Contributions à l'accélération de la méthode de Newton

par

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Dédié
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SOMMAIRE

Dans cette thèse par articles, nous nous intéressons à l'augmentation de l'ordre de convergence de la méthode itérative de Newton. Notre objectif a été d'établir en termes mathématiques, la meilleure façon d'augmenter l'ordre de convergence de cette méthode. À cet effet, cette thèse regroupera cinq articles qui soulignent le cheminement de notre travail de doctorat et la logique de notre recherche.

Dans un premier article, qui sert d'introduction et de mise en contexte de cette thèse, nous revisitons les méthodes itératives de point fixe et la méthode de Newton pour le calcul des zéros d'une fonction suffisamment régulière. Nous présentons les conditions nécessaires et suffisantes pour la convergence d'ordre supérieure de celles-ci. À l'aide de ces conditions, nous montrons comment augmenter de façon récursive l'ordre de convergence. Pour la méthode de point fixe, nous présentons une généralisation de la méthode de Schröder de première espèce. Plus spécifiquement, deux autres méthodes sont aussi présentées pour la méthode de Newton. L'une d'entre elles est montrée équivalente à la méthode de Schröder de deuxième espèce.

Notons que le début de nos recherches a été inspiré par l'article [8], qui présente deux familles de méthodes itératives d'ordre m pour le calcul de la racine n-ième. Celles-ci sont initialement apparues dans les articles [9] et [25]. Ces méthodes par contre sont spécifiquement adaptées au calcul de la racine n-ième d'un nombre et leur utilisation y est

limitée. L'un des objectif de notre premier chapitre a été de les généraliser. Nous voulions avoir deux familles distinctes de méthodes itératives qui pourraient approximer les zéros d'une fonction suffisamment régulière, mais nous voulions aussi, qu'une fois appliquées au problème du calcul de la racine, ces méthodes coïncident avec les méthodes que nous présente cet article. C'est ce que nous avons fait dans le premier chapitre.

Le second chapitre est une généralisation du chapitre précèdent, dans le plan complexe. De plus, plusieurs exemples numériques et illustrations de bassins d'attraction sont inclus dans cet article.

Dans le chapitre suivant, nous notons que c'est en 1669 , pour le calcul d'un zéro simple d'une fonction analytique $f(z)$, qu'Isaac Newton [57] a introduit sa fameuse méthode itérative d'ordre 2 . Quelques années plus tard, en 1694 , Edmond Halley [20] a lui introduit une autre fonction d'itération d'ordre 3. Depuis, quoique plusieurs mathématiciens aient essayé de trouver des méthodes plus rapides que la méthode de Newton, la méthode de Halley a été redécouverte de nombreuses fois [44]. Pourquoi? Ceci est le sujet de notre troisième chapitre. Nous montrons que la séquence de Halley, une suite de fonctions résultantes de l'augmentation de l'ordre de convergence de la méthode de Newton, est la façon la plus efficace d'augmenter l'ordre de convergence de la méthode de Newton en termes d'utilisation de dérivées d'ordre supérieur. Nous illustrons pourquoi ce fait est probablement la raison pour laquelle la méthode itérative de Halley a été si souvent redécouverte. À des fins illustratives nous présentons aussi un algorithme pour reconnaître la séquence de Halley afin d'éviter d'autres redécouvertes. Nous appliquons cet algorithme à certains exemples.

Dans le quatrième chapitre, nous montrons comment le développement de Taylor d'une fonction analytique peut être utilisé pour accoître l'ordre de convergence de méthode itérative. Ceci nous permet d'établir de nouveaux liens entre plusieurs differents processus d'accélération, notamment entre celui de Halley et celui de Chebyshev.

Dans le cinquième chapitre, nous montrons que la façon la plus efficace d'augmenter l'ordre de convergence de la méthode de Newton, en termes de fonctions polynomiales, nous donne la méthode de Schröder de première espèce. En particulier nous obtenons la fameuse méthode itérative de Euler-Chebyshev à l'ordre 3. Nous obtenons aussi le fait que les méthodes itératives de Schröder de première et deuxième espèce sont les meilleures façons d'augmenter l'ordre de convergence de la méthode itérative de Newton.

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J'aimerai remercier le personnel de la faculté des Sciences, pour leur assistance, leur professionnalisme et leur compétence. Pour finir je voudrais remercier ma famille pour son soutien.

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Gnang Calvin

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INTRODUCTION

Méthodes de point fixe

La méthode de Newton-Rahpson a été introduite en 1669 [57]. Aujourd'hui, elle peut être appliquée au calcul des zéros d'une fonction analytique. La forme moderne de la méthode de Newton est

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} = N_f(z_k).$$

L'objectif est de choisir un point initial z_0 et d'utiliser le processus

$$z_{k+1} = N_f(z_k) \qquad \text{pour} \qquad k = 0, 1, 2, 3, \ldots,$$

afin de générer une séquence $\{z_k\}_{k=0}^{+\infty}$ qui converge vers α. La méthode de Newton est, en général, d'ordre de convergence égale à 2.

En 1694 , Edmond Halley [17], a introduit la fonction d'itération d'ordre 3

$$z_{k+1} = z_k - \frac{2f(z_k)f'(z_k)}{2f'(z_k)^2 - f(z_k)f''(z_k)} = H_f(z_k),$$

qui s'est, par la suite révélée être la méthode de Newton appliquée à la fonction $G(z) = f(z)/\sqrt{f'(x)}$.

Ordre de convergence

On dit qu'une méthode de point fixe $x_{n+1} = g(x_n)$ qui converge vers α est d'ordre $p \geq 1$ si

$$\lim_{n \to +\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = K_p(\alpha) > 0,$$

$K_p(\alpha)$ est la constante asymtotique.

Pour une valeur de x_n proche de α, le terme $\mid x_n - \alpha \mid^p$ devient de plus en plus petit si p est large. Si donc on regarde

$$x_{n+1} - \alpha \approx K_p(\alpha, I)\,(x_n - \alpha)^p$$

ce terme indique la vitesse à laquelle l'erreur d'approximation décroît. Plus la valeur de p est large, plus proche x_{n+1} sera de α, si on assume que la sequence est convergente. Typiquement, augmenter l'ordre de convergence requiert le calcul de dérivées additionnelles, ce calcul peut rendre les méthodes d'ordre supérieur peu pratiques.

Cadre

Il est important de noter qu'en 1870, Schröder [48] a proposé une séquence de fonctions maintenant connue sous le nom de méthode de Schröder de seconde espèce. Le p-ième membre de cette séquence produit une fonction d'itération d'ordre p donné par

$$S_p(z) = z - \frac{R_{p-2}(z)}{R_{p-1}(z)} \quad \text{for} \quad p \geq 2,$$

avec

$$\begin{cases} R_0(z) &=& 1/f(z), \\ R_p(z) &=& \sum_{j=1}^{p} (-1)^{j+1} \frac{f^{(j)}(z)}{j!f(z)} R_{p-j}(z) \quad \text{for} \quad p \geq 1. \end{cases}$$

Si par exemple nous calculons l'expression pour $R_1(z)$ et $R_2(z)$, on obtient

$$R_1(z) = \frac{f'(z)}{f^2(z)} \quad \text{and} \quad R_2(z) = \frac{(f'(z))^2}{f^3(z)} \left[1 - \frac{f(z)f''(z)}{2\,(f'(z))^2}\right].$$

On observe que les deux premiers termes de cette séquence sont respectivement, la fonction d'itération de Newton $S_2(z) = N_f(z)$, et la fonction d'itération de Halley $S_3(z) = H_f(z)$.

Depuis 1870, plusieurs mathématiciens ont proposé d'autres séquences, la majorité d'entre elles ont malheureusement été des redécouvertes de la méthode de Schröder de seconde espèce [44].

En 1964, Traub [51] remarque le fait que cette séquence de fonction est souvent redécouverte. En effet, plusieurs redécouvertes ont été publiées: par exemple 1946 [21], 1966 [54], 1969 [52], 1972 [31], 1975 [16], 1991 [28], 1994 [19], 1996 [17], 1997 [32]. En moyenne une fois chaque décennie, cette suite de fonction est redécouverte [44, 45].

Observons que les formes sous lesquelles cette suite de fonction se présente peuvent être méconnaissables.

Par exemple, Householder's [26, 27] a présenté la suite

$$T_p(z) = z + (p-1)\Big[\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}\Big].$$

Ford and Pennline [17] avec l'assistance de Gerlach [19] ont présenté la suite

$$G_p(z) = z - f(z)\frac{Q_p(z)}{Q_{p+1}(z)},$$

avec

$$\begin{cases} Q_2(z) &=& 1, \\ Q_p(z) &=& f'(z)Q_{p-1}(z) - \frac{1}{p-2}f(z)Q'_{p-1}(z) \quad \text{for} \quad p \geq 3. \end{cases}$$

Le fait que

$$\underbrace{S_p(z)}_{1870} = \underbrace{T_p(z)}_{1953} = \underbrace{G_p(z)}_{1996}$$

a été observé en 2010 par Petković et al. [45] .

Tout en observant ces nombreuses redécouvertes, les auteurs n'ont par contre pas été capables d'expliquer le pourquoi de ces redécouvertes, ni de présenter une façon d'éviter d'autres éventuelles redécouvertes.

Nous avons réussi à montrer que cette suite de fonctions présentée par Schröder est la façon la plus efficace d'augmenter l'ordre de la méthode itérative de Newton en termes d'utilisation de certaines dérivées d'ordre supérieure. Par ce fait, nous pouvons expliquer ces nombreuses redécouvertes.

Nous avons aussi été capable d'établir un algorithme qui caractérise complètement la séquence de Schröder de deuxième espèce. Notamment:

Theorème: Soit $f(z)$ tel que $f(\alpha) = 0$ et $f'(\alpha) \neq 0$ tel que la méthode de Newton N_f est d'ordre 2. La suite de fonction $\{T_p(z)\}_{p=2}^{+\infty}$ définie par

$$T_p(z) = z - V_p(z) \quad \text{pour} \quad p \geq 2,$$

dont le p-th element est d'ordre p, corresponds a la suite de Schröder de seconde espèce si et seulement si

$$\begin{cases} V_2(z) & = & \frac{f(z)}{f'(z)}, \\ \\ V_{p+1}(z) & = & \left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) \quad \text{pour} \quad p \geq 2. \end{cases}$$

Nous pouvons par exemple appliquer cet algorithme aux suites de fonctions présentées antérieurement.

Exemple 1: Si nous regardons la suite de Householder's [26, 27] soit

$$T_p(z) = z + (p-1)\left[\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}\right]$$

en utilisant notre algorithme nous posons

$$V_p(z) = -(p-1)\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}.$$

donc

$$V_p'(z) = -(p-1)\left[1 - \frac{(1/f(z))^{(p-2)}(1/f(z))^{(p)}}{\left[(1/f(z))^{(p-1)}\right]^2}\right],$$

après direct substitution on a

$$\left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) = -p\frac{(1/f(z))^{(p-1)}}{(1/f(z))^{(p)}} = V_{p+1}(z). \quad \square$$

Exemple 2: Si nous regardons la suite de Ford and Pennline [17] inpirée de Gerlach [19]

$$G_p(z) = z - f(z)\frac{Q_p(z)}{Q_{p+1}(z)},$$

avec

$$\left\{\begin{array}{rcl} Q_2(z) & = & 1, \\ \\ Q_p(z) & = & f'(z)Q_{p-1}(z) - \frac{1}{p-2}f(z)Q_{p-1}'(z) \quad \text{pour} \quad p \geq 3. \end{array}\right.$$

On a

$$V_p(z) = f(z)\frac{Q_p(z)}{Q_{p+1}(z)},$$

et

$$\begin{aligned} V_p'(z) &= \left(f(z)\frac{Q_p(z)}{Q_{p+1}(z)}\right)' \\ &= f'(z)\frac{Q_p(z)}{Q_{p+1}(z)} + f(z)\frac{Q_p'(z)Q_{p+1}(z) - Q_p(z)Q_{p+1}'(z)}{Q_{p+1}^2(z)} \\ &= (1-p) + p\frac{Q_p(z)Q_{p+2}(z)}{Q_{p+1}^2(z)}, \end{aligned}$$

après avoir remplacé $f(z)Q_{p+1}'(z)$ et $f'(z)Q_p(z)$ en utilisant leur relations de recurrence pour avoir $Q_{p+2}(z)$ and $Q_{p+1}(z)$.

$$1 - \frac{1}{p}\left[1 - V_p'(z)\right] = \frac{Q_p(z)Q_{p+2}(z)}{Q_{p+1}^2(z)},$$

il advient donc que

$$\begin{aligned}\left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) &= \left[\frac{Q_p(z)Q_{p+2}(z)}{Q_{p+1}^2(z)}\right]^{-1} f(z)\frac{Q_p(z)}{Q_{p+1}(z)} \\ &= f(z)\frac{Q_{p+1}(z)}{Q_{p+2}(z)} \\ &= V_{p+1}(z). \quad \square\end{aligned}$$

Nos travaux ont fortement été influencés directement par ceux de Petkovic et al [45] et indirectement par Kalantari [36].

Notre contribution

L'une des contributions de nos recherches a été d'abord d'établir un algorithme qui caractérise complètement la séquence de Schröder de deuxième espèce. Par ce fait, nous pouvons éviter d'éventuelles redécouvertes.

Nous avons aussi réussi à montrer que cette suite de fonctions présentée par Schröder est la façon la plus efficace d'augmenter l'ordre de la méthode itérative de Newton en termes d'utilisation de certaines dérivées d'ordre supérieure. Par ce fait, nous sommes capables d'expliquer ces nombreuses redécouvertes.

Nous montrons également que tout processus d'accélération de la méthode Newton qui se développe en terme polynomiales de la fonction $f(z)$, peux être tronqué en la deuxième séquence de Schröder, de première espèce.

Ce travail repose toutefois sur l'étude des conditions nécessaires et suffisante, d'augmentation de l'ordre de convergence des méthodes de point fixe et plus particulièrement de la méthode de Newton.

Commentaires sur l'article présenté au chapitre 1

Dans cet article, nous revisitons la méthode du point fixe et la méthode de Newton pour le calcul d'un zéro simple d'une fonction suffisamment régulière.

Nous présentons les conditions nécessaires et suffisantes qui garantissent la convergence d'ordre supérieure de ces méthodes. À partir de ces conditions, nous montrons comment récursivement augmenter l'ordre de convergence.

Pour la méthode du point fixe, nous présentons une généralisation de la méthode de Schröder de première espèce. Pour la méthode de Newton, deux méthodes sont aussi présentées. L'une d'entre elles est montrée équivalente à la méthode de Schröder de deuxième espèce.

Des exemples numériques sont inclus dans cet article. Notamment, l'application de cette théorie au calcul de la racine n-ième d'un nombre.

Cet article a été rédigé conjointement avec M. François Dubeau et publié dans la revue SIAM Review en 2014, sous le titre 'Fixed point and Newton's methods for Solving a nonlinear equation: From linear to high-order convergence.'

Commentaires sur l'article présenté au chapitre 2

Dans cet article, nous revisitons la méthode du point fixe et la méthode de Newton pour le calcul d'un zéro simple d'une fonction analytique c'est une généralisation du chapitre précédent au cas complexe. Des exemples numériques sont inclus dans cet article. En particulier, tout comme Kalantari nous avons produit les bassins d'attraction de differentes méthodes, en particulier les méthodes de Schröder de première et deuxième espèce

Cet article a été rédigé conjointement avec M. François Dubeau et publié dans la revue Journal of Complex Analysis, sous le titre 'Fixed Point and Newton's Methods in the Complex Plane.'

Commentaires sur l'article présenté au chapitre 3

Dans cet article nous posons la question suivante: En créant une suite de fonctions dont le n-ième élément est d'ordre n, comment peut-on s'assurer que ce ne soit pas une redécouverte de la séquence de Schröder.

Nous montrons aussi que la suite de fonction présentée par Schröder est la façon la plus efficace d'augmenter l'ordre de la méthode itérative de Newton en terme d'utilisation de certaines dérivées d'ordre supérieur. Nous présentons aussi un algorithme qui caractérise complètement la suite de fonctions de Schröder, et par ce fait nous permet de la reconnaître. Finalement, nous appliquons cet algorithme à plusieurs exemples de redécouvertes et en présentons même de nouvelles.

Cet article a été rédigé conjointement avec M. François Dubeau et publié dans la revue Journal of Computational and Applied Mathematics, sous le titre 'On the rediscovery of Halley's iterative method for computing the zero of an analytic function.'

En tant qu'auteur principal, j'y ai développé les notions d'algorithmes pour éviter les redécouvertes. La formulation de plusieurs théorèmes et le développement de leurs preuves ont été faites conjointement avec M. Dubeau.

Commentaires sur l'article présenté au chapitre 4

Il s'est avéré que la fonction d'itération de Halley peut être obtenue en appliquant la méthode de Newton à une nouvelle fonction $F(z)$ au lieu de $f(z)$. Nous posons donc les questions suivantes: Quelles sont toutes les modifications possibles de la fonction $f(z)$ en une nouvelle fonction $F(z)$ qui nous permettront d'obtenir une nouvelle méthode plus rapide (comme celle de Halley , par exemple)? Quel est le lien entre toutes ces fonctions possibles? Nous regardons aussi le processus d'accélération présenté par Euler-Chebyshev et établissons un nouveau lien entre celui-ci et celui de Halley.

Cet article a été rédigé conjointement avec M. François Dubeau et publié dans la revue Journal of Mathematical Analysis, sous le titre 'Unifying old and new ways to increase order of convergence of fixed point and Newton's method'. La formulation de plusieurs théorèmes et élaboration de leurs preuves ont été fait conjointement avec M. Dubeau.

Commentaires sur l'article présenté au chapitre 5

Dans cet article nous cherchons la réponse à la question suivante: quelle est la meilleure façon d'augmenter l'ordre de convergence la méthode itérative de Newton?

Nous montrons que, la façon la plus effective d'augmenter l'ordre de convergence de la méthode de Newton, en termes de fonctions polynomiales, est la méthode de Schröder de première espèce.

Ainsi, joint au résultat du chapitre 3, nous obtenons le fait que les méthodes itératives de Schröder de première et de deuxième espèces sont les meilleures façons d'augmenter l'ordre de convergence de la méthode itérative de Newton.

Cet article a été rédigé conjointement avec M. François Dubeau et accepté pour publication dans la revue Elemente der Mathematik, sous le titre 'Schröder's processes and the best ways of increasing order of Newton's method.'

J'y ai proposé les notions de développement de fonctions itérantes en termes de fonctions polynomiales et leurs comparaisons avec le processus de Schröder de première espèce. La formulation des théorèmes et leurs preuves ont été faites conjointement avec M. Dubeau.

Commentaires sur la terminologie

Dans cette thèse nous avons parlé d'augmentation de l'ordre de convergence de la méthode de Newton. En effet en ce qui concerne la méthode de Schröder de deuxième espèce nous appliquons la méthode de Newton à une fonction modifiée pour en augmenter l'ordre. Ainsi nous augmentons bel et bien l'ordre de la méthode de Newton. En général la méthode de Newton est reconnue pour être d'ordre 2 mais il s'avère que l'ordre de convergence d'une méthode dépend de la fonction sur laquelle elle agit. Pour la méthode de Schröder de première espèce on ne peut pas l'écrire comme la méthode de Newton appliquée sur une fonction précise. Ainsi dans ce cas nous avons fait un abus de language en disant qu'on augmente l'ordre de la méthode de Newton. Il faudrait plutôt dire que nous augmentons l'ordre en générant une suite de méthodes dites Newton généralisées (generalized Newton), ou de type Newton (Newton type).

CHAPITRE 1

Fixed point and Newton's methods for Solving a non-linear equation: From linear to high-order convergence.

Abstract

In this paper we revisit the necessary and sufficient conditions for linear and high order convergence of fixed point and Newton's methods. Based on these conditions, we extend Schröder's process of the first kind to increase the order of convergence of the fixed point method. We also obtain two processes to increase the order of convergence of Newton's method. One of them is Schröder's process of the second kind, for which several forms are also presented. A link between Schröder's two processes is given. Examples and numerical experiments are included.

1.1 Introduction

In this paper we consider fixed point and Newton's methods to find a simple solution of a nonlinear equation. We not only present the sufficiency of conditions for convergence of fixed point and Newton's methods but we also prove the necessity of these conditions. Based on these conditions, we show how to obtain processes to recursively increase order of convergence. For the fixed point method, we present a generalization of Schröder's process of the first kind. Two methods are presented to increase the order of convergence of Newton's method when applied to this function. One of them coincides with Schröder's process of the second kind, which has several forms in the literature. We also explain the link between the two processes of Schröder. Finally, we point out ways to combine methods to obtain, for example, a Super-Halley process of order 3 and other possible higher order generalizations of this process. It is important to keep in mind that throughout the paper we consider real valued functions which are regular enough to be differentiated sufficiently many times. As a consequence, the proofs are based on a very basic tool: Taylor's expansion.

The plan of the paper is the following. In Section 2.3, we consider fixed point methods and necessary and sufficient conditions for convergence. This leads to a generalization of Schröder's process of the first kind. Section 2.4 is devoted to Newton's method. Based on the necessary and sufficient conditions, we propose two ways of increasing the order of convergence of Newton's method when applied to this function. Using examples given in Sections 2.3 and 2.4, numerical experiments are reported in Section 1.4. Schröder's process of the second kind, and its multiple different forms, is the object of Section 1.5. In Section 1.6, we explain the link between the two processes of Schröder. In Section 1.7, considering a linear combination of a fixed point process and its associated Newton's process, we obtain a general recursive method to increase the order of convergence of a fixed point process. Finally, remarks on further research topics are mentioned in Section 2.6.

To close the introduction, let us mention that several excellent books discuss fixed point and Newton's methods, see for example [22, 30, 37, 51]. Several proofs presented in this paper are in these books, they are included here for completeness.

1.2 Fixed point method

A fixed point method uses an iteration function (IF) which is a (regular) function mapping its domain of definition into itself. With an IF $\Phi(x)$ and an initial value x_0, we are interested in the convergence of the sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$. It is well known that if the sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ converges, it converges to a fixed point of $\Phi(x)$.

Let $\Phi(x)$ be an IF, p be a positive integer, and $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ be such that the following limit exists (and is finite)

$$\lim_{k \to +\infty} \frac{x_{k+1} - \alpha}{(x_k - \alpha)^p} = K_p(\alpha; \Phi).$$

We say that the convergence of the sequence to α is of (integer) order p if and only if $K_p(\alpha; \Phi) \neq 0$, and $K_p(\alpha; \Phi)$ is called the asymptotic constant. We also say that $\Phi(x)$ is of order p. If the limit $K_p(\alpha; \Phi)$ exists but is zero, we can say that $\Phi(x)$ is of order at least p.

From a numerical point of view, since α is not known, it is useful to define the ratio

$$\widetilde{K}_p(\alpha, k) = \frac{x_{k+1} - x_{k+2}}{(x_k - x_{k+1})^p}. \tag{1.2.1}$$

Following [4], it can be shown that

$$\lim_{k \to +\infty} \widetilde{K}_p(\alpha, k) = K_p(\alpha; \Phi),$$

and

$$\lim_{k\to+\infty} \frac{\ln\left|\widetilde{K}_1(\alpha, k+1)\right|}{\ln\left|\widetilde{K}_1(\alpha, k)\right|} = p.$$

The first result concerns necessary and sufficient conditions to have linear convergence. Almost any textbook on numerical analysis reports the sufficiency of the condition in (i) of the following result. We have included the necessity of the condition in (ii).

Theorem 1.2.1. *Let $\Phi(x)$ be an IF, and $\Phi^{(1)}(x)$ stand for its first derivative.*
(i) If $|\Phi^{(1)}(\alpha)| < 1$ then there exists a neighbourhood of α such that for any x_0 in that neighbourhood the sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ converges to α.
(ii) If there exists a neighbourhood of α such that for any x_0 in that neighbourhood the sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ converges to α, and $x_k \neq \alpha$ for all k, then $\left|\Phi^{(1)}(\alpha)\right| \leq 1$.
(iii) For any sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ which converges to α the limit $K_1(\alpha; \Phi)$ exists, and $K_1(\alpha; \Phi) = \Phi^{(1)}(\alpha)$.

Proof. See Appendix 1.9 □

For higher order convergence we have the following result about the necessary and sufficient conditions. Several textbooks, like [41, 37] and others, mention sufficiency of the condition but not necessity. The necessity of the condition is rarely reported, it appears for example in [51].

Theorem 1.2.2. *Let p be an integer ≥ 2 and let $\Phi(x)$ be a regular function such that $\Phi(\alpha) = \alpha$. The IF $\Phi(x)$ is of order p if and only if $\Phi^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-1$, and $\Phi^{(p)}(\alpha) \neq 0$. Moreover, the asymptotic constant is given by*

$$K_p(\alpha; \Phi) = \lim_{k\to+\infty} \frac{x_{k+1} - \alpha}{(x_k - \alpha)^p} = \frac{\Phi^{(p)}(\alpha)}{p!}.$$

Proof. See Appendix 1.10 □

It follows that for a regular IF and $p \geq 2$, the limit $K_p(\alpha;\Phi)$ exists if and only if $K_l(\alpha;\Phi) = 0$ for $l = 1,\ldots,p-1$.

We say that α is a root of $f(x)$ of multiplicity q if and only if $f^{(j)}(\alpha) = 0$ for $j = 0,\ldots,q-1$, and $f^{(q)}(\alpha) \neq 0$. Moreover, α is a root of $f(x)$ of multiplicity q if and only if there exists a continuous function $v_f(x)$ such that $v_f(\alpha) \neq 0$ and $f(x) = v_f(x)(x-\alpha)^q$.

We will use the big O notation $g(x) = O(f(x))$ (and the small o notation $g(x) = o(f(x))$) around $x = \alpha$ when $c \neq 0$ (when $c = 0$, respectively) where

$$\lim_{x\to\alpha} \frac{g(x)}{f(x)} = c. \tag{1.2.2}$$

If α is a root of multiplicity q of $f(x)$, then $g(x) = O(f(x))$ is equivalent to $g(x) = O((x-\alpha)^q)$. Also, if α is a simple root of $f(x)$, then α is a root of multiplicity q of $f^q(x)$. Hence $g(x) = O(f^q(x))$ is equivalent to $g(x) = O((x-\alpha)^q)$.

As a consequence, for a general regular IF $\Phi(x)$ and a simple root α of $f(x)$ we can say that: (a) $\Phi(x)$ is of order p if and only if $\Phi(x) = \alpha + O((x-\alpha)^p)$, or equivalently, if $\Phi(\alpha) = \alpha$ and $\Phi^{(1)}(x) = O((x-\alpha)^{p-1})$, and (b) if α is a simple root of $f(x)$, then $\Phi(x)$ is of order p if and only if $\Phi(x) = \alpha + O(f^p(x))$, or equivalently, if $\Phi(\alpha) = \alpha$ and $\Phi^{(1)}(x) = O(f^{p-1}(x))$.

SchrÄ̈oder's process of the first kind is a systematic and recursive way to construct an IF of arbitrary order p to find a simple zero α of $f(x)$. The IF has to fulfil at least the sufficient condition of Theorem 2.3.2. Let us present a generalization of this process.

Theorem 1.2.3. *Let α be a simple root of $f(x)$, and $c_0(x)$ be a regular function such that $c_0(\alpha) = \alpha$. Let $\Phi_p(x)$ be the IF defined by*

$$\Phi_p(x) = \sum_{l=0}^{p-1} c_l(x) f^l(x) \tag{1.2.3}$$

where the $c_l(x)$ *are such that*

$$c_l(x) = -\frac{1}{l}\left(\frac{1}{f^{(1)}(x)}\frac{d}{dx}\right)c_{l-1}(x) \tag{1.2.4}$$

for $l = 1, 2, \ldots$ *Then* $\Phi_p(x)$ *is of order* p*, and its asymptotic constant is*

$$K_p(\alpha, \Phi_p) = \frac{\Phi^{(p)}(\alpha)}{p!} = \frac{1}{p}c^{(1)}_{p-1}(\alpha)\left[f^{(1)}(\alpha)\right]^{p-1} = -c_p(\alpha)\left[f^{(1)}(\alpha)\right]^{p}. \tag{1.2.5}$$

Proof. We verify that $\Phi_p(\alpha) = \alpha$. Moreover, taking the first derivative and using (2.3.4), we obtain

$$\Phi^{(1)}_p(x) = c^{(1)}_{p-1}(x)f^{p-1}(x). \tag{1.2.6}$$

As a consequence we not only have $\Phi^{(1)}_p(\alpha) = 0$ but we also have $\Phi^{(l)}_p(\alpha) = 0$ for $l = 1, \ldots, p-1$. It follows that

$$\Phi^{(p)}_p(\alpha) = \left(c^{(1)}_{p-1}(x)f^{p-1}(x)\right)^{(p-1)}\Bigg|_{x=\alpha} = (p-1)!\, c^{(1)}_{p-1}(\alpha)\left[f^{(1)}(\alpha)\right]^{p-1}, \tag{1.2.7}$$

and using (2.3.4), we get (2.3.5).

This same result can also be obtained by considering Taylor's expansion of an inverse function [13, 51, 27]. For $c_0(x) = x$ in (2.3.3), we recover Schröder's process of the first kind of order p [48, 51, 27], which is also associated to Chebyshev and Euler [2, 49, 44]. The first term $c_0(x)$ could be seen as a preconditioning to decrease the asymptotic constant of the method, but its choice is not obvious. We present one such example below.

Example 1.2.1. *We illustrate Theorem 5.3.2 with a non-trivial* $c_0(x)$ *on the n-th root computation problem, namely find* $\alpha = r^{1/n}$ *for a strictly positive real number* r*. We consider*

$$f(x) = \frac{x^n}{r} - 1, \tag{1.2.8}$$

Tableau 1.1: Iteration functions for the computation of $r^{1/n}$ based on Theorem 5.3.2.

$f(x) = \frac{x^n}{r} - 1$	$c_0(x) = x\left(\frac{x^n}{r}\right)^M$
Iteration function	Asymptotic constant
$\Phi_p(x)$	$K_p(r^{1/n};\Phi_p)$
$\Phi_p(x) \quad = \quad c_0(x)\sum_{l=0}^{p-1}\binom{1/n+M}{l}\left(\frac{r}{x^n}-1\right)^l$	$(-1)^{p+1}\binom{1/n+M}{p} n^p r^{(1-p)/n}$
Example : $M = 0$ see [15], [51], [25]	
$\Phi_p(x) \quad = \quad E_p(x) \quad = \quad x\sum_{l=0}^{p-1}\binom{1/n}{l}\left(\frac{r}{x^n}-1\right)^l$	$(-1)^{p+1}\binom{1/n}{p} n^p r^{(1-p)/n}$
Example : $M = M^* = \lfloor (p-1)/2 \rfloor$	
$\Phi_p(x) \quad = \quad c_0^*(x)\sum_{l=0}^{p-1}\binom{1/n+M^*}{l}\left(\frac{r}{x^n}-1\right)^l$	$(-1)^{p+1}\binom{1/n+M^*}{p} n^p r^{(1-p)/n}$

and we set

$$c_0(x) = x\left(\frac{x^n}{r}\right)^M, \tag{1.2.9}$$

where M is an integer. We obtain

$$c_l(x) = c_0(x)\left(\frac{r}{x^n}\right)^l\binom{1/n+M}{l} \tag{1.2.10}$$

for $l = 0, 1, 2, \ldots$ Then

$$\Phi_p(x) = c_0(x)\sum_{l=0}^{p-1}\binom{1/n+M}{l}\left(\frac{r}{x^n}-1\right)^l \tag{1.2.11}$$

with

$$\Phi_p^{(p)}(r^{1/n}) = (-1)^{p+1}n^p r^{(1-p)/n}p!\binom{1/n+M}{p}, \tag{1.2.12}$$

and

$$K_p(r^{1/n}; \Phi_p) = (-1)^{p+1} n^p r^{(1-p)/n} \begin{pmatrix} 1/n + M \\ p \end{pmatrix}. \qquad (1.2.13)$$

The processes obtained are presented in Table 1.1.

For $M = 0$, we have $c_0(x) = x$ and the process corresponds to Schröder's process of the first kind presented in Durand [15] and Traub [51]. This process has also been rediscovered by Hernandez and Romero [25]. For $M \neq 0$, we obtain new IFs of order p with the smallest asymptotic constant for $M = M^* = \lfloor (p-1)/2 \rfloor$ given in Table 1.1. (See also [13].)

1.3 Newton's iteration function

Considering $c_0(x) = x$ and $p = 2$ in (2.3.3), we obtain

$$\Phi_2(x) = x - \frac{f(x)}{f^{(1)}(x)}$$

which is Newton's IF of order 2 to solve $f(x) = 0$. The sufficiency of the condition for high order convergence of Newton's method when applied to this function has been presented in [19]. The necessity of the condition is included in the next result.

Theorem 1.3.1. *Let $p \geq 2$ and let $\Psi(x)$ be a regular function such that $\Psi(\alpha) = 0$ and $\Psi^{(1)}(\alpha) \neq 0$. The Newton iteration $N_\Psi(x) = x - \frac{\Psi(x)}{\Psi^{(1)}(x)}$ is of order p if and only if $\Psi^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $\Psi^{(p)}(\alpha) \neq 0$. Moreover, the asymptotic constant is*

$$K_p(\alpha; N_\Psi) = \frac{p-1}{p!} \frac{\Psi^{(p)}(\alpha)}{\Psi^{(1)}(\alpha)}.$$

Proof. See Appendix 1.11

We can look for a recursive method to construct a function $\Psi_p(x)$ which fulfil the sufficient condition of Theorem 5.4.1. A consequence will be that $N_{\Psi_p}(x)$ will be of order p, and $N_{\Psi_p}(x) = \alpha + O(f^p(x))$. One such method is given in the next theorem.

Theorem 1.3.2. *Let α be a simple root of $f(x)$. Let $\Psi_p(x)$ be defined by*

$$\Psi_p(x) = \sum_{l=0}^{p-1} d_l(x) f^l(x) \tag{1.3.1}$$

where $d_0(x)$ and $d_1(x)$ are two regular functions such that

$$\begin{cases} d_0(\alpha) & = & 0, \\ d_0^{(1)}(\alpha) + d_1(\alpha) f^{(1)}(\alpha) & \neq & 0, \end{cases} \tag{1.3.2}$$

and

$$d_l(x) = -\frac{1}{l f^{(1)}(x)} \times \tag{1.3.3}$$
$$\left[d_{l-1}^{(1)}(x) + \frac{1}{(l-1) f^{(1)}(x)} \left[d_{l-2}^{(1)}(x) + (l-1) d_{l-1}(x) f^{(1)}(x) \right]^{(1)} \right]$$

for $l = 2, 3, \ldots$ Then

$$N_{\Psi_p}(x) = x - \frac{\Psi_p(x)}{\Psi_p^{(1)}(x)}$$

is of order p, with

$$\Psi_p^{(p)}(\alpha) = -p!\; d_p(\alpha) \left[f^{(1)}(\alpha) \right]^p,$$

and

$$K_p(\alpha; N_{\Psi_p}) = -\frac{(p-1) d_p(\alpha) \left[f^{(1)}(\alpha) \right]^p}{d_0^{(1)}(\alpha) + d_1(\alpha) f^{(1)}(\alpha)}.$$

Proof. We have $\Psi_p(\alpha) = d_0(\alpha) = 0$. A direct differentiation leads to

$$\Psi_p^{(1)}(x) = \sum_{l=0}^{p-2} \left[d_l^{(1)}(x) + (l+1) d_{l+1}(x) f^{(1)}(x) \right] f^l(x) + d_{p-1}^{(1)}(x) f^{p-1}(x),$$

and

$$\Psi_p^{(1)}(\alpha) = d_0^{(1)}(\alpha) + d_1(\alpha) f^{(1)}(\alpha) \neq 0.$$

Differentiating again, we obtain

$$\begin{aligned}
\Psi_p^{(2)}(x) &= \sum_{l=0}^{p-3}\left\{\left[d_l^{(1)}(x)+(l+1)d_{l+1}(x)f^{(1)}(x)\right]^{(1)}\right. \\
&\quad \left.+(l+1)\left[d_{l+1}^{(1)}(x)+(l+2)d_{l+2}(x)f^{(1)}(x)\right]f^{(1)}(x)\right\}f^l(x) \\
&\quad +\left\{\left[d_{p-2}^{(1)}(x)+(p-1)d_{p-1}(x)f^{(1)}(x)\right]^{(1)}+(p-1)d_{p-1}^{(1)}(x)f^{(1)}(x)\right\}f^{p-2}(x) \\
&\quad +d_{p-1}^{(2)}(x)f^{p-1}(x) \\
&= \left\{\left[d_{p-2}^{(1)}(x)+(p-1)d_{p-1}(x)f^{(1)}(x)\right]^{(1)}+(p-1)d_{p-1}^{(1)}(x)f^{(1)}(x)\right\}f^{p-2}(x) \\
&\quad +d_{p-1}^{(2)}(x)f^{p-1}(x).
\end{aligned}$$

Then we not only have $\Psi_p^{(2)}(\alpha)=0$ but also $\Psi_p^{(l)}(\alpha)=0$ for $l=2,\ldots,p-1$. Moreover

$$\begin{aligned}
\Psi_p^{(p)}(\alpha) &= \left(\Psi_p^{(2)}(x)\right)^{(p-2)}\Big|_{x=\alpha} \\
&= (p-2)!\ \left[f^{(1)}(\alpha)\right]^{p-2}\times \\
&\quad \left[\left[d_{p-2}^{(1)}(x)+(p-1)d_{p-1}(x)f^{(1)}(x)\right]^{(1)}+(p-1)d_{p-1}^{(1)}(x)f^{(1)}(x)\right]\Bigg|_{x=\alpha},
\end{aligned}$$

and the result follows from (2.4.6).

Let us observe that if we set $\Psi_p(x)=\Phi_p(x)-x$ with $\Phi_p(x)$ given by (2.3.3), then $\Psi_p(x)$ verifies the assumptions of Theorem 2.4.4.

Example 1.3.1. *We illustrate Theorem 2.4.4 by considering the n-th root computation problem. We consider*

$$f(x)=\frac{r}{x^n}-1 \tag{1.3.4}$$

and choose $d_0(x)=0$ and $d_1(x)=-\begin{pmatrix}1/n\\1\end{pmatrix}\frac{x^n}{r}$. We obtain

$$d_l(x)=(-1)^l\begin{pmatrix}1/n\\l\end{pmatrix}\left(\frac{x^n}{r}\right)^l$$

for $l=1,2,3,\ldots,$ and

$$\Psi_p(x)=\sum_{l=1}^{p-1}\begin{pmatrix}1/n\\l\end{pmatrix}\left(\frac{x^n}{r}-1\right)^l.$$

The asymptotic constant of $N_{\Psi_p}(x)$ *is*

$$K_p(r^{1/n}; N_{\Psi_p}) = -(p-1)n^p r^{(1-p)/n} \left(\begin{array}{c} 1/n \\ p \end{array} \right). \tag{1.3.5}$$

$N_{\Psi_p}(x)$*, which appears in Table 1.2, corresponds to Dubeau's method of order* p *[9].*

Tableau 1.2: Iteration functions for the computation of $r^{1/n}$ based on Theorem 2.4.4.

$f(x) = \frac{r}{x^n} - 1$	
Iteration function	Asymptotic constant
$N_{\Psi_p}(x)$	$K_p(r^{1/n}; N_{\Psi_p})$
$N_{\Psi_p}(x) \quad = \quad x - \frac{r}{nx^{n-1}} \frac{\sum_{l=1}^{p-1} \left(\begin{array}{c} 1/n \\ l \end{array} \right) \left(\frac{x^n}{r} - 1 \right)^l}{\sum_{l=1}^{p-1} l \left(\begin{array}{c} 1/n \\ l \end{array} \right) \left(\frac{x^n}{r} - 1 \right)^{l-1}}$	$-(p-1) \left(\begin{array}{c} 1/n \\ p \end{array} \right) n^p r^{(1-p)/n}$

A second method to recursively obtain a function which satisfies the necessary and sufficient conditions of Theorem 5.4.1 has been presented in [19, 17]. The technique can also be based on Taylor's expansion as indicated in [6].

Theorem 1.3.3. *[19] Let* $f(x)$ *be such that* $f(\alpha) = 0$ *and* $f^{(1)}(\alpha) > 0$*. If* $F_p(x)$ *is defined by*

$$\left\{ \begin{array}{rcl} F_2(x) & = & f(x), \\ F_p(x) & = & \frac{F_{p-1}(x)}{\left[F_{p-1}^{(1)}(x)\right]^{\frac{1}{p-1}}} \quad \textit{for} \quad p \geq 3, \end{array} \right. \tag{1.3.6}$$

then $F_p(\alpha) = 0$*,* $F_p^{(1)}(\alpha) > 0$*,* $F_p^{(l)}(\alpha) = 0$ *for* $l = 2, \ldots, p-1$*. It follows that* $N_{F_p}(x)$ *is of order* p*, or at least of order* p*.*

We also have the following simplification for $N_{F_p}(x)$.

Theorem 1.3.4. *[17] If $F_p(x)$ is given by (2.4.1), then*

$$N_{F_p}(x) = x - \frac{f(x)}{f^{(1)}(x) - \frac{1}{p-1} f(x) \frac{Q_p^{(1)}(x)}{Q_p(x)}} = x - f(x)\frac{Q_p(x)}{Q_{p+1}(x)}, \tag{1.3.7}$$

where

$$\begin{cases} Q_2(x) & = & 1, \\ Q_p(x) & = & f^{(1)}(x)Q_{p-1}(x) - \frac{1}{p-2} f(x) Q_{p-1}^{(1)}(x) \quad \textit{for} \quad p \geq 3. \end{cases} \tag{1.3.8}$$

Example 1.3.2. *To illustrate Gerlach's process we consider the n-th root computation using*

$$f(x) = \frac{x^n}{r} - 1,$$

for which

$$\frac{f(x)}{f^{(1)}(x)} = -\frac{x}{n}\left(\frac{r}{x^n} - 1\right).$$

Table 1.3 presents the first 4 *IFs. Unfortunately, no general formula for the IF and its asymptotic constant are known in this case, contrary to the preceding methods. However, the asymptotic constant can be estimated with (2.3.1).*

1.4 A numerical example

To illustrate the different methods presented in Sections 2.3 and 2.4, and explicitly given in Tables 1.1, 1.2, and 1.3, we have considered the n-th root problem, find $\alpha = 35^{1/n}$ for $n = 2$, 5, and 10. The results are reported in Tables 1.4, 1.5, and 1.6. For these examples, in all cases, Gerlach's process $N_{F_p}(x)$ has the smallest asymptotic constant for a given order. Schröder's process of the first kind $\Phi_p(x) = E_p(x)$ is slightly improved by choosing $M^* = \left\lfloor \frac{p-1}{2} \right\rfloor$ instead of $M = 0$. Finally $N_{\Psi_p}(x)$ seems to be the worse IF, but let us observe we did not try to choose the best $d_0(x)$ and $d_1(x)$ in (2.4.4) to minimize its asymptotic constant.

Tableau 1.3: Iteration functions for the computation of $r^{1/n}$ based on Theorems 1.3.3 and 1.3.4.

$$f(x) = \tfrac{x^n}{r} - 1 = F_2(x)$$

$$
\begin{aligned}
N_{F_2}(x) &= x + \frac{x}{n} \times \left(\frac{r}{x^n} - 1\right) \\
N_{F_3}(x) &= x + \frac{x}{n} \times \frac{\left(\frac{r}{x^n} - 1\right)}{1 + \frac{1}{n^2}\binom{n}{2}\left(\frac{r}{x^n} - 1\right)} \\
N_{F_4}(x) &= x + \frac{x}{n} \times \frac{\left(\frac{r}{x^n} - 1\right)\left(1 + \frac{1}{n^2}\binom{n}{2}\left(\frac{r}{x^n} - 1\right)\right)}{1 + \frac{2}{n^2}\binom{n}{2}\left(\frac{r}{x^n} - 1\right) + \frac{1}{n^3}\binom{n}{3}\left(\frac{r}{x^n} - 1\right)^2} \\
N_{F_5}(x) &= x + \frac{x}{n} \times \frac{\left(\frac{r}{x^n} - 1\right)\left(1 + \frac{2}{n^2}\binom{n}{2}\left(\frac{r}{x^n} - 1\right) + \frac{1}{n^3}\binom{n}{3}\left(\frac{r}{x^n} - 1\right)^2\right)}{1 + \frac{3}{n^2}\binom{n}{2}\left(\frac{r}{x^n} - 1\right) + \left[\frac{2}{n^3}\binom{n}{3} + \frac{1}{n^4}\binom{n}{2}^2\right]\left(\frac{r}{x^n} - 1\right)^2 + \frac{1}{n^4}\binom{n}{4}\left(\frac{r}{x^n} - 1\right)^3}
\end{aligned}
$$

Tableau 1.4: Computation of $35^{1/2} = 5.9160797830996160426...$ with $x_0 = 6$.

p	k	$IF_p(x)$			
		$\Phi_p(x) = E_p(x)$		$N_{\Psi_p}(x)$	$N_{F_p}(x)$
3		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 1$		
	1	$8.18 \ 10^{-6}$	$8.35 \ 10^{-6}$	$1.70 \ 10^{-5}$	$4.13 \ 10^{-6}$
	2	$7.82 \ 10^{-18}$	$8.33 \ 10^{-18}$	$1.40 \ 10^{-16}$	$5.04 \ 10^{-19}$
	3	$6.83 \ 10^{-54}$	$8.25 \ 10^{-54}$	$7.82 \ 10^{-50}$	$9.16 \ 10^{-58}$
	4	$4.55 \ 10^{-162}$	$8.05 \ 10^{-162}$	$1.37 \ 10^{-149}$	$5.49 \ 10^{-174}$
	5	$1.34 \ 10^{-486}$	$7.44 \ 10^{-486}$	$7.32 \ 10^{-449}$	$1.18 \ 10^{-522}$
	$\lvert K_3(35^{1/2}, IF_3) \rvert$	$1.43 \ 10^{-2}$	$1.43 \ 10^{-2}$	$2.857 \ 10^{-2}$	$0.7143 \ 10^{-2}$
4		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 1$		
	1	$1.42 \ 10^{-7}$	$8.73 \ 10^{-8}$	$4.54 \ 10^{-7}$	$2.91 \ 10^{-8}$
	2	$1.24 \ 10^{-30}$	$1.05 \ 10^{-31}$	$3.85 \ 10^{-28}$	$4.33 \ 10^{-34}$
	3	$7.09 \ 10^{-123}$	$2.23 \ 10^{-127}$	$2.00 \ 10^{-112}$	$2.13 \ 10^{-137}$
	4	$7.63 \ 10^{-492}$	$4.48 \ 10^{-510}$	$1.45 \ 10^{-449}$	$1.24 \ 10^{-550}$
	5	$1.02 \ 10^{-1967}$	$7.33 \ 10^{-2041}$	$3.98 \ 10^{-1798}$	$1.42 \ 10^{-2203}$
	$\lvert K_4(35^{1/2}, IF_4) \rvert$	$3.012 \ 10^{-3}$	$1.811 \ 10^{-3}$	$9.055 \ 10^{-3}$	$0.604 \ 10^{-3}$
5		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 2$		
	1	$2.77 \ 10^{-9}$	$1.25 \ 10^{-9}$	$1.21 \ 10^{-8}$	$2.05 \ 10^{-10}$
	2	$1.68 \ 10^{-46}$	$9.14 \ 10^{-49}$	$7.42 \ 10^{-43}$	$1.85 \ 10^{-53}$
	3	$1.55 \ 10^{-223}$	$1.96 \ 10^{-244}$	$6.44 \ 10^{-214}$	$1.10 \ 10^{-268}$
	4	$6.33 \ 10^{-1168}$	$8.81 \ 10^{-1223}$	$3.18 \ 10^{-1069}$	$8.06 \ 10^{-1345}$
	5	$7.24 \ 10^{-5840}$	$1.62 \ 10^{-6114}$	$9.24 \ 10^{-5346}$	$1.74 \ 10^{-6725}$
	$\lvert K_5(35^{1/2}, IF_5) \rvert$	$7.143 \ 10^{-4}$	$3.061 \ 10^{-4}$	$2.857 \ 10^{-3}$	$5.102 \ 10^{-5}$

1.5 The Schröder's process of the second kind

Several authors have investigated different ways to increase the rate of convergence of Newton's method. It happens that some of these ways lead exactly to the same process presented differently, and are equivalent to Schröder's process of the second kind. In this section we will present 6 approaches which all lead to the same process. In fact at least 11 different approaches are equivalent as it is reported in [44, 45].

Tableau 1.5: Computation of $35^{1/5} = 2.0361680046403980174...$ with $x_0 = 2.25$.

p	k	$IF_p(x)$			
		$\Phi_p(x) = E_p(x)$		$N_{\Psi_p}(x)$	$N_{F_p}(x)$
3		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 1$		
	1	$9.15 \ 10^{-3}$	$8.85 \ 10^{-3}$	$5.81 \ 10^{-2}$	$3.97 \ 10^{-3}$
	2	$1.09 \ 10^{-6}$	$6.72 \ 10^{-7}$	$5.17 \ 10^{-4}$	$3.00 \ 10^{-8}$
	3	$1.86 \ 10^{-18}$	$2.92 \ 10^{-19}$	$4.00 \ 10^{-10}$	$1.30 \ 10^{-23}$
	4	$9.36 \ 10^{-54}$	$2.41 \ 10^{-56}$	$1.85 \ 10^{-28}$	$1.07 \ 10^{-69}$
	5	$1.19 \ 10^{-159}$	$1.36 \ 10^{-167}$	$1.83 \ 10^{-83}$	$5.90 \ 10^{-208}$
	$\lvert K_3(35^{1/5}, IF_3) \rvert$	1.447	$9.648 \ 10^{-1}$	2.894	$4.824 \ 10^{-1}$
4		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 1$		
	1	$2.60 \ 10^{-3}$	$1.65 \ 10^{-3}$	$2.39 \ 10^{-2}$	$2.50 \ 10^{-4}$
	2	$1.12 \ 10^{-10}$	$7.94 \ 10^{-12}$	$2.61 \ 10^{-6}$	$4.63 \ 10^{-16}$
	3	$3.91 \ 10^{-40}$	$4.23 \ 10^{-45}$	$3.48 \ 10^{-22}$	$5.43 \ 10^{-63}$
	4	$5.79 \ 10^{-158}$	$3.41 \ 10^{-178}$	$1.10 \ 10^{-85}$	$1.03 \ 10^{-250}$
	5	$2.79 \ 10^{-629}$	$1.44 \ 10^{-710}$	$1.10 \ 10^{-339}$	$1.32 \ 10^{-1001}$
	$\lvert K_4(35^{1/5}, IF_4) \rvert$	2.488	1.066	7.462	$1.185 \ 10^{-1}$
5		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 2$		
	1	$7.91 \ 10^{-4}$	$4.48 \ 10^{-4}$	$2.02 \ 10^{-2}$	$3.22 \ 10^{-6}$
	2	$1.44 \ 10^{-15}$	$2.09 \ 10^{-17}$	$5.90 \ 10^{-8}$	$4.04 \ 10^{-30}$
	3	$2.86 \ 10^{-74}$	$4.54 \ 10^{-84}$	$1.33 \ 10^{-35}$	$1.25 \ 10^{-149}$
	4	$8.84 \ 10^{-368}$	$2.22 \ 10^{-417}$	$7.75 \ 10^{-174}$	$3.60 \ 10^{-747}$
	5	$2.50 \ 10^{-1385}$	$6.20 \ 10^{-2084}$	$5.18 \ 10^{-865}$	$7.04 \ 10^{-3735}$
	$\lvert K_5(35^{1/5}, IF_3) \rvert$	4.642	1.152	$1.860 \ 10^{+1}$	$1.16 \ 10^{-2}$

Tableau 1.6: Computation of $35^{1/10} = 1.4269435884576509836...$ with $x_0 = 1.5$.

		$IF_p(x)$			
p	k	$\Phi_p(x) = E_p(x)$		$N_{\Psi_p}(x)$	$N_{F_p}(x)$
3		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 1$		
	1	$3.67\ 10^{-3}$	$3.08\ 10^{-3}$	$2.71\ 10^{-2}$	$1.43\ 10^{-3}$
	2	$6.78\ 10^{-7}$	$2.37\ 10^{-7}$	$4.78\ 10^{-4}$	$1.19\ 10^{-8}$
	3	$4.37\ 10^{-18}$	$1.08\ 10^{-19}$	$3.07\ 10^{-9}$	$6.75\ 10^{-24}$
	4	$1.17\ 10^{-51}$	$1.03\ 10^{-56}$	$8.12\ 10^{-25}$	$1.24\ 10^{-69}$
	5	$2.22\ 10^{-152}$	$8.93\ 10^{-168}$	$1.50\ 10^{-71}$	$7.84\ 10^{-207}$
	$\lvert K_3(35^{1/10}, IF_3)\rvert$	$1.400\ 10^{+1}$	8.103	$2.799\ 10^{+1}$	4.052
4		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 1$		
	1	$1.07\ 10^{-3}$	$6.05\ 10^{-4}$	$1.01\ 10^{-2}$	$5.78\ 10^{-5}$
	2	$9.43\ 10^{-11}$	$3.62\ 10^{-12}$	$2.46\ 10^{-6}$	$1.59\ 10^{-17}$
	3	$5.63\ 10^{-39}$	$4.65\ 10^{-45}$	$7.84\ 10^{-21}$	$9.05\ 10^{-68}$
	4	$7.17\ 10^{-152}$	$1.26\ 10^{-176}$	$8.09\ 10^{-79}$	$9.53\ 10^{-269}$
	5	$1.88\ 10^{-603}$	$6.82\ 10^{-703}$	$9.12\ 10^{-311}$	$1.17\ 10^{-1072}$
	$\lvert K_4(35^{1/10}, IF_4)\rvert$	$7.112\ 10^{+1}$	$2.698\ 10^{+1}$	$2.133\ 10^{+2}$	1.420
5		$M = 0$	$M = M^* = \lfloor \frac{p-1}{2} \rfloor = 2$		
	1	$3.36\ 10^{-4}$	$1.57\ 10^{-4}$	$9.50\ 10^{-3}$	$3.10\ 10^{-6}$
	2	$1.66\ 10^{-15}$	$7.46\ 10^{-18}$	$1.07\ 10^{-7}$	$7.70\ 10^{-28}$
	3	$4.94\ 10^{-72}$	$1.84\ 10^{-84}$	$2.19\ 10^{-32}$	$7.26\ 10^{-136}$
	4	$1.14\ 10^{-354}$	$1.67\ 10^{-417}$	$7.78\ 10^{-156}$	$5.42\ 10^{-676}$
	5	$7.52\ 10^{-1768}$	$1.03\ 10^{-2082}$	$4.43\ 10^{-773}$	$1.26\ 10^{-3376}$
	$\lvert K_5(35^{1/10}, IF_5)\rvert$	$3.887\ 10^{+2}$	$7.940\ 10^{+1}$	$1.555\ 10^{+3}$	2.686

1.5.1 Determinant based methods

Methods based on a determinant's identity have been presented in the past under different forms. We will point out their equivalence by introducing the appropriate notation.

Let $\Delta_0(x) = 1$, and for $p \geq 1$

$$\Delta_p(x) = \begin{bmatrix} f^{(1)}(x) & \dots & \dots & \dots & \frac{f^{(p)}(x)}{p!} \\ f(x) & f^{(1)}(x) & \dots & \dots & \frac{f^{(p-1)}(x)}{(p-1)!} \\ 0 & f(x) & f^{(1)}(x) & \dots & \frac{f^{(p-2)}(x)}{(p-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & f(x) & f^{(1)}(x) \end{bmatrix}.$$

Expanding this determinant along the first line, we obtain

$$\Delta_p(x) = \sum_{j=1}^{p} (-1)^{j+1} \frac{f^{(j)}(x)}{j!} f^{j-1}(x) \Delta_{p-j}(x).$$

We can prove by mathematical induction that

$$\Delta_p(x) = \frac{(-1)^p f^{p+1}(x)}{p!} \left(\frac{1}{f(x)} \right)^{(p)}, \tag{1.5.1}$$

and

$$f(x)\Delta_{p-1}^{(1)}(x) = pf^{(1)}(x)\Delta_{p-1}(x) - p\Delta_p(x). \tag{1.5.2}$$

Schröder proposed the following IF

$$S_p(x) = x - \frac{R_{p-2}(x)}{R_{p-1}(x)} \tag{1.5.3}$$

where $R_p(x) = \frac{\Delta_p(x)}{f^{p+1}(x)}$. This method is also known as Schröder's process of the second kind [48, ?]. Schröder [48] derived this iteration formula by applying suitable development to partial fractions to rational function. The above derivation can be made using König's theorem [39], as presented in[45]. Wang [54] (see also [55]) proposed the IF

$$W_p(x) = x - \frac{\Gamma_{p-2}(x)}{\Gamma_{p-1}(x)} \tag{1.5.4}$$

where $\Gamma_p(x) = \frac{\Delta_p(x)}{f^p(x)}$. Kalantari [33, 34, 32] proposed the IF

$$B_p(x) = x - f(x)\frac{\Delta_{p-2}(x)}{\Delta_{p-1}(x)}. \tag{1.5.5}$$

Hence, $S_p(x) = W_p(x) = B_p(x)$.

Because $\Delta_0(x) = 1 = Q_2(x)$, $\Delta_1(x) = f^{(1)}(x) = Q_3(x)$, using (1.5.2) it follows by induction that $\Delta_{p-2}(x) = Q_p(x)$ for $p \geq 2$. Hence $N_{F_p}(x) = B_p(x)$ (see also [35]) and Gerlach's process corresponds to Schröder's process of the second kind.

Householder [27] analysed the following IF, also known as the König's IF,

$$H_p(x) = x - (p-1)\frac{\left(\frac{1}{f(x)}\right)^{(p-2)}}{\left(\frac{1}{f(x)}\right)^{(p-1)}} \tag{1.5.6}$$

for $p \geq 2$ which is also equivalent to $S_p(x)$ because of (1.5.1).

1.5.2 Jovanović's method

Jonanović [31] suggested the following process

$$\begin{cases} J_2(x) &= x - \frac{f(x)}{f^{(1)}(x)}, \\ J_p(x) &= x - \frac{x - J_{p-1}(x)}{1 - \frac{1}{p-1}J^{(1)}_{p-1}(x)} \quad \text{for} \quad p > 2. \end{cases} \tag{1.5.7}$$

We prove by mathematical induction that $J_p(x) = B_p(x)$. Clearly $J_2(x) = B_2(x) = N_f(x)$. Assume that $J_l(x) = B_l(x)$ for $l = 2, \ldots, p-1$. From

$$1 - J^{(1)}_{p-1}(x) = (x - J_{p-1}(x))^{(1)} = \frac{d}{dx}\left(f(x)\frac{\Delta_{p-3}(x)}{\Delta_{p-2}(x)}\right),$$

and using (1.5.2), we obtain

$$1 - J_{p-1}^{(1)}(x) = -(p-2) + (p-1)\frac{\Delta_{p-3}(x)\Delta_{p-1}(x)}{\left[\Delta_{p-2}(x)\right]^2}.$$

It follows that $J_p(x) = B_p(x)$.

1.6 A Link between the two processes of Schr$\ddot{o}$der

We have shown that

$$N_{F_p}(x) = S_p(x) = W_p(x) = B_p(x) = H_p(x) = J_p(x) = \alpha + O(f^p(x)).$$

Also

$$E_p(x) = \alpha + O(f^p(x)).$$

A link between the two processes $E_p(x)$ and $S_p(x)$ of Schröder can be shown by using the equivalent form $J_p(x)$ of $S_p(x)$.

We have $J_2(x) = E_2(x) = N_f(x)$. We proceed by induction for $p \geq 3$. Suppose

$$J_{p-1}(x) = E_{p-1}(x) + O(f^{p-1}(x)) = E_{p-1}(x) + \phi_{p-1}(x)f^{p-1}(x)$$

for a regular function $\phi_{p-1}(x)$. Then

$$J_{p-1}^{(1)}(x) = E_{p-1}^{(1)}(x) + \left[\phi_{p-1}(x)f^{p-1}(x)\right]^{(1)}$$

where, from (1.2.6), we have

$$E_{p-1}^{(1)}(x) = c_{p-2}^{(1)}(x)f^{p-2}(x) = O(f^{p-2}(x)),$$

and

$$\begin{aligned}
\left[\phi_{p-1}(x)f^{p-1}(x)\right]^{(1)} &= (p-1)\phi_{p-1}(x)f^{(1)}(x)f^{p-2}(x) + \phi_{p-1}^{(1)}(x)f^{p-1}(x) \\
&= (p-1)\phi_{p-1}(x)f^{(1)}(x)f^{p-2}(x) + O(f^{p-1}(x)) \\
&= O(f^{p-2}(x)).
\end{aligned}$$

Substituting in $J_p(x)$, we have

$$J_p(x) = x - \frac{x - [E_{p-1}(x) + \phi_{p-1}(x) f^{p-1}(x)]}{1 - \frac{1}{p-1}\left[E_{p-1}^{(1)}(x) + [\phi_{p-1}(x) f^{p-1}(x)]^{(1)}\right]}$$

Using the expansion $\frac{1}{1-y} = 1 + y + O(y^2)$, and the fact that $2p-3 \geq p$ for $p \geq 3$, we obtain

$$\begin{aligned} J_p(x) &= E_{p-1}(x) + \frac{1}{p-1} c_1(x) c_{p-2}^{(1)}(x) f^{p-1}(x) \\ &\quad + \phi_{p-1}(x)\left[1 + c_1(x) f^{(1)}(x)\right] f^{p-1}(x) + O(f^p(x)). \end{aligned}$$

Hence $J_p(x) = E_p(x) + O(f^p(x))$ because $c_1(x) = -\frac{1}{f^{(1)}(x)}$, and

$$\frac{1}{p-1} c_1(x) c_{p-2}^{(1)}(x) = -\frac{1}{p-1}\left(\frac{1}{f^{(1)}(x)} \frac{d}{dx}\right) c_{p-2}(x) = c_{p-1}(x).$$

This result shows that Schröder's process of the first kind can be obtained from Schröder's process of the second kind by expanding the denominator in $J_p(x)$, multiplying and truncating to keep powers of $f(x)$, or powers of $f(x)/f^{(1)}(x)$, up to $p-1$ (see also [32], [45]).

The reader can verify the link by using the formulae given below for the two processes. The verification of this result has already been done using symbolic computation up to order 20 [45].

Example 1.6.1. *The first 4 IFs of Schröder's process of the first kind are :*

$E_2(x) = x - \frac{f(x)}{f^{(1)}(x)}$, *which corresponds to Newton's IF of order 2;*

$E_3(x) = x - \frac{f(x)}{f^{(1)}(x)} - \frac{1}{2!}\frac{f^{(2)}(x)}{f^{(1)}(x)}\left[\frac{f(x)}{f^{(1)}(x)}\right]^2$, *which corresponds to the order 3 Chebyshev's IF [2];*

$$E_4(x) = x - \frac{f(x)}{f^{(1)}(x)} - \frac{1}{2!}\frac{f^{(2)}(x)}{f^{(1)}(x)}\left[\frac{f(x)}{f^{(1)}(x)}\right]^2 - \frac{1}{3!}\left[\frac{3\left[f^{(2)}(x)\right]^2 - f^{(1)}(x)f^{(3)}(x)}{\left[f^{(1)}(x)\right]^2}\right]\left[\frac{f(x)}{f^{(1)}(x)}\right]^3;$$

and

$$\begin{aligned} E_5(x) = x &- \frac{f(x)}{f^{(1)}(x)} - \frac{1}{2!}\frac{f^{(2)}(x)}{f^{(1)}(x)}\left[\frac{f(x)}{f^{(1)}(x)}\right]^2 \\ &- \frac{1}{3!}\left[\frac{3\left[f^{(2)}(x)\right]^2 - f^{(1)}(x)f^{(3)}(x)}{\left[f^{(1)}(x)\right]^2}\right]\left[\frac{f(x)}{f^{(1)}(x)}\right]^3 \\ &- \frac{1}{4!}\left[\frac{\left[f^{(1)}(x)\right]^2 f^{(4)}(x) - 10f^{(1)}(x)f^{(2)}(x)f^{(3)}(x) + 15\left[f^{(2)}(x)\right]^3}{\left[f^{(1)}(x)\right]^4}\right]\left[\frac{f(x)}{f^{(1)}(x)}\right]^3. \end{aligned}$$

The formulas for $E_4(x)$ and $E_5(x)$ appear in Traub's book [51].

Example 1.6.2. *The first 4 IFs of Schröder's process of the second kind are:*

$S_2(x) = N_{F_2}(x) = x - \frac{f(x)}{f^{(1)}(x)}$, *which is Newton's method of order 2;*

$S_3(x) = N_{F_3}(x) = x - \frac{f(x)}{f^{(1)}(x)}\left[\frac{1}{1 - \frac{1}{2}\frac{f^{(2)}(x)}{f^{(1)}(x)}\frac{f(x)}{f^{(1)}(x)}}\right]$, *which is Halley's method of order 3 [18];*

$$S_4(x) = N_{F_4}(x) = x - \frac{f(x)}{f^{(1)}(x)}\left[\frac{1 - \frac{1}{2}\frac{f^{(2)}(x)}{f^{(1)}(x)}\frac{f(x)}{f^{(1)}(x)}}{1 - \frac{f^{(2)}(x)}{f^{(1)}(x)}\frac{f(x)}{f^{(1)}(x)} + \frac{1}{3!}\frac{f^{(3)}(x)}{f^{(1)}(x)}\left[\frac{f(x)}{f^{(1)}(x)}\right]^2}\right],$$

and

$$\begin{aligned} S_5(x) &= N_{F_5}(x) = x - \frac{f(x)}{f^{(1)}(x)} \times \\ &\left[\frac{1 - \frac{f^{(2)}(x)}{f^{(1)}(x)}\frac{f(x)}{f^{(1)}(x)} + \frac{1}{3!}\frac{f^{(3)}(x)}{f^{(1)}(x)}\left[\frac{f(x)}{f^{(1)}(x)}\right]^2}{1 - \frac{3}{2}\frac{f^{(2)}(x)}{f^{(1)}(x)}\frac{f(x)}{f^{(1)}(x)} + \left[\frac{1}{3}\frac{f^{(3)}(x)}{f^{(1)}(x)} + \frac{1}{4}\left[\frac{f^{(2)}(x)}{f^{(1)}(x)}\right]^2\right]\left[\frac{f(x)}{f^{(1)}(x)}\right]^2 - \frac{1}{4!}\frac{f^{(4)}(x)}{f^{(1)}(x)}\left[\frac{f(x)}{f^{(1)}(x)}\right]^3}\right]. \end{aligned}$$

Iterative formulas $S_4(x)$ and $S_5(x)$ were derived by I. Kiss [38]

1.7 Combined iteration functions

Another way to obtain higher order fixed point methods to find a root α of $f(x)$ is to combine two fixed point methods of the same order $p-1$ in such a way that the new method $\widetilde{\Phi}_p(x)$ has an asymptotic constant $K_{p-1}(\alpha;\widetilde{\Phi}_p)=0$. Then the resulting method will be of order p, or at least of order p.

1.7.1 The Chebyshev-Halley family of order 3 [5]

Let us apply Newton's method to $f(x)/\left[f^{(1)}(x)\right]^{\beta}$ where β is any real number (we suppose $f^{(1)}(\alpha)>0$), then we get

$$N_{f/\left[f^{(1)}\right]^{\beta}}(x)=x-\frac{f(x)/\left[f^{(1)}(x)\right]^{\beta}}{\left[f(x)/\left[f^{(1)}(x)\right]^{\beta}\right]^{(1)}}=x-\frac{f(x)}{f^{(1)}(x)}\left[\frac{1}{1-\beta L_f(x)}\right],$$

where $L_f(x)=\frac{f(x)f^{(2)}(x)}{\left[f^{(1)}(x)\right]^2}$. It can be shown that

$$K_2\left(\alpha;N_{f/\left[f^{(1)}\right]^{\beta}}\right)=\frac{N^{(2)}_{f/\left[f^{(1)}\right]^{\beta}}(\alpha)}{2!}=\frac{1-2\beta}{2!}\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}.$$

Now let us combine two IFs of order 2, one with $\beta\neq 0$ and one with $\beta=0$ (which corresponds to Newton's IF), we obtain a new IF of order 3

$$\begin{aligned}
G_\beta(x) &= \frac{1}{2\beta}\left[N_{f/\left[f^{(1)}\right]^{\beta}}(x)-(1-2\beta)N_f(x)\right]\\
&= x-\frac{f(x)}{f^{(1)}(x)}\left[\frac{1-(\beta-1/2)L_f(x)}{1-\beta L_f(x)}\right].
\end{aligned}$$

It is exactly the Chebyshev-Halley family of IFs of order 3. For $\beta=1$ we get Super-Halley IF, for $\beta=1/2$ we get Halley's IF, and for $\beta=0$, which is a limit case, we get

Chebyshev's IF. Its asymptotic constant is

$$K_3(\alpha; G_\beta) = \frac{G_\beta^{(3)}(\alpha)}{3!} = \frac{1}{3!}\left[-\frac{f^{(3)}(\alpha)}{f^{(1)}(\alpha)} + 3(1-\beta)\left[\frac{f^{(2)}(\alpha)}{f^{(1)}(\alpha)}\right]^2\right]. \tag{1.7.1}$$

1.7.2 Combination of a fixed point method and its corresponding Newton's method

Assume $\Phi_{p-1}(x)$ is an IF of order $p-1$ such that $K_{p-1}(\alpha; \Phi_{p-1}) = \Phi_{p-1}^{(p-1)}(\alpha)/(p-1)!$, and let us set

$$\Psi_{p-1}(x) = \Phi_{p-1}(x) - x = (\Phi_{p-1} - I)(x)$$

where $I(x) = x$. We have $\Psi_{p-1}(\alpha) = 0$, $\Psi_{p-1}^{(1)}(\alpha) = \Phi_{p-1}^{(1)}(\alpha) - 1 = -1$, $\Psi_{p-1}^{(l)}(\alpha) = \Phi_{p-1}^{(l)}(\alpha) = 0$ for $l = 2, \ldots, p-2$, and $\Psi_{p-1}^{(p-1)}(\alpha) = \Phi_{p-1}^{(p-1)}(\alpha)$. Using the corresponding Newton's method, $N_{\Psi_{p-1}}(x) = N_{\Phi_{p-1}-I}(x)$, it follows that

$$K_{p-1}(\alpha; N_{\Psi_{p-1}}) = \frac{(p-2)\Psi_{p-1}^{(p-1)}(\alpha)}{(p-1)!\Psi_{p-1}^{(1)}(\alpha)} = -(p-2)\frac{\Phi_{p-1}^{(p-1)}(\alpha)}{(p-1)!} = -(p-2)K_{p-1}(\alpha; \Phi_{p-1}).$$

Hence the linear combination of iteration functions

$$\widetilde{\Phi}_p(x) = \frac{1}{p-1}\left[(p-2)\Phi_{p-1}(x) + N_{\Psi_{p-1}}(x)\right]$$

is an iteration function of order p, or at least of order p.

1.7.3 Families of Super-Halley methods

Combining the last two subsections, considering $N_f(x)$ and applying Newton's method to $N_f(x) - x = -f(x)/f^{(1)}(x)$, we obtain

$$\frac{1}{2}\left[N_f(x) + N_{N_f - I}(x)\right] = G_1(x),$$

which is Super-Halley IF of order 3, noted $SH_3(x)$ (see [56, 24, 1]). We could then consider different ways of defining generalized Super-Halley IFs of any order p. For example :

(a) since $N_f(x) = E_2(x)$, by using Schröder's process of the first kind $E_{p-1}(x)$ of order $p-1$ and considering the combination for $p > 2$

$$SH_p(x) = \frac{1}{p-1}\left[(p-2)E_{p-1}(x) + N_{E_{p-1}-I}(x)\right];$$

(b) since $N_f(x) = S_2(x)$, by using Schröder's process of the second kind $S_{p-1}(x)$ of order $p-1$ and considering for $p > 2$ the combination

$$SH_p(x) = \frac{1}{p-1}\left[(p-2)S_{p-1}(x) + N_{S_{p-1}-I}(x)\right];$$

(c) or finally, by using induction

$$\begin{cases} SH_2(x) &= N_f(x), \\ SH_p(x) &= \frac{1}{p-1}\left[(p-2)SH_{p-1}(x) + N_{SH_{p-1}-I}(x)\right] \quad \text{for} \quad p > 2. \end{cases}$$

1.8 Concluding remarks

In this paper we presented fixed point and Newton's methods to compute a simple root of a non-linear function. We pointed out that the usual sufficient conditions for convergence are also necessary. Based on the conditions for high order convergence, we obtained systematic methods to increase the order of convergence of fixed point and Newton's methods. Among the resulting methods are the two processes of Schröder, for which a link has been explicitly shown. Finally, a combination of fixed point method and its corresponding natural Newton's method has been used to increase the order of convergence.

Some extensions of this work are possible. For example we have recently extended Section 2.4 to cover the modified Newton's method for finding a multiple root of a non-linear function [6]. Also, since high order methods require high order derivatives of the function, it could be interesting to try automatic differentiation tools to implement these methods. Finally, generalization of these high order processes to system of non-linear equations would be of great theoretical and practical interest.

1.9 Proof of Theorem 2.3.1

(i) [37, 41] By continuity, there is an interval $I_\rho(\alpha) = (\alpha - \rho, \alpha + \rho)$ such that $\left|\Phi^{(1)}(x)\right| \leq \frac{1+\left|\Phi^{(1)}(\alpha)\right|}{2} = L < 1$. Then if $x_k \in I_\rho(\alpha)$, we have

$$|x_{k+1} - \alpha| \leq L\,|x_k - \alpha| \leq |x_k - \alpha| \leq \rho$$

and $x_{k+1} \in I_\rho(\alpha)$. Moreover

$$|x_k - \alpha| \leq L^k\,|x_0 - \alpha|\,,$$

and the sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ converges to α because $0 \leq L < 1$.
(ii) If $\left|\Phi^{(1)}(\alpha)\right| > 1$, there exists an interval $I_\rho(\alpha) = (\alpha - \rho, \alpha + \rho)$, with $\rho > 0$, such that $\left|\Phi^{(1)}(x)\right| \geq \frac{1+\left|\Phi^{(1)}(\alpha)\right|}{2} = L > 1$. Let us suppose that the sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ is such that $x_k \neq \alpha$ for all k. If x_k and $x_{k+1} \in I_\rho(\alpha)$, then we have

$$|x_{k+1} - \alpha| = |\Phi(x_k) - \Phi(\alpha)| = \left|\Phi^{(1)}(\eta_k)(x_k - \alpha)\right| \geq L\,|x_k - \alpha|\,.$$

Let $0 < \epsilon < \rho$, and suppose x_k, x_{k+1}, $\ldots$, x_{k+l} are in $I_\epsilon(\alpha) \subset I_\rho(\alpha)$. Because

$$|x_{k+l} - \alpha| \geq L^l\,|x_k - \alpha|\,,$$

eventually $L^{l+1}\,|x_k - \alpha| \geq \epsilon$, and $x_{k+l+1} \notin I_\epsilon(\alpha)$. Then the infinite sequence cannot converge to α.

(iii) [37, 41] For any sequence $\{x_{k+1} = \Phi(x_k)\}_{k=0}^{+\infty}$ which converges to α we have

$$\lim_{k\to+\infty} \frac{x_{k+1}-\alpha}{x_k-\alpha} = \lim_{k\to+\infty} \Phi^{(1)}(\eta_k) = \Phi^{(1)}(\alpha).$$

since η_k converges to α.

1.10 Proof of Theorem 2.3.2

(i) [37, 41] The (local) convergence is given by part (i) of Theorem 2.3.1. Moreover we have

$$x_{k+1} - \alpha = \Phi(x_k) - \Phi(\alpha) = \frac{\Phi^{(p)}(\eta_k)}{p!}(x_k-\alpha)^p,$$

and hence

$$\lim_{k\to+\infty} \frac{x_{k+1}-\alpha}{(x_k-\alpha)^p} = \lim_{k\to+\infty} \frac{\Phi^{(p)}(\eta_k)}{p!} = \frac{\Phi^{(p)}(\alpha)}{p!} = K_p(\alpha;\Phi).$$

(ii) [51] If the IF is of order $p \geq 2$, assume that $\Phi^{(j)}(\alpha) = 0$ for $j = 1, 2, \ldots, l-1$ with $l < p$. We have

$$x_{k+1} - \alpha = \Phi(x_k) - \Phi(\alpha) = \frac{\Phi^{(l)}(\eta_k)}{l!}(x_k-\alpha)^l.$$

Then

$$\frac{\Phi^{(l)}(\eta_k)}{l!} = \frac{x_{k+1}-\alpha}{(x_k-\alpha)^l} = \frac{x_{k+1}-\alpha}{(x_k-\alpha)^p}(x_k-\alpha)^{p-l},$$

and so

$$\frac{\Phi^{(l)}(\alpha)}{l!} = \lim_{k\to+\infty} \frac{\Phi^{(l)}(\eta_k)}{l!} = K_p(\alpha;\Phi) \lim_{k\to+\infty} (x_k-\alpha)^{p-l} = \begin{cases} 0 & \text{if} \quad l < p, \\ K_p(\alpha;\Phi) & \text{if} \quad l = p. \end{cases}$$

1.11 Proof of Theorem 5.4.1

(i) [19] If $\Psi^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $\Psi^{(p)}(\alpha) \neq 0$ we have

$$x_{k+1} - \alpha = (x_k - \alpha) - \frac{\Psi(x_k)}{\Psi^{(1)}(x_k)} = \frac{(x_k-\alpha)^p}{\Psi^{(1)}(x_k)} \left[\frac{\Psi^{(p)}(\eta_{1,k})}{(p-1)!} - \frac{\Psi^{(p)}(\eta_{2,k})}{p!} \right],$$

it follows that

$$\lim_{k\to+\infty}\frac{x_{k+1}-\alpha}{(x_k-\alpha)^p}=\lim_{k\to+\infty}\frac{\frac{\Psi^{(p)}(\eta_{1,k})}{(p-1)!}-\frac{\Psi^{(p)}(\eta_{2,k})}{p!}}{\Psi^{(1)}(x_k)}=\frac{p-1}{p!}\frac{\Psi^{(p)}(\alpha)}{\Psi^{(1)}(\alpha)}.$$

(ii) Conversely, if $N_\Psi(x)$ is of order p we have $N_\Psi^{(j)}(\alpha)=0$ for $f=1,\ldots,p-1$ and $N_\Psi^{(p)}(\alpha)\neq 0$. Hence α is a root of multiplicity $p-1$ of $N_\Psi^{(1)}(x)$ and we can write $N_\Psi^{(1)}(x)=v_{N_\Psi}(x)(x-\alpha)^{p-1}$ where $v_{N_\Psi}(x)$ is a regular function such that $v_{N_\Psi}(\alpha)\neq 0$. But

$$N_\Psi^{(1)}(x)=\frac{\Psi(x)\Psi^{(2)}(x)}{\left[\Psi^{(1)}(x)\right]^2}=v_{N_\Psi}(x)(x-\alpha)^{p-1},$$

and, since we have $\Psi(x)=v_\Psi(x)(x-\alpha)$, where $v_\Psi(x)$ is a regular function such that $v_\Psi(\alpha)\neq 0$, we obtain

$$\Psi^{(2)}(x)=\frac{v_{N_\Psi}(x)}{v_\Psi(x)}\left[\Psi^{(1)}(x)\right]^2(x-\alpha)^{p-2}.$$

It follows that α is a root of multiplicity $p-2$ of $\Psi^{(2)}(x)$, and so $\Psi^{(j)}(\alpha)=0$ for $j=2,\ldots,p-1$, and $\Psi^{(p)}(\alpha)\neq 0$.

CHAPITRE 2

Fixed point and Newton's method in the complex field.

Abstract

In this paper we revisit the necessary and sufficient conditions for linear and high order convergence of fixed point and Newton's methods in the complex field. Schröder's process of the first kind and second kind and revisited and extended. Examples and numerical experiments are included.

2.1 Introduction

In this paper we revisit fixed point and Newton's methods to find a simple solution of a non-linear equation in the complex plane. This paper is an adapted version of [11] for complex valued functions. We present only proofs of theorems we have to modify compared to the real case. We present sufficient and necessary conditions for the convergence of fixed point and Newton's methods. Based on these conditions we show how to obtain direct processes to recursively increase the order of convergence. For the fixed point method, we present a generalization of the Schröder's method of the first kind. Two methods are also presented to increase the order of convergence of the Newton's method. One of them coincide with the the Schröder's process of the second kind which has several forms in the literature. The link between the two Schröder's processes can be found in [7]. As for the real case, we can combine methods to obtain, for example, the Super-Halley process of order 3 and other possible higher order generalizations of this process. We refer to [11] for details about this subject.

The plan of the paper is as follows. In Section 2.2 we recall Taylor's expansions for analytic functions and the error term for truncated expansions. In Section 2.3 we consider the fixed point method and its necessary and sufficient conditions for convergence. These results lead to a generalization of the Schröder's process of the first kind. Section 2.4 is devoted to Newton's method. Based on the necessary and sufficient conditions, we propose two ways to increase the order of convergence of the Newton's method. Examples and numerical experiments are included in Section 2.5.

2.2 Analytic function

Since we are working with complex numbers, we will be dealing with analytic functions. Suppose $g(z)$ is an analytic function and α is in its domain, we can write

$$g^{(k)}(z) = \sum_{j=0}^{\infty} \frac{g^{(k+j)}(\alpha)}{j!} (z-\alpha)^j .$$

for any $k = 0, 1, \ldots$. Then, for $q = 1, 2, \ldots$ we have

$$g^{(k)}(z) = \sum_{j=0}^{q-1} \frac{g^{(k+j)}(\alpha)}{j!} (z-\alpha)^j + w_{g^{(k)},q}(z)\,(z-\alpha)^q .$$

where $w_{g^{(k)},q}(z)$ is the analytic function

$$w_{g^{(k)},q}(z) = \sum_{j=0}^{\infty} \frac{g^{(k+q+j)}(\alpha)}{(q+j)!} (z-\alpha)^j .$$

Moreover, the series for $g^{(k)}(z)$ and $w_{g^{(k)},q}(z)$ have the same radius of convergence for any k, and

$$w^{(j)}_{g^{(k)},q}(\alpha) = \frac{j!}{(q+j)!} g^{(k+q+j)}(\alpha)$$

for $j = 0, 1, 2, \ldots$.

2.3 Fixed point method

A fixed point method use an iteration function (IF) which is an analytic function mapping its domain of definition into itself. Using an IF $\Phi(z)$ and an initial value z_0, we are interested by the convergence of the sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$. It is well known that if the sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ converges, it converges to a fixed point of $\Phi(z)$.

Let $\Phi(z)$ be an IF, p be a positive integer, and $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ be such that the following limit exists

$$\lim_{k\to+\infty} \frac{z_{k+1}-\alpha}{(z_k-\alpha)^p} = K_p(\alpha;\Phi).$$

Let us observe that for $p_1 < p < p_2$ we have

$$\lim_{k\to+\infty} \frac{z_{k+1}-\alpha}{(z_k-\alpha)^{p_1}} = 0 \quad \text{and} \quad \lim_{k\to+\infty} \frac{z_{k+1}-\alpha}{(z_k-\alpha)^{p_2}} = \infty.$$

We say that the convergence of the sequence to α is of (integer) order p if and only if $K_p(\alpha;\Phi) \neq 0$, and $K_p(\alpha;\Phi)$ is called the asymptotic constant. We also say that $\Phi(z)$ is of p. If the limit $K_p(\alpha;\Phi)$ exists but is zero, we can say that $\Phi(z)$ is of order at least p.

From a numerical point of view, since α is not known, it is useful to define the ratio

$$\widetilde{K}_p(\alpha, k) = \frac{z_{k+1}-z_{k+2}}{(z_k - z_{k+1})^p}. \tag{2.3.1}$$

Following [4], it can be shown that

$$\lim_{k\to+\infty} \widetilde{K}_p(\alpha, k) = K_p(\alpha;\Phi),$$

and

$$\lim_{k\to+\infty} \frac{\ln\left|\widetilde{K}_1(\alpha, k+1)\right|}{\ln\left|\widetilde{K}_1(\alpha, k)\right|} = p.$$

We say that α is a root of $f(z)$ of multiplicity q if and only if $f^{(j)}(\alpha) = 0$ for $j = 0, \ldots, q-1$, and $f^{(q)}(\alpha) \neq 0$. Moreover, α is a root of $f(z)$ of multiplicity q if and only if there exists an analytic function $w_{f,q}(z)$ such that $w_{f,q}(\alpha) = \frac{f^{(q)}(\alpha)}{q!} \neq 0$ and $f(z) = w_{f,q}(z)(z-\alpha)^q$.

We will use the Big O notation $g(z) = O(f(z))$, respectively the small o notation $g(z) = o(f(z))$, around $z = \alpha$ when $c \neq 0$, respectively $c = 0$, when

$$\lim_{z\to\alpha} \frac{g(z)}{f(z)} = c. \tag{2.3.2}$$

For α a root of multiplicity q of $f(z)$, it is equivalent to write $g(z) = O(f(z))$ or $g(z) = O((z-\alpha)^q)$. Observe also that if α is a simple root of $f(z)$, then α is a root of multiplicity q of $f^q(z)$. Hence $g(z) = O(f^q(z))$ is equivalent to $g(z) = O((z-\alpha)^q)$.

The first result concerns the necessary and sufficient conditions for achieving linear convergence.

Theorem 2.3.1. *Let $\Phi(z)$ be an IF, and let $\Phi^{(1)}(z)$ stand for its first derivative. Observe that although the first derivative is usually denoted by $\Phi'(z)$, we will write $\Phi^{(1)}(z)$ to maintain uniformity throughout the text.*
(i) If $|\Phi^{(1)}(\alpha)| < 1$, then there exists a neighbourhood of α such that for any z_0 in that neighbourhood the sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ converges to α.
(ii) If there exists a neighbourhood of α such that for any z_0 in that neighbourhood the sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ converges to α, and $z_k \neq \alpha$ for all k, then $\left|\Phi^{(1)}(\alpha)\right| \leq 1$.
(iii) For any sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ which converges to α, the limit $K_1(\alpha;\Phi)$ exists and $K_1(\alpha;\Phi) = \Phi^{(1)}(\alpha)$.

Proof.
(i) By continuity, there is a disk $D_\rho(\alpha) = \{\alpha \in \mathbb{C} \mid |z-\alpha| < \rho\}$ such that $|w_{\Phi,1}(z)| \leq \frac{1+\left|\Phi^{(1)}(\alpha)\right|}{2} = L < 1$. Then if $z_k \in D_\rho(\alpha)$, we have

$$|z_{k+1} - \alpha| = |\Phi(z_k) - \Phi(\alpha)| = |w_{\Phi,1}(z_k)(z_k - \alpha)| \leq L\,|z_k - \alpha| \leq |z_k - \alpha| < \rho,$$

and $z_{k+1} \in D_\rho(\alpha)$. Moreover

$$|z_k - \alpha| \leq L^k\,|z_0 - \alpha|\,,$$

and the sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ converges to α because $0 \leq L < 1$.
(ii) If $\left|\Phi^{(1)}(\alpha)\right| > 1$, there exists a disk $D_\rho(\alpha)$, with $\rho > 0$, such that $|w_{\Phi,1}(z)| \geq \frac{1+\left|\Phi^{(1)}(\alpha)\right|}{2} = L > 1$. Let us suppose that the sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ is such that $z_k \neq \alpha$ for all k. If z_k and $z_{k+1} \in D_\rho(\alpha)$, then we have

$$|z_{k+1} - \alpha| = |\Phi(z_k) - \Phi(\alpha)| = |w_{\Phi,1}(z_k)(z_k - \alpha)| \geq L\,|z_k - \alpha|\,.$$

Let $0 < \epsilon < \rho$, and suppose z_k, z_{k+1}, …, z_{k+l} are in $D_\epsilon(\alpha) \subset D_\rho(\alpha)$. Because

$$|z_{k+l} - \alpha| \geq L^l\,|z_k - \alpha|$$

eventually $L^{l+1}|z_k - \alpha| \geq \epsilon$ and $z_{k+l} \notin D_\epsilon(\alpha)$. Then the infinite sequence cannot converge to α.

(iii) For any sequence $\{z_{k+1} = \Phi(z_k)\}_{k=0}^{+\infty}$ which converges to α we have

$$\lim_{k\to+\infty} \frac{z_{k+1} - \alpha}{z_k - \alpha} = \lim_{k\to+\infty} w_{\Phi,1}(z_k) = \Phi^{(1)}(\alpha). \quad \square$$

For higher order convergence we have the following result about necessary and sufficient conditions.

Theorem 2.3.2. *Let p be an integer ≥ 2 and let $\Phi(z)$ be an analytic function such that $\Phi(\alpha) = \alpha$. The IF $\Phi(z)$ is of order p if and only if $\Phi^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-1$, and $\Phi^{(p)}(\alpha) \neq 0$. Moreover, the asymptotic constant is given by*

$$K_p(\alpha; \Phi) = \lim_{k\to+\infty} \frac{z_{k+1} - \alpha}{(z_k - \alpha)^p} = \frac{\Phi^{(p)}(\alpha)}{p!}.$$

Proof.

(i) The (local) convergence is given by the part (i) of the Theorem 2.3.1. Moreover we have

$$z_{k+1} - \alpha = \Phi(z_k) - \Phi(\alpha) = w_{\Phi,p}(z_k)(z_k - \alpha)^p,$$

and hence

$$\lim_{k\to+\infty} \frac{z_{k+1} - \alpha}{(z_k - \alpha)^p} = \lim_{k\to+\infty} w_{\Phi,p}(z_k) = \frac{\Phi^{(p)}(\alpha)}{p!} = K_p(\alpha; \Phi).$$

(ii) If the IF is of order $p \geq 2$, assume that $\Phi^{(j)}(\alpha) = 0$ for $j = 1, 2, \ldots, l-1$ with $l < p$. We have

$$z_{k+1} - \alpha = \Phi(z_k) - \Phi(\alpha) = w_{\Phi,l}(z_k)(z_k - \alpha)^l,$$

where

$$w_{\Phi,l}(\alpha) = \lim_{k\to+\infty} w_{\Phi,l}(z_k) = \frac{\Phi^{(l)}(\alpha)}{l!}.$$

But

$$w_{\Phi,l}(z_k) = \frac{z_{k+1} - \alpha}{(z_k - \alpha)^l} = \frac{z_{k+1} - \alpha}{(z_k - \alpha)^p}(z_k - \alpha)^{p-l},$$

and hence

$$w_{\Phi,l}(\alpha) = \lim_{k\to+\infty} w_{\Phi,l}(z_k) = K_p(\alpha;\Phi) \lim_{k\to+\infty} (z_k-\alpha)^{p-l} = \begin{cases} 0 & \text{if} \quad l < p, \\ \\ K_p(\alpha;\Phi) & \text{if} \quad l = p. \end{cases}$$

So $\Phi^{(l)}(\alpha) = 0$. □

It follows that for an analytic IF and $p > 2$, the limit $K_p(\alpha;\Phi)$ exists if and only if $K_l(\alpha;\Phi) = 0$ for $l = 1, \ldots, p-1$.

As a consequence, for an analytic IF $\Phi(z)$ we can say that: (a) $\Phi(z)$ is of order p if and only if $\Phi(z) = \alpha + O((z-\alpha)^p)$, or equivalently, if $\Phi(\alpha) = \alpha$ and $\Phi^{(1)}(z) = O((z-\alpha)^{p-1})$, and (b) if α is a simple root of $f(z)$, then $\Phi(z)$ is of order p if and only if $\Phi(z) = \alpha + O(f^p(z))$, or equivalently, if $\Phi(\alpha) = \alpha$ and $\Phi^{(1)}(z) = O(f^{p-1}(z))$.

Schröder's process of the first kind is a systematic and recursive way to construct an IF of arbitrary order p to find a simple zero α of $f(z)$. The IF has to fulfil at least the sufficient condition of Theorem 2.3.2. Let us present a generalization of this process.

Theorem 2.3.3. *[11] Let α be a simple root of $f(z)$, and let $c_0(z)$ be an analytic function such that $c_0(\alpha) = \alpha$. Let $\Phi_p(z)$ be the IF defined by the finite series*

$$\Phi_p(z) = \sum_{l=0}^{p-1} c_l(z) f^l(z) \tag{2.3.3}$$

where the $c_l(z)$ are such that

$$c_l(z) = -\frac{1}{l}\left(\frac{1}{f^{(1)}(z)}\frac{d}{dz}\right) c_{l-1}(z) \tag{2.3.4}$$

for $l = 1, 2, \ldots$ Then $\Phi_p(z)$ is of order p, and its asymptotic constant is

$$K_p(\alpha,\Phi_p) = \frac{\Phi^{(p)}(\alpha)}{p!} = \frac{1}{p} c_{p-1}^{(1)}(\alpha)\left[f^{(1)}(\alpha)\right]^{p-1} = -c_p(\alpha)\left[f^{(1)}(\alpha)\right]^p. \tag{2.3.5}$$

For $c_0(z) = z$ in (2.3.3), we recover the Schröder's process of the first kind of order p [48, 51, 27], which is also associated to Chebyshev and Euler [2, 49, 44]. The first term $c_0(z)$ could be seen as a preconditioning to decrease the asymptotic constant of the method, but its choice is not obvious.

2.4 Newton's iteration function

Considering $c_0(z) = z$ and $p = 2$ in (2.3.3), we obtain

$$\Phi_2(z) = z - \frac{f(z)}{f^{(1)}(z)}$$

which is the Newton's IF of order 2 to solve $f(z) = 0$. The sufficiency and the necessity of the condition for high-order convergence of the Newton's method are presented in the next result.

Theorem 2.4.1. *Let $p \geq 2$ and let $\Psi(z)$ be an analytic function such that $\Psi(\alpha) = 0$ and $\Psi^{(1)}(\alpha) \neq 0$. The Newton iteration $N_\Psi(z) = z - \frac{\Psi(z)}{\Psi^{(1)}(z)}$ is of order p if and only if $\Psi^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $\Psi^{(p)}(\alpha) \neq 0$. Moreover, the asymptotic constant is*

$$K_p(\alpha; N_\Psi) = \frac{p-1}{p!}\frac{\Psi^{(p)}(\alpha)}{\Psi^{(1)}(\alpha)}.$$

Proof.
(i) If $\Psi^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $\Psi^{(p)}(\alpha) \neq 0$ we have

$$\begin{aligned} z_{k+1} - \alpha &= (z_k - \alpha) - \frac{\Psi(z_k)}{\Psi^{(1)}(z_k)} \\ &= \frac{(z_k - \alpha)\Psi^{(1)}(z_k) - \Psi(z_k)}{\Psi^{(1)}(z_k)}. \end{aligned}$$

But

$$\Psi^{(1)}(z_k) = \Psi^{(1)}(\alpha) + w_{\Psi^{(1)},p-1}(z_k)(z_k - \alpha)^{p-1},$$

and

$$\Psi(z_k) = \Psi^{(1)}(\alpha)(z_k - \alpha) + w_{\Psi,p}(z_k)(z_k - \alpha)^p.$$

It follows that

$$z_{k+1} - \alpha = \frac{w_{\Psi^{(1)},p-1}(z_k) - w_{\Psi,p}(z_k)}{\Psi^{(1)}(z_k)} (z_k - \alpha)^p ,$$

so

$$\begin{aligned}
\lim_{k\to+\infty} \frac{z_{k+1} - \alpha}{(z_k - \alpha)^p} &= \lim_{k\to+\infty} \frac{w_{\Psi^{(1)},p-1}(z_k) - w_{\Psi,p}(z_k)}{\Psi^{(1)}(z_k)} \\
&= \left\{ \frac{\Psi^{(p)}(\alpha)}{(p-1)!} - \frac{\Psi^{(p)}(\alpha)}{p!} \right\} / \Psi^{(1)}(\alpha) \\
&= \frac{(p-1)}{p!} \frac{\Psi^{(p)}(\alpha)}{\Psi^{(1)}(\alpha)}.
\end{aligned}$$

(ii) Conversely, if $N_\Psi(z)$ is of order p we have $N_\Psi^{(j)}(\alpha) = 0$ for $j = 1, \dots, p-1$, and $N_\Psi^{(p)}(\alpha) \neq 0$. Hence α is a root of multiplicity $p-1$ of $N_\Psi^{(1)}(z)$ and we can write

$$N_\Psi^{(1)}(z) = w_{N_\Psi^{(1)},p-1}(z)(z - \alpha)^{p-1}.$$

We also have

$$\Psi(z) = w_{\Psi,1}(z)(z - \alpha).$$

But

$$N_\Psi^{(1)}(z) = \frac{\Psi(z)\Psi^{(2)}(z)}{\left[\Psi^{(1)}(z)\right]^2},$$

so we obtain

$$\begin{aligned}
\Psi^{(2)}(z) &= N_\Psi^{(1)}(z) \frac{\left[\Psi^{(1)}(z)\right]^2}{\Psi(z)} \\
&= \frac{w_{N_\Psi^{(1)},p-1}(z)}{w_{\Psi,1}(z)} \left[\Psi^{(1)}(z)\right]^2 (z - \alpha)^{p-2}
\end{aligned}$$

where

$$\lim_{z\to\alpha} \frac{w_{N_\Psi^{(1)},p-1}(z)}{w_{\Psi,1}(z)} \left[\Psi^{(1)}(z)\right]^2 = \frac{N_\Psi^{(p)}(\alpha)\Psi^{(1)}(\alpha)}{(p-1)!} \neq 0.$$

It follows that α is a root of multiplicity $p-2$ of $\Psi^{(2)}(z)$. Hence $\Psi^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $\Psi^{(p)}(\alpha) \neq 0$. $\square$

We can look for a recursive method to construct a function $\Psi_p(z)$ which will satisfy the conditions of Theorem 5.4.1. A consequence will be that $N_{\Psi_p}(z)$ will be of order p, and $N_{\Psi_p}(z) = \alpha + O(f^p(z))$. A first method has been presented in [19, 17]. The technique can also be based on Taylor's expansion as indicated in [6].

Theorem 2.4.2. *[19] Let $f(z)$ be analytic such that $f(\alpha) = 0$ and $f^{(1)}(\alpha) \neq 0$. If $F_p(z)$ is defined by*

$$\begin{cases} F_2(z) &= f(z), \\ F_p(z) &= \frac{F_{p-1}(z)}{\left[F^{(1)}_{p-1}(z)\right]^{\frac{1}{p-1}}} \quad \text{for} \quad p \geq 3, \end{cases} \tag{2.4.1}$$

then $F_p(\alpha) = 0$, $F_p^{(1)}(\alpha) \neq 0$, $F_p^{(l)}(\alpha) = 0$ for $l = 2, \ldots, p-1$. It follows that $N_{F_p}(z)$ is of order at least p.

Let us observe that in this theorem it seems that the method depends on a choice of a branch for the $(p-1)$-th root function. In fact the newton iterative function does not depends of this choice because we have

$$N_{F_p}(z) = z - \frac{F_{p-1}(z)/F^{(1)}_{p-1}(z)}{1 - \frac{1}{p-1}\frac{F_{p-1}(z)F^{(2)}_{p-1}(z)}{[F^{(1)}_{p-1}(z)]^2}} = z - \frac{F_{p-1}(z)/F^{(1)}_{p-1}(z)}{1 - \frac{1}{p-1}\left[1 - \left(\frac{F_{p-1}(z)}{F^{(1)}_{p-1}(z)}\right)^{(1)}\right]}.$$

In fact the next theorem show that a branch for the $(p-1)$-th root function is not necessary.

Theorem 2.4.3. *[17] Let $F_p(z)$ given by (2.4.1), we can also write*

$$N_{F_p}(z) = z - \frac{f(z)}{f^{(1)}(z) - \frac{1}{p-1} f(z) \frac{Q_p^{(1)}(z)}{Q_p(z)}} = z - f(z)\frac{Q_p(z)}{Q_{p+1}(z)}, \tag{2.4.2}$$

where

$$\begin{cases} Q_2(z) &= 1, \\ Q_p(z) &= f^{(1)}(z)Q_{p-1}(z) - \frac{1}{p-2}f(z)Q_{p-1}^{(1)}(z) \quad \textit{for} \quad p \geq 3. \end{cases} \tag{2.4.3}$$

Unfortunately, there exist no general formulae for $N_{F_p}(z)$ and its asymptotic constant $K_p(\alpha; N_{F_p})$ exist. However, the asymptotic constant can be numerically estimated with (2.3.1).

A second method to construct a function $\Psi_p(z)$ which will satisfy the conditions of Theorem 5.4.1 is given in the next theorem.

Theorem 2.4.4. *[11] Let α be a simple root of $f(z)$. Let $\Psi_p(z)$ be defined by*

$$\Psi_p(z) = \sum_{l=0}^{p-1} d_l(z) f^l(z) \tag{2.4.4}$$

where $d_0(z)$ and $d_1(z)$ are two analytic functions such that

$$\begin{cases} d_0(\alpha) &= 0 \\ d_0^{(1)}(\alpha) + d_1(\alpha)f^{(1)}(\alpha) &\neq 0 \end{cases} \tag{2.4.5}$$

and

$$d_l(z) = -\frac{1}{l f^{(1)}(z)} \times \left[d_{l-1}^{(1)}(z) + \frac{1}{(l-1)f^{(1)}(z)} \left[d_{l-2}^{(1)}(z) + (l-1)d_{l-1}(z)f^{(1)}(z) \right]^{(1)} \right] \tag{2.4.6}$$

for $l = 2, 3, \ldots$. Then

$$N_{\Psi_p}(z) = z - \frac{\Psi_p(z)}{\Psi_p^{(1)}(z)}$$

is of order p, with

$$\Psi_p^{(p)}(\alpha) = -p!\, d_p(\alpha) \left[f^{(1)}(\alpha) \right]^p,$$

and

$$K_p(\alpha; N_{\Psi_p}) = -\frac{(p-1)d_p(\alpha) \left[f^{(1)}(\alpha) \right]^p}{d_0^{(1)}(\alpha) + d_1(\alpha)f^{(1)}(\alpha)}.$$

Let us observe that if we set $\Psi_p(z) = \Phi_p(z) - z$ with the $\Phi_p(z)$ given by (2.3.3), then $\Psi_p(z)$ verifies the assumptions of Theorem 2.4.4.

Remark 2.4.1. *For a given pair of $d_0(z)$ and $d_1(z)$ in Theorem 2.4.4, the linearity of the expression 2.4.6 with respect to $d_0(z)$ and $d_1(z)$ for computing the $d_l(z)$'s allows us to decompose the computation for $\Psi_p(z)$ in two computations, one for the pair $d_0(z)$ and $d_1(z) = 0$, and the other for the pair $d_0(z) = 0$ and $d_1(z)$, then add the two $\Psi_p(z)$'s hence obtained.*

2.5 Examples

Let us consider the problem of finding the 3rd roots of unity

$$\alpha_k = e^{2(k-1)\pi i/3} \quad \text{for} \quad k = 0, 1, 2,$$

for which we have $\alpha^3 = 1$. Hence we would like to solve

$$f(z) = 0,$$

for

$$f(z) = z^3 - 1.$$

As examples of the preceding results, we present methods of order 2 and 3 obtained from Theorems 5.3.2, 5.4.2, and 2.4.4. For each methods we consider we also present the basins of attraction of the roots.

The drawing process for the basins of attraction follows Varona [53]. Typically for the upcoming figures, in squares $[2.5, 2.5]^2$, we assign a color to each attraction basin of each root. That is, we color a point depending on whether within a fixed number of iteration (here 25) we lie with a certain precision (here 10^{-3}) of a given root. If after 25 iterations we do not lie within 10^{-3} of any given root we assign to the point a very dark shade of purple. The more purple, the more point have failed to achieve the required precision within the predetermined number of iteration.

2.5.1 Examples for Theorem 5.3.2

We start with iterative methods of order 2. From Theorem 5.3.2, we first want $c_0(\alpha) = \alpha$. We observe that the simplest such function is $c_0(z) = z$. Such a choice has the advantage that derivative of higher order then 2 of this function $c_0(z)$ will be 0, thus simplifying further computation. This is in fact the choice of function $c_0(z)$ which leads to Newton's method and Chebysev family of iterative methods. We observe however that it is generally possible to consider different choices of functions, although most will might be numerically convenient as we will illustrate here. We need $c_0(\alpha) = \alpha$, in such, we can also look at $c_0(z) = za(z)$ where $a(\alpha) = 1$. In the examples that follow we will look at such functions $a(z)$.

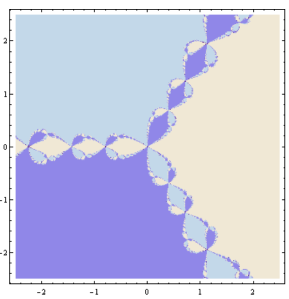
In Table 2.1, we have considered 3 functions of this kind. We have developed explicit expressions for $f(z) = z^3 - 1$. Figure2.1 presents different graphs for the basins of attraction for these methods. We observe that some of them have a lot of purple points.

Now let us consider method of order 3 with $c_0(z) = z^{3m+1}$ with $(m \in \mathbb{Z})$. In this case we obtain

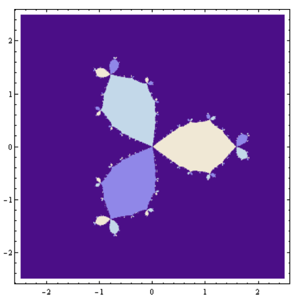
$$\Phi_3(z) = \frac{z^{3m-2}}{18}\left[(3m-2)(3m-5)z^3 - 2(3m+1)(3m-5) + (3m+1)(3m-2)z^{-3}\right],$$

and its asymptotic constant is

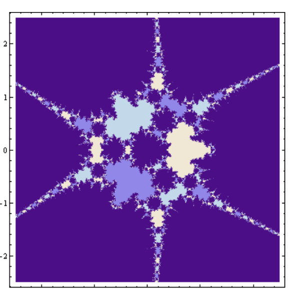
$$K_3(\alpha;\Phi_3) = \frac{(3m+1)(3m-2)(3m-5)}{6}\alpha.$$



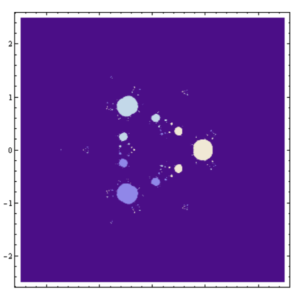
(a) For $m = 0$, $c_0(z) = z$, and its asymptotic constant is α^2.



(b) For $m = 1$, $c_0(z) = z^4$, and its asymptotic constant is $-2\alpha^2$



(c) For $c_0(z) = z(e^{(z^3-1)})$, the asymptotic constant is $-\frac{13}{2}\alpha^2$.



(d) For $c_0(z) = z\cos(z^3 - 1)$, the asymptotic constant is $\frac{11}{2}\alpha^2$.

Figure 2.1: Basins of attraction for methods of order 2 of Table 2.1.

Tableau 2.1: Method of order 2 based on Theorem 5.3.2.

$c_0(z) = za(z)$	$\Phi_2(z)$	$K_2(\alpha; \Phi_2)$
$z\, z^{3m} \quad (m \in \mathbb{Z})$	$\left[(3m+1) - (3m-2)\, z^3\right] \frac{z^{3m-2}}{3}$	$-\frac{(3m+1)(3m-2)}{2}\alpha^2$
$z \exp\left(z^3 - 1\right)$	$\frac{1+5z^3-3z^6}{3z_2} \exp\left(z^3 - 1\right)$	$-\frac{13}{2}\alpha^2$
$z \cos\left(z^3 - 1\right)$	$\frac{6z^3+1}{3^2} \cos\left(z^3 - 1\right) + z\left(z^3 - 1\right) \sin\left(z^3 - 1\right)$	$\frac{11}{2}\alpha^2$

Examples of basins of attraction are given in Figure 2.2 for $m = 0, 1, 2$. The smallest asymptotic constant is for $m = 1$.

2.5.2 Examples for Theorem 5.4.2

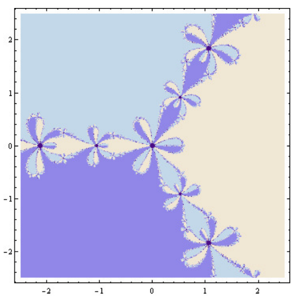
Gerlach's process described in Theorems 5.4.2 and 2.4.3 leads to Newton's method for $p = 2$ and Halley's method for $p = 3$. For our problem we have

$$N_{F_2}(z) = z - \frac{(z^3 - 1)}{3z^2},$$

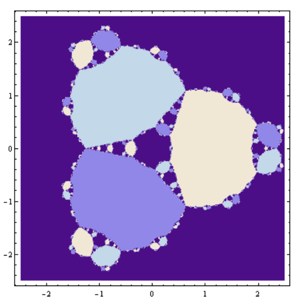
and

$$N_{F_3}(z) = z - \frac{\left(z^3 - 1\right)/3z^2}{1 - \frac{1}{2}\left[1 - \left(\frac{z^3-1}{3z^2}\right)^{(1)}\right]} = z\frac{z^3 + 2}{2z^3 + 1}.$$

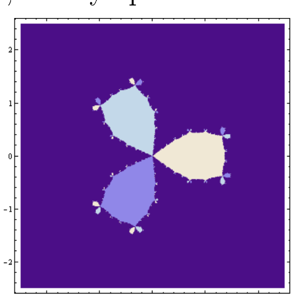
These methods are well known standard methods. For comparison, their basins of attraction are given in Figure 2.3.



(a) $m = 0$, its asymptotic constant is $\frac{5}{3}\alpha$ (Chebysev's method).

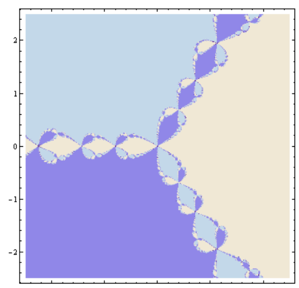


(b) $m = 1$, its asymptotic constant is $-\frac{4}{3}\alpha$.

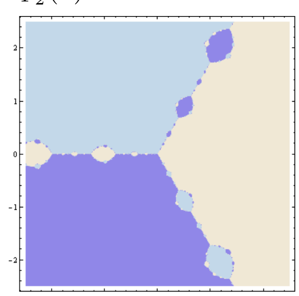


(c) $m = 2$, its asymptotic constant is $\frac{14}{3}\alpha$.

Figure 2.2: Methods of order 3 for computing the cubic root with $c_0(z) = z^{3m+1}$ for $M = 0, 1, 2$.



(a) $N_{F_2}(z)$ is Newton's method.



(b) $N_{F_3}(z)$ is Halley's method.

Figure 2.3: First two methods for computing the third root with Theorem 5.4.2.

2.5.3 Examples for Theorem 2.4.4

To illustrate Theorem 2.4.4, we set $d_0(z) = 0$ and $d_1(z) = z^k$ for $k \in \mathbb{Z}$, and let us consider methods of order 2 and 3 to solve $z^3 - 1 = 0$. Table 2.2 presents the quantities $\Psi_p(z)$, $N_{\Psi_p}(z)$, $d_p(z)$, and $K_p(\alpha; \Psi_p)$ for $p = 2, 3$ for this example.

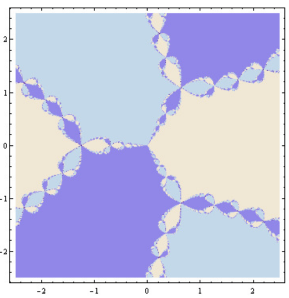
Tableau 2.2: Method of order 2 and 3 based on Theorem 5.3.2.

p	$\Psi_p(z)$	$N_{\Psi_p}(z)$	$d_p(z)$	$K_p(\alpha; \Psi_p)$
2	$z^k(z^3-1)$	$z\,\frac{(k+2)z^3-(k-1)}{(k+3)z^3-k}$	$-\frac{k+1}{3}\,z^{k-3}$	$(k+1)\,\alpha^2$
3	$\frac{z^k}{3}\left[(2-k)\,z^3+(2k-1)-(k+1)\,z^{-3}\right]$	$z\,\frac{\left[(k+2)(2-k)z^3+(k-1)(2k-1)-(k-4)(k+1)z^{-3}\right]}{\left[(k+3)(2-k)z^3+k(2k-1)-(k-3)(k+1)z^{-3}\right]}$	$\frac{3k^2-3k-8}{54z^2}\,z^{k-4}$	$-\frac{3k^2-3k-8}{3}\alpha$

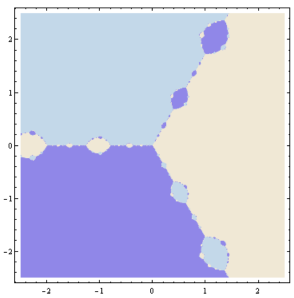
We observe that the asymptotic constant of the method of order 2 for $k = -1$ is zero, it means that this method as an order of convergence higher than 2, and in fact it corresponds to the Halley's method which is of order 3. We observe that methods of order 3 for the values of $k = -1$ and $k = 2$ both correspond to Halley's method for our specific problem. Examples of basins of attraction are given in Figure 2.4 for methods of order 2 and in Figure 2.5 for methods of order 3 using values of $k = -2, -1, 0, 1, 2, 3$.

2.6 Concluding remarks

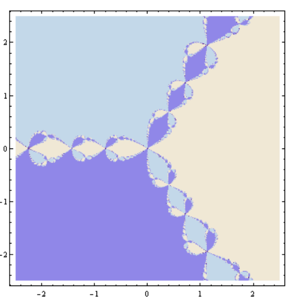
We have presented fixed point and Newton's methods to compute a simple root of a non-linear analytic function in the complex plane. Based on the necessary and sufficient conditions for convergence we revisited and extended both Schröder's methods. Like Kalantari [36] and Varona [53] we have illustrated those methods with their bassins.



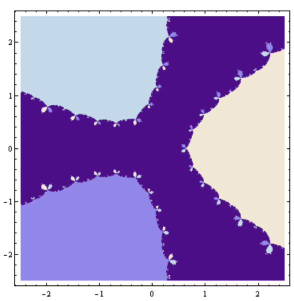
(a) $k = -2$ and $d_1(z) = z^{-2}$.



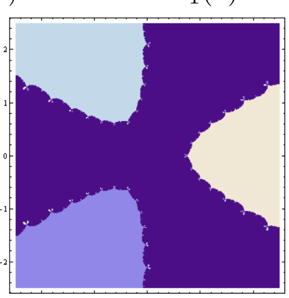
(b) $k = -1$ and $d_1(z) = z^{-1}$.



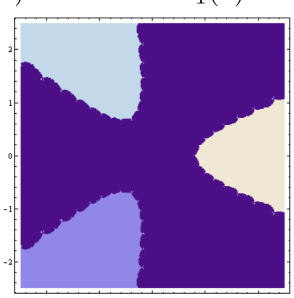
(c) $k = 0$ and $d_1(z) = 1$.



(d) $k = 1$ and $d_1(z) = z$.

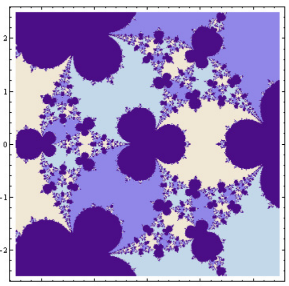


(e) $k = 2$ and $d_1(z) = z^2$.

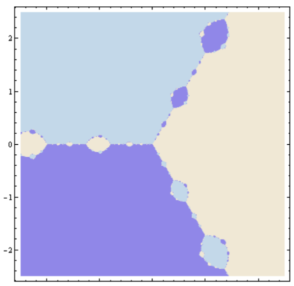


(f) $k = 3$ and $d_1(z) = z^3$.

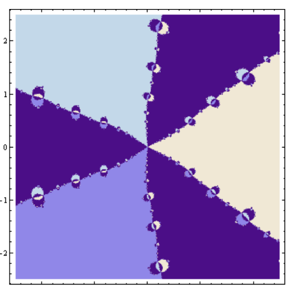
Figure 2.4: Methods of order 2 to illustrate Theorem 2.4.4.



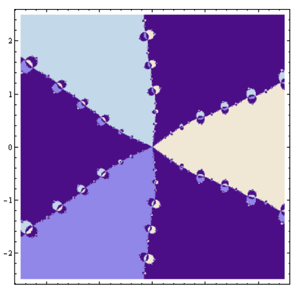
(a) $k = -2$ and $d_1(z) = z^{-2}$.



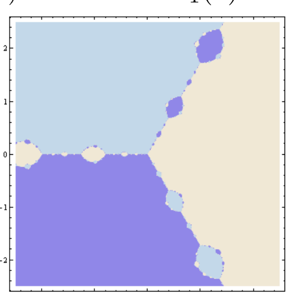
(b) $k = -1$ and $d_1(z) = z^{-1}$.



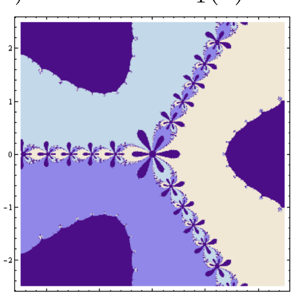
(c) $k = 0$ and $d_1(z) = 1$.



(d) $k = 1$ and $d_1(z) = z$.



(e) $k = 2$ and $d_1(z) = z^2$.



(f) $k = 3$ and $d_1(z) = z^3$.

Figure 2.5: Methods of order 3 to illustrate Theorem 2.4.4.

CHAPITRE 3

On the rediscovery of Halley's iterative method for computing the zero of an analytic function.

Abstract

We show that Halley's basic sequence, resulting from accelerating the order of convergence of Newton's method, is the most efficient way of doing so in terms of usage of certain derivatives. This fact could explain why this process of accelerating the convergence of Newton's method is so frequently rediscovered. Then we present an algorithmic way of recognizing Halley's family and we apply this algorithm to examples of rediscoveries.

3.1 Introduction

Newton's method of order 2 to solve a nonlinear equation appeared in 1669 [57]. Later, in 1694, Halley presented an improvement of order 3 of this method [20, 18]. In 1870 Schröder introduced an infinite sequence of methods based on rational approximations whose p-th member is of order p. The first two elements of this sequence where Newton's method ($p = 2$), and Halley's method ($p = 3$). This family is said to be the Schröder's method of 2nd kind, or also the Halley's basic sequence. This sequence, under different forms, has been rediscovered in 1946 [21], and 1953 [26]. In [51] Traub says that Halley's method has been very often rediscovered. That statement was made in 1964, since then, this sequence has been rediscovered several times: 1966 [54], 1969 [52], 1972 [31], 1975 [16], 1991 [28], 1994 [19], 1996 [17], 1997 [32]. In fact, at least once every decade Halley's basic sequence of iterative methods is rediscovered [44, 45]. This no longer seems like a simple coincidence.

Naturally we are faced with the following questions when we are looking for a new method: are we also on the verge of a rediscovery? How can we prevent oneself from a rediscovery ? In the following we first show that the Halley's basic sequence of iterative methods results from accelerating the order of convergence of Newton's method and we show that it is the most efficient way of doing so in terms of usage of certain derivatives. Then we present an algorithmic way of recognizing Halley's family. We apply this algorithm to examples of rediscoveries and present several forms under which the Halley's basic sequence can appear.

3.2 Preliminaries

Suppose $g(z)$ is an analytic function and α is in its domain, we can write

$$g^{(k)}(z) = \sum_{j=0}^{\infty} \frac{g^{(k+j)}(\alpha)}{j!} (z-\alpha)^j .$$

for any $k = 0, 1, \ldots$. Then, for $q = 1, 2, \ldots$ we have

$$g^{(k)}(z) = \sum_{j=0}^{q-1} \frac{g^{(k+j)}(\alpha)}{j!} (z-\alpha)^j + w_{g^{(k)},q}(z)\,(z-\alpha)^q .$$

where $w_{g^{(k)},q}(z)$ is the analytic function

$$w_{g^{(k)},q}(z) = \sum_{j=0}^{\infty} \frac{g^{(k+q+j)}(\alpha)}{(q+j)!} (z-\alpha)^j .$$

Moreover, the series for $g^{(k)}(z)$ and $w_{g^{(k)},q}(z)$ have the same radius of convergence for any k, and

$$w^{(j)}_{g^{(k)},q}(\alpha) = \frac{j!}{(q+j)!} g^{(k+q+j)}(\alpha)$$

for $j = 0, 1, 2, \ldots$.

We will use the Big $\mathcal{O}$ notation, $g(z) = \mathcal{O}(f(z))$, around $z = \alpha$ when there exists a constant $c \neq 0$ such that

$$\lim_{z \to \alpha} \frac{g(z)}{f(z)} = c.$$

If a root α is of multiplicity q for $f(z)$, it is equivalent to write $g(z) = \mathcal{O}(f(z))$ or $g(z) = \mathcal{O}((z-\alpha)^q)$.

Finally, the order of convergence to α of a sequence $\{z_k\}_{k=0}^{+\infty}$ is p if and only if there exists a non-zero constant C_p such that

$$\lim_{k \to +\infty} \frac{z_{k+1} - \alpha}{(z_k - \alpha)^p} = C_p.$$

3.3 On the Halley's accelerating process

3.3.1 High order Newton's method

An important result about high order Newton's method is given by the following theorem which presents necessary and sufficient conditions for obtaining a given order of convergence (see also [11] for this result for real valued functions).

Theorem 3.3.1. *Newton's method*

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

applied to an analytic function $f(z)$, with $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, is of order $p \geq 2$ if and only if $f''(z) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$.

Proof.
Sufficiency. If $f^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $f^{(p)}(\alpha) \neq 0$ we have

$$\begin{aligned} N_f(z) - N_f(\alpha) &= (z-\alpha) - \frac{f(z)}{f'(z)} \\ &= \frac{(z-\alpha)f'(z) - f(z)}{f'(z)}. \end{aligned}$$

But

$$f'(z) = f'(\alpha) + w_{f',p-1}(z)(z-\alpha)^{p-1},$$

and

$$f(z) = f'(\alpha)(z-\alpha) + w_{f,p}(z)(z-\alpha)^p.$$

It follows that

$$N_f(z) - N_f(\alpha) = \frac{w_{f',p-1}(z) - w_{f,p}(z)}{f'(z)}\,(z-\alpha)^p\,,$$

so

$$\begin{aligned}\lim_{z\to\alpha}\frac{z_k-\alpha}{(z-\alpha)^p} &= \lim_{z\to\alpha}\frac{w_{f',p-1}(z)-w_{f,p}(z)}{f'(z)}\\ &= \left[\frac{f^{(p)}(\alpha)}{(p-1)!}-\frac{f^{(p)}(\alpha)}{p!}\right]/f'(\alpha)\\ &= \frac{(p-1)}{p!}\frac{f^{(p)}(\alpha)}{f'(\alpha)}.\end{aligned}$$

Necessity. Conversely, suppose $N_f(z)$ is of order p. Let us assume that $N_f^{(j)}(\alpha)=0$ for $j=1,\ldots,l-1$ for $l<p$. We have

$$N_f(z)=N_f(\alpha)+w_{N_f,l}(z)\,(z-\alpha)^l\,.$$

But

$$\begin{aligned}\frac{N_f^{(l)}(\alpha)}{l!} &= \lim_{z\to\alpha} w_{N_f,l}(z)\\ &= \lim_{z\to\alpha}\frac{N_f(z)-N_f(\alpha)}{(z-\alpha)^l}\\ &= \lim_{z\to\alpha}\frac{N_f(z)-N_f(\alpha)}{(z-\alpha)^p}\,(z-\alpha)^{(p-l)}\\ &= 0,\end{aligned}$$

because $\lim_{z\to\alpha}\frac{N_f(z)-N_f(\alpha)}{(z-\alpha)^p}$ is a finite value. It follows that α is a root of multiplicity $p-1$ of $N_f'(z)$, and we can write

$$N_f'(z)=w_{N_f',p-1}(z)(z-\alpha)^{p-1}.$$

We also have

$$f(z)=w_{f,1}(z)(z-\alpha).$$

But

$$N_f'(z)=\frac{f(z)f''(z)}{\left[f'(z)\right]^2}$$

we obtain

$$\begin{aligned} f''(z) &= N'_f(z)\frac{[f'(z)]^2}{f(z)} \\ &= \frac{w_{N'_f,p-1}(z)}{w_{f,1}(z)}\left[f'(z)\right]^2(z-\alpha)^{p-2} \end{aligned}$$

where

$$\lim_{z\to\alpha}\frac{w_{N'_f,p-1}(z)}{w_{\Psi,1}(z)}\left[f'(z)\right]^2 = \frac{N_f^{(p)}(\alpha)f'(\alpha)}{(p-1)!} \neq 0.$$

It follows that α is a root of multiplicity $p-2$ of $f''(z)$. Hence $f^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $f^{(p)}(\alpha) \neq 0$. □

3.3.2 Increasing the order of convergence with the minimal amount of new information

Suppose $f(\alpha) = 0$, $f'(\alpha) \neq 0$, and $f''(\alpha) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$, then N_f is of order p. To increase the order of convergence to $p+1$ we look for $F(z)$ such that $F(\alpha) = 0$, $F'(\alpha) \neq 0$, and $F''(\alpha) = \mathcal{O}\left((z-\alpha)^{p-1}\right)$. Then N_F will be of order $p+1$.

Since α is supposed to be a simple root of $F(z)$, we can write

$$F(z) = f(z)\frac{F(z)}{f(z)} = f(z)\widetilde{F}(z).$$

with $\widetilde{F}(\alpha) \neq 0$. We want that $F(z)$ contains the minimal amount of new information. As suggested in [51], this new information comes from $f'(z)$. Hence we consider that $\widetilde{F}(z)$ depends only on $f'(z)$, so

$$\widetilde{F}(z) = G\left(f'(z)\right).$$

Now

$$F(z) = f(z)G\left(f'(z)\right),$$

and

$$F'(z) = f'(z)G\left(f'(z)\right) + f(z)G'\left(f'(z)\right)f''(z).$$

The requirement that α be a simple root of $F(z)$ implies

$$F'(\alpha) = f'(\alpha)G\left(f'(\alpha)\right) \neq 0,$$

so $G\left(f'(\alpha)\right) \neq 0$. Because the value $f'(\alpha)$ can be arbitrary in $\mathbb{C}$ except 0, we assume that $G(\xi) \neq 0$ for any $\xi \in \mathbb{C}$ except eventually at $\xi = 0$. We would also like that $F''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right)$. We have

$$\begin{aligned}F''(z) &= f''(z)G\left(f'(z)\right) + 2f'(z)G'\left(f'(z)\right)f''(z)\\ &\quad + f(z)G'\left(f'(z)\right)f'''(z) + f(z)G''\left(f'(z)\right)\left[f''(z)\right]^2.\end{aligned}$$

Since $f(z) = \mathcal{O}\left((z-\alpha)\right)$ and $f''(z) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$, we observe that

$$f(z)G''\left(f'(z)\right)\left[f''(z)\right]^2 = \mathcal{O}\left((z-\alpha)^{2p-3}\right)$$

regardless of $G''\left(f'(z)\right)$. Let us remark that $\mathcal{O}\left((z-\alpha)^{2p-3}\right)$ is equivalent or lower than $\mathcal{O}\left((z-\alpha)^{p-1}\right)$ for $p \geq 2$. So this term can be ignored. We already know that $G\left(f'(\alpha)\right) \neq 0$. Suppose $G'\left(f'(\alpha)\right) = 0$, then we will have $G'\left(f'(z)\right) = \mathcal{O}\left(z-\alpha\right)$ and

$$2f'(z)G'\left(f'(z)\right)f''(z) + f(z)G'\left(f'(z)\right)f'''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right),$$

meaning that

$$F''(z) = f''(z)G\left(f'(z)\right) + \mathcal{O}\left((z-\alpha)^{p-1}\right) = \mathcal{O}\left((z-\alpha)^{p-2}\right),$$

because

$$f''(z)G\left(f'(z)\right) = \mathcal{O}\left((z-\alpha)^{p-2}\right).$$

Therefore we need $G'\left(f'(\alpha)\right) \neq 0$ to get $F''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right)$. Now let us write

$$F''(z) = G'\left(f'(z)\right)\left[f''(z)\frac{G\left(f'(z)\right)}{G'\left(f'(z)\right)} + 2f'(z)f''(z) + f(z)f'''(z)\right] + \mathcal{O}\left((z-\alpha)^{p-1}\right).$$

Under the hypothesis on $f(z)$ it can be proved, see Lemma 3.3.3 in the next section, that

$$(p-2)f'(z)f''(z) - f(z)f'''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right).$$

So we deduce

$$F''(z) = G'\left(f'(z)\right) f''(z)\Big[\frac{G\left(f'(z)\right)}{G'\left(f'(z)\right)} + pf'(z)\Big] + \mathcal{O}\left((z-\alpha)^{p-1}\right).$$

Since $f''(z) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$, we obtain that $F''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right)$ if and only if

$$\frac{G\left(f'(z)\right)}{G'\left(f'(z)\right)} + pf'(z) = \mathcal{O}\left(z-\alpha\right),$$

and therefore

$$\frac{G\left(f'(\alpha)\right)}{G'\left(f'(\alpha)\right)} + pf'(\alpha) = 0.$$

We want this to be true regardless of the value $f'(\alpha) = \xi \neq 0$, then we have to find $G(\xi)$ such that

$$\frac{G(\xi)}{G'(\xi)} + p\xi = 0$$

for all $\xi \neq 0$. Solving this equation leads to

$$G(\xi) = \frac{c}{\xi^{1/p}},$$

for an arbitrary constant c. Conversely if the above equation holds then

$$\frac{G\left(f'(z)\right)}{G'\left(f'(z)\right)} + pf'(z) = 0$$

for all z in a neighbourhood of α, and $F''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right)$. So we have proved the following result.

Theorem 3.3.2. *Let $f(z)$ be such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and suppose N_f be of order p. There exists a function $G(\xi)$ such that N_F is of order $p+1$ for*

$$F(z) = f(z)G\left(f'(z)\right)$$

if and only if

$$G(\xi) = \frac{c}{\xi^{1/p}}. \quad \square$$

So using

$$F(z) = c\frac{f(z)}{[f'(z)]^{1/p}},$$

we have

$$N_F(z) = z - \frac{F(z)}{F'(z)} = z - \frac{f(z)}{f'(z)\left[1 - \frac{1}{p}\left[1 - \left(\frac{f(z)}{f'(z)}\right)'\right]\right]},$$

which is of order $p+1$. Let us mention that this is the result obtained in [19] for real valued functions, and it is called the Halley's acceleration process.

3.3.3 A fundamental lemma

To increase the order of convergence of the Newton's method, we need a way to increase the order of the zero of the second derivative of the appropriate function $F(z)$ related to the original function $f(z)$. Looking in this direction, the next lemma is very useful. It describes a way to combine the minimum number of successive terms like $f^{(k)}(z)$ $(k = 0, 1, 2, 3, \ldots)$ to get an expression of the form $\mathcal{O}\left((z-\alpha)^{p-2}\right)$. It happens that this minimum number is 3.

Lemma 3.3.3. *Let α be a simple root of an analytic function $f(z)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. For any $p \geq 2$, $f''(z) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$ iff*

$$(p-2)f'(z)f''(z) - f(z)f'''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right) = w(z)(z-\alpha)^{p-1}.$$

Moreover

$$w(\alpha) = -\frac{f'(\alpha)f^{(p+1)}(\alpha)}{(p-1)!}.$$

Proof. Let us observe that for $p = 2$ there is nothing to prove. Indeed $f''(z) = \mathcal{O}(1)$ and the condition holds because $f(z)f'''(z) = \mathcal{O}(z-\alpha) = w(z)(z-\alpha)$. So we consider $p \geq 3$.

Sufficiency. Let us consider the following expansions

$$\begin{aligned}
f(z) &= f'(\alpha)(z-\alpha) + w_{f,p}(z)(z-\alpha)^p, \\
f'(z) &= f'(\alpha) + w_{f',p-1}(z)(z-\alpha)^{p-1}, \\
f''(z) &= \frac{f^{(p)}(\alpha)}{(p-2)!}(z-\alpha)^{p-2} + w_{f'',p-1}(z)(z-\alpha)^{p-1}, \\
f'''(z) &= \frac{f^{(p)}(\alpha)}{(p-3)!}(z-\alpha)^{p-3} + w_{f''',p-2}(z)(z-\alpha)^{p-2},
\end{aligned}$$

where $w_{f^{(i)},j}(\alpha) = \dfrac{f^{(i+j)}(\alpha)}{j!}$ for any $i, j = 0, 1, \ldots$ By a direct substitution we get

$$\begin{aligned}
&(p-2)f'(z)f''(z) - f(z)f'''(z) \\
&= f'(\alpha)\big[(p-2)w_{f'',p-1}(z) - w_{f''',p-2}(z)\big](z-\alpha)^{p-1} + \mathcal{O}\left((z-\alpha)^{2p-3}\right) \\
&= \mathcal{O}\left((z-\alpha)^{p-1}\right),
\end{aligned}$$

because $p-1 \le 2p-3$ for $p \ge 2$. For the value of $w(\alpha)$, we have

$$w(\alpha) = \lim_{z\to\alpha} f'(\alpha)\big[(p-2)w_{f'',p-1}(z) - w_{f''',p-2}(z)\big] = -\frac{f'(\alpha)f^{(p+1)}(\alpha)}{(p-1)!}.$$

Necessity. We suppose for any $p \ge 3$ that

$$(p-2)f'(z)f''(z) - f(z)f'''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right) = w(z)(z-\alpha)^{p-1},$$

and we have to prove that $f''(z) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$, which is equivalent to $f^{(l)}(\alpha) = 0$ for $l = 2, \ldots, p-1$. Since

$$0 = \lim_{z\to\alpha}(p-2)f'(z)f''(z) - f(z)f'''(z) = (p-2)f'(\alpha)f''(\alpha)$$

it follows that $f''(\alpha) = 0$. If $p = 3$ we are done. Let us consider $p > 3$, and suppose $2 \le l \le p-2$ and $f^{(j)}(\alpha) = 0$ for $j = 2, \ldots, l$, we are going to show that $f^{(l+1)}(\alpha) = 0$.

Hence for $l = p-2$ we will have the result. We consider the following expansions

$$\begin{aligned}
f(z) &= f'(\alpha)(z-\alpha) + w_{f,l+1}(\alpha)(z-\alpha)^{l+1}, \\
f'(z) &= f'(\alpha) + w_{f',l}(z)(z-\alpha)^{l}, \\
f''(z) &= w_{f'',l-1}(z)(z-\alpha)^{l-1}, \\
f'''(z) &= w_{f''',l-2}(z)(z-\alpha)^{l-2},
\end{aligned}$$

By a direct substitution we get

$$\begin{aligned}
&(p-2)f'(z)f''(z) - f(z)f'''(z) \\
&= \Big[(p-2)w_{f'',l-1}(z) - w_{f''',l-2}(z)\Big] f'(\alpha)(z-\alpha)^{l-1} \\
&\quad + \Big[(p-2)w_{f',l}(z)w_{f'',l-1}(z) - w_{f,l+1}(z)w_{f''',l-2}(z)\Big](z-\alpha)^{2l-1} \\
&= \mathcal{O}\left((z-\alpha)^{p-1}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\Big[(p-2)w_{f'',l-1}(z) - w_{f''',l-2}(z)\Big] f'(\alpha) \\
&\quad = -\Big[(p-2)w_{f',l}(z)w_{f'',l-1}(z) - w_{f,l+1}(z)w_{f''',l-2}(z)\Big]\big(z-\alpha\big)^{l} \\
&\qquad + \mathcal{O}\left((z-\alpha)^{p-l}\right).
\end{aligned}$$

So

$$0 = \lim_{z\to\alpha} \Big[(p-2)w_{f'',l-1}(z) - w_{f''',l-2}(z)\Big] f'(\alpha) = [p-1-l]\frac{f'(\alpha)f^{(l+1)}(\alpha)}{(l-1)!}.$$

Since $l \leq p-2$ it follows that $f^{(l+1)}(\alpha) = 0$ and we have the result. □

3.4 Generating algorithm

Following [51], a *basic sequence* of methods is an infinite sequence whose p-th member is a method of order p. Suppose we start with a function $f(z)$ such that $f(\alpha) = 0$, $f'(\alpha) \neq 0$, and $f''(\alpha) \neq 0$, the p-th member of the *Halley's basic sequence* is

$$H_p(z) = z - \frac{F_p(z)}{F_p'(z)}.$$

For $p = 2$ we have

$$\frac{F_2(z)}{F_2'(z)} = \frac{f(z)}{f'(z)},$$

and for $p \geq 2$, we write

$$\frac{F_{p+1}(z)}{F_{p+1}'(z)} = \left[1 - \frac{1}{p}\left[1 - \left(\frac{F_p(z)}{F_p'(z)}\right)'\right]\right]^{-1} \frac{F_p(z)}{F_p'(z)}.$$

Consider any other basic sequence for which the p-th member is

$$T_p(z) = z - V_p(z),$$

where $V_p(z)$ is whatever expression. The methods $H_p(z)$ and $T_p(z)$ will be the same if and only if

$$V_p(z) = \frac{F_p(z)}{F_p'(z)} \quad \text{for} \quad p \geq 2.$$

Hence we have the following result.

Theorem 3.4.1. *Let $f(z)$ be such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ for which N_f is of order 2. The basic sequence $\{T_p(z)\}_{p=2}^{+\infty}$ given by*

$$T_p(z) = z - V_p(z) \quad \textit{for} \quad p \geq 2,$$

for which the p-th element is of order p, corresponds to the Halley's basic sequence if and only if

$$\begin{cases} V_2(z) & = \frac{f(z)}{f'(z)}, \\ V_{p+1}(z) & = \left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) \quad \textit{for} \quad p \geq 2. \end{cases} \qquad \square$$

We will now present some examples of application of this algorithm.

Example 3.4.1. *The Householder's basic sequence [26, 27] given by*

$$T_p(z) = z + (p-1)\Big[\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}\Big]$$

is in fact the Halley's basic sequence by using the above algorithm. Here we have

$$V_p(z) = -(p-1)\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}.$$

Then

$$V_p'(z) = -(p-1)\left[1 - \frac{(1/f(z))^{(p-2)}(1/f(z))^{(p)}}{\left[(1/f(z))^{(p-1)}\right]^2}\right],$$

and a direct substitution leads to

$$\left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) = -p\frac{(1/f(z))^{(p-1)}}{(1/f(z))^{(p)}} = V_{p+1}(z). \quad \square$$

Example 3.4.2. *Jovanović [31] suggested the following basic sequence*

$$\begin{cases} J_2(z) &= z - \frac{f(z)}{f'(z)}, \\ \\ J_p(z) &= z - \frac{z - J_{p-1}(z)}{1 - \frac{1}{p-1}J'_{p-1}(z)} \quad \text{for} \quad p > 2. \end{cases}$$

which is again the Halley's basic sequence. Indeed we have

$$V_p(z) = \frac{z - J_{p-1}(z)}{1 - \frac{1}{p-1}J'_{p-1}(z)} = z - J_p(z),$$

and

$$V_p'(z) = 1 - J_p'(z).$$

Hence

$$\left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) = \frac{z - J_p(z)}{1 - \frac{1}{p}J_p'(z)} = V_{p+1}(z). \quad \square$$

Example 3.4.3. *Ford and Pennline [17] have presented the following form of the Halley's basic sequence developed by Gerlach [19]*

$$G_p(z) = z - f(z)\frac{Q_p(z)}{Q_{p+1}(z)},$$

where
$$\begin{cases} Q_2(z) &=& 1, \\ \\ Q_p(z) &=& f'(z)Q_{p-1}(z) - \frac{1}{p-2}f(z)Q'_{p-1}(z) \quad \textit{for} \quad p \geq 3. \end{cases}$$
We have
$$V_p(z) = f(z)\frac{Q_p(z)}{Q_{p+1}(z)},$$
and
$$\begin{aligned} V'_p(z) &= \left(f(z)\frac{Q_p(z)}{Q_{p+1}(z)}\right)' \\ &= f'(z)\frac{Q_p(z)}{Q_{p+1}(z)} + f(z)\frac{Q'_p(z)Q_{p+1}(z) - Q_p(z)Q'_{p+1}(z)}{Q^2_{p+1}(z)} \\ &= (1-p) + p\frac{Q_p(z)Q_{p+2}(z)}{Q^2_{p+1}(z)}, \end{aligned}$$
where we have successively replaced $f(z)Q'_{p+1}(z)$ and $f'(z)Q_p(z)$ using the preceding relations to generate $Q_{p+2}(z)$ and $Q_{p+1}(z)$. Hence
$$1 - \frac{1}{p}\left[1 - V'_p(z)\right] = \frac{Q_p(z)Q_{p+2}(z)}{Q^2_{p+1}(z)},$$
and it follows that
$$\begin{aligned} \left[1 - \frac{1}{p}\left[1 - V'_p(z)\right]\right]^{-1} V_p(z) &= \left[\frac{Q_p(z)Q_{p+2}(z)}{Q^2_{p+1}(z)}\right]^{-1} f(z)\frac{Q_p(z)}{Q_{p+1}(z)} \\ &= f(z)\frac{Q_{p+1}(z)}{Q_{p+2}(z)} \\ &= V_{p+1}(z). \quad \square \end{aligned}$$

3.5 Looking for sequences to define $V_p(z)$

3.5.1 Looking for a sequence $\{\Delta_p(z)\}_{p=0}^{+\infty}$ such that $V_p(z) = f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}$

The last example of the preceding section suggests a general method to find a sequence $\{\Delta_p(z)\}_{p=0}^{+\infty}$ such that $V_p(z) = f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}$ corresponds to the Halley's basic sequence.

The condition on the first two terms of the sequence will be

$$V_2(z) = f(z)\frac{\Delta_0(z)}{\Delta_1(z)} = \frac{f(z)}{f'(z)},$$

so $\Delta_1(z) = f'(z)\Delta_0(z)$. Moreover we would like that $\Delta_1(z)$ be generated by $\Delta_0(z)$ and $\Delta_0'(z)$, so we write

$$\Delta_1(z) = f'(z)\Delta_0(z) = \left(f'(z) + g(z)\right)\Delta_0(z) + h(z)\Delta_0'(z).$$

where $g(z)$ and $h(z)$ have to be specified to get the Halley's family. Let us introduce a new function $\delta_0(z)$ such that $\Delta_0(z)\delta_0(z) = 1$. We can write

$$\begin{aligned}
h(z)\Delta_0'(z) &= h(z)\big(\Delta_0(z)\delta_0(z)\big)\Delta_0'(z) \\
&= h(z)\Delta_0(z)\left[(\delta_0(z)\Delta_0(z))' - \delta_0'(z)\Delta_0(z)\right] \\
&= -h(z)\Delta_0(z)\delta_0'(z)\Delta_0(z) \\
&= -h(z)\frac{\delta_0'(z)}{\delta_0(z)}\Delta_0(z).
\end{aligned}$$

Then we have

$$\Delta_1(z) = \left(f'(z) + g(z) - h(z)\frac{\delta_0'(z)}{\delta_0(z)}\right)\Delta_0(z).$$

It follows that

$$g(z) - h(z)\frac{\delta_0'(z)}{\delta_0(z)} = 0,$$

and

$$\Delta_1(z) = \left(f'(z) + h(z)\frac{\delta_0'(z)}{\delta_0(z)}\right)\Delta_0(z) + h(z)\Delta_0'(z).$$

We would also like to have that $\Delta_p(z)$ be generated by $\Delta_{p-1}(z)$ and $\Delta_{p-1}'(z)$. Let us see what happens for $\Delta_2(z)$. To be an element of the Halley's basic sequence, we must have

$$f(z)\frac{\Delta_1(z)}{\Delta_2(z)} = \left[1 - \frac{1}{2}\left[1 - \left(f(z)\frac{\Delta_0(z)}{\Delta_1(z)}\right)'\right]\right]^{-1} f(z)\frac{\Delta_0(z)}{\Delta_1(z)},$$

where

$$1-\frac{1}{2}\left[1-\left(f(z)\frac{\Delta_0(z)}{\Delta_1(z)}\right)'\right]=\left(1-\frac{1}{2}\left(1-\frac{f(z)}{h(z)}\right)\right)$$
$$+\frac{\Delta_0}{2\Delta_1^2}\left[\left(f'(z)-\frac{f(z)}{h(z)}\left(f'(z)+h(z)\frac{\delta_0'(z)}{\delta_0(z)}\right)\right)\Delta_1(z)-f(z)\Delta_1'(z)\right].$$

For obtaining an expression depending only on $\Delta_1(z)$ and $\Delta_1'(z)$ we set

$$1-\frac{1}{2}\left(1-\frac{f(z)}{h(z)}\right)=0,$$

which means that $h(z)=-f(z)$. Hence we obtain

$$f(z)\frac{\Delta_1(z)}{\Delta_2(z)}=f(z)\frac{\Delta_1(z)}{\frac{1}{2}\left[\left(2f'(z)-f(z)\frac{\delta_0'(z)}{\delta_0(z)}\right)\Delta_1(z)-f(z)\Delta_1'(z)\right]},$$

so

$$\Delta_2(z)=\left(f'(z)-\frac{f(z)}{2}\frac{\delta_0'(z)}{\delta_0(z)}\right)\Delta_1(z)-\frac{f(z)}{2}\Delta_1'(z).$$

Consequently, to get the Halley's basic sequence, starting with $\Delta_0(z)\delta_0(z)=1$, we show by induction that

$$\Delta_p(z)=\left(f'(z)-\frac{f(z)}{p}\frac{\delta_0'(z)}{\delta_0(z)}\right)\Delta_{p-1}(z)-\frac{f(z)}{p}\Delta_{p-1}'(z)$$

for $p\geq 1$. Indeed it is easy to show that

$$f(z)\frac{\Delta_{p-1}(z)}{\Delta_p(z)}=\left[1-\frac{1}{p}\left[1-\left(f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}\right)'\right]\right]^{-1}f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}.$$

So we have proved the following result.

Theorem 3.5.1. *Let $f(z)$ be such that $f(\alpha)=0$ and $f'(\alpha)\neq 0$ for which N_f is of order 2. The basic sequence $\{T_p(z)\}_{p=2}^{+\infty}$ given by*

$$T_p(z)=z-f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}\quad \textit{for}\quad p\geq 2,$$

where $\Delta_0(z)$ and $\delta_0(z)$ are two arbitrary analytic functions such that

$$\begin{cases}\Delta_0(z)\delta_0(z) &= 1,\\ \Delta_p(z) &= \left(f'(z)-\frac{f(z)}{p}\frac{\delta_0'(z)}{\delta_0(z)}\right)\Delta_{p-1}(z)-\frac{f(z)}{p}\Delta_{p-1}'(z)\quad \textit{for}\quad p\geq 1,\end{cases}$$

corresponds to the Halley's basic sequence. □

Corollary 3.5.2. *Suppose $f(z)$ satifies the preceding assumption. Let σ be an integer, $\widetilde{\delta}_\sigma(z)$ be an analytic function, and $\left\{\widetilde{\Delta}_p(z)\right\}_{p=\sigma}^{+\infty}$ defined by*

$$\begin{cases} \widetilde{\Delta}_\sigma(z)\widetilde{\delta}_\sigma(z) &= 1, \\ \widetilde{\Delta}_p(z) &= \left(f'(z) - \frac{f(z)}{p-\sigma}\frac{\delta'_\sigma(z)}{\delta_\sigma(z)}\right)\widetilde{\Delta}_{p-1}(z) - \frac{f(z)}{p-\sigma}\widetilde{\Delta}'_{p-1}(z) \quad \textit{for} \quad p \geq \sigma+1, \end{cases}$$

then

$$T_p(z) = z - f(z)\frac{\widetilde{\Delta}_{p+\sigma-2}(z)}{\widetilde{\Delta}_{p+\sigma-1}(z)},$$

is of order p for $p \geq 2$, and corresponds to the p-th element of the Halley's basic sequence.

Proof. It is enough to remark that, for $\delta_0(z) = \widetilde{\delta}_\sigma(z)$ we have $\Delta_p(z) = \widetilde{\Delta}_{p+\sigma}(z)$ for $p \geq 0$ to conclude. $\square$

Example 3.5.1. *Ford and Pennline basic sequence of Example 3.4.3 corresponds to the case $\sigma = 2$ and $\widetilde{\delta}_2(z) = 1$.* $\square$

3.5.2 Looking for a sequence $\{B_p(z)\}_{p=0}^{+\infty}$ such that $V_p(z) = \frac{B_{p-2}(z)}{B_{p-1}(z)}$

As a corollary of the last theorem, we have the following result.

Corollary 3.5.3. *Let $f(z)$ be such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ for which N_f is of order 2. The basic sequence $\{T_p(z)\}_{p=2}^{+\infty}$ given by*

$$T_p(z) = z - \frac{B_{p-2}(z)}{B_{p-1}(z)} \quad \textit{for} \quad p \geq 2,$$

where $B_0(z)$ and $\beta_0(z)$ are two arbitrary functions such that

$$\begin{cases} B_0(z)\beta_0(z &= 1, \\ pB_p(z) &= \left(\frac{f'(z)}{f(z)} - \frac{\beta'_0(z)}{\beta_0(z)}\right)B_{p-1}(z) - B'_{p-1}(z) \quad \textit{for} \quad p \geq 1, \end{cases}$$

corresponds to the Halley's basic sequence.

Proof. Let $\{\Delta_p(z)\}_{p=0}^{+\infty}$ be generated with $\delta_0(z) = 1$. Let us define $\beta_p(z) = f^p(z)\beta_0(z)$. If we set $B_p(z) = \frac{\Delta_p(z)}{\beta_p(z)}$, we obtain the result by direct substitution. □

Example 3.5.2. *Let* $B_0(z) = 1 = \beta_0(z)$, *we get*

$$pB_p(z) = \frac{f'(z)}{f(z)}B_{p-1}(z) - B'_{p-1}(z). \quad \square$$

Example 3.5.3. *Let* $B_0(z) = f(z)$ *and* $\beta_0(z) = 1/f(z)$, *we get*

$$pB_p(z) = 2\frac{f'(z)}{f(z)}B_{p-1}(z) - B'_{p-1}(z). \quad \square$$

Example 3.5.4. *Let* $B_0(z) = 1/f(z)$ *and* $\beta_0(z) = f(z)$, *we get*

$$pB_p(z) = -B'_{p-1}(z),$$

and hence

$$p!B_p(z) = (-1)^p \left(1/f(z)\right)^{(p)}.$$

So this basic sequence corresponds to the Householder's sequence of Example 3.4.1 because

$$V_p(z) = \frac{B_{p-2}(z)}{B_{p-1}(z)} = -(p-1)\frac{B_{p-2}(z)}{B'_{p-2}(z)} = -(p-1)\frac{\left(1/f(z)\right)^{(p-2)}}{\left(1/f(z)\right)^{(p-1)}}. \quad \square$$

Example 3.5.5. *The Schröder [48] process of the second kind is defined by*

$$S_p(z) = z - \frac{R_{p-2}(z)}{R_{p-1}(z)} \quad \textit{for} \quad p \geq 2,$$

where

$$\begin{cases} R_0(z) &=& 1/f(z), \\ R_p(z) &=& \sum_{j=1}^{p} (-1)^{j+1} \frac{f^{(j)}(z)}{j!f(z)} R_{p-j}(z \quad \textit{for} \quad p \geq 1. \end{cases}$$

It is easy to show by induction that

$$R_p(z) = \frac{(-1)^p}{p!}\left(\frac{1}{f(z)}\right)^{(p)}.$$

Then we obtain directly

$$\frac{R_{p-2}(z)}{R_{p-1}(z)} = \frac{\frac{(-1)^{p-2}}{(p-2)!}\left(\frac{1}{f(z)}\right)^{(p-2)}}{\frac{(-1)^{p-1}}{(p-1)!}\left(\frac{1}{f(z)}\right)^{(p-1)}} = -(p-1)\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}$$

which corresponds to the Householder's basic sequence of Example 3.4.1, which has been proved to be the Halley's basic sequence. We could also replace $\Delta_p(z)$, with $\delta_0 = 1$, by $f^{p+1}(z)R_p(z)$ in Theorem 3.5.1 to obtain that $R_p(z) = \frac{\Delta_p(z)}{f^{p+1}(z)}$. So it corresponds to the Halley's basic sequence because

$$\frac{R_{p-2}(z)}{R_{p-1}(z)} = \frac{\Delta_{p-2}(z)/f^{p-1}(z)}{\Delta_{p-1}(z)/f^{p}(z)} = f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}. \quad \square$$

Remark 3.5.1. *These expressions can also be written in other different forms. For example if we set $\beta_j(z) = f(z)\beta_{j-1}(z) = \ldots = f^j(z)\beta_0(z)$ for $j \in \mathbb{Z}$, we get*

$$\frac{\beta_j'(z)}{\beta_j(z)} = j\frac{f'(z)}{f(z)} + \frac{\beta_0'(z)}{\beta_0(z)},$$

so we can also write

$$pB_p(z) = \left((j+1)\frac{f'(z)}{f(z)} - \frac{\beta_j'(z)}{\beta_j(z)}\right) B_{p-1}(z) - B_{p-1}'(z). \quad \square$$

3.6 Other methods

Several other expressions of the Halley's basic sequence are presented in the literature, for example determinant-based methods. In [44, 45] their equivalence has been showed, so we only present these methods here for completeness of the paper. The sequence $\{\Delta_p(z)\}_{p=0}^{+\infty}$ used in this section is generated with $\delta_0(z) = 1$.

3.6.1 Determinant-based methods

Determinant-based families of methods where developed using the sequence of determinants given by: $M_0(z) = 1$, and for $p \geq 1$

$$M_p(z) = \begin{vmatrix} f'(z) & \dots & \dots & \dots & \frac{f^{(p)}(z)}{p!} \\ f(z) & f'(z) & \dots & \dots & \frac{f^{(p-1)}(z)}{(p-1)!} \\ 0 & f(z) & f'(z) & \dots & \frac{f^{(p-2)}(z)}{(p-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & f(z) & f'(z) \end{vmatrix}.$$

It can be shown, see for example [44, 35], for $p \geq 1$ that

$$M_p(z) = \sum_{j=1}^{p} (-1)^{j+1} \frac{f^{j-1}(z) f^{(j)}(z)}{j!} M_{p-j}(z),$$

and

$$M_p(z) = \frac{(-1)^p \, f^{p+1}(z)}{p!} \left(\frac{1}{f(z)} \right)^{(p)},$$

which lead to the relation

$$M_p(z) = f'(z) M_{p-1}(z) - \frac{1}{p} f(z) M'_{p-1}(z).$$

It follows that $M_p(z) = \Delta_p(z)$ for any $p \geq 0$.

Example 3.6.1. *Hamilton [21], and later Kalantari [33, 34, 32], proposed the basic sequence*

$$K_p(z) = z - f(z) \frac{M_{p-2}(z)}{M_{p-1}(z)} \quad \textit{for} \quad p \geq 2.$$

It is clearly the Halley's basic sequence [35]. □

Example 3.6.2. *Wang [54], see also [55], proposed that*

$$W_p(z) = z - \frac{\Gamma_{p-2}(z)}{\Gamma_{p-1}(z)} \quad for \quad p \geq 2,$$

where $\Gamma_p(z) = \frac{M_p(z)}{f^p(z)}$. *Clearly this is again the Halley's basic sequence.* □

Example 3.6.3. *Varjuhin and Kasjanjuk [52] considered the following determinants*

$$N_p(z) = \begin{vmatrix} f(z) & zf(z) & \dots & \dots & z^{p-2}f(z) \\ f''(z) & (zf(z))'' & \dots & \dots & \left(z^{(p-2)}f(z)\right)'' \\ f^{(3)}(z) & (zf(z))^{(3)} & \dots & \dots & \left(z^{(p-2)}f(z)\right)^{(3)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ f^{(p-1)}(z) & (zf(z))^{(p-1)} & \dots & \dots & \left(z^{(p-2)}f(z)\right)^{(p-1)} \end{vmatrix},$$

and

$$D_p(z) = \begin{vmatrix} f'(z) & (zf(z))' & \dots & \dots & \left(z^{(p-2)}f(z)\right)' \\ f''(z) & (zf(z))'' & \dots & \dots & \left(z^{(p-2)}f(z)\right)'' \\ f^{(3)}(z) & (zf(z))^{(3)} & \dots & \dots & \left(z^{(p-2)}f(z)\right)^{(3)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ f^{(p-1)}(z) & (zf(z))^{(p-1)} & \dots & \dots & \left(z^{(p-2)}f(z)\right)^{(p-1)} \end{vmatrix}.$$

They suggested the method

$$Y_p(z) = z - \frac{N_p(z)}{D_p(z)},$$

which is of order p. These determinants seem similar to the M_p*'s, and it has been proved [44] that*

$$N_p(z) = f(z)M_{p-2}(z)\prod_{m=0}^{p-2}(p-1-m)!$$

and

$$D_p(z) = M_{p-1}(z) \prod_{m=0}^{p-2} (p-1-m)!$$

So

$$\frac{N_p(z)}{D_p(z)} = f(z)\frac{M_{p-2}(z)}{M_{p-1}(z)} = f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)}.$$

Consequently $Y_p(z)$ corresponds to $H_p(z)$. □

3.6.2 Even more methods

Two other methods which seemed at first complicated but in fact where only other forms of the Halley's basic sequence. These methods where analyzed in [44].

Example 3.6.4. *Farmer and Loizou [16] considered*

$$\begin{cases} \Lambda_2(z) & = & \frac{f(z)}{f'(z)}, \\ \Lambda_p(z) & = & \frac{f(z)/f'(z)}{1-\sum_{i=2}^{p-1}\frac{f^{(i)}(z)}{i!f(z)}\prod_{q=p-i}^{p-2}\Lambda_{q+1}(z)} \quad \textit{for} \quad p \geq 3, \end{cases}$$

and

$$L_p(z) = z - \Lambda_p(z).$$

It can be shown [44] that

$$\Lambda_p(z) = f(z)\frac{M_{p-2}(z)}{M_{p-1}(z)} = f(z)\frac{\Delta_{p-2}(z)}{\Delta_{p-1}(z)},$$

and hence $L_p(z)$ corresponds to $H_p(z)$. □

Example 3.6.5. *Igarashi and Nagasaka [28, 29] set*

$$t_i(z) = \frac{f^{(i)}(z)}{i!f'(z)} \quad \textit{for} \quad i = 0, 1, \ldots,$$

and for $p \geq 2$

$$\begin{cases} b_{p,1}(z) &= t_{p-1}(z), \\ b_{p,j}(z) &= b_{p,j-1}(z)h_j(z) + t_{p-j}(z) \quad \textit{for} \quad j = 2, \ldots, p-1, \\ h_p(z) &= -\frac{t_0(z)}{b_{p,p-1}(z)}. \end{cases}$$

They considered

$$I_p(z) = z + h_p(z),$$

which is of order p. Since it has been proved that [44] that $h_p(z) = -\Lambda_p(z)$, so hence $I_p(z)$ corresponds to $H_p(z)$. □

3.7 Theory and practice

A well known result in complex function theory is that the zeros of an analytic function are isolated (see for example [47]).

Lemma 3.7.1. *If $\phi(z)$ in an analytic function such that $\phi(\alpha) = 0$, then either $\phi(z)$ is identically zero in a neighbourhood of α or there a punctured disk about α in which $\phi(z)$ has no zeros.*

A practical test to determine if $H_p(z)$ and $T_p(z)$ could be the same method would be to start with the same initial condition z_0, and verify if $H_p(z_k) = z_{k+1} = T_p(z_k)$ and $\lim_{k\to+\infty} z_k = \alpha$, which means that both methods generate the same convergent sequence. Since both functions are analytic in a neighbourhood of α, the lemma suggests that $H_p(z) = T_p(z)$ for all z in a neighbourhood of α. This numerical test, which is not a proof, can help to suggest if the basic sequence $\{T_p(z)\}_{p=2}^{+\infty}$ corresponds to the Halley's basic sequence $\{H_p(z)\}_{p=2}^{+\infty}$.

CHAPITRE 4

Unifying old and new ways to increase order of convergence of fixed point and Newton's method.

Abstract

Halley, Euler, Chebyshev are examples of mathematicians that have presented procedures for increasing the order of convergence of Newton's method. Through Taylor's expansions and a thorough analysis of the necessary and sufficient conditions that will entail for fixed point and Newton's iterative methods to be of higher order convergence, we are able to present a unified way which include old and new processes to make these methods faster.

4.1 Introduction.

In 1669 [57], for finding α, a simple root of an analytic function $f(z)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, Sir Isaac Newton introduced the iteration function (IF)

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

Some years later, in 1694 [18, 20, 57], Edmond Halley, introduced a "faster" IF

$$H_f(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)} = z - \left[1 - \frac{f''(z)f(z)}{2\left(f'(z)\right)^2}\right]^{-1} \frac{f(z)}{f'(z)}.$$

Interestingly enough, one can observe that if we apply Newton's method to the function $F(z) = f(z)/\sqrt{f'(z)}$, we get

$$N_F(z) = z - \frac{f(z)/\sqrt{f'(z)}}{[f(z)/\sqrt{f'(z)}]'} = H_f(z).$$

So, in considering this new function $F(z)$ and looking at its application to Newton's method, one could manage to make Newton's method faster. We ask the following question: What are all the possible functions $F(z)$, like Halley's, that we can find, for which Newton's method applied to the function $F(z)$ will be faster?

We also observe that Halley's IF can be rewritten as

$$H_f(z) = z - \frac{f(z)}{f'(z)} G(z)$$

where

$$G(z) = \left[1 - \frac{f''(z)f(z)}{2\left(f'(z)\right)^2}\right]^{-1}.$$

In noticing that, we also ask: What are all possible functions such as $G(z)$ that will entail an increase of order of Newton's method when applied to these functions? How do these functions related to our previously mentioned $F(z)$?

Moreover, using the first two terms of the Taylor's expansion of $1/(1-z)$ for $G(z)$, i.e. $1+z$, we get

$$\widetilde{G}(z) = 1 + \frac{f''(z)f(z)}{2\left(f'(z)\right)^2},$$

and

$$\widetilde{H}_f(z) = z - \frac{f(z)}{f'(z)}\widetilde{G}(z) = z - \frac{f(z)}{f'(z)} - \frac{f''(z)}{2f'(z)}\left(\frac{f(z)}{f'(z)}\right)^2 = C_f(z),$$

which is the so called Euler-Chebyshev's IF of order 3 [3].

Even if iteration functions of high order of convergence require high order derivatives and are more complex, these methods are not only of academic interest but can also sometime be of practical interest [13, 11].

The plan of the paper is the following. Some preliminaries are given in Sections 5.2. Section 4.3 presents a general result about the local convergence of fixed point methods. It also garanties convergence for iterative methods of higher order. Section 4.4 presents a thorough analysis of the necessary and sufficient conditions that will entail for fixed point and Newton's iterative methods to be of higher order convergence. These results allow us to consider two different procedures for increasing the order. The first approach, presented in Section 4.5, consists in modifying the fixed point iteration function by adding an additional term that conveniently enough increased the order of convergence of the iteration function. The second procedure, explained in Section 4.6, consists in modifying our original function $f(z)$ into a new one $F(z)$ which caused the order of convergence of Newton's method to increase. We also establish links between those two procedures in Section 4.7. Section 4.8 shows a way of linking very famous iterative methods like Halley's and Euler-Chebyshev's through our new results. In the last section, as an interesting example, we consider the Super-Halley family of iteration functions of order 3.

4.2 Preliminaries

4.2.1 Order of convergence

If our goal here is to make iterative methods faster, so let us start by mathematically defining first what we may mean by faster. Let us consider any sequence $\{z_{k+1} = I(z_k)\}_{k=0}^{\infty}$ generated from an initial condition z_0 and an IF $I(z)$. We say that the *order of convergence* of the IF $I(z)$, is p, a positive integer, if and only if there exists a non-zero constant $K_p(\alpha; I)$ such that

$$\lim_{k\to+\infty} \frac{z_{k+1}-\alpha}{(z_k-\alpha)^p} = \lim_{k\to+\infty} \frac{I(z_k)-I(\alpha)}{(z_k-\alpha)^p} = K_p(\alpha; I).$$

For linear convergence, $p = 1$, it is required that $|K_1(\alpha; I)| < 1$ [51]. For values of z_k close to α, the term $\mid z_k - \alpha \mid^p$ becomes considerably smaller if p is large, so looking at

$$|z_{k+1}-\alpha| \approx |K_p(\alpha, I)|\,|z_k-\alpha|^p$$

does indicate how fast the error of approximation decreases.

For instance, Newton's iteration function is of order $p = 2$ with $K_2(\alpha; N_f) = \frac{f''(\alpha)}{2f'(\alpha)}$, while Halley's and Euler-Chebyshev's iteration functions are of order $p = 3$ with asymptotic constant respectively given by

$$K_3(\alpha; H_f) = \frac{1}{3!}\left[\frac{3}{2}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 - \frac{f'''(\alpha)}{f'(\alpha)}\right],$$

and

$$K_3(\alpha; C_f) = \frac{1}{3!}\left[3\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 - \frac{f'''(\alpha)}{f'(\alpha)}\right],$$

If two IFs have the same order, provided convergence occurs, the one with the smallest asymptotic constant will typically be faster. But, as illustrated above, the asymptotic constant depends on the function $f(z)$, so a given IF may be faster for a given function

$f(z)$ and slower for another function $f(z)$ compared to another IF. There already exit numerical examples to illustrate this fact in [4].

In trying to increase the order of convergence of Newton's method, many mathematicians have found infinite sequences of IFs with increasing order of convergence. We will call such an infinite sequence, say $\{I_p(z)\}_{p=0}^{+\infty}$ whose pth member $I_p(z)$ is an IF of order p, a *basic sequence* of IFs [51].

4.2.2 Analytic function and Taylor's expansion.

Higher order convergence results for fixed-point and Newton's methods that we will introduce in the next sections are extensively based on Taylor's expansions. The notation $g^{(j)}(z)$ stands for the jth derivative $\dfrac{d^j}{dz^j}g(z)$.

Lemma 4.2.1. *[47] For any analytic function $g(z)$, for any $p \geq 1$ there exists a unique analytic function $w_{g,p}(z)$ such that*

$$g(z) = \sum_{j=0}^{p-1} \frac{g^{(j)}(\alpha)}{j!}\,(z-\alpha)^j + w_{g,p}(z)\,(z-\alpha)^p\,.$$

More precisely

$$w_{g,p}(z) = \sum_{j=0}^{+\infty} \frac{g^{(p+j)}(\alpha)}{(p+j)!}\,(z-\alpha)^j \quad \textit{and} \quad w_{g,p}(\alpha) = \frac{g^{(p)}(\alpha)}{p!}.$$

For a function $f(z)$ with a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, we can write

$$f(z) = w_{f,1}(z)(z-\alpha) \quad \text{where} \quad w_{f,1}(\alpha) = f'(\alpha).$$

We observe that

$$\frac{f(z)}{f'(z)} = \frac{w_{f,1}(z)}{f'(z)}\,(z-\alpha)$$

or

$$(z-\alpha) = \frac{f(z)}{w_{f,1}(z)} = \frac{f'(z)}{w_{f,1}(z)}\frac{f(z)}{f'(z)}.$$

We can rewrite the preceding lemma as follows.

Lemma 4.2.2. *Let α be a simple root of an analytic function $f(z)$, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Let $g(z)$ be any analytic function, and any $p \geq 1$.*

(a) There exists a unique analytic function $W_{g,p}(z)$ such that

$$g(z) = \sum_{j=0}^{p-1} \frac{g^{(j)}(\alpha)}{j!}(z-\alpha)^j + W_{g,p}(z)\left(\frac{f(z)}{f'(z)}\right)^p$$

where

$$W_{g,p}(z) = w_{g,p}(z)\left(\frac{f'(z)}{w_{f,1}(z)}\right)^p \quad \text{and} \quad W_{g,p}(\alpha) = \frac{f^{(p)}(\alpha)}{p!}.$$

(b) There exists a unique analytic function $\widetilde{W}_{g,p}(z)$ such that

$$g(z) = \sum_{j=0}^{p-1} \frac{g^{(j)}(\alpha)}{j!}(z-\alpha)^j + \widetilde{W}_{g,p}(z) f^p(z),$$

where

$$\widetilde{W}_{g,p}(z) = \frac{w_{g,p}(z)}{(w_{f,1}(z))^p} \quad \text{and} \quad \widetilde{W}_{g,p}(\alpha) = \frac{f^{(p)}(\alpha)}{p!\,(f'(\alpha))^p}.$$

For simplicity, we will use the big $\mathcal{O}$ notation for two functions $u(z)$ and $v(z)$, hence $u(z) = \mathcal{O}(v(z))$ around $z = \alpha$ to mean that there exists a constant $c \neq 0$ such that

$$\lim_{z\to\alpha} \frac{u(z)}{v(z)} = c.$$

Based on the assumption that α is a simple root of $f(z)$, Â¸the following three expressions, namely $\mathcal{O}\left((z-\alpha)^l\right)$, $\mathcal{O}\left(\left(\frac{f(z)}{f'(z)}\right)^l\right)$, and $\mathcal{O}\left(f^l(z)\right)$, are equivalent for any positive integer l.

4.3 Linear convergence for fixed-point method.

A first result concerns the necessary and sufficient conditions for achieving linear convergence, which explains why $|K_1(\alpha; I)| < 1$ is needed. Moreover this result implies the convergence of any higher order methods since for these methods $K_1(\alpha; I) = 0$.

Theorem 4.3.1. *[11, 12] Let $I(z)$ be an iteration function.*
(i) If $|I'(\alpha)| < 1$, then there exists a neighbourhood of α such that for any z_0 in that neighbourhood the sequence $\{z_{k+1} = I(z_k)\}_{k=0}^{+\infty}$ converges to α.
(ii) If there exists a neighbourhood of α such that for any z_0 in that neighbourhood the sequence $\{z_{k+1} = I(z_k)\}_{k=0}^{+\infty}$ converges to α, and $z_k \neq \alpha$ for all k, then $|I'(\alpha)| \leq 1$.
(iii) For any sequence $\{z_{k+1} = I(z_k)\}_{k=0}^{+\infty}$ which converges to α, the limit $K_1(\alpha; I)$ exists and $K_1(\alpha; I) = I'(\alpha)$.

Using this result, a simple way to obtain a fixed-point method to find α could be

$$I(z) = z + \lambda f(z)$$

for which we have $I(\alpha) = \alpha$. The parameter λ is fixed in such a way that

$$|I'(\alpha)| = |1 + \lambda f'(\alpha)| < 1$$

Since obviously α is not known, let us replace λ by $\frac{\tilde{\lambda}}{f'(z)}$ to get

$$I(z) = z + \tilde{\lambda}\frac{f(z)}{f'(z)},$$

and

$$I'(\alpha) = 1 + \tilde{\lambda}.$$

As long as $\tilde{\lambda}$ is such that $\left|1 + \tilde{\lambda}\right| < 1$, we get a linearly convergent IF. For $\tilde{\lambda} = -1$, we get the Newton's IF and $I'(\alpha) = N_f'(\alpha) = 0$ which, intuitively, indicates it is a "more" than linear convergent IF.

4.4 High-order convergence for fixed point and Newton's methods.

Since Newton's method is of order $p = 2$, we will consider now general results for method of order $p \geq 2$.

4.4.1 Necessary and sufficient conditions for fixed point method.

The next result indicates that the order of an IF coincides with the order of its first non-zero derivative at α.

Theorem 4.4.1. *[11, 12] Let the integer $p \geq 2$ and let $I(z)$ be an iteration function such that $I(\alpha) = \alpha$. The iteration function $I(z)$ is of order p if and only if $I^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-1$, and $I^{(p)}(\alpha) \neq 0$. Moreover, the asymptotic constant is given by*

$$K_p(\alpha; I) = \frac{I^{(p)}(\alpha)}{p!}.$$

This result says that the Taylor's expansion of an IF $I(z)$ of order p is

$$I(z) = \alpha + w_{I,p}(z)(z-\alpha)^p = \alpha + \mathcal{O}\left(f^p(z)\right) \quad \text{with} \quad w_{I,p}(\alpha) = \frac{I^{(p)}(\alpha)}{p!},$$

and we also have

$$I'(z) = w_{I',p-1}(z)(z-\alpha)^{p-1} = \mathcal{O}\left(f^{p-1}(z)\right) \quad \text{with} \quad w_{I',p-1}(\alpha) = \frac{I^{(p)}(\alpha)}{(p-1)!}.$$

Remark 4.4.1. *Two IFs $I_1(z)$ and $I_2(z)$ which are of the same order p, to compute α, differ only at most by $\mathcal{O}\left(f^p(z)\right)$ term, which means that $I_1(z) - I_2(z) = \mathcal{O}\left(f^{\tilde{p}}(z)\right)$ where $\tilde{p} \geq p$. Moreover if $I_1(z)$ is an IF of order p and if $I_2(z) = I_1(z) + \mathcal{O}\left(f^p(z)\right)$, than $I_2(z)$ is an IF of order at least p.*

Remark 4.4.2. *To determine an IF of order 2 to find α we can proceed as follows. Firstly, let us modify $f(z)$ by setting $\widetilde{f}(z) = h(z)f(z)$ where $h(z)$ is a regular function to be determined such that $\widetilde{f}'(\alpha) = 1$, that is to say $h(\alpha)f'(\alpha) = 1$ or $h(\alpha) = 1/f'(\alpha)$. Secondly, let us set $I(z) = z - \widetilde{f}(z)$. Since $g(\alpha) = \alpha$ and $g'(\alpha) = 0$, we can conclude that the IF $I(z)$ is at least of order 2. In fact we can take any function $h(z)$ such that $h(\alpha) = 1/f'(\alpha)$. In particular, for $h(z) = 1/f'(z)$ we get the Newton's IF for which $K_2(\alpha; N_f) = \frac{1}{2}N_f''(\alpha) = \frac{1}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]$. There exist several other choices for $h(z)$. For example, let us consider*

$$h(z) = \frac{1}{f'(z)\left[1 + \lambda(z)\frac{f(z)}{f'(z)}\right]} \quad \textit{then} \quad I(z) = z - \left[1 + \lambda(z)\frac{f(z)}{f'(z)}\right]^{-1}\frac{f(z)}{f'(z)},$$

or

$$h(z) = \frac{1}{f'(z)}\left[1 - \lambda(z)\frac{f(z)}{f'(z)}\right] \quad \textit{then} \quad I(z) = z - \left[1 - \lambda(z)\frac{f(z)}{f'(z)}\right]\frac{f(z)}{f'(z)},$$

where $\lambda(z)$ is an arbitrary function. Those two IFs are in general of order 2, like the Newton's IF, because $I''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} + 2\lambda(\alpha) = N_f''(\alpha) + 2\lambda(\alpha)$ which means that $K_2(\alpha; I) = K_2(\alpha; N_f) + \lambda(\alpha)$. They will be of order higher that 2 only for functions such that $\lambda(\alpha) = -\frac{f''(\alpha)}{2f'(\alpha)}$. Finally if we take $\lambda(z) = -\frac{f''(z)}{2f'(z)}$ we get respectively Halley's method and Euler-Chebyshev's method.

4.4.2 Necessary and sufficient conditions for Newton's method

Let us now focus on Newton's IF

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

One can observe that $N_f(\alpha) = \alpha$, and

$$N_f'(z) = \frac{f''(z)f(z)}{f'(z)^2},$$

so that $N_f'(\alpha) = 0$. Furthermore we also have

$$N_f''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}.$$

If α is a simple root of $f(z)$, then Newton's IF is of order at least 2, and it could be of order 3, if $f''(\alpha) = 0$, according to the above theorem. The necessity and the sufficiency of the condition for high-order convergence of Newton's method are presented in the next result.

Theorem 4.4.2. *[11, 12] Let $p \geq 2$ and let $f(z)$ be an analytic function such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. The Newton's iteration function $N_f(z) = z - \frac{f(z)}{f'(z)}$ is of order p if and only if $f^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $f^{(p)}(\alpha) \neq 0$. Moreover,*

$$N_f^{(p)}(\alpha) = (p-1)\frac{f^{(p)}(\alpha)}{f'(\alpha)},$$

and the asymptotic constant is

$$K_p(\alpha; N_f) = \frac{N_f^{(p)}(\alpha)}{p!} = \frac{(p-1)}{p!}\frac{f^{(p)}(\alpha)}{f'(\alpha)}.$$

This result says however that the Taylor's expansion of $N_f(z)$ of order p is

$$N_f(z) = \alpha + w_{N_f,p}(z)(z-\alpha)^p = \alpha + \mathcal{O}\left(f^p(z)\right) \quad \text{with} \quad w_{N_f,p}(\alpha) = \frac{N_f^{(p)}(\alpha)}{p!}.$$

For the function $f(z)$ we have

$$f(z) = f'(\alpha)(z-\alpha) + w_{f,p}(z)(z-\alpha)^p = f'(\alpha)(z-\alpha) + \mathcal{O}\left(f^p(z)\right)$$

with

$$w_{f,p}(\alpha) = \frac{f^{(p)}(\alpha)}{p!},$$

and

$$f''(z) = w_{f'',p-2}(z)(z-\alpha)^{p-2} = \mathcal{O}\left(f^{p-2}(z)\right) \quad \text{with} \quad w_{f'',p-2}(\alpha) = \frac{f^{(p)}(\alpha)}{(p-2)!}.$$

Moreover, if $N_f(z)$ is of order p, and $\widetilde{f}(z) = f(z) + \mathcal{O}\left(f^p(z)\right)$, then $N_{\widetilde{f}}(z)$ is also at least of order p.

Remark 4.4.3. *Let us observe that if $I(x)$ is an IF of order p, if we set $F(z) = z - I(z)$ we have $F(\alpha) = 0$, $F'(\alpha) = 1$, $F^{(j)}(\alpha) = -I^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $F^{(p)}(\alpha) = -I^{(p)}(\alpha) \neq 0$, so $N_F(z)$ is also of order p. But since we have $K_p(\alpha; N_F) = -(p-1)K_p(\alpha; I)$ $N_F(z)$ is slower that $I(z)$ for any $p > 2$. Conversely, suppose $f(z)$ is such that*

$$f(z) = f'(\alpha)(z-\alpha) + w_{f,p}(z)(z-\alpha)^p \qquad w_{f,p}\alpha) = \frac{f^{(p)}(\alpha)}{p!},$$

then $N_f(z)$ is of order p, and $K_p(\alpha; N_f) = -\frac{(p-1)}{p!}\frac{f^{(p)}(\alpha)}{f'(\alpha)}$. Suppose we set $\widetilde{f}(z) = f(z)/f'(z)$, then $\widetilde{f}'(\alpha) = 1$. So the IF defined by $I(z) = z - \widetilde{f}(z)$ is nothing but $N_f(z)$, then is of order p with $K_p(\alpha; I) = K_p(\alpha; N_f)$.

4.5 Acceleration of fixed point method

In this section we consider different ways to increase the order of convergence of fixed point methods.

4.5.1 Acceleration based on Theorem 4.4.1

The first result is a direct application of Theorem 4.4.1.

Theorem 4.5.1. *[11, 12] Let $p \geq 2$, let $S_p(z)$ be defined by*

$$S_p(z) = \sum_{j=0}^{p-1} c_j(z) f^j(z),$$

where the $c_j(z)$'s are defined by: $c_0(z)$ is analytic such that $c_0(\alpha) = \alpha$ (observe that we can always take the case $c_0(z) = z$), and

$$jf'(z)c_j(z) + c'_{j-1}(z) = 0$$

for $j = 1, 2, \ldots, p$. Then $S_p(z)$ is at least of order p. Moreover

$$S_p^{(p)}(\alpha) = (p-1)c'_{p-1}(\alpha)\left(f'(\alpha)\right)^{p-1} = -p!c_p(\alpha)\left(f'(\alpha)\right)^p,$$

and its asymptotic constant is

$$K_p(\alpha, S_p) = -c_p(\alpha)\left[f'(\alpha)\right]^p.$$

Proof. The $c_j(z)$'s are defined in such a way that

$$S'_p(z) = c'_{p-1}(z)f^{p-1}(z).$$

Then the result follows. □

Remark 4.5.1. *We observe that*

$$c_j(z) = \left(-\frac{1}{jf'(z)}\frac{d}{dz}\right)c_{j-1}(z),$$

and recursively we get

$$c_j(z) = \frac{(-1)^j}{j!}\left(\frac{1}{f'(z)}\frac{d}{dz}\right)^j c_0(z).$$

So we have

$$S_p(z) = \sum_{j=0}^{p-1}(-1)^j\frac{f^j(z)}{j!}\left(\frac{1}{f'(z)}\frac{d}{dz}\right)^j c_0(z).$$

It is also possible to get this result using the Taylor's expansion if the inverse of $f(z)$ [51, 13].

Remark 4.5.2. *If $c_0(z) = z$ in this result, we get the Schröder's IF of the first kind [48, 51].*

Remark 4.5.3. *If $c_0(z) = z$, we have $S_2(z) = N_f(z)$, and for $p \geq 3$ we can write*

$$S_p(z) = z - \frac{f(z)}{f'(z)}G_{p-1}(z) \quad \textit{with} \quad G_{p-1}(z) = 1 - \sum_{j=2}^{p-1}c_j(z)f'(z)f^{j-1}(z).$$

Remark 4.5.4. *For $c_0(z) = z$ and $p = 3$, we have $c_2(z) = -\frac{f''(z)}{2f'(z)^3}$, and*

$$S_3(z) = z - \frac{f(z)}{f'(z)}G_2(z) \qquad \textit{with} \qquad G_2(z) = 1 + \frac{f''(z)f(z)}{2\left(f'(z)\right)^2},$$

which is the Euler-Chebyshev's IF of order 3. Moreover, this result, combined to Remark 4.4.1, suggests that for any function $h(z)$ whose expansion around 0 is

$$h(z) = 1 + z + \mathcal{O}\left(z^2\right),$$

we can use

$$L_f(z) = \frac{f''(z)f(z)}{2\left(f'(z)\right)^2}$$

to define

$$\widetilde{G}_2(z) = h(L_f(z)) = G_2(z) + \mathcal{O}\left(f^2(z)\right).$$

Then, for

$$\widetilde{S}_3(z) = z - \frac{f(z)}{f'(z)}\widetilde{G}_2(z) = z - \frac{f(z)}{f'(z)}\left[G_2(z) + \mathcal{O}\left(f^2(z)\right)\right] = S_3(z) + \mathcal{O}\left(f^3(z)\right),$$

which means that $\widetilde{S}_3(z)$ is also of order 3. Some examples of such $h(z)$ functions again are $1+\arctan(z)$, $2-e^{-z}$, $1-\ln(1-z)$, $e^{-z}+2\sin(z)$, $\frac{1}{1-z}$, $\cos(z)+\sin(z)$. For example

$$S_3(z) = z - \frac{f(z)}{f'(z)}\Big[\cos\left(\frac{f''(z)f(z)}{2f'(z)^2}\right) + \sin\left(\frac{f''(z)f(z)}{2f'(z)^2}\right)\Big]$$

will be of order 3. Other examples are mentioned in [18].

Remark 4.5.5. *Suppose that $N_f(z)$ is of order k, hence $f''(z) = \mathcal{O}\left(f^{k-2}(z)\right)$. We have $c_0(z) = z$, $c_1(z) = -1/f'(z)$. Moreover*

$$c_2(z) = -\frac{1}{2f'(z)}c_1'(z) = -\frac{f''(z)}{2\left(f'(z)\right)^3} = \mathcal{O}\left(f^{k-2}(z)\right).$$

By induction, we show that

$$c_l(z) = -\frac{1}{lf'(z)}c_{l-1}'(z) = \mathcal{O}\left(f^{k-l}(z)\right)$$

for $l = 2, \dots, k$. Then each term $c_l(z) f^l(z) = \mathcal{O}\left(f^k(z)\right)$. It follows that for $p = 3, \dots, k$

$$I_p(z) = N_f(z) + \sum_{l=2}^{p-1} c_l(z) f^l(z) = N_f(z) + \mathcal{O}\left(f^k(z)\right).$$

Hence $I_p(z)$ and $N_f(z)$ are at least of the same order k. It follows that $I_p(z)$ could not be of lower order then k from this construction. We cannot loose order of convergence by applying this process.

4.5.2 Acceleration based on Taylor's expansion

Based on the expression of the asymptotic constant given in Theorem 4.4.1, the main idea to increase the order of convergence of a given IF of order p is by adding a term to cancel out the pth term of its Taylor's expansion.

Theorem 4.5.2. *Let $I_p(z)$ be an iteration function of order $p \geq 1$. Then $I_{p+1}(z)$ is an iteration function of order $p+1$ if and only if there exists an analytic function $\Delta I_{p,p+1}(z)$ such that*

$$I_{p+1}(z) = I_p(z) + \Delta I_{p,p+1}(z)$$

with $\Delta I^{(j)}_{p,p+1}(\alpha) = 0$ for $j = 0, \dots, p-1$, and $\Delta I^{(p)}_{p,p+1}(\alpha) = -I^{(p)}_p(\alpha)$.

Proof. Since we must have

$$I^{(j)}_{p+1}(\alpha) = I^{(j)}_p(\alpha) + \Delta I^{(j)}_{p,p+1}(\alpha)$$

for $j = 0, \dots, p$, the result follows. □

Remark 4.5.6. *We could also try to modify $I_p(z)$ by multiplying it by a function $H_p(z)$. It turns out that we must have $H_p(\alpha) = 1$, so we can rewrite $H_p(z)$ as*

$$H_p(z) = 1 + \widetilde{H}_p(z)$$

to obtain

$$I_{p+1}(z) = I_p(z)H_p(z) = I_p(z) + I_p(z)\widetilde{H}_p(z) = I_p(z) + \Delta I_{p,p+1}(z)$$

as we already did in the preceding theorem.

For $\Delta_{p,p+1}(z)$ of the preceding result we can write

$$\Delta I_{p,p+1}(z) = w_{\Delta I_{p,p+1},p}(z)(z-\alpha)^p$$

with

$$w_{\Delta I_{p,p+1},p}(\alpha) = \frac{\Delta I_{p,p+1}^{(p)}(\alpha)}{p!} = -\frac{I_p^{(p)}(\alpha)}{p!}.$$

As suggested in Section 4.2.2, we can rewrite this expression under different forms to get the next result.

Theorem 4.5.3. *Let α a simple root of an analytic function $f(z)$ and $p \geq 1$. $\Delta I_{p,p+1}(z)$ is an analytic function such that $\Delta I_{p,p+1}^{(j)}(\alpha) = 0$ for $j = 0, \ldots, p-1$, if and only if there exists an analytic function $H_p(z)$ such that*

$$\Delta I_{p,p+1}(z) = H_p(z) \left(\frac{f(z)}{f'(z)}\right)^p.$$

Moreover

$$H_p(\alpha) = \frac{\Delta I_{p,p+1}^{(p)}(\alpha)}{p!}.$$

The first general result to increase the order of convergence of an IF can now be stated as follows.

Theorem 4.5.4. *Let $I_p(z)$ be an iteration function of order $p \geq 1$ for computing the simple root of an analytic function $f(z)$. Then $I_{p+1}(z)$ is an iteration function of order $p+1$ if and only if there exists an analytic function $H_p(z)$ such that*

$$I_{p+1}(z) = I_p(z) + H_p(z) \left(\frac{f(z)}{f'(z)}\right)^p$$

with

$$H_p(\alpha) = -\frac{I_p^{(p)}(\alpha)}{p!}.$$

Example 4.5.1. *For the Newton's IF* $N_f(z) = z - \frac{f(z)}{f'(z)}$ *of order* 2*, we observe that* $N_f''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}$*. This allows us to conclude that for any function* $H_2(z)$ *such that*

$$H_2(\alpha) = -\frac{f''(\alpha)}{2f'(\alpha)}$$

we get $N_f(z) + H_2(z)\left(\frac{f(z)}{f'(z)}\right)^2$*, which is an IF of order* 3*. Particularly for* $H_2(z) = -\frac{f''(z)}{2f'(z)}$*, we have*

$$z - \frac{f(z)}{f'(z)} - \frac{f''(z)f(z)^2}{2f'(z)^3} = C_f(z),$$

which is the Euler-Chebyshev's IF [3] of order 3*.*

We now present some general ways to apply the idea presented above. They consist of adding appropriate terms to cancel out the pth term of the Taylor's expansion of the IF. The next three theorems provided us with ways to recursively create new IFs that increase the order of convergence from p to $p+1$.

Theorem 4.5.5. *Let* $I_p(z)$ *be an iteration function of order* $p \geq 1$*. Then*

$$I_{p+1}(z) = I_p(z) - \frac{(p-q)!}{p!} I_p^{(q)}(z) \left(\frac{f(z)}{f'(z)}\right)^q$$

is an iteration function at least of order $p+1$ *for any* $q = 1, \ldots, p$*.*

Proof. We have

$$\begin{aligned} H_p(z) &= -\frac{(p-q)!}{p!} I_p^{(q)}(z) \left(\frac{f'(z)}{f(z)}\right)^{p-q} \\ &= -\frac{(p-q)!}{p!} w_{I_p^{(q)}, p-q}(z) \left(\frac{f'(z)}{w_{f,1}(z)}\right)^{p-q}, \end{aligned}$$

so

$$H_p(\alpha) = -\frac{I_p^{(p)}(\alpha)}{p!},$$

and $I_{p+1}(z)$ is of order $p+1$. □

Theorem 4.5.6. *Let $I_p(z)$ be an iteration function of order $p \geq 2$. Then*

$$I_{p+1}(z) = I_p(z) - \frac{(p-q)!}{p!} I_p^{(q)}(z) \left[z - I_p(z)\right]^q$$

is iteration function of at least order $p+1$ for any $q = 1, \ldots, p$.

Proof. We have

$$\begin{aligned} H_p(z) &= -\frac{(p-q)!}{p!} I_p^{(q)}(z) \left[z - I_p(z)\right]^q \left(\frac{f'(z)}{f(z)}\right)^p \\ &= -\frac{(p-q)!}{p!} w_{I_p^{(q)},p-q}(z) \left[1 - w_{I_p,p}(z)\,(z-\alpha)^{p-1}\right]^q \left(\frac{f(z)}{w_{f,1}(z)}\right)^p. \end{aligned}$$

So

$$H_p(\alpha) = -\frac{I_p^{(p)}(\alpha)}{p!},$$

and $I_{p+1}(z)$ is of order $p+1$. □

Theorem 4.5.7. *[50] Let $I_p(z)$ be an iteration function of order $p \geq 2$. If $\widetilde{H}_p(z)$ is a regular function such that $\widetilde{H}_p(\alpha) = 0$ and $\widetilde{H}_p'(\alpha) = 1$, then the iteration function*

$$I_{p+1}(z) = I_p(z) - \frac{1}{p} I_p'(z) \widetilde{H}_p(z),$$

is at least of order $p+1$.

Proof. We have

$$\begin{aligned} H_p(z) &= -\frac{1}{p} I_p'(z) \widetilde{H}_p(z) \left(\frac{f'(z)}{f(z)}\right)^p \\ &= -\frac{1}{p!} w_{I_p',p-1}(z) w_{\widetilde{H}_p,1}(z) \left(\frac{f(z)}{w_{f,1}(z)}\right)^p. \end{aligned}$$

where $w_{\widetilde{H}_p,1}(\alpha) = \widetilde{H}_p'(\alpha) = 1$. So

$$H_p(\alpha) = -\frac{I_p^{(p)}(\alpha)}{p!},$$

and $I_{p+1}(z)$ is of order $p+1$. □

Remark 4.5.7. *let $\lambda(z)$ be any regular function, and let us consider*

$$\widetilde{H}_p(z) = \left[1 - \lambda(z) I_p'(z)\right]^{-1} (z - I_p(z)),$$

which is such that $\widetilde{H}_p(\alpha) = 0$ and $\widetilde{H}_p'(\alpha) = 1$ because $I_p'(\alpha) = 0$. Then the IF given by

$$I_{p+1}(z) = I_p(z) - \frac{1}{p} I_p'(z) \left[1 - \lambda(z) I_p'(z)\right]^{-1} (z - I_p(z)),$$

is at least order $p+1$ for any $\lambda(z)$. For $\lambda(z) = 1/p$ we get

$$I_{p+1}(z) = z - \left[1 - \frac{1}{p} I_p'(z)\right]^{-1} (z - I_p(z)),$$

which was presented in [31]. For $\lambda(z) = 0$, we get

$$I_{p+1}(z) = I_p(z) - \frac{1}{p} I_p'(z) (z - I_p(z)),$$

which was presented in [40].

4.5.3 Acceleration of Newton's method as a fixed point method.

Since $N_f(z)$ is an IF of order at least 2, we can rewrite the results of the preceding section in terms of $N_f(z)$. Let us apply Theorem 4.5.4 to the iteration function $N_f(z)$ of order $p \geq 2$. In this case for $N_f(z)$, we have

$$N_f^{(p)}(\alpha) = (p-1) \frac{f^{(p)}(\alpha)}{f'(\alpha)},$$

and its Taylor's expansion is

$$N_f(z) = \alpha + w_{N_f,p}(z)(z-\alpha)^p \quad \text{with} \quad w_{N_f,p}(\alpha) = \frac{N_f^{(p)}(\alpha)}{p!} = \frac{(p-1)}{p!} \frac{f^{(p)}(\alpha)}{f'(\alpha)}.$$

Again we add an appropriate $\mathcal{O}(f^p(z))$ term to cancel out the pth term of the corresponding Taylor's expansion. Theorems 4.5.4 and 4.5.5 are now restated as follows.

Theorem 4.5.8. *Let $N_f(z)$ be of order $p \geq 2$ for computing the simple root of an analytic function $f(z)$. Then $\widetilde{N}_f(z)$ is of order $p+1$ if and only if there exists an analytic function $H_p(z)$ such that*

$$\widetilde{N}_f(z) = N_f(z) + H_p(z)\left(\frac{f(z)}{f'(z)}\right)^p$$

with

$$H_p(\alpha) = -\frac{N_f^{(p)}(\alpha)}{p!} = -(p-1)\frac{f^{(p)}(\alpha)}{p!f'(\alpha)}.$$

Theorem 4.5.9. *If $N_f(z)$ is of order $p \geq 2$, then*

$$\widetilde{N}_f(z) = N_f(z) - \frac{(p-q)!}{p!}N_f^{(q)}(z)\left(\frac{f(z)}{f'(z)}\right)^q$$

is of order $p+1$ for $q = 1, \ldots, p$.

We also have the following result.

Theorem 4.5.10. *If $N_f(z)$ is of order $p \geq 2$, then*

$$\widetilde{N}_f(z) = N_f(z) - (p-1)\frac{(p-q)!}{p!}\frac{f^{(q)}(z)}{f'(z)}\left(\frac{f(z)}{f'(z)}\right)^q$$

is of order $p+1$ for $q = 2, \ldots, p$.

Proof. We have

$$\begin{aligned} H_p(z) &= -(p-1)\frac{(p-q)!}{p!}\frac{f^{(q)}(z)}{f'(z)}\left(\frac{f'(z)}{f(z)}\right)^{p-q} \\ &= -(p-1)\frac{(p-q)!}{p!}\frac{w_{f^{(q)},p-q}(z)}{f'(z)}\left(\frac{f'(z)}{w_{f,1}(z)}\right)^{p-q}. \end{aligned}$$

So

$$H_p(\alpha) = -(p-1)\frac{f^{(p)}(\alpha)}{p!f'(\alpha)},$$

and $\widetilde{N}_f(z)$ is of order $p+1$. $\square$

Remark 4.5.8. *Observe that for $q = 2$, if $p = 2$ we retrieve Euler-Chebyshev's iterative method, but if $p \geq 3$ we have*

$$\widetilde{N}_f(z) = N_f(z) - \frac{f''(z)}{pf'(z)} \left(\frac{f(z)}{f'(z)} \right)^2.$$

One thing that this illustrates is that, if Newton's method is of order higher than 2, Euler-Chebyshev would no longer be an acceleration process. Modifications will need to be made according.

Finally, Theorems 4.5.6 and 4.5.7 can be rewritten also as follow.

Theorem 4.5.11. *Let $N_f(z)$ be of order $p \geq 2$. Then*

$$\widetilde{N}_f(z) = N_f(z) - \frac{(p-q)!}{p!} N_f^{(q)}(z) \left[z - N_f(z) \right]^q$$

is iteration function of at least order $p+1$ for any $q = 1, \ldots, p$.

Remark 4.5.9. *For the case $p = 2$, $q = 1$ and $N_f(z)$ Newton's method, we have Traub's difference-differential relation [51]*

Theorem 4.5.12. *Let $N_f(z)$ be of order $p \geq 2$. If $\widetilde{H}_p(z)$ is a regular function such that $\widetilde{H}_p(\alpha) = 0$ and $H_p'(\alpha) = 1$, the iteration function*

$$\widetilde{N}_f(z) = I_p(z) - \frac{f(z)f''(z)}{p\left(f'(z)\right)^2} \widetilde{H}_p(z),$$

is at least of order $p+1$.

4.6 Acceleration based on Theorem 4.4.2

In this section we consider different ways to increase the order of convergence of Newton's method when applied to a new function, by modifying the function $f(z)$.

4.6.1 A direct approach

We will now apply Theorem 4.4.2 to study the modifications that can be made on the original function $f(z)$ in order to improve convergence of Newton's method. In doing so, we are able to recover some famous acceleration processes.

For $N_f(z)$ of order $p \geq 2$, let us look at a general modified function $F_{p+1}(z) = f(z)g_p(z)$ for which Newton's method will be of order $p+1$ to compute α. We need that $F_{p+1}(\alpha) = 0$, $F'_{p+1}(\alpha) \neq 0$, and the function $F^{(j)}_{p+1}(z) = 0$ not only for $j = 2, \ldots, p-1$, but also for $j = p$, so $F^{(p)}_{p+1}(\alpha) = 0$.

Theorem 4.6.1. *Let $N_f(z)$ be of order $p \geq 2$, and let $F_{p+1}(z) = f(z)g_p(z)$. $N_{F_{p+1}}(z)$ is of order $p+1$ if and only if $g_p(\alpha) \neq 0$, $g_p^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-2$, and*

$$g_p^{(p-1)}(\alpha) = -\frac{f^{(p)}(\alpha)}{pf'(\alpha)}g_p(\alpha).$$

Proof. $N_{F_{p+1}}(z)$ is of order $p+1$ if and only if $F_{p+1}(\alpha) = 0$, $F'_{p+1}(\alpha) \neq 0$, and $F^{(j)}_{p+1}(\alpha) = 0$ for $j = 2, \ldots, p$. For $F_{p+1}(z) = f(z)g_p(z)$ we have $F_{p+1}(\alpha) = 0$. Also

$$F'_{p+1}(z) = f'(z)g_p(z) + f(z)g'_p(z)$$

so $F'_{p+1}(\alpha) = f'(\alpha)g_p(\alpha) \neq 0$ if and only if $g_p(\alpha) \neq 0$. For $p = 2$ we have

$$F''_3(z) = f''(z)g_2(z) + 2f'(z)g'_2(z) + f(z)g''_2(z)$$

so

$$F''_3(\alpha) = f''(\alpha)g_2(\alpha) + 2f'(\alpha)g'_2(\alpha),$$

hence $F''_3(\alpha) = 0$ if and only if $g'_2(\alpha) = -\frac{f''(\alpha)}{2f'(\alpha)}g_2(\alpha)$. For $p \geq 3$, we have $f''(\alpha) = 0$ so $g'_p(\alpha) = 0$. Suppose that $g_p^{(j)}(\alpha) = 0$ for $j = 1, \ldots, l-2$ and $l \geq 3$. Then

$$F^{(l)}_{p+1}(z) = f^{(l)}(z)g_p(z) + \sum_{j=1}^{l-2} \begin{pmatrix} l \\ j \end{pmatrix} f^{(l-j)}(z)g_p^{(j)}(z) + lf'(z)g_p^{(l-1)}(z) + f(z)g_p^{(l)}(z)$$

and

$$F_{p+1}^{(l)}(\alpha) = f^{(l)}(\alpha)g_p(\alpha) + l f'(\alpha) g_p^{(l-1)}(\alpha).$$

Hence $F_{p+1}^{(l)}(\alpha) = 0$ if and only if $g_p^{(l-1)}(\alpha) = -\frac{f^{(l)}(\alpha)}{l f'(\alpha)} g_p(\alpha)$. Since $f^{(l)}(\alpha) = 0$ for $l = 2, \ldots, p-1$, we have $g_p^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-2$. Also $F_{p+1}^{(p)}(\alpha) = 0$ if and only if $g_p^{(p-1)}(\alpha) = -\frac{f^{(p)}(\alpha)}{p f'(\alpha)} g_p(\alpha)$. □

This first general result, which will be used to increase the order of convergence of Newton's method $N_f(z)$ by modifying the function $f(z)$, can thus be restated as follows.

Theorem 4.6.2. *Let $N_f(z)$ be of order $p \geq 2$ for computing the simple root of an analytic function $f(z)$. Then for any function $F_{p+1}(z) = f(z)g_p(z)$, $N_{F_{p+1}}(z)$ is of order $p+1$ if and only if one of the following equivalent expressions holds :*
(1) there exists an analytic function $w_{g_p,p-1}(z)$ such that

$$g_p(z) = g_p(\alpha) + w_{g_p,p-1}(z)(z-\alpha)^{p-1},$$

with

$$w_{g_p,p-1}(\alpha) = \frac{g_p^{(p-1)}(\alpha)}{(p-1)!} = -\frac{f^{(p)}(\alpha)}{p! f'(\alpha)} g_p(\alpha);$$

(2) there exists an analytic function $W_{g_p,p-1}(z)$ such that

$$g_p(z) = g_p(\alpha) + W_{g_p,p-1}(z) \left(\frac{f(z)}{f'(z)} \right)^{p-1},$$

with

$$W_{g_p,p-1}(z) = w_{g_p,p-1}(z) \left(\frac{f'(z)}{w_{f,1}(z)} \right)^{p-1},$$

and

$$W_{g_p,p-1}(\alpha) = \frac{g_p^{(p-1)}(\alpha)}{(p-1)!} = -\frac{f^{(p)}(\alpha)}{p! f'(\alpha)} g_p(\alpha);$$

(3) there exists an analytic function $\widetilde{W}_{g_p,p-1}(z)$ such that

$$g_p(z) = g_p(\alpha) + \widetilde{W}_{g_p,p-1}(z) f^{p-1}(z),$$

with

$$\widetilde{W}_{g_p,p-1}(z) = \frac{w_{g_p,p-1}(z)}{w_{f,1}^{p-1}(z)},$$

and

$$\widetilde{W}_{g_p,p-1}(\alpha) = \frac{g_p^{(p-1)}(\alpha)}{(p-1)!\,(f'(\alpha))^{p-1}} = -\frac{f^{(p)}(\alpha)}{p!\,(f'(\alpha))^p} g_p(\alpha).$$

Remark 4.6.1. *We could also try to add a term $\delta f(z)$ to $f(z)$ in order to cancel the pth derivative of $f(z)$. So, let us consider*

$$F(z) = f(z) + \delta f(z)$$

such that $\delta f(z)$ is analytic and

$$\delta f(z) = w_{\delta f,p}(z)(z-\alpha)^p$$

with

$$w_{\delta f,p}(\alpha) = -\frac{f^{(p)}(\alpha)}{p!}.$$

Hence we would cancel out the pth derivative of $f(z)$, and we would have $F(\alpha) = 0$, $F'(\alpha) = f(\alpha)$, and $F^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p$. We can always write this expression as a product because

$$F(z) = f(z) + \delta f(z) = f(z)\left[1 + \frac{\delta f(z)}{f(z)}\right] = f(z)\,[1 + \widetilde{g}(z)] = f(z)g(z),$$

where $g(z) = 1 + \widetilde{g}(z)$, and

$$\widetilde{g}(z) = \frac{\delta f(z)}{f(z)} = \frac{w_{\delta f,p}(z)}{w_{f,1}(z)}(z-\alpha)^{p-1},$$

with

$$\widetilde{g}^{(p-1)}(\alpha) = (p-1)!\frac{w_{\delta f,p}(\alpha)}{w_{f,1}(\alpha)} = -\frac{f^{(p)}(\alpha)}{pf'(\alpha)}.$$

The first example we present will provide us with a recursive way to progressively increase the order of convergence of Newton's method to arbitrary values. It coincides with

Halley's method at the order 3. It was first presented by Gerlach [19] in 1994, although it turned out, in [44, 45, 14], to be a rediscovery of the Schröder's method of the second kind dating from 1870 [48]. We present a new proof of this result.

Theorem 4.6.3. *Let $N_f(z)$ be of order $p \geq 2$. Then for*

$$F_{p+1}(z) = f(z)g_p(z) \quad \textit{with} \quad g_p(z) = \frac{1}{\sqrt[p]{f'(z)}},$$

$N_{F_{p+1}}(z)$ is of order $p+1$.

Proof. For $g_p(z) = \frac{1}{\sqrt[p]{f'(z)}}$ and $l \geq 1$ we have

$$g_p^{(l)}(z) = \frac{\Gamma\left(f', f'', \ldots, f^{(l)}; z\right)}{(f'(z))^{\frac{1}{p}+l}} - \frac{f^{(l+1)}(z)}{p\,(f'(z))^{\frac{1}{p}+1}}.$$

Since $f^{(l)}(\alpha) = 0$ for $l = 2, \ldots, p-1$, it follows that

$$\Gamma\left(f', f'', \ldots, f^{(l)}; \alpha\right) = 0 \quad \text{for} \quad l = 1, \ldots, p-1,$$

and

$$g_p^{(l)}(\alpha) = 0 \quad \text{for} \quad l = 1, \ldots, p-2.$$

Finally

$$g_p^{(p-1)}(\alpha) = -\frac{f^{(p)}(\alpha)}{p\,(f'(\alpha))^{\frac{1}{p}+1}},$$

so

$$g_p^{(p-1)}(\alpha) = -\frac{f^{(p)}(\alpha)}{pf'(\alpha)} g_p(\alpha),$$

and $N_{F_{p+1}}(z)$ is of order $p+1$ for $F_{p+1}(z) = f(z)g_p(z)$. $\square$

Remark 4.6.2. *Let us point out that $g_p(z) = \frac{1}{\sqrt[p]{f'(z)}}$ satisfies the differential equation*

$$\frac{g_p^{(p-1)}(z)}{g_p(z)} = -\frac{f^{(p)}(z)}{pf'(z)}$$

only for $z = \alpha$, while it is a solution of the differential equation

$$\frac{g_p'(z)}{g_p(z)} = -\frac{f''(z)}{pf'(z)}$$

around α [19, 14].

The next result shows that the order of convergence can also be increased using higher order derivatives of $f(z)$.

Theorem 4.6.4. *Let $N_f(z)$ be of order $p \geq 2$. Set*

$$\begin{aligned} F_{p+1}(z) &= f(z) - \frac{(p-q)!}{p!} f^{(q)}(z) \left(\frac{f(z)}{f'(z)}\right)^q \\ &= f(z) \left[1 - \frac{(p-q)!}{p!} \frac{f^{(q)}(z)}{f'(z)} \left(\frac{f(z)}{f'(z)}\right)^{q-1}\right], \end{aligned}$$

then $N_{F_{p+1}}(z)$ is of order $p+1$ for $q = 2, \ldots, p$.

Proof. The added term is an appropriate expression because we have

$$\begin{aligned} g_p(z) &= 1 - \frac{(p-q)!}{p!} \frac{f^{(q)}(z)}{f'(z)} \left(\frac{f(z)}{f'(z)}\right)^{q-1} \\ &= 1 - \frac{(p-q)!}{p!} \frac{w_{f^{(q)},p-q}(z)\,(w_{f,1}(z))^{q-1}}{(f'(z))^q} (z-\alpha)^{p-1}. \end{aligned}$$

Then, $g_p(\alpha) = 1$, $g_p^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-2$, and

$$g_p^{(p-1)}(\alpha) = -\frac{(p-q)!}{p} \frac{w_{f^{(q)},p-q}(\alpha)\,(w_{f,1}(\alpha))^{q-1}}{(f'(\alpha))^q} = -\frac{f^{(p)}(\alpha)}{pf'(\alpha)}.$$

□

4.6.2 An indirect approach

In the preceding section, we produced a process of increasing the order of convergence recursively from p to $p+1$. In the next theorem we present a way of finding an iteration function $F_p(z)$, or equivalently $g_{p-1}(z)$ such that $F_p(z) = f(z)g_{p-1}(z)$, that directly has order of convergence equal to p.

Theorem 4.6.5. *[11, 12] Let $p \geq 2$, and $F_p(z)$ be defined by*

$$F_p(z) = \sum_{j=0}^{p-1} b_j(z) f^j(z),$$

where the $b_j(z)$'s are such that: $b_0(z)$ and $b_1(z)$ are two analytic functions such that

$$\begin{cases} b_0(\alpha) &= 0, \\ b_0'(\alpha) + b_1(\alpha)f'(\alpha) &\neq 0, \end{cases}$$

If we set

$$b_j(z) = -\frac{b_{j-2}''(z) + 2(j-1)b_{j-1}'(z)f'(z) + (j-1)b_{j-1}(z)f''(z)}{j(j-1)f'(z)^2}$$

for $j \geq 2$, then $N_{F_p}(z)$ will be of order p. Moreover

$$F_p^{(p)}(\alpha) = -p!b_p(\alpha)\left(f'(\alpha)\right)^p,$$

and

$$K_p(\alpha; N_{F_p}) = -\frac{(p-1)b_p(\alpha)\left(f'(\alpha)\right)^p}{b_0'(\alpha) + b_1(\alpha)f'(\alpha)}.$$

Note that $b_0(z) = 0$ and any function $b_1(z)$ such that $b_1(\alpha) \neq 0$ are a trivial choices.

Example 4.6.1. *For $b_0(z) = 0$ and $b_1(z) = 1$, we get $N_{F_2}(z) = N_f(z)$, and we have $b_2(z) = -\frac{f''(z)}{2f'(z)^2}$, so $F_3(z) = f(z) + b_2(z)f(z)^2 = f(z)g_2(z)$, where*

$$g_2(z) = 1 - \frac{f''(z)f(z)}{2\left(f'(z)\right)^2},$$

and $N_{F_3}(z)$ is of order 3. First observe that the above function $g_2(z)$ is a particular case of Theorem 4.6.4 for $p = q = 2$. Also since we can add to $F_3(z)$ any function $\mathcal{O}(f^3(z))$ and the order of convergence will be conserved, Newton's method applied to any function $\widetilde{F}_3(z)$ defined by

$$\widetilde{F}_3(z) = F_3(z) + \mathcal{O}(f^3(z)) = f(z)\Big[1 - \frac{f''(z)f(z)}{2f'(z)^2} + \mathcal{O}(f^2(z))\Big],$$

will be of order 3*. So, as we observed in Example 4.5.4, for any functions* $h(z)$ *such that*

$$h(z) = 1 - z + \mathcal{O}(z^2),$$

if we use

$$L(z) = \frac{f''(z)f(z)}{2f'(z)^2}$$

to define

$$\widetilde{g}_2(z) = h(L(z)) = g_2(z) + \mathcal{O}(f^2(z)),$$

this leads to

$$\widetilde{F}_3(z) = f(z)\widetilde{g}_2(z) = f(z)g_2(z) + \mathcal{O}(f^3(z)) = F_3(z) + \mathcal{O}(f^3(z)),$$

for which Newton's is of order 3*. There are several such functions* $h(z)$ *like :* $1 - \arctan(z)$, $2 - e^z$, $1 - \ln(1+z)$, $e^z - 2\sin(z)$, $\frac{1}{1+z}$, $\cos(z) - \sin(z)$*. This illustrates the fact that we have quite a variety of functions that will increase the order of convergence from* 2 *to* 3*. If one wanted to be exotic, one could note that Newton's method applied to a function such as*

$$\widetilde{F}_3(z) = f(z)\Big[\cos\Big(\frac{f''(z)f(z)}{2f'(z)^2}\Big) - \sin\Big(\frac{f''(z)f(z)}{2f'(z)^2}\Big)\Big]$$

would thus be an iterative method of order 3*. This provides us also with several functions that satisfy the point-wise condition* $g_2''(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)}g_2(\alpha)$*, but not the differential equation* $g_2''(z) = -\frac{f''(z)}{f'(z)}g_2(z)$ *as done in [19].*

4.7 Linking Section 4.5 and Section 4.6

We had presented two ways of increasing the order of Newton's method. Firstly, we observed that by looking at

$$z - \frac{f(z)}{f'(z)}G(z)$$

we can find a convenient choice of function $G(z)$. Secondly we have considered a new $F(z)$ and looking at $N_F(z)$ instead of $N_f(z)$. In this section we answer the following questions: How are the functions $F(z)$ and $G(z)$ related? How do we go from one to the other?

Suppose $N_f(z)$ is an iteration function of order $p \geq 2$. Then

$$\widetilde{N}_f(z) = N_f(z) + H_p(z)\left(\frac{f(z)}{f'(z)}\right)^p$$

is of order $p+1$, where

$$H_p(\alpha) = -\frac{N_f^{(p)}(\alpha)}{p!} = -(p-1)\frac{f^{(p)}(\alpha)}{p!f'(\alpha)}.$$

Looking at

$$\widetilde{N}_f(z) = z - \frac{f(z)}{f'(z)}\left[1 - H_p(z)\left(\frac{f(z)}{f'(z)}\right)^{p-1}\right],$$

let us define

$$G_p(z) = 1 - H_p(z)\left(\frac{f(z)}{f'(z)}\right)^{p-1}$$

then $G_p(\alpha) = 1$, $G_p^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-2$, and

$$G_p^{(p-1)}(\alpha) = -(p-1)!H_p(\alpha) = (p-1)\frac{f^{(p)}(\alpha)}{pf'(\alpha)}.$$

So if we set

$$g_p(z) = \frac{p - G_p(z)}{p-1},$$

then $g_p(\alpha) = 1$, $g_p^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-2$, and

$$g_p^{(p-1)}(\alpha) = -\frac{G_p^{(p-1)}(\alpha)}{p-1} = -\frac{f^{(p)}(\alpha)}{pf'(\alpha)}.$$

It follows from Theorem 4.6.1 that for $F_{p+1}(z) = f(z)g_p(z)$, $N_{F_{p+1}}(z)$ will be of order $p+1$. We can thus conclude the following result.

Theorem 4.7.1. *Suppose $N_f(z)$ is of order $p \geq 2$. For any function $G_p(z)$ such that*

$$\widetilde{N}_f(z) = z - \frac{f(z)}{f'(z)} G_p(z)$$

is an iteration function of order $p+1$, if we set

$$g_p(z) = \frac{p - G_p(z)}{p-1},$$

and $F_{p+1}(z) = f(z)g_p(z)$, then $N_{F_{p+1}}(z)$ is also of order $p+1$.

Remark 4.7.1. *Let us observe that this result does not say that $\widetilde{N}_f(z) = N_{F_{p+1}}(z)$.*

Example 4.7.1. *In particular, for the case $p = 2$, we had obtained that in Example 4.5.1*

$$G_2(z) = 1 + \frac{f''(z)}{2f'(z)} \left(\frac{f(z)}{f'(z)} \right).$$

This means we can set

$$g_2(z) = 2 - G_2(z) = 1 - \frac{f''(z)}{2f'(z)} \left(\frac{f(z)}{f'(z)} \right).$$

Hence, once we have determined such a function $G_p(z)$ as in Section 4.6 we can deduce a corresponding function $g_p(z)$ such as in Section 4.5.

Conversely, suppose that $N_f(z)$ is of order $p \geq 2$. Let $F_{p+1}(z) = f(z)g_p(z)$ with $g_p(z)$ such that

$$\begin{cases} g_p(\alpha) & \neq & 0 \\ g_p(z) & = & g_p(\alpha) + w_{g_p,p-1}(z)(z-\alpha)^{p-1} \quad \text{with} \quad w_{g_p,p-1}(\alpha) = \frac{g_p^{((p-1)}(\alpha)}{(p-1)!}, \end{cases}$$

and for which $N_{F_{p+1}}(z)$ is of order $p+1$. We have

$$\begin{aligned}
N_{F_{p+1}}(z) &= z - \frac{F_{p+1}(z)}{F'_{p+1}(z)} \\
&= z - \frac{f(z)}{f'(z)}\left[\frac{1}{1+\frac{g'_p(z)}{g_p(z)}\frac{f(z)}{f'(z)}}\right] \\
&= z - \frac{f(z)}{f'(z)}\left[1 - \frac{g'_p(z)}{g_p(z)}\frac{f(z)}{f'(z)} + \mathcal{O}\left(\left(\frac{g'_p(z)}{g_p(z)}\frac{f(z)}{f'(z)}\right)^2\right)\right] \\
&= z - \frac{f(z)}{f'(z)}\left[1 - \frac{g'_p(z)}{g_p(z)}\frac{f(z)}{f'(z)}\right] + \mathcal{O}\left(\left(\frac{g'_p(z)}{g_p(z)}\right)^2\left(\frac{f(z)}{f'(z)}\right)^3\right).
\end{aligned}$$

But

$$\frac{g'_p(z)}{g_p(z)} = \frac{w_{g'_p,p-2}(z)}{g_p(z)}(z-\alpha)^{p-2}$$

and

$$\frac{f(z)}{f'(z)} = \frac{w_{f,1}(z)}{f'(z)}(z-\alpha),$$

so

$$\mathcal{O}\left(\left(\frac{g'_p(z)}{g_p(z)}\right)^2\left(\frac{f(z)}{f'(z)}\right)^3\right) = \mathcal{O}\left((z-\alpha)^{2(p-2)+3}\right) = \mathcal{O}\left((z-\alpha)^{2p-1}\right).$$

Observing that $2p-1 \geq p+1$ for $p \geq 2$, if we set

$$G_p(z) = 1 - \frac{g'_p(z)}{g_p(z)}\frac{f(z)}{f'(z)}$$

then

$$\widetilde{N}_{F_{p+1}}(z) = z - \frac{f(z)}{f'(z)}G_p(z)$$

is also of order $p+1$. This gives us the converse of the previous result.

Theorem 4.7.2. *Let $N_f(z)$ be of order $p \geq 2$, and suppose $F_{p+1}(z) = f(z)g_p(z)$ is such that $N_{F_{p+1}}(z)$ is of order $p+1$. Set*

$$G_p(z) = 1 - \frac{g'_p(z)}{g_p(z)}\frac{f(z)}{f'(z)},$$

then

$$\widetilde{N}_{F_{p+1}}(z) = z - \frac{f(z)}{f'(z)} G_p(z)$$

is also of order $p+1$.

Remark 4.7.2. *Let us observe that this result does not say that $\widetilde{N}_{F_{p+1}}(z) = N_{F_{p+1}}(z)$.*

Remark 4.7.3. *Observe if we have a Gander type method [18] for*

$$z - \frac{f(z)}{f'(z)} G(r(z))$$

for $r(z) = \frac{f(z)f''(z)}{f'(z)^2}$. For any function $G(z)$ such that $G(0) = 1$,$G'(0) = 1/2$, and $|G''(0)| < \infty$, Gander's method produces a family of third order method. For example, we can take

$$G(z) = \frac{1 - (\beta - 1/2)z}{(1 - \beta z)}$$

one can obtain Werner's family [56], Chebyshev's method ($\beta = 0$), Halley's method ($\beta = 1/2$) and super Halley ($\beta = 1$).

4.8 Linking Euler-Chebyshev's and Halley's works.

We will now link the work of Euler and Chebyshev with that of Halley. In Theorem 4.5.10 and Remark 4.5.8, we have increased the order of the iteration function $N_f(z)$ of order p by considering, for $q = 2$,

$$\widetilde{N}_f(z) = z - \frac{f(z)}{f'(z)} - \frac{1}{p}\frac{f''(z)}{f'(z)}\left(\frac{f(z)}{f'(z)}\right)^2 = z - \frac{f(z)}{f'(z)}\left[1 + \frac{1}{p}\frac{f''(z)f(z)}{(f'(z)^2)}\right],$$

to come up with an new iteration function $\widetilde{N}_f(z)$ which has one degree of higher order of convergence than our original $N_f(z)$. For the particular value $p = 2$ we had Euler-Chebyshev's third order method. One could ask the following question: is it possible to rewrite $\widetilde{N}_f(z)$ as $N_F(z)$, Newton's method for an appropriate function $F(z) = f(z)g(z)$?

One thing we can certainly do, is apply Theorem 4.7.1 directly on the function

$$G_p(z) = 1 + \frac{1}{p}\frac{f''(z)f(z)}{f'(z)^2}.$$

This will give us a possible solution. More interestingly however, we could ask ourself the following question instead: what is the function $g_p(z)$ to which we could apply Theorem 4.7.2 to have this above defined $G_p(z)$. That is, we want $g_p(z)$ such that

$$G_p(z) = 1 - \frac{g_p'(z)}{g_p(z)}\frac{f(z)}{f'(z)}.$$

In other words

$$1 + \frac{1}{p}\frac{f''(z)}{f'(z)}\frac{f(z)}{f'(z)} = 1 - \frac{g_p'(z)}{g_p(z)}\frac{f(z)}{f'(z)}.$$

That is

$$\frac{g'(z)}{g(z)} = -\frac{1}{p}\frac{f''(z)}{f'(z)}.$$

Solving this differential equation, we get solution $g_p(z) = 1/\sqrt[p]{f'(z)}$. Therefore $F_{p+1}(z) = f(z)/\sqrt[p]{f'(z)}$, which corresponds to Schröder's process of the second kind presented in Theorem 4.6.3, and we have

$$N_F(z) = z - \frac{f(z)/f'(z)}{1 - \frac{f(z)f''(z)}{p(f'(z))^2}}.$$

For the case $p = 2$ we recover Halley's work from Euler-Chebyshev.

Remark 4.8.1. *Other links between the Schröder's process of the first kind, for which the Euler-Chebyshev's method is the term of order three of the basic sequence, and the Schröder's process of the second king, for which the Halley's method is the term of order three of the basic sequence, are presented in [7].*

4.9 On the Chebyshev-Halley family of order 3.

The Super-Halley family of IFs can be obtained from the preceding results. For example, if we apply Theorem 4.5.7 and Remark 4.5.7 for $p = 2$, with $N_f(z)$ and

$$\widetilde{H}_2(z) = \frac{f(z)/f'(z)}{1 - \lambda(z)\frac{f(z)f''(z)}{(f'(z))^2}}$$

we obtain

$$\begin{aligned} I_\lambda(z) &= N_f(z) - \frac{1}{2}N_f'(z)\left[\frac{f(z)/f'(z)}{1 - \lambda(z)\frac{f(z)f''(z)}{(f'(z))^2}}\right] \\ &= z - \frac{f(z)}{f'(z)} - \frac{1}{2}\left[\frac{f(z)/f'(z)}{1 - \lambda(z)\frac{f(z)f''(z)}{(f'(z))^2}}\right]\frac{f(z)f''(z)}{(f'(z))^2} \\ &= z - \left[\frac{1 - \left(\lambda(z) - \frac{1}{2}\right)\frac{f(z)f''(z)}{(f'(z))^2}}{1 - \lambda(z)\frac{f(z)f''(z)}{(f'(z))^2}}\right]\frac{f(z)}{f'(z)}, \end{aligned}$$

since $N_f'(z) = \frac{f''(z)f(z)}{(f'(z))^2}$, and $\widetilde{H}_2(\alpha) = 0$ and $\widetilde{H}_2'(\alpha) = 1$.

This family can also be obtained using Theorem 4.5.8. Indeed, use

$$H_2(z) = -\frac{1}{2}\left[\frac{f''(z)/f'(z)}{1 - \lambda(z)\frac{f(z)f''(z)}{[f'(z)]^2}}\right],$$

which is such that

$$H_2(\alpha) = -\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)},$$

then

$$\begin{aligned} I_\lambda(z) &= N_f(z) + H_2(z)\left(\frac{f(z)}{f'(z)}\right)^2 \\ &= z - \frac{f(z)}{f'(z)} - \frac{1}{2}\left[\frac{f''(z)/f'(z)}{1 - \lambda(z)\frac{f(z)f''(z)}{[f'(z)]^2}}\right]\left(\frac{f(z)}{f'(z)}\right)^2 \\ &= z - \left[\frac{1 - \left(\lambda(z) - \frac{1}{2}\right)\frac{f(z)f''(z)}{(f'(z))^2}}{1 - \lambda(z)\frac{f(z)f''(z)}{(f'(z))^2}}\right]\frac{f(z)}{f'(z)}. \end{aligned}$$

Hence all the IFs of this family are at least of order 3. For $\lambda(z) = 0$ it is the Euler-Chebyshev's method, for $\lambda(z) = 1/2$ it is the Halley's method, and $\lambda(z) = 1$ it is known as the Super-Halley method.

It is straightforward to compute the asymptotic constant and get

$$K_3(\alpha; I_\lambda) = \frac{1}{3!}\left[3(1-\lambda(\alpha))\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 - \frac{f'''(\alpha)}{f'(\alpha)}\right].$$

From this expression, the Super-Halley IF is of order at least 4 for the $f(z)$ which are quadratic polynomials because $K_3(\alpha; I_1) = 0$, a result obtained in [23, 24]. Any member of the family can be superior to the others for a specific function $f(z)$ since the asymptotic constant depends on $f(z)$. So comparisons based on numerical experiments between members of this family are useless considering the expressions for their asymptotic constants which depends on the original function $f(z)$. Examples are given for Chebyshev, Halley, and Super-Halley IFs to illustrate this fact in [4, 5].

4.10 Conclusion.

In this paper we presented of thorough analysis of the necessary and sufficient conditions that will entail for fixed point and Newton's iterative methods to be of higher order convergence. We did so by considering two different procedures. The first one consisted in modifying the fixed point iteration function by adding an additional term that conveniently enough increased the order of convergence of the iteration function. The second procedure consisted in modifying our original function $f(z)$ into a new one $F(z)$ which caused the order of convergence of Newton's method to increase. We have also established a link between those two procedures. Interestingly enough the results presented are obtained using simple Taylor's expansions. Finally, as a particular example, we have considered the Super-Halley family of iteration functions of order 3 for which order of convergence of at least 3 is easy to established from the established preceding results.

One aspect that is not considered here is the basin of attraction of the method. But these basins of attraction depends also on the function under consideration.

CHAPITRE 5

Schröder processes and the best ways of increasing order of Newton's method.

Abstract

We seek the answer to the following question: What is the best way of increasing the order of convergence of Newton's method? We show that the most efficient way of increasing the order of convergence of Newton's method are respectively Schröder's process of the first and second kind. One, in terms of polynomial expansions and the other in term of transformation.

5.1 Introduction

Finding roots of an equation is a fundamental problem in applied mathematics. Many mathematicians worked to develop efficient numerical methods to solve such problem. One of the first and very popular way of numerically finding root of a general non-linear equation was presented in 1669 [57] by Sir Isaac Newton. He introduced an iterative method that can be applied to finding a simple root α of an analytic (or sufficiently regular) function $f(z)$, i.e. such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. It uses an initial point z_0 and a sequence generated by

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)},$$

which need the evaluation of $f(z_k)$ and $f'(z_k)$ for each k. If the sequence $\{z_k\}_{k=0}^{+\infty}$ converges to α, we get a fixed point to the iteration function (IF)

$$N_f(z) = z - \frac{f(z)}{f'(z)},$$

that is to say that $\alpha = N_f(\alpha)$, and consequently $f(\alpha) = 0$. The function $N_f(z)$ is called Newton's IF. Later, in 1694 [20], Edmond Halley, introduced another IF

$$H_f(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)} = z - \left[1 - \frac{f(z)f''(z)}{2f'(z)^2}\right]^{-1} \frac{f(z)}{f'(z)},$$

which has the property to be “faster" than Newton's IF. The last expression indicates that we can obviously look at $H_f(z)$ as a modified $N_f(z)$ using a rational expression in terms of $f(z)$ or $f(z)/f'(z)$. Less obvious is the fact that $H_f(z) = N_F(z)$ for $F(z) = f(z)/\sqrt{f'(z)}$, and it is Newton's IF applied to a modified function $F(z)$. Subsequently, as reported in [3], a third IF, called Euler-Chebyshev's IF, was introduced

$$C_f(z) = z - \frac{f(z)}{f'(z)} - \frac{f''(z)}{2f'(z)}\left(\frac{f(z)}{f'(z)}\right)^2 = z - \left[1 + \frac{f(z)f''(z)}{2f'(z)^2}\right] \frac{f(z)}{f'(z)},$$

which was as “fast" as Halley's IF. It is written as a polynomial expression in terms of $f(z)$ or $f(z)/f'(z)$. Again the last expression indicates that it can also be seen as a modified

$N_f(z)$. Moreover, it cannot be rewritten as a Newton's IF applied to any modified function $F(z)$, as it is possible for Halley's IF. Both Halley's and Euler-Chebyshev's IF which are constructed to be "faster" than Newton's IF, require an additional information, namely the knowledge of $f''(z)$.

Let us point out that Newton's, Halley's, and Chebyshev's IF correspond to one-point iteration processes. Such one-point iteration processes require the evaluations of $f(z)$, and some of its derivatives at only one point at each iteration. To increase the order of convergence for a one-point method of order p to get a one-point method of order $p+1$ we need, at least, to add the information given by $f^{(p)}(z)$ [51]. For example to increase the order of the Newton's method, which is of order 2, to get a one-point method of order 3 we need to add at least $f''(z)$ to the expression.

The goal of this paper is to study the "best" way of increasing the order of convergence of Newton's method as a one-point iteration process. There are at least two possible things we can do to add the information and obtain efficiently a higher order one-point IF. We can either use a polynomial expression to modify the Newton's method, in that case we get the Schröder's method of the first kind (for Newton's method of order 2 we get the Chebyshev's method of order 3), or apply Newton's method to a modified function, or equivalently use a rational expression to modify Newton's method, in that case we get the Schröder's method of the second kind (for Newton's method of order 2 we get the Halley's method of order 3).

We will see in this paper that the "best" ways of increasing the order of Newton's IF, in terms of one-point IF, which extend Halley's and Euler-Chebyshev's IF, correspond to two processes established by Schröder in 1870 called Schröder's processes of the first kind and of the second kind [48]. In order to get our result, we need to define what the terms "fast", "best", or "efficient", mean mathematically. It is done in Section 5.2. In Section 5.3, we consider a direct modification of the Newton's IF itself by adding appropriate terms. In Section 5.4, we apply Newton's IF to a modified function $F(z)$. Finally, concluding remarks are presented in the last Section 5.5.

This paper focus on the (historical) theoretical development of ways to increase the order of one-point methods using a basic definition of efficiency. In fact we point out a nice Schröder's achievement. Let us mention that Schröder's processes are not so well-known. If they had been so well-known they would not have been rediscovered so many times [44, 45].

From the application view point, some other topics about Newton's method are important, but they are not relevant for the purpose of the present paper. For example the choice of the initial starting point z_0 to assure convergence of the process (z_0 be in the basin of attraction), also related to the asymptotic constant of the method, depends on the function $f(z)$ [4]. The case of high order Newton's method for multiple zeros is interesting and studied elsewhere [6]. Also high order methods in the multidimensional framework deserves its own study, see [10]. Finally we don't consider multi-point methods, as the secant method which is one of the simplest example.

5.2 Preliminaries

5.2.1 Order of convergence

Let us consider IF to find a simple zero α of a function $f(z)$, that is to say $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. We say that the *order of convergence* of an IF $I(z)$ applied to find α is p, a positive integer, if and only if there exists a non-zero *asymptotic constant* $K_p(\alpha; I)$ such that

$$\lim_{k\to+\infty} \frac{z_{k+1}-\alpha}{(z_k-\alpha)^p} = \lim_{k\to+\infty} \frac{I(z_k)-I(\alpha)}{(z_k-\alpha)^p} = K_p(\alpha; I).$$

For values of z_k close to α, the term $(z_k - \alpha)^p$ becomes considerably smaller if p is large, so looking at

$$z_{k+1} - \alpha \approx K_p(\alpha, I)\,(z_k - \alpha)^p$$

hints at how fast the error of approximation decrease and the speed of convergence increases as p increases. So this concept of order of convergence provides us with a reasonable characterization of speed of an iterative process, or equivalently how fast, locally or close to α, an IF $I(z)$ may be.

Let us say also that this speed of convergence can be observed only for z sufficiently close to α. Let us mention that for two IF of the same order p, I_1 and I_2, if

$$|K_p(\alpha; I_2)| > |K_p(\alpha; I_1)| > 0$$

I_1 will converge locally more rapidly around α than I_2.

In this paper we look on infinite sequences of IF's with increasing order of convergence. Such sequence, say $\{I_p(z)\}_{p=0}^{+\infty}$ whose pth member $I_p(z)$ is an IF of order p, is called a *basic sequence* of IF's [51].

5.2.2 Efficiency of informational usage

The informational usage of an IF is the number of new pieces of information required at each iteration. Since the information to be used are the values of $f(z)$ and some of its derivatives, the informational usage is the total number of function and derivatives evaluated per iteration. Following [51], we will use the *informational efficiency* EFF which is the order p divided by the informational usage d

$$\mathrm{EFF} = p/d.$$

There exist other measures of efficiency, for example the *efficiency index* and the *computational efficiency* [51]. We will not consider those measures here.

If $I_p(z)$ is a one-point IF of order p, given an arbitrary analytic function $f(z)$, then IF requires the knowledge of at least the $(p-1)$th first derivatives of $f(z)$ in its formulation [51]. So $d \geq p$ and EFF ≤ 1.

This efficiency is maximized for process of order p, if its formulation requires no more then the $(p-1)$th first derivatives of the function $f(z)$, that is to say $d = p$, and EFF $= 1$. This illustrates the following thing: there are many ways of increasing the order of convergence, but to do so, as efficiently as possible, we need to always limit ourselves to computing at most one additional derivative each time we increase the order. It will be shown that the Schröder's processes are constructed such that EFF $= 1$, so the informational efficiency is maximized.

5.2.3 Taylor's expansion

For an IF of order p, let us observe that we can write

$$I(z_k) \approx I(\alpha) + K_p(\alpha, I)\,(z_k - \alpha)^p\,,$$

which looks like the Taylor's expansion of $I(z)$.

Let us recall some facts about Taylor's expansion of an arbitrary analytic function $f(z)$. Using the notation $f^{(j)}(z)$ which stands for the jth derivative of $f(z)$, and α in the domain of $f(z)$, we can write

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(\alpha)}{j!}\,(z-\alpha)^j = \sum_{j=0}^{q-1} \frac{f^{(j)}(\alpha)}{j!}\,(z-\alpha)^j + w_{f,q}(z)\,(z-\alpha)^q$$

for any $q = 1, 2, , \ldots$, where $w_{f,q}(z)$ is the analytic function

$$w_{f,q}(z) = \sum_{j=0}^{\infty} \frac{f^{(q+j)}(\alpha)}{(q+j)!}\,(z-\alpha)^j\,.$$

Moreover, the series for $f(z)$ and $w_{f,q}(z)$ have the same radius of convergence for any q, and $w_{f,q}(\alpha) = \frac{f^{(q)}(\alpha)}{j!}$.

We say that α is a root of $f(z)$ of multiplicity $q \geq 1$ if and only if $f^{(j)}(\alpha) = 0$ for $j = 0, \ldots, q-1$, and $f^{(q)}(\alpha) \neq 0$. So α is a root of $f(z)$ of multiplicity q if and only if there exists an analytic function $w_{f,q}(z)$ such that $w_{f,q}(\alpha) = \frac{f^{(q)}(\alpha)}{q!} \neq 0$ and $f(z) = w_{f,q}(z)(z-\alpha)^q$.

The Big O notation $g(z) = O(f(z))$, respectively the small o notation $g(z) = o(f(z))$, around $z = \alpha$ when $c \neq 0$, respectively $c = 0$, means that

$$\lim_{z \to \alpha} \frac{g(z)}{f(z)} = c.$$

For α a root of multiplicity q of $f(z)$, it is equivalent to write $g(z) = O(f(z))$ or $g(z) = O((z-\alpha)^q)$. Observe also that if α is a simple root of $f(z)$, then α is a root of multiplicity q of $f^q(z)$. Hence $g(z) = O(f^q(z))$ is equivalent to $g(z) = O((z-\alpha)^q)$.

5.3 Modifying Newton's IF as a fixed point

As mentioned before, the first thing we can do is directly modify the iteration function $N_f(z)$ itself into some new iteration function $S(z)$. Let us start by looking for the conditions on an IF to be of order p.

5.3.1 Conditions on IF

As we have seen, the concept of order of convergence is related to Taylor's expansion of an IF. More precisely we have the following result.

Theorem 5.3.1. *[12] Let p be an integer ≥ 2 and let $I(z)$ be an analytic function such* *that* $I(\alpha) = \alpha$. *The IF* $I(z)$ *is of order* p *if and only if* $I^{(j)}(\alpha) = 0$ *for* $j = 1, \ldots, p-1$, *and* $I^{(p)}(\alpha) \neq 0$.

This result says that Taylor's expansion of an IF $I(z)$ of order p is

$$I(z) = \alpha + w_{I,p}(z)(z-\alpha)^p \quad \text{with} \quad w_{I,p}(\alpha) = \frac{I^{(p)}(\alpha)}{p!},$$

and we also have

$$I'(z) = w_{I',p-1}(z)(z-\alpha)^{p-1} \quad \text{with} \quad w_{I',p-1}(\alpha) = \frac{I^{(p)}(\alpha)}{(p-1)!}.$$

Let us remark that for a function $f(z)$ with a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, since we can write

$$f(z) = w_{f,1}(z)(z-\alpha) \quad \text{where} \quad w_{f,1}(\alpha) = f'(\alpha),$$

the following two expressions, $\mathcal{O}\left((z-\alpha)^l\right)$ and $\mathcal{O}\left(f^l(z)\right)$, are equivalent for any positive integer l.

As consequences of Theorem 5.3.1, for an analytic IF $I(z)$, we could say that: (a) $I(z)$ is of order p if and only if $I(z) = \alpha + O((z-\alpha)^p)$, or equivalently, if $I(\alpha) = \alpha$ and $I'(z) = O((z-\alpha)^{p-1})$, and (b) if α is a simple root of $f(z)$, then $I(z)$ is of order p if and only if $I(z) = \alpha + O(f^p(z))$, or equivalently, if $I(\alpha) = \alpha$ and $I'(z) = O(f^{p-1}(z))$.

Now for the Newton's IF one can observe that $N_f(\alpha) = \alpha$. Furthermore $N_f'(z) = \frac{f''(z)f(z)}{f'(z)^2}$, and $N_f'(\alpha) = 0$. So if α is a simple root of $f(z)$, that is $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, we can conclude that Newton's method applied to $f(z)$ is an iteration function of at least of order 2.

5.3.2 Polynomial expansion and Schröder's process of the first kind

Now let us directly modify the iteration function $N_f(z)$ itself into some new iteration function $S(z)$. Here out of all possible such operations, we are looking at the "best" way, we shall look for the modification with the simplest algebraic formulation. Noting that the simplest algebraic operation we could performed on $N_f(z)$ would be an addition, we want to know what would be the simplest function $\Delta(z)$, so that for $S(z) = N_f(z)+\Delta(z)$, the order of convergence of $S(z)$ would be increased and furthermore we are required to not use anything more then an additional derivative of $f(z)$ in the formulation of $\Delta(z)$.

Suppose we start with an iteration function $N_f(z)$ of order 2, that is $N_f(\alpha) = \alpha$, $N_f'(\alpha) = 0$ and $N_f''(\alpha) \neq 0$. If $S_3(z)$ is an iteration function of order 3 defined by $S_3(z) = N_f(z) + \Delta(z)$, we must have $S_3(\alpha) = \alpha$, $S_3'(\alpha) = 0$ and $S_3''(\alpha) = 0$, according to Theorem 5.3.1. This means for $\Delta(z) = S_3(z) - N_f(z)$, α is a root of multiplicity 2. That is $\Delta(z)$, can be written as $\delta(z)f(z)^2$, for some analytic function $\delta(z)$, since α is a simple root of $f(z)$. Our goal here would be to figure out what would be the simplest and most efficient choice of function $\delta(z)$ to increase the order of convergence. This leads to the following theorem.

Theorem 5.3.2. *[12] Let α be a simple root of $f(z)$. Let $S_p(z)$ be the IF defined by the finite series*

$$S_p(z) = \sum_{j=0}^{p-1} c_j(z) f^j(z),$$

where $c_0(z) = z$, and the $c_j(z)$ are defined by

$$j f'(z) c_j(z) + c_{j-1}'(z) = 0 \quad \textit{or} \quad c_j(x) = -\frac{1}{j}\left(\frac{1}{f'(x)}\frac{d}{dx}\right) c_{j-1}(z)$$

for $j = 1, 2, \ldots$ Then $S_p(z)$ is, at least, of order p.

The sequence of iteration function $S_p(z)$, as introduced by the Theorem 5.3.2, is known as *Schröder's process of the first kind* introduced in 1870 [48, 51].

It is very important to observe that $c_1(z) = 1/f'(z)$, so $c_1(z)$ is a function of the parameter $f'(z)$ only. Since $c_{j+1}(z) = -c_j'(z)/[(j+1)f'(z)]$, $c_j(z)$ depends on $f'(z)$, $f''(z)$,...,$f^{(j)}(z)$, more precisely

$$c_j(z) = \frac{\text{polynomial expression in terms of } f''(z), \ldots, f^{(j)}(z)}{(f'(z))^{2j-1}}.$$

In particular, it is important to note that $c_j(z)$ is a function whose algebraic expression does not depend explicitly on the parameter $f(z)$ but depends only on $f^{(l)}(z)$ for $l = 1, \ldots, j$. Also by definition $c_{j+1}(z)$ only has one order of derivative of $f(z)$ higher then $c_j(z)$.

We have shown that $S_p(z)$ is an iteration function of at least order p. We know that if $S_p(z)$ is to be an iteration function of order p, given an arbitrary analytic function $f(z)$, then efficiency requires knowledge of at least the first $(p-1)$th derivatives of $f(z)$ in the formulation [51]. This illustrates the fact that $S_p(z)$, as written above, is as efficient as possible, because it does not require computing derivatives of unnecessarily higher order then $p-1$. Each time the order is increased, only one additional derivative is computed as required to maximize efficiency. We will show furthermore that the coefficient $c_j(z)$, given by Schröder's process of the first kind, minimize the dependence on the number of parameters z, $f(z)$, $f'(z)$,...,$f^{(j)}(z)$.

5.3.3 Consequence

Let us present the consequence of the result of the last section.

Theorem 5.3.3. *Suppose we have an iterative method of order $p \geq 2$,*

$$\widetilde{S}_p(z) = \sum_{j=0}^{p-1} \tilde{c}_j(z) f(z)^j,$$

where $\tilde{c}_0(z) = z = c_0(z)$, *and* $\tilde{c}_j(z)$ *depends on* z, $f'(z)$, $f''(z)$,..., $f^{(j)}(z)$ *for* $j = 1, 2, \ldots,$ *then*

$$\tilde{c}_j(z) = c_j(z)$$

for $j = 0, 1, 2, \ldots,$ *and* $\widetilde{S}_p(z) = S_p(z)$.

Note that in this last theorem $\tilde{c}_j(z)$ are only dependent of the parameter z and derivatives of $f(z)$, they do not directly depend on the function $f(z)$ itself.

Proof. First, we observe that in order for $\widetilde{S}_p(z)$ to be an iterative method of any order for finding the simple root α we need that $\widetilde{S}_p(\alpha) = \alpha$. We have $\widetilde{S}_1(z) = S_1(z)$. Now, suppose there exists a first index $p^* > 1$, such that $\widetilde{c}_{p^*-1}(z) \neq c_{p^*-1}(z)$. Then, because both processes, $S_{p^*}(z)$ and $\widetilde{S}_{p^*}(z)$, are of the same order p^*, we observe that

$$\tilde{S}_{p^*}(z) - S_{p^*}(z) = \mathcal{O}(f^{p^*}(z)).$$

But

$$\tilde{S}_{p^*}(z) - S_{p^*}(z) = \left(\tilde{c}_{p^*-1}(z) - c_{p^*-1}(z)\right) f^{p^*-1}(z),$$

which means that

$$\tilde{c}_{p^*-1}(z) - c_{p^*-1}(z) = \mathcal{O}(f(z)).$$

In other words we have

$$\Gamma_{p^*-1}\Big(z, f'(z), f''(z), .., f^{(p^*-1)}(z)\Big) = \tilde{c}_{p^*-1}(z) - c_{p^*-1}(z) = \mathcal{O}(f(z)).$$

This is only possible if

$$\Gamma_{p^*-1}\Big(z, f'(z), f''(z), .., f^{(p^*-1)}(z)\Big) = 0$$

as we will show. But this contradict the fact that $\tilde{c}_{p^*-1}(z) \neq c_{p^*-1}(z)$, and hence $\widetilde{S}_{p^*}(z) = S_{p^*}(z)$.

To show that $\Gamma_{p^*-1}\Big(z, f'(z), f''(z), .., f^{(p^*-1)}(z)\Big) = 0$, let us observe that the formula for $\tilde{c}_{p^*-1}(z)$ and $c_{p^*-1}(z)$ must hold for any function $f(z)$ having a simple root at α. So let us consider

$$f(z) = \sum_{l=1}^{p^*-1} \frac{\lambda_l}{l!} (z-\beta)^l ,$$

with arbitrary β, $\lambda_1 \neq 0$, λ_2, ..., λ_{p^*-2}, and λ_{p^*-1}. For this expression of $f(z)$ we have $f(\beta) = 0$, $f'(\beta) = \lambda_1 \neq 0$, and $f^{(l)}(\beta) = \lambda_l$ for $l = 2, \ldots, p^* - 1$. Then

$$\Gamma_{p^*-1}\Big(z, f'(z), f''(z), .., f^{(p^*-1)}(z)\Big)_{\big|z=\beta} = 0,$$

which leads to

$$\Gamma_{p^*-1}\Big(\beta, \lambda_1, \lambda_2, \ldots, \lambda_{p^*-1}\Big) = 0.$$

So, this fact means that this function is identically zero as required, because β, λ_1, λ_2, ..., λ_{p^*-2}, and λ_{p^*-1} can take any values. □

So we have also the following consequence.

Corollary 5.3.4. *Suppose $Q_p(z)$ is an acceleration process of order $p \geq 2$, that expands into polynomial terms of $f(z)$, then*

$$Q_p(z) = S_p(z) + \delta(z) f^p(z)$$

and can be truncated to $S_p(z)$ for every p.

So, these results say that $S_p(z)$ is the most efficient IF of order p, in polynomial terms of $f(z)$, that increases the order of convergence of Newton's IF.

5.4 Modifying $f(z)$ into a new function $F(z)$

The second way is to modify the function $f(z)$ itself into a new function $F(z)$, whose application to Newton's will be faster. We will need to establish conditions that will entail any new function $F(z)$ to increase the order of convergence of $N_f(z)$ by one. Furthermore if our modification $F(z)$ is to be the "best" possible choice of such a function, we are required to not use anything more then an additional derivative of $f(z)$ in the formulation of $F(z)$, as discussed before. We shall look for such a function $F(z)$ with the simplest possible formulation.

5.4.1 Finding the form

We first look at the necessary and sufficient conditions to get high order of convergence of the Newton's method.

Theorem 5.4.1. *[12] Let $p \geq 2$ and let $F_p(z)$ be an analytic function such that $F_p(\alpha) = 0$ and $F_p'(\alpha) \neq 0$. Newton IF $N_{F_p}(z) = z - \frac{F_p(z)}{F_p'(z)}$ is of order p if and only if $F_p^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $F_p^{(p)}(\alpha) \neq 0$.*

Starting with $F_2(z) = f(z)$, we can look for a recursive way to construct a function $F_p(z)$ which will satisfy the conditions of Theorem 5.4.1. A consequence is that $N_{F_p}(z)$ will be of order p, and $N_{F_p}(z) = \alpha + O(f^p(z))$. Our goal here however is to establish the most efficient way of constructing such a function.

Suppose $F_p(z)$ is an analytic function such that $F_p(\alpha) = 0$ and $F_p'(\alpha) \neq 0$. If $N_{F_p}(z)$ is of order p, then Theorem 5.4.1 implies that $F_p''(z) = \mathcal{O}\left((z-\alpha)^{p-2}\right)$. To increase the order of convergence from p to $p+1$, we must look for a function $F_{p+1}(z)$ such that $F_{p+1}(\alpha) = 0$, $F_{p+1}'(\alpha) \neq 0$, and $F_{p+1}''(z) = \mathcal{O}\left((z-\alpha)^{p-1}\right)$. Because the conditions of Theorem 5.4.1 are an if and only if, we really have no other choice but to go about it this way.

Several functions $F_{p+1}(z)$ might a priori be used. These functions may depend on z, $f(z)$, $f'(z)$, $f''(z)$, ... Our goal here is to find a new function $F_{p+1}(z)$, related to $f(z)$, that would use the least possible amount of such parameters. Since α is supposed to be a simple root of both $F_{p+1}(z)$ and of $F_p(z)$, we can write

$$F_{p+1}(z) = F_p(z) g_p(z).$$

The function $g_p(z)$ will contains information from $f(z)$ or $F_p(z)$, in terms of some of its derivatives $f^{(l)}(z)$ for $l = 1, 2, 3, \ldots$, and we know, in term of its dependence on the initial function $F_p(z)$, at least $F_p'(z)$ is needed to increase the order of $N_{F_{p+1}}(z)$. That is because to increase the order of convergence at each step, at least knowledge of an additional derivative is required as discussed previously. So if we could write $g_p(z)$ as a function depending only of $F_p'(z)$, that is $g_p(z) = G_p(F_p'(z))$, we would have effectively used the least amount of new information of $F_p(z)$ in term of its derivatives. Let us point out that the fact that such a solution may even exist is not obvious. However, the following result has been proved to solve the problem.

Theorem 5.4.2. *[14] Let $F_p(z)$ be an analytic function such that $F_p(\alpha) = 0$ and $F_p'(\alpha) \neq 0$, and suppose $N_{F_p}(z)$ is of order p. There exists a unique function $G_p(\xi)$, up to a multiplicative constant ρ_p, such that $N_{F_{p+1}}(z)$ is of order $p+1$, for*

$$F_{p+1}(z) = F_p(z) G_p\left(F_p'(z)\right),$$

and this function is

$$G_p(\xi) = \frac{\rho_p}{\xi^{1/p}}.$$

So using

$$F_{p+1}(z) = F_p(z) G_p(F_p'(z)) = \rho_p \frac{F_p(z)}{\left[F_p'(z)\right]^{1/p}},$$

we obtain

$$N_{F_{p+1}}(z) = z - \frac{F_{p+1}(z)}{F_{p+1}'(z)} = z - \frac{F_p(z)}{F_p'(z)} \left[1 - \frac{F_p''(z) F_p(z)}{p F_p'(z)^2}\right]^{-1},$$

which will be of order $p+1$.

Because $F_2(z) = f(z)$, $N_{F_2}(z) = N_f(z)$ is of order 2. When we apply Newton's IF to the function $F_3(z) = F_2(z)/\sqrt{F_2'(z)} = f(z)/\sqrt{f'(z)}$, we have an IF of order 3, this just so happen to exactly be Halley's IF.

What we have obtained, additionally, is that the function $G_p(\xi)$ is in fact unique and this allows us to conclude that if $F_p(z)$ is a function for which Newton's IF applied to $F_p(z)$, that is $N_{F_p}(z)$, is of order p, then for

$$F_{p+1}(z) = F_p(z)G_p(F_p'(z)) \quad \text{with} \quad G_p(z) = \frac{1}{\sqrt[p]{z}},$$

not only is $N_{F_{p+1}}(z)$ of order $p+1$, but $G_p(z) = \frac{1}{\sqrt[p]{z}}$ is the simplest such function we could use. So the specific family obtained by this process is in fact the most efficient way of recursively increasing the order of convergence of Newton's IF in terms of usage of derivatives of the function $f(z)$ if we were to only modify the function $f(z)$.

5.4.2 Implication of results

The above described process provides us with a recursive way of progressively determining functions $\{F_p(z)\}_{p=2}^{+\infty}$, with $F_2(z) = f(z)$, for which $H_p(z) = N_{F_p}(z)$ is of order p. The family $\{H_p(z)\}_{p=2}^{+\infty}$ will be called *Halley's basic sequence* because $H_3(z) = H_f(z)$.

Recall that very early we had observed: if there was such a thing as the "best" way of making Newton's IF, dating from 1669, faster, we shouldn't be surprised that mathematicians have rediscover it, several times. In fact this acceleration process has been rediscovered several times since 1870 [44, 45].

Halley's basic sequence has one fundamental property it always satisfies. So providing a formulaic approach that can be used to recognize Halley's basic sequence, regardless of the formulation, we would have a complete characterization of this basic sequence. It could be used to effectively recognize it, regardless of form, and avoid any further rediscovery. This is the goal of the next result.

Theorem 5.4.3. *[14] Let $f(z)$ be an analytic function such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ for which $N_f(z)$ is of order 2. The basic sequence $\{G_p(z)\}_{p=2}^{+\infty}$ given by*

$$G_p(z) = z - V_p(z) \quad \textit{for} \quad p \geq 2,$$

for which the pth element is of order p, corresponds to the Halley's basic sequence if and only if

$$\begin{cases} V_2(z) & = & \frac{f(z)}{f'(z)}, \\ V_{p+1}(z) & = & \left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) \quad \textit{for} \quad p \geq 2. \end{cases}$$

5.4.3 Schröder's method of the second kind

In 1870, Schröder [48] proposed 2 basic sequences. The second one, known as *Schröder's method of the second kind*, is based on rational approximations. Its pth member of order p is given by

$$T_p(z) = z - \frac{R_{p-2}(z)}{R_{p-1}(z)} \quad \text{for} \quad p \geq 2,$$

where

$$\begin{cases} R_0(z) & = & 1/f(z), \\ R_p(z) & = & \sum_{j=1}^{p} (-1)^{j+1} \frac{f^{(j)}(z)}{j! f(z)} R_{p-j}(z) \quad \text{for} \quad p \geq 1. \end{cases}$$

Computing the expressions for $R_1(z)$ and $R_2(z)$, we observe that the first two elements of this sequence are respectively, Newton's IF $T_2(z) = N_f(z)$ of order 2, and Halley's IF $T_3(z) = H_f(z)$ of order 3.

In fact, Schröder's method of the second kind is actually Halley's basic sequence. To get this result, the next lemma provide us with an equivalent formulation for $R_p(z)$.

Lemma 5.4.4. *[14]* $R_p(z) = \dfrac{(-1)^p}{p!}\left(\dfrac{1}{f(z)}\right)^{(p)}$ *for* $p = 0, 1, 2, \ldots$

Then we have the result.

Theorem 5.4.5. *Halley's basic sequence and Schröder's method of the second kind coincide.*

Proof. We have directly that

$$V_p(z) = \frac{R_{p-2}(z)}{R_{p-1}(z)} = \frac{\frac{(-1)^{p-2}}{(p-2)!}\left(\frac{1}{f(z)}\right)^{(p-2)}}{\frac{(-1)^{p-1}}{(p-1)!}\left(\frac{1}{f(z)}\right)^{(p-1)}} = -(p-1)\frac{(1/f(z))^{(p-2)}}{(1/f(z))^{(p-1)}}.$$

Consequently

$$V_p'(z) = -(p-1)\left[1 - \frac{(1/f(z))^{(p-2)}(1/f(z))^{(p)}}{\left[(1/f(z))^{(p-1)}\right]^2}\right],$$

and a direct substitution leads to

$$\left[1 - \frac{1}{p}\left[1 - V_p'(z)\right]\right]^{-1} V_p(z) = -p\frac{(1/f(z))^{(p-1)}}{(1/f(z))^{(p)}} = V_{p+1}(z),$$

which establishes the result. □

So, Schröder's process of the second kind, as proposed in 1870, is the most efficient way of increasing the order of convergence of Newton's IF if we were to modify the function $f(z)$ into a new one $F(z)$.

5.5 Conclusion

In seeking to increase the order of convergence of Newton's method as efficiently as possible, based on necessary and sufficient conditions for high order of convergence, we found that while Schröder process of the first kind is the best way of increasing the order of convergence of Newton's method by addition to obtain a polynomial expression in terms of $f(z)$, Schröder's process of the second kind was the best way of increasing the order of Newton's method, when applied to a new function, by modifying the function $f(z)$ into a new function $F(z)$. Those two processes introduced by Schröder in 1870 are thus respectively the best one in some terms of minimal usage of derivatives of $f(z)$ and simplicity of formulation. Finally, let us observe that those two processes are related by polynomial and rational interpolation as shown in [7].

CONCLUSION

Nous avons développé dans nos travaux un nouvel algorithme pour reconnaître le processus d'accélération de Schröder de première espèce. Celui-ci pourrait éviter toute autre éventuelle redécouverte.

Nous avons aussi prouvé que les deux processus de Schröder sont respectivement les meilleures façons d'augmenter l'ordre de convergence de la méthode itérative de Newton. L'un en termes d'application de la méthode de Newton à une nouvelle fonction, l'autre en termes de développement polynomial de la fonction itérante par rapport à la fonction originale.

Nous avons aussi unifié le travail de mathématiciens comme Euler, Chebyshev et Halley, en présentant une façon générale d'augmenter la convergence de méthodes itératives, basées sur le développement Taylor de fonctions analytiques.

L'extension de plusieurs de ces résultats dans les espaces de Banach est faite dans [10].

L'une des perspectives de recherche pourrait être de voir s'il y a un lien entre la constante asymptotique et les bassins d'attraction. Egalement de voir comment on peut modifier une fonction afin d'agrandir le bassin d'attraction pour une racine précise. Ainsi voir comment nos travaux peuvent contribuer à étendre et améliorer les résutats connus dans ce domaine, en particulier les travaux de Traub [51] et Kalantari [36]. Finalement, un domaine d'interêt qui pourrait être exploré concerne l'application de nos travaux aux méthodes de recherche simultannées des racines d'une fonction. Lorsque plusieurs zéros d'une fonction sont requis ces méthodes présentent des avantages par rapport à la méthode de Newton utilisée pour la recherche d'un zéro isolé à la fois. Pour en savoir plus sur ces methodes de recherche des zéros de polynomes voir par example [43, 42, 55, 16] .

On pourrait aussi voir comment les bassins d'attraction changeraient si on augmentait le nombre d'iterations qu'on a effectué dans le chapitre 2.

BIBLIOGRAPHIE

[1] I.K. Argyros and F. Szidarovszky (1993), *The Theory and Applications of Iteration Methods*, CRC Press, Boca Raton, FL .

[2] E. Bodewig (1949), On types of convergence and on the behavior of approximations in the neighborhood of a multiple root of an equation, *Quart. Appl. Math.*, 7 , 325-333.

[3] P.L. Chebyshev (1951), Collected Works, *Izd. Akad. Nauk SSSR*, 5, 5-27 (in Russian).

[4] F. Dubeau (2013), On comparisons of Chebyshev-Halley iteration functions based on their asymptotic constants, *International J. of Pure and Applied Mathematics*, 85(5), 965-981.

[5] F. Dubeau and C. Gnang (2013), On the Chebyshev-Halley family of iteration functions and the n-*th* root computation problem, *Inter. J. Pure Applied Math.*, 85 , 1051-1059.

[6] F. Dubeau (2013), On the modified Newton's method for multiple root, *Journal of Mathematical Analysis*, 4(2), 9-15.

[7] F. Dubeau (2014), Polynomial and rational approximations and the link between Schröder's processes of the first and second kind, *Abstract and Applied Analysis*, 2014, Article ID 719846, 5 pages.

[8] F. Dubeau (2009), Newton's method and high-order algorithms for the nth root computation, *JCAM*, 224 , 66-76.

[9] F. Dubeau (1996), Algorithms for n-th root approximation, *Computing*, 57 , 365-369.

[10] F. Dubeau (2021), High order fixed point and Newton's methods in Banach space *Numerical Functional Analysis and Optimization*, https://doi.org/10.1080/01630563.2021.1873365

[11] F. Dubeau and C. Gnang (2014), Fixed point and Newton's for solving a nonlinear equation: from linear to high order convergence, *SIAM Review*, 56.4 , 691-708.

[12] F. Dubeau and C. Gnang (2018), Fixed-point and Newton's method in the complex plane, *Journal of Complex Analysis*, **2018**, Article ID 7289092, 11 pages.

[13] F.Dubeau and C.Gnang (2014), Inverse function, Taylor's expansion and extended Schröder's processes, *Inter. J. Comput. Math.*, http://dx.doi.org/10.1080/00207160.2013.833611.

[14] F. Dubeau and C. Gnang (2018), On the rediscovery of Halley's iterative method for computing the zero of an analytic function, *Journal of Computational and Applied Mathematics*, **335**, 129-141.

[15] E. Durand (1960), *Solutions Numériques des Équations Algébriques*, Tome 1, Masson et Cie, Paris.

[16] M.R. Farmer and G. Loizou (1975), A class of iteration functions for improving, simultaneously, approximations to the zeros of a polynomial, *BIT* **15**(3): 250-258. http://dx.doi.org/10.1007/BF01933657

[17] W.F. Ford and J. A. Pennline (1996), Accelerated convergence in Newton's method, *SIAM Review* **38**(4): 658-659. http://dx.doi.org/10.1137/S0036144594292972

[18] W. Gander (1985), On Halley's iteration method, *Amer. Math. Monthly* **92**(2): 131–134. http://dx.doi.org/10.2307/2322644

[19] J. Gerlach (1994), Accelerated convergence in Newton's method, *SIAM Review* **36**(2): 272-276. http://dx.doi.org/10.2307/2322644

[20] E. Halley (1694), A new and general method of finding the roots of equations, *Philos. Trans. Roy. Soc. London*, **18**: 136–148.

[21] H.J. Hamilton (1946), Roots of equations by functional iteration, *Duke Math. J.* **13**(1): 113-121. http://dx.doi.org/10.1215/S0012-7094-46-01312-9

[22] P. Henrici (1964), *Elements of Numerical Analysis*, John Wiley and Sons, New York.

[23] M.A. Hernandez and M.A. Salanova, (1993), A family of Chebyshec-Halley type methods , *Inter. J. Computer Math*,47 59-63.

[24] J.M. Gutierrez and M.A. Hernandez (1997), A family of Chebyshev-Halley type methods in Banach spaces, *Bull. Austral. Math. Soc.*, 66 , 113-130.

[25] M.A. Hernandez and N. Romero (2004), High order algorithms for approximating n*th* roots, *Inter. J. Comput. Math.*, 81 , 1001-1014.

[26] A.S. Householder (1953), *Principles of Numerical Analysis*, McGraw Hill, Columbus OH.

[27] A.S. Householder (1970), *The Numerical Treatment of a Single Nonlinear Equation*, McGraw Hill, New York.

[28] M. Igarashi and H. Nagasaka (1991), Relationships between the iteration times and the convergence order for Newton-Raphson like methods, *J. Inform. Process* **32**: 1349–1354 (in Japanese).

[29] M. Igarashi and T. Ypma (1995), Relationships between order and efficiency of a class of methods for multiple zeros of polynomials, *J. Comput. Math. Appl.*, 60 101-113.

[30] E. Isaacson and H.B. Keller (1966), *Analysis of Numerical Methods*, John Wiley and Sons, New York.

[31] B. Jovanović (1972), A method for obtaining iterative formulas of higher order, *Mat. Vesnik* **24**: 365-369.

[32] B. Kalantari and I. Kalantari, R. Zaare-Nahandi (1997), A basic family of iteration functions for polynomial root-finding and its characterizations, *J. Comput. Appl. Math.* **80**(2): 209–226. http://doi.org/10.1016/S0377-0427(97)00014-9

[33] B. Kalantari (1999), On the order of convergence of a determinantal family of root-finding methods, *BIT* **39**(1): 96-109. http://dx.doi.org/10.1023/A:1022321325108

[34] B. Kalantari (2000), Generalization of Taylor's theorem and Newton's method via a new family of determinantal interpolation formulas and its applications, *J. Comput. Appl. Math.* **126**(1-2): 287–318. http://doi.org/10.1016/S0377-0427(99)00360-X

[35] B. Kalantari and J. Gerlach (2000), Newton's method and generation of determinantal family of iteration functions, *J. Comput. Appl. Math.* **116**(1): 195-200. http://doi.org/10.1016/S0377-0427(99)00361-1

[36] B. Kalantari (2008), *Polynimial root-finding and polynomiography*, World Scientific, New Jersey.

[37] D. Kincaid and W. Cheney (1991), *Numerical Analysis*, Brooks/Cole Pub. Co., California.

[38] I.Kiss (1954), Über die Verallgemeinerun des Newtonschen, *Angew. Math. Mech.* 34 , 68-69.

[39] J.König (1954), Über eine eigenshratf der potenzreihen, *Math. Anal.* 23 ,447-449.

[40] G.V. Milovanović (1974), A method to accelerated iterative processes in Banach spaces, *Univ.Beiograd. Publ. Eleektrotehn. Fak. Ser. Mat. Fiz. Mat. Vesnik* **461-497**: 67-71.

[41] G.M. Phillips and P.J. Taylor (1973), *Theory and Applications of Numerical Analysis*, Academic Press, New York.

[42] I. Petković and D. Herceg (2017), Symbolic computation and computer graphics as tools for developing and studying new root-finding methods, *Applied Mathematics and Coomputation* **295**: 95-113.

[43] I. Petković and D. Herceg (2020), Computer tools for the construction and analysis of some efficient root-finding simultaneous methods, *Filomet* **34**(7): 2151-2169. http://doi.org/10.2298/FIL2007151P

[44] M. S. Petković and D. Herceg (1999), On rediscovered iteration methods for solving equations, *J. Comput. Appl. Math.* **107**(2): 275-284. http://doi.org/10.1016/j.cam.2009.09.012

[45] M.S. Petković and L. D. Petković, D. Herceg (2010), On Schröder's families of root-finding methods, *J. Comput. Appl. Math.* **233**(8): 1755-1762. http://doi.org/10.1016/j.cam.2009.09.012

[46] G.M. Phillips and P.J. Taylor (1973), *Theory and applications of Numerical Analysis*, Academic Press, new York.

[47] E.B. Saff and A.D. Snider (2003), *Fundamentals of Complex Analysis, with Applications to Engineering and Science*, 3rd ed., Prentice hall, New Jersey.

[48] E. Schröder (1870), Uber unendlich viele Algorithmen zur Auflosung der Gleichungen, *Math. Ann.* **2**: 317–365. English translation by G.W. Stewart (1992), *On Infinitely Many Algorithms for Solving Equations* (1992), Institute for Advanced Computer Studies, TR-92-121 and TR-2990, University of Maryland.

[49] M.Shub and S.Smale (1985), Computation complexity: On the geometry of polynomials and the theory of cost I, *Ann. Sci. Ecole Norm. Sup.*, 18 , 107-142.

[50] D. Simeunović (2005), On a method for obtaining iterative formulas of higher order, *Mathematica Moravica*, **9**, 53-58.

[51] J.F. Traub (1964), *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, New Jersey.

[52] V.A. Varjuhin and S.A. Kasjanjuk (1969), Ob iteracionnyh metodah utočnenija korneĭ uravneniĭ, *Ž. Vycisl. Mat. i Mat. Fiz.* **9**: 684–687 (in Russian).

[53] J.L. Varona (2002), Graphic and numerical comparaison between iterative methods, *The Mathematical Intelligencer*, vol. 24, no 1, 7-46.

[54] X. Wang (1966), *J. Hangzhou Univ. (Natural Science)*, **3**: 63-70 (in Chinese).

[55] X. Wang and S. Zheng (1984), A family of parallel iterations for finding all roots of a polynomial simultaneously with rapid convergence, *J. Comput. Math.*, 4 , 305-309.

[56] W. Werner (1981), Some improvements of classical iterative methods for the solution of nonlinear equations, in *Numerical Solution of Nonlinear equations*, (Proc., Bremen, 1980), E.L. Allgower, K. Glashoff and H.O. Peitgen, eds, *Lecture Notes in Math.*, 878, 427-440.

[57] T.J. Ypma (1995), Historical development of the Newton-Raphson method, *SIAM Review* **37**(4): 531-551. http://doi.org/10.1137/1037125