this theory-building project (Piaget, 1936/1952). In order for me to genuinely embrace the hard work that is required for theory-building, I had to first package this project as a treasure hunt, for lack of a better word. When one engages in theory-building, one needs to move beyond the obvious and seek the obscure. In other words, a theory-builder must gather the explicit elements, and then dig deep and investigate in order to find the implicit elements and the implicit relationships. For me, I interpreted this digging as a treasure hunt because I was seeking the implicit elements and the implicit relationships that exist within any given mathematical situation. Allow me to provide an example.

In the case of this particular theoretical project, after selecting a mathematics word problem, there are two concepts, primary and secondary, which I found to exist. The primary concept is the ability of the student to make meaning from the given context. The first step in this primary concept, I refer to as "decoding the context." The resulting meaning does not have to be mathematical; it is whatever meaning the student can construct based on his or her interpretation of the given semispherical context (Halliday, 1978). The second step in this primary concept, I refer to as "assign the mathematical principle and procedure." During this assignment, the student analyzes the given instructions or question, then coordinates the given instruction or question to the meaning that was made from the context, and finally determines the appropriate mathematical principle and or procedure that is necessary to execute the instructions or answer the question. The third step in the primary concept, I refer to as "encoding of the mathematics." During this encoding process, the student must determine how to connect the relevant information from the context with the selected mathematical principle and or then determine how to apply this relevant information to the selected mathematical procedure. The last step in the primary concept, I refer to as “verification,” and it is during this step that the student checks for any errors in the application of the selected mathematical principle or in the execution of the selected mathematical procedure. These four steps represent what I refer to as the primary concept. Undoubtedly, these four steps could be perceived as the assignments for the five embodiments of the student: researcher, analyzer, designer, executor, and critic.

Now, within this primary concept, it is important to bring attention to the following point, there is what I identify as a secondary concept. The secondary concept is the “encoding of the mathematics”—the determination of the respective mathematical principle and or the determination of the respective mathematical procedure. When I first began teaching 14 years ago, I had privileged this secondary concept as the preeminent indication of a student’s mathematical success. Now, I do not. Based on my own pedagogical transformation and the inclusion and emphasis of semiotics in my mathematics classrooms, I now subordinate the encoding of the mathematics to a preeminent goal: contextual meaning-making. If a student cannot make meaning from the given context, then the student will not arrive at the third step, which is the beginning of encoding of the mathematics, the secondary concept. This theoretical project represents my particular preference and the manifestation of this preference was detailed in the data analysis section (see Chapter 6). It was in the data analysis section where I dissected my analysis according to the student’s ability to understand the given situation, understand the question, and then solve the problem.

With my position on the importance of contextual meaning-making stated clearly, I now reconsider a slightly different interpretation of the five strands from the *Adding It Up: Helping*

Children Learn Mathematics report (Kilpatrick et al., 2001). What would happen if you interpreted the conceptual understanding strand as referring to the preeminent goal of my primary concept—contextual meaning-making? Then, what would happen if you interpreted the procedural fluency strand as referring to the four steps within the primary concept that I have just detailed: (a) decoding the context, (b) assign the mathematical principle and procedure, (c) encode the mathematics, and (d) verification? Next, what would happen if you interpreted the adaptive reasoning and strategic competence strands as referring to the reflective abstraction and metacognition necessary to self-direct oneself through these four steps? I posit that if one could make these adjustments in her or his interpretation of the phenomena referred to in the *Adding It Up: Helping Children Learn Mathematics* report to first refer to my primary concept, and then to refer to my secondary concept, encoding the mathematics, then it may be easier to understand why my students experienced success, and consequently experienced a heightened productive disposition, even though most of them did not successfully solve the mathematics problem that they were given. For my students, what I refer to as the preeminent goal of the primary concept seemed to also actually be their preeminent goal.

The CHAT–Teaching Experiment Triplet Heuristic

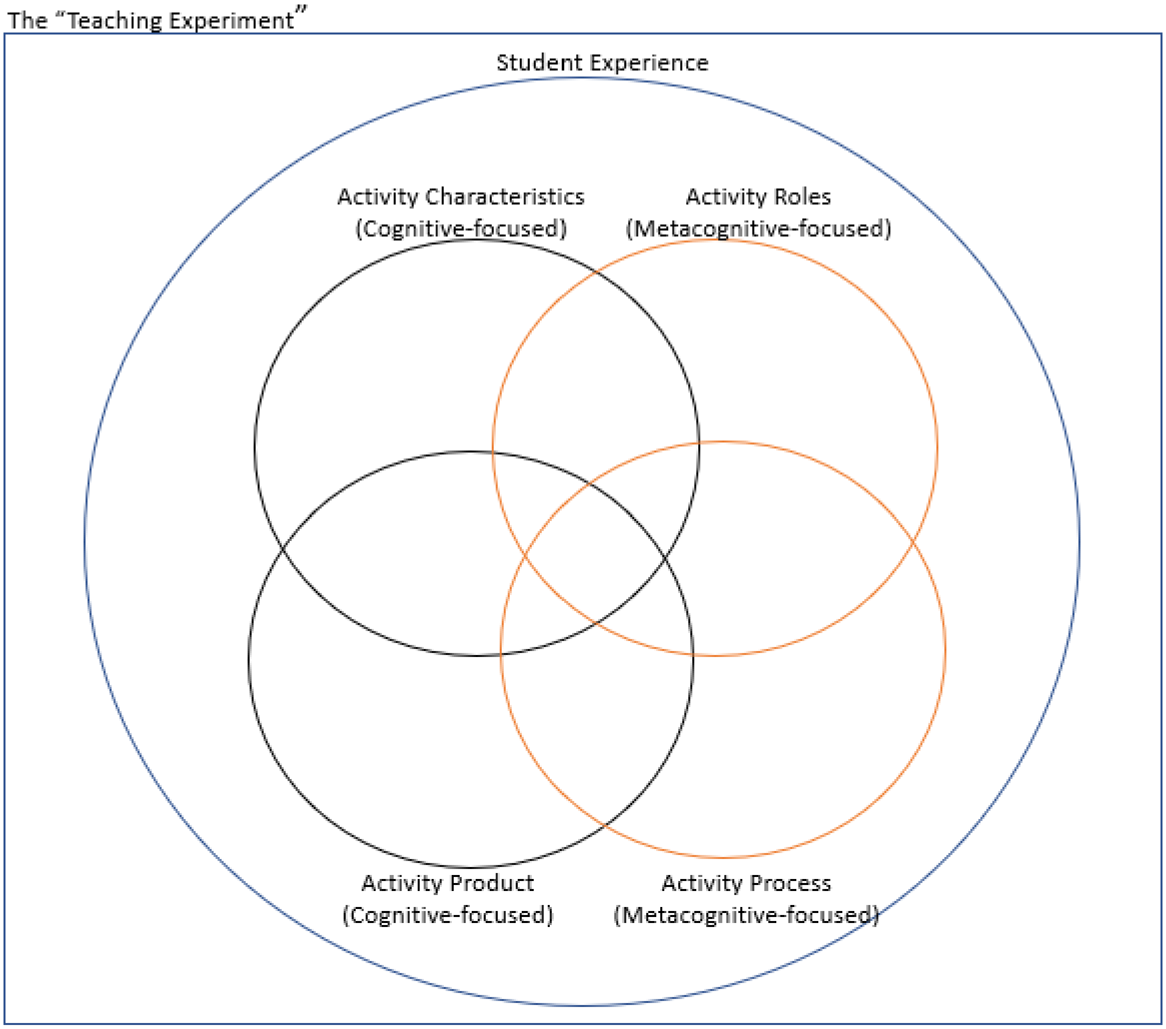
When taken together, the foci of this theory-building project can become overwhelming. I say so, because it was overwhelming for me. Nevertheless, when I transferred these before-mentioned foci into a visual image or conceptual model, then my own understanding became more fluid and more substantial. Below, I offer the three conceptual models that I constructed during this project and enhanced thereafter. I refer to the three conceptual models as the CHAT–Teaching Experiment Triplet Heuristic:

- In **Figure 7.1**, I present the CHAT–Teaching Experiment Heuristic Overview Model. This model shows the four aspects of a mathematics activity and how these four aspects constitute the mathematics experience of the student.

- In **Figure 7.2**, I present the CHAT–Teaching Experiment Heuristic Theoretical Model. This model shows the cognitive aspects as I discerned them through my re-engineered teaching experiment, and how these cognitive aspects fit within the traditional CHAT heuristic. It is important to note the semiospheric stratification of the activity that occurs through the use of the various elements of semiotics that I have placed in the center of the traditional heuristic. Based on my work, it is this semiospheric stratification that serves as the engine of the student's entire mathematics experience.

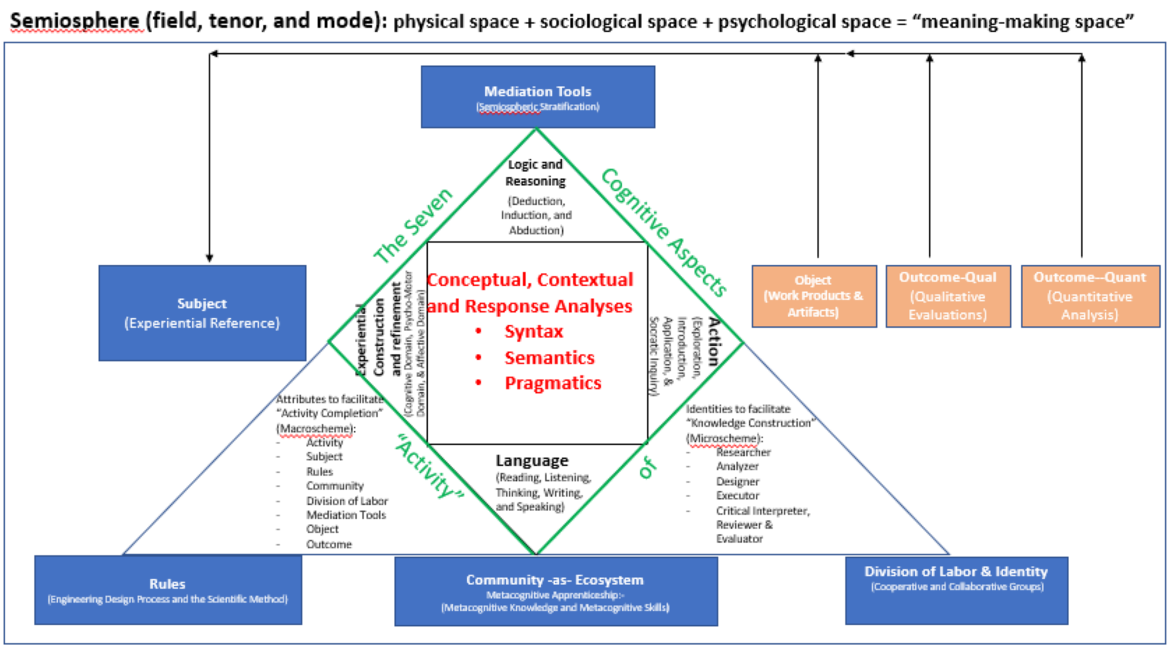
- In **Figure 7.3**, I present the CHAT–Teaching Experiment Heuristic Methodological Model. This model shows the research design of this study. It represents the re-engineering of the teaching experiment that I detailed in Chapter 4.

Each of these models complements the other. In fact, it may be easy for some to dismiss this theory-building project not only because of its complexity, but also because it is packaged within a simulation. I caution the unsuspecting reader to not forget that I am not only a doctoral student, but I am also a practicing mathematics teacher. I embody the theory and practice that both Steffe and Lynham stressed so emphatically (Lynham, 2002; Steffe, 1984). Therefore, although the COVID-19 pandemic denied me the opportunity to conduct an empirical study to demonstrate my new CHAT–teaching experiment methodology, it does not mean that this work is not based upon practical implementation. Evidence of this practical implementation is represented in my CHAT-Teaching Experiment Triplet Heuristic, and it serves as the theoretical representations of what I have learned from my tutoring and classroom students. It is important to note that these three visual models are my anticipated representations of what may develop after a rigorous empirical study. I have attempted to represent explicitly many of the relationships that I believe exist implicitly based upon my experiences during this theory-building project.



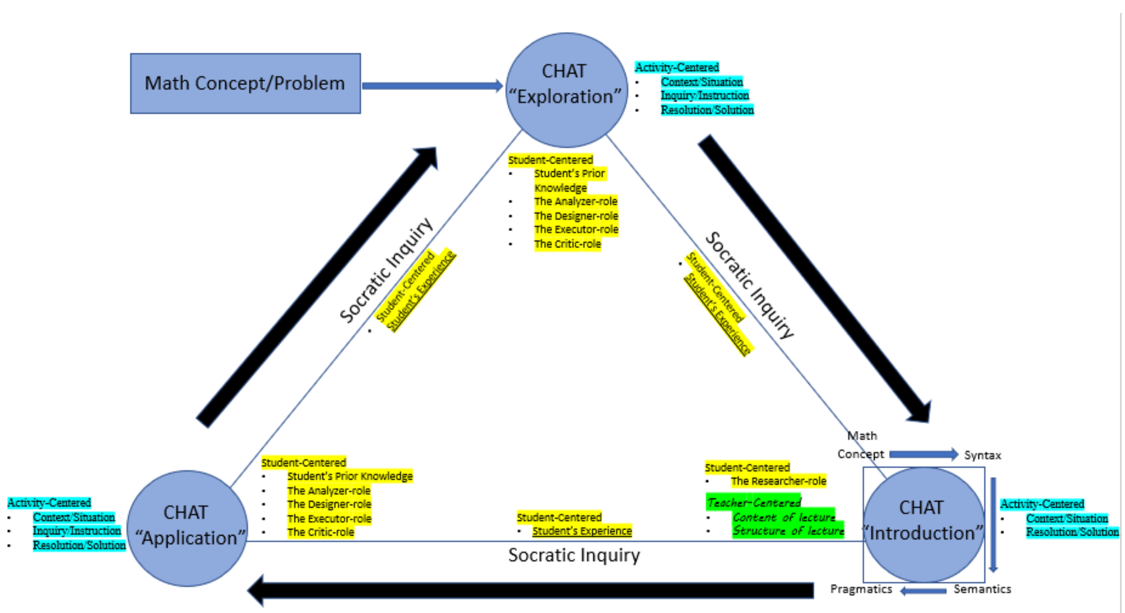
META-DOCTRINE
(Click to See Video)

Figure 7.1 CHAT–Teaching Experiment Heuristic Overview Model



META-DOCTRINE
(Click to See Video)

Figure 7.2 CHAT–Teaching Experiment Heuristic Theoretical Model



META-DOCTRINE
(Click to See Video)

Figure 7.3 CHAT–Teaching Experiment Methodological Model

Responses to the Research Questions

Through this theory-building project and through the creation of the CHAT–Teaching Experiment Triplet Heuristic, I have gained the breadth and depth of knowledge necessary to provide cogent responses to my guiding research questions. My answers to the research questions represent what I have learned through this process. For ease of recollection, I provide my research questions below with my respective responses.

1. **How does teaching mathematics as a language system affect the construction of mathematical knowledge (learning of mathematics) by African American students?**
   As I presented in the prior paragraphs, the teaching of mathematics as a language system brings to the forefront the specificity needed to construct deep and substantial meaning from the given context, through analysis, before any effort is made to fulfill the given question or the given instructions. It can be argued that mathematics problem-solving has always included the obvious prerequisite of understanding the given situation before an attempt at solving the problem is made. I do not challenge this point. As my three-member classroom group analysis suggested, when the student understands the discourse of the given problem, an explicit effort at a syntactic analysis is not necessary. However, the focus of my work is not when the student understands mathematical context, mathematical principles, and mathematical procedures. The focus of my work is what needs to be done when the student does not understand the mathematical context, mathematical principle, and/or mathematical procedure. The student who will gain the most benefit from my work is that student who at a particular moment does not understand the respective mathematics, and it does not matter if the student typically performs in the top quarter of mathematics performers or in the bottom quarter. My work

posits two main points. First, students need to be apprenticed in descriptive and inferential meaning-making with mathematical principles and mathematical procedures, as well as through contextual analysis. Last, students need to understand that attaining mathematics proficiency requires a process-oriented metacognitive mindset. Impulsive reactions to mathematics are more destructive to a productive disposition than they are generative. META-DOCTRINE (Click to See Video)

2. **What can be understood about the construction of mathematical knowledge (learning of mathematics) by African American students when different language systems beyond numbers and operations (visual imagery, movement, written/oral language, for example) are integrated into the mathematics curriculum? (In short, how can semiotics assist with the interpretation and learning of mathematics by African American students?)**

The explicit attention on semiotics when engaged in mathematical contexts directly leads the conversation to the semiosphere that is captured or encased within the given mathematical context. How any one particular student constructs meaning from a mathematical context is contingent upon that particular student's prior knowledge and prior experience in life, not just in mathematics. Therefore, when deliberately and explicitly engaged in descriptive and inferential meaning-making, the semiospheric aspects of a mathematical context, the processes and the products are more substantial and personal for the student. As discussed earlier, the process of descriptive and inferential meaning-making is one concept and producing the mathematical solution to the given problem is another concept. The two working in tandem therefore allow the student to engage in a more rigorous knowledge construction experience. META-DOCTRINE (Click to See Video)

3. **What are the dispositions of African American students toward mathematics when different language systems (visual imagery, movement, written/oral language, etc.) are integrated into their learning? (In short, how can semiotics impact the disposition of African American students toward mathematics?)**

My response to this question will be a review of my message at the beginning of this chapter. Although the disposition of the student is linked to his or her ability to actually produce a correct solution to a given mathematics problem, his or her disposition is also linked to his or her ability to successfully engage in the contextual analysis of the given mathematics problem. When working with students who do not routinely perform in the first quartile or when working with students who are convinced that they will never be successful in mathematics, I have found it wiser and more beneficial for the student to have the student separate the contextual analysis from the mathematical execution. When students are apprenticed in the tools that semiotics offers and apprenticed in a process-oriented metacognitive mindset, then the students act with greater intentionality and less impulsiveness. Consequently, their productive dispositions are heightened (Kilpatrick et al., 2001). META-DOCTRINE (Click to See Video)

Before closing this discussion, there was one more question that I presented to the reader in Chapter 1. I offered it as my foundational question, and I repeat it here:

What insight can an analysis of mathematics as a language provide in the formulation of a counter-narrative to this hegemonic propaganda?[28]

I think it appropriate that I answer this question by restating the macro-theory that resulted from my work: META-DOCTRINE (Click to See Video)

> The mathematics learning ecosystem, and by extension, the student, excels when it serves as a semiotics apprenticeship that guides students through metacognitive knowledge and metacognitive skills toward proficiency in inferencing, storytelling, and practicing.

Implications and "Next Steps"

I would like to present this last section in two parts. The first part will speak to the opportunities that are available for further theoretical development. Specifically, there is strong theoretical ground for advancing the CHAT aspect of the CHAT–Teaching Experiment Triplet Heuristic. The second part will speak to the opportunities that exist for further practical implementation in the mathematics classroom. Specifically, there are a multitude of facets of the CHAT–Teaching Experiment Triplet Heuristic that can be made manifest in mathematics classrooms across the nation.

First, I present my ideas for the further theoretical development of the CHAT–Teaching Triplet Heuristic. The cultural and historical elements of CHAT were not emphasized in this theory-building project. This was not an oversight, because their presence was implicit. Recall that this project centered on the experiences of an African American male teaching mathematics to African American students. As an impassioned professional with my lived experience, I took hold of the racial and ethnic cohesion between myself and my African American students. This

[28]It is helpful to recall my discussion on hegemony in Chapter 1. It is in Chapter 1 that the following discussion point was made: Hegemony pertains to an individual's subscription to ideas that manage his or her behavior for the benefit of an oppressive system or institution (Adamson, 2014).

deliberate yet tacit action allowed me to develop a mathematics caring relationship (MCR) with my students (Hackenberg, 2010), which allowed me to take aim at the *activity* aspects of CHAT.

I will not presume that my lived experience as a 40-year-old African American male math teacher is equivalent to the lived experiences of pre-teen-age African American students. However, I will say that I have lived through and continue to live through similar hegemonic institutional practices as the pre-teen-age African American students who I teach (Hilliard, 1994; Kozol, 2005; Ladson-Billings, 1997; Martin, 2000; Steele 1992). So, in some form, I embody many of the cultural and historical considerations that impact my students. In many cases, my students are not or were not aware of the depth and impact of these cultural and historical considerations. For example, many of my students are not familiar with the term *Achievement Gap*. So, although not emphasized within this theory-building project, the cultural and historical elements of CHAT were implicitly present in every decision that I made. In fact, it would not be an overreach for me to say that it was due to the cultural and historical elements of the mathematics education dilemma that exists here in the United States that I began this doctoral journey and theory-building project from the beginning. Nevertheless, in order for the CHAT–Teaching Triplet Heuristic to be as useful and impactful as possible, emphasis must be placed on the cultural and historical elements of CHAT and its ability to investigate the cultural and historical phenomena of the student, the researcher, the activity, and the learning environment. One opportunity for such an emphasis would be for the participants to provide their own cultural and historical profiles and synopses of their relationships with mathematics. Such involvement from the participants would move the cultural and historical elements of CHAT from an implied subordination to an explicit equivalence with the other aspects of the CHAT–Teaching

Experiment Triplet Heuristic. The reader can see the Afterword for further discussion on this topic.

Next, I present my ideas for some of the opportunities that exist for further practical implementation of the CHAT–Teaching Experiment Triplet Heuristic within the mathematics classroom. I thought it valuable to provide some considerations for the classroom mathematics teacher. It is one achievement to build theory in mathematics education; it is a much higher achievement to implement the theory into practice (Lynham, 2002). With practical implementation in mind, I offer a suggested lesson plan for the middle school or high school Algebra I teacher. The topic of linear equations is a popular topic in Algebra I courses, and so, in what follows, I provide an approach for incorporating the core components of the CHAT–Teaching Experiment Triplet Heuristic within the mathematics classroom.

General Information for a lesson on Linear Equations

> The setting is a Title I school. The students attend a middle school or high school; students self-identify as male and female. The population is diverse with the majority of the students being African American and Latinx. Some of the students have taken Algebra I before, but did not demonstrate proficiency on a standardized assessment. There are 30 students in the classroom. The areas of interest will be the important vocabulary, relationships, and representations of linear equations.

Classroom procedure

> The Algebra I class meets three times a week. The class meets for 90-minutes on Mondays and Wednesdays, and meets for 45 minutes on Fridays. There is one teacher in the classroom. There is no assistance from a paraprofessional or student-aide. During the first 2 weeks of the semester, the students are given access to their free online Algebra I textbook and their free online collaborative white board. The students are also given access to their district-sanctioned email accounts. A description of the events for each of the six class periods is provided in Table 7.1 and Table 7.2.

Table 7.1
Events for Class Periods #1, #2, and #3

Class Period #1	**Class Period #2**	**Class Period #3**
During Class Period #1, the teacher demonstrates how to navigate through the online Algebra I textbook. The teacher also demonstrates how to read the informational text within a mathematics textbook and shows the consistent structure throughout each lesson and chapter. The teacher guides the students to reading the text and locating mathematically important vocabulary words. Often in texts, such words are highlighted or are in bold type, but other times they are not. The teacher also guides the students to locating important formula or equations in the text. Often time such formulas or equations are outlined by a box, but other times they are not. The teacher emphasizes the importance of the students to translate mathematical formulas and equations into verbal expressions to increase the understanding and utility of the formula or equation. Last, the teacher demonstrates to the students how to read an example mathematics problem. The approach extends beyond looking at the example. The approach requires the student to identify each of the mathematical actions that are taken and place them in the appropriate sequence. In addition, the student must explain, in mathematical terms, why each mathematical action was taken and how it was performed. This type of an analysis of a mathematics example problem can be challenging for some students, because it requires the student to infer connections and relationships that may not be explicitly stated in the example problem.	During Class Period #2, the teacher demonstrates a syntactic analysis and a semantic analysis of the material in the text. For the syntactic analysis, for example, the teacher selects one of the various representations of a linear equation, pictorial, graphical, tabular, numerical, or conceptual word problem, and discusses its structural elements. In a word problem, the structural elements include the nouns, verbs, and numerical quantities. In a graph, the structural elements include the title of the graph, the axes, the labels for the axes, the numerical markings on each axis, and the image displayed on the graph. For the semantic analysis of a word problem, the teacher must discuss the meaning of the selected words and detail their relationships. For the semantic analysis of a graph, the teacher must generate a discussion of the meaning of the title, the axes, the labels for the axes, and the numerical markings on each axis. The teacher must also generate a discussion on the meaning of the image displayed on the graph. Important considerations include the linearity of the image, its direction, its slope, and the meaning of its slope.	During Class Period #3, the teacher presents the task of creating a visual model that represents the important elements and relationships of a linear function. The recommended image would include a combination of the various representative forms to allow for a greater degree of comprehension by the students. For example, the teacher could guide the students in the placement of a vocabulary word on an equation that has been correctly placed on the picture of a line that has been correctly placed on a Cartesian coordinate graph. This example contains four valid representations of a linear function and provides these representations in context and in relationship with one another. Descriptive classroom discourses should accompany this activity. At the conclusion of Class Period #3, or throughout the week, additional practice can be provided to the students in the form of homework. The teacher may find it beneficial to advocate that the students work together on the homework and engage in mathematics discussion with one another through an appropriate digital platform.

Table 7.2
Events for Class Periods #4, #5, and #6

Class Period #4	Class Period #5	Class Period #6
During Class Period #4, the teacher poses questions or instructions that require the students to give attention to the structural elements and relationships of a given linear mathematics situation. This activity simulates the inquiry or instruction that accompanies a mathematics problem. The inquiry or instruction could be to engage the students in conceptual understanding, procedural fluency, inquire about the definitions of various vocabulary words, or to correctly transfer relationships from one linear representation into another linear representation. The greater the variety of inquiry or instruction type, the more prepared the students will be when they have to engage future mathematics problems involving linear functions.	During Class Period #5, the teacher demonstrates how to generate a sequence of mathematical action steps to solve a problem involving a linear function. Brevity is ideal when constructing such action steps so that they are easy to generate, understand and explain by the students. It has been my experience that the inclusion of the respective vocabulary words and mathematical actions in the action steps is an effective indicator of a student's conceptual understanding.	During Class Period #6, the teacher demonstrates how to execute the previously created sequence of action steps. The teacher may find it most beneficial to guide the students in writing their mathematics in an orderly and algebraic manner. Providing only the arithmetic calculations in a mathematical solution without the appropriate algebraic structure can make it difficult for students to refer to their work or refer to another student's work. An agreed upon mathematical structure facilitates classroom discourse. In addition to facilitating classroom discourse, an agreed upon mathematical structure facilitates critical reflection that the teacher must also demonstrate. At the conclusion of Class Period #6, or throughout the week, additional practice can be provided to the students in the form of homework. The teacher may find it beneficial to include tasks that require the students to critically review a sample of work, work together, seek mistakes within the sample, and discuss corrective measures. The students can accomplish this work together through an appropriate digital platform.

What I have detailed is six intentional objectives, over six different class periods. The goal of the teacher is not for the students to demonstrate mastery of each objective at the end of the respective class period. The goal of the teacher is to establish a common foundation and a clear semiotic, discursive, and collaborative culture in the classroom. Each of these six objectives represents some form of the multi-dimensional discourse on roles and the activity dis-aggregates that I detailed earlier. Demonstrating these aspects in context helps to address the concern of a lack of time that is on the mind of many teachers. After such a common foundation and culture are established in the classroom, the expectation can be set by the teacher for the students to practice these roles and the activity dis-aggregates until they achieve proficiency and eventually mastery. What is important to note here is that these first 6 days are focused on building the culture and the community of the classroom, not on the successful completion of mathematics problems. Due to this focus, using the exploration, introduction, and application sequence (see Chapter 4) of the re-engineered teaching experiment is not warranted. Nevertheless, after the establishment of the culture and community within the classroom, the teacher may find great benefit in using the exploration, introduction, and application sequence throughout the remainder of the school year. Once the culture and community are established within the classroom, the teacher most expectedly will experience the students becoming independent learners and self-guided inquirers due to the fact that each portion of the multi-dimensional discourse on roles and the activity dis-aggregates has been demonstrated. The teacher might also experience subsequent benefits when he or she engages the students in mathematics discussions once such a culture and community are established because a common discourse will develop.

As the curriculum advances during the weeks and months, the teacher should not need to make changes to the epistemic culture and epistemic community that have been established. The specific topic of the mathematics curriculum is irrelevant once such an epistemic culture and epistemic community are established. However, over time, the teacher will realize that students are not developing at the same rate, which is to be expected. Over time, the teacher will be able to collect empirical data in the form of formative assessments and summative assessments and identify students who require more or less specific apprenticing in certain aspects of the multi-dimensional discourse on roles and the activity dis-aggregates. The teacher will be able to produce customized activities, whether individualized or grouped, for these students as needed.

Data Collection

I suggest allowing the students to collectively create and develop a note-taking template, a template for the syntactic analysis, a template for the semantic analysis, and a template for the sequence of action steps. Engaging the students in the creation of these templates builds a sense of community within the classroom and cultivates a sense of ownership within the students. The teacher may also find it valuable to have the students keep a mathematics journal so that the students can record their own mathematics learning experiences.

Final Summative Remarks

A firm commitment to a rigorous research study established my position that research can and should be more than simply seeking answers to research questions. I now understand that a research study should and can be the manifestation of the researcher’s identity fueled by his or her commitment to inquiry. As such, this CHAT infused re-engineered teaching experiment to examine a student’s interpretation of mathematics as a language system in order to engage mathematical situations more effectively satisfies the requirements of a rigorous and comprehensive design. It is hoped that the theoretical results of this re-engineered teaching experiment demonstrate the rigor of the approach and the effectiveness of a mathematics learning environment that highlights the semiotic aspects of mathematics. The ultimate aim of this work was to serve as a source, whether theoretical or practical, for a narrative of some of the contributing factors that correlate with mathematics proficiency for African American students.

I have enjoyed my growth and development within this theory-building project. To be clear, this has been an arduous process and has been the culmination of my 14-year teaching career. My ability to combine the learning from my doctoral training with the classroom experience as a mathematics educator has allowed me to manifest the necessary connection of theory and practice that substantiate a theory-building effort (Lynham, 2002). This project has allowed me to find new meaning in the phrase mathematics proficiency. The inclusion of semiotics as a tool in the mathematics classroom allowed my students to acknowledge and appreciate the semiosphere as an aspect of not only mathematical contexts but also of their lives. The resulting theories and their composites, although informal constructs, will be great guides for me as I continue to teach mathematics to young aspiring mathematicians. My adaptations of the teaching experiment to better accommodate the experiences of the students whom I teach have led to a more rigorous methodology and one which can provide greater insight into the mathematical experiences of the student and consequently, can provide greater insight into the construction of mathematical knowledge by the student.

META-DOCTRINE
(Click to See Video)

One of those adaptations is the explicit investigation of a student's metacognition. Some may argue that the teaching experiment has always examined the metacognitive knowledge and metacognitive skills of the student. In that case, my response is that my effort has been only to make such examination more explicit, and to establish that a metacognitive apprenticeship is a viable tool in the mathematics classroom. I also posit that such an apprenticeship is evidence of the overlay between sociocultural theory and radical constructivism. Now that this phase of this theory-building project is concluded, I feel motivated to repeat yet another before-mentioned statement:

> It is hoped that the efforts demonstrated within this theoretical project will chart a course for pivotal transformation in mathematics teaching and learning for all. It is also my hope that my dissertation work will be effective as one insightful and instructive example for releasing mathematics education from the repressive tentacles of hegemonic forces.

AFTERWORD

This final elaboration of my dissertation addresses three concerns, not an any particular order, noted during my oral defense; I address each concern in turn here. The first concern: the absence of an explicit conversation on the "blackness"[29] of the students who were referenced in this theory-building project, in general, and more specifically, on the blackness of the students who I teach. The state of blackness, of both the teacher and the student, does indeed impact the mentoring and apprenticing experiences that exist within the mathematics learning environment. The second concern: the absence of an explicit conversation on the fourth cueing system of semiotics, graphophonics. This theory-building project privileged three of the semiotic cueing systems, syntactic, semantic, and pragmatic. Nonetheless, the lack of a discussion on the fourth cueing system should not be interpreted as a lack of awareness on my part but rather should be interpreted as a lack of resources and time. In any event, my perspective on discourse and the graphophonic cueing system is provided for the reader's review. The third concern: the cognitive transition that is necessary to change the understanding of a concrete phenomenon into an abstract or conceptual understanding of its underlying and implicit relationships. Throughout this dissertation, I take time to articulate the students' experiences with meaning-making and knowledge-construction at a foundational level; however, I do not take an equivalent amount of time to articulate the measures which are necessary to elevate a student's understanding from a foundational level to the higher-order and more abstract conceptual level. I provide such insights briefly here. My thoughts and perspectives for each of these concerns are below, and I hope the reader finds the elaborations on these three concerns insightful.

[29] Blackness is used here to bring attention to the forever present anti-Blackness that permeates all aspects of social and political structures in the United States, implicitly and explicitly, including the mathematics classroom.

The "CH" part of CHAT = The "Blackness" of African American Students

In Chapter 7 of this dissertation, I discussed the opportunity for future research with the CHAT–Teaching Experiment Heuristic to intentionally transition the cultural and historical elements of CHAT from an implied subordination, as was done in this dissertation, to an explicit equivalence with the other aspects of the CHAT–Teaching Experiment Heuristic. Within Chapter 7, the example that I offered that could achieve this goal would be to have the participants provide their own cultural and historical profiles and synopses of their relationships with mathematics. Now, I would like to take an additional opportunity to expound on what this idea of having the participants provide their own cultural and historical profiles and synopses. To incorporate a cultural and historical interview aspect to the CHAT–Teaching Experiment Heuristic that focuses on the student's relationship and perception of mathematics would be a valuable addition.

Research shows that the relationship of many African Americans with mathematics has been deformed and strained due to stereotype threat (Steele & Aronson, 1995). Stereotype threat theory states that an individual will be at risk of confirming a negative stereotype about one's identity group (Steele & Aronson). In this case, the negative stereotype and falsehood is that African Americans are deficient in mathematics and that this deficiency is manifested in an achievement gap and accented through discursive practices. My theory-building project did not provide an explicit focus on this aspect of my students' identity or even my identity as teacher or researcher. But because my students are African American and because I am also African American, we each have been exposed and are exposed to this particular negative stereotype due to our blackness. How one responds to this negative stereotype or how one is inoculated against this negative stereotype is unique to the individual. Based upon my 14 years of teaching experience, my qualitative assessment generates a specific perception of the student when the student says, "I've always struggled in math" or when the parent of one of the students says, "I've never been good in math." My intuition suggests to me that stereotype threat may have a large adverse impact on not only my student, but also may have a large adverse impact on the student's family. So, as the teacher, I must be aware of such phenomenon, and I must be patient with and tolerant of the student's initial self-projections.

As I detailed in this dissertation, a powerful counter action that I employed with my students was repetition. Many of my students had to practice the same activity multiple times and I had to model the same line of thinking multiple times in order to penetrate through the student's initial self-projection of not being successful with mathematics. Only after such repetition by the student and by me, the teacher, was the student convinced that another path for his or her development was possible. It required commitment to repetition. In fact, it required a commitment to activity repetition until cognitive saturation resulted in cognitive redundancy.

What is meant by this cognitive redundancy is that my students needed to practice an activity until the breadth and depth of knowledge took the student through the stages of cognitive dis-equilibrium, assimilation, cognitive vacuum, and accommodation (Piaget, 1976). It also means that I, the teacher, had to model the same line of thinking multiple times until my student developed the breadth and depth of knowledge that took him or her through the stages of cognitive dis-equilibrium, assimilation, cognitive vacuum, and accommodation.

Upon reflection, I posit that the blackness of my students produced a longer and deeper commitment to this requisite amount of repetition by my students, in particular, and from African American students, in general. Consequently, this need also means that the teacher of African

American students must embody a longer and deeper commitment to the requisite amount of repetitive modeling of the same line of thinking. Now, there is a unique aspect when the teacher of African American students is himself or herself also African American. In this case, like in my case, when the African American teacher of African American students has also experienced stereotype threat then there is a purity in the authenticity of the length and depth of the teacher's commitment to the patience, tolerance, and repetitive modeling necessary to nurture African American students through the activity repetitions until their initial self-projections have been penetrated.

Symbolic Interactionism = "Discourse" and the 4-cueing systems of Semiotics

As I detailed in Chapter 3, semiotics centers on the existence of signs and symbols. Throughout this entire dissertation, I focused on only the syntactic, semantic, and pragmatic systems of semiotics. But there are indeed four cueing systems within semiotics; the fourth cueing system is graphophonic (Jones & Norris, 2005; Norris, 2002, 2004, 2008, 2014). The graphophonic cueing system includes non-verbal communication, verbal communication, and kinesics, where kinesics pertains to communication through physical body movement. Admittedly, I did not refer to the graphophonic cueing system in this project, but the reader should not view my privileging two of the four semiotic cueing systems as a lack of awareness. A lack of time and a lack of archived student work were the reasons for my choices. However, much of the observational data collected and subsequently analyzed, both for the individual student as well as for the three-member groups were non-verbal, verbal, and kinetic forms of communication. In short, the graphophonic cueing system was the engine, the very heart of this entire theory-building project, and future work in this area must maintain this trait.

Now, I would like to take a moment now to expound on my view of discourse in the mathematics classroom. I view discourse as potential and actual interaction, and therefore as potential and actual experience. Meaning-making potential and knowledge-construction opportunities are more fully realized when attention is given to all four semiotic cueing systems. Mathematics proficiency is influenced according to the human immersion and utility of all four of these cueing systems. As detailed in Chapter 3, the work of George Herbert Mead and extended through the work of Herbert Blumer established symbolic interactionism as the study of the human utility of signs and symbols. A broader and deeper understanding of human communication occurs when signs and symbols are understood as a nexus of all four of the cueing systems. This point is integral within the mathematics classroom. This nexus of all four cueing systems requires great attention by the mathematics teacher, student, and researcher because in the mathematics classroom, the discourse requires that action be taken, not just that comprehension be gained.

The Cognitive Continuum and the Cognitive Jump

This theory-building project resulted in an affirming paradigm for the mathematics learning of African American students. Such an affirmative paradigm becomes more potent when the discussion includes a response to the following question, “How can the understanding of a concrete phenomenon evolve into the realization of the more conceptual and abstract underlying relationships?” According to my work in this theory-building project, I provide the following response. When one attends to the details of a given situation and then commits oneself to seeking the underlying relationships within the given situation through inferencing and storytelling, then one can transition one’s mind from understanding the given information to appreciating the conceptual and abstract elements. Metaphors and analogies can also be used to facilitate this process (Presmeg, 1998). I call such a cognitive transformation the cognitive jump and it requires practice and time. The reader should recall the multiple iterations that my students required in order to gain a level of conceptual understanding that allowed them to feel accomplished. This cognitive process can be seen below in Figure A.1.

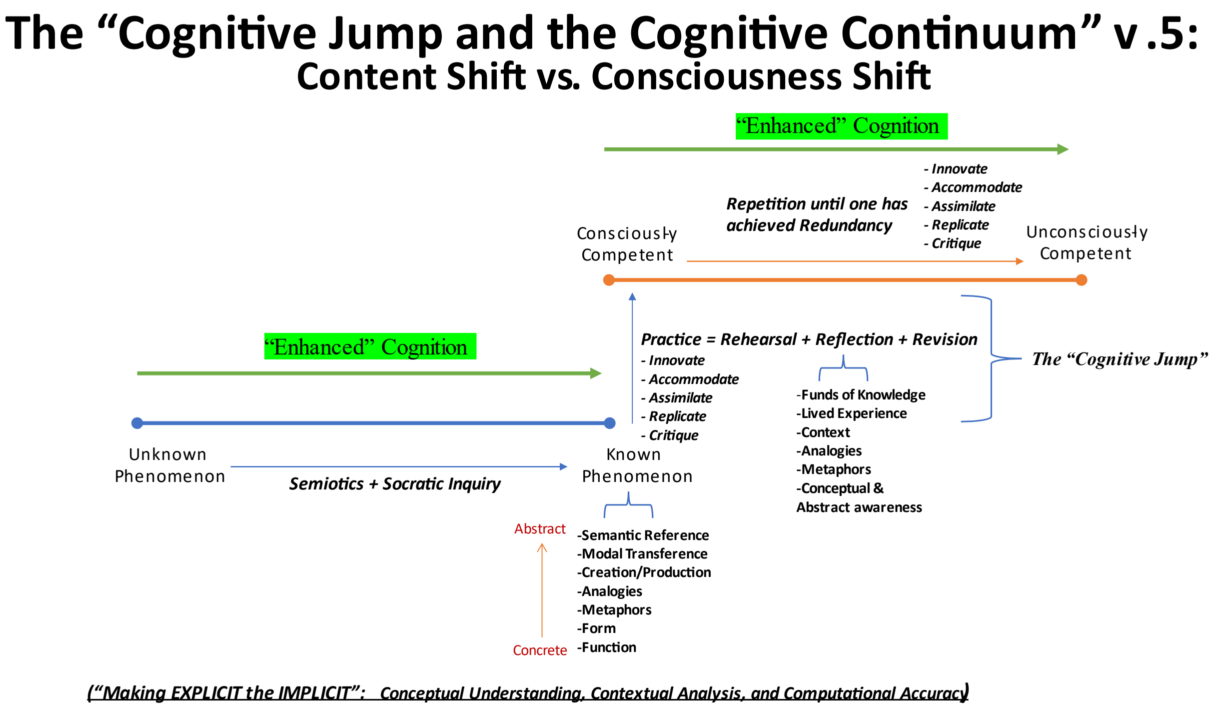


Figure A.1. Cognitive Jump and Cognitive Continuum

The reader should take notice of both the horizontal aspects of the figure, the cognitive continuum, as well as the vertical aspect of the figure, the cognitive jump. The reader should also take notice of the two separate locations where metaphors and analogies can serve to guide the student from a phenomenon's more immediate and tangible elements toward the phenomenon's more high-order and conceptual aspects. This step function semblance, or ladder-effect, provides a visual representation of my interpretation of the cognitive path that my students experience. My awareness of such a path and my ability to provide my students with the time-appropriate tools necessary for them to more effectively navigate this path facilitate the learning experiences of my students. When the student more effectively navigates this cognitive path, the student experiences greater foundational and higher-order meaning-making and knowledge-construction.

As I end these elaborations on my dissertation, I want to stress the importance of continuing this work. This effort should not and will not end on the manifestations of this simulation. The power of simulations exists in their ability to test hypotheses and forecast future events. I offer this theory-building project as a launch point from which both endeavors can be achieved.

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APPENDIXES

Inquiry Letter for Exempt-Status

Elijah Porter II
6162 Holt Road
Lithonia, GA 30058

October 1, 2020

To Whom It May Concern:

I, Elijah Porter II, Panther # 001-94-1286, am requesting advisory guidance on my use of data within my dissertation. I am a full-time mathematics instructor at the middle school and high school levels. I am currently pursuing a Ph.D. in Teaching and Learning with a concentration in Mathematics Education with an anticipated graduation date of Spring 2021. My advisor is Dr. David Stinson.

Currently, I am nearing the end of the dissertation writing stage of my work. I have completed the initial draft of my dissertation and have received feedback from Dr. Stinson. During spring semester 2020, I had to re-structure my dissertation design due to the COVID19-pandemic and the national mandate for social distancing. A face-to-face participant-engaged dissertation design was no longer possible, and it affected my data collection and analyses processes. The alternative structure of my dissertation required that I utilize as resource material my non-formal journal notes, analytic memos, observation experiences, student-teacher conferences, teaching templates, and reflections, which I had accumulated over my fifteen-year teaching career. Such material items are typical tools for professional educators to generate and use in order to improve their instruction. Although it was never my intention to use this resource to frame a dissertation, the COVID19-pandemic and social distancing gave me no other option.

In order to produce a database that could supply a dissertation, I used the resource material from my classroom teaching and my tutoring sessions of my most recent three years. I used this resource material to generate composite participants for my dissertation work. I am including samples of this resource material with this letter and request your review of this resource material and your guidance in how I may move forward with the submission of the resulting dissertation. Your insight on this matter is necessary and appreciated because the notes and recollections of a professional teacher are not IRB-sanctioned data sources. As such, an appropriate protocol for the writing and submission of a dissertation rooted in such resource material is requested.

I have attached an outline that specifies the type and sequence of the resource material that I have used. I have also included a naming convention, which aligns each resource item with the outline for your reference.

I look forward to progressing through the next steps of my doctoral process in accordance to your guidance. Thank you for your time and consideration.

Regards
Elijah Porter II

Department Chair Support Correspondence

Gertrude Tinker Sachs <gtinkersachs@gsu.edu>
Tue 10/20/2020 4:23 PM

To:

- David Stinson;
- Elijah Porter, II

Cc:

Deonne Janelle McNeill <dmcneill@gsu.edu>

Dear Elijah, Dr. Stinson,

I have read your proposed dissertation research protocol and approve it.

Congratulations and good luck!
Dr. Tinker Sachs
Department Chair
Department of Middle and Secondary Education

DM

Deonne Janelle McNeill <dmcneill@gsu.edu>
Tue 10/20/2020 3:20 PM

To:

- David Stinson;
- Elijah Porter, II;
- Gertrude Tinker Sachs <gtinkersachs@gsu.edu>

Hi Elijah,

Please upload Dr. Tinker Sachs email to the IRB application before submitting back to the IRb for review.

Thank you,
Deonne McNeill, M.S.

Georgia State University
IRB Compliance Specialist
Office: 404-413-3503
Fax: 404-413-3504
Email: dmcneill@gsu.edu

Mailing Address:
Office of Research Integrity
P.O. Box 3999
Atlanta, Georgia 30302-3999

IRB-Exempt Application

Exempt Study (Version 1.4)

1.0 General Information

***Please enter the full title of your study:**

SEMIOTICS AND SYMBIOSIS—"GAP-CLOSING":
HOW SIGNS, SYMBOLS AND STRUCTURE IMPACT THE TEACHING AND LEARNING OF MATHEMATICS FOR
MIDDLE SCHOOL AFRICAN AMERICAN STUDENTS

***Please enter a short title for your own personal reference.**

SEMIOTICS AND SYMBIOSIS—"GAP-CLOSING"
* This field allows you to enter an abbreviated version of the Study Title to quickly identify this study.

2.0 Add Department(s)

2.1 Your department is listed below. Click "add" to add an additional department or select the check box next to the department and select "remove" to remove it. PLEASE DO NOT LEAVE "GSU - Georgia State University" AS YOUR PRIMARY DEPARTMENT.:

Primary Dept?	Department Name
○	**GSU** - Georgia State University
◉	**GSU** - Middle & Secondary Education

3.0 Assign Study Personnel

3.1 *Please add a Principal Investigator for the study:

Stinson, David

3.2 If applicable, please select the Research Staff personnel (If you are adding a GSU student, staff or faculty member and their name does not appear in the list of personnel, ask that person to log-in to iRIS with his/her campus ID and password which will populate their name in the list. If you are adding personnel from outside GSU and their name does not appear in the list, they can be added with the form available at https://irbaccountrequest.gsu.edu/

A) Additional Investigators

Porter, Elijah
Student PI

B) Research Support Staff

3.3 *Please add a Study Contact:

Porter, Elijah
Stinson, David

The Study Contact(s) will receive all important system notifications along with the Principal Investigator. (e.g. The project contact(s) are typically either the Study Coordinator or the Principal Investigator themselves).

4.0 Please answer the questions below regarding research personnel:

4.1 * **Human Subjects Training is a requirement for approval. Have you and your research team members completed Human Subjects Training?**

(•) Yes ( ) No

4.2 * **Below is the PI you selected. Please confirm that the PI is eligible to serve as the Principal Investigator for this study. In general, the PI must be a current, full time faculty or staff member (no students may serve as a PI). For more information see 4.5.3 of the IRB Manual.**

David Stinson

Is the PI eligible?

(•) Yes ( ) No

4.3 * **Below is the department you selected. Please confirm that the department listed is the correct department for the study. (GSU - Georgia State University can NOT be listed as the department)**

GSU - Georgia State University, GSU - Middle &Secondary Education

(•) Yes ( ) No

5.0 Exempt Category Questions

5.1 * **Does the study involve any information that will be submitted to the Food and Drug Administration (FDA)?**

( ) Yes (•) No

5.2 * **Does the study involve prisoners?**

( ) Yes (•) No

5.3 * **Which of the following does the research involve? (check all that apply)**

- [ ] Surveys or interviews
- [ ] Educational tests (Cognitive, diagnostic, aptitude, achievement)

- [ ] Observation of public behavior
- [ ] Benign behavioral intervention
- [ ] Research or demonstration project that is conducted by or supported by a federal department or federal agency to study public benefit programs
- [ ] Taste and food quality evaluation
- [x] Normal educational practices in a normal education setting
- [ ] Secondary research of data or biospecimens

5.4 * Does the study involve interviews with children?

(•) Yes ( ) No

*** Are the interviews a component of the normal classroom practices and not solely for the purpose of generating data for research?**

(•) Yes ( ) No

5.5 * Does the study involve surveys or questionnaires of minors?

( ) Yes (•) No

5.6 * Describe how procedures are normal education practices or assessment of educators providing education (including regular or special education instructional strategies and the effectiveness of comparison of education techniques, curricula, or classroom management).

The results of normal Student-Teacher conferences and discussions which occurred were used. These discussions require the student to specify his or her thought processes and the selected mathematical procedures used to solve assigned mathematical problems.

5.7 * Describe how the research will not adversely impact students' opportunity to learn required educational content and any precautions in place.

The research was based upon normal daily mathematics customs, routines, and activities.

There were no deviations in the students' learning opportunities.

5.8 * Will the study involve data from student education records (e.g. class work, grades, attendance records, communications, projects, classroom tests, standardized tests, journals, SAT/ACT scores, etc.)? This list is not exhaustive. Please see section 1.6 of the IRB manual for more information on FERPA records.

○ Yes ◉ No

5.9 * Will participants be compensated or given the opportunity for extra credit?

○ Yes ◉ No

5.10 * Is the research conducted in an established or commonly accepted educational setting?

◉ Yes ○ No

*** Describe the research location. (It is the PI's responsibility to ensure adequate permission for each site is obtained.)**

My mathematics classroom. As well as the residences of my tutoring students with adult supervision.

5.17 * Describe in lay terms the purpose of the research.

The purpose of this research study is to explore the influence that a linguistic approach to the teaching and learning of mathematics has on the cognitive activity and knowledge-construction of middle school African American students. This study will fill gaps in research on mathematics learning and teaching as it relates to the perception and comprehension that African American students have of mathematics.

5.18 * Describe how human subjects will be involved. If there is interaction with participants, describe all the proposed procedures for research. If the study also involves analysis of secondary research, describe the content and source of the data or biospecimens.

This study represents my reflections and analyses of my experiences with my mathematics students of the past three years, which occurred in my mathematics classroom. This study also represents my reflections and analyses of my experiences with middle school students who I tutored during the past three years. The tutoring sessions occurred in the homes of the separate students who I tutor in mathematics. All of the participants are middle
school African American male and female students (6th, 7th, and 8th grade) located in the southeastern United States.
For this teaching experiment, as is common with other teaching experiments,
discursive inquiry will be a major source for gathering data. The student PI will take part in the instructional activities, and
events of the participants as a means of learning about their learning experience (Steffe, 1991). The research team composed of the student PI, and the Principal PI, will be
the only individuals to have access to the data. I (student PI) will be the sole investigator in the data collection for this study.
There will be two categories of activities. The first category will be called, "Direct Instruction". This will include three "instructional periods" led by the student-PI. The second category will be called, "Teachable Moments". This will include three "discursive inquiry conferences" between the student-PI and the participants. The following paragraphs summarize each "instructional period".

1.) Period #1 – Instruction: Linguistic Analysis
In this instructional period, the researcher will first present the fundamental parts of speech to the student: noun, verb, adjective, adverb, preposition, coordinating conjunction, subordinating conjunction, and complement. Then, the student will demonstrate his/her understanding of the fundamental parts of speech as presented by the researcher during the Instructional Period. Last, the student will present any questions, comments, or concerns about his/her understanding of the fundamental parts of speech so that the student-PI can address any mistakes, misconceptions, and inconsistencies.

Period #2 – Instruction: Mathematics Concept
In this instructional period, the researcher will present the investigative techniques to be used when pursuing the following attributes of an unfamiliar word or new mathematical concept: definition, description, features, characteristics, relationships, representations, significance and purpose. Then, the student will demonstrate his/her functionality with the investigative techniques to be used when seeking the attributes of an unfamiliar word or new mathematical concept. Last, the student will present any questions, comments, or concerns about his /her understanding of the investigative techniques so that the student-PI can address any mistakes, misconceptions, and inconsistencies.

3.) Period #3 – Instruction: Bar Modeling/Model Construction
In this instructional period, the researcher will present the mathematical modeling technique of Bar Modeling to be used when modeling the foundational principles of mathematics: addition, subtraction, multiplication, division, fractions, ratios, and proportions. Then, the student will demonstrate and explain the mathematical modeling technique of Bar Modeling. Last, the student will present any questions, comments, or concerns about the mathematical modeling technique of Bar Modeling so that the student-PI can address any mistakes, misconceptions, and inconsistencies.

The following paragraphs summarize the "discursive inquiry conferences".

Application and Discursive Inquiry. During these conferences, the student will make attempts to solve various mathematics word problems. Also, the student-PI and student will engage in dialogue to evaluate the student's conceptual understanding, procedural fluency, strategic thinking, and adaptive reasoning with regards to the new mathematics word problems which were provided. Specifically, the student-PI will pose open-ended questions to the student in the following areas:
The student's linguistic analyses of the word problems
The student's problem-solving approach
The student's construction of bar models to solve the word problems

During this entire research effort, the research team will observe restrictions imposed by Georgia State University, and relevant government or public health authorities in the conduct of research activities.

5.19 * Is the study funded?

○ Yes ◉ No

A research protocol must be uploaded in the study document section during submission. If a study plan is included in the thesis, dissertation, or prospectus, it can be uploaded to meet this requirement.

5.20 * Is this study or any part of this study contributing to a dissertation or thesis?

(•) Yes ( ) No

Is the research being conducted ONLY for a thesis, dissertation, or capstone project?

(•) Yes ( ) No

5.73 * Does the research involve the use of radiation or lasers on or at a GSU campus of facility?

( ) Yes (•) No

5.74 * Does the study involve the use of non-human animals (e.g., dogs, mice, non-human primates, etc.)?

( ) Yes (•) No

5.75 * Will the study involve the use or possible exposure to infectious material (e.g., blood, bodily fluids, mucosal swabs, tissue samples, etc.)?

( ) Yes (•) No

6.0 Data Collection

6.1

*** Is information, which may also include information about biospecimens, recorded or obtained in a way that makes it identifiable? This includes having identifiable information or a link, code, or key that links the identifying information to the data. Identifying information includes (but is not limited to) name, social security or student ID number, date of birth, contact information including email address or phone number, photographs, and audio or video recordings.**

( ) Yes (•) No

6.4

*** Describe confidentiality information. State where and how any data/biospecimens will be collected /obtained, stored, and transported; who will have access to the data and what will be done with it after the study is over; protections for storing or sharing hard-copy and electronic data (flash drive, cloud storage, Dropbox, etc.) If a code sheet will be used to separate identifying information from the participant data describe the means of protecting this document. If identifiable data are inadvertently collected, please state how it will be managed.**

My reflective journals, field notes, and analytic memos from classroom and tutoring activities were used as the data for this project. My reflective journals, field notes, and analytic memos were crucial to recording the
participants' actions during the activities.
I will delete/disregard any information that may cause discomfort or uneasiness.
The referencing data, including my (the student-PI) observation notes, and journals will be kept in a secure place. Only the research team will have access (student-PI, and Principal PI). Pseudonyms, composites, and profiles will be used so as to protect the identity of the participants. Security and identity are essential elements of this work, therefore, any and all declarations of names are fictitious and are used solely to allow for easier reference throughout the work.
The student-PI will transport and secure all data in research office.
Only the research team (student-PI and Principal PI) will have access to the referencing data. The information provided will be stored on the student-PI's
password-and firewall-protected computer.
When the study is completed, the data will be stored on a flash drive and kept in the research office under lock and key. The flash drive will be erased within one year of the completion of the study.

6.5

*** Will consent be obtained from participants?**

Exempt consent has fewer requirements than the Expedited/Full informed consent. For more information about consent for exempt studies please see Exempt Consent Document .

○ Yes ◉ No

6.6

*** Provide a justification for not obtaining consent.**

My reflections and analyses were based upon the information that the students generated during the normal activities of a mathematics classroom environment or a mathematics tutoring session. My reflections and analytic memos were then used as the data source for this project. No actual student work was used within the framework of this project. This entire project was based upon and resulted from my experiences, reflections, and analyses of my student interactions as a mathematics classroom teacher and mathematics tutor. The purpose of this effort was to understand their knowledge-construction and their mathematical procedures. None of their student work will be included in the project. Only my reflections and analytic memos about their student work will be included in the project.

7.0 COI

7.1

*** Does the PI, Co-Investigators, or other research staff including their spouse and dependents have a significant financial conflict of interest defined as:**

- An equity interest that, when aggregated for the Investigator or research staff and their spouse or dependents meets all of the following tests: Exceed $5,000 in value as determined through reference to public prices or other reasonable measures of fair market value, represents more than a 5% ownership interest in any single entity, and value is affected by the outcome of the research; or

- Salary, royalties or other payments that, when aggregated for the Investigator or research staff and their spouse and dependents over the next 12 months, are expected to exceed $5,000 and value is affected by the outcome of the research.

○ Yes ◉ No

7.2

*** Does the PI, Co-Investigators, or other research staff including their spouse and dependents have:**

- A board or executive relationship related to the research regardless of compensation.

- Proprietary interest related to the research including by not limited to a patent, trademark, copyright, or licensing agreement.

○ Yes ◉ No

8.0 Endorsement

8.1

*** The Principal Investigator and research team are responsible and accountable for the research design and implementation. With the exempt categories, the IRB is making a determination of exempt. While less information is reviewed by the IRB, it is still important for the researchers to ensure all aspects of the study are completed in a way that protects human subjects and minimizes risk. Additionally, the research team is responsible for ensuring all qualifications and permissions are in place before conducting a study.**

◉ I agree

8.2

*** Please affirm the following endorsement statement:**

• I will notify GSU IRB through the iRIS system of any non-compliance, deviations, unanticipated problems, or suspensions/terminations;
• I will complete a status check at the appropriate time;
• I will gain IRB approval before altering the research study;
• I will notify the IRB if there are any changes in my contact information.

◉ I agree

IRB Outcome Letter

INSTITUTIONAL REVIEW BOARD

Mail: P.O. Box 3999 In Person: 3rd Floor
Atlanta, Georgia 30302-3999 58 Edgewood
Phone: 404/413-3500 FWA: 00000129

November 24, 2020

Principal Investigator: David Stinson

Key Personnel: Porter, Elijah; Stinson, David

Study Department: Georgia State University, Middle & Secondary Education

Study Title: SEMIOTICS AND SYMBIOSIS—"GAP-CLOSING":

HOW SIGNS, SYMBOLS AND STRUCTURE IMPACT THE TEACHING AND LEARNING OF MATHEMATICS FOR

MIDDLE SCHOOL AFRICAN AMERICAN STUDENTS

Submission Type: Exempt Protocol Category 1,4

IRB Number: H21208

Reference Number: 362660

Determination Date: 11/20/2020

Status Check Due By: 11/19/2023

The above referenced study has been determined by the Institutional Review Board (IRB) to be exempt from federal regulations as defined in 45 CFR 46 and has evaluated for the following:

1. Determination that it falls within one or more of the eight exempt categories allowed by the institution; and
2. Determination that the research meets the organization's ethical standards

If there is a change to your study, you should notify the IRB through an Amendment Application before the change is implemented. The IRB will determine whether your research continues to qualify for exemption or if a new submission of an expedited or full board application is required.

A Status Check must be submitted three years from the determination date indicated above. When the study is complete, a Study Closure Form must be submitted to the IRB.

This determination applies only to research activities engaged in by the personnel listed on this document.

It is the Principal Investigator's responsibility to ensure that the IRB's requirements as detailed in the Institutional Review Board Policies and Procedures For Faculty, Staff, and Student Researchers (available at gsu.edu/irb) are observed, and to ensure that relevant laws and regulations of any jurisdiction where the research takes place are observed in its conduct.

Any unanticipated problems resulting from this study must be reported immediately to the University Institutional Review Board. For more information, please visit our website at www.gsu.edu/irb.

Sincerely,

Alison Alesi, IRB Member

META-DOCTRINE
(Click to See Video)

PowerPoint slides from Doctoral Defense Presentation

March 31, 2021

(10:00am Eastern)

SEMIOTICS AND SYMBIOSIS— "GAP-CLOSING":

HOW SIGNS, SYMBOLS, AND STRUCTURE IMPACT THE TEACHING AND LEARNING OF MATHEMATICS FOR MIDDLE SCHOOL AFRICAN AMERICAN STUDENTS

by

ELIJAH PORTER, II

College of Education and Human Development
Georgia State University

Atlanta, GA

March 31, 2021

Agenda

1) The "Complexities of Mathematics"
2) The "Pillars of Mathematics Proficiency"
3) My "Proficient Critical Thinker" – The "Pillars of Mathematics Proficiency"
    a.) A "Cultivator's/Grower's Mindset"
    b.) The "Ecosystem of a growing seed"
    c.) The "Cross-Section of Learning"
4) My "Proficient Critical Thinker" – The "Student"
    a.) The "Experiential Reference"
    b.) The "Two separate Experiential Paths"
    c.) Model for the "Academic Episteme"
5.) My "Proficient Critical Thinker" – The "Researcher"
    a.) The CHAT-Teaching Experiment heuristic: Overview
    b.) The CHAT-Teaching Experiment heuristic: Theoretical View
    c.) The CHAT-Teaching Experiment heuristic: Methodological View
6.) My "Proficient Critical Thinker" – The "Teacher"
    a.) The "Metacognition – Cognition Duality"
    b.) The "Primary Concept – Secondary Concept Duality"
    c.) The "Efficacy – Agency Duality"
7.) The "Collaborative Team of the Learning Environment"
8.) Answers to my Research Questions
9.) Closing Remarks – "The Cognitive Jump and The Cognitive Continuum"

Important Vocabulary

1.) **Mathematics** – the use of representations, relationships, and order to describe a situation or phenomenon

2.) **Semiotics** – the study of using representations and relationships to describe a situation or phenomenon

3.) **Symbiosis** – a relationship based upon cooperative and collaborative teamwork for the benefit of all involved

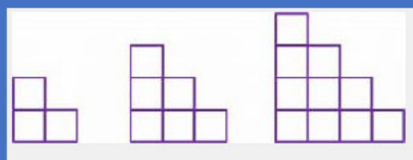
4.) **Cultural-Historical Activity Theory (CHAT)** – the theory that learning occurs according to one's identity, social group, language and activity, and the historical shifts of each

5.) **The Teaching Experiment** – the science that promotes the idea that a person's learning can be studied through the person's experiences

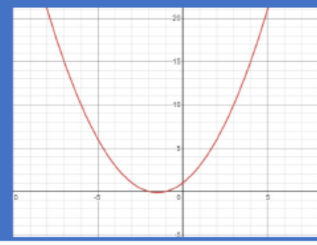
Defining statement: This work focused on the study of one's mathematical learning according to one's mathematical experiences. I privileged the importance of the representations and relationships used to describe the person's identity, symbiotic group, language, activity, and the historical shifts of each.

What do you see?

Pictorial



Graphical



Tabular

x	$g(x)$
-2	0
-1	0
0	1
1	3
2	6
3	10
4	15

Numerical

$$g(x) = \frac{1}{2}\left(x + \frac{3}{2}\right)^2 - \frac{1}{8}$$

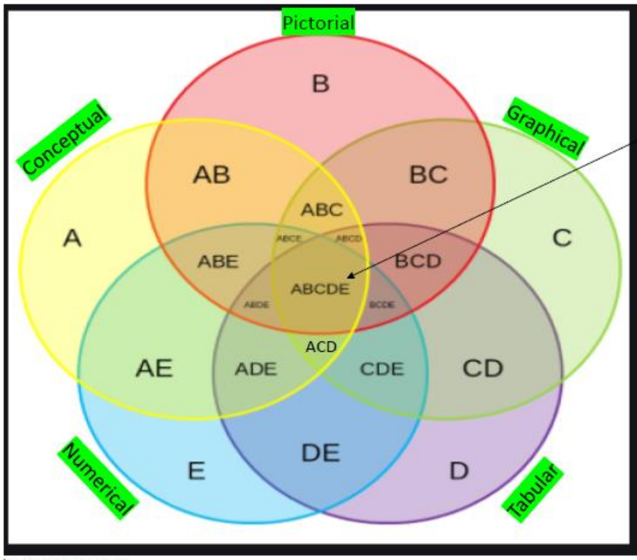
$$g(x) = \frac{1}{2}(x+1)(x+2)$$

$$g(x) = 0.5x^2 + 1.5x + 1$$

Conceptual

One-half times the product of one more than a number and two more than a number.

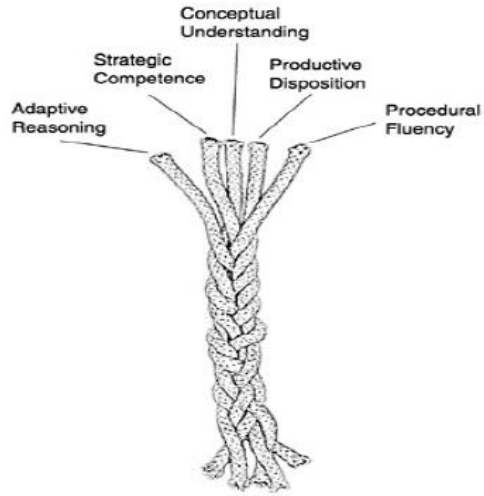
The 5-Modalities of Mathematics



The commonality within all 5-Modalities is the idea that there are Vital Signs or Essential Elements of a mathematical concept which can be found in any of the representatives. One example of a Vital Sign or Essential Element is "function type".

Adapted from:
https://images.app.goo.gl/kixrnipTEWxc5wky9

Mathematics Proficiency according to *Adding It Up* report

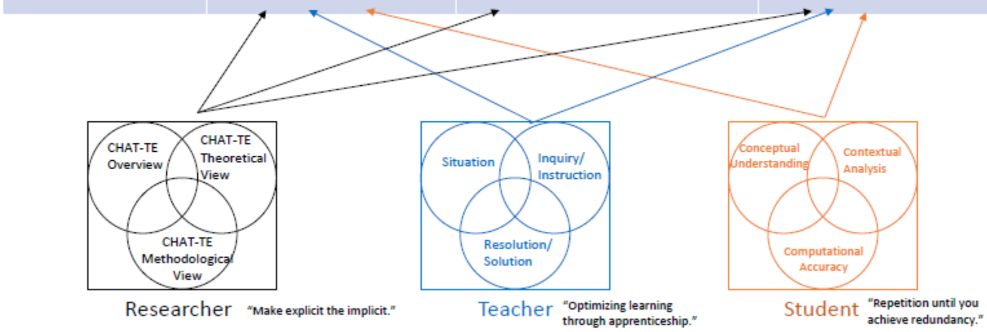


Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding It Up: Helping Children Learn Mathematics*. Washington, D.C.: National Academies Press.

My "Proficient" Critical Thinker

Item	Critical Questions	Semiotic Element	Role (Individual and Team)
The Situation	What do you see?	Syntax	Researcher {Conceptual Understanding}
The Situation	What does it say?	Syntax, Semantics	Analyzer {Adaptive Reasoning}
The Situation	What does it mean? (Conceptually and Contextually)	Semantics (Semantic Reference, Modal Transference, Creation/Production, Form, and Function)	Interpreter {Conceptual Understanding}
Inquiry/Instruction	What must be done?	Pragmatics(Problem-solving)	Designer {Strategic Competence}
Inquiry/Instruction	How shall it be done?	Pragmatics (Problem-solving)	Designer {Procedural Fluency}

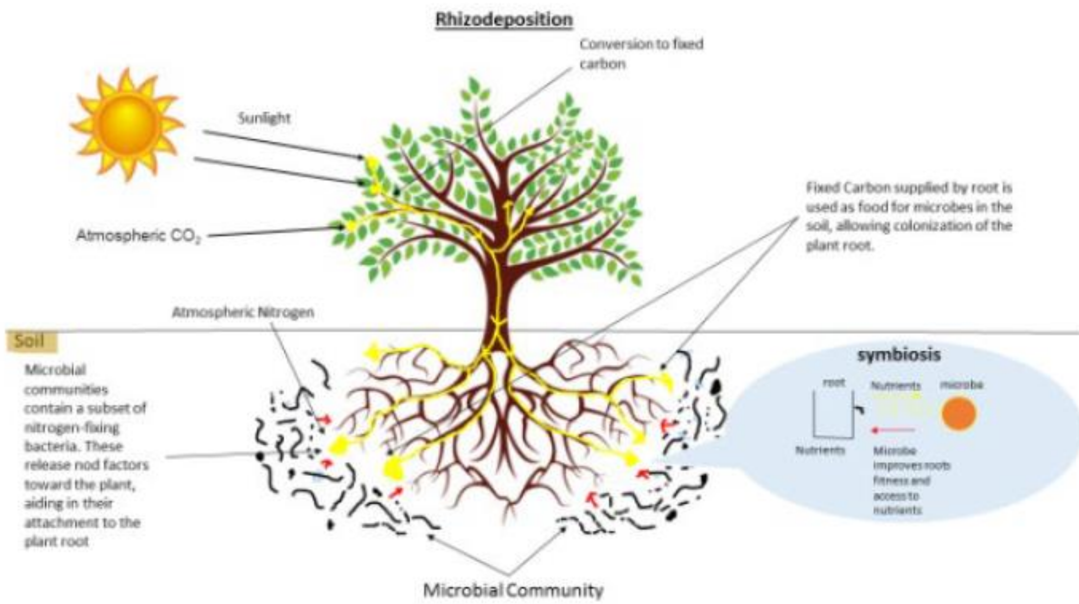
Evaluator {Productive Disposition}



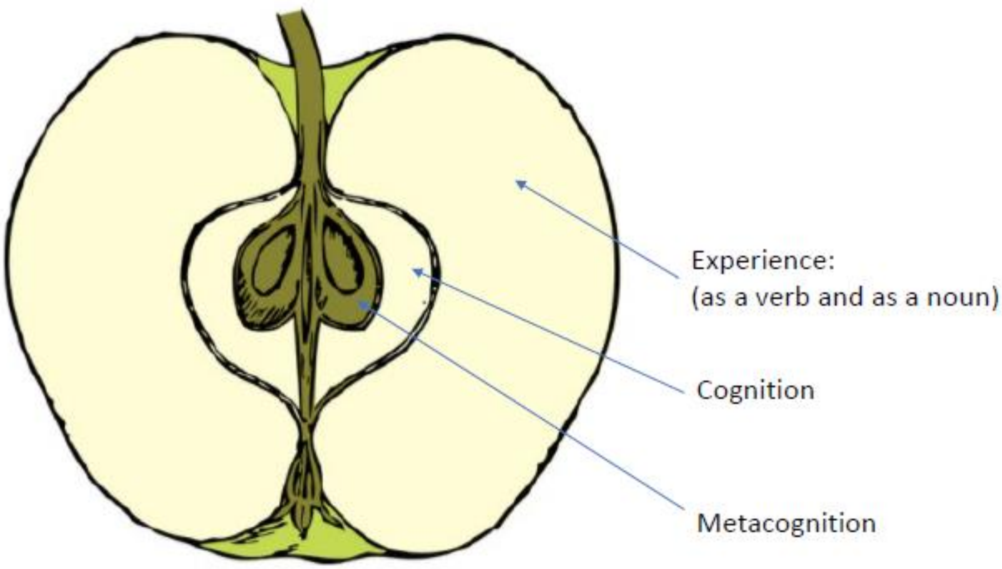
Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding It Up: Helping Children Learn Mathematics*. Washington, D.C.: National Academies Press

Image of “Farmers of Proficiency”

Image of Tree



Cross-Section of "The Apple"



My "Proficient" Critical Thinker

Item	Critical Questions	Semiotic Element	Role (Individual and Team)
The Situation	What do you see?	Syntax	Researcher {Conceptual Understanding}
The Situation	What does it say?	Syntax, Semantics	Analyzer {Adaptive Reasoning}
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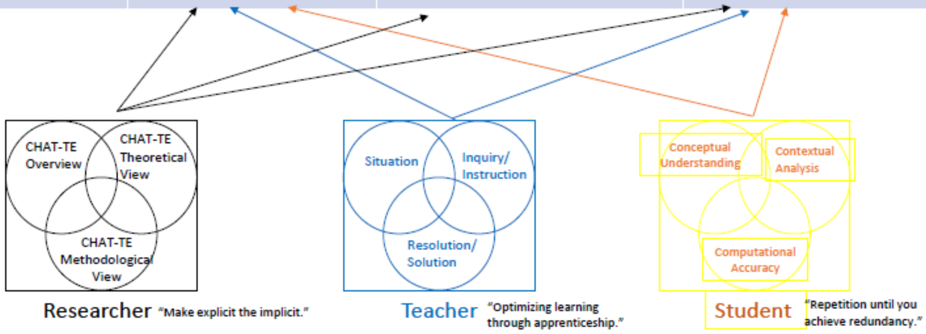
Inquiry/Instruction	How shall it be done?	Pragmatics (Problem-solving)	Designer {Procedural Fluency}

Evaluator {Productive Disposition}



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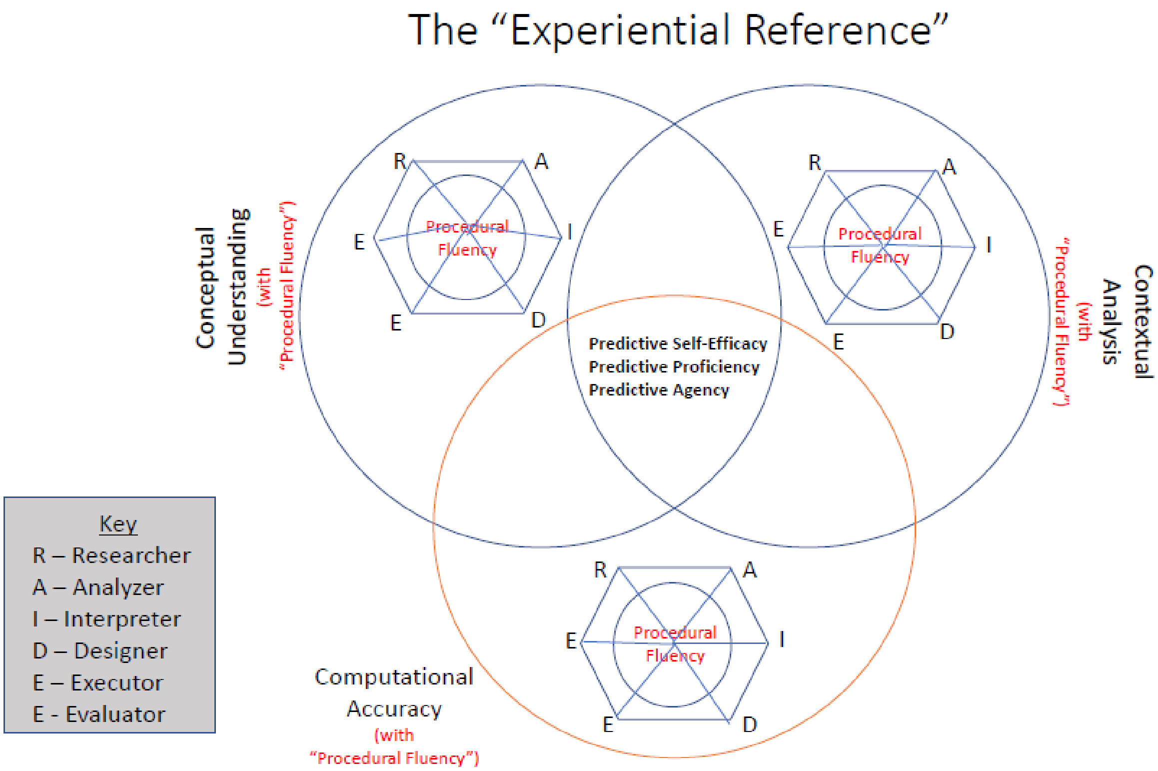
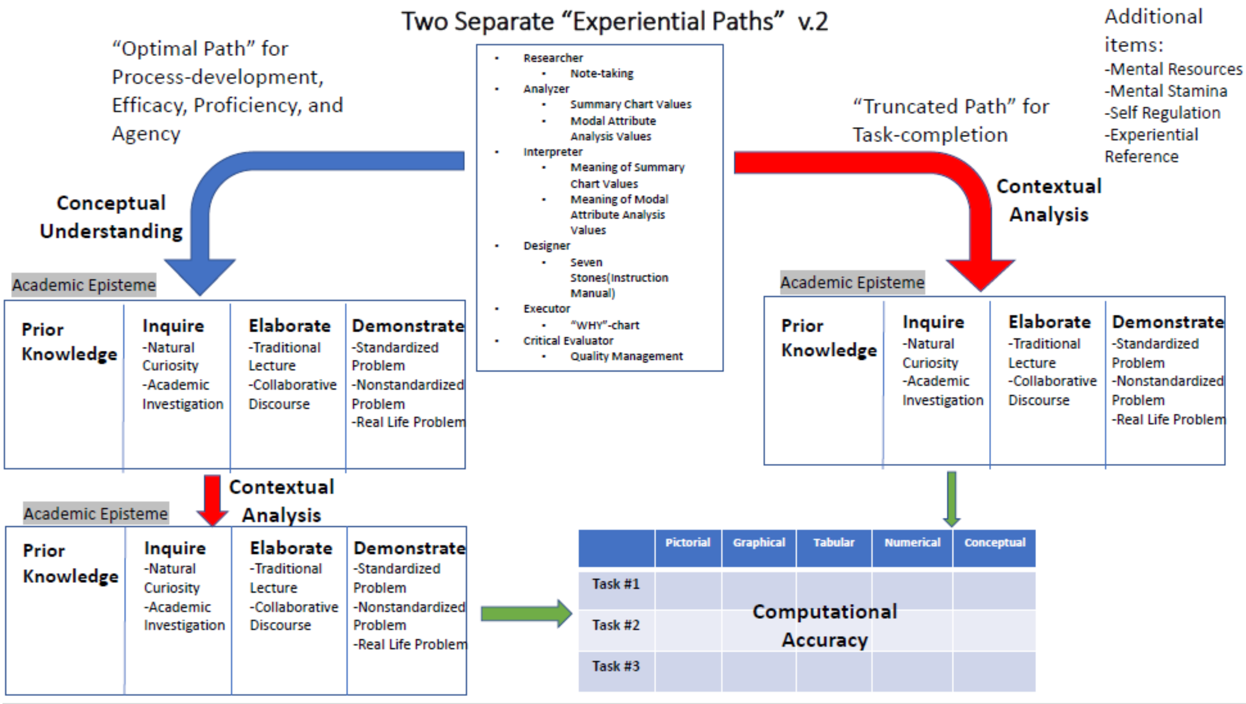

The "Experiential Reference"
Conceptual Understanding
(with "Procedural Fluency")
Contextual Analysis
(with "Procedural Fluency")
Computational Accuracy
(with "Procedural Fluency")
Predictive Self-Efficacy
Predictive Proficiency
Predictive Agency
Procedural Fluency
R
A
I
D
E
E
Key
R – Researcher
A – Analyzer
I – Interpreter
D – Designer
E – Executor
E - Evaluator

Experiential Reference

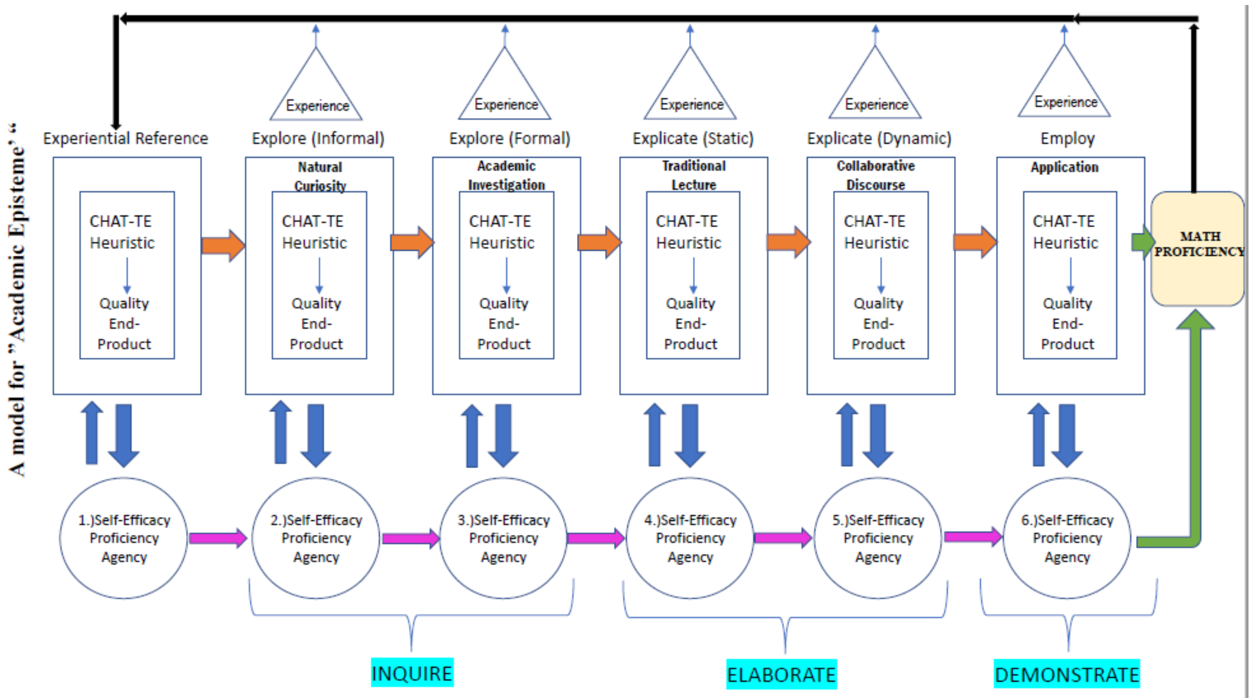
A Resulting Integrative Statement would be the following: *__The intersection of Conceptual Understanding, Contextual Analysis, and Computational Accuracy generates a person's feeling of confidence that he/she has the ability to proficiently achieve a mathematical goal.__*
Each Cognitive Expanse is fueled by the 6-Embodiments of Proficiency: Researcher, Analyzer, Interpreter, Designer, Executor, and Evaluator.
The "Mathematics Experiential Reference" seems composed of three Cognitive Expanses: Conceptual Understanding, Contextual Analysis, and Computational Accuracy.



Two Separate "Experiential Paths" v.2
"Optimal Path" for Process-development, Efficacy, Proficiency, and Agency
Researcher
Note-taking
Analyzer
Summary Chart Values
Modal Attribute Analysis Values
Interpreter
Meaning of Summary Chart Values
Meaning of Modal Attribute Analysis Values
Designer
Seven Stones(Instruction Manual)
Executor
"WHY"-chart
Critical Evaluator
Quality Management
"Truncated Path" for Task-completion
Additional items:
-Mental Resources
-Mental Stamina
-Self Regulation
-Experiential Reference
Conceptual Understanding
Contextual Analysis
Academic Episteme
Prior Knowledge
Inquire
-Natural Curiosity
-Academic Investigation
Elaborate
-Traditional Lecture
-Collaborative Discourse
Demonstrate
-Standardized Problem
-Nonstandardized Problem
-Real Life Problem
Academic Episteme
Prior Knowledge
Inquire
-Natural Curiosity
-Academic Investigation
Elaborate
-Traditional Lecture
-Collaborative Discourse
Demonstrate
-Standardized Problem
-Nonstandardized Problem
-Real Life Problem
Contextual Analysis
Academic Episteme
Prior Knowledge
Inquire
-Natural Curiosity
-Academic Investigation
Elaborate
-Traditional Lecture
-Collaborative Discourse
Demonstrate
-Standardized Problem
-Nonstandardized Problem
-Real Life Problem
Pictorial
Graphical
Tabular
Numerical
Conceptual
Task #1
Task #2
Task #3
Computational Accuracy

Experiential Paths

A Resulting Integrative Statement would be the following: **Most of my students initially travel along the Contextual Analysis Path and then they determine that restricting themselves to a non-evolved Prior Knowledge and a non-evolved 6-Embodiments of Proficiency does not lead to a successful completion of the task. Such a disruption made the students more receptive to travelling the Conceptual Understanding Path.**
The 6-Embodiments of Proficiency and one's Prior Knowledge cultivate a broader and deeper degree of knowledge when the person travels along the Conceptual Understanding Path due to the amount of reflecting and revising that is possible before addressing the given problem.
The Conceptual Understanding Path allows for the activation, integration, and enhancement of one's Prior Knowledge according to new ideas, skills, reflections, and revisions. The Contextual Analysis Path allows for the activation of one's Prior Knowledge according to the given problem.
For every activity, there are two potential experiential paths. One path is process-focused and emphasizes conceptual understanding. The second path is product-focused and attempts to emphasize contextual analysis.
The 6-Embodiments of Proficiency are an integral part of a student's mathematical identity.


A model for "Academic Episteme' "
Experience
Experiential Reference
Explore (Informal)
Explore (Formal)
Explicate (Static)
Explicate (Dynamic)
Employ
Natural Curiosity
Academic Investigation
Traditional Lecture
Collaborative Discourse
Application
CHAT-TE Heuristic
Quality End-Product
MATH PROFICIENCY
1.)Self-Efficacy Proficiency Agency
2.)Self-Efficacy Proficiency Agency
3.)Self-Efficacy Proficiency Agency
4.)Self-Efficacy Proficiency Agency
5.)Self-Efficacy Proficiency Agency
6.)Self-Efficacy Proficiency Agency
INQUIRE
ELABORATE
DEMONSTRATE

Academic Episteme

A Resulting Integrative Statement would be the following: Mathematics Proficiency is a culminating event, not a singular occurrence.
The "Academic Episteme" is a network of Metacognition Systems.
Experience (the noun and the verb) is a multi-dimensional phenomena due to the constant feedback in the system.
There is constant feedback throughout the system.
There are two aspects to the "Academic Episteme": a.) The Cognitive; and b.) The Affective.

My “Proficient” Critical Thinker

Item	Critical Questions	Semiotic Element	Role (Individual and Team)
The Situation	What do you see?	Syntax	Researcher {Conceptual Understanding}
The Situation	What does it say?	Syntax, Semantics	Analyzer {Adaptive Reasoning}
The Situation	What does it mean? (Conceptually and Contextually)	Semantics (Semantic Reference, Modal Transference, Creation/Production, Form, and Function)	Interpreter {Conceptual Understanding}
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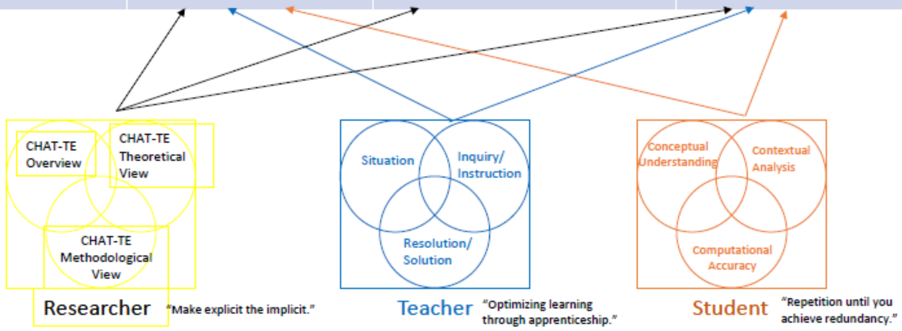
Evaluator {Productive Disposition}



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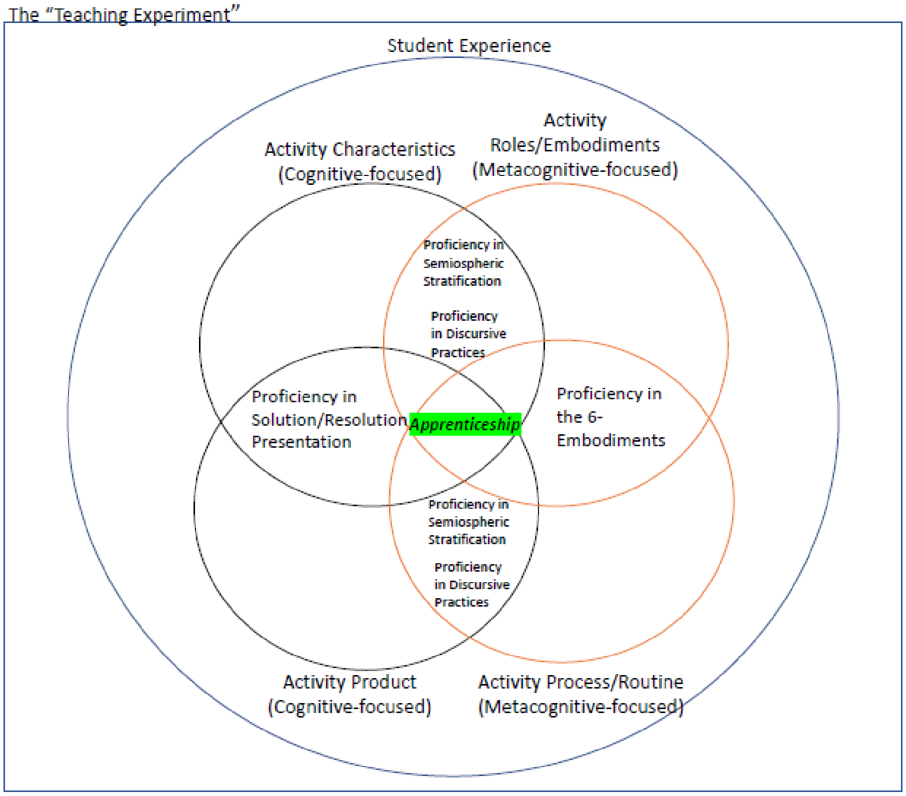
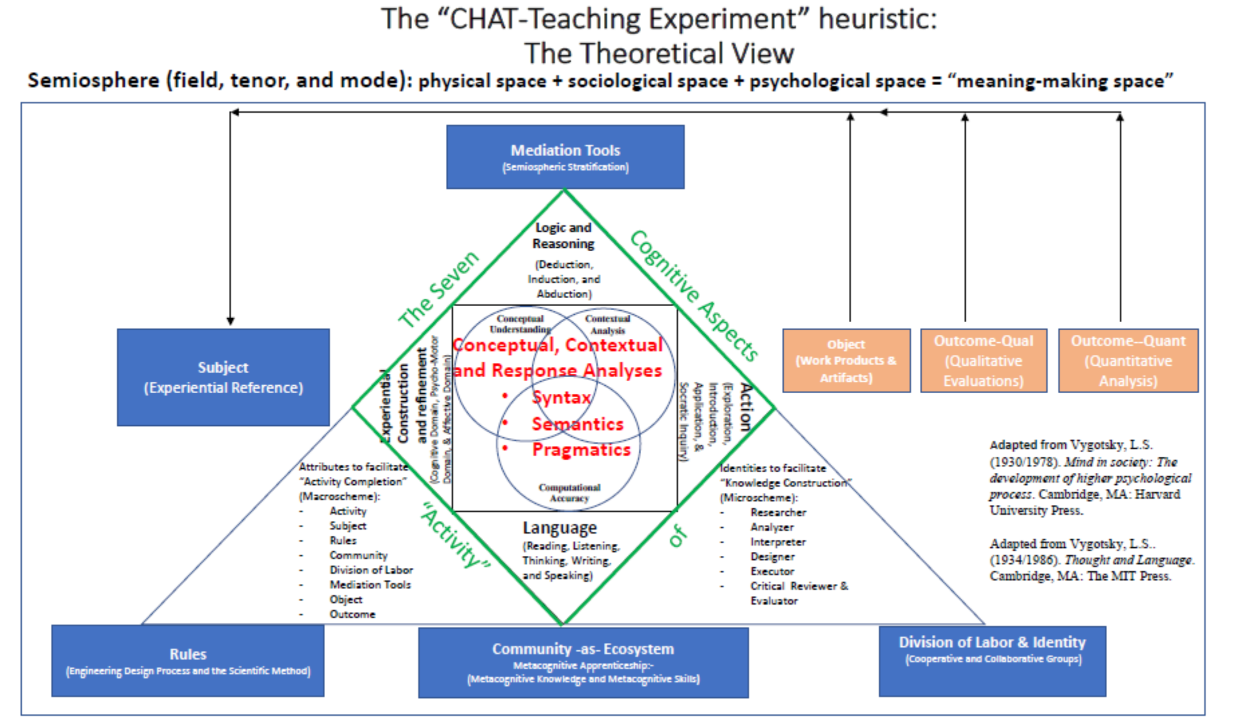


The "Activity Dissection":
(The CHAT-Teaching
Experiment heuristic:
Overview-V.2)

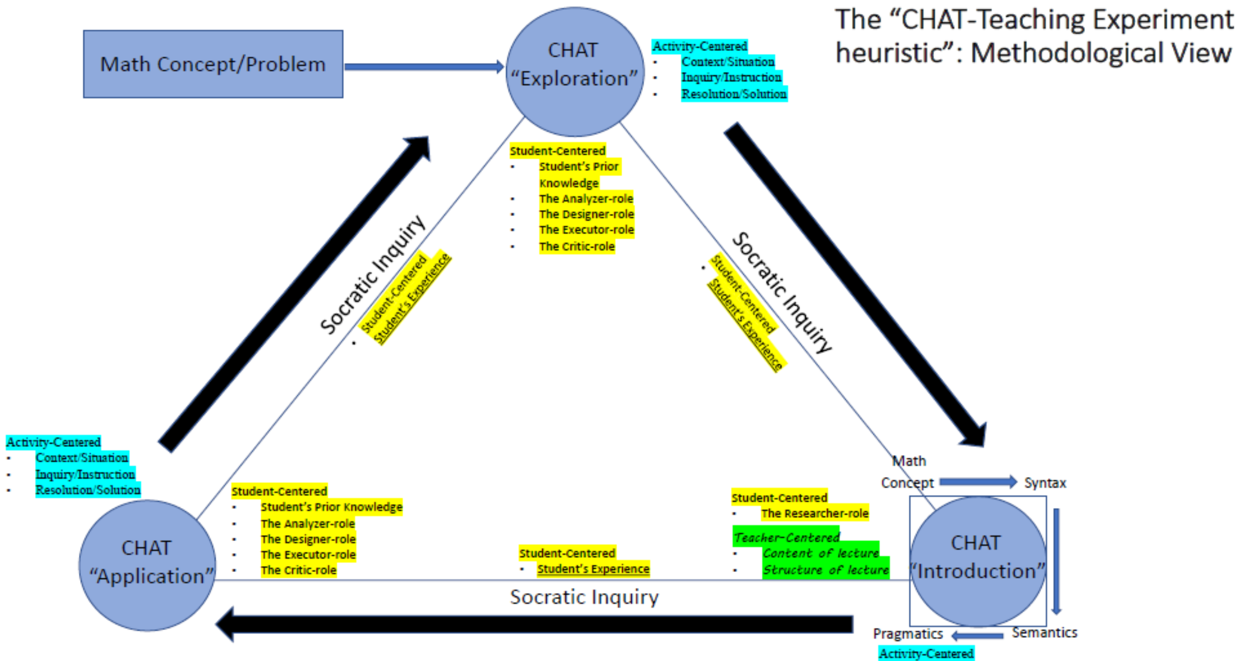
CHAT-TE Overview

A Resulting Integrative Statement would be the following: The CHAT-Teaching Experiment heuristic Overview hypothesizes that each person has uniquely-different metacognitive and cognitive experiences which result from human activity and its four sub-parts.
The Teaching Experiment allows for investigations into the metacognitive and cognitive aspects of one's experiences due to the activity.
The Teaching Experiment acknowledges the uniquely different human experiences which result from each of the four activity sub-parts.
CHAT allows for the further dissection of the actual activity into four sub-parts: the characteristics, the roles, the process and the product.
The CHAT-Teaching Experiment heuristic interprets human activity as two intertwined elements: The actual activity and the human experience; where human experience refers to one's real-time interactions and impact and one's prior interactions & impact.


The "CHAT-Teaching Experiment" heuristic:
The Theoretical View
Semiosphere (field, tenor, and mode): physical space + sociological space + psychological space = "meaning-making space"
Mediation Tools
(Semiospheric Stratification)
Subject
(Experiential Reference)
Object
(Work Products & Artifacts)
Outcome-Qual
(Qualitative Evaluations)
Outcome--Quant
(Quantitative Analysis)
The Seven
Cognitive Aspects
of
"Activity"
Logic and Reasoning
(Deduction, Induction, and Abduction)
Experiential Construction and refinement
(Cognitive Domain, Psycho-Motor Domain, & Affective Domain)
Action
(Exploration, Introduction, Application, & Socratic Inquiry)
Language
(Reading, Listening, Thinking, Writing, and Speaking)
Conceptual Understanding
Contextual Analysis
Computational Accuracy
Conceptual, Contextual and Response Analyses
• Syntax
• Semantics
• Pragmatics
Attributes to facilitate "Activity Completion" (Macroscheme):
- Activity
- Subject
- Rules
- Community
- Division of Labor
- Mediation Tools
- Object
- Outcome
Identities to facilitate "Knowledge Construction" (Microscheme):
- Researcher
- Analyzer
- Interpreter
- Designer
- Executor
- Critical Reviewer & Evaluator
Adapted from Vygotsky, L.S. (1930/1978). Mind in society: The development of higher psychological process. Cambridge, MA: Harvard University Press.
Adapted from Vygotsky, L.S.. (1934/1986). Thought and Language. Cambridge, MA: The MIT Press.
Rules
(Engineering Design Process and the Scientific Method)
Community -as- Ecosystem
Metacognitive Apprenticeship:-
(Metacognitive Knowledge and Metacognitive Skills)
Division of Labor & Identity
(Cooperative and Collaborative Groups)

CHAT-TE Theoretical View

A Resulting Integrative Statement would be the following: **The CHAT-Teaching Experiment heuristic Theoretical View hypothesizes that the task-completion, meaning-making, and knowledge-construction due to an activity generates a unique experiential reference that can be investigated by a researcher and used later by the person.**
There are 7 fundamental cognitive aspects of the actual activity: logic and reasoning, action, language, experiential construction, syntax, semantics, and pragmatics
There are 6 embodiments of the knowledge-construction aspect: the researcher, the analyzer, the interpreter, the designer, the executor, and the evaluator
There are 8 components of the task-completion aspect: the activity, the person, the rules, the community, the division of labor, the mediation tools, the product, and the impact
The CHAT-Teaching Experiment heuristic allows for the human experience to be partitioned into two aspects: a task-completion aspect and a knowledge-construction aspect



Adapted from Engelhardt, P. V., Corpuz, E. G., Ozimek, D. J., & Rebello, N. S. (2004). The Teaching Experiment—What it is and what it isn't. In *AIP Conference Proceedings* (Vol. 720, No. 1, (pp. 157-160). American Institute of Physics.

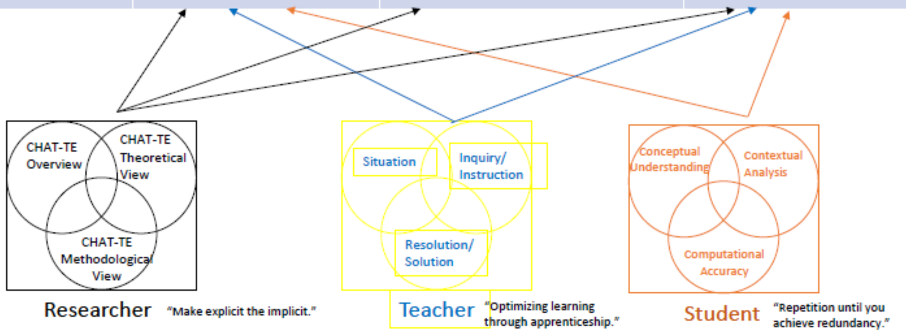
CHAT-TE Methodological View

A Resulting Integrative Statement would be the following: **The CHAT-Teaching Experiment heuristic Methodological View hypothesizes that an activity has many different facets which must be investigated, and the necessary methods and instruments for the investigations are varied.**
The CHAT-Teaching Experiment heuristic requires the researcher to consider that the necessary concept or skill may be about syntax, semantics, or pragmatics and not about the specific topic of the given problem.
The CHAT-Teaching Experiment heuristic requires the researcher to privilege the student's perspective of his/her experience. Such insight is achieved using the Socratic Method of Inquiry.
The CHAT-Teaching Experiment heuristic requires the researcher to consider the following three aspects of a student's problem-solving approach to a given activity and its underlying task: the student's independent explorations, the introduction/presentation of the relevant concept, and the application of the concept or skills.

My "Proficient" Critical Thinker

Item	Critical Questions	Semiotic Element	Role (Individual and Team)
The Situation	What do you see?	Syntax	Researcher {Conceptual Understanding}
The Situation	What does it say?	Syntax, Semantics	Analyzer {Adaptive Reasoning}
The Situation	What does it mean? (Conceptually and Contextually)	Semantics (Semantic Reference, Modal Transference, Creation/Production, Form, and Function)	Interpreter {Conceptual Understanding}
Inquiry/Instruction	What must be done?	Pragmatics(Problem-solving)	Designer {Strategic Competence}
Inquiry/Instruction	How shall it be done?	Pragmatics (Problem-solving)	Designer {Procedural Fluency}

Evaluator {Productive Disposition}



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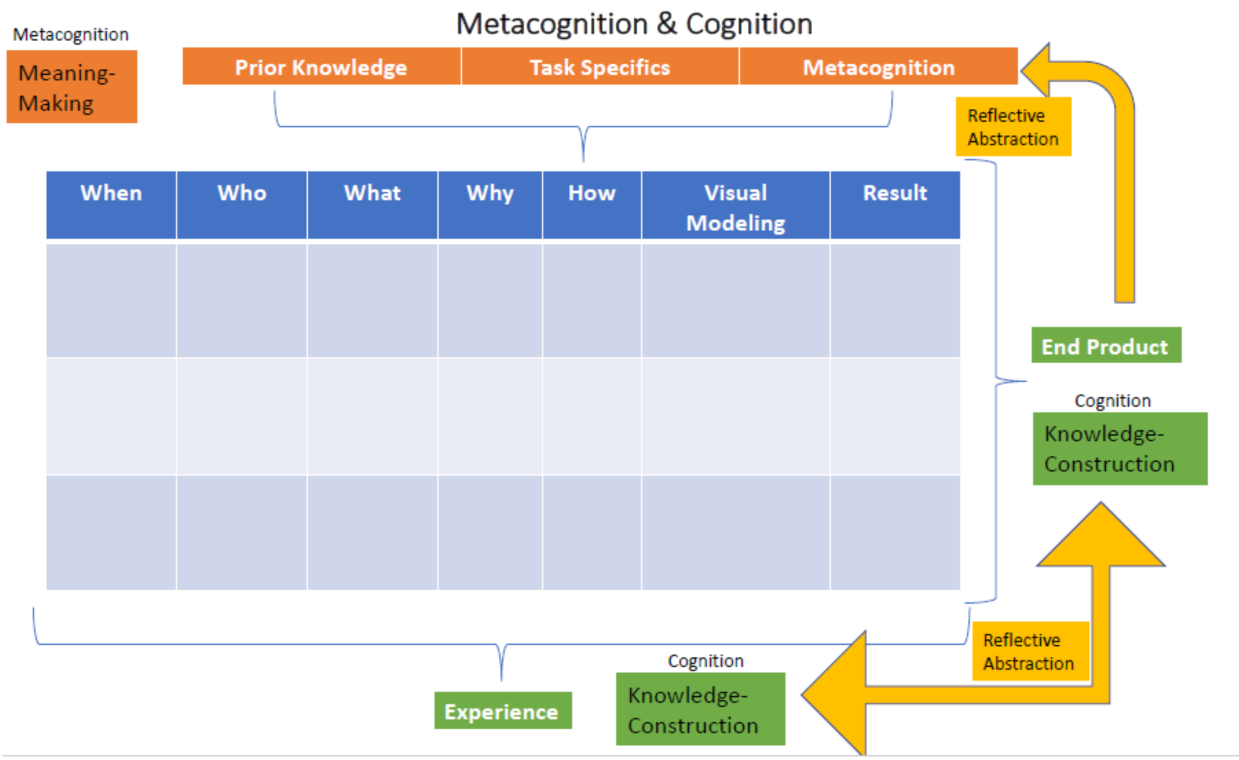
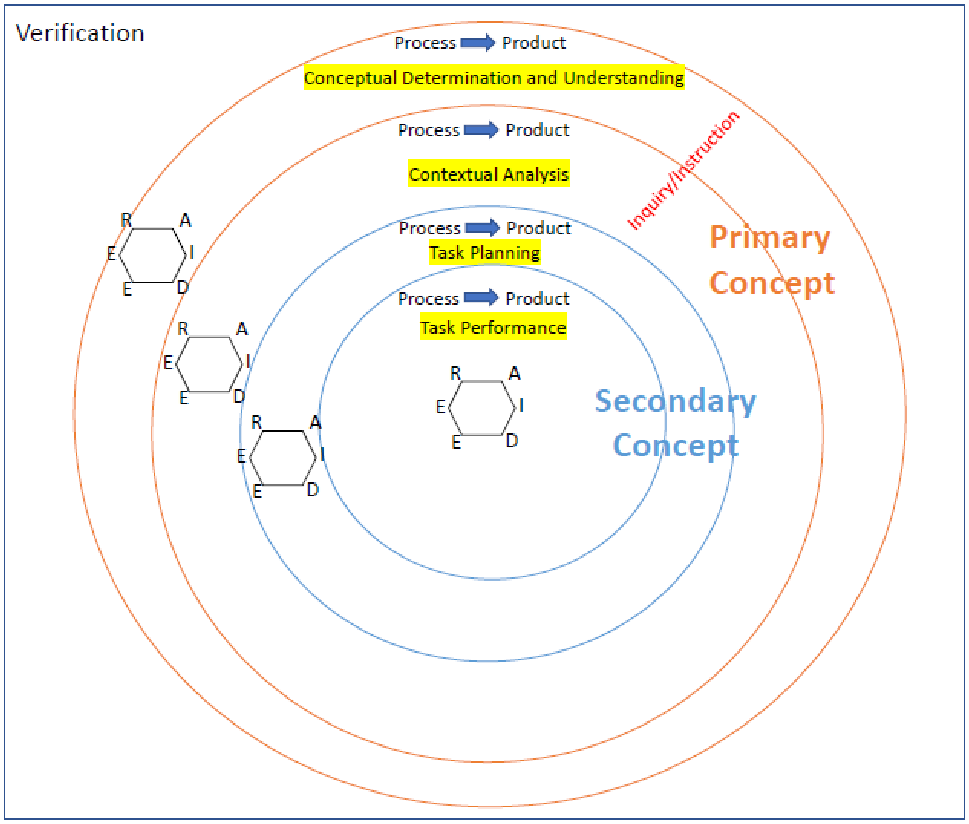

Metacognition & Cognition
Metacognition
Meaning-Making
Prior Knowledge
Task Specifics
Metacognition
Reflective Abstraction
When
Who
What
Why
How
Visual Modeling
Result
End Product
Cognition
Knowledge-Construction
Reflective Abstraction
Cognition
Knowledge-Construction
Experience

Metacognition & Cognition

A Resulting Integrative Statement would be the following: **Modeling, the Inquiry Questions, and Reflective Abstractions have a dominant impact on the activity process, the activity product, and the overall human experience.**
The Inquiry questions can be recursive by focusing within a particular question type; or the Inquiry questions can be multi-dimensional by interacting across multiple question types.
Reflective Abstraction impacts experience, metacognition, cognition, the activity, and one's prior knowledge.
Modeling and Inquiry questions generate both experience (noun and verb) and cognition.
Metacognition is process-oriented; while Cognition is product-oriented.

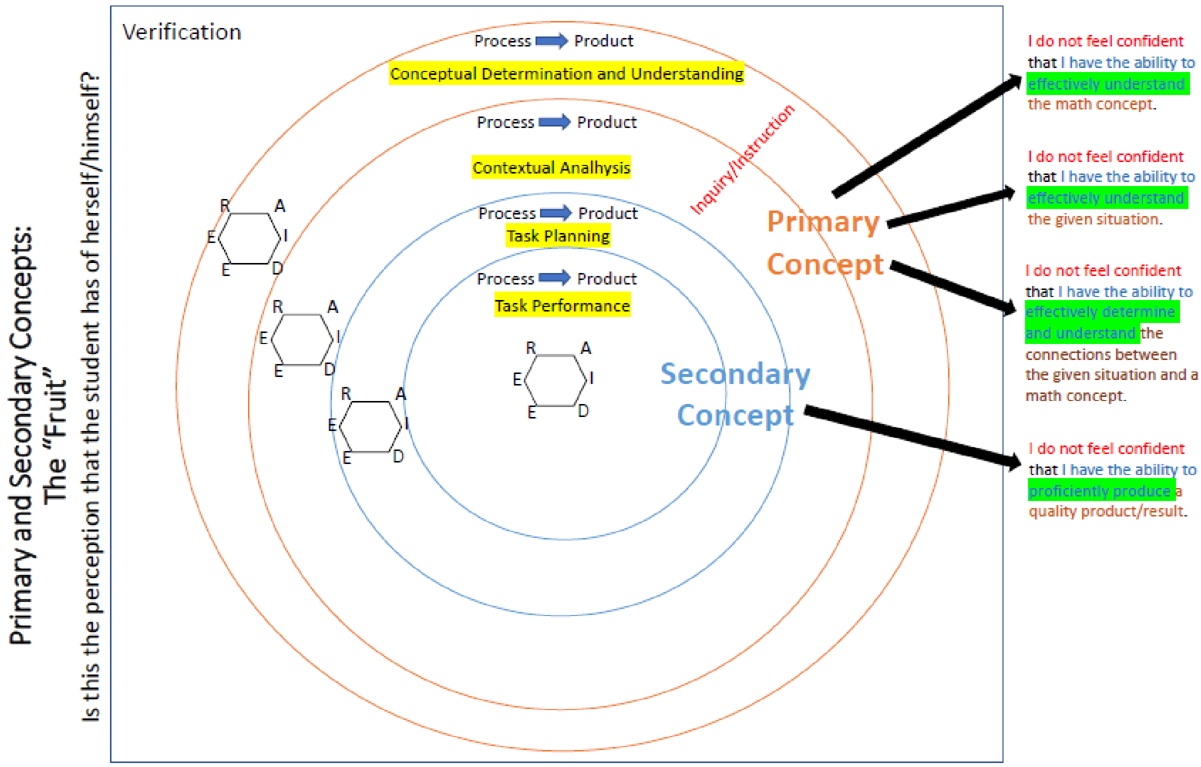
Primary and Secondary Concepts:
The "Fruit"

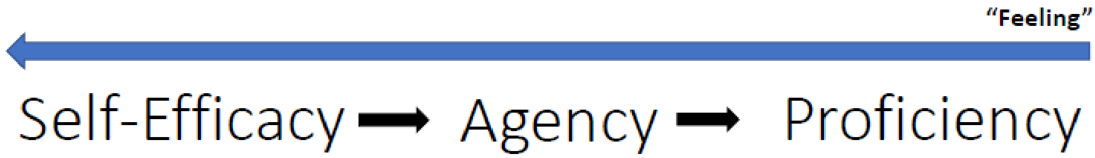
Is this the perception of the student that the teacher has?

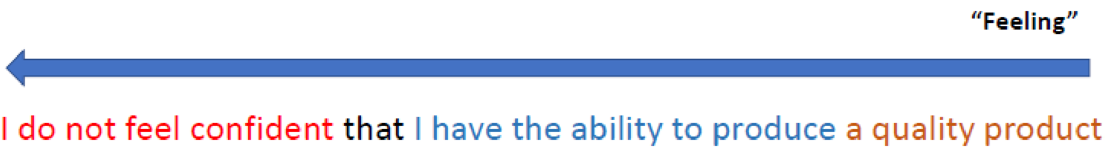


The "Primary and Secondary Concepts"

A Resulting Integrative Statement would be the following: **The 6-Embodiments of Proficiency exist in each band, because each band possesses a "process" and a "product".**
Each band requires metacognition and cognition, because each band possesses a "process" and a "product".
The Secondary Concept consists of the task planning and task performance.
The Primary Concept consists of the Conceptual Understanding, Contextual Analysis, and the Inquiry/Instruction of the task.
There are two objectives within a task: a.) The Primary Concept; and b.) The Secondary Concept


Primary and Secondary Concepts:
The "Fruit"
Is this the perception that the student has of herself/himself?
Verification
Process Product
Conceptual Determination and Understanding
Process Product
Contextual Analhysis
Process Product
Task Planning
Process Product
Task Performance
Inquiry/Instruction
Primary Concept
Secondary Concept
R A I D E E
I do not feel confident that I have the ability to effectively understand the math concept.
I do not feel confident that I have the ability to effectively understand the given situation.
I do not feel confident that I have the ability to effectively determine and understand the connections between the given situation and a math concept.
I do not feel confident that I have the ability to proficiently produce a quality product/result.



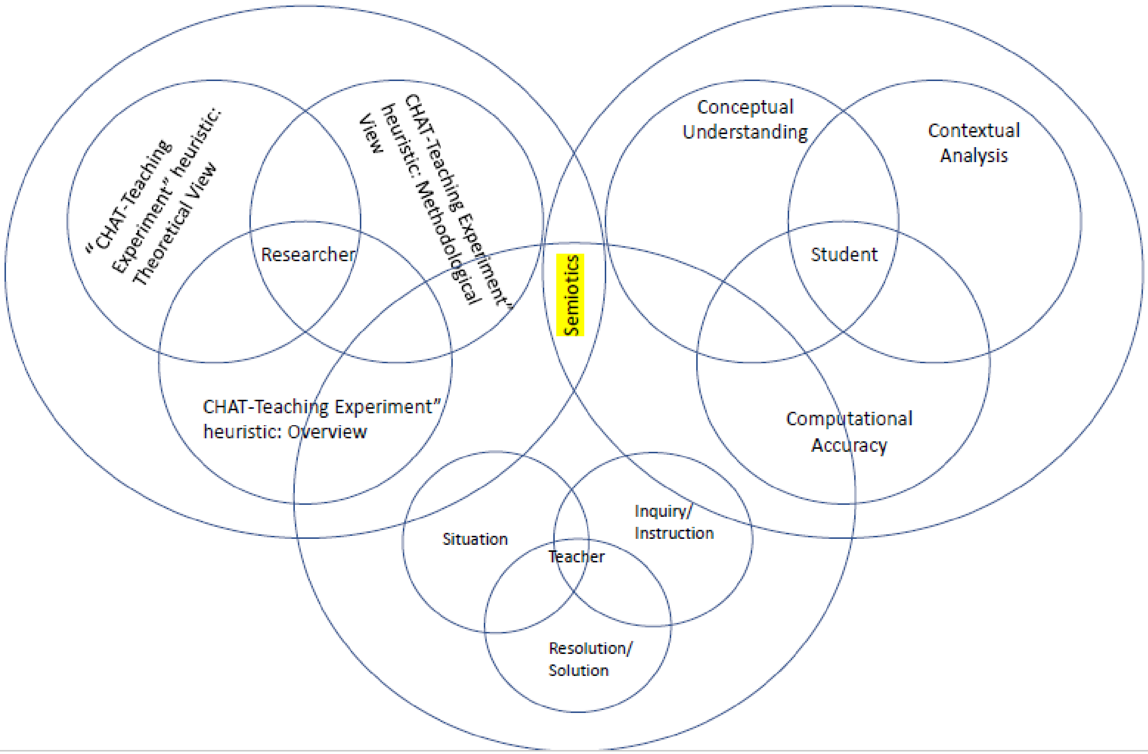


*The necessary "feeling" seems to develop through the following process:
1.) The production of a quality product;
2.) The awareness that the individual, himself/herself actually produced the quality product;
3.) The accumulation of multiple experiences of having produced a quality product

The Efficacy – Agency – Proficiency Triad

A Resulting Integrative Statement would be the following: **A student with a positive disposition says, "I feel confident that I have the ability to produce a quality product."**
"Proficiency" can be conveyed through the following words: "Produce a quality project."
"Agency" can be conveyed through the following words: "I have the ability to (insert action) ."
"Self-efficacy" can be conveyed through the following words: "I do not feel confident."
A productive disposition consists of self-efficacy, agency, and proficiency and each of these is impacted by one's feelings.

The Researcher, the Student, & the Teacher



Researcher, Student, & Teacher

A Resulting Integrative Statement would be the following: **An effective learning environment requires a cooperative collaboration of the following mindsets: teacher, student, and researcher**
Since each critical element has an impact on the learning environment, it is important that each critical element understands the work, perspective, and impact of the other two critical elements.
Although there are many overlaps, there is one fundamental commonality across the three critical elements – Semiotics.
Each critical element has its own unique three-points of focus.
There are at least three critical elements in the understanding of a learning environment: a.) The teacher; b.) The student(s); and c.) The researcher (The researcher does not necessarily have to be a separate person, the "researcher" could be a role that the teacher and/or student(s) embody).

Research Questions

Response to Research Question #1

1. How does teaching mathematics as a language system affect the construction of mathematical knowledge (learning of mathematics) by African American students?

The teaching of mathematics as a language system can facilitate the student's effort to construct deep and substantial meaning from the given context. Sometimes, it is necessary for the student to conduct a syntactic analysis of the given problem to ensure an understanding of the details of the given problem, and the necessary inferences. Students need the following:

a. to be apprenticed in descriptive and inferential meaning making with mathematical principles and mathematical procedures
b. to understand that attaining mathematics proficiency requires a process-oriented metacognitive mindset
c. not to engage in impulsive reactions to mathematics because impulsive acts are more destructive to a productive disposition than they are generative.

Response to Research Question #2

2. **What can be understood about the construction of mathematical knowledge (learning of mathematics) by African American students when different language systems beyond numbers and operations (visual imagery, movement, written/oral language, for example) are integrated into the mathematics curriculum? (In short, how can semiotics assist with the interpretation and learning of mathematics by African American students?)**

How any one particular student constructs meaning from a mathematical context is contingent upon that particular student's prior knowledge and prior experience in life, not just in mathematics. <u>Two achievements result</u> when a student uses representations beyond numbers and operations to express her/his understanding:

a) leading of the conversation to the meaning-making potential that is captured within the given mathematical context.

b) Sharing of deeper and more descriptive expressions of one's understanding

Response to Research Question #3

3. What are the dispositions of African American students toward mathematics when different language systems (visual imagery, movement, written/oral language, etc.) are integrated into their learning? (In short, how can semiotics impact the disposition of African American students toward mathematics?)

Although the disposition of the student is linked to his or her ability to actually produce a correct solution to a given mathematics problem, his or her disposition is also linked to his or her ability to successfully understand the given mathematics problem. When working with students who are convinced that they will never be successful in mathematics, I have found that students act with greater intentionality and less impulsiveness when the students achieve the following three goals:

a) separate problem-comprehension from problem-solving
b) learn the tools that semiotics offers
c) commit to a process-oriented metacognitive mindset

Response to the Foundational Question

What insight can an analysis of mathematics as a language provide in the formulation of a counter-narrative to this hegemonic propaganda?

The mathematics learning ecosystem, and by extension, the student, excels when it serves as a semiotics apprenticeship that guides students through metacognitive knowledge and metacognitive skills toward proficiency in inferencing, storytelling, and practicing.

Concluding Remarks

My "Proficient" Critical Thinker

Item	Critical Questions	Semiotic Element	Role (Individual and Team)
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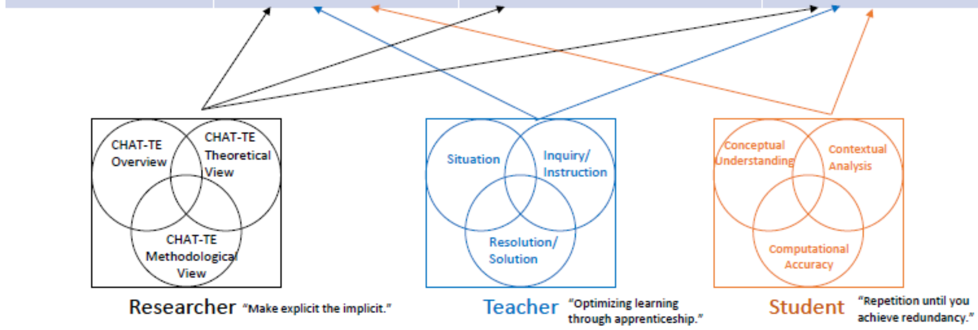
Evaluator {Productive Disposition}



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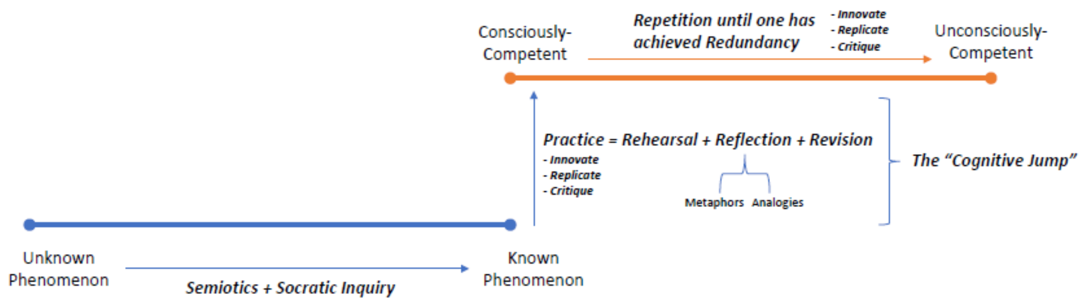
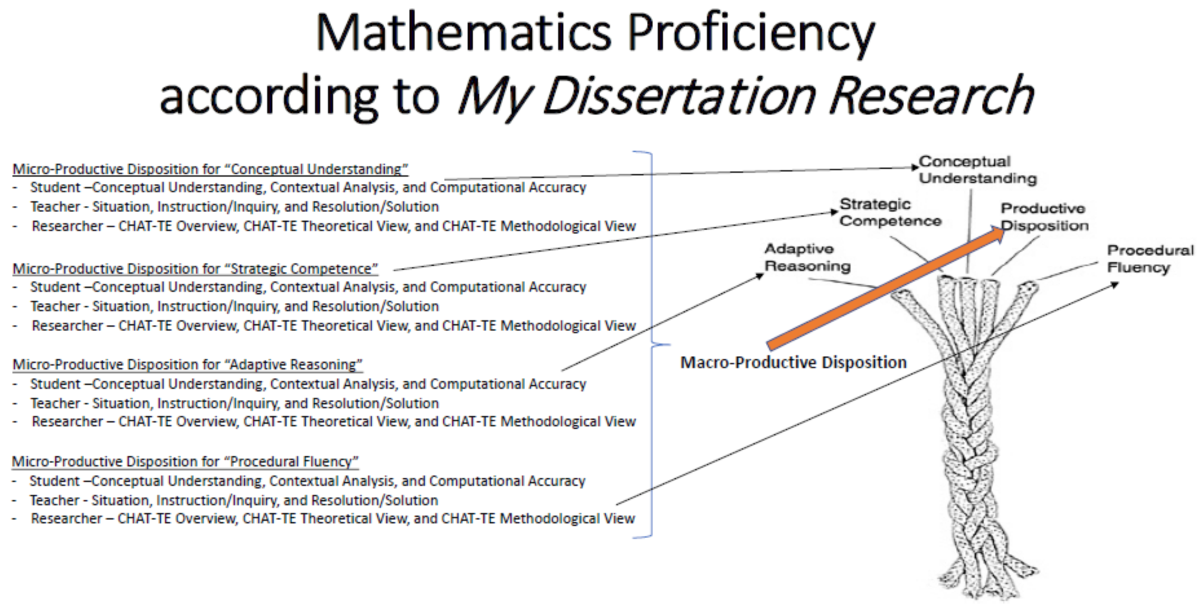
The "Cognitive Jump and the Cognitive Continuum" v.2:

Content Shift vs. Consciousness Shift



(*"Making EXPLICIT the IMPLICIT": Conceptual Understanding, Contextual Analysis, and Computational Accuracy*)



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The End.