MODIFIED (Q, r) INVENTORY CONTROL POLICY FOR AN ASSEMBLE-TO-ORDER ENVIRONMENT

A Dissertation

by

ROBERTO LUIS SEIJO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

Major Subject: Industrial Engineering

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Approved by:

Co-Chairs of Committee, Gary M. Gaukler
César O. Malavé
Committee Members, Barry Lawrence
Don R. Smith
Head of Department, Brett A. Peters

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ABSTRACT

Modified (Q, r) Inventory Control Policy for an Assemble-to-Order Environment.

(August 2009)

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The traditional (Q, r) inventory control model assumes that the date at which the order is entered is the same as the date at which it is requested or expected to be delivered. Hence, the penalty cost is incurred when the customer places the order if inventory is unavailable. This is a reasonable assumption for retail systems and most distribution centers (DC), but not for an assemble-to-order (ATO) environment. In this scenario, there is a delivery time which is usually pre-negotiated and in addition to considering the manufacturing process time and in some cases the outbound transportation time, it also has some safety time built-in. This safety time is defined by the manufacturer and represents information related to when the penalty is incurred. The main objective of this research is to develop a modified (Q, r) policy that incorporates the safety time, and to evaluate this policy in terms of expected inventory cost and expected penalty cost / late orders. The problem is addressed following the heuristic approach discussed by Hadley and Whitin (1963).

Two main models are developed based on the following assumptions: 1) early shipments are allowed by the customer, and 2) no early shipments are allowed. The behavior of both models is analyzed mathematically and by means of numerical examples. It is shown that from a manufacturer perspective, the first model is preferred over the traditional (Q, r) model. However, it poses a threat for the long term business relationship with the customer because the service level deteriorates, and for the implications that early shipments have on the customer inventory. The behavior of the second model is strictly related to the problem being addressed. Its merits with respect to the traditional and the "early shipment" model are discussed. This discussion is centered on the coefficient of variation of the lead-time demand, the ratio (IC/π), and the location of the supplier. A final model which is a hybrid of the previous two shipping policies is developed.

The models developed in the course of this research are generalizations of the traditional (Q, r) model.

DEDICATION

To Mayra, my wife, for letting me love you and sharing the dreams…

To Susan, Roberto and Daniel, my children, for being the most precious gift that the Lord has given me…you make me proud!

To Mamita and Papito, my late parents, for being proud of me and smiling from Heavens…

To my sisters, brother, nephews and nieces, and my family in law…for believing!

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I thank first of all, God, for letting me make this dream come true and giving me the strength and comfort when I needed it most. I thank my wife, Mayra, for her friendship, companionship, patience, understanding and encouragement, and most important for her unconditional love and for being my accomplice during this journey. I also thank with all my heart my children, Susan, Roberto, and Daniel, for making the most out of the endless sacrifices, for understanding and just being good kids.

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I would like to thank the institution of Texas A&M University, the Industrial Engineering and Systems Department and their staff and faculty, for giving me the opportunity to complete my Ph.D. and become a proud aggie. My special gratitude goes to Judy Meeks and Dr. Guy Curry, from the ISEN Graduate Program for all the opportunities provided; and to Dr. Rich Feldman, for listening when I just needed somebody to listen.

The acknowledgements would not be completed without mentioning my fellow graduate students; our new family in Aggieland, the Hispanic Community at Saint Mary's Catholic Church; and the University of Puerto Rico, Mayagüez Campus.

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INTRODUCTION

1.1. Introduction/Motivation

Globalization has increased competition around the world forcing many companies to reduce cost, reduce inventory, and increase availability of products. This situation raises the management question of how much inventory is enough to support the operations. Management is concerned with establishing an inventory control policy that allows them to minimize the cost of the system without negatively impacting the service level offered to the customer.

This research has been motivated by a manufacturing company that follows an assemble-to-order (ATO) strategy. In this scenario, the manufacturer does not keep finished goods inventory and the final assembly of the product is hold until a firm customer order is received. A delivery time that is usually pre-negotiated is promised to the customer. It is common practice to define a delivery time larger than the time of all the processes required to deliver the final product (assembly, test/inspection, packing, shipping, and in some cases the outbound transportation time). ATO manufacturers add a safety time when defining the delivery time. This is done to cover for uncertainty, as in the case of a make-to-stock product (MTS) the safety stock is used for protection against demand variability. The length of this safety time depends on the industry and the particular manufacturing characteristics of the company, such as:

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This dissertation follows the style of *IIE Transactions*.

- Level of customization, production volume, and demand variability/uncertainty,
- Components availability from both, inventoried as well as buy-to-order items,
- Capacity constraints and manufacturing flexibility,
- Market considerations, competition, and cost of not meeting customer expectations.

Existence of such a safety time is for example implicitly assumed in Wemmerlov (1984) when he specifies that components not available at time of order booking must be acquired or made available in time for the final assembly.

Most components used to assemble the final product are inventory items. As result of the safety time, the unavailability of any component at time of the customer order entry does not necessarily translate in missing the delivery/promised date and incurring penalty cost. Once the customer order is entered, the safety time becomes information that the manufacturer has in advance related to when the penalty is incurred. By recognizing this, we propose to modify the (Q, r) inventory control policy that is used for the inventory item to account for the safety time. This is a generalization of a well-known policy that allows it to be useful for any application in which the penalty is not incurred at time of the customer order entry if inventory is unavailable. The model studied by Hadley and Whitin (1963) is the special case of the models studied in this research when there is no safety time. These modified policies have application not only in the ATO arena but also for distribution centers (DC) in which the customer enters the order in advance such that the order entry date is not the same as the request date.

The intuitive conclusion is that these modified policies should perform better than the traditional. Let us say that the lead-time from the supplier is a constant l, the safety time is a constant d, and $l < d$, then there is no chance of incurring penalty since the replenishment order of the inventory component is always received ahead of the delivery date of the final product. Moreover, delaying the placement of the replenishment order by $d - l$ units of time reduces the inventory cost without incurring penalty. For the case of a random supply lead-time L, it is appealing to think that d acts as cushion or safety against demand variability, resulting not only in lower total cost but less inventory and higher service level. However, we will demonstrate that this "common sense" expected result is not necessarily true.

1.2. Problem Description

The traditional (Q, r) inventory control policy does not model scenarios in which the penalty cost is not incurred at time of the customer order entry if inventory is unavailable. It assumes that the date at which the order is entered is the same as the date at which it is requested or expected to be delivered. Hence, if a penalty is incurred, it occurs when the customer places the order. This is a reasonable assumption for retail systems and most distribution centers (DC), but not for an assemble-to-order (ATO) environment. In this scenario, there is a delivery time that has a safety time built-in, and the unavailability of any item/component delays the delivery of the product causing a penalty to be incurred, not at time of the customer order entry date but several units of time later as given by the safety time.

Glasserman and Wang (1998), under a base-stock policy assumed that the customer order is fulfilled as soon as the components required to assembly the final product are available. The time elapsed from when a customer order is received until it is delivered is zero if the order is filled from on-hand inventory, and strictly positive otherwise. Any order filled within the delivery lead-time is assumed to be on-time. These assumptions could be characterized by a scenario in which early shipments are allowed by the customer. This scenario is not compatible with the traditional (Q, r) policy since this implies the occurrence of stock-outs/backorders without incurring penalty. This is the result of the fact that the traditional model does not account for the safety time as a parameter. Scenarios in which early shipment is allowed can be modeled by modifying the traditional policy to account for the safety time as a parameter.

It can be argued that shipping early has negative implications with the customer inventory if he/she can not convert into sales the products received ahead of time as soon as they arrive; and a knowledgeable customer might impose a policy where shipping is only allowed at the delivery date. Allocation of inventory to the customer order is required in this scenario and can also be modeled with the (Q, r) policy by accounting for the safety time as a parameter.

The heuristic approach explained by Hadley and Whitin (1963) is followed in order to incorporate the safety cushion time under the two shipping extreme scenarios: 1) early shipment is allowed, and 2) early shipment is not allowed. It is important to understand the behavior of both models with respect to increases in the safety time in order to assure convexity. This is a consequence of the assumption that the expected number of backorders is negligible with respect to the expected number of units being inventoried at any time,

1.3. Literature Review

Most of the literature related to inventory control management and supply chain management (SCM) assumes that the product of interest is a make-to-stock (MTS) item. An example could be a product that is expected to sell “off the shelf” by a retailer or a wholesaler. The main focus is to solve the strategic problem of safety stock placement in order to minimize the holding cost of the supply chain or inventory system. Consideration to product availability is given by means of setting a target level for some customer service metric or by means of a shortage/penalty cost which is imposed when the customer order is received and the product is not available on the shelf. Another stream of literature related to MTS focuses on the benefit of information sharing in the supply chain.

Rosling (1989), Gallego and Zipkin (1999), Cachon and Zipkin (1999), Daniel and Rajendran (2006), Gallego and Özer (2005), and Barnes-Schuster et al. (2006) are examples of MTS scenarios where consideration to product availability is given by means of a penalty cost. Inderfurth (1991), Lee and Billington (1993), Ettl et al. (2000), Graves and Willems (2000), and Magnanti et al. (2006) considered some kind of service target level. Gavirneni et al. (1999) and Moinzadeh (2002) studied the benefits of advanced information for making replenishment decisions. The research presented in this dissertation differentiates from the work mentioned above in the modeling of the safety time.

Another stream of literature relates to assemble-to-order (ATO) products. In this case, final assembly of the product is not executed until a firm customer order with all specifications is received. Please refer to Wemmerlöv (1984) for more information related to ATO manufacturing and its implications for materials management. It is usually assumed that any item, component or subassembly used in the assembly of the final product is stocked following a base-stock policy. Glasserman and Wang (1998) studied the trade-off between inventory levels and delivery lead-time offered to customers in achieving a target level of service. Song et al. (1999) performed an exact analysis on a wide range of performance measures in the ATO system. Song and Yao (2002) addressed the trade-off for the case of multiple components and a single product. In addition to the inventory policy being used, the main difference that this study has with the papers mentioned in this paragraph is their focus to quantify the trade-off between inventory and the delivery time under service level guarantees. Specifically, this research does not focus in the trade-off but in understanding the behavior of the total cost (ordering, inventory and penalty) as the delivery time increases (as result of an increase in the safety time) constrained by the penalty cost. In fact, the service level is calculated as result of the optimal policy being selected.

Another venue of research interest in the ATO scenario is related to contract manufacturing. Hsu et al. (2006) developed and analyzed an optimization model to determine the optimal stocking quantities for components of an ATO product in an environment where demand is uncertain and the price for the final product and the cost of components depend on their delivery lead-times. Fu et al. (2006) studied the case in which the contract manufacturer anticipates an order of a single product with uncertain quantity and may need to procure components or even assemble some quantities of the final product before receiving the confirmation of the actual order quantity. The main difference between the previous cited work and this study is in the nature of the problem being addressed and the subsequent implications, in the sense that for our case, the ATO manufacturer is not a contract manufacturer.

In the ATO scenario, consideration to product availability is commonly given by means of customer service metrics. Nevertheless, there are practical situations in which it is of interest to consider a penalty cost as part of the analysis as we will do in this research. For example, consider the following two cases: 1) as in Fu et al. (2006) the manufacturer incurs in high penalty cost imposed by the customer, and 2) as an attempt to meet delivery dates the manufacturer incurs in expediting cost that include overtime and the use of the expensive air transportation (inbound and outbound) as result of the supplier's and customer's locations. Because of the high cost nature of these penalties, management could be interested in modeling the penalty and then determining the appropriate service level as an output rather than setting it as an input parameter. However, since the customer order entry and the request dates are not the same and there is a safety time built in the delivery date, the penalty is only incurred if the final product is not delivered on-time and not if there is a component shortage at time of order entry.

In summary, most of the literature related to ATO products addresses the problem of inventory management based on service metrics rather than modeling the penalty cost. Recognizing that the safety time is information that the manufacturer has, and by considering its inclusion as a parameter of the inventory policy we pretend to contribute to the literature in particular by generalizing the well known (Q, r) policy.

1.4. System Description

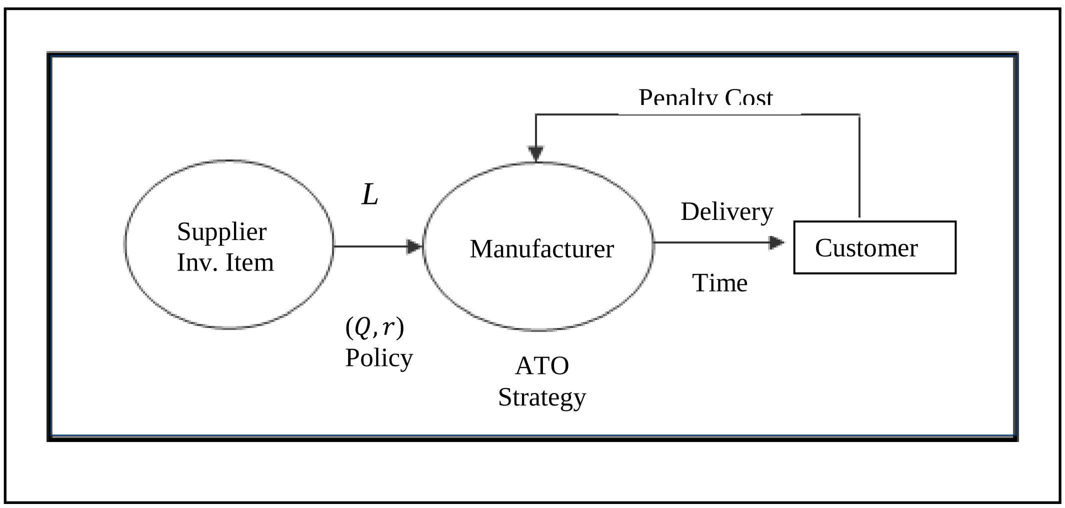


Figure 1: Supply chain representation.

This paper focuses on the simplest ATO scenario which is modeled as one un-capacitated production step that requires the use of an externally sourced component which is kept in stock based on a (Q, r) policy as shown in Figure 1. The externally sourced component has random lead-time L with known probability distribution function $g(l)$. The demand distribution for the final product is assumed known and stationary; each customer order requests only one unit of the final product. The manufacturer publishes a delivery time (DT) which is defined as the sum between the constant manufacturing time (MT) required for the assembly of the final product and some constant safety time (d).

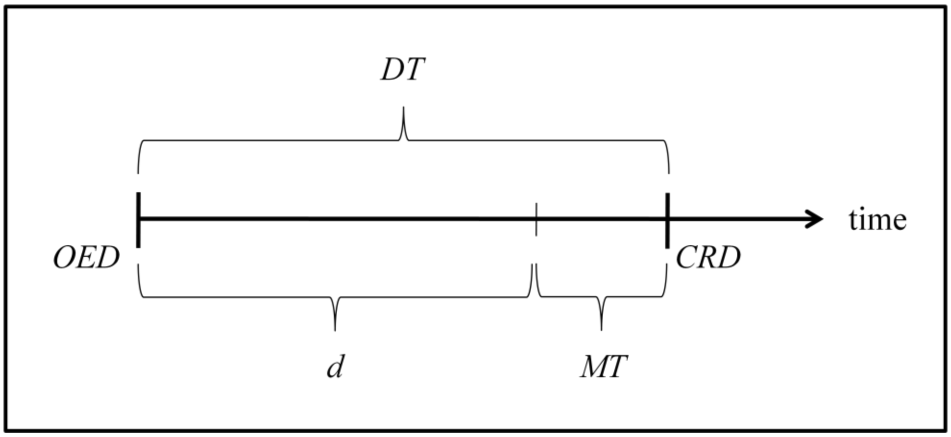


Figure 2: Safety time (d).

The customer takes into account the delivery time when placing orders, such that the delivery time is also the difference between the date by which the customer requests the shipment of the final product (Customer Request Date - CRD) and the date by which the customer enters or places the order (Order Entry Date - OED). We can define the safety time (d) as follows:

$$d = DT - MT$$

$$= (CRD - OED) - MT. \quad (1.1)$$

When a firm customer order is received, the penalty cost is incurred if the component is not available within d units of time as shown in Figure 2. Note that under this scenario we are assuming that the manufacturing time can not be shortened and that there are no emergency replenishment orders whenever a penalty is foreseeable in the future.

1.5. Objective/Scope

The main objective of this research is to derive a generalized (Q, r) inventory control policy that incorporates the safety time, and to evaluate this policy in terms of expected inventory cost and expected penalty cost / late orders. The goal is to develop an inventory policy that performs better than the traditional (Q, r) for applications in which the penalty is not incurred at time of the customer order entry but some units of time in the future if inventory is unavailable. Without loss of generalization, the focus is to model an ATO scenario with some safety time added to the manufacturing time in order to define the delivery time.

The safety time is modeled for both extremes of the shipping spectrum, when early shipment is allowed by the customer and when early shipment is not allowed. The behavior of both models is analyzed with respect to the safety time. This is done mathematically and by means of numerical examples. Conclusions are drawn from the perspective of the manufacturer as well as the customer.

Special attention is given to the case when there is a minimum lead-time (l_{min}) or a period of time in which there is no chance for receiving a replenishment order. Understanding the behavior of each model during this period of time is of practical value since this period could be related to the location of the supplier. In addition, for the modified (Q, r) model with the "no-early" shipment policy importance is given to determine under what circumstances it is beneficial to delay the replenishment order.

Finally, a hybrid model is developed which is a generalization of the (Q, r) model with respect to the safety time and the shipping policy being used. The traditional model, as well as the "early shipment" and the "no-early shipment" modified models are special cases of the hybrid model.

1.6. Summary

This chapter includes the motivation, a description of the problem and the system being modeled, and the objectives and scope of this research. It also presents a summary of supply chain and inventory management literature related to make-to-stock (MTS) and assemble-to-order (ATO) products in order to highlight the contribution of this research.

It is mentioned that the traditional (Q, r) inventory control policy does not model scenarios in which the penalty is not incurred at time of the customer order entry if inventory is unavailable. An ATO manufacturer is a good example of a scenario with this characteristic since there is a delivery time that usually considers some safety time. Modeling this safety time in the context of the (Q, r) policy is the main objective of this research. This research looks to generalize the traditional (Q, r) policy under two different shipping perspectives: 1) shipping early to the customer is allowed, and 2) shipping early is not allowed.

The next chapter discusses the modified (Q, r) model with the "shipping early" policy. The behavior of the model with respect to the safety time is explained mathematically and by means of numerical examples.

CHAPTER II

MODIFIED (Q, r) MODEL WITH THE EARLY SHIPMENT POLICY

2.1. Introduction

This chapter describes the modified (Q, r) inventory control policy under a scenario in which early shipments are allowed by the customer. This follows from the assumption that the customer does not penalize the manufacturer for delivering early. One could think of a scenario in which the customer does not monitor early shipments as part of their inventory control procedures.

The safety time (d) is modeled under the perspective that the manufacturer releases the customer order to the shop floor as soon as the order is received if inventory of the component is available, and the final product is shipped d units of time ahead of the delivery date. If inventory is not available, the order is released as soon as the replenishment material is received and the penalty is incurred only if the component becomes available d units of time after receiving the customer order.

It is appealing to think that shipping ahead should not only reduce the inventory of the manufacturer but improve the service level provided to the customer. As we will see in this chapter the reduction of the total cost of the system comes with a trade-off in service performance. The customer will also face an increase in inventory which is the direct result of the early shipment if he/she cannot convert into sales the product received ahead of the delivery date as soon as it arrives.

2.2. Methodology

The modified (Q, r) inventory control policy for the case when early shipments are allowed to the customer is modeled following the heuristic approach discussed by Hadley and Whitin (1963). This approach assumes that the expected number of backorders/stock-outs is negligible with respect to the number of components being inventoried at any time. This assumption must be true for any particular safety time (d) in order for the model to remain valid. This observation requires the analysis of the behavior of the model with respect to d in order to assure the convexity of the model with respect to Q and r in the region defined by Brooks and Lu (1968).

The first and second order conditions are used to show that there is a $\hat{d}$, such that for any d in the range given by $[0, \hat{d})$ the model is convex with respect to Q and r. Partial derivatives and total derivatives with respect to d are calculated in order to determine the behavior of the model as d increases in the range mentioned before. Special attention is given to show the behavior of the model for the case when there is a minimum lead-time l_{min} and to show how the policy can be used for any $d \geq \hat{d}$.

Three sets of numerical examples are run assuming that the lead-time follows the following distributions: expo (β), normal $(\mu_{LT}, \sigma_{LT}^2)$, and uniform (a, b). For each case study, the lead-time demand is assumed to follow a normal (μ, σ^2) distribution function.

2.3. Description of the Model

The major difference between the modified (Q, r) model with the "early shipment" policy and the traditional (Q, r) model is related to the penalty cost structure.

From Figure 3, it can be observed that the inventory item/component does not need to be readily available at time of the customer order entry. The chance for incurring the penalty cost in the modified model is related to $P(L \geq d)$ which is the probability for the replenishment order taking longer than the safety time to arrive, and as such, the delivery of the final product is delayed and a penalty incurred.

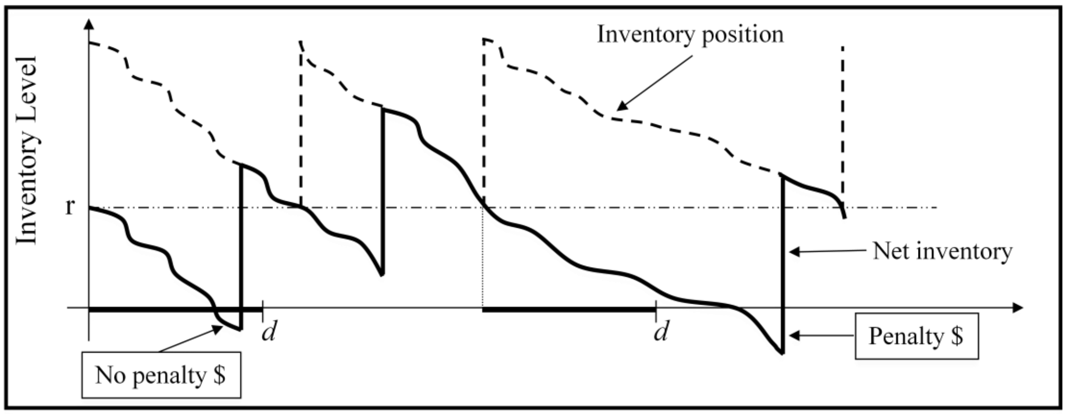


Figure 3: Behavior of the "early shipment" model.

The following notation is used in order to define the cost function (K):

- X is the random lead-time demand;
- $f(x;l)dx$ is the probability that the number of units demanded during the lead-time lies between x and $x + dx$;
- $g(l)dl$ is the probability that the lead-time for the replenishment/procurement order lies between l and $l + dl$;
- d is the cushion or safety time by which no penalty is incurred if material is not available at time of the customer order entry;

- $h(x)$ is the marginal distribution for the lead-time demand,

$$h(x) = \int_{l=d}^{\infty} f(x; l)g(l)dl\,; \tag{2.1}$$

- μ is the expected lead-time demand,

$$\mu = \int_{x=0}^{\infty} xh(x)dx\,; \tag{2.2}$$

- $0 < Q < \infty$ is the procurement/replenishment order size;
- $u \leq r < \infty$ is the reorder level that triggers the procurement order by means of the inventory position, where u is a non-negative number such that $h(x)$ is non-increasing for any $x \geq u$;
- λ is the average annual demand which is constant over time;
- A is the cost of placing an order with the supplier;
- IC is the average cost of carrying inventory per unit per unit time;
- π is the penalty cost per unit incurred when the requested customer date is missed;
- $G(d)$ is the complementary cumulative distribution of $g(d)$, or $P(L \geq d)$,

$$G(d) = \int_{l=d}^{\infty} g(l)dl\,; \tag{2.3}$$

- $\bar{\eta}_d(r)$ is the expected number of units incurring the penalty cost per cycle.

The same assumptions as in Hadley and Whitin (1963) are being followed in order to define the model. These are enumerated as follows:

1) The unit cost C of the item is a constant independent of Q;
2) There is never more than a single order outstanding;

3) The cost of operating the information processing system is independent of Q and r;

4) The reorder point r is positive.

The only major change is that π has been assumed to be the penalty cost incurred when the customer order for the finished product is overdue or late based on its request/delivery date since backorders/stock-outs at the inventory items are allowed at time of booking the customer order. Despite the fact that backorders are allowed, Hadley's and Whitin's assumption that the average number of backorders is negligible as compared to the average inventory at any time is being followed. Hence, the expected on-hand inventory is equal to the expected net inventory in order to calculate the cost structure related to the inventory cost.

In order to define the average annual penalty cost, note that the penalty is incurred only when $l \geq d$ and the lead-time demand x is greater than the reorder point r. The number of customer orders incurring penalty in a cycle is:

$$\eta_d(x,r) = \begin{cases} 0, & x - r < 0, l \geq d; \\ x - r, & x - r \geq 0, l \geq d; \\ 0, & l < d. \end{cases} \tag{2.4}$$

The expected number of orders incurring penalty per cycle is:

$$\begin{aligned} \bar{\eta}_d(r) &= \left[\int_{x=0}^{\infty} \eta_d(x,r) h(x) dx\right] P(L \geq d) \\ &= \left[\int_{x=r}^{\infty} (x - r) h(x) dx\right] G(d) \\ &= \left[\int_{x=r}^{\infty} x h(x) dx - r H(r)\right] G(d), \end{aligned} \tag{2.5}$$

where $H(x)$ is the complementary cumulative distribution of $h(x)$. Note that this is similar to Hadley and Whitin (1963) except for $G(d)$. In other words, the expected number of orders incurring penalty per cycle is the expected number of backorders per cycle times $P(L \geq d)$. The expected annual penalty cost is defined as

$$\frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} xh(x)dx - rH(r)\right]G(d), \tag{2.6}$$

and the expected annual cost K for the shipping early policy can then be defined as follows:

$$K = \frac{\lambda}{Q}A + IC\left[\frac{Q}{2} + r - \mu\right] + \frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} xh(x)dx - rH(r)\right]G(d). \tag{2.7}$$

The behavior with respect to the safety time d of the modified (Q, r) model with the "early shipment" policy is analyzed in the following section.

2.4. Analysis/Behavior of the Model

The safety time (d) is defined by the manufacturer and represents the time when the penalty is incurred once the customer order is entered. In that sense, d is information that the manufacturer possesses and understanding the behavior of the cost function K with respect to d and its implications with the customer can be of strategic importance.

2.4.1. Convexity with Respect to Q and r, and Optimality

In order to understand the behavior of the model let us first define the following lemmas related to the convexity of the cost function for any particular d , namely K_d, and some important relationships at optimality.

Lemma 2.4.1.1: *For any d, the cost function K_d is jointly convex in Q and r in the range* $0 < Q < \infty$ *and* $u \leq r < \infty$.

Proof: It can easily be concluded from (2.7) that the cost function given by the expected ordering cost and the expected inventory cost is jointly convex with respect to Q and r in the range $0 < Q < \infty$ and $u \leq r < \infty$. Brooks and Lu (1968) showed that for a number μ, the expected backorders per year is convex in the region given by $0 < Q < \infty$ and $\mu \leq r < \infty$, if the probability density function for the lead-time demand (X) is non-increasing for $x \geq \mu$. In our case, since $G(d)$ is not a function of Q nor r, and u has been defined as a non-negative number such that $h(x)$ is non-increasing for any $x \geq u$, it must be concluded that the expected number of backorders per year as given in (2.7) must be jointly convex with respect to Q and r in the region $0 < Q < \infty$ and $u \leq r < \infty$. Then, K_d is jointly convex with respect to Q and r in the region of interest since the addition of convex functions gives a convex function. Note that $u = \mu$ for the special case in which $h(x)$ is the probability density function of a normal lead-time demand. ■

Lemma 2.4.1.2: *In the range in which K_d is jointly convex in Q and r, the iterative procedure from Hadley and Whitin (1963) can be used to obtain the optimal policy* $(Q_d^*,\ r_d^*)$.

Proof: Let us assume that for a particular d, (Q_d^*, r_d^*) is the optimal policy satisfying $0 < Q < \infty$ and $u \leq r < \infty$. Then, (Q_d^*, r_d^*) must satisfy the set of equations (2.8) and (2.9) derived from the 1st order conditions.

$$\frac{\partial K}{\partial Q} = 0 \quad \xrightarrow{yields} \quad Q = \sqrt{\frac{2\lambda}{IC}[A + \pi \bar{n}_d(r)]} \tag{2.8}$$

$$\frac{\partial K}{\partial r} = 0 \xrightarrow{yields} H(r)G(d) = \frac{QIC}{\pi\lambda} \tag{2.9}$$

(2.10) is obtained by rearranging (2.9) in terms of Q.

$$Q = \frac{\pi\lambda}{IC} G(d)H(r) \tag{2.10}$$

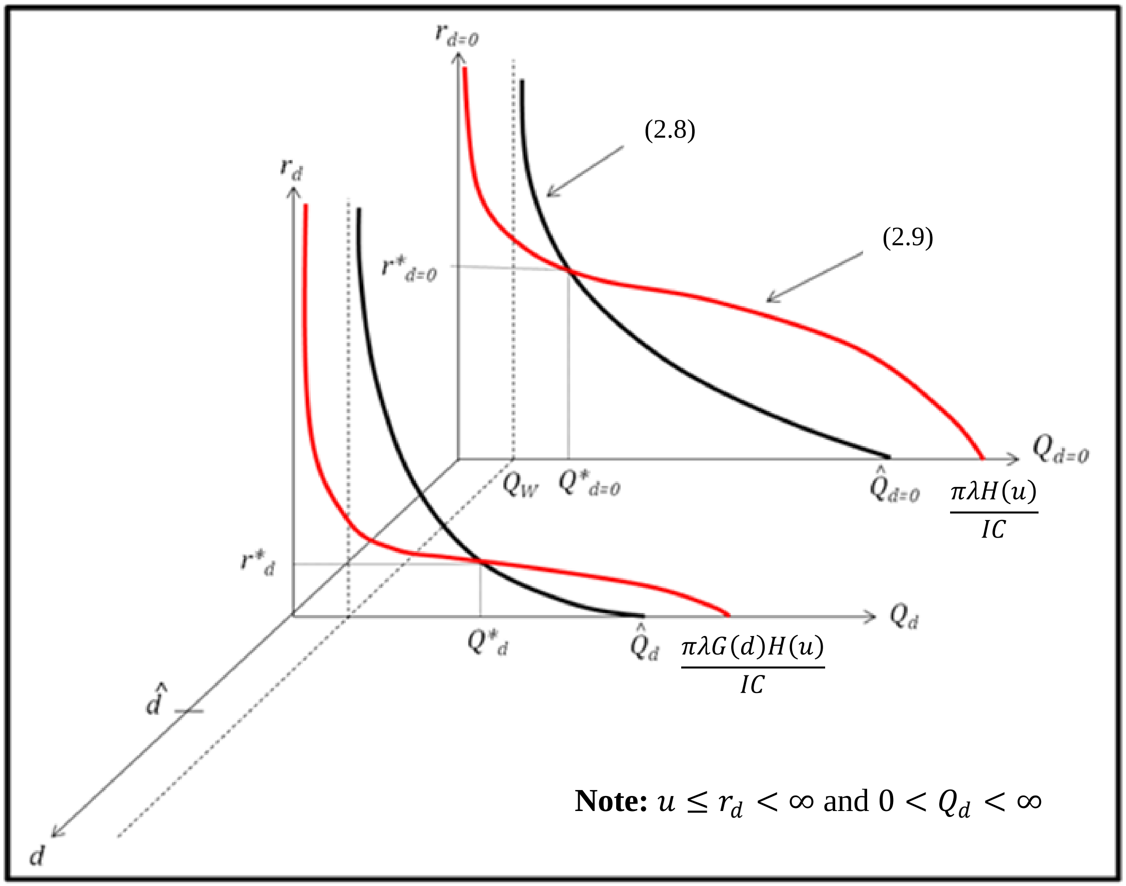


Figure 4: Behavior of Q^* and r^* with respect to d.

Note that both, (2.8) and (2.10) are functions of r representing a curve in a plane with axis given by Q and r as shown in Figure 4; and the specific optimal policy (Q_d^*, r_d^*) must be found at the intersection of both curves. Hadley and Whitin (1963) showed that

Q and r are inversely related. This relationship holds for this case since $G(d)$ is not a function of Q nor r. In addition, let us observe from (2.8) and (2.10) that for $u \leq r < \infty$,

$$Q_W = \sqrt{2\lambda A/IC} \leq Q \leq \frac{\pi\lambda G(d)H(r=u)}{IC}. \tag{2.11}$$

From these observations it follows that Q_W can be set as the initial Q and (2.9) can be used to find the corresponding initial r. Then, (2.8) can be used with this initial r in order to find a new Q. This iterative procedure continues until it converges to (Q_d^*, r_d^*). Note that this is the iterative procedure from Hadley and Whitin (1963) and its convergence was proved by them. ■

Lemma 2.4.1.3: *There is a* $\hat{d}$ *such that for any* $d \geq \hat{d}$*, there is no optimal policy* (Q_d^*, r_d^*) *in the range given by* $0 < Q < \infty$ *and* $u \leq r < \infty$.

Proof: Let us assume that for a particular d, (Q_d^*, r_d^*) is the optimal policy satisfying $0 < Q < \infty$ and $u \leq r < \infty$. Then, from (2.11) and (2.8) it follows that

$$\hat{Q}_d \leq \frac{\pi\lambda}{IC} G(d)H(u), \tag{2.12}$$

where,

$$\hat{Q}_d = \sqrt{2\lambda\left[A + \pi\left[\int_{x=u}^{\infty} xh(x)dx - uH(u)\right]G(d)\right]/IC}. \tag{2.13}$$

Because of $G(d)$, both sides of (2.12) are decreasing functions of d. It can be observed that as d increases $\hat{Q}_d$ is bounded by Q_W and there must be a $\hat{d}$ such that

$$\hat{Q}_{\hat{d}} > \frac{\pi\lambda G(\hat{d})H(u)}{IC}, \tag{2.14}$$

and (2.12) is not satisfied for any $d \geq \hat{d}$. Then, there is no optimal policy (Q_d^*, r_d^*) in the region given by $0 < Q < \infty$ and $u \leq r < \infty$. ∎

Note that $\hat{d}$ is defined as the smallest d not satisfying (2.12) and can be computed as $\hat{d} \coloneqq min\{d \geq 0 \big| \hat{Q}_d > \pi\lambda G(d)H(u)/IC\}$.

Lemma 2.4.1.4: *If the optimal policy* (Q_d^*, r_d^*) *exists and satisfies* $0 < Q < \infty$ and $u \leq r < \infty$*, then* (2.15) *is satisfied.*

$$Q_d^* > \frac{H(r_d^*)}{h(r_d^*)} \tag{2.15}$$

Proof: From Lemma 2.4.1.1 it is known that the cost function K_d is jointly convex in Q and r in the range $0 < Q < \infty$ and $u \leq r < \infty$. Then, the $\nabla^2 K_d$ evaluated at the optimal policy (Q_d^*, r_d^*) must be positive definite.

$$\begin{bmatrix} {}^{IC}\!/_{Q_d^*} & {}^{IC}\!/_{Q_d^*} \\ {}^{IC}\!/_{Q_d^*} & \frac{\pi\lambda}{Q_d^*} h(r_d^*)G(d) \end{bmatrix} \tag{2.16}$$

The leading principal minors of the matrix (2.16) must be positive. This is true for the leading principal minor given by the upper left-hand corner of the matrix and must be true for the leading principal minor given by the matrix itself.

$$det \begin{bmatrix} {}^{IC}\!/_{Q_d^*} & {}^{IC}\!/_{Q_d^*} \\ {}^{IC}\!/_{Q_d^*} & \frac{\pi\lambda}{Q_d^*} h(r_d^*)G(d) \end{bmatrix} = \frac{IC\pi\lambda}{{Q_d^*}^2} h(r_d^*)G(d) - \frac{IC^2}{{Q_d^*}^2} > 0 \tag{2.17}$$

We get (2.18) from (2.17).

$$\frac{Q_d^*}{H(r_d^*)G(d)} > \frac{1}{h(r_d^*)G(d)} \tag{2.18}$$

From (2.18) we must conclude that (2.15) is a necessary condition for K_d to be convex. Since we have previously assumed that K_d is convex, then (2.15) is satisfied. ∎

Lemma 2.4.1.5: *If the optimal policy* (Q_d^*, r_d^*) *exists and satisfies* $0 < Q < \infty$ and $u \leq r < \infty$, *then* (2.19) *is satisfied.*

$$Q_d^* > \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)} \tag{2.19}$$

Proof: From Lemma 2.4.1.2 it is known that the iterative procedure explained in Hadley and Whitin (1963) can be used to find the optimal policy which is found at the intersection of the curves given by (2.8) and (2.10).

$$\frac{\pi\lambda}{IC} H(r_d^*) G(d) = \sqrt{\frac{2\lambda}{IC}\left[A + \pi\overline{\eta_d}(r_d^*)\right]} \tag{2.20}$$

We get (2.21) from (2.20).

$$\frac{Q_d^*}{2} = \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)} + \frac{\pi\lambda}{Q_d^* IC} \tag{2.21}$$

From (2.21) we can observe that (2.19) must be true. ∎

2.4.2. Behavior of the Optimal Policy and Cost Function with Respect to d

Let us continue with the understanding of the behavior of the model by calculating the partial derivative of K with respect to d:

$$\frac{\partial K}{\partial d} = -\frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} xh(x)dx - rH(r)\right] g(d) \leq 0 \qquad \forall\, d, Q, r. \tag{2.22}$$

Note that depending upon the lead-time distribution function $g(l)$, $g(d) \geq 0$ and $\partial K/\partial d \leq 0$. This observation leads to the following lemma:

Lemma 2.4.2.1: *Let us define K^* as a function of d that represents the behavior of the optimal cost function as d increases in the range given by $[0, \hat{d})$, then K^* is a non-increasing function.*

Proof: Note from (2.22) that $\partial K \,/\, \partial d < 0$ for any range of $d's$ in which $g(d) > 0$. Then, defining the safety time d such that $d < \hat{d}$, defining $(Q^*_{d-\Delta d}, r^*_{d-\Delta d})$ as the optimal policy for the safety time $d - \Delta d$ and arbitrarily using this policy to calculate the cost function K_d, we must conclude from (2.22) that $K_d < K^*_{d-\Delta d}$, and that there is an optimal policy (Q^*_d, r^*_d) such that $K^*_d \leq K_d < K^*_{d-\Delta d}$. Now, defining $d + \Delta d$ such that $d - \Delta d < d < d + \Delta d < \hat{d}$ and arbitrarily using the optimal policy (Q^*_d, r^*_d) for $d + \Delta d$, we must conclude from (2.22) as well that $K_{d+\Delta d} < K^*_d$, and that there is an optimal policy $(Q^*_{d+\Delta d}, r^*_{d+\Delta d})$ such that $K^*_{d+\Delta d} \leq K_{d+\Delta d} < K^*_d$. This iterative procedure can be continued as d increases and K^* becomes a strictly decreasing function of d when $g(d) > 0$.

Note from (2.22) that $\partial K/\partial d = 0$ for any range of $d's$ in which $g(d) = 0$, then the same iterative procedure as in the previous paragraph can be followed in order to show that K^* is constant. It follows that the optimal policy (Q^*_d, r^*_d) for any d in the range in which $g(d) = 0$ is constant as well.

We have shown that K^* can be constant or strictly decreasing depending on the value of $g(d)$, but never an increasing function of d. From these observations we must conclude that K^* behaves as a non-increasing function of d. ■

Let us look into the following generalized practical scenario in order to understand the implications of Lemma 2.4.2.1. Let us define d_1, such that $d_1 \geq 0$ and $P(L \geq d_1) = 1$ but $P(L \geq d_1 + \Delta d) < 1$. In addition, $g(d) = 0$ in the range $[0, d_1)$ and $g(d) > 0$ in the range $\left[d_1, \hat{d}\right]$. Assuming that a customer order that triggers a replenishment order is received at time $t = 0$, there is no chance for receiving the replenishment order before d_1 units of time since the minimum lead-time (l_{min}) is occurring at time $t = d_1$. For practical purposes, $d_1 = 0$ can be related to a supplier located next to the manufacturer, who can deliver as soon as the replenishment order is placed because he/she keeps available inventory of the component. On the other hand, a d_1 relatively larger than zero could be related to a supplier located far away from the manufacturer such that the replenishment order is received the earliest d_1 units of time after being placed.

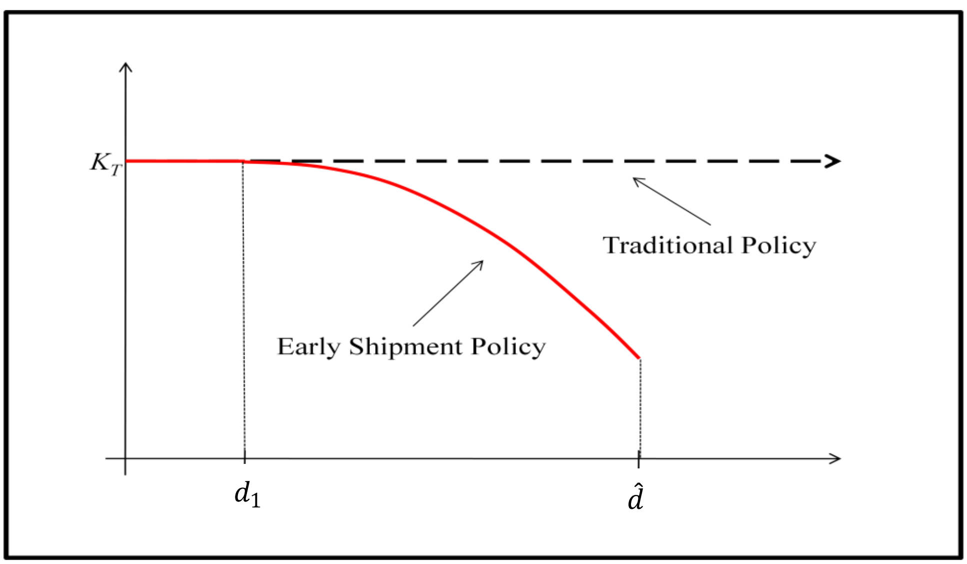


Figure 5: "Early shipment" (Q, r) policy vs. traditional (Q, r) policy.

Note from the proof of Lemma 2.4.2.1 that K^* must be constant for the range given by $[0, \mathrm{d}_1]$ since $g(d) = 0$, and the optimal policy for any given d in that range is the same as in the traditional policy $(Q^*_{d=0}, r^*_{d=0})$. In addition, K^* is strictly decreasing for the range given by $(d_1, \hat{d})$ since $g(d) > 0$. The behavior for this generalized practical scenario is shown in Figure 5.

From the previous lemma and generalized scenario, we can conclude that the manufacturer will prefer the modified model over the traditional for any $d > l_{min}$.

We now have all the information needed to understand the behavior of Q^* and r^* as d increases, which is explained in the following theorem:

Theorem 2.4.2.2: *Let Q^* and r^* be functions of d representing the behavior of Q^*_d and r^*_d as d increases in the range $[0, \hat{d})$, then $dr^*/dd \leq 0$ and the direction of dQ^*/dd depends on the problem being addressed.*

Proof: We get (2.23) by rearranging (2.9) in terms of $H(r^*)$.

$$H(r^*) = \frac{IC}{\lambda\pi} Q^* G(d)^{-1} \tag{2.23}$$

We know from its definition that $H(r^*) = \int_{x=r^*}^{\infty} h(x)dx$. Realizing that both sides of the resulting equation (2.23) are functions of d, we proceed to calculate the derivative with respect to d at both sides in order to get (2.24).

$$-h(r^*)\frac{\mathrm{d}r^*}{\mathrm{d}d} = \frac{IC}{\pi\lambda}\left[\frac{Q^*}{G(d)^2}g(d) + \frac{1}{G(d)}\frac{\mathrm{d}Q^*}{\mathrm{d}d}\right]$$

$$\frac{\mathrm{d}r^*}{\mathrm{d}d} = -\frac{H(r^*)G(d)}{Qh(r^*)}\left[\frac{Q^*}{G(d)^2}g(d) + \frac{1}{G(d)}\frac{\mathrm{d}Q^*}{\mathrm{d}d}\right]$$

$$\frac{\mathrm{d}r^*}{\mathrm{d}d} = -\frac{H(r^*)}{Q^*h(r^*)}\left[Q^*\frac{g(d)}{G(d)} + \frac{\mathrm{d}Q^*}{\mathrm{d}d}\right] \tag{2.24}$$

Q^* is given by (2.8), deriving it with respect to d we get (2.25).

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d} = \frac{\pi\lambda}{Q^*IC}\left[-\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]g(d) - H(r^*)G(d)\frac{\mathrm{d}r^*}{\mathrm{d}d}\right]$$

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d} = -\frac{1}{H(r^*)G(d)}\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]g(d) - \frac{\mathrm{d}r^*}{\mathrm{d}d} \tag{2.25}$$

We get equation (2.26) by replacing from (2.25) the $\mathrm{d}r^*/\mathrm{d}d$ with (2.24).

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d} = \frac{g(d)}{G(d)}\left[\frac{H(r^*)}{h(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]}{H(r^*)}\right]\left[1 - \frac{H(r^*)}{Q^*h(r^*)}\right]^{-1} \tag{2.26}$$

We get (2.27) by replacing from (2.24) the $\mathrm{d}Q^*/\mathrm{d}d$ with (2.26).

$$\frac{\mathrm{d}r^*}{\mathrm{d}d} = -\frac{H(r^*)}{Q^*h(r^*)}\frac{g(d)}{G(d)}\left[Q^* + \left[\frac{H(r^*)}{h(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]}{H(r^*)}\right]\left[1 - \frac{H(r^*)}{Q^*h(r^*)}\right]^{-1}\right] \tag{2.27}$$

Note that it is trivial to conclude that $\mathrm{d}r^*/\mathrm{d}d = 0$ and $\mathrm{d}Q^*/\mathrm{d}d = 0$ whenever $g(d) = 0$. This observation should not take us by surprise because it was already shown during the second part of the proof related to Lemma 2.4.2.1. However, for the rest of this proof the non-trivial case when $g(d) > 0$ is addressed. Note from Lemma 2.4.1.4 that $[1 - H(r^*)/Q^*h(r^*)]^{-1} > 0$, then the sign of $\mathrm{d}r^*/\mathrm{d}d$, and $\mathrm{d}Q^*/\mathrm{d}d$ are both dependent on (2.28).

$$\left[\frac{H(r^*)}{h(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]}{H(r^*)}\right] \tag{2.28}$$

It can easily be observed that $\mathrm{d}r^*/\mathrm{d}d < 0$ and $\mathrm{d}Q^*/\mathrm{d}d \geq 0$ if (2.28) is greater than or equal to zero for any particular d in the given range $[0, \hat{d})$. In order to show that this is possible, let us rearrange (2.9) to define $H(r_d^*)/h(r_d^*)$ as in (2.29).

$$\frac{H(r_d^*)}{h(r_d^*)} = Q_d^* \frac{IC}{\pi\lambda G(d)h(r_d^*)} \tag{2.29}$$

From (2.29) we can conclude that (2.30) is true if (2.28) is greater than or equal to zero.

$$Q_d^* \frac{IC}{\pi\lambda G(d)h(r_d^*)} \geq \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)} \tag{2.30}$$

Note from the previous expression that it is possible for Lemma 2.4.1.5 to hold since $IC/[\pi\lambda G(d)h(r_d^*)] < 1$ as can be deduced from (2.29) and Lemma 2.4.1.4. Then, (2.28) can be greater than or equal to zero, $\mathrm{d}r^*/\mathrm{d}d < 0$ and $\mathrm{d}Q^*/\mathrm{d}d \geq 0$.

Let us verify now the possibility for (2.28) to be less than zero for any d in the range $[0, \hat{d})$. Following the same approach as above, we get (2.31).

$$Q_d^* \frac{IC}{\pi\lambda G(d)h(r_d^*)} < \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)} \tag{2.31}$$

Note that it is possible for Lemma 2.4.1.5 to hold and we can not completely rule out the possibility for (2.28) to be less than zero,$\mathrm{d}r^*/\mathrm{d}d \leq 0$ and $\mathrm{d}Q^*/\mathrm{d}d < 0$. On the other hand, (2.32) must stand true if we assume that (2.28) is less than zero such that $\mathrm{d}r^*/\mathrm{d}d > 0$ and $\mathrm{d}Q^*/\mathrm{d}d < 0$.

$$Q_d^* + \left[\frac{H(r_d^*)}{h(r_d^*)} - \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)}\right]\left[1 - \frac{H(r_d^*)}{Q_d^* h(r_d^*)}\right]^{-1} < 0$$

$$Q_d^* - \frac{H(r_d^*)}{h(r_d^*)} + \frac{H(r_d^*)}{h(r_d^*)} - \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)} < 0$$

and,

$$Q_d^* < \frac{\left[\int_{x=r_d^*}^{\infty} xh(x)dx - r_d^* H(r_d^*)\right]}{H(r_d^*)} \tag{2.32}$$

Note that (2.32) is a contradiction of Lemma 2.4.1.5, so we must conclude that it is not possible for $\mathrm{d}r^*/\mathrm{d}d > 0$ and $\mathrm{d}Q^*/\mathrm{d}d < 0$. Summarizing, $\mathrm{d}r^*/\mathrm{d}d < 0$ and $\mathrm{d}Q^*/\mathrm{d}d \geq 0$ if (2.28) is equal or greater than zero; and $\mathrm{d}r^*/\mathrm{d}d \leq 0$ and $\mathrm{d}Q^*/\mathrm{d}d < 0$ if (2.28) is less than zero. From these remarks we must conclude r^* is a non-increasing function of d and Q^* is dependent on the problem being addressed. ■

2.4.3. Penalty, Inventory, and Ordering Costs

We have previously said that the manufacturer will prefer the modified model over the traditional for any $d > l_{min}$. However, it is important to understand the implications at the customer. In order to do so, we need to focus our attention on the behavior of the penalty and inventory costs.

Equation (2.33) gives us the behavior of the penalty cost per year as d increases.

$$\frac{\mathrm{d}}{\mathrm{d}d}\left[\frac{\pi\lambda}{Q^*}\bar{\eta}_d(r^*)\right] = -\frac{\pi\lambda}{Q^*}\left[\begin{array}{c}\left[\int_{x=r^*}^{\infty} xh(x)dx - r^* H(r^*)\right] g(d) + H(r^*)G(d)\dfrac{\mathrm{d}r^*}{\mathrm{d}d} \\ + \dfrac{\left[\int_{x=r^*}^{\infty} xh(x)dx - r^* H(r^*)\right]}{Q^*} G(d)\dfrac{\mathrm{d}Q^*}{\mathrm{d}d}\end{array}\right] \tag{2.33}$$

We can observe that (2.33) depends on $\mathrm{d}r^*/\mathrm{d}d$ and $\mathrm{d}Q^*/\mathrm{d}d$. We know from Theorem 2.4.2.2 that $\mathrm{d}r^*/\mathrm{d}d \leq 0$, so our attention will focus on $\mathrm{d}Q^*/\mathrm{d}d$. In particular, the following possible behaviors for the penalty cost per year are shown: 1) it is a non-decreasing function of d if $\mathrm{d}Q^*/\mathrm{d}d \geq 0$, and 2) it is a decreasing function of d if $\mathrm{d}Q^*/\mathrm{d}d < 0$.

For the first case, let us assume that (2.33) is negative when $\mathrm{d}Q^*/\mathrm{d}d \geq 0$; in other words, the penalty cost per year is a decreasing function of d. We can get (2.34) by replacing from (2.33) the $\mathrm{d}Q^*/\mathrm{d}d$ with (2.26) and $\mathrm{d}r^*/\mathrm{d}d$ with (2.27).

$$\left[\frac{H(r^*)}{h(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]}{H(r^*)}\right]\left[1 - \frac{\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]}{Q^*H(r^*)}\right] < 0 \quad (2.34)$$

Note that when $\mathrm{d}r^*/\mathrm{d}d < 0$ and $\mathrm{d}Q^*/\mathrm{d}d \geq 0$, (2.28) is greater than or equal to zero and (2.34) is not satisfied since the left-hand side is non-negative. This is a contradiction from our original assumption and we must conclude that the penalty cost per year is a non-decreasing function of d if $\mathrm{d}r^*/\mathrm{d}d < 0$ and $\mathrm{d}Q^*/\mathrm{d}d \geq 0$.

For the second case, we can observe that (2.34) is true if (2.28) is less than zero. We also know from the proof of Theorem 2.4.2.2 that $\mathrm{d}Q^*/\mathrm{d}d < 0$ when (2.28) is negative. Then, we must conclude that the penalty cost per year is a decreasing function of d when $\mathrm{d}r^*/\mathrm{d}d < 0$ and $\mathrm{d}Q^*/\mathrm{d}d < 0$.

Explaining this behavior, in order to minimize the increase in penalty cost the model increases the value of Q^* whenever the expected number of orders incurring penalty per year increases as d increases. On the other hand, the model lowers the value of Q^*if the expected number of orders incurring penalty per year decreases; as result of this action the system reduces inventory but the expected penalty cost per year still decreases.

Equation (2.35) gives us the behavior of the inventory cost per year as d increases:

$$\frac{\mathrm{d}}{\mathrm{d}d}\left[IC\left[\frac{Q^*}{2}+r^*-\mu\right]\right]=IC\left[\frac{1}{2}\frac{\mathrm{d}Q^*}{\partial d}+\frac{\partial r^*}{\partial d}\right] \tag{2.35}$$

Note that (2.35) is non-positive if $\mathrm{d}Q^*/\mathrm{d}d \leq 0$. For the case of $\mathrm{d}Q^*/\mathrm{d}d > 0$, let us assume that (2.35) is greater than or equal to zero. We can get (2.36) by replacing from (2.35) $\mathrm{d}Q^*/\mathrm{d}d$ with (2.26) and $\mathrm{d}r^*/\mathrm{d}d$ with (2.27).

$$-\frac{1}{2}\frac{H(r^*)}{h(r^*)}+\left[\int_{x=r^*}^{\infty}xh(x)dx-r^*H(r^*)\right]\left[\frac{1}{Q^*h(r^*)}-\frac{1}{H(r^*)}\right]\geq 0 \tag{2.36}$$

From Lemma 2.4.1.4, the right term from the left-hand side of the previous inequality is negative. Then, (2.36) is not satisfied and this leads to a contradiction from our original assumption and (2.35) is negative if $\mathrm{d}Q^*/\mathrm{d}d > 0$. From these observations we must conclude that the inventory cost is a non-increasing function of d. This means that for inventory purposes, the reduction of r^* offsets any increase in Q^*.

The behavior of the ordering cost ($\lambda A/Q^*$) is self-explanatory since it depends only on the value of Q^*.

2.4.4. Claim with Respect to $\mathbf{d}\boldsymbol{Q}^*/\mathbf{d}\boldsymbol{d}$

It is claimed that for most practical purposes (or instances) $\mathrm{d}Q^*/\mathrm{d}d \geq 0$. This claim is justified by focusing our attention on the case when the $\mathrm{d}Q^*/\mathrm{d}d < 0$ which happens when (2.28) negative. Let us rearranged (2.9) and (2.8) in terms of

$H(r^*)/h(r^*)$ and $\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]/H(r^*)$, respectively, in order to get (2.37).

$$\frac{Q^*IC}{\pi\lambda G(d)h(r^*)} - \left[\frac{{Q^*}^2 IC}{2\pi\lambda G(d)H(r^*)} - \frac{A}{\pi G(d)H(r^*)}\right] < 0 \tag{2.37}$$

Simplifying (2.37) we get (2.38).

$$\frac{IC}{\pi\lambda G(d)h(r^*)} < \frac{1}{2} - \frac{A}{2\left[A + \pi\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]G(d)\right]} \tag{2.38}$$

Just as a reminder, $0 < IC/[\pi\lambda G(d)h(r^*)] < 1$, then (2.38) must be satisfied when $\mathrm{d}Q^*/\mathrm{d}d < 0$. We have previously shown that the derivative of the penalty cost per year is negative when $\mathrm{d}Q^*/\mathrm{d}d < 0$. From this observation we can conclude that as d increases the penalty cost per cycle as given by $\pi\left[\int_{x=r^*}^{\infty} xh(x)dx - r^*H(r^*)\right]G(d)$ decreases and the window defined by the right-hand side of (2.38) is closing. This is the base for the claim and it is shown with numerical examples.

It is important to mention that as a result of this claim, the penalty cost per year is expected to be a non-decreasing function of d while the ordering cost is non-increasing.

2.4.5. Behavior When $d \geq \hat{d}$

It is known from Lemma 2.4.1.3 that for any $d \geq \hat{d}$ the convexity of the cost function (K_d) with respect to Q and r is not guaranteed. However, we can always use the policy ($Q_d = Q^*_{\hat{d}-\Delta d}, r_d = r^*_{\hat{d}-\Delta d}$) and set $K_d = K^*_{\hat{d}-\Delta d}$ for any $d \geq \hat{d}$. Note from (2.22) that K_d is non-increasing as d increases in the range $[\hat{d},\infty)$. In fact, it is strictly decreasing if $g(d) > 0$ as shown in Figure 6. In addition, note that the ordering and inventory holding costs are constant in that range and the reduction in K_d is driven by a reduction in the penalty cost.

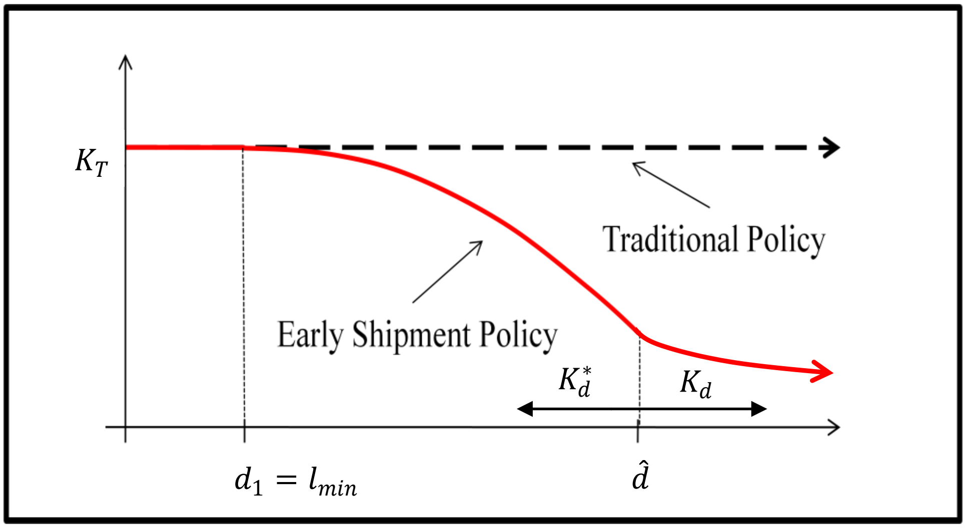


Figure 6: "Early shipment" (Q, r) policy with replenishment delay vs. traditional (Q, r) policy.

To use the modified (Q, r) policy for any choice of d, the following algorithm is used:

Start,

Define d;

Compute $\hat{d} \coloneqq min\{d \geq 0 | \hat{Q}_d > \pi\lambda G(d)H(u)/IC\}$;

If $d < \hat{d}$,

Find Q_d^*, r_d^*, and calculate K_d^*;

Else,

Find $Q^*_{\hat{d}-1}, r^*_{\hat{d}-1}$,

Set $Q^*_d = Q^*_{\hat{d}-1}, r^*_d = r^*_{\hat{d}-1}$, and calculate K^*_d,

End;

End.

2.5. Numerical Study

This section presents the results of a numerical study performed with the following objectives:

1) Demonstrate the performance of the modified policy,
2) Demonstrate the claim that for most practical scenarios the $\mathrm{d}Q^*/\mathrm{d}d \geq 0$.

Three sets of problems are run in which $g(l)$ is assumed to follow an expo (β), normal $(\mu_{LT}, \sigma^2_{LT})$, and uniform (a, b) distribution functions. For each lead-time distribution function, a set of problems are defined by varying one parameter at a time from a base case problem, and a final case is run by varying all parameters at the same time as shown in tables 1, 2, and 3. For all cases the marginal distribution of the lead-time demand, $h(x)$, is assumed normal (μ, σ^2).

For practical purposes the analysis was done assuming that $\Delta d = 1$ week. The following algorithm is followed in order to show the behavior of the model as d increases

Start,

Compute $\hat{d} := min\{d \geq 0 | \hat{Q}_d > \pi\lambda G(d)H(u)/IC\}$;

Set $d = 0$;

For any $d < \hat{d}$,

Find Q_d^*, r_d^*, and calculate K_d^*;

$d = d + 1$,

End;

End.

The total cost, inventory cost, and the "on-time performance" (OTP) are used for comparison purposes between the modified (Q, r) model with the "early shipment" policy and the traditional (Q, r) model for the cases when $d = 2$, 4, and 6 weeks. In addition, unless otherwise specified, the analysis is done for any d such that $d < \hat{d}$.

2.5.1. Expo (β) Lead-Times

Table1: Parameters for each case problem – Expo (β).

Case No.	h(x)		g (l)	λ	A	IC	π	P (X≤ 0)
	μ	σ	β					
1	750	50	12	3250	4000	10	20	3.67097E-51
2	800	50	12	3466.7	4000	10	20	6.38875E-58
3	750	150	12	3250	4000	10	20	2.86652E-07
4	750	50	6	6500	4000	10	20	3.67097E-51
5	750	50	12	3250	1	10	20	3.67097E-51
6	750	50	12	3250	4000	1	20	3.67097E-51
7	750	50	12	3250	4000	10	40	3.67097E-51
8	800	150	6	6933.3	1	1	40	4.8213E-08

Table 1 shows the parameters used for the set of problems in which the lead-time distribution $g(l)$ was assumed to be exponential. Note that λ is calculated assuming 52 production weeks in a year, $\lambda = 52 * \mu/\beta$. In addition, the $P(X \leq 0)$ is determined in order to confirm that it is negligible since the lead-time demand is assumed normal.

Figure 7 shows the rate of change as d increases for the optimal cost function K^*, the penalty cost, inventory cost, and ordering cost. For scale purposes the rate of change is shown rather than the actual value for each cost metric. However, it can be concluded from the charts that for all case studies K^*, the inventory cost, and the ordering cost are decreasing functions of d, while the penalty cost is increasing.

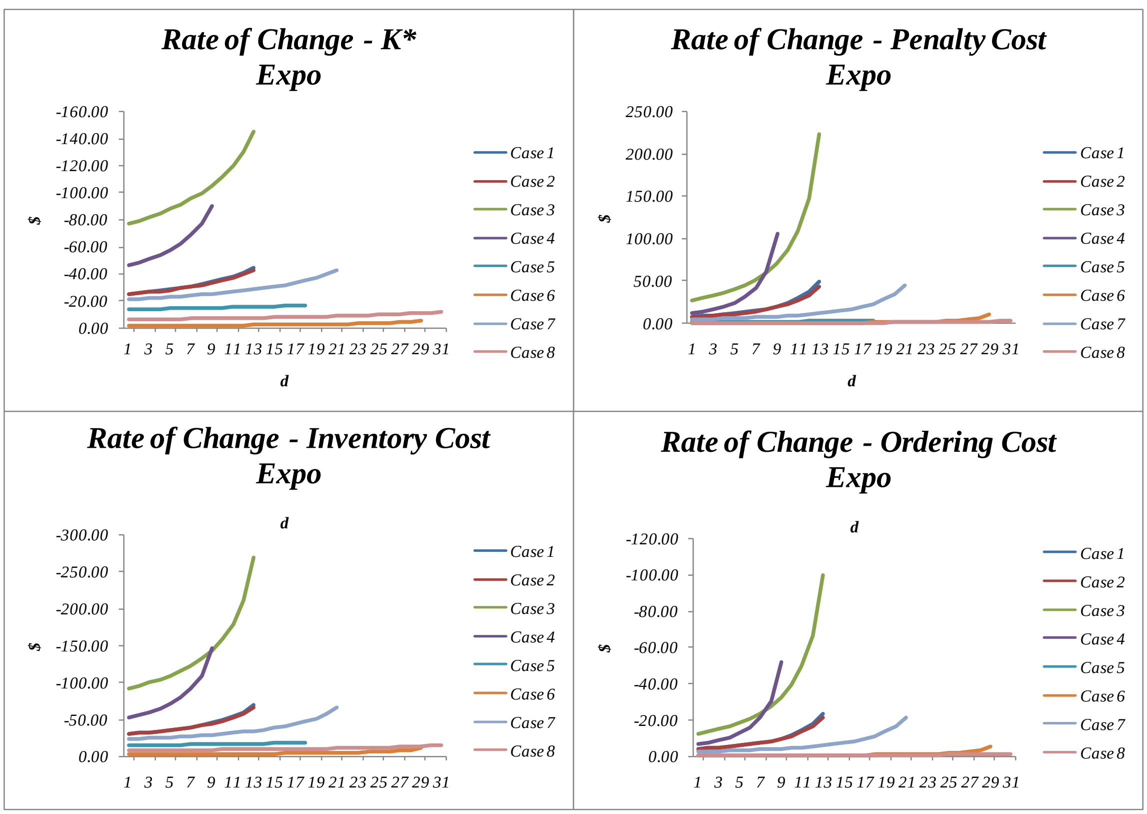


Figure 7: $\Delta K^*/\Delta d$, Δ Penalty Cost$/\Delta d$, Δ Inventory Cost$/\Delta d$, and Δ Ordering Cost$/\Delta d$ for the exponential lead-times.

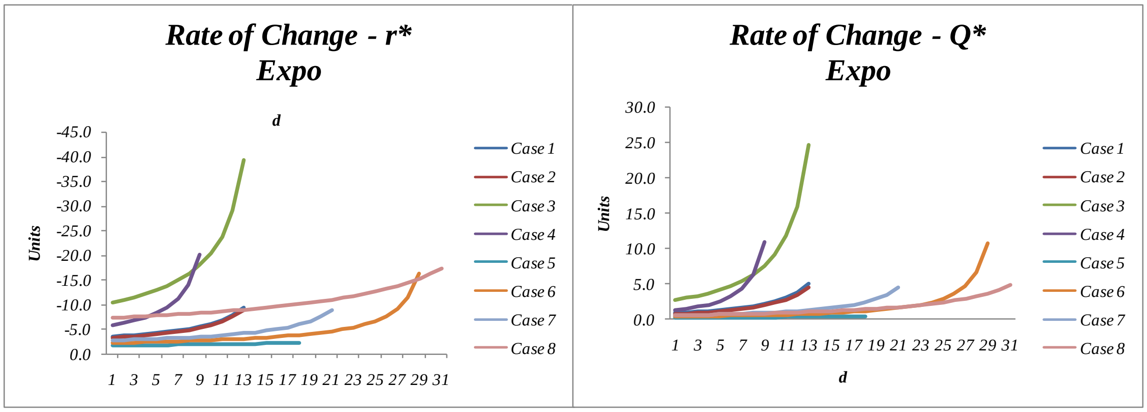


Figure 8: $\Delta r^*/\Delta d$ and $\Delta Q^*/\Delta d$ for the exponential lead-times.

Figure 8 shows the rate of change for r^* and Q^*as d increases, and it can be concluded that r^* is a decreasing function of d while Q^* is increasing. These results are compatible with the explanation given in section 2.4, and for the particular case of the rate of change of Q^*, it supports the claim that for most practical scenarios $\mathrm{d}Q^*/\mathrm{d}d \geq 0$.

Figure 9 relates to the base case of the exponential lead-time distribution and is intended to help the reader visualize the behavior of the model as d increases. Note that $\hat{d} = 9$, and $(Q_d = Q_8^*, r_d = r_8^*)$ is arbitrarily chosen for any $d \geq \hat{d}$. Observe the change of behavior for the ordering, inventory, and penalty costs. In particular, note that the penalty cost is increasing until $\hat{d} - \Delta d = 8$, and then becomes a decreasing function of d. From this observation we can conclude that the service is deteriorating for any $d < \hat{d}$ and improving for any $d \geq \hat{d}$. Despite these changes in the behavior of the ordering, inventory, and penalty costs, the total cost is a non-increasing function of d. In this case in particular, the cost function is strictly decreasing because $g(l)$ is assumed exponential.

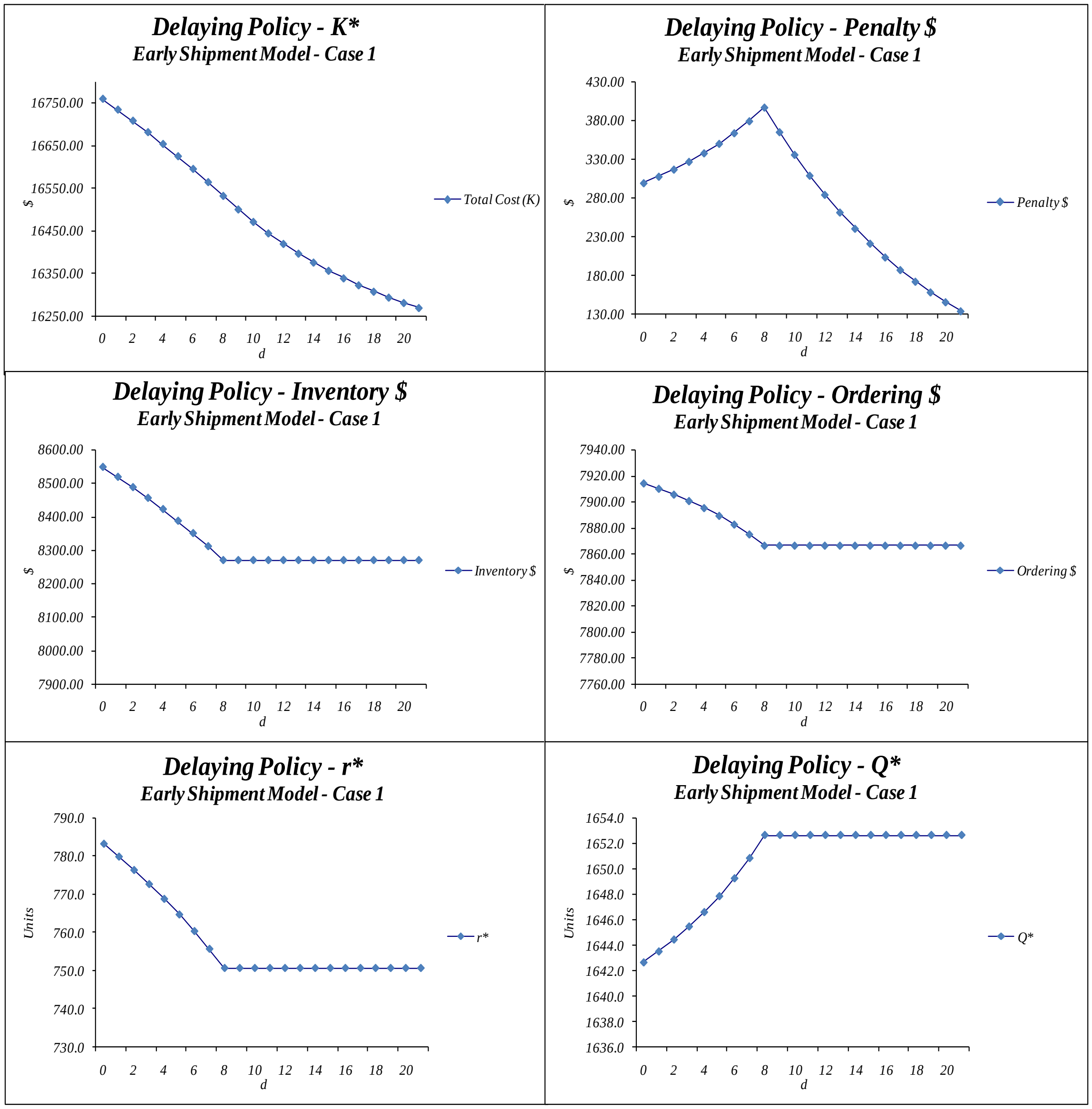


Figure 9: Case 1 for the exponential lead-times.

2.5.2. Normal $(\mu_{LT}, \sigma_{LT}^2)$ Lead-Times

Table 2 shows the parameters for the case studies in which the lead-time was assumed to follow a normal distribution. Note that λ is calculated as follows:

$\lambda = 52 * \mu/\mu_{LT}$. The $P(X \leq 0)$ and $P(L \leq 0)$ are calculated in order to confirm that the chances of having negative demand or negative lead-time l are negligible.

Table 2: Parameters for each case problem – Normal (μ_{LT}, σ_{LT}^2).

Case No.	h (x)		g (l)		λ	A	IC	π	P (X≤ 0)	P (L ≤ 0)
	μ	σ	μ_{Lt}	σ^2_{Lt}						
1	750	50	12	1	3250	4000	10	20	3.67E-51	1.78E-33
2	800	50	12	1	3466.7	4000	10	20	6.39E-58	1.78E-33
3	750	150	12	1	3250	4000	10	20	2.87E-07	1.78E-33
4	750	50	6	1	6500	4000	10	20	3.67E-51	9.87E-10
5	750	50	12	6.25	3250	4000	10	20	3.67E-51	2.74E-02
6	750	50	12	1	3250	1	10	20	3.67E-51	1.78E-33
7	750	50	12	1	3250	4000	1	20	3.67E-51	1.78E-33
8	750	50	12	1	3250	4000	10	40	3.67E-51	1.78E-33
9	800	150	6	4	6933.3	1	1	40	4.82E-08	6.68E-02

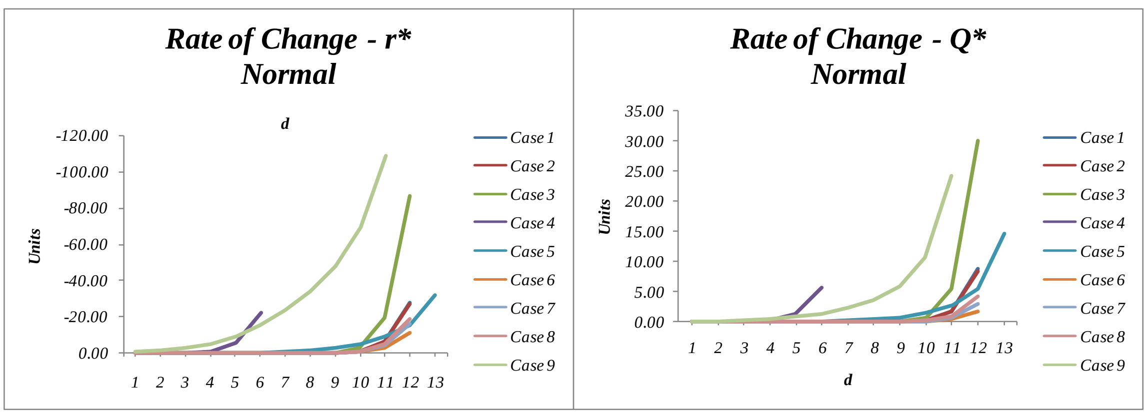


Figure 10: $\Delta r^*/\Delta d$ and $\Delta Q^*/\Delta d$ for the normal lead-times.

Figures 10 and 11 show the behavior of the model as d increases for each case study. Note that as in the case of the exponential distribution the behavior is as expected. For the rate of change of Q^*, Figure 10 supports the claim that for most practical scenarios $\mathrm{d}Q^*/\mathrm{d}d \geq 0$.

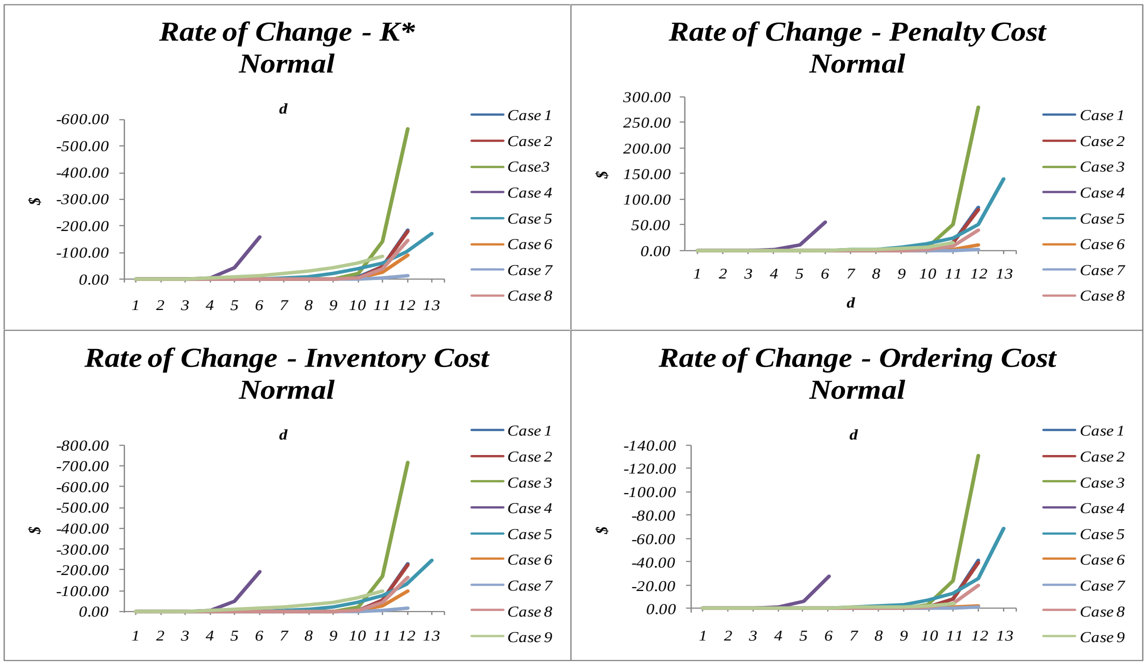


Figure 11: $\Delta K^*/\Delta d$, Δ Penalty Cost/Δd, Δ Inventory Cost/Δd, and Δ Ordering Cost/Δd for the normal lead-times.

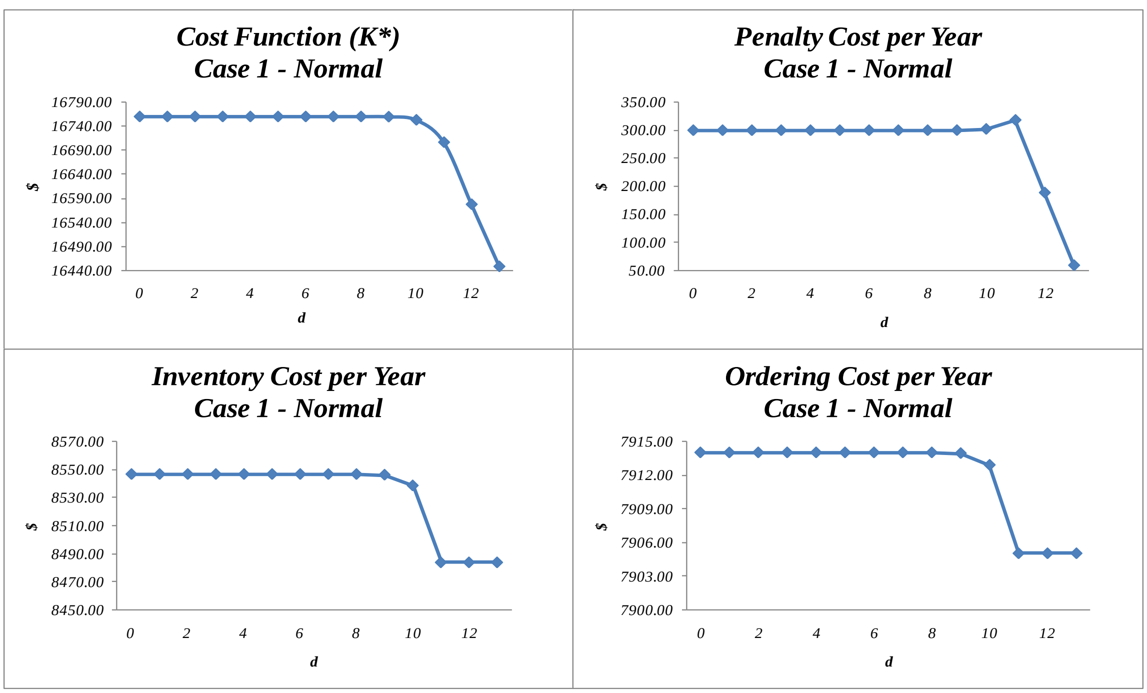
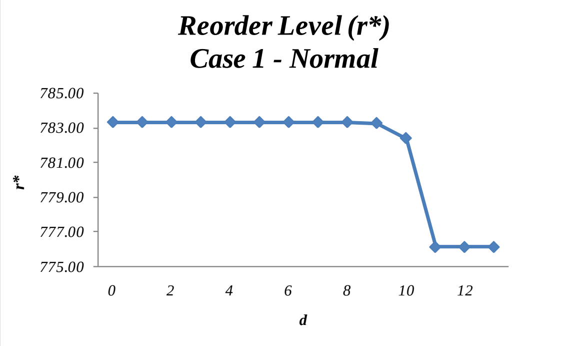


Figure 12: Cost function (K^*), penalty cost, inventory cost, and ordering cost for Case 1 – Normal.

Figures 12 and 13 show the behavior of the model for the base case of the normal distribution function. Since $g(d) > 0$ but negligible for any $d \leq 9$, the behavior is almost similar to the traditional model for any d within that period of time. This is the result of setting-up a small σ_{LT}^2 in order to guarantee no negative lead-time l. From this observation we can conclude that the modified (Q, r) model with the "early shipment" policy is preferred when $d > l_{min}$ and $g(l)$ is not negligible.



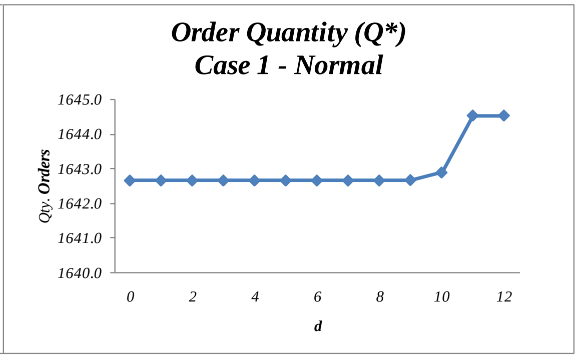


Figure 13: r^* and Q^* for Case 1 – Normal.

2.5.3. Uniform (a, b) Lead-Times

Table 3: Parameters for each case problem – Uniform (a, b)

Case No.	h (x)		g (l)		λ	A	IC	π	P (X ≤ 0)
	μ	σ	a	b					
1	750	50	8	16	3250.0	4000	10	20	3.67E-51
2	800	50	8	16	3466.7	4000	10	20	6.39E-58
3	750	150	8	16	3250.0	4000	10	20	2.87E-07
4	750	50	0	16	4875.0	4000	10	20	3.67E-51
5	750	50	8	16	3250.0	1	10	20	3.67E-51
6	750	50	8	16	3250.0	4000	1	20	3.67E-51
7	750	50	8	16	3250.0	4000	10	40	3.67E-51
8	800	150	0	16	5200.0	1	1	40	4.82E-08

Table 3 shows the parameters for the case studies in which the lead-time was assumed to follow a uniform distribution. The $P(X \leq 0)$ is calculated in order to confirm that the probability of having negative demand is negligible. Note that λ is calculated as follows: $\lambda = 52 * \mu * 2/(b + a)$.

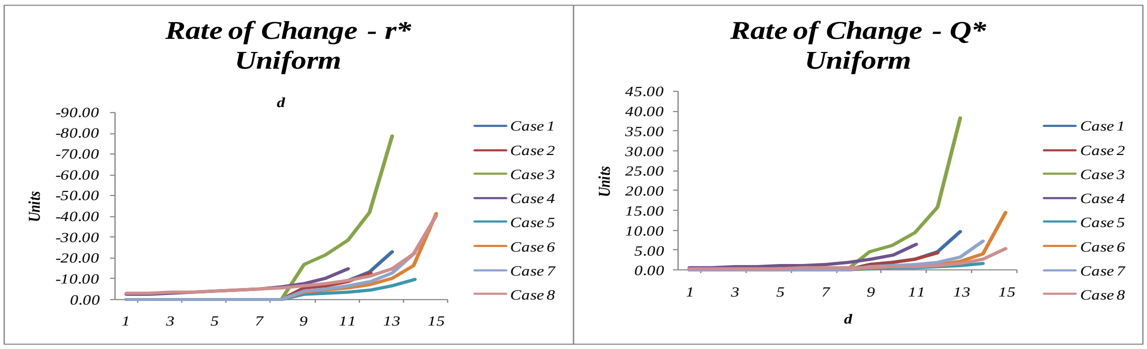


Figure 14: $\Delta r^*/\Delta d$ and $\Delta Q^*/\Delta d$ for the uniform lead-times.

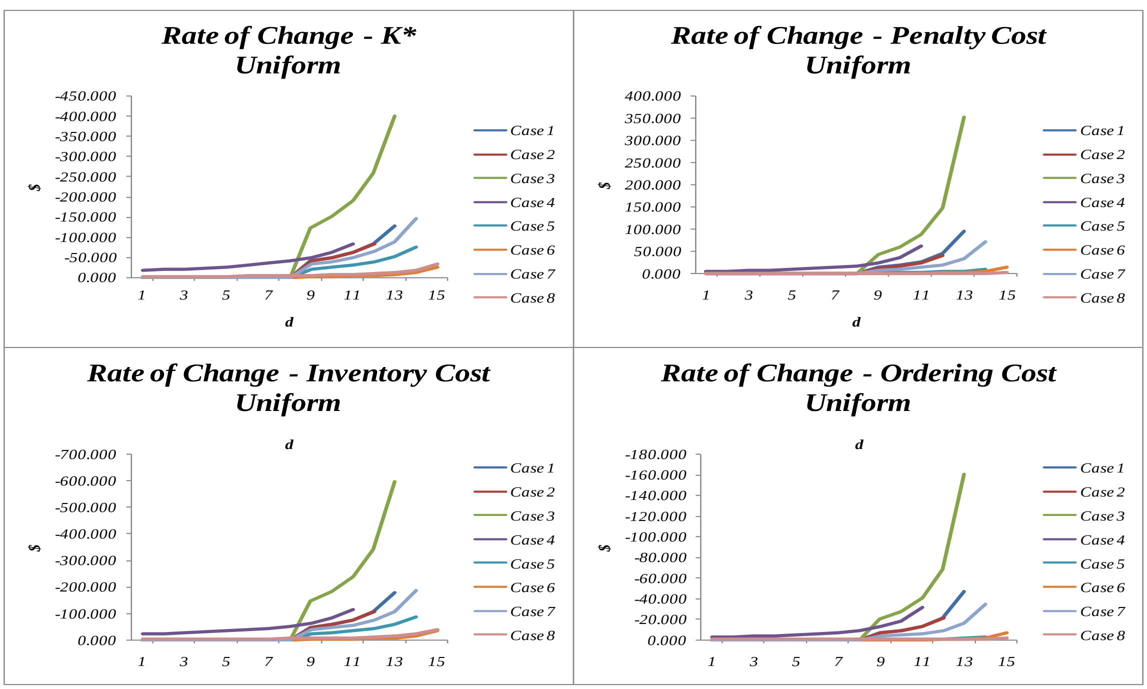


Figure 15: $\Delta K^*/\Delta d$, Δ Penalty Cost$/\Delta d$, Δ Inventory Cost$/\Delta d$, and Δ Ordering Cost$/\Delta d$ for the uniform lead-times.

Figures 14 and 15 show the behavior of the model as d increases for each case study. Note that as in the case of the exponential and normal distributions the behavior is as expected. For the rate of change of Q^*, Figure 14 supports the claim that for most practical scenarios $\mathrm{d}Q^*/\mathrm{d}d \geq 0$.

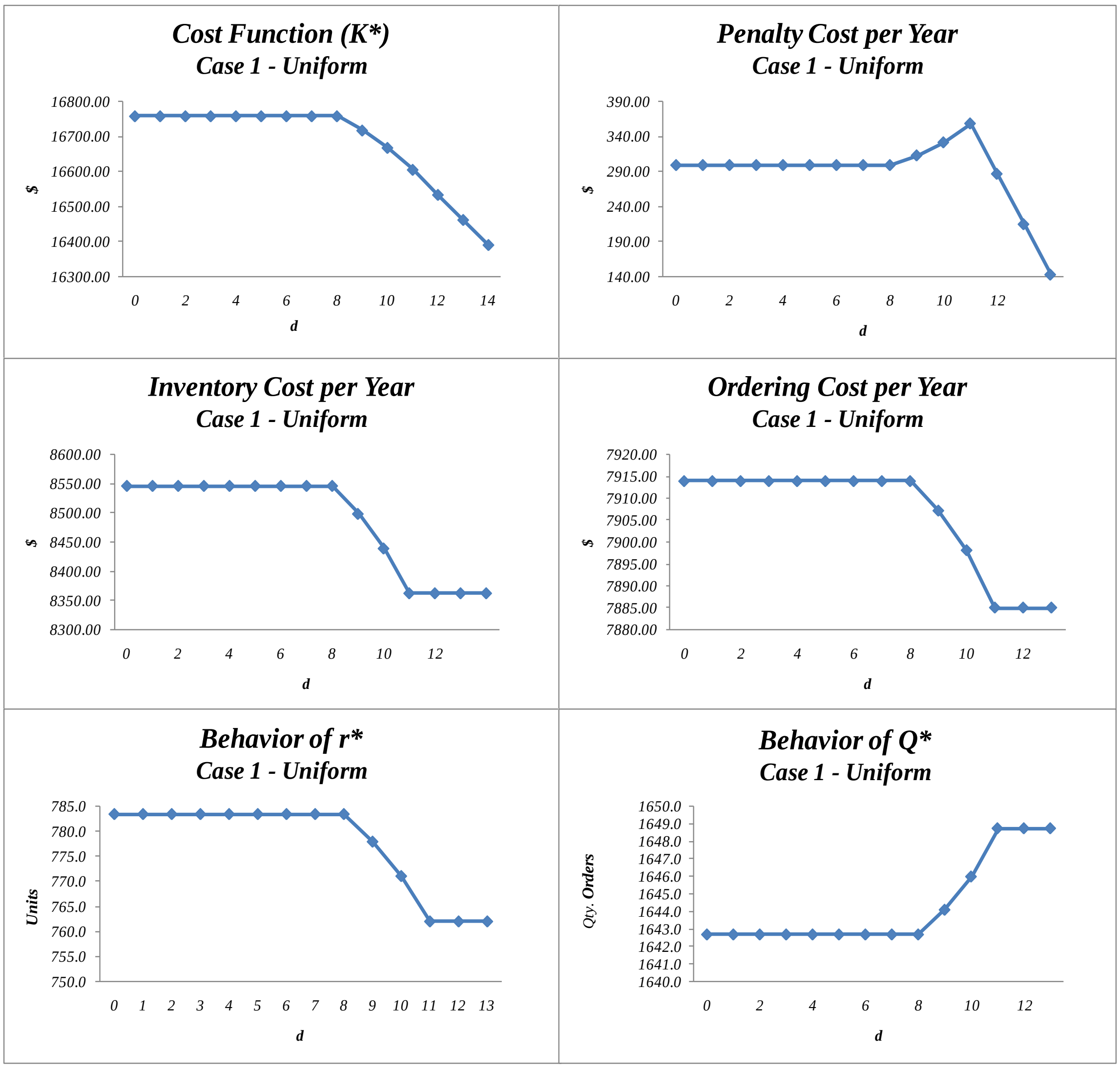


Figure 16: Cost function (K^*) for base case problem – Uniform.

Figure 16 shows the behavior of the model for the base case of the uniform distribution function. Note that $l_{min} = d_1 = a = 8$ is the minimum lead-time and the "early shipment" model has the same behavior as the traditional model (given by $K^*_{d=0}$) for any given $d \leq l_{min}$ since $g(d) = 0$. Observe that $\hat{d} = 12$ and K^*_d is strictly decreasing in the range given by $(8,12)$ since $g(d) > 0$. This behavior is compatible with the generalized practical scenario discussed in section 2.4.

2.5.4. Modified $(\boldsymbol{Q}, \boldsymbol{r})$ Model with the "Early Shipment" Policy vs. Traditional $(\boldsymbol{Q}, \boldsymbol{r})$ Model

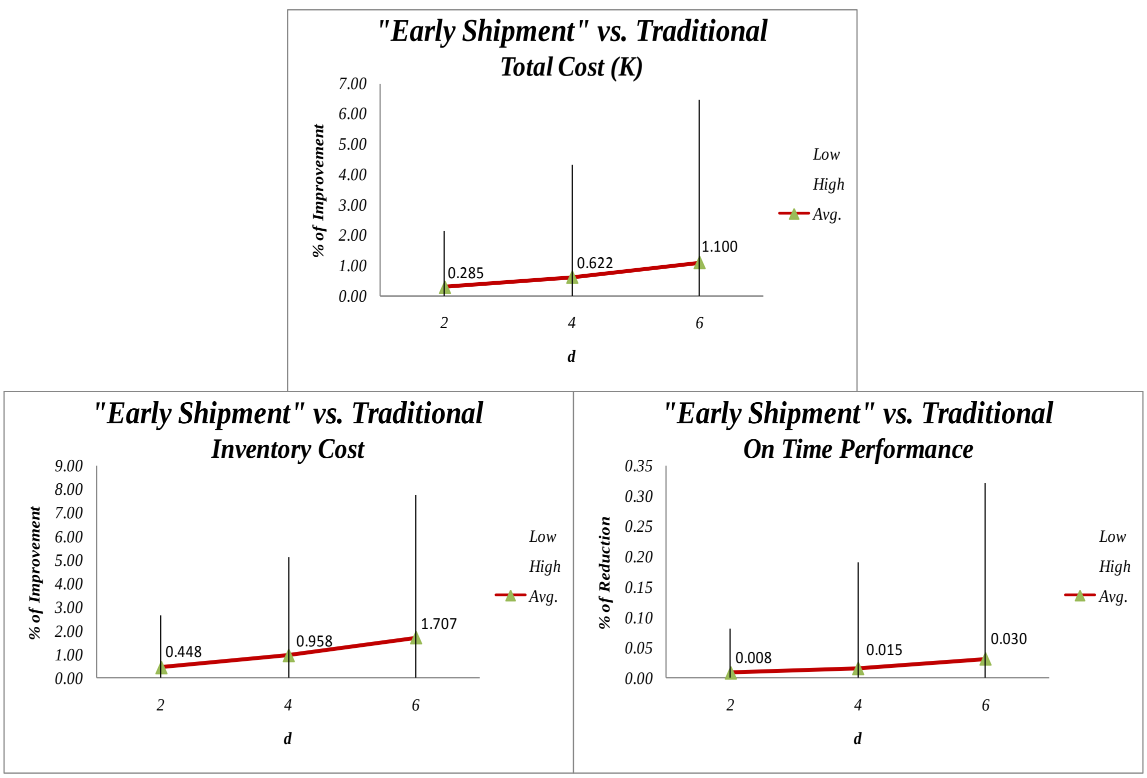


Figure 17: "Early shipment" model vs. the traditional model – Expo, Normal, Uniform.

Figure 17 shows a comparison between the modified (Q, r) model with the "early shipment" policy and the traditional (Q, r) model for the case when $d = 2$, 4, and 6. In average, the modified model reduced the cost as d increases by 0.29%, 0.62%, and 1.10%, respectively; and the inventory cost by 0.45%, 0.96%, and 1.71%, respectively. This analysis considers the data from the exponential, uniform and normal lead-time distributions.

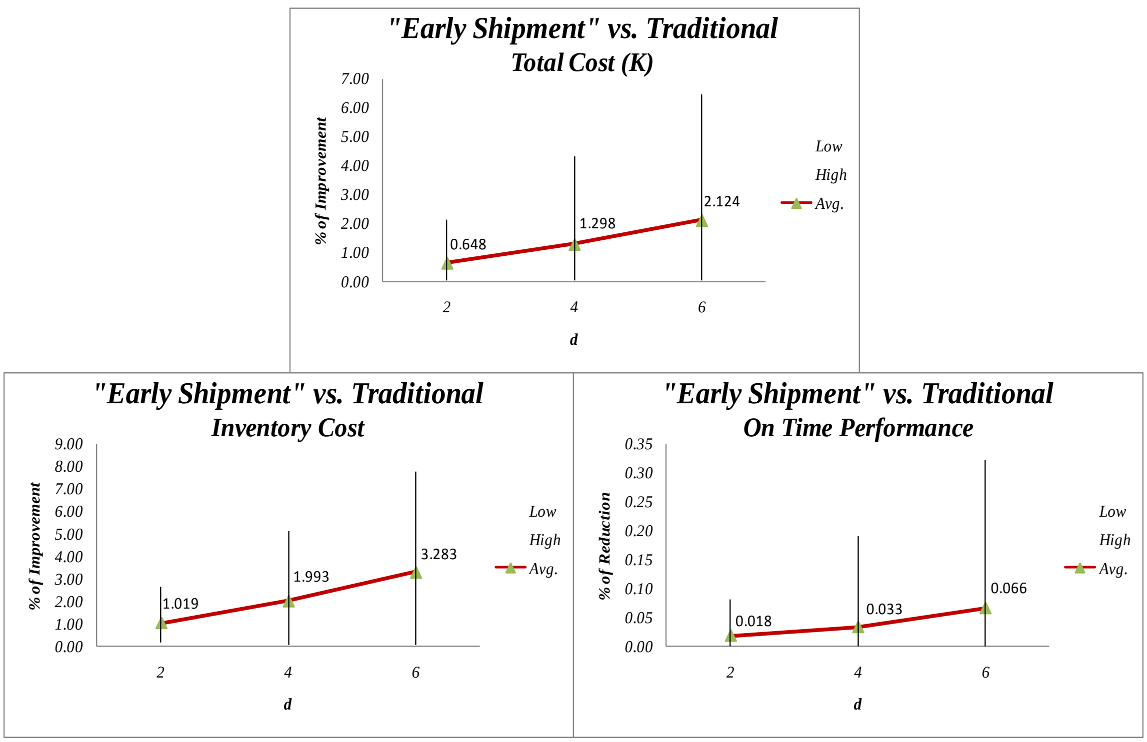


Figure 18: "Early shipment" model vs. the traditional model; $d > l_{min}$ and $g(d)$ is not negligible.

As previously mentioned the modified policy will tend to be preferable when $d > l_{min}$ and $g(d)$ is not negligible. Among other examples, this could be the case of a local supplier that holds inventory of the component being supplied. It is of interest to eliminate from the analysis any data from the uniform and normal lead-time distribution functions that does not meet those criteria. Figure 18 shows this analysis and it can be observed that the expected improvement in the total cost was 0.65%, 1.30%, and 2.12%, respectively; and 1.02%, 1.99%, and 3.28% for the inventory cost.

Note from figures 17 and 18 that the deterioration in service performance is minimal, which is the result of the assumption that the expected number of backorders is negligible with respect to the expected inventory at any time.

2.6. Summary

This chapter discusses the modified (Q, r) model with the "early shipment" policy. A detailed description of the model is given, and its behavior is discussed mathematically and shown by means of numerical examples.

The basic assumption for this model is that the customer does not penalize for early shipments. In that sense, the manufacturer can ship the final product up to d units of time ahead of the delivery date if inventory of the component is available at time of the customer order entry. If inventory is not available, penalty is only incurred if the component becomes available more than d units of time after receiving the customer order. The main difference with the traditional (Q, r) inventory control model is given by the penalty cost structure. Backorders/stock-outs are allowed and the chance for incurring penalty is only present when the lead-time (l) is greater or equal to the safety time (d).

Defining l_{min} as the minimum lead-time, the modified (Q, r) model with the "early shipment" policy has the same behavior as the traditional model for any $d \leq l_{min}$. However, for any $d > l_{min}$ the optimal cost function (K^*) tends to decrease as the safety time (d) increases, and the modified model is preferred over the traditional (Q, r) model. The main reason for this behavior is that r^* decreases, driving a reduction in inventory which in turn drives the reduction of K^*. The behavior of Q^* and the penalty cost per year depend on the problem being addressed, but a claim is established and shown by numerical examples that for most practical scenarios Q^* and the penalty cost per year will increase.

There is a $\hat{d}$, such that for any $d \geq \hat{d}$ the convexity of the model is not guaranteed anymore. However, the policy $(Q_d = Q^*_{\hat{d}-\Delta d}, r_d = r^*_{\hat{d}-\Delta d})$ can arbitrarily be used for any $d \geq \hat{d}$ and the cost function (K) is still a non-increasing function of d driven by a reduction in the penalty cost.

The next chapter discusses the modified (Q, r) model with the "no-early shipment" policy. The behavior of the model with respect to the safety time is explained mathematically and by means of numerical examples.

CHAPTER III

MODIFIED (Q, r) MODEL WITH THE NO-EARLY SHIPMENT POLICY

3.1. Introduction

This chapter describes the modified (Q, r) model under a scenario in which early shipments are not allowed by the customer. This follows from the assumption that the customer is knowledgeable of the inventory implications for allowing early shipments. There is a penalty imposed over the manufacturer for shipping orders ahead of time. One could think of a scenario in which the customer is aware of the implications that early shipments have on their inventory, and monitors early orders as part of their control procedures.

The safety time (d) is modeled under the perspective that the manufacturer allocates inventory of the required component to the customer order, releases the order to the shop floor d units of time after the order entry date, and ships the product at the delivery date. Penalty is incurred only if the component is not available at the time that the order is released to the shop floor.

As we will see in this chapter, the behavior of the model is dependent on the problem being addressed. The merits of the model with respect to the traditional and the modified (Q, r) model with the "early shipment" policy are discussed. In particular, the behavior of the model as d increases is discussed with respect to the coefficient of variation, the ratio (IC/π), and the period of time given by $[0, l_{min}]$ which could be related to the location of the supplier.

3.2. Methodology

The modified (Q, r) model for the case when early shipments are not allowed by the customer is modeled following the heuristic approach discussed by Hadley and Whitin (1963). The $\partial K/\partial d$ and $\partial^2 K/\partial d^2$ are used to determine that the behavior of the model with respect to the safety time (d) is dependent on the problem being addressed. Total derivatives with respect to d are calculated in order to determine the relationship between r^*, Q^*, and the penalty cost. Special attention is given to show the benefits of delaying the replenishment order, and to show the merits of the model with respect to the traditional model and the modified (Q, r) model with the "early shipment" policy for the case when there is a minimum lead-time l_{min}.

Numerical examples are run to identify which parameters are critical to the behavior of the model as d increases.

3.3. Description of the Model

This model differs from the traditional (Q, r) model not only in the penalty cost structure but in the inventory cost as well. This is the result of the allocation of inventory during the time period $(0, d]$ in order to guarantee the future on-time delivery of the customer order. Since the lead-time L is random, there are instances in which $l \geq d$ and $l < d$, the chance of incurring penalty is related to $P(L \geq d)$, and the demand during the time periods $(d, l]$ and $(l, d]$ are both of interest. In addition, the inventory position has been redefined as the net inventory plus in-transit inventory minus the allocations as shown in Figure 19.

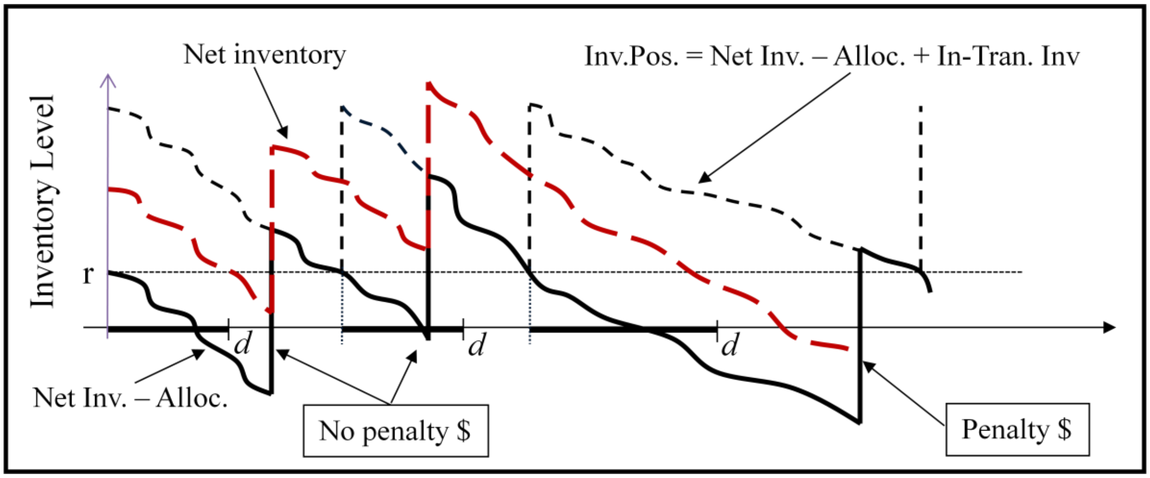


Figure 19: Behavior of the "no-early shipment" model.

The following notation is used in order to define the cost function (K):

- $X_{(t,t+\Delta]}$is the demand in any time interval $(t, t+\Delta]$;
- $f(x;\Delta)$ is the probability that the number of units demanded in a time interval of length $\Delta = (t+\Delta) - t$ lies between x and $x + dx$;
- $g(l)dl$ is the probability that the lead-time for the replenishment/procurement order lies between l and $l + dl$;
- d is the cushion or safety time by which no penalty is incurred if material is not available at time of the customer order entry;
- $h_1(x)$ is the marginal distribution for the demand during the time period $(d, l]$ and it is assumed unimodal,

$$h_1(x) = \int_{l=d}^{\infty} f(x; l-d) g(l) dl \, ; \tag{3.1}$$

- $\mu_{(d,l]}$ is the expected demand during the time period $(d, l]$,

$$\mu_{(d,l]} = \int_{x=0}^{\infty} x h_1(x) dx\,; \tag{3.2}$$

- $h_2(x)$ is the marginal distribution for the demand during the time period $(l, d]$ and it is assumed unimodal,

$$h_2(x) = \int_{l=0}^{d} f(x; d-l) g(l) dl\,; \tag{3.3}$$

- $\mu_{(l,d]}$ is the expected demand during the time period $(l, d]$,

$$\mu_{(l,d]} = \int_{x=0}^{\infty} x h_2(x) dx\,; \tag{3.4}$$

- $0 < Q < \infty$ is the procurement/replenishment order size;
- $u_d \leq r < \infty$ is the reorder level that triggers the procurement order by means of the inventory position, where u_d is a non-negative number such that $h_1(x)$ is non-increasing for any $x \geq u_d$;
- λ is the average annual demand which is constant over time;
- A is the cost of placing an order with the supplier;
- IC is the average cost of carrying inventory per unit per unit time;
- π is the penalty cost per unit incurred when the requested customer date is missed;
- $G(d)$ is the complementary cumulative distribution of $g(d)$, or $P(L \geq d)$,

$$G(d) = \int_{l=d}^{\infty} g(l) dl\,; \tag{3.5}$$

- $\bar{\eta}_d(r)$ is the expected number of units incurring the penalty cost per cycle.

The same assumptions as in Hadley and Whitin (1963) are being followed and are enumerated as follows:

1) The unit cost C of the item is a constant independent of Q;
2) There is never more than a single order outstanding;
3) The cost of operating the information processing system is independent of Q and r;
4) The reorder point r is positive.

The only major change is that π has been assumed to be the penalty cost incurred when the customer order for the finished product is overdue or late based on its request/delivery date since backorders/stock-outs of the inventory item are allowed at time of order entry. Despite the fact that backorders are allowed, Hadley's and Whitin's assumption that the average number of backorders is negligible as compared to the average inventory at any time is being followed. Hence, the expected on-hand inventory is equal to the expected net inventory in order to calculate the cost structure related to the inventory cost.

In order to calculate the expected annual inventory cost, let us assume that a customer order that triggers the decision of placing a replenishment order for the inventory component is entered at time $t = 0$. The customer order is released to the shop floor at time d and the replenishment order is received at time l if there is no delay in placing the replenishment order. Since the lead-time L is random, then there are instances in which $l \geq d$ and $l < d$.

For the instance in which $l \geq d$, the customer orders entered at or before time $t = 0$ have inventory allocated and are required to be released during the time interval $(0, d]$. The net inventory at time of booking the customer order is $r + X_{(0,d]}$; the chance of incurring penalty is related to the demand during the time period $(d, l]$; and the net inventory at time of receiving the replenishment order is $r - X_{(d,l]}$. For the instance in which $l < d$, there is no chance of incurring penalty and the net inventory at the time of arrival of the replenishment order is $r + X_{(l,d]}$.

The net inventory at time of arrival of the replenishment order can be summarized as per (3.6).

$$\xi_d(x, r) = \begin{cases} r - X_{(d,l]}, & l \geq d, 0 \leq X_{(d,l]} < \infty; \\ r + X_{(l,d]}, & l < d, 0 \leq X_{(l,d]} < \infty. \end{cases} \tag{3.6}$$

The expected net inventory at the time of arrival of a replenishment order can be calculated as follows:

$$E[\xi_d(x, r)] = \left[\int_{x=0}^{\infty}\int_{l=d}^{\infty}(r - x)f(x; l - d)g(l)dldx\right]P(L \geq d)$$

$$+\left[\int_{x=0}^{\infty}\int_{l=0}^{d}(r + x)f(x; d - l)g(l)dldx\right]P(L < d)$$

$$E[\xi_d(x, r)] = \left[r - \mu_{(d,l]}\right]G(d) + \left[r + \mu_{(l,d]}\right]\left(1 - G(d)\right) \tag{3.7}$$

$$= s_d G(d) + e_d\left(1 - G(d)\right).$$

Note that s_d is the safety stock needed to protect us against the variability of the demand when $l \geq d$, and e_d is the excess inventory that occurs naturally when $l < d$ since the replenishment order arrives before the material is really needed. As d increases, the time period $(d, l]$ decreases and less safety stock is needed; however, the time period $(l, d]$ increases and more excess inventory occurs. Since by definition $G(d)$ is a decreasing function of d, it must be concluded that as d increases the term $e_d\left(1 - G(d)\right)$ will become the major contributor for the expected net inventory. This observation justifies the implementation of a policy based on delaying the placement of the replenishment order with the objective of reducing the excess inventory. Under this policy, the decision of reordering is taken when the inventory position reaches the reorder level but the execution is delayed by m units of time. This policy is discussed in section 3.4.3.

Continuing with the inventory cost structure without any delay of the replenishment order, the expected annual inventory carrying cost is defined as follows:

$$IC\left[\frac{Q}{2} + s_d + e_d\right]$$

$$= IC\left[\frac{Q}{2} + r - \mu_{(d,l]}G(d) + \mu_{(l,d]}\left(1 - G(d)\right)\right]. \tag{3.8}$$

In order to define the expected annual penalty cost, note that the penalty is only incurred when $l \geq d$ and the demand $X_{(d,l]}$ is greater than the reorder point r. The number of customer orders incurring penalty in a cycle is given by

$$\eta_d(x,r) = \begin{cases} 0, & l < d; \\ 0, & l \geq d, x - r < 0; \\ x - r, & l \geq d, x - r \geq 0. \end{cases} \tag{3.9}$$

The expected number of customer orders incurring penalty per cycle can be calculated as follows:

$$\bar{\eta}_d(r) = \left[\int_{x=r}^{\infty}\int_{l=d}^{\infty}(x - r)f(x; l - d)g(l)dldx\right]P(L \geq d)$$

$$\bar{\eta}_d(r) = \left[\int_{x=r}^{\infty} x h_1(x) dx - r\, H_1(r)\right] G(d), \tag{3.10}$$

where $H_1(x)$ is the complementary cumulative of $h_1(x)$. The expected annual penalty cost is defined as

$$\frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} x h_1(x) dx - r\, H_1(r)\right] G(d) \tag{3.11}$$

The expected annual cost K for the modified (Q, r) model with the "no-early shipment" policy and no delay of the replenishment/procurement order is

$$K = \frac{\lambda}{Q} A + IC\left[\frac{Q}{2} + r - \mu_{(d,l]} G(d) + \mu_{(l,d]}\big(1 - G(d)\big)\right]$$
$$+ \frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} x h_1(x) dx - r\, H_1(r)\right] G(d). \tag{3.12}$$

In the following sections we analyze the behavior of the model with respect to the safety time (d) and discuss the benefits of implementing a policy based on delaying the replenishment order.

3.4. Analysis/Behavior of the Model

It has been said that the safety time (d) represents information that the manufacturer possesses that can be used for his/her benefit when setting the optimal (Q, r) policy. In that sense, understanding the behavior of the cost function K with respect to d, and its implications with the customer can be of strategic importance for the manufacturer's management.

3.4.1. Convexity with Respect to Q and r, and Optimality

The following lemmas are related to the convexity of the cost function and are necessary in order to understand the behavior of the model:

Lemma 3.4.1.1: *For any d, the cost function K_d is jointly convex in Q and r in the range* $0 < Q < \infty$ *and* $u_d \leq r < \infty$.

Proof: It can easily be concluded from (3.12) that the cost function given by the expected ordering cost and the expected inventory cost is jointly convex with respect to Q and r in the range $0 < Q < \infty$ and $u_d \leq r < \infty$. Brooks and Lu (1968) showed that for a number μ, the expected backorders per year is convex in the region given by $0 < Q < \infty$ and $\mu \leq r < \infty$, if the probability density function for the lead-time demand (X) is non-increasing for $x \geq \mu$. In our case, since $G(d)$ is not a function of Q nor r, and u_d has been defined as a non-negative number such that $h_1(x)$ is non-increasing for any $x \geq u_d$, it must be concluded that the expected number of backorders per year as given in (3.12) must be jointly convex with respect to Q and r in the region $0 < Q < \infty$ and $u_d \leq r < \infty$. Then, K_d is jointly convex with respect to Q and r in the region of interest since the addition of convex functions gives a convex function. Note that $u_d = \mu_{(d,l]}$ for the special case in which $h_1(x)$ is the probability density function of a normal demand during the time period $(d, l]$. ■

Lemma 3.4.1.2: *In the range in which K_d is jointly convex in Q and r, the iterative procedure from Hadley and Whitin (1963) can be used to obtain the optimal policy* $(Q_d^*,\ r_d^*)$.

Proof: Let us assume that for a particular d, (Q_d^*, r_d^*) is the optimal policy satisfying $0 < Q < \infty$ and $u_d \leq r < \infty$. Then, (Q_d^*, r_d^*) must satisfy the set of equations (3.13) and (3.14) derived from the 1st order conditions.

$$\frac{\partial K}{\partial Q} = 0 \quad \xrightarrow{yields} \quad Q = \sqrt{\frac{2\lambda}{IC}[A + \pi\bar{n}_d(r)]} \tag{3.13}$$

$$\frac{\partial K}{\partial r} = 0 \quad \xrightarrow{yields} \quad H_1(r)G(d) = \frac{QIC}{\pi\lambda} \tag{3.14}$$

(3.15) is obtained by rearranging (3.14) in terms of Q.

$$Q = \frac{\pi\lambda}{IC}G(d)H_1(r) \tag{3.15}$$

It can be observed that both, (3.15) and (3.13) are functions of r representing a curve in a plane with axis given by Q and r as shown in Figure 20; and the specific optimal policy (Q_d^*, r_d^*) must be found at the intersection of both curves. Hadley and Whitin (1963) showed that Q and r are inversely related. This relationship holds in our case since $G(d)$ is not a function of Q nor r. In addition, let us observe from (3.13) and (3.15) that for $u_d \leq r < \infty$,

$$Q_W = \sqrt{2\lambda A/IC} \leq Q \leq \frac{\pi\lambda G(d)H_1(r = u_d)}{IC}. \tag{3.16}$$

From these observations it follows that Q_W can be set as the initial Q and (3.15) can be used to find the corresponding initial r. Then, (3.14) can be used with this initial r in order to find a new Q. This iterative procedure continues until it converges to (Q_d^*, r_d^*). Note that this is the iterative procedure from Hadley and Whitin (1963) and its convergence was proved by them. ■

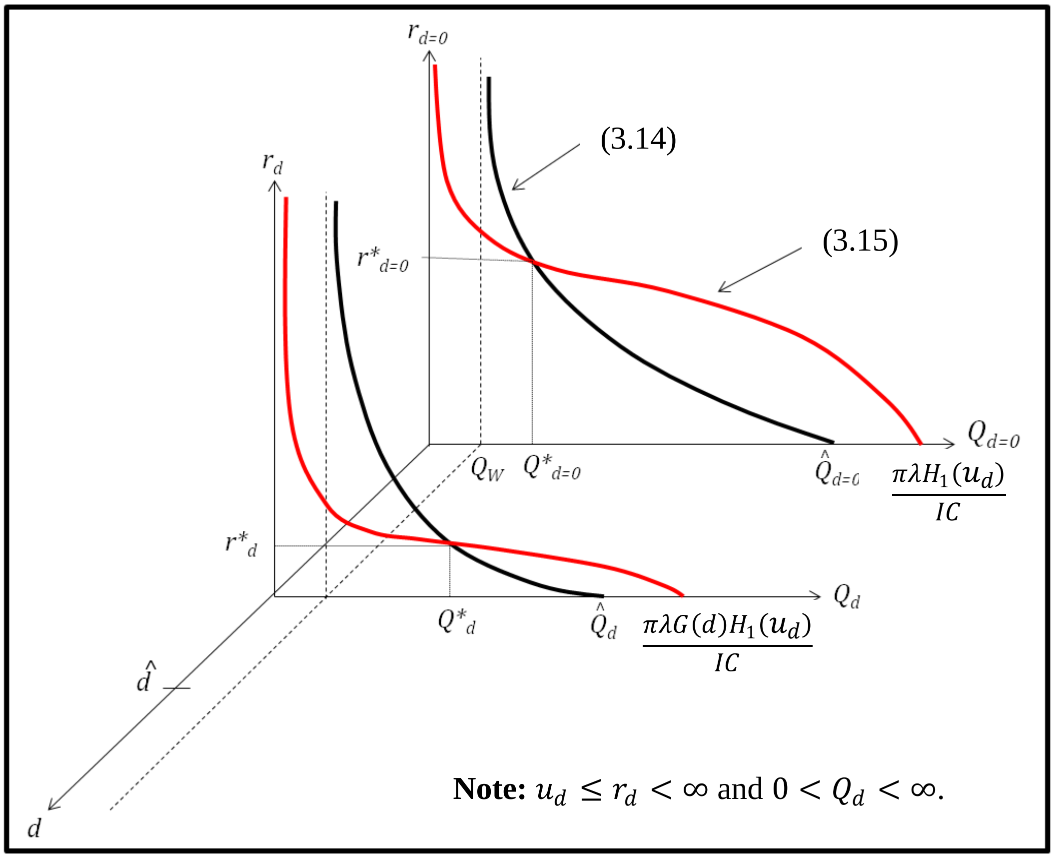


Figure 20: Behavior of Q^* and r^* with respect to d.

Lemma 3.4.1.3: *If* $h_1(x)$ *is the probability density function of a normal* $\left(\mu_{(d,l]}, \sigma^2_{(d,l]}\right)$, *then there is a* $\hat{d}$ *such that for any* $d \ge \hat{d}$, *there is no optimal policy* (Q^*_d, r^*_d) *in the range given by* $0 < Q < \infty$ *and* $\mu_{(d,l]} \le r < \infty$.

Proof: Let us assume that for a particular d, (Q^*_d, r^*_d) is the optimal policy satisfying $0 < Q < \infty$ and $\mu_{(d,l]} \le r < \infty$ when $h_1(x)$ is the probability density function of a normal $\left(\mu_{(d,l]}, \sigma^2_{(d,l]}\right)$. Then, from (3.16) and (3.13) it follows that

$$\hat{Q}_d \le \frac{\pi\lambda}{IC} G(d) H_1\left(\mu_{(d,l]}\right), \tag{3.17}$$

where,

$$\hat{Q}_d = \sqrt{2\lambda\left[A + \pi\left[\int_{x=\mu_{(d,l]}}^{\infty} x h_1(x)dx - \mu_{(d,l]} H_1\left(\mu_{(d,l]}\right)\right] G(d)\right]/IC}. \quad (3.18)$$

Both sides of (3.17) are decreasing functions of d. It can be observed that as d increases $\hat{Q}_d$ is bounded by Q_W and there must be a $\hat{d}$ such that

$$\hat{Q}_{\hat{d}} > \frac{\pi\lambda G(\hat{d}) H_1\left(\mu_{(\hat{d},l]}\right)}{IC}, \quad (3.19)$$

and (3.17) is not satisfied for any $d \geq \hat{d}$. Then, there is no optimal policy (Q_d^*, r_d^*) in the region given by $0 < Q < \infty$ and $\mu_{(d,l]} \leq r < \infty$. ■

Note that $\hat{d}$ is defined as the smallest d not satisfying (3.17) and can be computed as $\hat{d} \coloneqq min\left\{d \geq 0 \middle| \hat{Q}_d > \pi\lambda G(d) H_1\left(\mu_{(d,l]}\right)/IC\right\}$.

Lemma 3.4.1.4: *If the optimal policy* (Q_d^*, r_d^*) *exists and satisfies* $0 < Q < \infty$ and $u_d \leq r < \infty$, *then* (3.20) *is satisfied.*

$$Q_d^* > \frac{H_1(r_d^*)}{h_1(r_d^*)} \quad (3.20)$$

Proof: From Lemma 3.4.1.1 it is known that the cost function K_d is jointly convex in Q and r in the range $0 < Q < \infty$ and $u_d \leq r < \infty$. Then, the $\nabla^2 K_d$ evaluated at the optimal policy (Q_d^*, r_d^*) must be positive definite.

$$\begin{bmatrix} IC/Q_d^* & IC/Q_d^* \\ IC/Q_d^* & \frac{\pi\lambda}{Q_d^*} h_1(r_d) G(d) \end{bmatrix} \quad (3.21)$$

The leading principal minors of the matrix (3.21) must be positive. This is true for the leading principal minor given by the upper left-hand corner of the matrix and must be true for the leading principal minor given by the matrix itself.

$$det\begin{bmatrix} IC/Q_d^* & IC/Q_d^* \\ IC/Q_d^* & \frac{\pi\lambda}{Q_d^*}h_1(r_d^*)G(d) \end{bmatrix} = \frac{IC\pi\lambda}{{Q_d^*}^2}h_1(r_d^*)G(d) - \frac{IC^2}{{Q_d^*}^2} > 0 \qquad (3.22)$$

We can get (3.23) from the inequality given by (3.22).

$$\frac{Q_d^*}{H_1(r_d^*)G(d)} > \frac{1}{h_1(r_d^*)G(d)} \qquad (3.23)$$

From (3.23) we must conclude that (3.20) is a necessary condition for K_d to be convex. Since we have previously assumed that K_d is convex, then (3.20) is satisfied. ■

Lemma 3.4.1.5: *If the optimal policy* (Q_d^*, r_d^*) *exists and satisfies* $0 < Q < \infty$ and $u_d \leq r < \infty$, *then* (3.24) *is satisfied.*

$$Q_d^* > \frac{\left[\int_{x=r_d^*}^{\infty} xh_1(x)dx - r_d^* H_1(r_d^*)\right]}{H_1(r_d^*)} \qquad (3.24)$$

Proof: From Lemma 3.4.1.2 it is known that the iterative procedure explained in Hadley and Whitin (1963) can be used to find the optimal policy which is found at the intersection of (3.13) and (3.15). Then,

$$\frac{\pi\lambda}{IC}H_1(r_d^*)G(d) = \sqrt{\frac{2\lambda}{IC}[A + \pi\bar{n}_d(r_d^*)]}; \qquad (3.25)$$

and (3.26) is developed from (3.25).

$$\frac{Q_d^*}{2} = \frac{\left[\int_{x=r_d^*}^{\infty} xh_1(x)dx - r_d^* H_1(r_d^*)\right]}{H_1(r_d^*)} + \frac{\pi\lambda}{Q_d^* IC} \qquad (3.26)$$

Observe from (3.26) that (3.24) must be true. ■

3.4.2. Behavior of the Optimal Cost Function (K^*) with Respect to d without Delaying the Replenishment Order

It can be deduced from Lemma 3.4.1.3 that if $h_1(x)$ is the probability density function of a normal $\left(\mu_{(d,l]}, \sigma^2_{(d,l]}\right)$, then there is a $\hat{d}$ such that for any $d < \hat{d}$ the optimal policy (Q_d^*, r_d^*) exists in the region given by $0 < Q < \infty$ and $\mu_{(d,l]} \leq r < \infty$. It can be observed that for every $d < \hat{d}$ there is a minimum cost K_d^*, and K^* can be defined as a function of d representing $K_d^* \; \forall \; d < \hat{d}$. It is important to understand the behavior of K^* since the excess inventory (e_d) is increasing as d increases as mentioned in section 3.3. In particular, defining d^* such that $K_{d^*}^*$ is the minimum value of K^*, it is appealing to delay the placement of the replenishment order for any $d > d^*$ in order to minimize e_d.

Let us first focus our attention on the behavior of the cost function K with respect to d assuming no delay of the replenishment order. Let us calculate the $\partial K/\partial d = 0$ and $\partial^2 K/\partial d^2$ as given by (3.27) and (3.28), respectively.

$$\frac{\partial K}{\partial d} = 0 \xrightarrow{yields} \frac{\pi\lambda}{Q}\frac{\partial}{\partial d}\bar{n}_d(r) = IC\left[\frac{\partial}{\partial d}\left[\mu_{(d,l]}G(d)\right] - \frac{\partial}{\partial d}\left[\mu_{(l,d]}\left(1 - G(d)\right)\right]\right] \quad (3.27)$$

$$\frac{\partial^2 K}{\partial d^2} = IC\left[\begin{array}{c}\left[\mu_{(d,l]} + \mu_{(l,d]}\right]\frac{\partial}{\partial d}g(d) - \left[\frac{\partial^2}{\partial d^2}\mu_{(d,l]} + \frac{\partial^2}{\partial d^2}\mu_{(l,d]}\right]G(d) \\ + \frac{\partial^2}{\partial d^2}\mu_{(l,d]} + 2\left[\frac{\partial}{\partial d}\mu_{(d,l]} + \frac{\partial}{\partial d}\mu_{(l,d]}\right]g(d)\end{array}\right]$$

$$+\frac{\pi\lambda}{Q}\begin{bmatrix} -\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(r)\right]\frac{\partial}{\partial d}g(d) \\ +\frac{\partial^2}{\partial d^2}\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(r)\right]G(d) \\ -2\frac{\partial}{\partial d}\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(r)\right]G(d) \end{bmatrix} \quad (3.28)$$

Where,

$$\frac{\partial}{\partial d}\left[\mu_{(d,l]}G(d)\right] = -\mu_{(d,l]}g(d) + G(d)\frac{\partial}{\partial d}\mu_{(d,l]} < 0 \quad \forall\, d; \quad (3.29)$$

$$\frac{\partial}{\partial d}\left[\mu_{(l,d]}\big(1 - G(d)\big)\right] = \mu_{(l,d]}g(d) + \big(1 - G(d)\big)\frac{\partial}{\partial d}\mu_{(l,d]} > 0 \quad \forall\, d; \quad (3.30)$$

and

$$\frac{\partial}{\partial d}\bar{n}_d(r) = -\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(r)\right]g(d)$$

$$+\frac{\partial}{\partial d}\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(r)\right]G(d) < 0 \quad \forall\, r, d. \quad (3.31)$$

It is known from the 1st order conditions that (3.27) is satisfied if K is convex with respect to$d$. Note that under the assumption that $X_{(d,l]}$ and $X_{(l,d]}$ follow normal distributions, $N\big(\mu_{(d,l]}, \sigma^2_{(d,l]}\big)$ and $N\big(\mu_{(l,d]}, \sigma^2_{(l,d]}\big)$, respectively, any change in the coefficient of variation (CV) of the lead-time demand as result of a change in the standard deviation will have an impact in the left-hand side of (3.27) but not in the right-hand side. This suggests that satisfying (3.27) is dependent on CV. A similar case could be established for parameter π. In addition, it is impossible to make inferences from the hessian matrix $\nabla^2 K$ since (3.28) is a function of $\partial g(d)/\partial d$. It must be concluded from these observations that the behavior of K as a function of d is strictly dependent on the problem being addressed. Hence, the behavior of the optimal cost function K^* is also dependent on the problem being addressed as shown in figures 21 and 22.

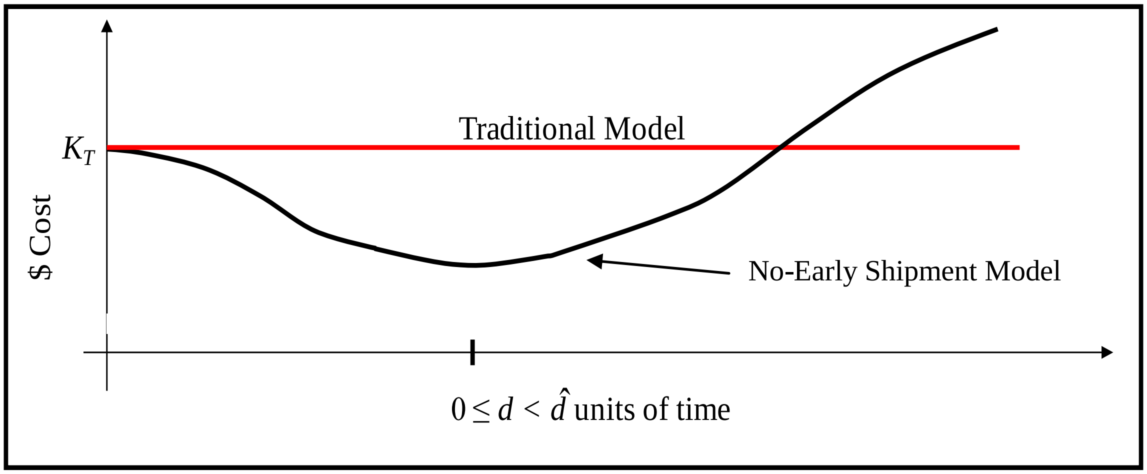


Figure 21: Best case behavior of the "no-early shipment/inventory allocation" model without delaying policy vs. traditional model.

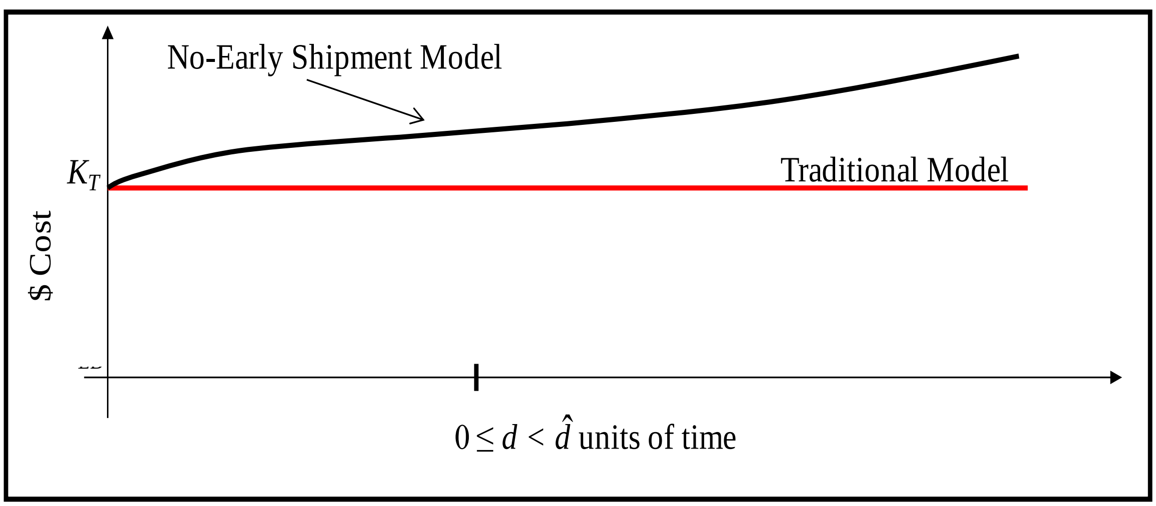


Figure 22: Worst case behavior of the "no-early shipment/inventory allocation" model without delaying policy vs. traditional model.

At first glance, we might be tempted to conclude that a generalized comparison between the modified and the traditional model is impossible. However, this statement is only true if the placement of the replenishment order is not delayed. It is shown in the following section that delaying the replenishment order is always desired whenever the optimal cost function K^* increases as d increases.

3.4.3. Behavior of the Optimal Cost Function (K^*) with Respect to d When the Replenishment Order Is Delayed

In order to show the benefit of delaying the replenishment order, let us define $m = |d - d^*|$ as the units of time by which the replenishment order is delayed; and $d' = d - m$ as the amount of time left before the penalty is incurred, measured from the date when the replenishment order is placed. The only change in the model is the use of d' instead of d.

Let us define $d^*, 0 \leq d^* < \hat{d}$, as the optima that minimizes K^* such that $K^*_{d^*} < K^*_d$ for any d. Observe that if $d > d^*$, $d' = d^*$ and $K^*_{d'} = K^*_{d^*}$ if the replenishment order is delayed by m units of time. It can easily be concluded that delaying when $d < d^*$ is of no benefit since $d' < d < d^*$ and $K^*_{d'} \geq K^*_{d^*}$. From these observations it is concluded that delaying the replenishment order is only desired for any $d > d^*$. Note that since $d' = d^* < \hat{d}$, the convexity of $K_{d'}$ is guaranteed for any $d \geq \hat{d}$.

For the best case behavior presented in Figure 21, delaying the replenishment order by the appropriate amount of time gives the optimal policy $\left(Q^*_{d'} = Q^*_{d^*}, r^*_{d'} = r^*_{d^*}\right)$ for any $d > d^*$and the modified model performs better than the traditional one as shown in Figure 23. However, for the worst case as shown in Figure 22, $d^* = 0$ and delaying for any $d > 0$ gives the same optimal policy as in the traditional (Q, r) model.

It is concluded from the previous remarks that even though the behavior of the modified (Q, r) model with the "no-early shipment" policy is dependent on the problem being addressed, delaying the replenishment order when necessary guarantees the same or a better performance than the traditional model.

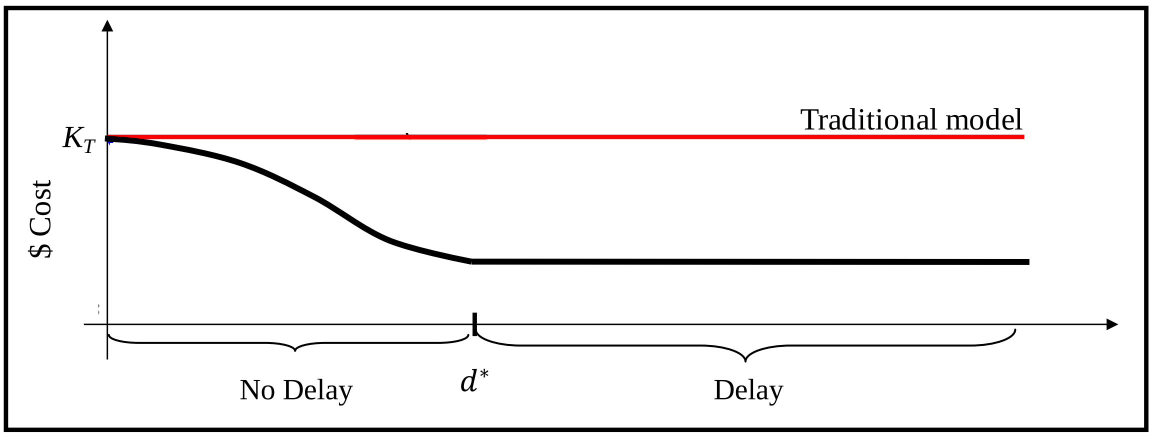


Figure 23: Best case behavior of the "no-early shipment/inventory allocation" model delaying the replenishment order.

The modified (Q, r) model with the "no-early" shipment policy can be implemented for any choice of d using the following algorithm:

Start,

Define d;

Calculate $\hat{d} \coloneqq min\{d \geq 0 | \hat{Q}_d > \pi\lambda G(d) H_1(\mu_{(d,l]})/IC\}$;

Set $Dmax$ equal to d or $\hat{d} - 1$, whichever is lower;

Set $d^* = 0$ and find $Q^*_{d^*}$, $r^*_{d^*}$, and $K^*_{d^*}$;

Set $D = 1$;

For $D \leq Dmax$,

Find Q^*_D, r^*_D, and K^*_D,

If $K^*_D \leq K^*_{d^*}$,

Set $m_D = 0, d^* = D, Q^*_{d^*} = Q^*_D,\ r^*_{d^*} = r^*_D,\ K^*_{d^*} = K^*_D$;

Else,

Set $m_D = D - d^*, Q^*_D = Q^*_{d^*},\ r^*_D = r^*_{d^*},\ K^*_D = K^*_{d^*}$,

End;

$D = D + 1$,

End;

For $d \geq \hat{d}$,

Set $m_d = d - d^*, Q^*_d = Q^*_{d^*},\ r^*_d = r^*_{d^*},\ K^*_d = K^*_{d^*}$,

End;

End.

Defining $d^* = 0$ as the initial minimizer, this algorithm increases d by Δd units of time until a new local minimizer is found. It delays the replenishment order by $m_d = d - d^*$ units of time for any $K^*_d > K^*_{d^*}$. Once a new local minimizer is found at a new d, it identifies $d^* = d$ and starts the cycle again until the particular d of interest or $\hat{d} - 1$ is reached first. If $d \geq \hat{d}$, the algorithm delays the replenishment order for d by $m_d = d - d^*$, where d^* is the local minimizer found last.

3.4.4. Behavior of the Optimal Cost Function (K^*) During the Time Period $[0, d_1)$ – Minimum Lead-Time (l_{min})

Let us define the following generalized scenario: there exists $d_1 = l_{min}$, such that $d_1 \geq 0$ and $P(L \geq d_1) = 1$ but $P(L \geq d_1 + \Delta d) < 1$. Observe that $g(d) = 0$ in the range $[0, d_1)$. Assuming that a customer order that triggers a replenishment order is received at time $t = 0$, there is no chance for receiving the replenishment order before d_1 units of time since the minimum lead-time (l_{min}) is occurring at time $t = d_1$.

As mentioned in the previous chapter, understanding the behavior of the optimal cost function (K^*) for this generalized scenario has important practical implications since the period of time $[0, d_1)$ could be related to the location of the supplier. For example, $d_1 = 0$ can be related to a supplier located next to the manufacturer, who delivers as soon as the replenishment order is placed because he/she keeps available inventory of the component. On the other hand, a d_1 relatively larger than zero could be related to a supplier located far away from the manufacturer such that the replenishment order is received the earliest d_1 units of time after being placed.

Observe that for this scenario $G(d) = 1$ for any $d \leq d_1$ and (3.12) becomes (3.32).

$$K = \frac{\lambda}{Q}A + IC\left[\frac{Q}{2} + r - \mu_{(d,l]}\right] + \frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} x h_1(x)dx - r\,H_1(r)\right] \qquad (3.32)$$

(3.32) has the same structure of the traditional model as defined by Hadley and Whitin (1963) except that the demand of interest is the demand during the time period $(d, l]$ instead of the lead-time demand. As d increases, the time period $(d, l]$ is closing such that the observed demand and its variability are both decreasing, and the behavior of (3.32) is given by the traditional model for cases in which the lead-time period $(0, l]$ is decreasing. In such a case, it can be concluded that K^* is a decreasing function of d for any $d \leq d_1 = l_{min}$. This observation is important because the existence of d_1 is guaranteeing the better performance of the modified (Q, r) model with the "no-early shipment" policy, suggesting that the manufacturer will tend to prefer the modified model over the traditional model.

3.4.5. Behavior of r^*, Q^*, and Their Relationship with the Penalty Cost

It is of interest to understand mathematically the behavior of r^* and Q^* as d increases, and their relationship to the penalty cost since this can facilitate the analysis of any numerical example. Let us define Q^* and r^* as functions of d representing Q_d^* and r_d^* $\forall\, d$.

The behavior of r^* as d increases is given by the following lemma:

Lemma 3.4.5.1: *r^* is a strictly decreasing function of d.*

Proof: This proof follows directly from the definition of the safety stock, $s_d = r_d - \mu_{(d,l]}$, as given in (3.7). We get (3.33) by rearranging the safety stock in terms of r^*.

$$r^* = \mu_{(d,l]} + s^* \tag{3.33}$$

As defined by (3.33), s^* is a function of d representing the safety stock related to r_d^* $\forall\, d$. $X_{(d,l]}$ is the demand of interest as given by the time interval $(d, l]$. This interval is closing as d increases and the observed demand with its variability must be decreasing.

Then, the expected demand and s^* are decreasing functions of d, and r^* must be strictly decreasing as d increases. ■

For the case of Q^*, let us calculate the derivative of (3.11) with respect to d.

$$\frac{\mathrm{d}}{\mathrm{d}d}\left[\frac{\pi\lambda}{Q^*}\bar{\eta}_d(r^*)\right] = \frac{\mathrm{d}}{\mathrm{d}d}\left[\frac{\pi\lambda}{Q^*}\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]G(d)\right]$$

$$= \pi\lambda\left[\begin{array}{c}\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]G(d)\dfrac{\mathrm{d}Q^{*-1}}{\mathrm{d}d} \\ +Q^{*-1}\left[\begin{array}{c}\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]\dfrac{\mathrm{d}}{\mathrm{d}d}G(d) \\ +G(d)\dfrac{\mathrm{d}}{\mathrm{d}d}\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]\end{array}\right]\end{array}\right]$$

$$= -\frac{\pi\lambda}{Q^*}\left[\begin{array}{c}\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]\left[\dfrac{G(d)}{Q^*}\dfrac{\mathrm{d}Q^*}{\mathrm{d}d} + g(d)\right] \\ -G(d)\left[\left[\int_{x=r^*}^{\infty} x\dfrac{\partial}{\partial d}h_1(x)dx - r^*\int_{x=r^*}^{\infty}\dfrac{\partial}{\partial d}h_1(x)dx\right] - H_1(r^*)\dfrac{\mathrm{d}r^*}{\mathrm{d}d}\right]\end{array}\right] \quad (3.34)$$

Observe that mathematical expressions are required for $\mathrm{d}r^*/\mathrm{d}d$ and $\mathrm{d}Q^*/\mathrm{d}d$ in order to understand (3.34). Let us rearrange (3.14) in terms of $H_1(r^*)$ and realize that both sides of the resulting equation (3.35) are functions of d.

$$H_1(r^*) = \frac{IC}{\lambda\pi}Q^*G(d)^{-1} \quad (3.35)$$

From its definition, $H_1(r^*) = \int_{x=r^*}^{\infty} h_1(x)dx$ and (3.36) is developed by calculating the derivative with respect to d at both sides of (3.35).

$$\int_{x=r^*}^{\infty}\frac{\partial}{\partial d}h_1(x)dx - h_1(r^*)\frac{\mathrm{d}r^*}{\mathrm{d}d} = \frac{IC}{\pi\lambda}\left[\frac{Q^*}{G(d)^2}g(d) + \frac{1}{G(d)}\frac{\mathrm{d}Q^*}{\mathrm{d}d}\right]$$

$$\frac{\mathrm{d}r^*}{\mathrm{d}d} = -\frac{H_1(r^*)}{Q^*h_1(r^*)}\left[Q^*\frac{g(d)}{G(d)} + \frac{\mathrm{d}Q^*}{\mathrm{d}d}\right] + \frac{\int_{x=r^*}^{\infty}\frac{\partial}{\partial d}h_1(x)dx}{h_1(r^*)} \quad (3.36)$$

Q^* is given by (3.14), and deriving it with respect to d we get (3.37).

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d} = \frac{\pi\lambda}{Q^* IC}\left[\left[\int_{x=r^*}^{\infty} x h_1(x)dx - r^* H_1(r^*)\right]\frac{\mathrm{d}}{\mathrm{d}d}G(d) + G(d)\frac{\mathrm{d}}{\mathrm{d}d}\left[\int_{x=r^*}^{\infty} x h_1(x)dx - r^* H_1(r^*)\right]\right]$$

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d} = -\frac{1}{H_1(r^*)G(d)}\left[\int_{x=r^*}^{\infty} x h_1(x)dx - r^* H_1(r^*)\right] g(d) - \frac{\mathrm{d}r^*}{\mathrm{d}d} \tag{3.37}$$

We get equation (3.38) by replacing from (3.37) the $\mathrm{d}r^*/\mathrm{d}d$ with (3.36).

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d} = \left[\begin{array}{c} \left[\frac{H_1(r^*)}{h_1(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} x h_1(x)dx - r^* H_1(r^*)\right]}{H_1(r^*)}\right]\frac{g(d)}{G(d)} \\ + \left[\frac{\left[\int_{x=r^*}^{\infty} x \frac{\partial}{\partial d} h_1(x)dx - r^* \int_{x=r^*}^{\infty} \frac{\partial}{\partial d} h_1(x)dx\right]}{H_1(r^*)} - \frac{\int_{x=r^*}^{\infty} \frac{\partial}{\partial d} h_1(x)dx}{h_1(r^*)}\right] \end{array}\right] * \left[1 - \frac{H_1(r^*)}{Q^* h_1(r^*)}\right]^{-1} \tag{3.38}$$

And (3.38) can be expressed as (3.39).

$$\left[\begin{array}{c} \left[\frac{H_1(r^*)}{h_1(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} x h_1(x)dx - r^* H_1(r^*)\right]}{H_1(r^*)}\right]\frac{g(d)}{G(d)} \\ + \left[\frac{\left[\int_{x=r^*}^{\infty} x \frac{\partial}{\partial d} h_1(x)dx - r^* \int_{x=r^*}^{\infty} \frac{\partial}{\partial d} h_1(x)dx\right]}{H_1(r^*)} - \frac{\int_{x=r^*}^{\infty} \frac{\partial}{\partial d} h_1(x)dx}{h_1(r^*)}\right] \end{array}\right] = \frac{\mathrm{d}Q^*}{\mathrm{d}d}\left[1 - \frac{H_1(r^*)}{Q^* h_1(r^*)}\right] \tag{3.39}$$

The relationship between the penalty and Q^* as r^* decreases is defined with the following lemma:

Lemma 3.4.5.2: As r^* *decreases,* $\mathrm{d}Q^*/\mathrm{d}d \geq 0$ *if the penalty cost does not decrease, or* $\mathrm{d}Q^*/\mathrm{d}d < 0$ *if the penalty cost decreases.*

Proof: We know form Lemma 3.4.5.1 that $\mathrm{d}r^*/\mathrm{d}d < 0$. Let us first assume that the penalty cost does not decrease such that $\mathrm{d}Q^*/\mathrm{d}d < 0$. Then, (3.40) is developed by setting (3.34) equal as or greater than zero.

$$\left[\begin{array}{c} -\dfrac{\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]}{Q^* H_1(r^*)}\dfrac{\mathrm{d}Q^*}{\mathrm{d}d} \\ -\dfrac{\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]}{H_1(r^*)}\dfrac{g(d)}{G(d)} \\ +\dfrac{\left[\int_{x=r^*}^{\infty} x\dfrac{\partial}{\partial d}h_1(x)dx - r^* \int_{x=r^*}^{\infty} \dfrac{\partial}{\partial d}h_1(x)dx\right]}{H_1(r^*)} \end{array}\right] \geq \frac{\mathrm{d}r^*}{\mathrm{d}d} \tag{3.40}$$

(3.41) is obtained by substituting from (3.40) the $\mathrm{d}r^*/\mathrm{d}d$ by (3.36).

$$\left[\begin{array}{c} \left[\dfrac{H_1(r^*)}{h_1(r^*)} - \dfrac{\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]}{H_1(r^*)}\right]\dfrac{g(d)}{G(d)} \\ +\left[\dfrac{\left[\int_{x=r^*}^{\infty} x\dfrac{\partial}{\partial d}h_1(x)dx - r^* \int_{x=r^*}^{\infty} \dfrac{\partial}{\partial d}h_1(x)dx\right]}{H_1(r^*)} - \dfrac{\int_{x=r^*}^{\infty} \dfrac{\partial}{\partial d}h_1(x)dx}{h_1(r^*)}\right] \end{array}\right]$$

$$\geq -\frac{1}{Q^*}\left[\frac{H_1(r^*)}{h_1(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]}{H_1(r^*)}\right]\frac{\mathrm{d}Q^*}{\mathrm{d}d} \tag{3.41}$$

The left-hand side of (3.41) can be substituted by (3.39) in order to get (3.42).

$$\frac{\mathrm{d}Q^*}{\mathrm{d}d}\left[1 - \frac{H_1(r^*)}{Q^* h_1(r^*)}\right] \geq -\frac{1}{Q^*}\left[\frac{H_1(r^*)}{h_1(r^*)} - \frac{\left[\int_{x=r^*}^{\infty} xh_1(x)dx - r^* H_1(r^*)\right]}{H_1(r^*)}\right]\frac{\mathrm{d}Q^*}{\mathrm{d}d} \tag{3.42}$$

For any particular d, (3.43) is developed from (3.42).

$$Q_d^* - \frac{H_1(r_d^*)}{h_1(r_d^*)} \leq \frac{\left[\int_{x=r_d^*}^{\infty} xh_1(x)dx - r_d^* H_1(r_d^*)\right]}{H_1(r_d^*)} - \frac{H_1(r_d^*)}{h_1(r_d^*)} \tag{3.43}$$

(3.43) is a contradiction of Lemma 3.4.1.5 and $\mathrm{d}Q^*/\mathrm{d}d \geq 0$ if the penalty cost does not decrease.

For the second part of this proof, let us assume that the penalty cost is decreasing such that $\mathrm{d}Q^*/\mathrm{d}d \geq 0$. Note that (3.44) is obtained by setting (3.34) as less than zero and by following the same approach as in the first part.

$$Q_d^* - \frac{H_1(r_d^*)}{h_1(r_d^*)} < \frac{\left[\int_{x=r_d^*}^{\infty} x h_1(x)dx - r_d^* H_1(r_d^*)\right]}{H_1(r_d^*)} - \frac{H_1(r_d^*)}{h_1(r_d^*)} \tag{3.44}$$

Then, it must be concluded by contradiction of Lemma 3.4.1.5 that $\mathrm{d}Q^*/\mathrm{d}d < 0$ if the penalty cost decreases. ■

Summarizing the relationship between r^*, Q^*, and the penalty cost as d increases: r^* strictly decreases, and Q^* and the penalty cost are both dependent on the problem being addressed. In particular, the model takes advantage and benefits the inventory by reducing Q^* if the penalty cost decreases as r^* decreases, or increases Q^* in order to minimize any increase in penalty cost.

The following has been shown so far:

1) There is a $\hat{d}$ such that for any $d < \hat{d}$, K_d is jointly convex with respect to Q and r in the region $0 < Q < \infty$ and $\mu_{(d,l]} \leq r < \infty$ when $h_1(x)$ is the probability density function of a normal $\left(\mu_{(d,l]}, \sigma^2_{(d,l]}\right)$;
2) The behavior of the optimal cost function (K^*) is dependent on the problem being addressed;
3) Delaying the replenishment order is always desired whenever K_d^* increases as d increases;

4) K^* is a decreasing function of d for any $d \leq d_1 = l_{min}$;

5) r^* is a strictly decreasing function of d while Q^* and the penalty cost are dependent on the problem being addressed.

The result of numerical examples used to help visualize the behavior of the model is presented in the following section.

3.5. Numerical Study

This section presents the result of a numerical study performed with the following objectives:

1) Identify the critical parameters for the behavior of the optimal cost function (K^*) with respect to the safety time (d);

2) Show the behavior of the model as discussed in the previous chapter, in particular:

    a. K^* when the replenishment order is delayed,

    b. K^* for any $d \leq d_1 = l_{min}$,

    c. Relationship between r^*, Q^*, and the penalty cost;

3) Demonstrate the performance of the modified (Q, r) model with the "no-early shipment" policy as compared to the traditional (Q, r) model.

The study was done under the assumption that the lead-time follows an expo (β), the lead-time demand follows a $N(\mu, \sigma^2)$, the demand during the interval of time $(d, l]$ follows a $N\left(\mu e^{-d/\beta}, \sigma^2 e^{-2d/\beta}\right)$, and the demand during the period of time $(l, d]$ follows a $N\left(\mu\left[e^{-d/\beta} + d/\beta - 1\right], \sigma^2\left[e^{-2d/\beta} + 2\, d/\beta - 1\right]\right)$. The following parameters were varied as shown in Table 4 with the objective of understanding the behavior of the cost function (K^*): σ^2, β, IC, and π. μ was fixed to 950, and A to 4000. λ is calculated assuming 52 production weeks in a year, $\lambda = 52 * \mu/\beta$. The $P(X \leq 0)$ is calculated in order to confirm that it is negligible due to the assumption that each demand of interest follows a normal distribution.

Table 4: Parameters for each case problem.

Case	Parameters					P (X≤ 0)	Case	Parameters					P (X≤ 0)
	Calculated	Experimental						Calculated	Experimental				
	λ	h(x)	g(l)	π	IC			λ	h(x)	g(l)	π	IC	
		σ	β						σ	β			
1	24700.0	50	2	20	1	8.53E-81	31	12350.0	150	4	2000	1	1.2E-10
2	24700.0	50	2	20	10	8.53E-81	32	12350.0	150	4	2000	10	1.2E-10
3	24700.0	50	2	2000	1	8.53E-81	33	8233.3	150	6	20	1	1.2E-10
4	24700.0	50	2	2000	10	8.53E-81	34	8233.3	150	6	20	10	1.2E-10
5	12350.0	50	4	20	1	8.53E-81	35	8233.3	150	6	2000	1	1.2E-10
6	12350.0	50	4	20	10	8.53E-81	36	8233.3	150	6	2000	10	1.2E-10
7	12350.0	50	4	2000	1	8.53E-81	37	24700.0	200	2	20	1	1.02E-06
8	12350.0	50	4	2000	10	8.53E-81	38	24700.0	200	2	20	10	1.02E-06
9	8233.3	50	6	20	1	8.53E-81	39	24700.0	200	2	2000	1	1.02E-06
10	8233.3	50	6	20	10	8.53E-81	40	24700.0	200	2	2000	10	1.02E-06
11	8233.3	50	6	2000	1	8.53E-81	41	12350.0	200	4	20	1	1.02E-06
12	8233.3	50	6	2000	10	8.53E-81	42	12350.0	200	4	20	10	1.02E-06
13	24700.0	100	2	20	1	1.05E-21	43	12350.0	200	4	2000	1	1.02E-06
14	24700.0	100	2	20	10	1.05E-21	44	12350.0	200	4	2000	10	1.02E-06
15	24700.0	100	2	2000	1	1.05E-21	45	8233.3	200	6	20	1	1.02E-06
16	24700.0	100	2	2000	10	1.05E-21	46	8233.3	200	6	20	10	1.02E-06
17	12350.0	100	4	20	1	1.05E-21	47	8233.3	200	6	2000	1	1.02E-06
18	12350.0	100	4	20	10	1.05E-21	48	8233.3	200	6	2000	10	1.02E-06
19	12350.0	100	4	2000	1	1.05E-21	49	24700.0	250	2	20	1	7.23E-05
20	12350.0	100	4	2000	10	1.05E-21	50	24700.0	250	2	20	10	7.23E-05
21	8233.3	100	6	20	1	1.05E-21	51	24700.0	250	2	2000	1	7.23E-05
22	8233.3	100	6	20	10	1.05E-21	52	24700.0	250	2	2000	10	7.23E-05
23	8233.3	100	6	2000	1	1.05E-21	53	12350.0	250	4	20	1	7.23E-05
24	8233.3	100	6	2000	10	1.05E-21	54	12350.0	250	4	20	10	7.23E-05
25	24700.0	150	2	20	1	1.2E-10	55	12350.0	250	4	2000	1	7.23E-05
26	24700.0	150	2	20	10	1.2E-10	56	12350.0	250	4	2000	10	7.23E-05
27	24700.0	150	2	2000	1	1.2E-10	57	8233.3	250	6	20	1	7.23E-05
28	24700.0	150	2	2000	10	1.2E-10	58	8233.3	250	6	20	10	7.23E-05
29	12350.0	150	4	20	1	1.2E-10	59	8233.3	250	6	2000	1	7.23E-05
30	12350.0	150	4	20	10	1.2E-10	60	8233.3	250	6	2000	10	7.23E-05

3.5.1. Critical Parameters

The main objective of this section is to determine which parameters influence the behavior of the optimal cost function (K^*) as d increases. In particular, the analysis is done with respect to the coefficient of variation (CV) and the ratio (IC/π). For practical purposes, the analysis is done assuming that $\Delta d = 1$ week and the same algorithm presented at the beginning of Section 2.5 is followed.

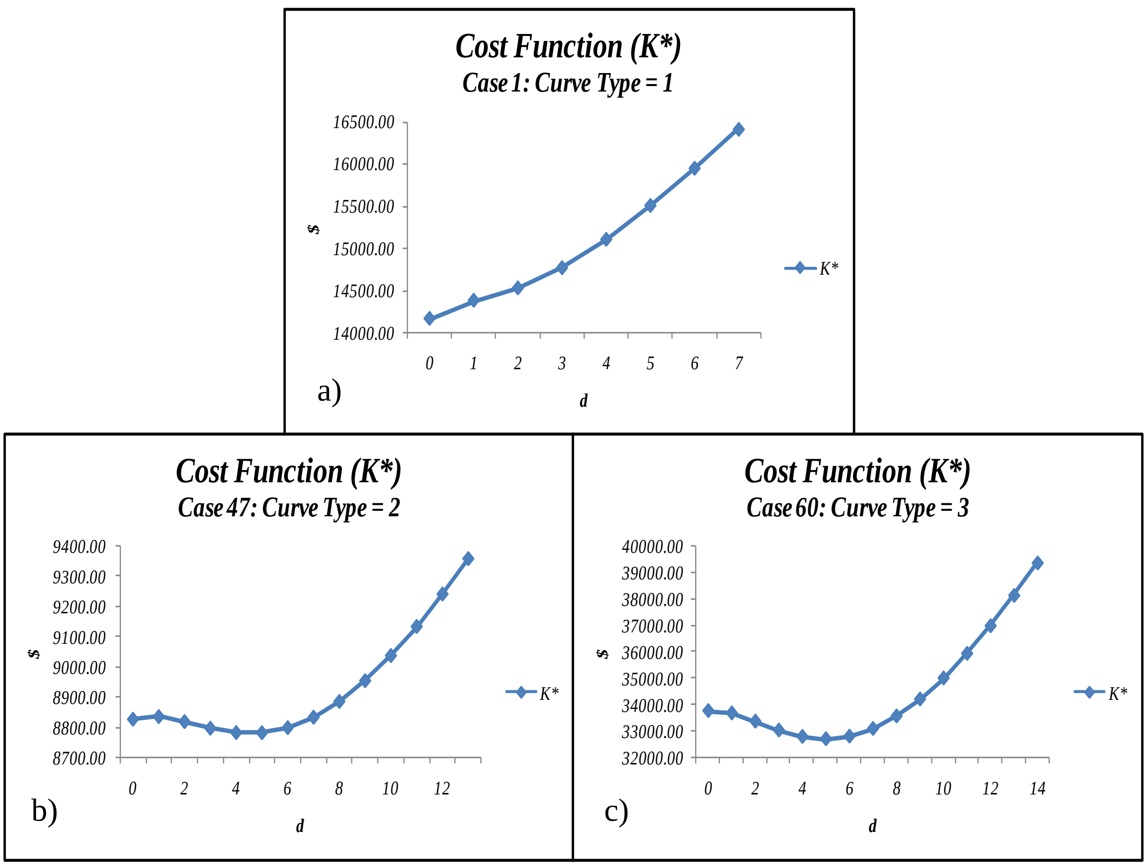


Figure 24: Categories for the behavior of K^*: a) Type 1, b) Type 2, and c) Type 3.

The behavior of K^* is categorized as follows: 1) as a strictly increasing function of d as shown by Figure 24.a, 2) as initially increasing, then decreasing and finally increasing again as shown in Figure 24.b, and 3) as initially decreasing and then increasing as shown in Figure 24.c. The characteristics of the behavior that are analyzed are the following: the type of curve (category), and the location of d^* and $\hat{d}$.

Curve type #1 represents the worst case scenario presented in Figure 22 of section 3.4.2. For this scenario, it was said that delaying for any $d > 0$ guarantees the same cost as the traditional model since $d^* = 0$. On the other hand, curve type #3 represents the best case scenario presented in Figure 21. In such a case, $d^* > 0$ and delaying for any $d > d^*$ guarantees the better performance of the modified (Q, r) model with the "no-early shipment" policy for any $d > 0$. Understanding which parameters influence the behavior of K^* is critical for this research because the chances for the modified model to be preferred by the manufacturer are higher under curve type #3.

The following set of charts are intended to show how critical σ of the lead-time demand is for the cost function (K^*). This is done by means of analyzing the behavior of K^* with respect to the CV since μ is left constant. Figure 25.a shows the range of the values observed for the curve type at each CV in addition to the average. Note that at low levels of CV the only curve observed is the type #1, but as the CV increases, curve type #2 and #3 are also observed as given by the "high" value of the range. It can also be deduced from the average that the number of times that curve type #3 is observed is increasing as the CV increases. This observation is validated with Figure 25.b.

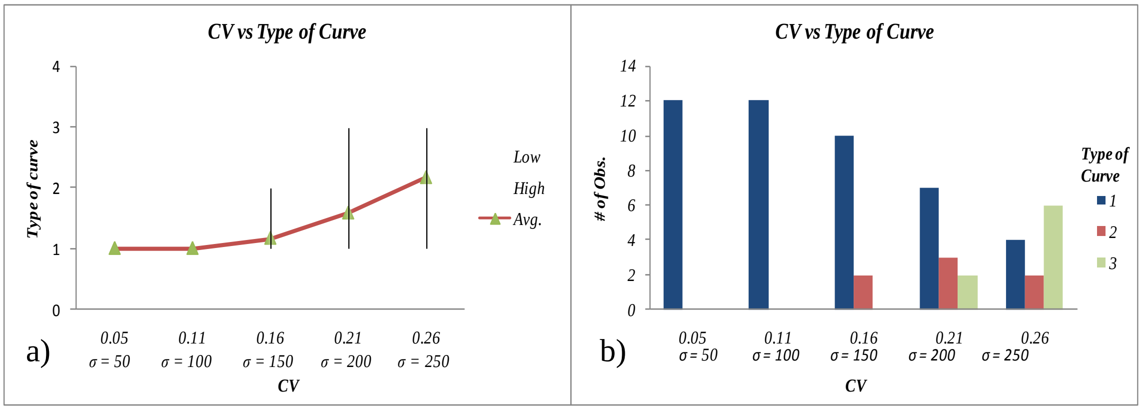


Figure 25: Coefficient of variation (CV) vs. type of curve.

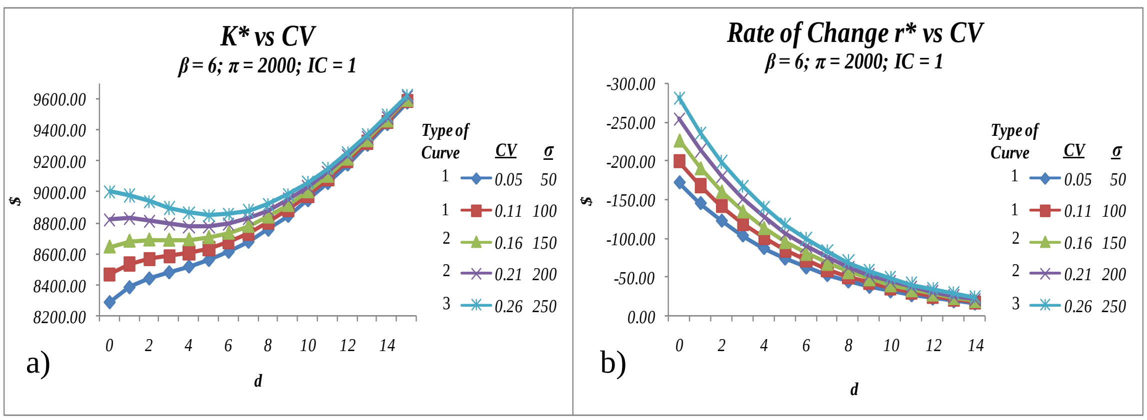


Figure 26: Relationship between r^* and the behavior of K^* with respect to the CV.

From the previous observations we must deduct that at low levels of CV, the behavior of K^* is more probable to follow a form similar to curve #1 but at high levels, it is more probable to follow the pattern or curve type #3 as shown in Figure 26.a. The explanation for this finding is mostly explained by the change in inventory given by the relationship between the inventory being allocated and r^*. Note that for any particular d and everything else being equal, r_d^* is set at a higher level as the CV is increased as result of an increase in the σ of the lead-time demand. In addition, the safety stock is needed for the demand during the period of time $(d, l]$, and the demand variability removed as d increases by Δd units of time is higher for higher CV's. Then, the rate of change of r^* with respect to d is more negative at any d for higher CV's as shown in Figure 26.b; meaning that r^* decreases faster at higher CV's. However the expected amount of inventory being allocated is the expected demand observed during the period of time $(0, d]$, which is increasing as d increases but constant with respect to the CV. Then, for any particular d, there must be a $\widehat{\mathrm{CV}}$ such that for any $\mathrm{CV} > \widehat{\mathrm{CV}}$ the absolute value for the rate of change of r^* with respect to d is higher than the rate of change of the expected demand during the period of time $(d, l]$. Since r^* is decreasing as d increases, it follows that if the reduction in r^* is large enough to counteract the increase in inventory as result of the allocation and of any possible increase in Q^*, the total inventory will decrease driving a decrease in K^*. Otherwise, the inventory will increase driving an increase in K^*.

It can be deduced from the previous paragraph that the behavior of the optimal cost function K^* as d increases is closely related to the behavior of the inventory holding cost. In fact, Figure 27 shows that there is a positive linear relationship between these two costs, confirming the previous explanation that K^* increases if Q^* increases, and vice versa.

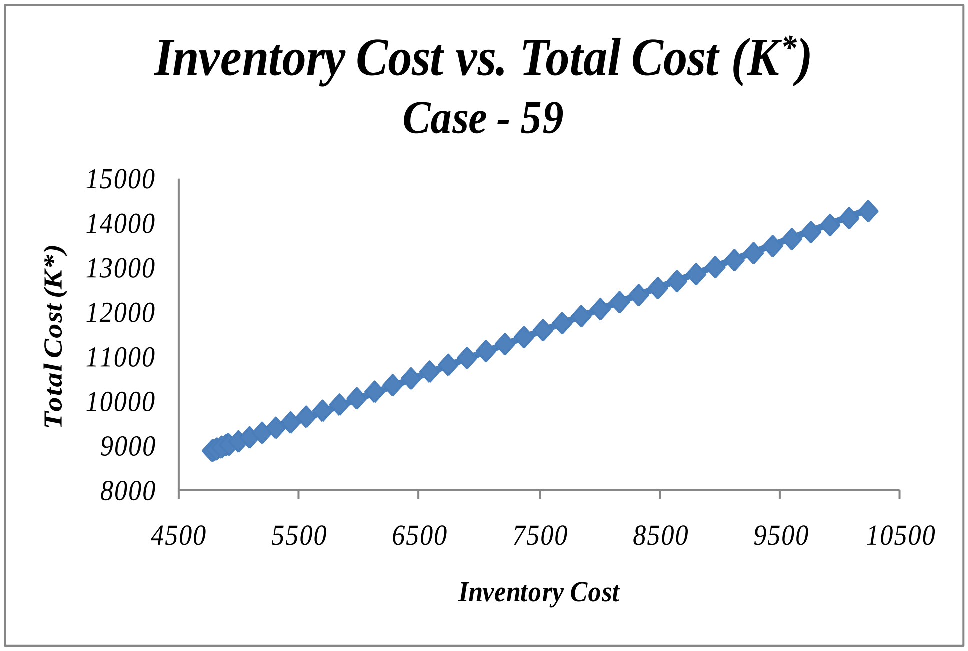


Figure 27: Linear relationship between the inventory holding cost and K^*.

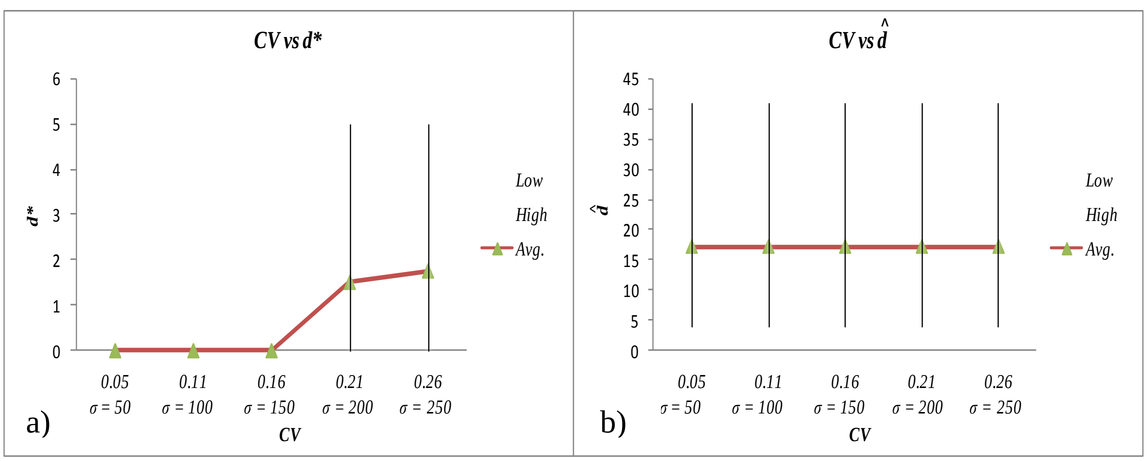


Figure 28: CV vs. d^* and $\hat{d}$.

If the curve type is influenced by the CV, then it must be concluded that d^* is also influenced by the CV as shown in Figure 28.a. Note that $d^* = 0$ at low levels of the CV but for high levels it is greater than zero and increasing as shown by the average. On the other hand, it can be concluded from Figure 28.b that $\hat{d}$ is not sensitive to the CV. Note that the minimum and maximum values as well as the average do not change as the CV increases.

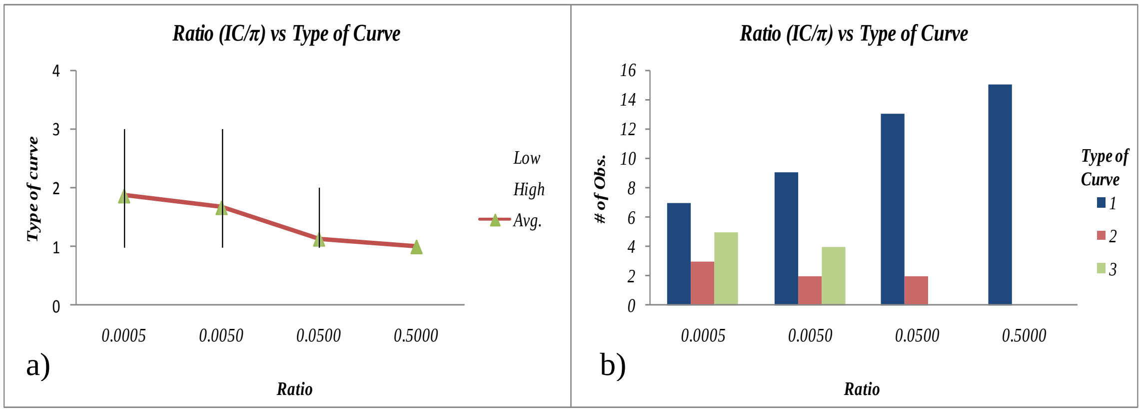


Figure 29: Ratio (IC/π) vs. type of curve.

From figures 25 - 28, it is concluded that the behavior of K^* as d increases is sensitive to the CV of the lead-time demand. The following set of charts is intended to show how sensitive is the optimal cost function (K^*) to the ratio given by IC/π.

Figure 29.a shows how the range and the average of the values observed for the curve type vary with respect to IC/π. Note that as the ratio increases, the high value (type of curve #3) of the range is closing towards the low value (type of curve #1) and the average is decreasing. From Figure 29.b we can clearly observe that there is shift towards curve #1 as the ratio increases.

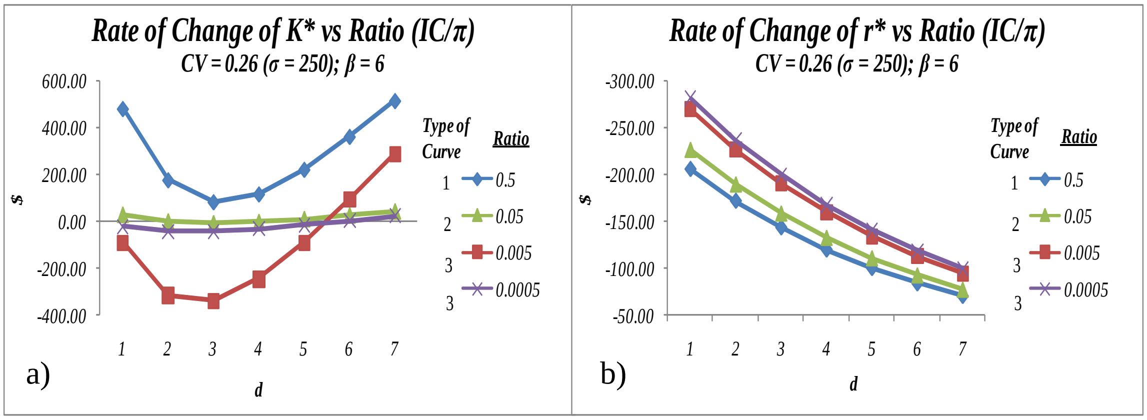


Figure 30: Relationship between r^* and the behavior of K^* with respect to the ratio (IC/π).

From the previous observations, we can conclude that at low levels of IC/π, K^* is more probable to behave as curve #3 but at high levels it is more probable to behave as curve #1. This is shown in Figure 30.a in terms of the rate of change of K^* as d increases due to scale reasons. Note that when $IC/\pi = 0.5$ the rate of change of K^* is positive for any d, meaning that K^* behaves as curve type #1; when $IC/\pi = 0.05$ the rate of change starts positive, then becomes negative and positive again, meaning that K^* behaves as curve type #2; and for $IC/\pi = 0.005$ and $IC/\pi = 0.0005$ the rate of change starts negative and then becomes positive, meaning that K^* follows a behavior similar to curve #3. Note that as the ratio decreases π becomes larger with respect to IC, and the rate of change of r^* with respect to d becomes more negative at any d as shown in Figure 30.b. This observation means that as d increases the reduction of r^* is faster as the ratio is lowered. This behavior is mostly explained by the change in inventory given by the relationship between the inventory being allocated and r^*. Note that for any particular d and everything else being equal, r_d^* is set at a higher level as the ratio IC/π

decreases as result of an increase in π. This follows from the relationship given by the right-hand side of (3.17). As d increases by Δd units of time, $\pi\lambda G(d)/IC$ is decreasing and the change given by $\delta = [G(d) - G(d + \Delta d)]\,\pi\lambda/IC$ is greater for bigger π's. Then, the rate of change of r^* with respect to d must be more negative at any d as the ratio IC/π decreases. Since r^* is decreasing as d increases, it follows that if the reduction in r^* is large enough to counteract the increase in inventory as result of the allocation and of any possible increase in Q^*, the total inventory will decrease driving a decrease in K^*. Otherwise, the inventory will increase driving an increase in K^*.

From Figure 31, it can be concluded that d^* and $\hat{d}$ are both influenced by IC/π. In the case of d^*, this result is expected since the type of curve is sensitive to the ratio, and d^* is related to the type of curve. While for $\hat{d}$, this result follows from (3.17). It is concluded from Figures 29 - 31 that the behavior of the cost function (K^*) is sensitive to the ratio given by IC/π.

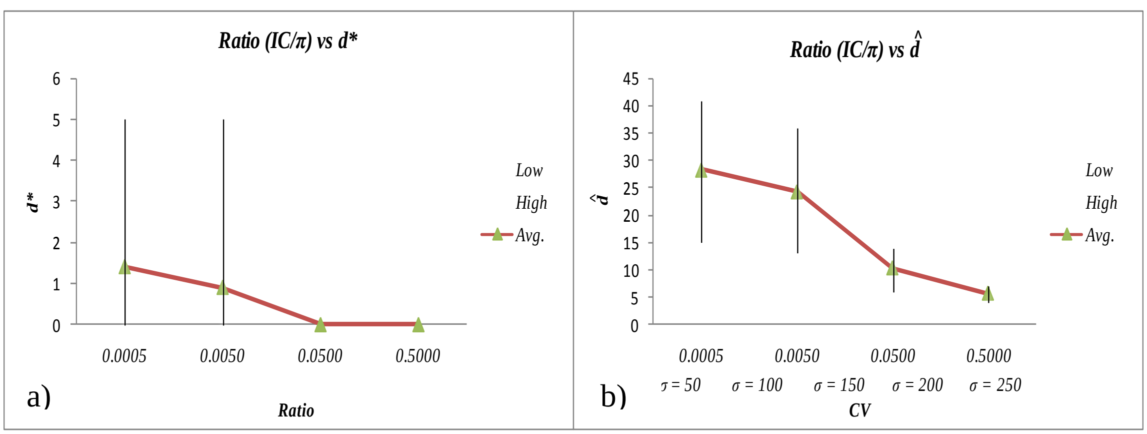


Figure 31: Ratio (IC/π) vs. d^* and $\hat{d}$.

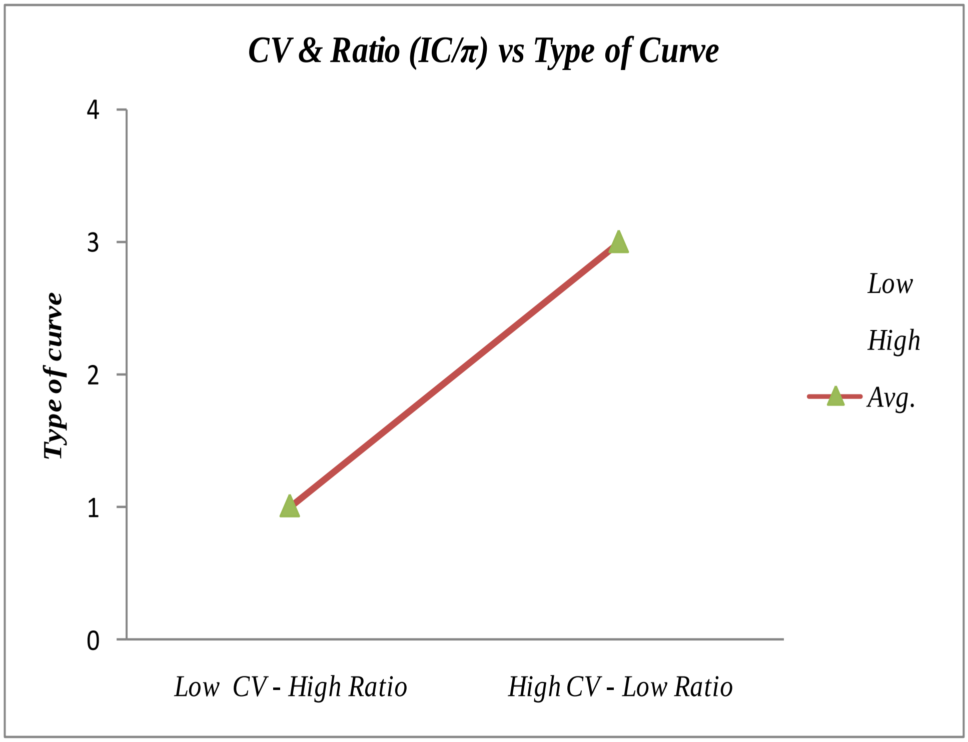


Figure 32: Joint effect of the coefficient of variation (CV) and the ratio (IC/π) on the behavior of K^*.

Figure 32 shows the joint effect of the CV and IC/π on the behavior of the optimal cost function (K^*). As it can be observed, curve type #3 should be expected when the CV is high and the ratio is low. For clarification purposes, 0.26 is the highest CV level experimented in this study due to the concern of not having negative demand. This so called "high" level could be considered low for many applications. In that sense, "high" or "low" is relative to the problem being addressed but irrespectively, the chances for preferring the modified (Q, r) model with the "no-early shipment" policy over the traditional model are increased as the CV increases and the ratio (IC/π) decreases.

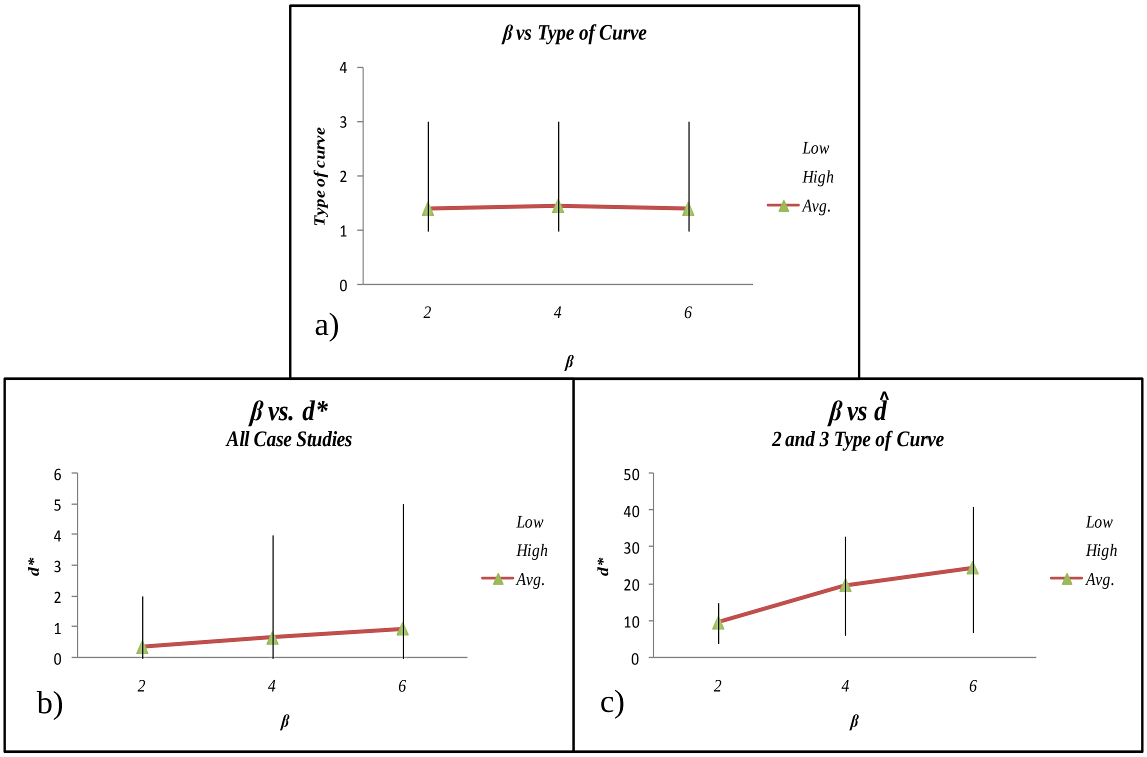


Figure 33: Influence of β.

Figure 33 shows the analysis with respect to β. It is observed from Figure 33.a that β has no influence on the curve type since the spread and the average values observed do not change as β increases. However, from 33.b and 33.c we can observe that d^* and $\hat{d}$ are both sensitive to β. This information can be interpreted as follows, the longer the expected lead-time (β), the farther away d^* is located from zero given that K^* behaves as curve #2 or #3. The relation of β with respect to $\hat{d}$ follows from (3.17). It seems that the higher the value of β, the longer it takes for (3.17) not to be satisfied.

Summarizing this section, it has been observed as explained in section 3.4 that the behavior of the modified (Q, r) model with the "no-early shipment" policy is dependent on the problem being addressed. In particular, the behavior of the optimal cost function (K^*) is sensitive to the CV and the ratio given by IC/π. As result, the location of d^* is also dependent on the particular problem being addressed. It was also shown that the highest the CV and the lowest the ratio, $d^* > 0$ and better the chances for preferring the modified model over the traditional.

3.5.2. Behavior of the Optimal Cost Function (K^*) with Respect to d When the Replenishment Order Is Delayed

It has been said that delaying the replenishment order is only desired for any $d > d^*$ if $K_d^* > K_{d^*}^*$. It is important to search for d^* since its location is dependent on the problem being addressed. The algorithm presented in Section 3.4.3 is used to implement the modified model under the delaying policy.

The behavior of the optimal cost function (K^*) is shown in Figure 34 for both scenarios, when no delay is done and when the replenishment order is delayed. Note from the "no delay" curve that $d = 0$ is a local minimizer for the range of $d's$ given by $[0,1]$, and $d^* = 5$ is the global minimizer. The algorithm is initialized with $d^* = 0$ as a local minimizer and delays the replenishment order for $d = 1$ by $m_1 = 1$ units of time and $K_{d=1}^* = K_{d^*=0}^*$. It does not delay for $d = 2$ since $K_{d=2}^* < K_{d^*=o}^*$. A search for a new minimizer is done as d increases. $d^* = 5$ is identified as a new minimizer and the algorithm delays the replenishment for any $d > 5$ by $m_d = d - 5$ units of time, and $K_d^* = K_{d^*=5}^*$. The algorithm continues this iterative process of finding a local minimizer and delaying until a particular d of interest or $\hat{d}$ – 1 is reached first. If $d \geq \hat{d}$, the algorithm delays the replenishment order for d by $m_d = d - d^*$, where d^* is the local minimizer found last.

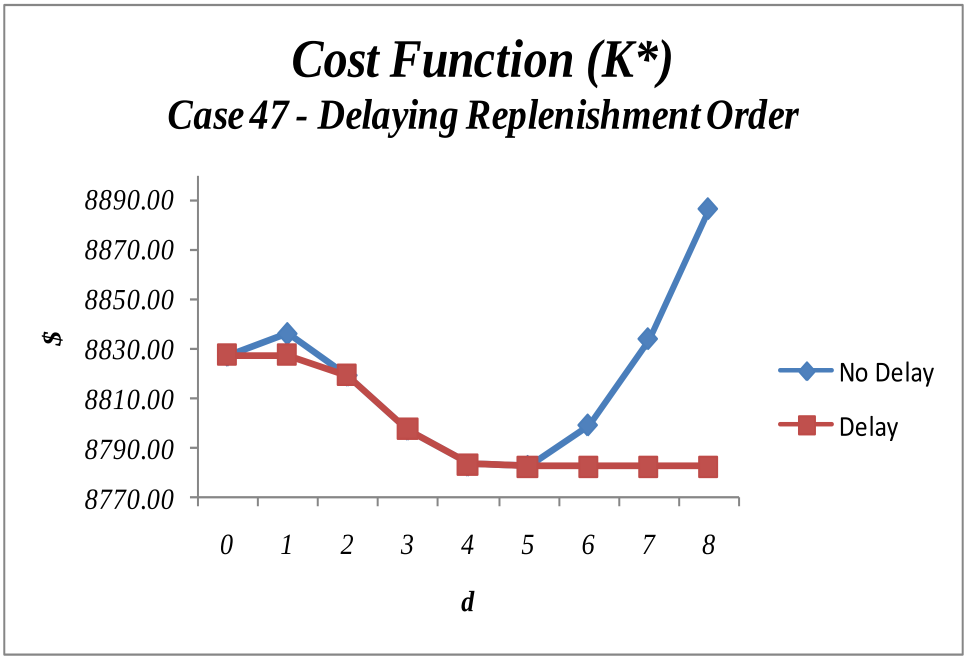


Figure 34: Delaying policy of the replenishment order.

The behavior of the traditional (Q, r) model as d increases is represented by a horizontal line from the value of $K^*_{d=0}$. Then, the delay policy for this case in particular guaranteed the better performance for any $d \geq 2$ since the behavior of K^* for the "no delay" scenario followed the curve type #2. It can be deduced from this observation that the modified (Q, r) model with the "no-early shipment" policy will perform better for any $d > 0$ whenever the behavior of K^* for the "no delay" scenario follows the curve type #3.

3.5.3. Behavior of the Optimal Cost Function (K^*) During the Time Period $[0, d_1)$ - Minimum Lead-Time (l_{min})

The main objective of this section is to show with a numerical example the behavior of the optimal cost function (K^*) for the scenario in which there is a minimum lead-time $l_{min} = d_1$. The explanation for the behavior of K^* for any $d \leq d_1$ was given in Section 3.4.4 and as previously mentioned, this scenario has practical implications because it could be related to the location of the supplier.

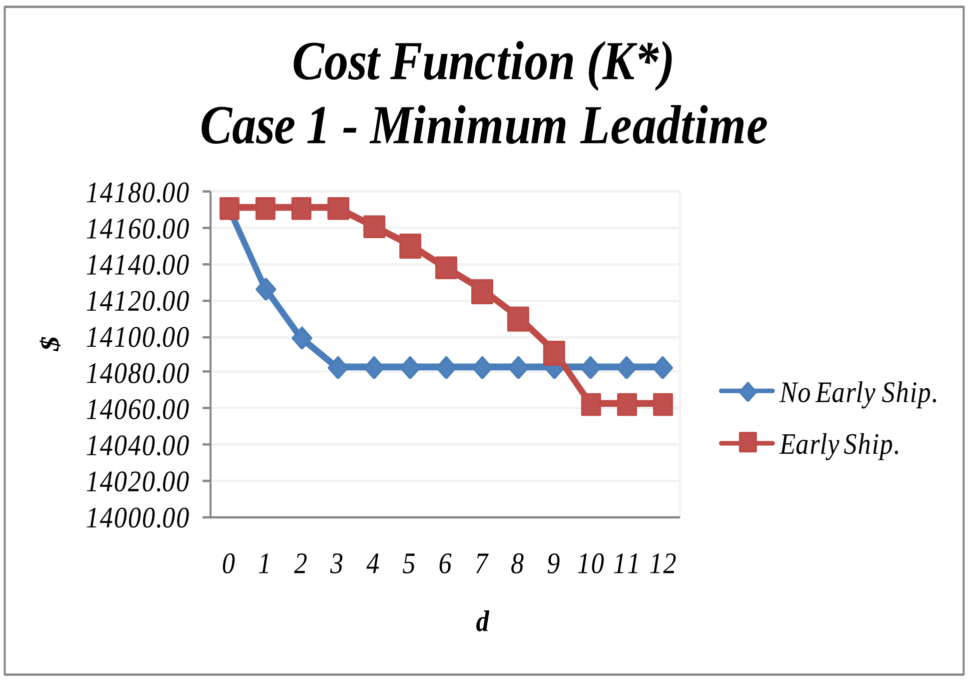


Figure 35: Case when there is a minimum lead-time (l_{min}).

Figure 35 shows the behavior of K^* as d increases under the assumption that $d_1 = 3$ units of time. The following was assumed with respect to the lead-time distribution function:

$$g(l) = \begin{cases} 0, & 0 \le l < d_1; \\ Expo\ (\beta), & d_1 \le l. \end{cases} \tag{3.45}$$

Note that K^* is a decreasing function of d for any $d \le d_1$. Observe that the behavior is constant for any $d > 3$. For this case in particular, $d^* = 3$ and the replenishment order is delayed for any subsequent d. Since the behavior of the traditional (Q, r) model is represented by a horizontal line from the value of $K^*_{d=0}$, it must be concluded that the modified (Q, r) model with the "no-early shipment" policy performs better than the traditional model for any $d > 0$ whenever $l_{min} = d_1 > 0$.

The behavior of the optimal cost function for the modified (Q, r) model with the "early shipment" policy is constant with respect to d for any $d \le d_1$ as explained in section 2.4.2. Then, the modified model with the "no-early shipment" policy must perform better than the "early shipment" policy for some range of $d's$ as shown in Figure 35.

A final observation about this scenario is that for any $d \le d_1$there is no excess inventory (e_d) since $g(d) = 0$. Then, there is no need for delaying the replenishment order for any d in that range.

From the previous observations we must conclude that the manufacturer will prefer the modified (Q, r) model with the "no-early shipment" policy over the traditional and the modified model with the "early shipment" policy for any $d \le d_1$. In terms of practical applications, for example, this scenario could be related to an overseas supplier and a manufacturer that promises a delivery time with a safety time that is less than the minimum lead-time from the supplier.

3.5.4. Merits of the "No-Early Shipment" Model vs. the "Early Shipment" Model

The main objective of this section is to show how the joint relationship between the CV and the ratio (IC/π) can benefit the "no-early shipment" policy over the "early shipment" policy. It was shown in the previous section that the modified (Q,r) model with the "no-early shipment" policy is the preferred model for any $d \leq l_{min} = d_1$. For the scenario in which $d_1 = 0$ (e.g.: a local supplier that holds inventory of the component being supplied) one might tend to expect a better performance from the less restrictive model ("early shipment" policy). However, this is not the case for the particular example discussed in this section.

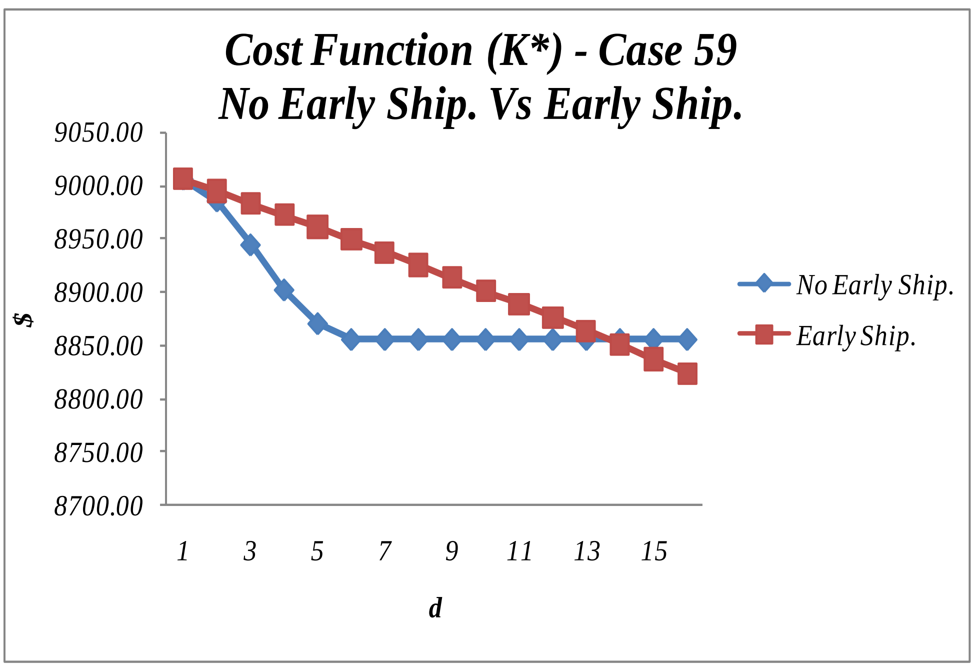


Figure 36: Comparison between the "no-early shipment" and the "early shipment" models for the case when the CV is high and the ratio (IC/π) is low.

Figure 36 shows the comparison between the "no-early shipment" model and the "early shipment", both for the scenario in which the CV is set at the high level and the ratio (IC/π) at the low level. Note that the "no-early shipment" model performed better for any d in the range $(0, 12]$. This result was driven by the levels at which the CV and the ratio(IC/π) were set, and the fact that as d increases r^* decreases faster in the "no-early shipment" model. As result, the initial dropped in inventory was faster for the "no-early shipment" model guaranteeing the better performance for the time period mentioned before.

The relationship between the behavior of K^* and the CV and the ratio (IC/π) was already explained in section 3.5.1. Let us focus then on the claim that r^* decreases faster in the "no-early shipment" model. Note that for the "early shipment" model the reduction of r^* as d increases is driven only by $G(d)$; the demand of interest is the lead-time demand, $X_{(0,l]}$, which is not related to d and as such, the demand variability is constant with respect to d. Meanwhile, for the "no-early" shipment model the reduction of r^* as d increases is driven by $G(d)$ and the demand of interest, $X_{(d,l]}$, which is the demand during the time period $(d, l]$; note that the variability of the observed demand decreases as d increases. Then, r^* must decrease faster in the "no-early shipment" model.

This finding is important because it shows the merit of the "no-early shipment" model not only for the case of a supplier located far away, but also for the case of a local supplier.

3.5.5. Behavior of r^*, Q^*, and their Relationship with the Penalty Cost

Let us focus our attention on the relationship between r^*, Q^*, and the penalty cost. Note from Table 5 that as in Lemma 3.4.5.1, r^* is strictly decreasing as d increases. Note that as r^* decreases the expected number of backorders per cycle increases until $\hat{d} = 6$, point at which (3.23) is not satisfied. Note that the penalty cost as well as Q^* are decreasing as d increases which is in accordance to Lemma 3.4.5.2.

Table 5: Case 18 – Relationship between r^*, Q^*, and the penalty cost.

d	*Q**	*r**	*Penalty $*	*Back Orders/Cy*	*Penalty Orders/Cy*	*Penalty Orders/Yr*	*Service %*	*$\pi\lambda G_d H/IC$*
0	3193.67	1063.0	500.27	6.47	6.47	25.01	99.7975	12350.00000
1	3184.85	815.5	413.38	6.84	5.33	20.67	99.8326	9618.18967
2	3177.86	624.7	344.27	7.30	4.43	17.21	99.8606	7490.65365
3	3172.35	477.4	289.67	7.88	3.72	14.48	99.8827	5833.72693
4	3168.06	363.8	247.19	8.62	3.17	12.36	99.8999	4543.31110
5	3164.85	276.0	215.32	9.63	2.76	10.77	99.9128	3538.33424

The results presented in Table 5 are only related to Case Study #18. However, the same relationships were observed for all case studies. It was never observed an increase in the penalty cost and Q^*. This finding could be related to the lead-time distribution function $g(l)$ used in our numerical examples. In that respect, the behavior of the service level offered to the customer as result of the implementation of this model is related to the problem being addressed. However, any increase or decrease in service performance is expected to be minimal as result of the assumption that the average number of backorders is negligible with respect to the average inventory at any time.

3.6. Summary

This chapter discusses the "no-early shipment" model. A detailed description of the model is given, and its behavior is discussed mathematically and shown by means of numerical examples.

The basic assumption for this model is that the customer penalizes the manufacturer for early shipments. In turn, the manufacturer ships only at the delivery date and allocates inventory as soon as the customer order is received. If inventory is not available, the penalty is incurred if the component becomes available more than d units of time after receiving the customer order. The main difference with the traditional (Q, r) inventory control policy and the "early shipment" model is given not only by the penalty cost structure but the inventory cost as well. With respect to the penalty cost, the demand of interest is the demand given during the time period $(d, l]$ rather than the lead-time demand. For the inventory, the demand during the time period $(d, l]$ and $(l, d]$ are both of interest. In addition to the safety stock needed when $l \geq d$, there is a natural excess inventory that occurs each time that $l < d$.

The behavior of the cost function (K^*) is dependent on the problem being addressed. The coefficient of variation (CV) and the ratio (IC/π) play a major role in the behavior of the model. As the safety time (d) increases, the relationship between the decrease in r^* and the increase in excess inventory drives the change of the inventory holding cost which in turn defines the behavior of K^*.

There is a $d^* \geq 0$ at which the cost $K^*_{d^*}$ is the minimum cost with respect to d. There is also a $\hat{d}$, such that for any $d \geq \hat{d} > d^*$ the convexity of the model is not guaranteed if the replenishment order is not delayed. The same optimal cost $K_{d^*}^*$ can be obtained for any $d \geq d^*$ if the replenishment order is delayed by $m = d - d^*$ units of time.

The model performs as the traditional (Q, r) model or better for any $d \geq 0$. For the special case when there is a minimum lead-time $l_{min} > 0$, the optimal cost function (K^*) is a decreasing function of d for any $d \leq l_{min}$, and the model performs better than the traditional (Q, r) inventory control policy for any $d > 0$ and better than the "early shipment" model for any $d \leq l_{min}$.

The next chapter discusses the "hybrid" model which is a generalization of the previous two models and the traditional policy.

CHAPTER IV

HYBRID MODEL

4.1. Introduction

The models discussed in the previous two chapters address two scenarios that could be considered as the extreme points of the shipping spectrum. In the first model, early shipments are allowed following from the assumption that the customer does not penalize for receiving early orders. In the second case, the manufacturer can only ship right at the delivery date following from the assumption that the customer penalizes for early shipments.

One could think of situations in which the customer does not monitor early shipments as part of their inventory control procedures or they can always convert the final product into sales as soon as received; another situation will be the case in which the customer is aware of the negative impact that early shipments have on their inventory and as result, an aggressive procedure for preventing early shipments is in placed and followed. Nevertheless, there could be situations in which the customer has some means of control in order to prevent early shipments from their suppliers but not necessarily to the extent that the final product must only ship at the delivery date. In that sense, there is a window in which early shipment is not penalized in some sort of direct or indirect way.

The explanation given above is the main reason for developing the model presented in this chapter, which is a hybrid and a generalization of all cases previously discussed. The traditional (Q, r) model, the modified (Q, r) model for the "early shipment" policy and the "no early shipment" policy are special cases of this hybrid model. Since the previous two sections already covered the understanding of the behavior of each policy with respect to d, our focus in this section is towards developing the policy rather than mathematically explaining the behavior and running numerical examples.

4.2. Description of the Model

The same assumptions as in Hadley and Whitin (1963) are being followed and are enumerated as follows:

1) The unit cost C of the item is a constant independent of Q;
2) There is never more than a single order outstanding;
3) The cost of operating the information processing system is independent of Q and r;
4) The reorder point r is positive.

The only major change is that π has been assumed to be the penalty cost incurred when the customer order for the finished product is overdue or late based on its request/delivery date since backorders/stock-outs at the inventory items are allowed at time of booking the customer order. Despite the fact that backorders are allowed, Hadley's and Whitin's assumption that the average number of backorders is negligible as compared to the average inventory at any time is being followed. Hence, the on-hand inventory is equal to the net inventory in order to calculate the cost structure related to the inventory cost.

The following notation is used in order to define the cost function (K):

- $X_{(t,t+\Delta]}$is the demand in any time interval $(t, t+\Delta]$;
- $f(x;\Delta)$ is the probability that the number of units demanded in a time interval of length $\Delta = (t+\Delta) - t$ lies between x and $x + dx$;
- $g(l)dl$ is the probability that the lead-time for the replenishment/procurement order lies between l and $l + dl$;
- d_{min} is the earliest time by which the customer order can be shipped without incurring any penalty for early shipment;
- d_{max} is the latest time at which the customer order can be shipped without incurring any penalty for late shipment;
- $h_1(x)$ is the marginal distribution for the demand during the time period $(d_{min}, l]$,

$$h_1(x) = \int_{l=d_{min}}^{\infty} f(x; l-d)g(l)dl\,; \tag{4.1}$$

- $\mu_{(d_{min},l]}$ is the expected demand during the time period $(d_{min}, l]$,

$$\mu_{(d_{min},l]} = \int_{x=0}^{\infty} xh_1(x)dx\,; \tag{4.2}$$

- $h_2(x)$ is the marginal distribution for the demand during the time period $(l, d_{min}]$,

$$h_2(x) = \int_{l=0}^{d_{min}} f(x; d-l)g(l)dl\,; \tag{4.3}$$

- $\mu_{(l,d_{min}]}$ is the expected demand during the time period $(l, d_{min}]$,

$$\mu_{(l,d_{min}]} = \int_{x=0}^{\infty} x h_2(x) dx\,; \tag{4.4}$$

- $0 < Q < \infty$ is the procurement/replenishment order size;
- $m_{d_{min}} \leq r < \infty$ is the reorder level that triggers the procurement order by means of the inventory position, where $m_{d_{min}}$ is a non-negative number such that $h_1(x)$ is non-increasing for any $x \geq m_{d_{min}}$;
- λ is the average annual demand which is constant over time;
- A is the cost of placing an order with the supplier;
- IC is the average cost of carrying inventory per unit per unit time;
- π is the penalty cost per unit incurred when the requested customer date is missed;
- $G(d_{min})$ is the complementary cumulative distribution of $g(d_{min})$, or $P(L \geq d_{min})$,

$$G(d_{min}) = \int_{l=d_{min}}^{\infty} g(l) dl\,; \tag{4.5}$$

- $G(d_{max})$ is the complementary cumulative distribution of $g(d_{max})$, or $P(L \geq d_{max})$,

$$G(d_{max}) = \int_{l=d_{max}}^{\infty} g(l) dl\,; \tag{4.6}$$

- $\bar{\eta}_{d_{max}}(r)$ is the expected number of units incurring the penalty cost per cycle.

In order to calculate the expected annual inventory cost, let us assume that a customer order that triggers the decision of placing a replenishment order for the outsourced component is entered at time $t = 0$. This means that inventory of the component needed is allocated to the customer order and released to the shop floor at time d_{min}; and the replenishment order is received at time l if there is no delay in reordering. Since the lead-time L is random, then there are instances in which $l \geq d_{min}$ and $l < d_{min}$.

For the instance in which $l \geq d_{min}$, note that customer orders entered before time $t = 0$ are required to be released during the time interval $(0, d_{min}]$ and have inventory already allocated to them. In that sense, the net inventory at time of receiving the replenishment order is $r - X_{(d_{min},l]}$ since the inventory at time $t = 0$ is the reorder level r plus the inventory already allocated to customer orders $\left(X_{(0,d_{min}]}\right)$.

For the instance in which $l < d_{min}$, note that as in the previous case there is inventory allocated for all customer orders that will be released to the shop floor during the time interval $(0, d_{min}]$. Because the lead-time l is less than the time period d_{min}, the net inventory at the time of arrival of the replenishment order is $r + X_{(l,d_{min}]}$.

The net inventory at time of arrival of the replenishment order can be summarized as per (4.7).

$$\xi_{d_{min}}(x,r) = \begin{cases} r - X_{(d_{min},l]}, & l \geq d_{min};\ 0 \leq X_{(d_{min},l]} < \infty \\ r + X_{(l,d_{min}]}, & l < d_{min};\ 0 \leq X_{(l,d_{min}]} < \infty \end{cases} \tag{4.7}$$

The expected net inventory at the time of arrival of a replenishment order can be calculated using (4.8).

$$E\left[\xi_{d_{min}}(x,r)\right] = \left[\int_{x=0}^{\infty}\int_{l=d_{min}}^{\infty}(r-x)f(x;l-d_{min})g(l)dldx\right]P(L \geq d_{min})$$

$$+\left[\int_{x=0}^{\infty}\int_{l=0}^{d_{min}}(r+x)f(x;d_{min}-l)g(l)dldx\right]P(L < d_{min})$$

$$E\left[\xi_{d_{min}}(x,r)\right] = \left[r-\mu_{(d_{min},l]}\right]G(d_{min}) + \left[r+\mu_{(l,d_{min}]}\right]\left(1-G(d_{min})\right) \quad (4.8)$$

$$= s_{d_{min}}G(d_{min}) + e_{d_{min}}\left(1-G(d_{min})\right)$$

Note that $s_{d_{min}}$ is the safety stock needed to protect us against the variability of the demand when $l \geq d_{min}$, and $e_{d_{min}}$ is the excess inventory that occurs naturally when $l < d_{min}$ since the replenishment order arrives before the material is really needed on the shop floor. The expected annual inventory carrying cost is defined by (4.9).

$$IC\left[\frac{Q}{2} + s_{d_{min}} + e_{d_{min}}\right]$$

$$= IC\left[\frac{Q}{2} + r - \mu_{(d_{min},l]}G(d) + \mu_{(l,d_{min}]}\left(1-G(d_{min})\right)\right] \quad (4.9)$$

In order to define the expected annual penalty cost, note that the penalty is incurred only when $l \geq d_{max}$ and the demand $X_{(d_{min},l]}$ is greater than the reorder point r. The number of customer orders incurring penalty in a cycle can be summarized as per (4.10).

$$\eta_{d_{max}}(x,r) = \begin{cases} 0, & l < d_{max} \\ 0, & l \geq d_{max}; x-r < 0 \\ x-r, & l \geq d_{max}; x-r \geq 0 \end{cases} \quad (4.10)$$

The expected number of customer orders incurring penalty per cycle can be calculated as follows:

$$\bar{\eta}_{d_{max}}(r) = \left[\int_{x=r}^{\infty}\int_{l=d_{min}}^{\infty}(x-r)f(x;l-d_{min})g(l)dldx\right]P(L \geq d_{max})$$

$$\bar{\eta}_{d_{max}}(r) = \left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(x)\right]G(d_{max}), \tag{4.11}$$

where $H_1(x)$ is the complementary cumulative of $h_1(x)$. The expected annual penalty cost is defined as per (4.12).

$$\frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(x)\right]G(d) \tag{4.12}$$

The expected annual cost K for the hybrid model is calculated by (4.12).

$$K = \frac{\lambda}{Q}A + IC\left[\frac{Q}{2} + r - \mu_{(d_{min},l]}G(d_{min}) + \mu_{(l,d_{min}]}\left(1 - G(d_{min})\right)\right]$$

$$+\frac{\pi\lambda}{Q}\left[\int_{x=r}^{\infty} xh_1(x)dx - r\,H_1(x)\right]G(d_{max}) \tag{4.13}$$

As with the previous models, the assumption is that d_{min} and d_{max} are known parameters. Note that the models previously discussed are special cases of this model:

- $d_{min} = d_{max} = 0$, the model converts into the traditional (Q, r) model.
- $d_{min} = 0$ and $d_{max} > 0$, the model converts into the modified (Q, r) model with the "early shipment" policy. (Chapter II).
- $d_{min} = d_{max} > 0$, the model converts into the modified (Q, r) model with the "early shipment" policy. (Chapter III).

4.3. Summary

This chapter presents a hybrid policy which is a generalization of the models discussed in the previous two Chapters. The traditional (Q, r) model, and the modified

(Q, r) model for the "early shipment" policy and the "no-early shipment" policy are special cases of this model. The next chapter provides a summary and discusses the conclusions and future research.

CHAPTER V

SUMMARY, CONCLUSIONS, AND FUTURE RESEARCH

5.1. Summary

The models discussed in this research are generalizations of the (Q, r) inventory control policy for the case in which the penalty cost is not incurred at time when the customer order is placed if inventory is unavailable. The traditional policy is the special case of this models when $d = 0$.

Without loss of generalization, the focus has been on the ATO scenario under two different shipping assumptions: 1) early shipments are allowed by the customer, and 2) early shipments are not allowed. Different from the prevailing literature related to inventory and supply chain management under ATO scenarios, we do not focus on the trade-off between inventory and delivery time under service level constraints. Rather, we focus on understanding the behavior of the total cost (ordering, inventory and penalty) as the delivery time increases (as result of an increase in the safety time) constrained by the penalty cost. This approach provides additional information in situations in which management is interested in modeling the penalty and then determining the appropriate service level as an output rather than setting it as an input parameter.

To do this, we derived the optimal modified (Q, r) policy following Hadley and Whitin's heuristic approach and modeling the safety time as the period of time by which backorders/stock-outs are allowed without incurring penalty. Numerical examples are run to show the behavior of the models.

5.2. Conclusions

This section presents the conclusions related to the modified (Q, r) models analyzed in this research. The merits of both models have been analytically and numerically shown. The conditions under which the models perform better than the traditional (Q, r) model have been identified. Conclusions are drawn from the perspective of the manufacturer and the customer, and the implications to their business relationship.

With respect to the modified (Q, r) model with the “early shipment” policy, it has been shown that the manufacturer will tend to prefer this model over the traditional (Q, r) model for cases in which $d > l_{min}$ and $g(d)$ is not negligible (e.g.: a local supplier that holds inventory of the component being supplied). He/she will be tempted to increase the delivery time by increasing the safety time (d) as this action will decrease the total cost of the system. However, we have seen that for all case studies considered in this research the expected number of orders incurring penalty increased as d was increased, meaning that the reduction of the total cost of the system comes with a trade-off in service performance. Nevertheless, this reduction was minimal and could be attributed to the fact that the model is valid for scenarios in which the expected backorders/stock-outs at any time are negligible with respect to the expected inventory.

Early shipments represent an increase of the customer inventory if he/she cannot convert into sales the products received ahead of time as soon as arrived. Under this scenario only, we can expect the customer to push for a reduction in the delivery time which will force the manufacturer to reduce the safety time (d). This action is a contradiction of the action that the manufacturer will try to pursue, and in a competitive market scenario the short-term benefit for the manufacturer could be detrimental to the long-term relationship with the customer. This conclusion opens the door for the modified (Q, r) model with the "no-early shipment" policy.

With respect to the modified (Q, r) model with the "no-early shipment" policy, it has been shown that the manufacturer will tend to prefer this model over the traditional (Q, r) model for any case in which $l_{min} > 0$ (e.g.: supplier located away from the manufacturer such that there is a time period for when receiving a replenishment order is impossible). This scenario guarantees a $d^* > 0$ and the delaying policy for the replenishment order will be implemented for any $d > d^*$. The manufacturer will be tempted to increase the delivery time by $d^* - d$ units of time whenever $d^* > d$.

Even though the service performance is related to the problem being addressed, any change in service should be minimal. Moreover, for all case studies considered in this research the expected number of orders incurring penalty was a non-increasing function of d. The customer should not expect an increase in inventory since the orders are not delivered ahead of time. These observations imply that the modified model with the "no-early shipment" policy has no major negative implications and should not jeopardize the long-term business relationship with the customer.

5.3. Future Research

It has been shown that for the modified (Q, r) model with the "early shipment" policy the optimal cost function (K^*) is a non-increasing function of d, and the manufacturer will pursue the increase of the safety time (d). The customer can look into contract negotiation if he/she is concerned with early shipments and is knowledgeable about the implied benefits at the manufacturer for increasing the delivery date. This is an area of future research, and the assumption would be that early shipments and an increase in the delivery date are allowed by the customer if the monetary benefit for the manufacturer is shared to cover for any negative monetary implication at the customer.

There is immediate research opportunity that could lead towards modeling a more realistic and complex scenario under both shipping perspectives: shipping early is allowed and shipping early is not allowed.

1) An intrinsic assumption of this research is that the component is unique to the product being manufactured and it is only used in one production stage. The scenario in which the component is common to several product lines and/or used in more than one production stage is an interesting generalization of the models described in this proposal since there will be multiple constant safety times;
2) Modeling a random safety time in order to understand the real benefit of the models since a customer order with a later order entry date can be requested earlier than another with an earlier order entry date.

It has been said that under an ATO scenario the safety time is added to the delivery time as a protection against uncertainty. The safety time has been assumed known and the problem was defined as finding the optimal (Q, r) policy that minimizes the cost of the system (ordering, inventory and penalty). Throughout the course of this research we have gotten to realize that there is an optimal safety time (d^*) that minimizes the optimal cost function (K^*). It can be said that the optimal safety time is a function of the inventory control model being assumed. In that respect, it would be interested to expand this research to other inventory control models with the intention of identifying the optimal safety time (d^*) that minimizes the cost of the system.

Finally, we can conclude that it is not trivial to always prefer the "early shipment" policy over the "non-early shipment" policy based on the results from section 3.5.4. This conclusion opens the door for performing an extensive numerical study with the hybrid model in order to determine the optimal early shipping date when there is no requirement or concern related to early shipment.

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APPENDIX A

"NO-EARLY SHIPMENT" MODEL – RESULTS OF CASE STUDIES

This appendix shows in form of a table the following results for each case study: type of curve, value of d^*, and value of $\widehat{\boldsymbol{d}}$.

Case	$\boldsymbol{\beta}$	$\boldsymbol{CV = \sigma/\mu}$	$\boldsymbol{IC/\pi}$	*Curve Type*	$\boldsymbol{d^*}$	$\widehat{\boldsymbol{d}}$
1	2	0.05	0.05	1	0	6
2	2	0.05	0.5	1	0	4
3	2	0.05	0.0005	1	0	15
4	2	0.05	0.005	1	0	13
5	4	0.05	0.05	1	0	11
6	4	0.05	0.5	1	0	6
7	4	0.05	0.0005	1	0	29
8	4	0.05	0.005	1	0	24
9	6	0.05	0.05	1	0	14
10	6	0.05	0.5	1	0	7
11	6	0.05	0.0005	1	0	41
12	6	0.05	0.005	1	0	36
13	2	0.11	0.05	1	0	6
14	2	0.11	0.5	1	0	4
15	2	0.11	0.0005	1	0	15
16	2	0.11	0.005	1	0	13
17	4	0.11	0.05	1	0	11
18	4	0.11	0.5	1	0	6
19	4	0.11	0.0005	1	0	29
20	4	0.11	0.005	1	0	24
21	6	0.11	0.05	1	0	14
22	6	0.11	0.5	1	0	7
23	6	0.11	0.0005	1	0	41
24	6	0.11	0.005	1	0	36
25	2	0.16	0.05	1	0	6

Case	β	$CV = \sigma/\mu$	IC/π	*Curve Type*	d^*	$\hat{d}$
26	2	0.16	0.5	1	0	4
27	2	0.16	0.0005	1	0	15
28	2	0.16	0.005	1	0	13
29	4	0.16	0.05	1	0	11
30	4	0.16	0.5	1	0	6
31	4	0.16	0.0005	2	0	29
32	4	0.16	0.005	1	0	24
33	6	0.16	0.05	1	0	14
34	6	0.16	0.5	1	0	7
35	6	0.16	0.0005	2	0	41
36	6	0.16	0.005	1	0	36
37	2	0.21	0.05	1	0	6
38	2	0.21	0.5	1	0	4
39	2	0.21	0.0005	3	2	15
40	2	0.21	0.005	3	1	13
41	4	0.21	0.05	1	0	11
42	4	0.21	0.5	1	0	6
43	4	0.21	0.0005	3	3	29
44	4	0.21	0.005	2	3	24
45	6	0.21	0.05	1	0	14
46	6	0.21	0.5	1	0	7
47	6	0.21	0.0005	2	5	41
48	6	0.21	0.005	2	4	36
49	2	0.26	0.05	1	0	6
50	2	0.26	0.5	1	0	4
51	2	0.26	0.0005	3	2	15
52	2	0.26	0.005	3	2	13
53	4	0.26	0.05	2	0	11
54	4	0.26	0.5	1	0	6
55	4	0.26	0.0005	3	4	29
56	4	0.26	0.005	3	3	24
57	6	0.26	0.05	2	0	14
58	6	0.26	0.5	1	0	7
59	6	0.26	0.0005	3	5	41
60	6	0.26	0.005	3	5	36

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