SOME RESULTS IN THE HYPERINVARIANT SUBSPACE PROBLEM AND FREE PROBABILITY

A Dissertation

by

GABRIEL H. TUCCI SCUADRONI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2009

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Kenneth Dykema
Committee Members,	Ronald Douglas
	Scott Miller
	Roger Smith
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ABSTRACT

Some Results in the Hyperinvariant Subspace Problem and Free Probability. (May 2009)

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This dissertation consists of three more or less independent projects. In the first project, we find the microstates free entropy dimension of a large class of $L^\infty[0,1]$–circular operators, in the presence of a generator of the diagonal subalgebra.

In the second one, for each sequence $\{c_n\}_n$ in $l_1(\mathbb{N})$, we define an operator A in the hyperfinite II_1-factor $\mathcal{R}$. We prove that these operators are quasinilpotent and they generate the whole hyperfinite II_1-factor. We show that they have non-trivial, closed, invariant subspaces affiliated to the von Neumann algebra, and we provide enough evidence to suggest that these operators are interesting for the hyperinvariant subspace problem. We also present some of their properties. In particular, we show that the real and imaginary part of A are equally distributed, and we find a combinatorial formula as well as an analytical way to compute their moments. We present a combinatorial way of computing the moments of A^*A.

Finally, let $\{T_k\}_{k=1}^\infty$ be a family of $*$–free identically distributed operators in a finite von Neumann algebra. In this paper, we prove a multiplicative version of the Free Central Limit Theorem. More precisely, let $B_n = T_1^*T_2^*\ldots T_n^*T_n\ldots T_2T_1$ then B_n is a positive operator and $B_n^{1/2n}$ converges in distribution to an operator Λ. We completely determine the probability distribution ν of Λ from the distribution μ of $|T|^2$. This gives us a natural map $\mathcal{G} : \mathcal{M}_+ \to \mathcal{M}_+$ with $\mu \mapsto \mathcal{G}(\mu) = \nu$. We study how this map behaves with respect to additive and multiplicative free convolution.

As an interesting consequence of our results, we illustrate the relation between the probability distribution ν and the distribution of the Lyapunov exponents for the sequence $\{T_k\}_{k=1}^{\infty}$ introduced by Vladismir Kargin.

To María Valentina Vega Veglio

ACKNOWLEDGMENTS

There are a number of people whom I would like to thank for having contributed to this dissertation in some way or another.

First, I would like to thank Dr. Kenneth Dykema for having provided me with interesting problems to work on and for letting me benefit from his mathematical knowledge and intuition. I am proud to be one of his students.

I thank the other members of my advisory committee, Dr. Ronald Douglas, Dr. Scott Miller and Dr. Roger Smith, for their time, comments on my research, and editorial advice. My special thanks to Dr. Roger Smith for helpful conversations, sharing of knowledge and advice since the beginning of my graduate studies.

Next, I would like to thank the many other excellent professors I have had in the math department at Texas A&M, specially Gilles Pisier, Thomas Schlumprecht, Ronald Douglas, Harold Boas, Bill Johnson and many others. I would also like to thank many great professors and friends I have had at the Universidad de la República in Uruguay, specially Beatriz Abadie and Fernando Abadie that influenced me in my first steps and played an important role in my decision to do a Ph.D.

Being a Ph.D. student took me to various places around the world. I am grateful for the invitations I received and for all the new friends I made. My stay at the Fields Institute in 2007 was a particularly great experience.

A special thanks goes to my mother, father, brother, sister and friends for always believing in me, even when it was difficult. And finally, I must thank my wife Valentina Vega for all her unconditional support!

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CHAPTER I

INTRODUCTION AND BACKGROUND

In August 2003 I was accepted as a Ph.D. student by the Department of Mathematics at Texas A&M University. The outcome of my studies is a total of three papers intended for publication, two of which have been published at the time of writing. Some of them are closely related, some are not. I have chosen to simply include each paper as a (self-contained) chapter of my dissertation. Each chapter contains a brief introduction to the subjects dealt with therein, and to the expert in the field that introduction may suffice. This first chapter, has the purpose of give a slightly more detailed introduction to the above mentioned subjects, is then meant as a service to the non-experts. People with a solid background in functional analysis should be able to follow the presentation and thereby also learn about some of the important applications of free probability and random matrices to operator algebras.

A. An Introduction to Free Probability

John von Neumann established the theory of so called *von Neumann algebras* in the 1930's. This theory was motivated by the spectral theorem of selfadjoint Hilbert space operators and by the needs of the mathematical foundation of quantum mechanics. A von Neumann algebra is an algebra of bounded linear operators acting on a Hilbert space which is closed with respect to the topology of pointwise convergence. If the von Neumann algebra has trivial center it is called a Factor. Factors are in a sense the building blocks of general von Neumann algebras. In a joint paper with F.J. Murray,

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The journal model is *IEEE Transactions on Automatic Control.*

a classification of the factors was given. Special attention was given to the type II_1 factors, which are continuous analogues of the finite dimensional matrix algebras. A type II_1 factor admits an abstract trace functional τ that takes values in $[0, 1]$ on projections. The hyperfinite II_1 was the first example of a type II_1 factor.

Countable discrete groups give rise to von Neumann algebras; in fact one can associate to a discrete group G a von Neumann algebra $\mathcal{L}(G)$ in a canonical way. On the Hilbert space $l_2(G)$ the group G has a natural unitary representation $g \mapsto L_g$,

$$(L_g\xi)(h) := \xi(g^{-1}h) \quad (\xi \in l_2(G), g, h \in G).$$

The group von Neumann algebra $\mathcal{L}(G)$ associated to G is by definition the closure of $\{L_g : g \in G\}$ in the topology of pointwise convergence. If the group under consideration is ICC (i.e. all its non–trivial conjugacy classes contain infinitely many elements), then the von Neumann algebra $\mathcal{L}(G)$ is a factor. Let $\delta_e \in l_2(G)$ stand for the characteristic function of $\{e\}$ and define

$$\tau(\cdot) := \langle \cdot\, \delta_e, \delta_e \rangle.$$

Then it is easy to check that τ is a trace, i.e. it satisfies $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{L}(G)$.

Let $\mathbb{F}_n$ denote the free group with n generators. von Neumann showed that type II_1 factors $\mathcal{L}(\mathbb{F}_n)$ are non–hyperfinite. It is still unknown if $\mathcal{L}(\mathbb{F}_n)$ and $\mathcal{L}(\mathbb{F}_m)$ are isomorphic or not for $n \neq m$. This question was the main motivation for D.V.Voiculescu to study the free relation and to develop free probability theory.

Free probability theory, as invented by D.V.Voiculescu in the 80s, is a highly noncommutative analogue of (classical) probability theory, the main purpose of which was to deal with von Neumann algebras of free groups. The present section will provide the reader with the basic definitions from free probability and with a sample of its most important achievements in operator algebras. The reader may also want to take a look at some of the references [47] and [53] for more information on the subject.

In free probability theory, the abelian von Neumann algebra $L^\infty(\Omega,\mu)$ associated with the probability space (Ω,μ), which is equipped with the state $f \mapsto \int f d\mu$, is replaced by a non-commutative probability space. A non-commutative probability space is a pair (A,φ) where A is a unital algebra and $\varphi : A \to \mathbb{C}$ is a linear map with $\varphi(1) = 1$. For (A,φ) to be a C^*–probability space, we require that A be a unital C^*–algebra and φ a state on A, and for (A,φ) to be a W^*–probability space, we furthermore require that A is a von Neumann algebra and φ a normal state on A.

Some frequently encountered W^*–probability spaces are the II_1-factors. A II_1-factor is a finite, infinite-dimensional von Neumann algebra with trivial center. Such a von Neumann algebra M has a unique faithful, normal, tracial state τ which gives rise to an inner product on M. We let $\|\cdot\|_2$ denote the corresponding norm on M. The Hilbert space completion of M with respect to this norm is denoted by $L^2(M,\tau)$, or simply $L^2(M)$.

Elements in the non–commutative probability space (A,φ) replace random variables $f \in L^\infty(\Omega,\mu)$ and are called non–commutative random variables. If $a \in A$ is such an element, the distribution of a is the linear map $\mu_a : \mathbb{C}[X] \to \mathbb{C}$ given by

$$\mu_a(X^k) = \varphi(a^k), \quad k \in \mathbb{N}$$

where $\mathbb{C}[X]$ denotes the set of complex polynomials in the indeterminate X.

If A is a C^*-algebra, φ a state on A, and $a = a^*$, then by the Riesz representation theorem, μ_a determines a probability measure on $\mathbb{R}$ which we will also denote by μ_a. That is, μ_a is the unique compactly supported Borel probability measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} t^k \, d\mu_a(t) = \varphi(a^k), \quad k \in \mathbb{N}.$$

The joint distribution of a family $(a_i)_{i\in I}$ of non-commutative random variables in (A, φ) is the linear map $\mu_{(a_i)} : \mathbb{C}\langle (X_i)_{i\in I}\rangle \to \mathbb{C}$ given by

$$\mu_{(a_i)}(X_{i_1} X_{i_2} \dots X_{i_n}) = \varphi(a_{i_1} a_{i_2} \dots a_{i_n}), \quad n \in \mathbb{N}, \quad i_1, i_2, \dots i_n \in I,$$

where $\mathbb{C}\langle (X_i)_{i\in I}\rangle$ denotes the set of polynomials in $|I|$ non-commuting indeterminates.

The notion of independence in classical probability theory is replaced by the notion of freeness in free probability theory. In order to motivate the definition given below, recall that random variables $f, g \in L^\infty(\Omega, \mu)$ are independent iff for all polynomials P and Q in one variable,

$$\int_\Omega P(f)Q(g) \, d\mu = \int_\Omega P(f) \, d\mu \int_\Omega Q(g) \, d\mu.$$

Equivalently, f and g are independent iff their joint distribution $\mu(f, g)$ on $\mathbb{R}^2$ is the tensor product $\mu_f \otimes \mu_g$ of the marginal distributions. Thus, the definition of independence can be recovered from the notion of a tensor product.

Definition A.1. Let (A, φ) be a non–commutative probability space.

1. Unital subalgebras of A, $(A_i)_{i\in I}$, are said to be freely independent or free if for all $n \geq 1$, for all $i_1, \dots, i_n \in I$ with $i_j \neq i_{j+1}$ and for all $a_j \in A_{i_j}$ with $\varphi(a_j) = 0$,

$$\varphi(a_1 a_2 \dots a_n) = 0.$$

2. Elements in A, $(a_i)_{i\in I}$ are said to be freely independent or free if the algebras $(\mathrm{Alg}(a_i, 1))_{i\in I}$ are free.

3. Elements $(a_i)_{i\in I}$ in the C^*–probability space (A, φ) are said to be $*$–free if the algebras $(\mathrm{Alg}(a_i, a_i^*, 1))_{i\in I}$ are free.

In analogy with the classical setting, if $(a_i)_{i\in I}$ in (A, φ) are free, then the joint distribution $\mu_{(a_i)_{i\in I}}$ is uniquely determined by the individual distributions μ_{a_i}.

Let us take a look at a free family of elements which plays a particularly important role in free probability: A semicircular system (or family) in a C^*–probability space is a family $(a_i)_{i\in I}$ of freely independent, self-adjoint elements from A, such that each variable a_i is distributed according to the $(0,1)$ semicircle law $d\sigma(t) = \frac{1}{2\pi}\sqrt{4-t^2}1_{[-2,2]}(t)\,dt$. That is,

$$\mu_{a_i}(X^k) = \frac{1}{2\pi}\int_{-2}^{2} t^k\sqrt{4-t^2}\,dt, \quad k \in \mathbb{N}.$$

The parameters 0 and 1 refer to the first and the second moment, respectively, of σ. In general, the semicircle law centered at a and of radius r is given by

$$d\sigma_{a,r}(t) = \frac{2}{\pi r^2}\sqrt{r^2-(t-a)^2}\,1_{[a-r,a+r]}(t)\,dt$$

Semicircular systems arise naturally as bounded operators on Fock space.

In the non–commutative setting, the semicircle law plays the role of the Gaussian distribution in classical probability. For instance, σ is the unique probability measure on $\mathbb{R}$, for which the R–transform (the free analogue of the logarithm of the Fourier transform) is the identity map on $\mathbb{C}$. For this reason, the semicircle law replaces the Gaussian distribution in the free central limit theorem [47].

Example A.2. Given a group Γ with neutral element e, consider the left regular (unitary) representation λ of Γ on $B(l^2(\Gamma))$ given by

$$\lambda(g)(\delta_h) = \delta_{gh}, \quad g, h \in \Gamma.$$

Let $L(\Gamma) := \lambda(\Gamma)'' \subset B(l^2(\Gamma))$ denote the group von Neumann algebra with tracial vector state $\tau(\cdot) = \langle \cdot\, \delta_e, \delta_e\rangle$. $L(\Gamma)$ is then finite, and in case $\Gamma \neq \{e\}$ is i.c.c., $L(\Gamma)$ is a factor and the tracial state τ is unique.

Consider now any family of groups, $(\Gamma_i, e_i)_{i\in I}$, and let $\Gamma = *_{i\in I}\Gamma_i$ denote their free product. Then $L(\Gamma)$ with tracial state $\tau(\cdot) = \langle \cdot\, \delta_e, \delta_e\rangle$ is isomorphic to $*_{i\in I}(L(\Gamma_i, \tau_i))$ where $\tau_i(\cdot) = \langle \cdot\, \delta_e, \delta_e\rangle$. For instance, let $\mathbb{F}_n = *_{i=1}^n\mathbb{Z}$ denote the free group on n generators. Then for the free group factor $L(\mathbb{F}_n)$ we have:

$$(L(\mathbb{F}_n), \tau) = *_{i=1}^n (L(\mathbb{Z}), \tau_i) \tag{1.1}$$

where $\tau_i(\cdot) = \langle \cdot\, \delta_1, \delta_1\rangle$ is the trace on $L(\mathbb{Z})$.

For $1 \le i \le n$, the unitary $\lambda(g_i)$, which corresponds to $\lambda(1)$ in the i'th copy of $L(\mathbb{Z})$ in $L(\mathbb{F}_n)$, has the Haar distribution on the unit circle $\mathbb{T}$ and it follows from (1.1) that $(\lambda(g_i))_{i=1}^n$ are $*$–free in $L(\mathbb{F}_n)$ and that

$$(L(\mathbb{F}_n), \tau) = *_{i=1}^n \Big(L^\infty(\mathbb{T}, \nu), \int_{\mathbb{T}} d\nu \Big)$$

where ν denotes Haar measure. Clearly, $L^\infty(\mathbb{T}, \nu) \cong L^\infty([-2, 2], \sigma)$, where $d\sigma(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\, 1_{[-2,2]}(t)\, dt$ is the (0,1) semicircle law. Hence,

$$L(\mathbb{F}_n) = *_{i=1}^n \Big(L^\infty([-2, 2], \sigma) \Big).$$

and $L(\mathbb{F}_n)$ is therefore generated by the semicircular system $(x_1, x_2, \ldots, x_n)$, where x_i in the i'th copy of $L^\infty([-2, 2], \sigma)$ is the identity map on $[-2, 2]$. Due to Voiculescu's results in [46], this links random matrices to the free group factors.

As we will see, free probability has already contributed significantly to our understanding of the free group factors. However, the question left open is whether $L(\mathbb{F}_n)$ is isomorphic to $L(\mathbb{F}_n)$ for $n \ne m$. It may or may not be that free probability will provide the solution to this famous problem.

B. Free Harmonic Analysis

The classical convolution of measures on the real line is strongly related to independence of random variables; namely, the convolution of distributions is the distribution of the sum of independent variables. Analogously, if x_1 and x_2 are freely independent elements in a non-commutative probability space, then the distributions of $x_1 + x_2$ and x_1x_2 are uniquely determined by the distributions μ_{x_1} and μ_{x_2}. However, it is rarely trivial to actually compute $\mu_{x_1+x_2}$ and $\mu_{x_1x_2}$. The R– and S–transforms, which we will define in this section, are tools from analytic function theory which may make the computations easier. Due to the connections between free probability and random matrices, these tools may also prove useful in determining the limit distributions of sums and products of independent random matrices as the matrix size tends to infinity.

Definition B.1. If x_1 and x_2 are freely independent elements in a non-commutative probability space with distributions μ_{x_1} and μ_{x_2}, respectively, the distribution of $x_1 + x_2$ is called the additive free convolution of μ_{x_1} and μ_{x_2} and is denoted by $\mu_{x_1} \boxplus \mu_{x_2}$. The distribution of x_1x_2 is called the multiplicative free convolution and is denoted by $\mu_{x_1} \boxtimes \mu_{x_2}$.

Note that $\boxplus$ and $\boxtimes$ may be viewed as binary operations on the set Σ of linear maps $\mu : \mathbb{C}[X] \to \mathbb{C}$ with $\mu(1) = 1$. For two such maps μ_1, μ_2 in Σ we will also denote by $\mu_1 \boxplus \mu_2$ and $\mu_1 \boxtimes \mu_2$ the distributions of the sum and the product of two freely independent elements with distributions μ_1 and μ_2, respectively.

Consider now a Borel probability measure μ on the real line. The Cauchy transform (or Stieltjes transform) of μ, $G_\mu : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}, \quad z \in \mathbb{C} \setminus \mathbb{R} \tag{1.2}$$

has an analytic inverse (with respect to composition) in a neighborhood of infinity. We denote this inverse, which is defined in a neighborhood of 0, by $G_\mu^{<-1>}$ and define the R–transform of μ, R_μ, by

$$R_\mu(z) = G_\mu^{<-1>}(z) - \frac{1}{z} \tag{1.3}$$

R_μ has a removable singularity at 0.

The following Theorem was proved by Voiculescu in [43].

Theorem B.2. *Let μ_1 and μ_2 be Borel probability measures on $\mathbb{R}$. Then the R–transform of $\mu_1 \boxplus \mu_2$ satisfies*

$$R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z)$$

in a neighborhood of 0.

It follows that if μ_1 and μ_2 are known and if we are able to invert the G_{μ_i}'s, we can easily find $R_{\mu_1 \boxplus \mu_2}$ and hopefully also $G_{\mu_1 \boxplus \mu_2}$. The inverse Stieltjes transform helps us find $\mu_1 \boxplus \mu_2$. If μ is a Borel probability measure on $\mathbb{R}$, then in the weak* topology on $\mathrm{Prob}(\mathbb{R})$,

$$d\mu(x) = \lim_{y \to 0^+} \Big(-\frac{1}{\pi} \mathrm{Im} G_\mu(x+iy)\, dx \Big)$$

It is then not hard to prove, using Theorem B.2, that if x_1 and x_2 are freely independent and both semicircular, then $x_1 + x_2$ is again semicircular.

The multiplicative analogue of the R–transform is the S–transform which we will now define. For an element $x \neq 0$ in a C^*–probability space define

$$\Psi_x(z) = \sum_{k=1}^{\infty} \varphi(x^k) z^k = \varphi([1 - zx]^{-1}), \quad |z| \leq \frac{1}{\|x\|}.$$

If x has first moment $\varphi(x) \neq 0$, then Ψ_x is invertible with respect to composition in a neighborhood of 0. We denote the inverse $\Psi_x^{<-1>}$ and define the S–transform of x

by

$$S_x(z) = \frac{1+z}{z}\Psi_x^{<-1>}(z).$$

S_x has a removable singularity at 0. Then next fundamental theorem was also proved by Voiculescu in [44].

Theorem B.3. *If x_1 and x_2 are freely independent elements in the C^*–probability space (A, φ) with $\varphi(x_i) \neq 0$ then $\varphi(x_1x_2) \neq 0$, and the S–transform of x_1x_2 satisfies*

$$S_{x_1x_2}(z) = S_{x_1}(z)S_{x_2}(z)$$

in a neighborhood of 0.

C. Free Entropy

The concept of entropy originated from thermodynamics and became a mathematical notion in the work of Gibbs and Boltzmann. Later it got importance in information theory and in the statistical problem of testing hypothesis. The entropy $-\int f(x)\log f(x)\,dx$ of a probability density f appears mostly in limit theorems. Even the *central limit theorem* is understandable in terms of entropy. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables of mean 0. Then the random variables $x_n = (\xi_1 + \xi_2 + \ldots + \xi_n)/\sqrt{n}$ have the same variance and their entropy is increasing (when n runs over the powers of 2). The limiting Gaussian variable has maximal entropy among the distributions of given variance.

In [48], Voiculescu invented the notions of free entropy and free entropy dimension. These are quantities which have given rise to some of the most significant applications of free probability to operator algebras.

The free entropy of an n-tuple $(x_1, \ldots, x_n)$ of self–adjoint elements in a tracial W^*–probability space $(\mathcal{M}, \tau)$ is the proper free analogue of Shannon's entropy of an n–tuple of real-valued random variables. Shannon's entropy, or the information entropy, as defined by Shannon in 1948, is a quantity supposed to describe how much randomness or uncertainty there is in a given probability distribution (a signal, in Shannon's terminology). Voiculescu aimed at defining free entropy in such a way that the appropriate translation of the properties of classical entropy would be satisfied by the free entropy.

In statistical thermodynamics, when computing the entropy of a system subject to a given set of macroscopic constraints (e.g. total energy, volume, temperature, pressure), one considers the set of microstates for the system consistent with those constraints. A microstate is one of a huge number of different accessible microscopic arrangements of a particular system (the macrostate) that the system visits in the course of its thermal fluctuations. It may be seen as one of a huge number of possible instantaneous photos of the system. The macrostate is then characterized by a probability distribution on its ensemble of microstates. This ensemble may in principle consist of a continuum of possible states, but let us assume that there are only finitely many, say N, of them and that the probability of finding the system in the i'th state is p_i . Then the entropy of the system is

$$S = -k_B \sum_{i=1}^{N} p_i \log p_i$$

where k_B is Boltzmann's constant.

In free probability, a microstate space for the n-tuple $(x_1, \ldots, x_n)$ of self–adjoints in $(\mathcal{M}, \tau)$ is the set of n–tuples of self–adjoint matrices which to some extend approximate the joint distribution of $x_1, \ldots, x_n$. More precisely, for $m \in \mathbb{N}$ and $\epsilon > 0$, let $\Gamma(x_1, \ldots, x_n\, ;\, m, k, \epsilon)$ denote the set of n-tuples $(A_1, \ldots, A_n) \in (M_k(\mathbb{C})_{sa})^n$ such that

$$|\tau(x_{i_1} \ldots x_{i_l}) - \mathrm{tr}_k(A_{i_1} \ldots A_{i_l})| < \epsilon$$

for all $l \in \{1, \ldots, m\}$ and for all $i_1, \ldots, i_l \in \{1, \ldots, n\}$. Here, $\mathrm{tr}_k = \frac{1}{k}\,\mathrm{Tr}_k$ denotes the normalized trace on $M_k(\mathbb{C})$ and m and ϵ specify the degree of approximation.

The euclidean structure on $(M_k(\mathbb{C})_{sa})^n$ equipped with the Hilbert–Schmidt scalar product

$$\langle A_1, \ldots, A_n, B_1, \ldots, B_n \rangle_2 = \sum_{i=1}^{n} \mathrm{Tr}_k(A_i B_i), \quad A_1, \ldots, A_n, B_1, \ldots, B_n \in M_k(\mathbb{C})_{sa}$$

gives rise to the notion of volumes of (measurable) subsets of $(M_k(\mathbb{C})_{sa})^n$. This volume is nothing but Lebesgue measure λ on $\mathbb{R}^{kn^2}$ when we identify $(M_k(\mathbb{C})_{sa})^n$ with $\mathbb{R}^{kn^2}$ (as vector spaces over $\mathbb{R}$). Now, let

$$\chi(x_1, \ldots, x_n\,;\, m, \epsilon) = \limsup_{k \to \infty} \left(\frac{1}{k^2} \log \lambda(\Gamma(x_1, \ldots, x_n\,;\, m, k, \epsilon)) + \frac{n}{2} \log k \right),$$

and increase the degree of approximation by putting

$$\chi(x_1, \ldots, x_n) = \inf_{m,\epsilon} \chi(x_1, \ldots, x_n\,;\, m, \epsilon)$$

This number, $\chi(x_1, \ldots, x_n) \in [-\infty, +\infty)$ (cf. [48]), is the free entropy of $(x_1, \ldots, x_n)$. In the following we will list some the properties of $\chi(\cdot)$, all of which were verified by Voiculescu. We refer to [53] for a comprehensive list.

1. *Upper bound:* $\chi(x_1, \ldots, x_n) \leq \frac{n}{2} \log \left(\frac{2\pi e C^2}{n} \right)$ where $C^2 = \tau(x_1^2 + \ldots + x_n^2)$.

2. *Semicontinuity:* Let $\left((x_1^{(p)}, \ldots, x_n^{(p)}) \right)_{p=1}^{\infty}$ and $(x_1, \ldots, x_n)$ be the n–tuples of self–adjoint elements in $(\mathcal{M}, \tau)$ such that

$$\sup_p \|x_i^{(p)}\| \leq \infty, \quad 1 \leq i \leq n,$$

and suppose $(x_1^{(p)}, \ldots, x_n^{(p)})$ converges in distribution to $(x_1, \ldots, x_n)$. Then

$$\limsup_p \chi(x_1^{(p)}, \ldots, x_n^{(p)}) \leq \chi(x_1, \ldots, x_n).$$

3. *Additivity and free independence:* If $\chi(x_i) > -\infty$, $1 \leq i \leq n$, then $x_1, \ldots, x_n$ are free if and only if
$$\chi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \chi(x_i).$$

4. *Semicircular maximum:* Among those $(x_1, \ldots, x_n) \in \mathcal{M}_{sa}^n$ with $\sum_{i=1}^n \tau(x_i^2) = n$ $\chi(x_1, \ldots, x_n)$ is maximized by n free $(0,1)$ semicircular elements only.

5. *A single variable:* For $x \in \mathcal{M}_{sa}$ with distribution μ_x in $\mathrm{Prob}(\mathbb{R})$,
$$\chi(x) = \int\int \log|s-t|\, d\mu_x(s)\, d\mu_x(t) + \frac{3}{4} + \frac{1}{2}\log(2\pi).$$

Of course, one may ask if for any n–tuple $(x_1, \ldots, x_n)$ there exist microstates corresponding to every degree of approximation (m, ϵ). The answer is 'yes' if and only if $W^*(x_1, \ldots, x_n)$ embeds into R^ω, the ultraproduct of the hyperfinite II_1–factor, i.e. iff $W^*(x_1, \ldots, x_n)$ has *Connes' embedding property.* So far there are no known examples of finite von Neumann algebras not having this property.

In addition to free entropy there is a notion of relative free entropy which we will need in the following. Suppose $x_1, \ldots, x_p, y_1, \ldots, y_q$ are self–adjoint elements in $(\mathcal{M}, \tau)$. Then the free entropy of $x_1, \ldots, x_p$ in the presence of $y_1, \ldots, y_q$, $\chi(x_1, \ldots, x_p : y_1, \ldots, y_q)$, is obtained by first considering the microstate spaces of $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ and then projecting onto the first p coordinates. This will give us a set $\Gamma(x_1, \ldots, x_p : y_1, \ldots, y_q\,;\, m, k, \epsilon)$. $\chi(x_1, \ldots, x_p : y_1, \ldots, y_q)$ is then obtained by replacing $\Gamma(x_1, \ldots, x_n\,;\, m, k, \epsilon)$ with $\Gamma(x_1, \ldots, x_p : y_1, \ldots, y_q\,;\, m, k, \epsilon)$ in the definition of $\chi(x_1, \ldots, x_n)$.

Given an n–tuple of self–adjoints operators $(x_1, \ldots, x_n)$, take a semicircular family $(s_1, \ldots, s_n)$ which is free from $W^*(x_1, \ldots, x_n)$. Then the *(modified) free entropy* *dimension* of $(x_1, \ldots, x_n)$ is given by

$$\delta_0(x_1, \ldots, x_n) = n + \limsup_{\epsilon \to 0} \frac{\chi(x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n \,:\, s_1, \ldots, s_n)}{|\log \epsilon|}.$$

D. Applications to Operator Algebras

Now, here are some of the aforementioned applications of free probability to operator algebras:

1. *Property non–*Γ. A II_1–factor $\mathcal{M}$ is said to have property Γ if there exists a non–trivial sequence of asymptotically central projections $\{P_m\}_{m=1}^{\infty}$ in $\mathcal{M}$, i.e. $\{P_m\}_{m=1}^{\infty}$ must satisfy

   $$\liminf_m \tau(P_m) > 0, \qquad \limsup_m \tau(P_m) < 1,$$

   and

   $$\forall x \in \mathcal{M} \;:\; \qquad \|\,[x, P_m]\,\|_2 \to 0.$$

   An example of a II_1–factor with property Γ is the hyperfinite II_1–factor R.

   In [50] Voiculescu showed that $n \geq 2$ self–adjoint elements $x_1, \ldots, x_n$ with non–degenerate free entropy ($\chi(x_1, \ldots, x_n) > -\infty$) generate a von Neumann algebra which is non–Γ.

2. *Absence of Cartan subalgebras.* A maximal abelian subalgebra (a MASA) A in a II_1–factor M is said to be Cartan if its normalizer,

   $$N(A) = \{u \in U(M) \,:\, uAu^* = A\},$$

   generates M, i.e. $N(A)'' = M$. For instance, when M is obtained as a crossed product, that is when $M = L^{\infty}(\Omega, \mu) \rtimes_{\alpha} \Gamma$, where μ is a probability measure on Ω, Γ is a discrete group, and α is a free, ergodic, measure–preserving action of Γ on (Ω, μ), then $L^\infty(\Omega, \mu)$ is a Cartan subalgebra of M. Many examples of II_1–factors are obtained in this way, and it was a longstanding open question whether the free group factors were in fact also crossed products. In the case of uncountably many generators this was answered in the negative by S. Popa. The case of finitely/countably many generators was taken care of by Voiculescu in [50]. He showed that if $n \geq 2$ self–adjoint variables $(x_1, \ldots, x_n)$ have non–degenerate free entropy, then $W^*(x_1, \ldots, x_n)$ is not generated by any normalizer of any of its diffuse *, hyperfinite W^*–subalgebras. A MASA in a II_1–factor is by maximality necessarily diffuse. Since $L(\mathbb{F}_n)$ is generated by a semicircular system of generators with free entropy $\frac{n}{2}\log(2\pi e)$, it follows that $L(\mathbb{F}_n)$ has no Cartan MASA for $2 \leq n < \infty$. This holds for $n = \infty$ as well (cf. [50], Theorem 5.3). In fact, Voiculescu could prove even stronger results when using δ_0 instead of χ (cf. [50], section 7).

3. *Primeness results.* A II_1–factor M is said to be prime if it can not be written as a tensor product $M_1 \otimes M_2$ of infinite–dimensional factors M_1 and M_2. S. Popa proved in [38] that there exist prime II_1–factors with non–separable predual. The separable case remained open until the mid 90s when L. Ge showed in [15] that $L(\mathbb{F}_n)$ is prime for $2 \leq n < \infty$. His reasoning was the same as in [48]: If M_1 and M_2 are II_1–factors and if $(x_1, \ldots, x_n)$ is a generating system of self–adjoints for $M = M_1 \otimes M_2$, then $\chi(x_1, \ldots, x_n) = -\infty$.

4. *Finite index subfactors of the interpolated free group factors.* It is still unknown what the possible finite index subfactors of the interpolated free group factors

*A von Neumann algebra is said to be diffuse if it has no non–zero minimal projections.

are (are they necessarily interpolated free group factors?) In [41], M. Stefan showed that such subfactors are at least prime: If M is a II_1–factor which is finitely generated as a von Neumann algebra, and if N is a finite index subfactor of M which is non–prime, then for any generating set $(x_1, \ldots, x_n)$ for M with $n \geq 3$, $\chi(x_1, \ldots, x_n) = -\infty$. This implies that finite index subfactors of $L(\mathbb{F}_n)$, $3 \leq n < \infty$, are prime. As a consequence of this, any finite index subfactor of $L(\mathbb{F}_r)$ is prime, $1 < r \leq \infty$ (cf. [41]). Later on, in [37], N. Ozawa strengthened this result by the use of C^*–algebra theory only. He showed that any non–injective subfactor of a hyperbolic group von Neumann algebra is prime.

5. *Further non–isomorphism results.* In [25], Jung introduced the notion of strong 1-boundedness: A finite von Neumann algebra M is said to be strongly 1-bounded if it has a set of generators $X = \{x_1, \ldots, x_n\}$, such that $\chi(x_1) > -\infty$ and for which

$$\limsup_{\epsilon} \chi(x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n \,:\, s_1, \ldots, s_n) + (n-1)|\log \epsilon| < \infty,$$

where $(s_1, \ldots, s_n)$ is a semicircular family free from X. The free entropy dimension δ_0 is an invariant for strongly 1-bounded von Neumann algebras, namely if M is strongly 1-bounded then $\delta_0(M) := \delta_0(X) \leq 1$ for any set of generators X. Therefore, the interpolated free group factors are not strongly 1-bounded since $L(\mathbb{F}_r)$ has a generating set of free entropy dimension r. Jung showed that von Neumann algebras with property Γ, those with Cartan subalgebras, and those which are prime, are strongly 1–bounded. He also showed that the following are not interpolated free group factors (because they are strongly 1–bounded):

- $M \rtimes_\alpha \Gamma$ where M is strongly 1-bounded and Γ is any group acting on M.
- $M_1 *_N M_2$ the amalgamated free product of strongly 1–bounded von Neu-

mann algebras M_1 and M_2 over a common diffuse subalgebra N.

Moreover, he generalized Stefan's result by showing that a finite index subfactor of $L(\mathbb{F}_r)$ is not strongly 1-bounded, hence not prime.

The results listed above show that free entropy has given rise to important results within the theory of II_1–factors. However, there are some fundamental questions that are still open. For instance:

- We may ask: If X and Y are free sets of selfadjoint elements in M, is

$$\chi(X, Y) = \chi(X) + \chi(Y)\,?$$

- If X and Y generate the same von Neumann algebra, is $\delta_0(X) = \delta_0(Y)$? That is, is $\delta_0(X)$ an invariant for $W^*(X)$?

It is not hard to see that if $(x_1, \ldots, x_n)$ is a semicircular family, then $\delta_0(x_1, \ldots, x_n) = n$. An affirmative answer to the invariance question for δ_0 above would therefore imply that $L(\mathbb{F}_n) \ncong L(\mathbb{F}_m)$ for $n \neq m$.

CHAPTER II

THE FREE ENTROPY DIMENSION OF SOME $L^\infty[0,1]$-CIRCULAR OPERATORS*

A. Introduction

Let $\mathcal{M}$ be a von Neumann algebra with a specified normal faithful tracial state τ. The free entropy dimension

$$\delta_0(X_1,\ldots,X_n) \tag{2.1}$$

for $X_1,\ldots,X_n \in \mathcal{M}$, was introduced by Voiculescu [49], [50], see also [53]. This quantity is sometimes called the microstates free entropy dimension to distinguish it from another version introduced by Voiculescu and because its definition utilizes matricial microstates for the operators $X_1,\ldots,X_n$. It is an open problem whether the quantity (2.1) is an invariant of the von Neumann algebra generated by $X_1,\ldots,X_n$, and it is of interest to find the free entropy dimension of various operators. See, for example [50], [52] [16], [22], [12], [24], [25], [26], [28] for some such results.

In [12], Dykema, Jung and Shlyakhtenko computed $\delta_0(T)=2$ for the quasinilpotent DT-operator T. This operator was introduced by Dykema and Haagerup in [11]. It can be realized as a limit in $*-$moments of strictly upper-triangular random matrices with i.i.d. complex Gaussian entries above the diagonal. Alternatively, as was seen in [11], T can be obtained in the free group factor $L(\mathbb{F}_2)$ from a semicircular element X and a free copy of $L^\infty([0,1])$ by using projections from the latter to cut out the upper triangular part of X. (Note that X may be replaced by a circular

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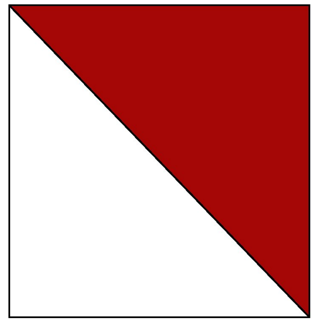

Fig. 1. The upper triangle, representing the quasinilpotent DT-operator T

element Z for this procedure.) Then we can visualize T as in Figure 1, where the shaded region has weight 1, the unshaded region has weight 0, and these weights are used to multiply entries of a Gaussian random matrix. It was proved in [10] that the von Neumann algebra generated by T contains all of $L^\infty([0,1])$, and is, thus, the free group factor $L(\mathbb{F}_2)$.

In this paper we consider more general operators than T, defined also as limits of random matrices or, equivalently, in the approach was taken in [9], by cutting a circular operator Z using projections in a $*$–free copy of $L^\infty([0,1])$. The class of operators considered there consisted of those $L^\infty([0,1])$-circular operators described as follows. Let η be an absolutely continuous measure with respect to Lebesgue measure on $[0,1]^2$ with Radon–Nikodym derivative $H \in L^1([0,1]^2)$ and assume the push–forward measures $\pi_{i*}\eta$ under the coordinate projections $\pi_1, \pi_2 : [0,1]^2 \to [0,1]$ are absolutely continuous with respect to Lebesgue measure and have essentially bounded Radon–Nikodym derivatives. For each such measure η with the associated function $H \in L^1([0,1]^2)$ we have the operator Z_H described in [9]; (however, this operator was denoted z_η in [9]). When η is Lebesgue measure on $[0,1]^2$, then $H = 1$ and Z_H is the usual circular operator. When η is the restriction of Lebesgue measure to the upper triangle pictured in Figure 1, then H is the characteristic function of this triangle and Z_H is the quasinilpotent DT-operator T.

Let $D \in L^\infty([0,1])$ be the identity map from $[0,1]$ to itself; thus, D generates $L^\infty([0,1])$. In this paper, with H as above, we compute the free entropy dimension $\delta_0(Z_H : D)$ of Z_H in the presence of D, in the case H satisfies certain additional hypothesis, showing that then

$$\delta_0(Z_H : D) = 1 + 2\,\mathrm{area}(\mathrm{supp}(H)), \tag{2.2}$$

where $\mathrm{supp}(H)$ is the measurable support of H and where the area is Lebesgue measure. We prove the upper bound $\leq$ in (2.2) for general H, (see Theorem C.2) using basic estimates inspired by [54]. We prove the lower bound $\geq$ in (2.2) for all H that are supported in the upper triangle as drawn in Figure 1 and whose restrictions to some band as drawn in Figure 2 are nonzero constant. (Actually, somewhat weaker conditions suffice — see Theorem C.1.) Our proof of the lower bound uses techniques similar to those used in [12].

The organization of the rest of this paper is as follows. In §B, we discuss some definitions and results that we need for the calculation. These include (§1) basic facts about the class of $L^\infty([0,1])$–circular operators that we consider, their construction in $L(\mathbb{F}_2)$ and a lemma about them; (§2) a result about certain matrix approximants to the quasinilpotent DT–operator which was lifted from [12] but that follows directly from work of Aagaard and Haagerup [1] and Śniady [40]; (§3) Jung's equivalent approach to free entropy dimension in terms of packing numbers [23]; (§4) Dyson's formula for the volumes of sets of matrices that are invariant under unitary conjugation. In §C, we prove the main result, namely the equation (2.2). Finally, in §D, we consider an example when $\delta_0(Z_H : D) < \delta_0(Z_H)$ and we ask a natural question.

Acknowledgment: The first named author thanks Kenley Jung for helpful comments.

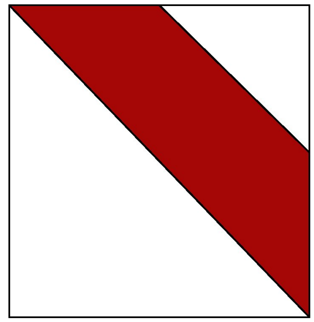

Fig. 2. A band above the diagonal

B. Definitions and Preliminaries

1. $L^\infty([0,1])$-circular Operators in Free Group Factors

In this section we recall how $L^\infty([0,1])$-circular operators in a certain class were constructed in [9], and we prove a lemma. We work in W*–noncommutative probability space $(\mathcal{M},\tau)$, with τ a faithful trace, and we fix a copy $\mathcal{A}=L^\infty[0,1]\subseteq\mathcal{M}$, such that the restriction of τ to $\mathcal{A}$ is given by integration with respect to Lebesgue measure on $[0,1]$. Let $D\in\mathcal{A}$ be the operator corresponding the function in $L^\infty[0,1]$ that is the identity map from $[0,1]$ to itself. Let $E:\mathcal{M}\to\mathcal{A}$ be the τ–preserving conditional expectation. Let $H\in L^1([0,1]^2)$, $H\geq 0$, and assume H has essentially bounded coordinate expectations $CE_1(H)$ and $CE_2(H)$, given by

$$CE_1(H)(x)=\int_0^1 H(x,y)dy,\qquad CE_2(H)(y)=\int_0^1 H(x,y)dx. \tag{2.3}$$

By Z_H, we will denote an $\mathcal{A}$–circular operator in $(\mathcal{M},E)$ with covariance (α_H,β_H) where α_H, $\beta_H:L^\infty[0,1]\to L^\infty[0,1]$ are given by

$$\alpha_H(f)(x)=\int_0^1 H(t,x)f(t)dt,\qquad \beta_H(f)(x)=\int_0^1 H(x,t)f(t)dt. \tag{2.4}$$

Suppose $Z \in \mathcal{M}$ is a $(0,1)$–circular element, namely a circular element satisfying $\tau(Z) = 0$ and $\tau(Z^*Z) = 1$, and suppose $\mathcal{A}$ and $\{Z\}$ are $*$–free. We will construct our operator Z_H from $\mathcal{A}$ and Z as in Theorem 6.5 of [9]. (Note that our notation differs slightly from that used in [9].)

Definition B.1. Let $\omega \in L^\infty([0,1]^2)$. We say that ω is in regular block form if ω is constant on all blocks in the regular $n \times n$ lattice superimposed on $[0,1]^2$, for some n, i.e. if there are $n \in \mathbb{N}$ and $\omega_{i,j} \in \mathbb{C}$, $(1 \le i,j \le n)$ such that $\omega(s,t) = \omega_{i,j}$ whenever $\frac{i-1}{n} \le s \le \frac{i}{n}$ and $\frac{j-1}{n} \le t \le \frac{j}{n}$, for all integers $1 \le i,j \le n$. (We then say ω is in $n \times n$ regular block form.) Then we set

$$M(\omega, Z) = \sum_{i,j=1}^{n} \omega_{i,j} p_i Z p_j$$

where $p_i = \mathbf{1}_{[\frac{i-1}{n}, \frac{i}{n}]} \in \mathcal{A}$. Note that we have $M(\omega, Z) \in W^*(\mathcal{A} \cup \{Z\}) \cong L(\mathbb{F}_3)$.

Recalling Lemma 6.4 and Theorem 6.5 of [9] we can state the following theorem.

Theorem B.2. *Let $\omega = \sqrt{H}$. Then there exists a sequence $\{\omega^{(n)}\}_n$ in $L^\infty([0,1]^2)$ such that*

(i) for each n, $\omega^{(n)}$ is in regular block form,

(ii) $\lim_n \|\omega - \omega^{(n)}\|_{L^2} = 0$

(iii) letting $H^{(n)} = (\omega^{(n)})^2$, both $\|CE_1(H^{(n)})\|_\infty$ and $\|CE_2(H^{(n)})\|_\infty$ remain bounded as n goes to ∞.

Moreover, there is an an $L^\infty[0,1]$–circular operator Z_H with covariance (α_H, β_H) as described in equations (2.4) *such that whenever $\{\omega^{(n)}\}_n$ is a sequence satisfying conditions (i)–(iii) above, the operators $M(\omega^{(n)}, Z)$ as given in Definition B.1 converge in the strong–operator–topology as $n \to \infty$ to Z_H.*

Remark B.3. Of particular interest is the operator Z_R when $R = 1_{\{(s,t)|s<t\}}$ is the characteristic function in the upper triangle in $[0,1]^2$. This Z_R is an instance of the DT$(\delta_0, 1)$-operator, also called the quasinilpotent DT–operator, and also denoted T. The construction of Z_R in Theorem B.2 above is approximately what was done in §4 of [11].

The following lemma will be used in §C to prove the upper bound on free entropy dimension. For emphasis, we will denote by $\lambda : L^\infty[0,1] \to \mathcal{M}$ the identification of $L^\infty[0,1]$ (with its trace given by Lebesgue measure) and $\mathcal{A} = \lambda(L^\infty[0,1]) \subseteq \mathcal{M}$.

Lemma 1. *Let $T = Z_R \in W^*(\{Z\} \cup \mathcal{A})$ be the quasinilpotent DT–operator as described in Remark B.3. Let N be an integer, $N \geq 2$. Assume for all $i, j \in \{1, \dots, N\}$ with $i \neq j$, $Y_{i,j} \in \mathcal{M}$ is a $(0,1)$–circular element such that the family*

$$\mathcal{A}, \quad \{Z\}, \quad (\{Y_{i,j}\})_{1 \leq i,j \leq N,\, i \neq j}$$

is $*$–free. Let $(e_{ij})_{1 \leq i,j \leq N}$ be a system of matrix units for $M_N(\mathbb{C})$. Consider the $*$–noncommutative probability space $(\mathcal{M} \otimes M_N(\mathbb{C}), \tau \otimes \mathrm{tr}_N)$, and let $\tilde{\lambda} : L^\infty[0,1] \to \mathcal{M} \otimes M_N(\mathbb{C})$ be the $*$–homomorphism given by

$$\tilde{\lambda}(f) = \sum_{j=1}^{N} \lambda(f \circ \rho_j) \otimes e_{jj},$$

where $\rho_j : [0,1] \to [0,1]$ is $\rho_j(t) = \frac{t}{N} + \frac{j-1}{N}$. Let $\widetilde{\mathcal{A}} = \tilde{\lambda}(L^\infty[0,1])$. Then the $\tau \otimes \mathrm{tr}_N$–preserving conditional expectation $\widetilde{E} : \mathcal{M} \otimes M_N(\mathbb{C}) \to \widetilde{\mathcal{A}}$ is given by

$$\widetilde{E}\Big(\sum_{1 \leq i,j \leq N} a_{ij} \otimes e_{ij}\Big) = \sum_{j=1}^{N} E(a_{jj}) \otimes e_{jj}.$$

Let $c_{ij} \in [0, \infty)$ $(1 \leq i, j \leq N,\ i \neq j)$ and let

$$\widetilde{Y} = \frac{1}{\sqrt{N}} \left(\sum_{k=1}^{N} T \otimes e_{kk} + \sum_{1 \leq i,j \leq N,\, i \neq j} c_{ij} Y_{ij} \otimes e_{ij} \right).$$

Then $\widetilde{Y}$ *is* $\widetilde{\mathcal{A}}$*–circular with covariance* (α_H, β_H) *as given in* (2.4)*, where*

$$H(s,t) = \begin{cases} 1, & \frac{k-1}{N} \le s \le t \le \frac{k}{N},\ 1 \le k \le N \\ (c_{ij})^2, & \frac{i-1}{N} \le s \le \frac{i}{N},\ \frac{j-1}{N} \le t \le \frac{j}{N},\ 1 \le i,j \le N,\ i \ne j. \end{cases}$$

Proof. Let

$$\widetilde{Z} = \frac{1}{\sqrt{N}}\bigg(\sum_{k=1}^{N} Z \otimes e_{kk} + \sum_{1 \le i,j \le N,\, i \ne j} Y_{ij} \otimes e_{ij}\bigg).$$

We will show that $\widetilde{Z}$ is $(0,1)$–circular and is $*$–free from $\widetilde{\mathcal{A}}$. Let $u_1, \ldots, u_N \in \mathcal{M}$ be Haar unitary elements such that the family

$$(\{u_k, u_k^*\})_{1 \le k \le N}, \quad \mathcal{A}, \quad \{Z\}, \quad (\{Y_{i,j}\})_{1 \le i,j \le N,\, i \ne j}$$

is $*$–free (after enlarging $(\mathcal{M}, \tau)$ if necessary). Let

$$U = \sum_{k=1}^{N} u_k \otimes e_{kk}.$$

It will suffice to show that $U^*\widetilde{Z}U$ is $(0,1)$–circular and is $*$–free from $U^*\widetilde{\mathcal{A}}U$. For this, by results following directly from Voiculescu's matrix model [46] (see [45]), it will suffice to show that each $u_k^* Z u_k$ and each $u_i^* Y_{ij} u_j$ is circular and that the family

$$(\{u_k^* Z u_k\})_{1 \le k \le N}, \quad (\{u_i^* Y_{ij} u_j\})_{1 \le i,j \le N,\, i \ne j}, \quad (u_k^* \mathcal{A} u_k)_{1 \le k \le N} \tag{2.5}$$

is $*$–free in $(\mathcal{M}, \tau)$. Let $Z = V|Z|$ and $Y_{ij} = V_{ij}|Y_{ij}|$ be the polar decompositions. Then (see [45]), V and V_{ij} are Haar unitaries, $|Z|$ and $|Y_{ij}|$ are quarter–circular elements, V and $|Z|$ are $*$–free and, for each i and j, V_{ij} and $|Y_{ij}|$ are $*$–free in $(\mathcal{M}, \tau)$. We have the polar decompositions

$$u_k^* Z u_k = (u_k^* V u_k)(u_k^* |Z| u_k)$$

$$u_i^* Y_{ij} u_j = (u_i^* V_{ij} u_j)(u_j^* |Y_{ij}| u_j).$$

Therefore, in order to show that $*$–freeness of the family (2.5) and circularity of $u_k^* Z u_k$ and $u_i^* Y_{ij} u_j$, it will suffice to show $*$–freeness of the family

$$(\{u_k^* |Z| u_k\})_{1\le k\le N}, \quad (\{u_k^* V u_k\})_{1\le k\le N},$$
$$(\{u_j^* |Y_{ij}| u_j\})_{1\le i,j\le N,\, i\ne j}, \quad (\{u_i^* V_{ij} u_j\})_{1\le i,j\le N,\, i\ne j}, \quad (u_k^* \mathcal{A} u_k)_{1\le k\le N}.$$

Let B be a Haar unitary generating $W^*(|Z|)$, let B_{ij} be a Haar unitary generating $W^*(|Y_{ij}|)$, and let C be a Haar unitary generating $\mathcal{A}$. It will suffice to show $*$–freeness of the family

$$(u_k^* B u_k)_{1\le k\le N}, \quad (u_k^* V u_k)_{1\le k\le N},$$
$$(u_j^* B_{ij} u_j)_{1\le i,j\le N,\, i\ne j}, \quad (u_i^* V_{ij} u_j)_{1\le i,j\le N,\, i\ne j}, \quad (u_k^* C u_k)_{1\le k\le N}$$

of Haar unitaries. This follows from the $*$–freeness of the family

$$B,\, C,\, V,\, (u_k)_{1\le k\le N}, \quad (B_{ij})_{1\le i,j\le N,\, i\ne j}, \quad (V_{ij})_{1\le i,j\le N,\, i\ne j}.$$

by an argument involving words in a free group. This shows that $\widetilde{Z}$ is $(0,1)$–circular and $*$–free from $\widetilde{\mathcal{A}}$.

Now we use the method of Theorem 6.5 of [9], described in Theorem B.2 above, but taking $\omega^{(n)}$ in $n\times n$ regular block form with n always a multiple of N, and with each such $\omega^{(n)}$ constant equal to c_{ij} on each off–diagonal block of the form $[\frac{i-1}{N},\frac{i}{N}]\times[\frac{j-1}{N},\frac{j}{N}]$ for $1\le i,j\le N$, $i\ne j$, where projections from $\widetilde{\mathcal{A}}$ are used to cut $\widetilde{Z}$ and make each $M(\omega^{(n)},\widetilde{Z})$. It is then clear that the operators $M(\omega^{(n)},\widetilde{Z})$ converge to $\widetilde{Y}$ as $n\to\infty$, and, from Theorem B.2, they also converge to an $\widetilde{\mathcal{A}}$–circular operator having the desired covariance (α_H,β_H). $\square$

2. Microstates for the Quasinilpotent DT-operator

Let $T = Z_R$ be the quasinilpotent DT–operator as described in Remark B.3 and let D be the corresponding operator described in §1. It was proved by Aagaard and Haagerup [1] that if we consider T a DT$(\delta_0, 1)$-operator and Y a circular operator that is $*$–free from T (and D), then the Brown measure of $T + \epsilon Y$ is equal to the uniform distribution on the closed disk centered at 0 and of radius $r_\epsilon = \log(1+\epsilon^{-2})^{-\frac{1}{2}}$. Note how slowly this disk shrinks as ϵ approaches to 0. Moreover, they also showed that the spectrum of $T + \epsilon Y$ is equal to the disk.

The next lemma is an immediate consequence of the above described Brown measure result of Aagaard and Haagerup and a result of Śniady [40]. A detailed proof can be formulated exactly as was done for Lemma 2.2 in [12]. In the following lemma and throughout this paper, for a matrix $A \in M_k(\mathbb{C})$ we let $|A|_2 = \mathrm{tr}_k(A^*A)^{1/2}$, where tr_k is the normalized trace of $M_k(\mathbb{C})$. Also, by the eigenvalue distribution of a matrix $A \in M_k(\mathbb{C})$ we mean the probability measure $\frac{1}{n}\sum_1^n \delta_{\lambda_j}$, where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of A listed according to general multiplicity.

Lemma 2. *Let $c > 0$. Then there exists sequences $\{g_k\}_k$ and $\{y_k\}_k$ such that for any $\epsilon > 0$, there exists a sequence $\{z_{k,\epsilon}\}_k$ such that*

- $g_k, y_k, z_{k,\epsilon} \in M_k(\mathbb{C})$,
- $\|g_k\|$, $\|y_k\|$ *and* $\|z_{k,\epsilon}\|$ *remain bounded as* $k \to +\infty$,
- $\limsup_k |y_k - z_{k,\epsilon}|_2 \le \epsilon c$,
- *the pair* (g_k, y_k) *converges in* $*$*–moments as* $k \to +\infty$ *to the pair* (D, T),
- *the eigenvalue distribution of* $z_{k,\epsilon}$ *converges weakly as* $k \to +\infty$ *to the measure* $\sigma_{\epsilon,c}$, *which is the uniformly distributed measure in the disk of center at 0 and radius* $r_{\epsilon,c} = c\log(1+\epsilon^{-2})^{-\frac{1}{2}}$ *in the complex plane.*

3. Packing Number Formulation of the Free Entropy Dimension

In this section we will review the packing number formulation of Voiculecu's microstates free entropy dimension due to K. Jung [23]. If $X = (x_1, \ldots, x_n)$ and $Z = (z_1, \ldots, z_m)$ are tuples of selfadjoint elements in a tracial von Neumann algebra, then the microstates free entropy dimension (as defined by Voiculescu [50]) is given by the formula

$$\delta_0(X) = n + \limsup_{\epsilon \to 0} \frac{\chi(x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n : s_1, \ldots, s_n)}{|\log \epsilon|}$$

and the microstates free entropy dimension in the presence of Z is defined by

$$\delta_0(X : Z) = n + \limsup_{\epsilon \to 0} \frac{\chi(x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n : z_1, \ldots, z_m, s_1, \ldots, s_n)}{|\log \epsilon|}$$

where $\{s_1, \ldots, s_n\}$ is a semicircular family free from X and Z. The packing formulation found in [23] is

$$\delta_0(X) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(X)}{|\log \epsilon|} \qquad \delta_0(X : Z) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(X : Z)}{|\log \epsilon|} \tag{2.6}$$

where

$$\mathbb{P}_\epsilon(X) = \inf_{m,\gamma} \limsup_k k^{-2} \log P_\epsilon(\Gamma(X; m, k, \gamma))$$

and

$$\mathbb{P}_\epsilon(X : Z) = \inf_{m,\gamma} \limsup_k k^{-2} \log P_\epsilon(\Gamma(X : Z; m, k, \gamma))$$

Here, $\Gamma(X : Z; m, k, \gamma) \subseteq (M_k(\mathbb{C})_{s.a.})^n$ is the microstates space of Voiculescu, and P_ϵ is the packing number with respect to the metric arising from the normalized trace. Let $Y = (y_1, \ldots, y_n)$ and $W = (w_1, \ldots, w_m)$ be arbitrary tuples of possibly non-selfadjoints elements in a tracial von Neumann algebra. Now the definition of $\mathbb{P}_\epsilon$ makes perfect sense for the set Y if we replace the microstates space in (2.6) with the non-selfadjoint $*-$microstates space $\Gamma(Y : W; m, k, \gamma) \subseteq (M_k(\mathbb{C}))^n$, which is the set of all $n-$tuples of $k \times k$ matrices whose $*-$moments up to order m approximate those of Y within tolerance of γ in the presence of W. It is also true that

$$\delta_0(\mathrm{Re}(y_1), \mathrm{Im}(y_1), \ldots, \mathrm{Re}(y_n), \mathrm{Im}(y_n) : W) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(Y : W)}{|\log \epsilon|}$$

see [12] for details.

Finally, we review the standard volume comparison inequality for packing numbers. Recall that for a metric space A we have

$$P_{4\epsilon}(A) \leq K_{2\epsilon}(A) \leq P_\epsilon(A),$$

where $P_\epsilon(A)$ is the ϵ–packing number, i.e. the maximal number of disjoint open balls of radius ϵ in A, and $K_\epsilon(A)$ is the minimal number of elements in a cover of A consisting of open balls of radius ϵ. If A is a subspace of a Euclidean space, then we have

$$\mathrm{vol}(\mathcal{N}_\epsilon(A)) \leq K_\epsilon(A) \cdot \mathrm{vol}(\mathcal{B}_{2\epsilon}),$$

where $\mathcal{N}_\epsilon(A)$ is the ϵ–neighborhood, $\mathcal{B}_r$ is a ball of radius r and vol is the volume, all in the ambient Euclidean space. We thus have the volume comparison test,

$$P_\epsilon(A) \geq K_{2\epsilon}(A) \geq \frac{\mathrm{vol}(\mathcal{N}_{2\epsilon}(A))}{\mathrm{vol}(\mathcal{B}_{4\epsilon})}. \tag{2.7}$$

4. Dyson's Formula

Every matrix of $M_k(\mathbb{C})$ has an upper-triangular matrix in its unitary orbit. Thus, letting $T_k(\mathbb{C})$ denote the set of upper-triangular matrices in $M_k(\mathbb{C})$, there is a probability measure ν_k on $T_k(\mathbb{C})$ such that

$$\lambda_k(\mathcal{O}) = \nu_k(\mathcal{O} \cap T_k)$$

for every $\mathcal{O} \subseteq M_k(\mathbb{C})$ that is invariant under unitary conjugation. Freeman Dyson identified such a measure [31], and showed that if we view $T_k(\mathbb{C})$ as a Euclidean space of real dimension $k(k+1)$ with coordinates corresponding to the real and imaginary part of the matrix entries lying on and above the diagonal, then ν_k is absolutely continuous with respect to Lebesgue measure on $T_k(\mathbb{C})$ and has density given at $A = (a_{ij})_{1\leq i,j\leq k} \in T_k(\mathbb{C})$ by

$$C_k \cdot \prod_{1\leq p<q\leq k} |a_{pp} - a_{qq}|^2 \quad \text{where} \quad C_k = \frac{\pi^{k(k+1)/2}}{\prod_{j=1}^{k} j!}. \tag{2.8}$$

We will use Dyson's formula in our main result to find lower bound on the volume of unitary orbits of an $\epsilon-$neighborhood of the microstates space.

C. Free Entropy Dimension Computations

Lemma 3. *Let (Ω, μ) a finite measurable space. Let $f \in L^1(\Omega)$ and $f \geq 0$. Then*

$$\lim_{\epsilon\to 0} \frac{\int_\Omega \log(\max(f(t), \epsilon))d\mu(t)}{|\log \epsilon|} = \mu(\mathrm{supp}(f)) - \mu(\Omega),$$

where $\mathrm{supp}(f) = f^{-1}((0, +\infty))$.

Proof. It is clear that we have $\log(\max(f(t), \epsilon)) \leq \log(f(t) + 1) + \log(\epsilon) \cdot \mathbf{1}_{f^{-1}([0,\epsilon))}$, and this yields

$$\limsup_{\epsilon\to 0} \frac{\int_\Omega \log(\max(f(t), \epsilon))d\mu(t)}{|\log \epsilon|} \leq -\liminf_{\epsilon\to 0} \mu(f^{-1}([0, \epsilon))) = \mu(\mathrm{supp}(f)) - \mu(\Omega).$$

On the other hand, given $\gamma > 0$, let $\delta > 0$ be such that $\mu(f^{-1}((0, \delta))) < \gamma$. Taking $0 < \epsilon < \delta$, we have $\mathbf{1}_{f^{-1}([0,\delta))} \cdot \log \epsilon + \mathbf{1}_{f^{-1}([\delta,+\infty))} \cdot \log \delta \leq \log \max(f(t), \epsilon)$ and integrating on both sides we obtain

$$\mu(f^{-1}([0, \delta))) \cdot \log \epsilon + \mu(f^{-1}([\delta, +\infty))) \cdot \log \delta \leq \int_\Omega \log(\max(f(t), \epsilon))d\mu(t).$$

Now dividing by $|\log\epsilon|$ and taking $\liminf$ on both sides we get

$$-\mu(f^{-1}([0,\delta))) \leq \liminf_{\epsilon\to 0} \frac{\int_\Omega \log(\max(f(t),\epsilon))d\mu(t)}{|\log\epsilon|}.$$

Using the fact that $\mu(f^{-1}([0,\delta))) < \mu(f^{-1}(0)) + \gamma$ and that γ is arbitrary we obtain

$$\mu(\mathrm{supp}(f)) - \mu(\Omega) \leq \liminf_{\epsilon\to 0} \frac{\int_\Omega \log(\max(f(t),\epsilon))d\mu(t)}{|\log\epsilon|},$$

proving the claim. □

As in §1, we work in $(\mathcal{M},\tau)$ and we have $\mathcal{A} = L^\infty[0,1]$ and a $(0,1)-$circular element Z such that $\mathcal{A}$ and Z are $*-$free, and with H as described there. We construct as in §1 an $L^\infty[0,1]$–circular operator $Z_H \in W^*(\mathcal{A}\cup\{Z\}) \cong L(\mathbb{F}_3)$. We also take $D = D^* \in \mathcal{A}$ to correspond to the identity function from $[0,1]$ to itself. The following is our main result.

Theorem C.1. *Let $H \geq 0$, $H \in L^1([0,1]^2)$ have essentially bounded coordinate expectations $CE_1(H)$ and $CE_2(H)$, as in equations* (2.3)*. Assume H has support contained in the upper-triangle U of $[0,1]^2$ and assume there exists $r \in \mathbb{N}$ such that*

$$\Delta := \bigcup_{i=1}^{r} U_i^{(r)} \subseteq \mathrm{supp(H)}, \qquad U_i^{(r)} = \{(x,y) : \frac{i-1}{r} \leq x < y \leq \frac{i}{r}\}$$

and that H restricted to Δ is constant equal to $c > 0$. Then

$$\delta_0(Z_H : D) \geq 1 + 2\cdot \mathrm{area}(\mathrm{supp}(H)).$$

In particular, $\delta_0(Z_H) \geq 1 + 2\cdot\mathrm{area}(\mathrm{supp}(H))$.

Proof. Without loss of generality we can assume $c = 1$. Fix $\epsilon > 0$. By hypothesis we may choose N arbitrarily large and so that $\bigcup_{i=1}^N U_i^{(N)} \subseteq \Delta$. Let $R > 1$, $m \in \mathbb{N}$ and $\gamma > 0$. There is $\delta > 0$ such that $\|Z_H - Y\|_2 < \delta$ implies $\Gamma_R(Y; m, k, \gamma/2) \subseteq$

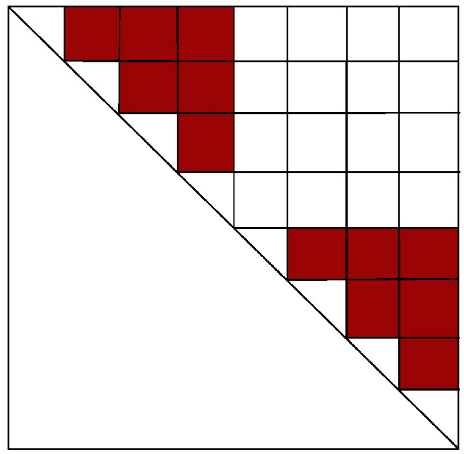

Fig. 3. Case $N = 2$ and $p = 4$

$\Gamma_R(Z_H; m, k, \gamma)$. Making use of Theorem B.2, there exist $M = Np$ and

$$\omega := \sum_{i=1}^{M} \mathbf{1}_{U_i^{(M)}} + \sum_{1\le i<j\le M} \alpha_{ij} \mathbf{1}_{E_{ij}^{(M)}}$$

where $E_{ij}^{(M)} = \{(x, y) : \frac{i-1}{M} \le x \le \frac{i}{M}, \frac{j-1}{M} \le y \le \frac{j}{M}\}$ with $\alpha_{ij} > 0$, such that $\|Z_H - Z_\omega\|_2 < \delta$ and, therefore, we have $\Gamma_R(Z_\omega; m, k, \gamma/2) \subseteq \Gamma_R(Z_H; m, k, \gamma)$. We define the sets of indices

$$\Theta = \{(i, j) : 1 \le i < j \le p,\ p + 1 \le i < j \le 2p, \ldots, (N-1)p + 1 \le i < j \le Np\}$$

and

$$\Phi = \{(i, j) : 1 \le i < j \le Np\} \setminus \Theta.$$

For example, in the case $N = 2$ and $p = 4$ the squares corresponding to elements of Θ are shaded in Figure 3.

Note that by the hypothesis of H we may insist, $\alpha_{ij} = 1$ whenever $(i, j) \in \Theta$. Let $\gamma' = \gamma/(MR)^{m-1}$.

Consider $(C_{11}, \ldots, C_{MM})$, $(C_{ij})_{1\le i<j\le M}$ a $*-$free family in $(\mathcal{M}, \tau)$, where each

C_{ii} is $\mathrm{DT}(\delta_0, \frac{1}{\sqrt{M}})$, and each C_{ij} with $i < j$ is circular with $\tau(|C_{ij}^2|) = \frac{1}{M}$. Let $\{g_k\}_k$ and $\{y_k\}_k$ the sequences constructed in Lemma 2 with $c = 1/\sqrt{M}$. There are $a_{ij}(k) \in M_k(\mathbb{C})$ for $(i,j) \in \Theta$ such that for each $(i,j) \in \Theta$ as before $a_{ij}(k)$ converge in distribution as $k \to +\infty$ to a $(0, \frac{1}{M})$-circular element and such that the family

$$\{g(k), y(k)\},\ (\{a_{ij}(k)\})_{(i,j)\in\Theta}$$

of sets of random variables is asymptotically $*$–free as $k \to \infty$. By an application of Corollary 2.14 of [51], for k large enough there exists a set $\Omega_k \subset \Gamma((C_{ij})_{(i,j)\in\Phi}; m, k, \gamma')$ such that for any $(\eta_{ij})_{(i,j)\in\Phi} \in \Omega_k$,

$$\{y_k, g(k)\},\ (a_{ij}(k))_{(i,j)\in\Theta},\ (\eta_{ij})_{(i,j)\in\Phi}$$

is an (m, γ')–$*$–free family of sets of random variables and

$$\liminf_k \left(k^{-2}. \log(\mathrm{vol}(\Omega_k)) + \left(\frac{N(N-1)p^2}{2}\right). \log(k) \right) \geq$$
$$\geq \chi((\mathrm{Re}\, C_{ij})_{(i,j)\in\Phi}, (\mathrm{Im}\, C_{ij})_{(i,j)\in\Phi}) > -\infty \quad (2.9)$$

where the volume is computed with respect to the Euclidean norm $k^{1/2}|\cdot|_2$. For each $(\eta_{ij})_{(i,j)\in\Phi} \in \Omega_k$ we define a matrix $R(k) \in M_{Mk}(\mathbb{C})$ by

$$R(k) = \begin{bmatrix} r_{11}(k) & r_{12}(k) & \dots & r_{1M}(k) \\ 0 & r_{22}(k) & \dots & r_{2M}(k) \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & r_{MM}(k) \end{bmatrix}, \qquad r_{ij}(k) = \begin{cases} y_k, & i = j \\ a_{ij}, & (i,j) \in \Theta \\ \alpha_{ij}\eta_{ij}, & (i,j) \in \Phi. \end{cases}$$

Let

$$G(k) = \mathrm{diag}(g(k), \tfrac{1}{M} + g(k), \dots, \tfrac{M-1}{M} + g(k)) \in M_{Mk}(\mathbb{C}).$$

As a consequence of Lemma 1,

$$(R(k), G(k)) \in \Gamma(Z_\omega, D; m, Mk, \gamma/2).$$

Set $\tilde{\alpha}_{ij} = \max(\alpha_{ij}, \epsilon)$ and let

$$\tilde{R}(k) = \begin{bmatrix} r_{11}(k) & r_{12}(k) & \dots & r_{1M}(k) \\ 0 & r_{22}(k) & \dots & r_{2M}(k) \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & r_{MM}(k) \end{bmatrix}, \qquad r_{ij}(k) = \begin{cases} y_k, & i = j \\ a_{ij}, & (i,j) \in \Theta \\ \tilde{\alpha}_{ij}\eta_{ij}, & (i,j) \in \Phi. \end{cases} \tag{2.10}$$

Then $\tilde{R}(k)$ lies in an ϵ-neighborhood of $\Gamma(Z_\omega : D; m, Mk, \gamma/2)$. Let $A_l(k) \in M_{kp}(\mathbb{C})$ for $l \in \{1, 2, \dots, N\}$ be defined by

$$A_l(k) = \begin{bmatrix} y_k & a_{f+1,f+2} & \dots & a_{f+1,f+p} \\ 0 & y_k & \dots & \vdots \\ \vdots & \ddots & \dots & a_{f+p-1,f+p} \\ 0 & \dots & 0 & y_k \end{bmatrix}$$

with $f = (l-1)p$. Note that we have

$$\tilde{R}(k) = \begin{bmatrix} A_1(k) & Y_{12}(k) & \dots & Y_{1N}(k) \\ 0 & A_2(k) & \dots & \vdots \\ \vdots & \dots & \ddots & Y_{N-1,N} \\ 0 & \dots & 0 & A_N(k) \end{bmatrix}, \tag{2.11}$$

where the $Y_{ij}(k) \in M_{pk}(\mathbb{C})$ are determined by equations (2.10) and (2.11). Then, by again making use of Lemma 1, we have $A_l(k) \in \Gamma_{p^2R}(\frac{1}{\sqrt{N}}T; m, pk, \gamma)$ for all $l \in \{1, 2, \dots, N\}$, where T is the the DT$(\delta_0, 1)$–operator. Let $\epsilon > 0$ and let $z_{k,\epsilon}$ be as in

Lemma 2. Let

$$B_{l,\epsilon}(k) = \begin{bmatrix} z_{k,\epsilon} & a_{f+1,f+2} & \dots & a_{f+1,f+p} \\ 0 & z_{k,\epsilon} & \dots & \vdots \\ \vdots & \ddots & \dots & a_{f+p-1,f+p} \\ 0 & \dots & 0 & z_{k,\epsilon} \end{bmatrix} \in M_{kp}(\mathbb{C}).$$

Note that the eigenvalue distribution of $B_{l,\epsilon}(k)$ converge weakly as $k \to +\infty$ to the measure $\sigma_{\epsilon,\frac{1}{\sqrt{N}}}$ of Lemma 2.

Since every complex matrix can be put into an upper-triangular form with respect to an orthonormal basis, we can find a $k \times k$ unitary matrix $v(k)$ such that $v(k)z_{k,\epsilon}v(k)^*$ is upper triangular. Since microstate spaces are invariant under conjugation by unitaries, also $(v(k) \otimes I_M)\tilde{R}(k)(v(k) \otimes I_M)^*$ lies in an ϵ-neighborhood of $\Gamma(Z_\omega : D; m, Mk, \gamma/2)$.

For each $1 \leq l \leq N$, we have

$$|(v(k) \otimes I_p)B_{l,\epsilon}(k)(v(k) \otimes I_p)^* - (v(k) \otimes I_p)A_l(k)(v(k) \otimes I_p)^*|_2 = |A_l(k) - B_{l,\epsilon}(k)|_2.$$

Since $\limsup_k |B_{l,\epsilon}(k) - A_l(k)|_2 \leq \frac{\epsilon}{\sqrt{N}}$, and taking $N > 4$, for k sufficiently large we have

$$|(v(k) \otimes I_p)B_{l,\epsilon}(k)(v(k) \otimes I_p)^* - (v(k) \otimes I_p)A_l(k)(v(k) \otimes I_p)^*|_2 \leq \epsilon/2.$$

Set $\widetilde{B}_l(k) = (v(k) \otimes I_p)B_{l,\epsilon}(k)(v(k) \otimes I_p)^*$ and $\widetilde{Y}_{ij}(k) = (v(k) \otimes I_p)Y_{ij}(k)(v(k) \otimes I_p)^*$ and denote by $\mathcal{G}_k$ the set of all $Mk \times Mk$ matrices of the form

$$\begin{bmatrix} \widetilde{B}_1(k) & \widetilde{Y}_{12}(k) & \dots & \widetilde{Y}_{1N}(k) \\ 0 & \widetilde{B}_2(k) & \ddots & \dots \\ \vdots & \ddots & \ddots & \widetilde{Y}_{N-1,N}(k) \\ 0 & \dots & 0 & \widetilde{B}_N(k) \end{bmatrix},$$

over all choices of $(\eta_{ij})_{(i,j)\in\Phi} \in \Omega_k$. Note that the matrices in $\mathcal{G}_k$ are upper triangular and their eigenvalue distributions are exactly the same as $z_{k,\epsilon}$. For k sufficiently large, the set $\mathcal{G}_k$ lies in a $2\epsilon-$neighborhood of $\Gamma(Z_\omega : D; m, Mk, \gamma/2)$ and, therefore, in a $2\epsilon-$neighborhood of $\Gamma(Z_H : D; m, Mk, \gamma)$. Let $\theta(\mathcal{G}_k)$ denote the unitary orbit of $\mathcal{G}_k$ in $M_{Mk}(\mathbb{C})$. We will now find lower bounds for the volumes of $\theta(\mathcal{G}_k)$ and thus, via the estimate (2.7), lower bounds for packing number of $\Gamma(Z_H : D; m, Mk, \gamma)$.

Denote by $\mathcal{H}_k \subset M_{Mk}(\mathbb{C})$ the set of all matrices of the form

$$\begin{bmatrix} 0 & \widetilde{Y}_{12}(k) & \dots & \widetilde{Y}_{1N}(k) \\ 0 & 0 & \ddots & \dots \\ \vdots & \ddots & \ddots & \widetilde{Y}_{N-1,N}(k) \\ 0 & \dots & 0 & 0 \end{bmatrix},$$

over all choices of $(\eta_{ij})_{(i,j)\in\Phi} \in \Omega_k$. Notice that $\mathcal{H}_k$ is isometric to the space of all matrices of the form $(w_{ij})_{1\le i,j\le M} \in M_{Mk}(\mathbb{C})$ with $w_{ij} \in M_k(\mathbb{C})$ and

$$w_{ij} = \begin{cases} 0, & (i,j) \notin \Phi \\ \tilde{\alpha_{ij}}\eta_{ij}, & (i,j) \in \Phi. \end{cases}$$

It follows that $\mathcal{H}_k$ must also have the same volume as the above subspace, computed in the ambient Hilbert space of block upper-triangular matrices with the indicated entries set to zero. Therefore,

$$\mathrm{vol}(\mathcal{H}_k) = \mathrm{vol}(\Omega_k) \cdot (M^{1/2})^{k^2 M(M-1)} \cdot \prod_{(i,j)\in\Phi} |\tilde{\alpha}_{ij}|^{2k^2}.$$

Let T_n the set of upper triangular matrices in $M_n(\mathbb{C})$; let $T_{n,<}$ denote the matrices in T_n that have zero diagonal, i.e. the strictly upper triangular matrices. Denote by $\mathcal{W}_k$ the set of $T_{Mk,<}$ consisting of all matrices x such that $|x|_2 < \epsilon$ and $x_{ij} = 0$ whenever $1 \le r < s \le N$ and $(r-1)pk < i \le rpk$, $(s-1)pk < j \le spk$. Thus, $\mathcal{W}_k$ consists of

$N \times N$ diagonal matrices whose diagonal entries are strictly upper triangular $pk \times pk$ matrices. Denote by $\mathcal{D}_k$ the subset of diagonal matrices x of $M_{Mk}(\mathbb{C})$ such that $|x|_2 < \epsilon$. It follows that if f_k is the matrix

$$f_k = \begin{bmatrix} \widetilde{B}_1(k) & 0 & \dots & 0 \\ 0 & \widetilde{B}_2(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \widetilde{B}_N(k) \end{bmatrix}$$

then $f_k + \mathcal{D}_k + \mathcal{W}_k + \mathcal{H}_k \subset \mathcal{N}_{3\epsilon}(\mathcal{G}_k)$, where the $3\epsilon-$neighborhood is taken in the ambient space T_{Mk} with respect to the metric induced by $|\cdot|_2$. Now observe that the space of diagonal $Mk \times Mk$ and $T_{Mk,<}$ are orthogonal subspaces. Let $\theta_{3\epsilon}(\mathcal{G}_k)$ denote the 3ϵ neighborhood of the unitary orbit of $\theta(\mathcal{G}_k)$ of $\mathcal{G}_k$. Let dX denote Lebesgue measure on T_{Mk} corresponding to the Euclidean norm $(Mk)^{1/2}|\cdot|_2$, which is coordinatized by the complex entries $X = \{x_{ij}\}_{1 \le i \le j \le Mk}$ of the matrix. Using Dyson's formula we have

$$\begin{aligned} \mathrm{vol}(\theta_{3\epsilon}(\mathcal{G}_k)) &\ge C_{Mk} \cdot \int_{f_k + \mathcal{D}_k + \mathcal{W}_k + \mathcal{H}_k} \prod_{1 \le i < j \le Mk} |x_{ii} - x_{jj}|^2 dX \\ &= C_{Mk} \cdot \mathrm{vol}(\mathcal{W}_k + \mathcal{H}_k) \cdot \int_{D(f_k + \mathcal{D}_k)} \prod_{1 \le i < j \le Mk} |x_{ii} - x_{jj}|^2 dx_{11} \cdots dx_{Mk,Mk} \\ &\ge C_{Mk} \cdot \mathrm{vol}(\mathcal{W}_k + \mathcal{H}_k) \cdot E_\epsilon(f_k) \end{aligned} \tag{2.12}$$

where the constant C_{Mk} is as in [12] and where $\mathrm{vol}(\theta_{3\epsilon}(\mathcal{G}_k))$ is computed in $M_{Mk}(\mathbb{C})$ and $\mathcal{W}_\mathrm{k} + \mathcal{H}_\mathrm{k}$ is computed in $T_{Mk,<}$, both being Euclidean volumes corresponding to the norms $(Mk)^{1/2}|\cdot|_2$, where the integral over $D(f_k + \mathcal{D}_k)$ is over the diagonal parts of these matrices, and where $E_\epsilon(f_k)$ is the integral defined on p. 252 of [12]. It is clear that $\theta_{3\epsilon}(\mathcal{G}_k) \subset \mathcal{N}_{4\epsilon}(\Gamma(Z_H : D; m, Mk, \gamma))$, so (2.12) gives a lower bound on $\mathrm{vol}(\mathcal{N}_{4\epsilon}(\Gamma(Z_H : D; m, Mk, \gamma)))$.

Using (2.12) and the standard volume comparison test (2.7), we have

$$\begin{aligned} P_{2\epsilon}(\Gamma(Z_H : D; m, Mk, \gamma)) &\geq \frac{\mathrm{vol}(\mathcal{N}_{4\epsilon}(\Gamma(Z_H; m, Mk, \gamma)))}{\mathrm{vol}(\mathcal{B}_{8\epsilon})} \\ &\geq C_{Mk} \cdot \mathrm{vol}(\mathcal{W}_k + \mathcal{H}_k) \cdot E_\epsilon(f_k) \cdot \frac{\Gamma((Mk)^2+1)}{\pi^{(Mk)^2}(8(Mk)^{1/2}\epsilon)^{2(Mk)^2}} \end{aligned}$$

where $\mathcal{B}_{8\epsilon}$ is a ball in $M_{Mk}(\mathbb{C})$ of radius 8ϵ with respect to $|\cdot|_2$, and we are taking volumes corresponding to the Euclidean norm $(Mk)^{1/2}|\cdot|_2$. Since $\mathcal{W}_k$ and $\mathcal{H}_k$ are orthogonal, we have that $\mathrm{vol}(\mathcal{W}_k + \mathcal{H}_k) = \mathrm{vol}(\mathcal{W}_k) \cdot \mathrm{vol}(\mathcal{H}_k)$, where each volume is taken in the subspace of appropriate dimension. But $\mathcal{W}_k$ is a ball of radius $(Mk)^{1/2}\epsilon$ in space of real dimension $Npk(pk-1)$, so

$$\mathrm{vol}(\mathcal{W}_k + \mathcal{H}_k) = \frac{\pi^{\frac{Npk(pk-1)}{2}}((Mk)^{1/2}\epsilon)^{Npk(pk-1)}}{\Gamma(\frac{Npk(pk-1)}{2}+1)} \cdot \mathrm{vol}(\mathcal{H}_k)$$

where $\mathrm{vol}(\mathcal{H}_k) = \mathrm{vol}(\Omega_k) \cdot (M^{1/2})^{k^2M(M-1)} \cdot \prod_{(i,j)\in\Phi} |\tilde{\alpha}_{ij}|^{2k^2}$. Using Stirling's formula and $M = Np$, we find

$$\begin{aligned} \mathbb{P}_\epsilon(Z_H : D; m, \gamma) &\geq \liminf_k (Mk)^{-2} \log P_\epsilon(\Gamma(Z_H : D; m, Mk, \gamma)) \\ &\geq \liminf_k (Mk)^{-2} \log(E_\epsilon(f_k)) \\ &+ \liminf_k \Bigg((Mk)^{-2} \log(C_{Mk}) + (Mk)^{-2} \log(\mathrm{vol}(\Omega_k)) + \\ &+ \Big(2 - \frac{1}{N}\Big)|\log \epsilon| + \Big(1 - \frac{1}{2N}\Big) \log k \\ &+ (\frac{M-1}{2M}) \log M + \frac{2}{M^2} \sum_{(i,j)\in\Phi} \log |\tilde{\alpha_{ij}}| \Bigg) + L_1 \end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}_\epsilon(Z_H : D; m, \gamma) &= \liminf_k (Mk)^{-2} \log(E_\epsilon(f_k)) \\ &+ \liminf_k \Big((Mk)^{-2} \log C_{Mk} + \frac{1}{2} \log Mk\Big) \\ &+ \liminf_k \left((Mk)^{-2} \log(\mathrm{vol}(\Omega_k)) + \Big(\frac{1}{2} - \frac{1}{2N}\Big) \log k\right) \\ &+ \Big(2 - \frac{1}{N}\Big)|\log \epsilon| + \frac{2}{M^2} \sum_{(i,j)\in\Phi} \log|\tilde{\alpha_{ij}}| + L_2\end{aligned}$$

where L_1 and L_2 are constants independent of ϵ, m and γ. As $\gamma \to 0$ and $m \to +\infty$, we have convergence

$$\frac{2}{M^2} \sum_{(i,j)\in\Phi} \log|\tilde{\alpha_{ij}}| \longrightarrow 2 \iint_{K_N} \log(\max(H(s,t), \epsilon)) ds dt$$

where

$$K_N = \bigcup_{j=1}^{N-1} \Big\{ \frac{j}{N} \le x \le \frac{j+1}{N} \le y \le 1 \Big\}.$$

Note that we have $\mathrm{area}(K_N) = \frac{N(N-1)}{2N^2}$. Now by (2.9), we have

$$\liminf_k \left((Mk)^{-2} \log(\mathrm{vol}(\Omega_k)) + \left(\frac{1}{2} - \frac{1}{2N}\right). \log(k)\right)$$
$$\ge M^{-2} \chi\Big(\{\mathrm{Re} C_{ij}\}, \{\mathrm{Im} C_{ij}\} : (i,j) \in \Phi\Big)$$

Then

$$\begin{aligned}\mathbb{P}_\epsilon(Z_H : D) \ge \liminf_k \, (Mk)^{-2} \log(E_\epsilon(f_k)) + \Big(2 - \frac{1}{N}\Big)|\log \epsilon| \\ + 2 \iint_{K_N} \log(\max(H(s,t), \epsilon)) ds dt + L_3\end{aligned}$$

The eigenvalue distribution of f_k equals that of $z_{k,\epsilon}$ and converges as $k \to +\infty$ to the measure $\sigma_{\epsilon, \frac{1}{\sqrt{N}}}$, we may apply Lemma 2.3 of [12] concerning the asymptotics of

$E_\epsilon(f_k)$ as $k \to \infty$. Using also Lemma 3, we get

$$\delta_0(Z_H : D) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(Z_H : D)}{|\log \epsilon|} \geq 1 + 2 \cdot \mathrm{area}(\mathrm{supp}(H) \cap K_N).$$

Taking N arbitrarily large completes the proof. □

The following Theorem gives us an upper bound on $\delta_0(Z_H : D)$ without any conditions on the support of H.

Theorem C.2. *Let $H \geq 0$, $H \in L^1([0,1]^2)$ have essentially bounded coordinate expectations $CE_1(H)$ and $CE_2(H)$, as in equations* (2.3). *Then*

$$\delta_0(Z_H : D) \leq \min\{\, 2\,,\ 1 + 2\,\mathrm{area}(\mathrm{supp}(H))\}.$$

Proof. First of all it is clear that $\delta_0(Z_H : D) \leq \delta_0(Z_H) \leq 2$.
By standard arguments we can find ω in regular block form such that both $\|Z_H - Z_\omega\|_2$ and $\mathrm{area}(\mathrm{supp}(H) \triangle \mathrm{supp}(w))$ are arbitrarily small. Using this, given $\delta > 0$ we can find projections p_1, q_1, p_2, $q_2, \ldots, p_n$, q_n in $W^*(D)$ such that if $i \neq j$, then $p_i \otimes q_i$ is orthogonal to $p_j \otimes q_j$ in $W^*(D) \overline{\otimes} W^*(D)$ and such that

$$\sum_{i=1}^{n} \tau(p_i)\tau(q_i) > 1 - \mathrm{area}(\mathrm{supp}(H)) - \delta/3 \tag{2.13}$$
$$\sum_{i=1}^{n} \|p_i Z_H q_i\|_2 < \delta/4. \tag{2.14}$$

Take $R > \max\{\|Z_H\|_2, \|D\|_2\}$. Using Lemma 2.9 of [27], given $\epsilon > 0$ there exist m_0, γ_0, k_0 such that for $m \geq m_0$, $\gamma < \gamma_0$, $k \geq k_0$ and for every (A, B) and $(\tilde{A}, \tilde{B}) \in \Gamma_R(Z_H, D; m, k, \gamma)$ there exists a unitary $U \in M_k(\mathbb{C})$ such that

$$\|U\tilde{B}U^* - B\|_2 < \epsilon. \tag{2.15}$$

For m and k sufficiently big and γ sufficiently small we can find spectral projections of B

$$P_1, Q_1, \ldots, P_n, Q_n \in M_k(\mathbb{C})$$

and spectral projections of $\tilde{B}$

$$\tilde{P}_1, \tilde{Q}_1, \ldots, \tilde{P}_n, \tilde{Q}_n \in M_k(\mathbb{C})$$

such that if $i \neq j$ then $P_i \otimes Q_i$ is orthogonal to $P_j \otimes Q_j$ in $M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$ and $\tilde{P}_i \otimes \tilde{Q}_i$ is orthogonal to $\tilde{P}_j \otimes \tilde{Q}_j$ satisfying

$$|\mathrm{tr}_k(P_i) - \tau(p_i)| < \frac{\delta}{3n}, \quad |\mathrm{tr}_k(Q_i) - \tau(q_i)| < \frac{\delta}{3n}, \quad \sum_{i=1}^{n} \|P_i A Q_i\|_2 < \frac{\delta}{2}$$

$$|\mathrm{tr}_k(\tilde{P}_i) - \tau(p_i)| < \frac{\delta}{3n}, \quad |\mathrm{tr}_k(\tilde{Q}_i) - \tau(q_i)| < \frac{\delta}{3n}, \quad \sum_{i=1}^{n} \|\tilde{P}_i \tilde{A} \tilde{Q}_i\|_2 < \frac{\delta}{2}.$$

Taking ϵ sufficiently small and using (2.15) together with the fact that we can always approximate these projections with polynomials in B and $\tilde{B}$ in the $|\cdot|_2$, we can also guarantee that

$$\|P_i - U\tilde{P}_i U^*\|_2 < \frac{\delta}{6nR}, \quad \|Q_i - U\tilde{Q}_i U^*\|_2 < \frac{\delta}{6nR} \quad (1 \leq i \leq n).$$

Therefore,

$$\sum_{i=1}^{n} \|P_i(U\tilde{A}U^*)Q_i\|_2 < \sum_{i=1}^{n} \left(\frac{3\delta\|\tilde{A}\|}{6nR} + \|\tilde{P}_i \tilde{A} \tilde{Q}_i\|_2 \right) < \delta. \tag{2.16}$$

Let $\Omega_R(H,k) = \{X \in M_k(\mathbb{C}) : \|X\|_2 \leq R, \ P_i X Q_i = 0 \text{ for } i = 1, \ldots, n\}$, this is a ball of radius R in a space of real dimension $d(k) = 2k^2(1 - \sum_{i=1}^{n} \mathrm{tr}_k(P_i)\mathrm{tr}_k(Q_i))$. By (2.16) it is clear that

$$\Gamma_R(Z_H : D\,; m, k, \gamma) \subseteq \theta(N_\delta(\Omega_R(H,k))) \tag{2.17}$$

where $\theta(N_\delta(\Omega_R(H,k)))$ is the unitary orbit of the $\delta-$neighborhood of $\Omega_R(H,k)$. Taking the P_δ packing number on both sides of (2.17), we get

$$P_\delta(\Gamma_R(Z_H : D\,; m, k, \gamma)) \leq P_\delta(\theta(N_\delta(\Omega_R(H,k)))) \leq P_\delta(U_k(\mathbb{C})) \cdot P_\delta(N_\delta(\Omega_R(H,k))).$$

Using Theorem 7 of [42], there exists a constant K_1 independent of k such that

$$P_\delta(U_k(\mathbb{C})) \leq \left(\frac{K_1}{\delta}\right)^{k^2}. \tag{2.18}$$

On the other hand, standard packing number estimations gives us

$$P_\delta(N_\delta(\Omega_R(H,k))) \leq P_\delta(\Omega_{R+\delta}(H,k)) \leq \left(\frac{K_2(R+\delta)}{\delta}\right)^{d(k)} \tag{2.19}$$

where K_2 is a constant independent of k. It follows that

$$P_\delta(\Gamma_R(Z_H : D\,; m, k, \gamma)) \leq \left(\frac{K_1}{\delta}\right)^{k^2} \cdot \left(\frac{K_2(R+\delta)}{\delta}\right)^{d(k)}.$$

Now using (2.13) yields

$$\begin{aligned}\frac{d(k)}{k^2} &= 2\Big(1 - \sum_{i=1}^{n} \mathrm{tr}_k(P_i)\mathrm{tr}_k(Q_i)\Big) \leq 2\Big(1 - \sum_{i=1}^{n} \tau(p_i)\tau(q_i) + 2\delta/3\Big) \\ &\leq 2\Big(\mathrm{area}(\mathrm{supp}(H)) + \delta\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_k \frac{1}{k^2}\log(P_\delta(\Gamma_R(Z_H : D\,; m,k,\gamma))) &\leq \log(K_1) + |\log(\delta)| + \\ &+ 2(\mathrm{area}(\mathrm{supp}(H)) + \delta)\cdot \log(K_2(R+\delta)) \\ &+ 2(\mathrm{area}(\mathrm{supp}(H)) + \delta)\cdot |\log(\delta)|.\end{aligned}$$

When $\gamma \to 0$ and $m \to +\infty$ we obtain

$$\mathbb{P}_\delta(Z_H : D) \leq (1 + 2\cdot \mathrm{area}(\mathrm{supp}(H)) + 2\delta)\cdot |\log\delta| + C$$

where C is a constant. It follows that

$$\delta_0(Z_H : D) = \limsup_{\delta \to 0} \frac{\mathbb{P}_\delta(Z_H : D)}{|\log \delta|} \leq 1 + 2 \cdot \mathrm{area}(\mathrm{supp}(H)).$$

$\square$

D. Concluding Remarks and Questions

Since the free entropy dimension of Z_H in the presence of D is a lower bound for the free entropy dimension of Z_H, from Theorems C.1 and C.2 we have that for any H as in Theorem C.1,

$$1 + 2\,\mathrm{area}(\mathrm{supp}(H)) = \delta_0(Z_H : D) \leq \delta_0(Z_H). \tag{2.20}$$

However, $1 + 2\,\mathrm{area}(\mathrm{supp}(H))$ is not the actual value of $\delta_0(Z_H)$ in all cases. For example, if $n \geq 2$ and if H is the characteristic function of $\cup_{i=1}^n T_i$, where $T_i = \{(x, y) \in [0, 1] : \frac{i-1}{n} \leq x < y \leq \frac{i}{n}\}$, then the moments of Z_H agree with the moments of a nonzero multiple of the quasinilpotent DT–operator T. Therefore, in this case we have

$$\delta_0(Z_H : D) = 1 + \frac{1}{n} < \delta_0(Z_H) = \delta_0(T) = 2. \tag{2.21}$$

Of course, if D belongs to the von Neumann algebra generated by Z_H, then equality holds in (2.20). It is an interesting question, when do we have $D \in W^*(\{Z_H\})$? More generally, what is the von Neumann algebra generated by Z_H? When is it a factor? Is it then an interpolated free group factor? A particular case of interest is when H is the characteristic function of the band

$$\{(x, y) \mid 0 \leq x < y < \min(1, x + \alpha)\},$$

for $\alpha \in (0, 1)$, as is drawn in Figure 2 (on page 20).

CHAPTER III

QUASINILPOTENT GENERATORS OF THE HYPERFINITE II_1 FACTOR*

A. Introduction

Consider a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if the projection of $\mathcal{H}$ onto $\mathcal{H}_0$ belongs to $\mathcal{M}$. The subspace $\mathcal{H}_0$ is said to be non-trivial if $\mathcal{H}_0 \neq 0$ and $\mathcal{H}_0 \neq \mathcal{H}$. For $T \in \mathcal{M}$, a subspace $\mathcal{H}_0$ is said to be T-invariant, if $T(\mathcal{H}_0) \subseteq \mathcal{H}_0$, i.e. if T and the projection $P_{\mathcal{H}_0}$ onto $\mathcal{H}_0$ satisfy

$$P_{\mathcal{H}_0} T P_{\mathcal{H}_0} = T P_{\mathcal{H}_0}.$$

$\mathcal{H}_0$ is said to be hyperinvariant for T (or T-hyperinvariant) if it is S-invariant for every $S \in \mathcal{B}(\mathcal{H})$ that commutes with T. If the subspace $\mathcal{H}_0$ is T-hyperinvariant, then $P_{\mathcal{H}_0} \in W^*(T) = \{T, T^*\}''$ (cf. [9]). However, the converse statement does not hold true. In fact, one can find $A \in M_3(\mathbb{C})$ and an A-invariant projection $P \in W^*(A)$ which is not A-hyperinvariant (cf. [9]).

The invariant subspace problem relative to the von Neumann algebra $\mathcal{M}$ asks whether every operator T has a non-trivial, closed, invariant subspace $\mathcal{H}_0$ affiliated with $\mathcal{M}$, and the hyperinvariant subspace problem asks whether one can always choose such an $\mathcal{H}_0$ to be hyperinvariant for T. Of course, if $\mathcal{M}$ is not a factor, then the answer to both of these questions is *yes*. Also, if $\mathcal{M}$ of finite dimension, i.e. $\mathcal{M} \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, then every operator in $\mathcal{M} \backslash \mathbb{C}1$ has a non-trivial eigenspace, and therefore a non-trivial T-invariant subspace. Recall from [7] that every operator in

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a II_1-factor defines a probability measure μ_T on $\mathbb{C}$, the Brown measure of T, with $\operatorname{supp}(T) \subseteq \sigma(T)$. In [19], Uffe Haagerup and Hanne Schultz made a huge advance in this problem. Namely, they proved that if the Brown measure of the operator T is not concentrated in one point, then the operator T has a non-trivial, closed, invariant subspace, affiliated with $\mathcal{M}$ and moreover, this subspace is hyperinvariant. More specifically, for each Borel set $B \subseteq \mathbb{C}$, they constructed a maximal, closed, T-invariant subspace, $\mathcal{K} = \mathcal{K}_T(B)$, affiliated with $\mathcal{M}$, such that the Brown measure of $T|_{\mathcal{K}}$ is concentrated on B and if we denote by P the projection onto this subspace, then $\tau(P) = \mu_T(B)$. Therefore, if μ_T is not a Dirac measure, then T has a non-trivial invariant subspace affiliated with $\mathcal{M}$. If the Borel set B is a closed ball of radius r centered at λ. Then $\mathcal{K}_T(B)$ is the set of vectors $\xi \in \mathcal{H}$, for which there is a sequence $\{\xi_n\}_n$ in $\mathcal{H}$ such that

$$\lim_n \|\xi_n - \xi\| = 0 \quad \text{and} \qquad \limsup_n \|(T - \lambda 1)^n \xi_n\|^{\frac{1}{n}} \leq r.$$

As regards the invariant subspace problem relative to the von Neumann algebra, the following question remains completely open: If T is an operator in a II_1-factor $\mathcal{M}$ and if the Brown measure μ_T is a Dirac measure, for example if T is quasinilpotent, does T has a non-trivial closed, invariant subspace affiliated with $W^*(T)$?

In [11], Dykema and Haagerup introduced the family of DT-operators and they studied many of their properties. The case of the quasinilpotent DT-operator arose as a natural candidate for an operator without an invariant subspace affiliated to the von Neumann algebra. Later on, in [10], Dykema and Haagerup finally showed that every quasinilpotent DT-operator T has a one-parameter family of non-trivial hyperinvariant subspaces. In particular, they proved that for $t \in [0, 1]$,

$$\mathcal{H}_t := \left\{ \xi \in \mathcal{H} \,:\, \limsup_n \left(\frac{k}{e} \|T^k \xi\| \right)^{\frac{2}{k}} \leq t \right\}$$

is a closed, hyperinvariant subspace of T.

In this paper, for each sequence $\{c_n\}_n \in l_1(\mathbb{N})$ we define an operator A in the hyperfinite II_1-factor. These operators are quasinilpotent, and under a certain mild restriction on the sequence $\{c_n\}_n$ they generates the whole hyperfinite II_1-factor. As a corollary of the proof that A is quasinilpotent we deduce that given $\{c_n\}_n \in l_1(\mathbb{N})$ then

$$\limsup_k \,(k!\ \sigma_k)^{1/k} = 0 \quad \text{where} \quad \sigma_k := \sum_{1\leq n_1<n_2<\ldots<n_k} |c_{n_1}c_{n_2}\ldots c_{n_k}|.$$

We also show that these operators have invariant subspaces affiliated with the von Neumann algebra. The projections onto these subspaces live in the diagonal masa

$$\mathcal{D} := \overline{\left(\bigotimes_{n=1}^{+\infty} \mathbb{D}_2(\mathbb{C})\right)}^{\mathrm{WOT}} \subset \mathcal{R}$$

where $\mathbb{D}_2(\mathbb{C})$ is the algebra of the 2×2 diagonal matrices. We also show that none of these projections is hyperinvariant. Moreover, we show that if p is a non-trivial hyperinvariant projection for A then

$$p \notin \bigcup_{n=1}^{+\infty} \Big(\bigotimes_{k=1}^{n} M_2(\mathbb{C})\Big).$$

In section §4 we show that these operators have trivial kernel and dense range. We prove also that given $r>0$ and any sequence $\{\gamma_n\}_{n=1}^{+\infty}$ of positive numbers, if we define the subspace $\mathcal{H}_r(A)$ by

$$\mathcal{E}_r(A) := \{\xi \in \mathcal{H} \,:\, \limsup_n \,\gamma_n \|A^n(\xi)\|^{1/n} \leq r\} \quad \text{and} \quad \mathcal{H}_r(A) = \overline{\mathcal{E}_r(A)}$$

then this subspace is either $\mathcal{H}$ or $\{0\}$. We are unable to determine if the operator A has a non-trivial hyperinvariant subspace, and for the evidence showed above, it is a possible counterexample to the hyperinvariant subspace problem.

In section §5, we show that the real and imaginary part of A, $a := \mathrm{Re}(A)$ and $b := \mathrm{Im}(A)$, are equally distributed. We find a combinatorial formula as well as an analytical way to compute their moments. We also compute some of their mixed moments. We prove also that when $c_n = \alpha^n$ where $0 < \alpha \leq \frac{1}{2}$ then $W^*(a)$ is a Cartan masa in the hyperfinite and we find countably many values of $\alpha \in (\frac{1}{2}, 1)$ in which $W^*(a)$ is not maximal abelian. However, for all the values of $\alpha \in (0, 1)$ this algebra is diffuse. In section §6, we find a combinatorial formula for the moments of A^*A in terms of alternating partitions of elements of two different colors. We also ask a question regarding these partitions.

Acknowledgment: I thank my advisor, Ken Dykema, for many helul discussions and comments.

B. Notation and Preliminaries

1. Infinite Tensor Products of Finite von Neumann Algebras

The Hilbert space tensor product of two Hilbert spaces is the completion of their algebraic tensor product. One can define a tensor product of von Neumann algebras (a completion of the algebraic tensor product of the algebras considered as rings), which is again a von Neumann algebra, and acts on the tensor product of the corresponding Hilbert spaces. The tensor product of two finite algebras is finite, and the tensor product of an infinite algebra and a non-zero algebra is infinite. The type of the tensor product of two von Neumann algebras (I, II, or III) is the maximum of their types. The Tomita commutation Theorem for tensor products states that

$$(M \overline{\otimes} N)' = M' \overline{\otimes} N'.$$

The tensor product of an infinite number of von Neumann algebras, if done naively, is usually a ridiculously large non-separable algebra. Instead one usually chooses a state on each of the von Neumann algebras, uses this to define a state on the algebraic tensor product, which can be used to product a Hilbert space and a (reasonably small) von Neumann algebra. Given finite factors $\{\mathcal{M}_n\}_{n=1}^{+\infty}$, denote τ_n the unique faithful normal trace on $\mathcal{M}_n$. We write $\bigotimes_{n=1}^{+\infty} \mathcal{M}_n$ for the algebraic tensor product, that is finite linear combination of elementary tensors $\bigotimes_{n=1}^{+\infty} x_n$, where $x_n \in \mathcal{M}_n$ and all but finitely many x_n are 1. We have the product state τ on $\bigotimes_{n=1}^{+\infty} \mathcal{M}_n$ defined on elementary tensors by

$$\tau\left(\bigotimes_{n=1}^{+\infty} x_n\right) = \prod_{n=1}^{+\infty} \tau_n(x_n).$$

Now let π be the representation of $\bigotimes_{n=1}^{+\infty} \mathcal{M}_n$ by left multiplication on the Hilbert space $L^2\Big(\bigotimes_{n=1}^{+\infty} \mathcal{M}_n\Big)$ in the usual way. The infinite von Neumann tensor product of the $\mathcal{M}_n$ is then the weak-closure of the image of π. This is necessarily a finite factor, as it has a trace, namely the extension of τ, which is the unique normalized trace. The Tomita commutation Theorem remains true in this infinite setting.

2. The Hyperfinite II_1-factor

Let $\mathcal{M}$ a finite von Neumann algebra and τ a faithful normal trace. Given an element x in such a von Neumann algebra, we will denote $\|x\|_2 = \tau(x^*x)^{1/2}$. Let $L^2(\mathcal{M})$ the Hilbert space obtained by the completion of $\mathcal{M}$ with respect to the $\|\cdot\|_2$. We shall follow the tradition in the subject of regarding $\mathcal{M}$ as a subset of $L^2(\mathcal{M})$ whenever it is convenient. The standard form is the representation of $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}))$ obtained by letting each x in $\mathcal{M}$ act by left multiplication on $L^2(\mathcal{M})$.

Murray and von Neumann defined the approximate finite dimensional property usually denoted by AFD. Namely, a II_1-factor $\mathcal{M}$ is said to be AFD when for any $x_1, \ldots, x_n \in \mathcal{M}$ and strong neighborhood V of 0 in $\mathcal{M}$ there exists a finite dimensional $*$-subalgebra $\mathcal{N}$ of $\mathcal{M}$ such that $x_i \in \mathcal{N} + V$ for each i. Let $M_2(\mathbb{C})$ be the algebra of 2×2 matrices. Then the infinite tensor product

$$\mathcal{R} := \overline{\left(\bigotimes_{n=1}^{+\infty} M_2(\mathbb{C}) \right)}^{\mathrm{WOT}} \tag{3.1}$$

produced with respect to the unique normalized trace on $M_2(\mathbb{C})$ is a II_1-factor, which is obviously AFD. In [32], Murray and von Neumann showed that up to isomorphism this is the unique AFD II_1-factor. In complete contrast with the C^*-case, the resulting object is independent of the size of the matrices algebras involved.

Given a discrete group we can always define a finite von Neumann algebra via the left or right regular representation. This algebra is called the group von Neumann algebra. The group von Neumann algebra of a discrete group with the infinite conjugacy class property is a factor of type II_1, and if the group is amenable and countable then the factor is AFD. There are many groups with these properties, as any group such that any finite subset generates a finite subgroup is amenable. For example, the group von Neumann algebra of the infinite symmetric group of all permutations of a countable infinite set that fix all but a finite number of elements is the hyperfinite type II_1 factor.

C. Quasinilpotent Generators

In this section we construct the operators described before. We define the 2×2 matrices V, Q and P by

$$V := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q := V^*V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P := VV^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\{c_n\}_n$ be a sequence in $l_1(\mathbb{N})$ and let us consider $A_n := c_1 V + c_2 I \otimes V + \ldots + c_n I^{\otimes(n-1)} \otimes V \in \mathcal{R}$ then $\|A_n\| \leq \sum_{k=1}^{n} |c_k|$ for all $n \geq 1$. The sequence $\{A_n\}_n$ is Cauchy in norm since by assumption $\sum_{n=1}^{+\infty} |c_n| < +\infty$. Therefore, it converges in the operator norm to an operator A in the hyperfinite II_1-factor with

$$A := \sum_{n=1}^{+\infty} c_n V_n \qquad \text{where} \qquad V_n := I^{\otimes(n-1)} \otimes V. \tag{3.2}$$

We will prove that this operator is quasinilpotent and that under certain mild hypothesis it generates the hyperfinite II_1 factor $\mathcal{R}$.

Theorem C.1. *The operator A in (3.2) is quasinilpotent.*

Proof. Let $A = \sum_{n=1}^{+\infty} c_n I^{\otimes(n-1)} \otimes V$ then using that $V^2 = 0$ we see that

$$A^k = k! \sum_{1 \leq n_1 < n_2 < \ldots < n_k} c_{n_1} c_{n_2} \ldots c_{n_k} V_{n_1} V_{n_2} \ldots V_{n_k} \quad \text{where} \quad V_n := I^{\otimes(n-1)} \otimes V.$$

Then $\|A^k\| \leq k! \sum_{1 \leq n_1 < n_2 < \ldots < n_k} |c_{n_1} c_{n_2} \ldots c_{n_k}|$. Let us define

$$\sigma_k := \sum_{1 \leq n_1 < n_2 < \ldots < n_k} |c_{n_1} c_{n_2} \ldots c_{n_k}| \quad \text{and} \quad \sigma_0 := 1.$$

Therefore,

$$\|A^k\| \leq k! \sigma_k. \tag{3.3}$$

Consider the function $f(z) := \prod_{n=1}^{+\infty} (1 + z c_n) = \sum_{n=0}^{+\infty} \sigma_n z^n$. Using the Weierstrass factorization Theorem [17] and the fact that the sequence $\{c_n\}_n$ is absolutely sumable we see that the function $f(z)$ is entire. From this function $f(z)$ we define formally $g(z)$ by

$$g(z) := \int_0^{+\infty} f(tz)e^{-t}\,dt. \tag{3.4}$$

Now we will prove that the function $g(z)$ is well defined, entire and its power series expansion is $g(z) = \sum_n n!\sigma_n z^n$. Therefore, $\limsup\,(n!\sigma_n)^{1/n} = 0$ and using (3.3), we deduce that A is quasinilpotent.

Take $R > 0$ then there exists N_0 such that $\sum_{k=N_0+1}^{+\infty} |c_k| < \frac{1}{2R}$. Then for $|w| < R$

$$\begin{gathered}
|f(tw)| = \prod_{n=1}^{+\infty} |1 + wtc_n| = \prod_{n=1}^{N_0} |1 + wtc_n| \cdot \prod_{n=N_0+1}^{+\infty} |1 + twc_n| \\
\leq p_R(t) \cdot \exp\left(\sum_{n=N_0+1}^{+\infty} |c_n tw| \right) = p_R(t) \cdot \exp\left(t|w| \sum_{n=N_0+1}^{+\infty} |c_n| \right) \leq \\
\leq p_R(t) \cdot \exp\left(t|w| \frac{1}{2R} \right) \leq p_R(t) \cdot \exp(t/2)
\end{gathered}$$

where $p_R(t) = \prod_{n=1}^{N_0} (1 + R\,t|c_n|)$ a polynomial of degree N_0.

Therefore,

$$|g(w)| \leq \int_0^{+\infty} p_R(t) \cdot e^{-t/2}\,dt = K_R \qquad \text{for all} \quad |w| < R. \tag{3.5}$$

Then g is a well defined function for all $w \in \mathbb{C}$. Moreover, for all closed curves γ contained in the disk of radius R centered at origin we have

$$\oint_\gamma g(z)dz = \oint_\gamma \left(\int_0^{+\infty} f(tz)e^{-t}\,dt \right) dz = \int_0^{+\infty} e^{-t} \left(\oint_\gamma f(tz)dz \right) dt = 0.$$

(Note that we are allowed to interchange the integrals by applying Fubini's Theorem since g is bounded (3.5).) Using Morera's Theorem we see that g is holomorphic in the disk of radius R, and since R is arbitrary, g is entire. The fact that the function g has the desired power series expansion comes from the fact that $k! = \int_0^{+\infty} t^k e^{-t} dt$. $\square$

From the proof of the last Theorem we observe that something a little bit more general was proved. We state it in the next corollary.

Corollary C.2. *Let $\{c_n\}_n$ be a sequence of complex number in $l_1(\mathbb{N})$. Then*

$$\limsup_k \,(k!\ \sigma_k)^{1/k} = 0$$

where $\sigma_k := \sum_{1 \le n_1 < n_2 < \ldots < n_k} |c_{n_1} c_{n_2} \ldots c_{n_k}|$.

In the next Theorem we will prove that under certain mild hypothesis the operator A generates the whole hyperfinite II_1-factor $\mathcal{R}$ as in (3.1).

Theorem C.3. *Let $\{c_n\}_n$ be a sequence of complex numbers in $l_1(\mathbb{N})$ such that $|c_i| \neq |c_j|$ whenever $i \neq j$ and $c_j \neq 0$ for all $j \geq 1$. Then the von Neumann algebra generated by A is $\mathcal{R}$. Moreover, if there exist $i \neq j$ so that $|c_i| = |c_j|$ then the von Neumann algebra generated by A is not the whole hyperfinite factor.*

Proof. By applying an automorphism, if necessary, we can assume without loss of generality that $c_1 > c_2 > \ldots > c_n > c_{n+1} > \cdots > 0$. Let us define

$$q_k := \sum_{n=1}^{+\infty} c_n^k \, I^{\otimes(n-1)} \otimes Q \qquad p_k := \sum_{n=1}^{+\infty} c_n^k \, I^{\otimes(n-1)} \otimes P \tag{3.6}$$

and

$$v_{n,m} := I^{\otimes(n-1)} \otimes V \otimes I^{\otimes(m-n-1)} \otimes V^* + I^{\otimes(n-1)} \otimes V^* \otimes I^{\otimes(m-n-1)} \otimes V \quad \text{for} \ \ n < m.$$

Then we can see that $A^*A = q_2 + \sum_{1\le n<m} c_n c_m\, v_{n,m} = q_2 + v$ and $AA^* = p_2 + \sum_{1\le n<m} c_n c_m\, v_{n,m} = p_2 + v$ where $v := \sum_{1\le n<m} c_n c_m\, v_{n,m}$. Observing that $p_2 + q_2 = \sum_{n=1}^{+\infty} c_n^2$ we see that $\{v, p_2, q_2\} \in W^*(A)$.

Then $q_2 A - A q_2 = \sum_{n=1}^{+\infty} c_n^3\, I^{\otimes(n-1)} \otimes V$ and therefore $q_6 \in W^*(A)$. Repeating the same argument we see that given $k \ge 1$ there exists $N(k) \ge k$ such that $q_{N(k)} \in W^*(A)$. Now observing that

$$\lim_k \left(\sum_{n=1}^{+\infty} c_n^{N(k)} \right)^{\frac{1}{N(k)}} = \max\{c_n \,:\, n \ge 1\} = c_1 > c_2 = \lim_k \left(\sum_{n=2}^{+\infty} c_n^{N(k)} \right)^{\frac{1}{N(k)}}$$

we obtain Q as a spectral projection of $q_{N(k)}$ for k sufficiently large. So, $Q \in W^*(A)$ and since $AQ - QA = c_1 V$ we have also that $V \in W^*(A)$. Repeating the same argument now using that $c_2 > c_3$ we obtain that $I \otimes Q,\ I \otimes V \in W^*(A)$. Analogously, $I^{\otimes(n-1)} \otimes Q,\ I^{\otimes(n-1)} \otimes V \in W^*(A)$ for all $n \ge 1$. Then

$$\{I^{\otimes(n-1)} \otimes Q,\ I^{\otimes(n-1)} \otimes P,\ I^{\otimes(n-1)} \otimes V,\ I^{\otimes(n-1)} \otimes V^* \ :\ n \ge 1\} \in W^*(A)$$

and we conclude that $W^*(A) = \mathcal{R}$.

If there exist $n \ne m$ such that $|c_n| = |c_m|$, then by applying an automorphism as we did before we can assume that $c_n = c_m$. It is a direct computation to check that the operators

$$S_{n,m} := P_n Q_m + Q_n P_m - \frac{c_n}{c_m} V_n V_m^* - \frac{c_m}{c_n} V_n^* V_m$$

commute with A. Note that if $c_n = c_m$ this operator is selfadjoint and commutes with A and hence the von Neumann algebra generated by A is not the whole hyperfinite. $\square$

The operator $A_n := c_1 V + c_2\, I \otimes V + \ldots + c_n\, I^{\otimes(n-1)} \otimes V$ is a nilpotent operator of order $n+1$ and $A_n^n := n!\, c_1 c_2 \ldots c_n\, V \otimes V \otimes \ldots \otimes V$. Since the projection $P^{\otimes(n)}$ is the orthogonal projection onto the range of A_n^n it is an hyperinvariant projection for the operator A_n. Since A commutes with A_n it is an invariant projection for A affiliated to the von Neumann algebra $\mathcal{R}$. However, we will see that none of these invariant projections for A are A–hyperinvariant.

Given $1 \leq n$ we will denote by $V_n := I^{\otimes(n-1)} \otimes V$ and analogously with Q_n, P_n and V_n^*. Let $n < m$ and consider the operator $S_{n,m}$ defined by

$$S_{n,m} := P_n Q_m + Q_n P_m - \frac{c_n}{c_m} V_n V_m^* - \frac{c_m}{c_n} V_n^* V_m. \tag{3.7}$$

As we mention before, $AS_{n,m} = S_{n,m}A$ for all $1 \leq n < m$ and we can see that

$$P^{\otimes(n)} S_{n,n+1} P^{\otimes(n)} = P^{\otimes(n)} \otimes Q \quad \text{and} \quad S_{n,n+1} P^{\otimes(n)} = P^{\otimes(n)} \otimes Q - \frac{c_{n+1}}{c_n} P^{\otimes(n-1)} \otimes V^* \otimes V.$$

So the projection $P^{\otimes(n)}$ is not invariant for $S_{n,n+1}$ and therefore, not A–hyperinvariant for $n \geq 1$.

Denote by $\mathbb{D}_2(\mathbb{C})$ the algebra of the 2×2 diagonal matrices. Then

$$\mathcal{D} := \overline{\left(\bigotimes_{n=1}^{+\infty} \mathbb{D}_2(\mathbb{C}) \right)}^{\text{WOT}} \subset \mathcal{R}$$

is a maximal abelian subalgebra (masa) of $\mathcal{R}$. Therefore, $\mathcal{D} \cong L^\infty[0,1]$ and under this identification the projection P corresponds to the characteristic function on $[0, 1/2]$ and the projection Q to the characteristic function on $[1/2, 1]$ and so on.

Given a word with letters in the alphabet $\{P, Q, V, V^*\}$ we can associate an element in $\mathcal{R}$ by adding a tensor product between each of the letters. For example, the word VPV^*Q corresponds to the element $V \otimes P \otimes V^* \otimes Q$ and so on. Note that if the word consists only of letters P and Q the associated element is a projection in the diagonal algebra $\mathcal{D}$, and under the identification with $L^\infty[0,1]$, the words in P

and Q correspond to dyadic intervals in $[0,1]$.

Now we will prove the following Proposition.

Proposition C.4. *Given any word w with letters in $\{P,Q\}$ the corresponding projection $p_w \in \mathcal{D}$ is not A-hyperinvariant. Moreover,*

$$\bigvee_{S\in\{A\}'\cap\mathcal{R}} \overline{\mathrm{Range}(Sp_w)} = \mathcal{H}.$$

Proof. Let's first consider the case $w = P$. Let $S_{1,n}$ be the operator defined in (3.7). Then

$$S_{1,n}P = P \otimes I^{\otimes(n-2)} \otimes Q - \frac{c_n}{c_1} V^* \otimes I^{\otimes(n-2)} \otimes V.$$

Hence for $n \geq 2$

$$\mathrm{Range}(P) \vee \mathrm{Range}(S_{1,n}P) = \mathrm{Range}(P + Q \otimes I^{\otimes(n-2)} \otimes P).$$

Since

$$\bigvee_{n\geq 2} \mathrm{Range}(Q \otimes I^{\otimes(n-2)} \otimes P) = \mathrm{Range}(Q) \tag{3.8}$$

we see that

$$\bigvee_{S\in\{A\}'\cap\mathcal{R}} \overline{\mathrm{Range}(SP)} = \mathcal{H}.$$

The case $w = Q$ follows similarly. Reasoning by induction in the length of the word let's assume that it is true for all the words of length n. Take any word v of length $n+1$. Without loss of generality we can assume that it ends with Q (the other case follows similarly). Then $v = w \otimes Q$ where w is a word of length n. Thus for $m \geq n+2$

$$S_{n+1,m}(w \otimes Q) = w \otimes Q \otimes I^{\otimes(m-n-2)} \otimes P - \frac{c_{n+1}}{c_m} w \otimes V \otimes I^{\otimes(m-n-2)} \otimes V^*$$

hence

$$\text{Range}(w \otimes Q) \vee \text{Range}(S_{n+1,m}(w \otimes Q)) = \text{Range}(w \otimes Q + w \otimes P \otimes I^{\otimes(m-n-2)} \otimes Q).$$

Using (3.8) again, and the induction hypothesis we obtain

$$\bigvee_{S \in \{A\}' \cap \mathcal{R}} \overline{\text{Range}(Sp_v)} = \mathcal{H}$$

and finishes the proof. $\square$

Theorem C.5. *Let $n \in \mathbb{N}$ and $p \in \mathcal{R}$ be a non-trivial A_n–hyperinvariant projection. Then it is not A–hyperinvariant.*

Before proving this Theorem let's state a well known result proved by Barraa in [2]. This is a generalization of a result proved by Domingo Herrerro for finite dimensional Hilbert spaces in [20].

Theorem C.6 (Barraa)**.** *Every non-trivial hyperinvariant subspace $\mathcal{M}$ for a nilpotent operator A of order n satisfies that*

$$\overline{\text{Range}(A^{n-1})} \subseteq \mathcal{M} \subseteq \text{Ker}(A^{n-1}).$$

Proof. of Theorem C.5: Let $\mathcal{M}$ be a non-trivial hyperinvariant subspace for A_n. Since A_n is nilpotent of order $n+1$ we have by Theorem C.6 that $\overline{\text{Range}(A_n^n)} \subseteq \mathcal{M} \subseteq \text{Ker}(A_n^n)$. Since $\text{Ker}(A_n^n) = \text{Range}(1 - Q^{\otimes n})$ and $\overline{\text{Range}(A_n^n)} = \text{Range}(P^{\otimes n})$, if we denote by p the projection onto $\mathcal{M}$ we have that $P^{\otimes n} \leq p \leq 1 - Q^{\otimes n}$. Using Proposition C.4 we know that

$$\bigvee_{S \in \{A\}' \cap \mathcal{R}} \overline{\text{Ran}(SP^{\otimes n})} = \mathcal{H}.$$

Then, there exists $S \in \{A\}' \cap \mathcal{R}$ and $h \in P^{\otimes n}(\mathcal{H})$ such that

$$0 \neq Sp(h) = SpP^{\otimes n}(h) = SP^{\otimes n}(h) \in Q^{\otimes n}(\mathcal{H})$$

therefore, $pSp(h) = 0$. Thus, p is not S-invariant and then not A-hyperinvariant.

$\square$

Remark C.7. Note that if $S \in \bigotimes_{k=1}^{n} M_2(\mathbb{C}) \subset \mathcal{R}$ and $SA_n = A_n S$ then $AS = SA$.

Theorem C.8. *Assume p is a non-trivial hyperinvariant projection for A. Then $p \notin \bigcup_{n=1}^{+\infty} \left(\bigotimes_{k=1}^{n} M_2(\mathbb{C}) \right)$.*

Proof. Assume that there exists $n \geq 1$ such that $p \in \bigotimes_{k=1}^{n} M_2(\mathbb{C})$. Since p is hyperinvariant, it is A_n-invariant. Moreover, by Remark C.7, p is invariant for all $S \in \bigotimes_{k=1}^{n} M_2(\mathbb{C})$ such that $SA_n = A_n S$. Hence, p is A_n-hyperinvariant which contradicts Theorem C.5. Thus, $p \notin \bigotimes_{k=1}^{n} M_2(\mathbb{C})$ for any n. $\square$

It will be convenient to introduce some notation at this point. Given an operator $A \in B(\mathcal{H})$ we denote by $\mathcal{S}(A)$ the similarity orbit of A. In other words, $\mathcal{S}(A) := \{WAW^{-1} : \text{ where } W \text{ is invertible}\} \subset B(\mathcal{H})$. As in Chapter 2 of [20] we say that two operators A and B are asymptotically similar if $A \in \overline{\mathcal{S}(B)}$ and $B \in \overline{\mathcal{S}(A)}$, where the closure is with respect to the operator norm. Or equivalently, iff $\overline{\mathcal{S}(B)} = \overline{\mathcal{S}(A)}$. Now we are ready to state the next result.

Proposition C.9. *Let $\{a_n\}_n$ and $\{b_n\}_n$ in $l_1(\mathbb{N})$ be such that a_n, $b_n \neq 0$ for all n. Let $A = \sum_{n=1}^{+\infty} a_n V_n$ and $B = \sum_{n=1}^{+\infty} b_n V_n$, where $V_n = I^{\otimes(n-1)} \otimes V$. Then A and B are asymptotically similar.*

Proof. To prove $B \in \overline{\mathcal{S}(A)}$ it is enough to construct invertible operators W_n such that $\lim_n \|B - W_n A W_n^{-1}\| = 0$. For this, consider the sequence $\lambda_n := \frac{a_n}{b_n}$ and the 2×2 matrices $D_{\lambda_n} := P + \lambda_n Q$. So if we define the invertible element W_n by

$$W_n := D_{\lambda_1} \otimes D_{\lambda_2} \otimes \ldots \otimes D_{\lambda_n} \in \mathcal{R}$$

it is easy to see that if $A_n = \sum_{k=1}^n a_k V_k$ and $B_n = \sum_{k=1}^n b_k V_k$ then $W_n A_n W_n^{-1} = B_n$ and $W_n A W_n^{-1} = B_n + A - A_n$. Since $\lim_n \|B - B_n\| = 0$ and $\lim_n \|A - A_n\| = 0$ we see that $\lim_n \|B - W_n A W_n^{-1}\| = 0$. A similar argument shows that $A \in \overline{\mathcal{S}(B)}$ and concludes the proof. $\square$

Remark C.10. Let $\{a_n\}_n$ in $l_1(\mathbb{N})$ and A as before. We will show that A is a commutant operator, i.e.: there exist B and W such that $A = [W, B]$. It is clear that we can choose $\{b_n\}_n \in l_1(\mathbb{N})$ such that $b_n > 0$ and $\sum_{n=1}^{+\infty} \frac{|a_n|}{b_n} < +\infty$. Let $B := \sum_{n=1}^{+\infty} b_n V_n$ and $W := \sum_{n=1}^{+\infty} \frac{a_n}{b_n} P_n$. Since $P_n V_n = V_n$ and $V_n P_n = 0$ it is easy to see that

$$WB - BW = [W, B] = A.$$

D. Haagerup's Invariant Subspaces

As we described in the introduction, given an operator T in a II_1 factor $\mathcal{M}$, Haagerup and Schultz [19] constructed for each Borel set B in the complex plane an invariant subspace affiliated to the von Neumann algebra generated by T, such that $\tau(P_B) = \mu(B)$. If the Borel set B is a closed ball of radius r centered at λ. Then $\mathcal{K}_T(B)$ is the set of vectors $\xi \in \mathcal{H}$, for which there is a sequence $\{\xi_n\}_n$ in $\mathcal{H}$ such that

$$\lim_n \|\xi_n - \xi\| = 0 \quad \text{and} \quad \limsup_n \|(T - \lambda 1)^n \xi_n\|^{\frac{1}{n}} \leq r.$$

For any sequence $\{\gamma_n\}_{n=1}^{+\infty}$ of positive numbers and $r > 0$, we define a subspace $\mathcal{H}_r(T)$ (similar to the one considered in [10] to prove that the quasinilpotent DT-operator has non-trivial hyperinvariant subspaces) by

$$\mathcal{E}_r(T) := \{\xi \in \mathcal{H} \,:\, \limsup_n \gamma_n \|T^n(\xi)\|^{1/n} \leq r\} \quad \text{and} \quad \mathcal{H}_r(T) = \overline{\mathcal{E}_r(T)}. \tag{3.9}$$

This subspace is closed, T-invariant, affiliated to the von Neumann algebra, and moreover, hyperinvariant. However, we will prove that for any sequence $\{\gamma_n\}_n$ and for any $r > 0$ this subspace is trivial. Let $0 < \alpha < 1$ and consider the operator

$$A := \sum_{n=1}^{+\infty} \alpha^n V_n \quad \text{where} \quad V_n = I^{\otimes(n-1)} \otimes V. \tag{3.10}$$

Proposition D.1. *Let $w = P^{\otimes k_1} \otimes Q^{\otimes r_1} \otimes \ldots \otimes P^{\otimes k_n} \otimes Q^{\otimes r_n}$ where $n \geq 1$ and $k_i,\, r_i \geq 0$ for $i = 1, \ldots, n$. Then for A as in (3.10) we have*

$$\lim_m \left(\frac{\|A^m \hat{w}\|_2}{\|A^m \hat{1}\|_2} \right)^{\frac{1}{m}} = \alpha^{k_1 + k_2 + \ldots + k_n}. \tag{3.11}$$

Proof. Let $m \geq 1$ then

$$(A^*)^m A^m = \sum_{1 \leq p_1,\, q_1, \ldots,\, p_m,\, q_m} \alpha^{p_1} \alpha^{q_1} \ldots \alpha^{p_m} \alpha^{q_m} V_{q_1}^* \ldots V_{q_m}^* V_{p_1} \ldots V_{p_m} \tag{3.12}$$

and $\|A^m \hat{w}\|_2^2 = \tau(w (A^*)^m A^m w) = \tau((A^*)^m A^m w)$.

Note that $(A^*)^m A^m = R_m + T_m$ where

$$R_m := (m!)^2 \sum_{1 \leq q_1 < q_2 < \ldots < q_m} \alpha^{2q_1} \alpha^{2q_2} \ldots \alpha^{2q_m} Q_{q_1} \ldots Q_{q_m} \quad \text{and} \quad T_m = (A^*)^m A^m - R_m. \tag{3.13}$$

Since $VP = 0$, $VQ = V$, $V^*P = V^*$, $V^*Q = 0$ and $\tau(V) = \tau(V^*) = 0$ it is easy to see that for any word w, with letters in $\{P, Q\}$, then $\tau(T_m w) = 0$. Hence,

$$\|A^m \hat{w}\|_2^2 = \tau((A^*)^m A^m w) = \tau(R_m w).$$

Now we will proceed by induction, the case $w = 1$ is obvious. Assume that the statement is true for any word w of length $r \geq 1$. We will prove it for $P \otimes w$ and for $Q \otimes w$ and we will be done. Consider first the case $P \otimes w$. Then

$$\|A^m (P \hat{\otimes} w)\|_2^2 = \tau(R_m(P \otimes w))$$

and

$$\begin{aligned}
\tau(R_m(P \otimes w)) &= (m!)^2 \tau\left(\sum_{1 \leq q_1 < q_2 < \ldots < q_m} \alpha^{2q_1} \ldots \alpha^{2q_m} Q_{q_1} \ldots Q_{q_m} (P \otimes w)\right) \\
&= (m!)^2 \tau\left(\sum_{2 \leq q_1 < q_2 < \ldots < q_m} \alpha^{2q_1} \ldots \alpha^{2q_m} Q_{q_1} \ldots Q_{q_m} (P \otimes w)\right) \\
&= \frac{1}{2}\alpha^{2m} (m!)^2 \tau\left(\sum_{1 \leq q_1 < q_2 < \ldots < q_m} \alpha^{2q_1} \ldots \alpha^{2q_m} Q_{q_1} \ldots Q_{q_m} w\right) \\
&= \frac{1}{2}\alpha^{2m} \tau(R_m w).
\end{aligned}$$

Therefore, $\|A^m (P \hat{\otimes} w)\|_2^2 = \frac{1}{2}\alpha^{2m} \|A^m \hat{w}\|_2^2$ and hence

$$\lim_m \left(\frac{\|A^m (P \hat{\otimes} w)\|_2}{\|A^m \hat{1}\|_2}\right)^{1/m} = \alpha \cdot \lim_m \left(\frac{\|A^m \hat{w}\|_2}{\|A^m \hat{1}\|_2}\right)^{1/m}.$$

We are done with the case $P \otimes w$ by the induction hypothesis and the fact that the number of P's in $P \otimes w$ is the same as the number in w plus one. Let us consider the case $Q \otimes w$. First note that

$$\tau(R_m(1 \otimes w)) = \tau(R_m(P \otimes w)) + \tau(R_m(Q \otimes w)) = \frac{1}{2}\alpha^{2m}\tau(R_m w) + \tau(R_m(Q \otimes w))$$

and

$$\begin{aligned}\tau(R_m(1\otimes w)) &= (m!)^2\tau\Bigg(\sum_{1=q_1<q_2<\ldots<q_m}\alpha^2\alpha^{2q_2}\ldots\alpha^{2q_m}Q_1Q_{q_2}\ldots Q_{q_m}(1\otimes w)\Bigg)+\\ &+ (m!)^2\tau\Bigg(\sum_{2\le q_1<q_2<\ldots<q_m}\alpha^{2q_1}\alpha^{2q_2}\ldots\alpha^{2q_m}Q_{q_1}Q_{q_2}\ldots Q_{q_m}(1\otimes w)\Bigg).\end{aligned}$$

Therefore,

$$\tau(R_m(1\otimes w))=\frac{\alpha^{2m}m^2}{2}\tau(R_{m-1}w)+\alpha^{2m}\tau(R_mw).$$

Thus,

$$\tau(R_m(Q\otimes w))=\frac{\alpha^{2m}}{2}\Big(m^2\tau(R_{m-1}w)+\tau(R_mw)\Big)\ge\frac{\alpha^{2m}}{2}m^2\tau(R_{m-1}w).$$

Hence,

$$\begin{aligned}\liminf_m\left(\frac{\|A^m(Q\hat{\otimes}w)\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}} &= \liminf_m\left(\frac{\tau(R_m(Q\otimes w))}{\tau(R_m)}\right)^{\frac{1}{2m}}\\ &\ge \liminf_m\left(\frac{\alpha^{2m}m^2\tau(R_{m-1}w)}{2\ \tau(R_m)}\right)^{\frac{1}{2m}}.\end{aligned}$$

Since $\tau(R_m)=\frac{m^2\alpha^{2m}}{2(1-\alpha^{2m})}\tau(R_{m-1})$ we obtain that

$$\begin{aligned}\liminf_m\left(\frac{\|A^m(Q\hat{\otimes}w)\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}} &\ge \liminf_m(1-\alpha^{2m})^{\frac{1}{2m}}\left(\frac{\tau(R_{m-1}w)}{\tau(R_{m-1})}\right)^{\frac{1}{2m}}\\ &= \lim_m\left(\frac{\|A^m\hat{w}\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}}.\end{aligned}$$

Observing now that since $\tau(R_m(Q \otimes w)) = \frac{\alpha^{2m}}{2}\Big(m^2\tau(R_{m-1}w) + \tau(R_m w)\Big)$ then

$$\begin{aligned}
\|A^m(Q \hat{\otimes} w)\|_2^2 &= \frac{\alpha^{2m}}{2}\left(m^2\|A^{m-1}\hat{w}\|_2^2 + \|A^m\hat{w}\|_2^2\right) \\
&\leq \frac{\alpha^{2m}}{2}\left(m^2\|A^{m-1}\hat{w}\|_2^2 + \|A\|_\infty^2\|A^{m-1}\hat{w}\|_2^2\right) \\
&= \frac{\alpha^{2m}}{2}\|A^{m-1}\hat{w}\|_2^2\Big(m^2 + \|A\|_\infty^2\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\limsup_m \left(\frac{\|A^m(Q \hat{\otimes} w)\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}} &\leq \limsup_m \left((1-\alpha^{2m})\,\frac{m^2 + \|A\|_\infty^2}{m^2}\right)^{\frac{1}{2m}} \left(\frac{\|A^m\hat{w}\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}} \\
&= \lim_m \left(\frac{\|A^m\hat{w}\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}}.
\end{aligned}$$

Hence,

$$\lim_m \left(\frac{\|A^m(Q \hat{\otimes} w)\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}} = \lim_m \left(\frac{\|A^m\hat{w}\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}}$$

which concludes the proof. □

Note that if w is a tensor word as before then ww^* is a tensor word with letters in P and Q only. We define the symbol $\#_P(ww^*)$ as the number of P's in the word ww^*. Since $\|A^m\hat{w}\|_2^2 = \tau(w^*(A^*)^m A^m w) = \tau((A^*)^m A^m ww^*)$ then Proposition D.1 says that

$$\lim_m \left(\frac{\|A^m\hat{w}\|_2}{\|A^m\hat{1}\|_2}\right)^{\frac{1}{m}} = \alpha^{\#_P(ww^*)}. \tag{3.14}$$

Proposition D.2. *Let $n \geq 1$ and $\xi = \sum_{i=1}^n c_i w_i$ be a vector with $c_i \in \mathbb{C}$ and w_i tensor words of length r_i for $i = 1, \dots, n$. Then*

$$\liminf_m \left(\frac{\|A^m\xi\|_2}{\|A^m 1\|_2}\right)^{\frac{1}{m}} \geq \frac{1}{\sqrt{2}}\alpha^r \qquad \textit{where} \quad r = \max\{r_i \,:\, i = 1, \dots, n\}. \tag{3.15}$$

Proof. Let $\xi = \sum_{i=1}^{n} c_i w_i$ and $r = \max\{r_i \,:\, i = 1, \ldots, n\}$ then

$$\begin{aligned} A^m\xi &= m! \sum_{1 \le p_1 < \ldots < p_m} \alpha^{p_1} \ldots \alpha^{p_m} V_{p_1} \ldots V_{p_m}\xi \\ &= m! \sum_{J_m} \alpha^{p_1} \ldots \alpha^{p_m} V_{p_1} \ldots V_{p_m}\xi + m! \sum_{r+1 \le p_1 < \ldots < p_m} \alpha^{p_1} \ldots \alpha^{p_m} V_{p_1} \ldots V_{p_m}\xi \\ &= m! \sum_{J_m} \alpha^{p_1} \ldots \alpha^{p_m} V_{p_1} \ldots V_{p_m}\xi + m! \sum_{r+1 \le p_1 < \ldots < p_m} \alpha^{p_1} \ldots \alpha^{p_m} \xi V_{p_1} \ldots V_{p_m} \end{aligned}$$

where $J_m := \{1 \le p_1 < p_2 < \ldots < p_m \,:\, \text{such that exists } i \text{ so that } p_i \le r\}$.

It is easy to see that for $r + 1 \le p_1 < \ldots < p_m$ the vectors $\xi V_{p_1} \ldots V_{p_m}$ are pairwise orthogonal and are orthogonal to $A_r^{(m)}\xi := m! \sum_{J_m} \alpha^{p_1} \ldots \alpha^{p_m} V_{p_1} \ldots V_{p_m}\xi$. Note also that $\|\xi V_{p_1} \ldots V_{p_m}\|_2^2 = \frac{\|\xi\|_2^2}{2^m}$ for $r + 1 \le p_1 < \ldots < p_m$.

Therefore,

$$\begin{aligned} \|A^m\xi\|_2^2 &= \|A_r^{(m)}\xi\|_2^2 + \frac{\|\xi\|_2^2 (m!)^2}{2^m} \sum_{r+1 \le p_1 < \ldots < p_m} \alpha^{2p_1} \ldots \alpha^{2p_m} \\ &\ge \frac{\|\xi\|_2^2 (m!)^2 \alpha^{2rm}}{2^m} \sum_{1 \le p_1 < \ldots < p_m} \alpha^{2p_1} \ldots \alpha^{2p_m} \\ &= \frac{\|\xi\|_2^2}{2^m} \alpha^{2rm} \|A^m 1\|_2^2. \end{aligned}$$

Hence,

$$\liminf_m \left(\frac{\|A^m\xi\|}{\|A^m 1\|} \right)^{\frac{1}{m}} \ge \frac{1}{\sqrt{2}} \alpha^r \qquad \text{where} \quad r = \max\{r_i \,:\, i = 1, \ldots, n\}.$$

$\square$

Theorem D.3. *The operator A has trivial kernel and dense range.*

Proof. Since this operator lives in a finite factor it is enough to prove that $\mathrm{Ker}(A) = \{0\}$. Let us consider the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ and the operator $\tilde{A} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ given by

$$\tilde{A} = \begin{pmatrix} \alpha A & \alpha \\ 0 & \alpha A \end{pmatrix}.$$

Decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 := \{\hat{Q} \otimes \xi + \hat{V}^* \otimes \eta \ : \ \xi, \eta \in \mathcal{H}\}$ and $\mathcal{H}_2 := \{\hat{P} \otimes \xi + \hat{V} \otimes \eta \ : \ \xi, \eta \in \mathcal{H}\}$ and Q and P are the orthogonal projections onto these subspaces respectively. Since

$$PAP = \sum_{n=2}^{+\infty} \alpha^n P \otimes I^{\otimes(n-1)} \otimes V, \quad PAQ = \alpha V$$

$$QAQ = \sum_{n=2}^{+\infty} \alpha^n Q \otimes I^{\otimes(n-1)} \otimes V, \quad QAP = 0$$

then A has trivial kernel if and only if $\tilde{A}$ has trivial kernel. Moreover,

$$\gamma_n := \tau_{\mathcal{R}}(\mathrm{Ker}(A^n)) = \tau_{\mathcal{R}\otimes M_2(\mathbb{C})}(\mathrm{Ker}(\tilde{A}^n)).$$

It is easy to see that

$$\tilde{A}^n = \begin{pmatrix} \alpha^n A^n & n\alpha^n A^{n-1} \\ 0 & \alpha^n A^n \end{pmatrix}$$

and since

$$\begin{pmatrix} \alpha^n A^n & n\alpha^n A^{n-1} \\ 0 & \alpha^n A^n \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \alpha^n (A^n(\xi_1) + nA^{n-1}(\xi_2)) \\ \alpha^n A^n(\xi_2) \end{pmatrix}$$

we see that $\mathrm{Ker}(\tilde{A}^n) = \left\{(\xi, -\frac{1}{n}A(\xi) + \eta) \ : \ \xi \in \mathrm{Ker}(A^{n+1}),\ \eta \in \mathrm{Ker}(A^{n-1})\right\}$.

Hence

$$\tau_{\mathcal{R}\otimes M_2(\mathbb{C})}(\mathrm{Ker}(\tilde{A}^n)) = \frac{1}{2}\Big(\tau_{\mathcal{R}}(\mathrm{Ker}(A^{n+1})) + \tau_{\mathcal{R}}(\mathrm{Ker}(A^{n-1}))\Big).$$

Therefore, $\gamma_n = \frac{1}{2}(\gamma_{n+1} + \gamma_{n-1})$ which implies that $\gamma_n = n\gamma_1$. Therefore, $\gamma_1 = 0$ and thus $\mathrm{Ker}(A) = \{0\}$.

□

Theorem D.4. *Let $r > 0$ and $\{\gamma_n\}_{n=1}^{+\infty}$ be a sequence of positive numbers and A be as in (3.10). The subspace $\mathcal{H}_r(A)$ defined by $\mathcal{H}_r(A) := \overline{\mathcal{E}_r(A)}$ where*

$$\mathcal{E}_r(A) := \{\xi \in \mathcal{H} \,:\, \limsup_n \gamma_n \left\|A^n\xi\right\|_2^{\frac{1}{n}} \leq r\}$$

is either $\mathcal{H}$ or $\{0\}$.

Proof. Decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 := \{\hat{Q}\otimes\xi + \hat{V}^*\otimes\eta \;:\; \xi, \eta \in \mathcal{H}\}$ and $\mathcal{H}_2 := \{\hat{P}\otimes\xi + \hat{V}\otimes\eta \;:\; \xi, \eta \in \mathcal{H}\}$ as we did in Theorem D.3. Then the operator A can be represented as $A : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ with

$$A = \begin{pmatrix} \alpha A & \alpha \\ 0 & \alpha A \end{pmatrix} \tag{3.16}$$

and hence

$$A^n = \begin{pmatrix} \alpha^n A^n & n\alpha^n A^{n-1} \\ 0 & \alpha^n A^n \end{pmatrix}.$$

Therefore, under the canonical isomorphism of $\mathcal{R} \simeq M_2(\mathbb{C}) \otimes \mathcal{R}$ we see that the operator A is identified with $\alpha(P+Q)\otimes A + \alpha V \otimes 1$ and A^n is identified with $\alpha^n(P+Q)\otimes A^n + n\alpha^n V \otimes A^{n-1}$. The subspace $\mathcal{H}_r(A)$ is hyperinvariant, hence, affiliated to the von Neumann algebra $\mathcal{R}$. Let β be the trace of this subspace $\beta =$

$\tau(\mathcal{H}_r(A))$. Define the subspaces $E_1 := \{\hat{P} \otimes A(\xi) + \hat{V} \otimes A(\eta) \ : \ \xi, \eta \in \mathcal{E}_r(A)\}$, $E_2 := \{\hat{Q} \otimes A(\xi) + \hat{V}^* \otimes A(\eta) \ : \ \xi, \eta \in \mathcal{E}_r(A)\}$, $H_1 = \overline{E_1}$ and $H_2 = \overline{E_2}$. The subspaces H_1 and H_2 are affiliated to $\mathcal{R}$ and since the kernel of A is trivial $\tau(H_1) = \tau(H_2) = \frac{\beta}{2}$. It is clear that the subspaces H_1 and H_2 are orthogonal. Now we will prove that $E_1, E_2 \subset \mathcal{E}_r(A)$ and hence $\mathcal{H}_r(A) = H_1 \oplus H_2$. Let ξ and η be vectors in $\mathcal{E}_r(A)$ and $h = \hat{P} \otimes A(\xi) + \hat{V} \otimes A(\eta) \in E_1$ then

$$\begin{aligned}
\|A^n(h)\|_2 &= \|\alpha^n(P+Q) \otimes A^n + n\alpha^n V \otimes A^{n-1}(\hat{P} \otimes A(\xi) + \hat{V} \otimes A(\eta))\|_2 \\
&= \|\alpha^n(\hat{P} \otimes A^{n+1}(\xi) + \hat{V} \otimes A^{n+1}(\eta))\|_2 \\
&\leq 2\alpha^n \cdot \sup\{\|\hat{P} \otimes A^{n+1}(\xi)\|_2, \|\hat{V} \otimes A^{n+1}(\eta)\|_2\} \\
&\leq \sqrt{2}\alpha^n \|A\| \cdot \sup\{\|A^n(\xi)\|_2, \|A^n(\eta)\|_2\}.
\end{aligned}$$

Therefore,

$$\limsup_n \gamma_n \|A^n(\hat{P} \otimes A(\xi) + \hat{V} \otimes A(\eta))\|_2^{\frac{1}{n}} \leq \alpha\, r < r.$$

Thus, $E_1 \subset \mathcal{E}_r(A)$. Analogously, let ξ and η be vectors in $\mathcal{E}_r(A)$ and $h = \hat{Q} \otimes A(\xi) + \hat{V}^* \otimes A(\eta) \in E_2$ then

$$\begin{aligned}
\|A^n(h)\|_2 &= \|\alpha^n(P+Q) \otimes A^n + n\alpha^n V \otimes A^{n-1}(\hat{Q} \otimes A(\xi) + \hat{V}^* \otimes A(\eta))\|_2 \\
&= \alpha^n \|\hat{Q} \otimes A^{n+1}(\xi) + \hat{V}^* \otimes A^{n+1}(\eta) + n\hat{V} \otimes A^n(\xi) + n\hat{P} \otimes A^n(\eta)\|_2 \\
&\leq \sqrt{2}\alpha^n (n + \|A\|) \cdot \sup\{\|A^n(\xi)\|_2, \|A^n(\eta)\|_2\}.
\end{aligned}$$

Therefore,

$$\limsup_n \gamma_n \|A^n(\hat{Q} \otimes A(\xi) + \hat{V}^* \otimes A(\eta))\|_2^{\frac{1}{n}} \leq \alpha\, r < r.$$

Hence, $E_2 \subset \mathcal{E}_r(A)$ and therefore, $\mathcal{H}_r(A) = H_1 \oplus H_2$. Since $V^*(E_1) \subseteq E_2$ and $V^*(E_2) = \{0\}$ we see that $\mathcal{H}_r(A)$ is V^*–invariant.

Representing now our operator A as

$$A = \begin{pmatrix} \alpha^2 A & \alpha^2 & \alpha & 0 \\ 0 & \alpha^2 A & 0 & \alpha \\ 0 & 0 & \alpha^2 A & \alpha^2 \\ 0 & 0 & 0 & \alpha^2 A^2 \end{pmatrix}$$

it is not hard to see that

$$A^n = \begin{pmatrix} \alpha^{2n} A^n & \alpha^{2n} n A^{n-1} & \alpha^{2n-1} n A^{n-1} & \alpha^{2n-1} n(n-1) A^{n-2} \\ 0 & \alpha^{2n} A^n & 0 & \alpha^{2n-1} n A^{n-1} \\ 0 & 0 & \alpha^{2n} A^n & \alpha^{2n} n A^{n-1} \\ 0 & 0 & 0 & \alpha^{2n} A^n \end{pmatrix}.$$

Define the subspaces $E_{11} := \{\hat{P} \otimes \hat{P} \otimes A^2(\xi_1) + \hat{P} \otimes \hat{V} \otimes A^2(\xi_2) + \hat{V} \otimes \hat{P} \otimes A^2(\xi_3) + \hat{V} \otimes \hat{V} \otimes A^2(\xi_4) \ : \ \xi_1, \xi_2, \xi_3, \xi_4 \in \mathcal{E}_r(A)\}$, $E_{12} := \{\hat{P} \otimes \hat{Q} \otimes A^2(\xi_1) + \hat{P} \otimes \hat{V}^* \otimes A^2(\xi_2) + \hat{V} \otimes \hat{Q} \otimes A^2(\xi_3) + \hat{V} \otimes \hat{V}^* \otimes A^2(\xi_4) \ : \ \xi_1, \xi_2, \xi_3, \xi_4 \in \mathcal{E}_r(A)\}$, $E_{21} := \{\hat{Q} \otimes \hat{P} \otimes A^2(\xi_1) + \hat{Q} \otimes \hat{V} \otimes A^2(\xi_2) + \hat{V}^* \otimes \hat{P} \otimes A^2(\xi_3) + \hat{V}^* \otimes \hat{V} \otimes A^2(\xi_4) \ : \ \xi_1, \xi_2, \xi_3, \xi_4 \in \mathcal{E}_r(A)\}$, $E_{22} := \{\hat{Q} \otimes \hat{Q} \otimes A^2(\xi_1) + \hat{Q} \otimes \hat{V}^* \otimes A^2(\xi_2) + \hat{V}^* \otimes \hat{Q} \otimes A^2(\xi_3) + \hat{V}^* \otimes \hat{V}^* \otimes A^2(\xi_4) \ : \ \xi_1, \xi_2, \xi_3, \xi_4 \in \mathcal{E}_r(A)\}$ and $H_{ij} = \overline{E_{i,j}}$ for $i, j = 1, 2$. Using the same argument as before it is not hard to see that $\mathcal{H}_r(A) = H_{11} \oplus H_{12} \oplus H_{21} \oplus H_{22}$ and that $\mathcal{H}_r(A)$ is $I \otimes V^*$–invariant.

Analogously, given $n \geq 1$ and $i_1, \ldots, i_n \in \{1, 2\}$ we define $\Omega_{i_1,\ldots,i_n}$ the set of words of length n in the alphabet $\{\hat{P}, \hat{Q}, \hat{V}^*, \hat{V}\}$ given by

$$\Omega_{i_1,\ldots,i_n} = \{a_1 \otimes \ldots \otimes a_n \ : \ a_k \in \{\hat{P}, \hat{V}\} \text{ if } i_k = 1 \text{ and } a_k \in \{\hat{Q}, \hat{V}^*\} \text{ if } i_k = 2\}.$$

Let $E_{i_1,\ldots,i_n}$ and $H_{i_1,\ldots,i_n}$ be the subspaces defined by

$$E_{i_1,\ldots,i_n} := \text{span}\{w \otimes A^n(\xi) \;:\; w \in \Omega_{i_1,\ldots,i_n},\, \xi \in \mathcal{E}_r(A)\} \quad \text{and} \quad H_{i_1,\ldots,i_n} = \overline{E_{i_1,\ldots,i_n}}.$$

We can see that

$$\mathcal{H}_r(A) = \bigoplus_{\{i_1,\ldots,i_n\}\in\{1,2\}^n} H_{i_1,\ldots,i_n}$$

and therefore, $\mathcal{H}_r(A)$ is $I^{\otimes(n-1)} \otimes V^*$–invariant. Since n is arbitrary we see that $\mathcal{H}_r(A)$ is A^*–invariant. Thus, the subspace $\mathcal{H}_r(A)$ is A–invariant and A^*–invariant and hence trivial.

□

Question 4. Does the operator A have non-trivial hyperinvariant subspaces?

E. Distribution of Re(A) and Im(A)

In this section we will prove that given $\{c_n\}_n \in l_1(\mathbb{N})$ with $c_n \geq 0$, then Re(A) and Im(A) have the same distribution and we will describe its moments. Let $X = A + A^*$ and $Y = A - A^*$, then $\text{Re}(A) = \frac{1}{2}X$ and $\text{Im}(A) = \frac{1}{2i}Y$. Thus,

$$X = \sum_{n=1}^{+\infty} c_n R_n \quad \text{where} \quad R_n = I^{\otimes(n-1)} \otimes R \quad \text{with} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.17}$$

$$Y = \sum_{n=1}^{+\infty} c_n T_n \quad \text{where} \quad T_n = I^{\otimes(n-1)} \otimes T \quad \text{with} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.18}$$

Note that $R^2 = 1$, $T^2 = -1$ and $\tau(R) = \tau(T) = 0$. From this observation, it is clear that $\tau(X^{2p+1}) = \tau(Y^{2p+1}) = 0$. Now we will find a combinatorial formula for $\tau(X^{2p})$

and prove that $\tau(Y^{2p}) = (-1)^p \tau(X^{2p})$ for $p \geq 0$. But first we will fix some notation. Given $p \geq 0$, $1 \leq k \leq p$ and $n_1 \geq n_2 \geq \ldots \geq n_k$ such that $\sum_{i=1}^k n_i = p$ we will denote by $\gamma(p\,; n_1, n_2, \ldots, n_k)$ the number of partitions of the set $\{1, 2, \ldots 2p\}$ in exactly k blocks $B_1, B_2, \ldots B_k$ with $\#B_i = 2n_i$. In the following Lemma we will prove some properties of these numbers that will permit us to compute them recursively.

Lemma 5. *Let $p \geq 1$ and $n_1 \geq n_2 \geq \ldots \geq n_k$ be such that $\sum_{i=1}^k n_i = p$. Let $\gamma(p\,; n_1, \ldots, n_k)$ be as before, then*

1. $\gamma(p\,; 1, 1, \ldots, 1) = (2p-1)(2p-3)\cdots 1$
2. $\gamma(p\,; p) = 1$
3. *If $n_1 > n_2$ then $\gamma(p\,; n_1, \ldots, n_k) = \binom{2p}{2n_1} \cdot \gamma(p - n_1\,; n_2, \ldots, n_k)$*
4. *If exists $r < k$ such that $n_1 = n_2 = \ldots = n_r$ and $n_1 > n_{r+1}$ then*
$$\gamma(p\,; n_1, n_2, \ldots, n_k) = \frac{1}{r!}\binom{2p}{2n_1}\cdots\binom{2p-2(r-1)n_1}{2n_1}\cdot\gamma(p - rn_1\,; n_{r+1}, \ldots, n_k).$$

Proof. (1) Each element in $\{1, 2, \ldots, 2p\}$ has to be paired with another. For the first element we have $(2p-1)$ possibilities. Now we remove these two elements and we have $2p-2$ remaining. Each remaining element has to be paired with another, having $(2p-3)$ possibilities. Continuing with this process we get (1). (2) is trivial. (3) In this case, we have only one block of size $2n_1$ and we have exactly $\binom{2p}{2n_1}$ possible different blocks like this. We remove this block and we have $2p-2n_1$ elements and we continue with our partition process to get (3). (4) is similar to (3). $\square$

Given $p \geq 0$ we have that

$$X^{2p} = \sum_{1 \leq i_1, j_1, \ldots, i_p, j_p} c_{i_1} c_{j_1} \ldots c_{i_p} c_{j_p} \, R_{i_1} R_{j_1} \ldots R_{i_p} R_{j_p}$$

and using that $R^2 = 1$ and $\tau(R) = 0$ it is not difficult to see that

$$\tau(X^{2p}) = \sum_{k=1}^{p} \left\{ \sum_{(n_1, n_2, \ldots, n_k)} \left(\gamma(p\,; n_1, n_2, \ldots, n_k) \cdot \sum_{(p_1, p_2, \ldots, p_k)} c_{p_1}^{2n_1} c_{p_2}^{2n_2} \ldots c_{p_k}^{2n_k} \right) \right\} \quad (3.19)$$

where the second sum runs over $n_1 \geq n_2 \geq \ldots \geq n_k$ such that $\sum_{i=1}^{k} n_i = p$ and the last one over all the possible $1 \leq p_1, \ldots, p_k < +\infty$ with $p_i \neq p_j$ if $i \neq j$.

Analogously,

$$Y^{2p} = \sum_{1 \leq i_1, j_1, \ldots, i_p, j_p} c_{i_1} c_{j_1} \ldots c_{i_p} c_{j_p} \, T_{i_1} T_{j_1} \ldots T_{i_p} T_{j_p}$$

and using that $\tau(T^{2n}) = (-1)^n$ we get

$$\tau(Y^{2p}) = (-1)^p \sum_{k=1}^{p} \left\{ \sum_{(n_1, n_2, \ldots, n_k)} \left(\gamma(p\,; n_1, n_2, \ldots, n_k) \cdot \sum_{(p_1, p_2, \ldots, p_k)} c_{p_1}^{2n_1} c_{p_2}^{2n_2} \ldots c_{p_k}^{2n_k} \right) \right\}.$$

Hence, $\tau(X^{2p+1}) = \tau(Y^{2p+1}) = 0$ and $\tau(Y^{2p}) = (-1)^p \tau(X^{2p})$. Since $\mathrm{Re}(A) = \frac{1}{2}X$ and $\mathrm{Im}(A) = \frac{1}{2i}Y$ we see that $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ have the same distribution. More precisely, we can state the following Proposition.

Proposition E.1. *Let A be as before and let $a = \mathrm{Re}(A)$ and $b = \mathrm{Im}(A)$. Then*

$$\tau(a^n) = \tau(b^n) = \begin{cases} 0 & \text{if } \; n = 2p+1 \\ (\frac{1}{2})^{2p} \tau(X^{2p}) & \text{if } \; n = 2p \end{cases}$$

where $\tau(X^{2p})$ is as in equation (3.19).

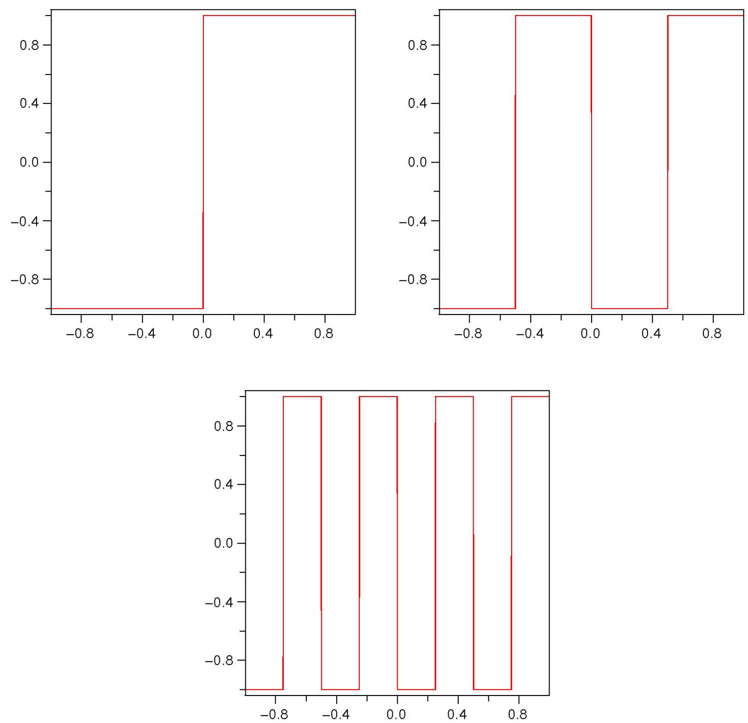

Fig. 4. Functions $f_1(x)$, $f_2(x)$ and $f_3(x)$

Another way of looking at the operator X,

$$X = \sum_{n=1}^{+\infty} c_n \, R_n \quad \text{where} \quad R_n = I^{\otimes(n-1)} \otimes R \quad \text{with} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.20}$$

is as a measurable function in $[-1,1]$. The operators $\{R_n\}_n$ are selfadjoint and commute with each other. Therefore, we can think them as independent random variables in $[-1,1]$. Moreover, if we think R_n as a function f_n in $[-1,1]$ then these functions satisfy that $\tau(R_n) = \int_{-1}^{1} f_n(x)dx = 0$ and $f_n^2(x) = 1$ and we can picture them as,

$$f_1(x) := \begin{cases} -1 & \text{if} \quad -1 < x < 0 \\ 1 & \text{if} \qquad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and $f_2(x) = f_1(2x+1) + f_1(2x-1)$ and in general,

$$f_{n+1}(x) = f_n(2x+1) + f_n(2x-1) \quad \text{for } x \in [-1,1] \text{ and } n \geq 1.$$

The functions f_1, f_2 and f_3 can be seen in Figure 4.

Hence, we can represent the operator X by the measurable function $f(x)$, given by $f(x) = \sum_{n=1}^{+\infty} c_n \, f_n(x)$ and

$$\tau(X^n) = \int_{-1}^{1} f(x)^n \, dx \quad \text{for all} \quad n \geq 0. \tag{3.21}$$

Note that in the case $c_n = (\frac{1}{2})^n$ we get that $f(x) = x$ on $[-1,1]$. Hence, $\tau(X^{2p}) = \int_{-1}^{1} x^{2p} \, dx = \frac{2}{2p+1}$ and therefore

$$\tau(a^n) = \tau(b^n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ (\frac{1}{2})^n \frac{2}{n+1} & \text{if } n \text{ even.} \end{cases}$$

Remark E.2. The spectrum of the operator X, $\sigma(X)$, is the image of the function f. Therefore, if $c_1 > \sum_{n=2}^{+\infty} c_n$ then $0 \notin \sigma(X)$ and the operator X is invertible and so a and b. This is the case, for example, of $c_n = \alpha^n$ when $0 < \alpha < \frac{1}{2}$ (see Figure 5). Note also that $\sigma(a) = \sigma(b) \subseteq [-s/2, s/2]$ where $s = \sum_{n=1}^{+\infty} c_n$.

In probability theory, the characteristic function of any random variable completely defines its probability distribution. On the real line it is given by the following

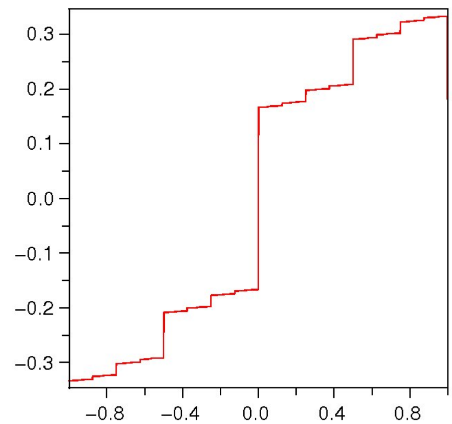

Fig. 5. Function $\sum_{k=1}^{10} \left(\frac{1}{4}\right)^k f_k(x)$

formula, where Z is any random variable with the distribution in question:

$$\varphi_Z(t) := \mathbb{E}(e^{itZ})$$

where t is a real number, i is the imaginary unit, and $\mathbb{E}$ denotes the expected value. Characteristic functions are particularly useful for dealing with functions of independent random variables. In particular, if Z_1 and Z_2 are independent random variables then $\varphi_{Z_1+Z_2}(t) = \varphi_{Z_1}(t)\varphi_{Z_2}(t)$. Characteristic functions can also be used to find moments of random variables. Provided that n-th moment exists, characteristic function can be differentiated n times and the following formula holds

$$\mathbb{E}(Z^n) = \left(\frac{1}{i}\right)^n \left[\frac{d^n}{dt^n}\varphi_Z(t)\right]_{t=0}. \tag{3.22}$$

We will compute the characteristic function of $X_\alpha := \sum_{n=1}^{+\infty} \alpha^n R_n$. For each $n \geq 1$,

$\tau(R_n^k) = 0$ if k is odd and 1 if k is even. Therefore,

$$\varphi_{\alpha^n R_n}(t) = \sum_{k=0}^{+\infty} (-1)^k \frac{t^{2k}}{(2k)!} \alpha^{2nk} = \cos(\alpha^n t)$$

hence

$$\varphi_{X_\alpha}(t) = \prod_{n=1}^{+\infty} \cos(\alpha^n t).$$

Then we can use (3.22) and the last equation to compute the even moments of X_α. For example, using this formula we can see that $\tau(X_\alpha^2) = \frac{\alpha^2}{1-\alpha^2}$.

Proposition E.3. *The self–adjoint operators a and b do not commute but $\tau(a^n b^m) = \tau(a^n)\tau(b^m) = \tau(a^n)\tau(a^m)$ for all n and m.*

Proof. The last equality is trivial since a and b have the same distribution. To prove the first equality it is enough to prove that $\tau(X^n Y^m) = \tau(X^n)\tau(Y^m)$ for all n and m. Since,

$$X^n = \sum_{1 \le l_1, \ldots, l_n} c_{l_1} \ldots c_{l_n} R_{l_1} \ldots R_{l_n} \quad \text{and} \quad Y^m = \sum_{1 \le k_1, \ldots, k_m} c_{k_1} \ldots c_{k_m} T_{k_1} \ldots T_{k_m},$$

to prove $\tau(X^n Y^m) = \tau(X^n)\tau(Y^m)$ it is enough to prove that

$$\tau(R_{l_1} \ldots R_{l_n} T_{k_1} \ldots T_{k_m}) = \tau(R_{l_1} \ldots R_{l_n})\tau(T_{k_1} \ldots T_{k_m})$$

and this is true since $\tau(R^l T^h) = \tau(R^l)\tau(T^h)$ for all l and h. $\square$

The family of operators $\{R_n\}_{n=1}^{+\infty}$ is a commuting family of selfadjoint operators. If we denote by

$$\mathbb{N}_2(\mathbb{C}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subset M_2(\mathbb{C}),$$

then it is not difficult to see that

$$\mathcal{A} := W^*(\{R_n\}_{n=1}^{+\infty}) = \overline{\left(\bigotimes_{n=1}^{+\infty} \mathbb{N}_2(\mathbb{C})\right)}^{\mathrm{WOT}} \tag{3.23}$$

which is a Cartan masa in the hyperfinite II_1–factor $\mathcal{R}$. It is clear that $W^*(a) = W^*(X) \subseteq \mathcal{A}$. A natural question is when is $W^*(a) = \mathcal{A}$? Is $W^*(a)$ always a diffuse abelian subalgebra of $\mathcal{A}$?

Remark E.4. Consider the projections

$$p_n := \frac{1}{2}\, I^{\otimes(n-1)} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad q_n := \frac{1}{2}\, I^{\otimes(n-1)} \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then $R_n = p_n - q_n$ and $X = \sum_{n=1}^{+\infty} c_n R_n$. If for all $n \geq 1$, $c_n \geq \sum_{k=n+1}^{+\infty} c_k$ then the function f is increasing and we can recover p_n and q_n as spectral projections of X and hence $W^*(X) = \mathcal{A}$. This is the case, for example, of $c_n = \alpha^n$ when $0 < \alpha \leq \frac{1}{2}$.

The following Theorem answers the questions asked before.

Theorem E.5. *Let $0 < \alpha < 1$ and $X_\alpha = \sum_{n=1}^{+\infty} \alpha^n R_n$. Then the abelian algebra $W^*(X_\alpha)$ is always diffuse. If $0 < \alpha \leq \frac{1}{2}$ then $W^*(X_\alpha)$ is the Cartan masa $\mathcal{A}$ as in (3.23). However, if there exist a polynomial $p(x) = a_1 x^{n_1} + a_2 x^{n_2} + \ldots + a_k x^{n_k}$ with coefficients $a_i \in \{1, -1\}$ such that $p(\alpha) = 0$ (for example $\alpha = \frac{\sqrt{5}-1}{2}$) then $W^*(X_\alpha) \subsetneq \mathcal{A}$.*

Proof. The case $0 < \alpha \leq \frac{1}{2}$ was discussed in Remark E.4.

Consider the Bernoulli space $(M, \mu) = \left(\prod_{n=1}^{+\infty} \{1, -1\}, (\frac{1}{2}(\delta_1 + \delta_{-1}))^{\otimes \mathbb{N}}\right)$. We can model the selfadjoint element X_α as the measurable function $g_\alpha : (M, \mu) \to \mathbb{R}$ defined by $g_\alpha(\{\epsilon_n\}) = \sum_{n=1}^{+\infty} \epsilon_n \alpha^n$. In order to prove that $W^*(X_\alpha)$ is diffuse it is equivalent to prove that $W^*(g_\alpha) \subseteq L^\infty(M,\mu)$ is diffuse. Assume this is not true, hence there exists $\beta \in \mathbb{R}$ such that $\mu(g_\alpha^{-1}(\{\beta\})) = \gamma > 0$. Denote by E the set $E := g_\alpha^{-1}(\{\beta\})$. For each $n \geq 1$, we define

$$E_n^+ := \{x = \{\epsilon_k\}_k \in E \,:\, \epsilon_n = 1\} \quad \text{and} \quad E_n^- := \{x = \{\epsilon_k\}_k \in E \,:\, \epsilon_n = -1\}.$$

It is clear that for each $n \geq 1$ the sets E_n^+ and E_n^+ are measurable sets, $E_n^+ \cup E_n^- = E$ and $E_n^+ \cap E_n^- = \emptyset$. Hence, for each n, either E_n^+ or E_n^- has measure bigger or equal than $\gamma/2$. If $\mu(E_n^+) > \gamma/2$ then define $F_n := \{(\epsilon_1, \ldots, \epsilon_{n-1}, -1, \epsilon_{n+1}, \ldots) \,:\, \{\epsilon_k\} \in E_n^+\}$ and if $\mu(E_n^-) \geq \gamma/2$ then define $F_n := \{(\epsilon_1, \ldots, \epsilon_{n-1}, 1, \epsilon_{n+1}, \ldots) \,:\, \{\epsilon_k\} \in E_n^-\}$. By definition, $\mu(F_n) \geq \gamma/2$ and if $x \in F_n$ then $g_\alpha(x)$ is either $\beta + 2\alpha^n$ or $\beta - 2\alpha^n$. Assume there exists $x \in F_n \cap F_m$ then $\beta \pm 2\alpha^m = \beta \pm 2\alpha^n$ and hence $\alpha^n = \pm\alpha^m$ then $n = m$. Therefore, we constructed a sequence of disjoint measurable sets $\{F_n\}_n$ each of measure $\mu(F_n) \geq \gamma/2$ which is clearly impossible. Therefore, $W^*(X_\alpha)$ is diffuse.

Let $p(x)$ be a polynomial $p(x) = a_1 x^{n_1} + a_2 x^{n_2} + \ldots + a_k x^{n_k}$ with coefficients $a_i \in \{1, -1\}$ and $\alpha \in (0,1)$ be such that $p(\alpha) = 0$. (Note that there are infinitely many countable α in $(\frac{1}{2}, 1)$ with this property but none in $(0, \frac{1}{2}])$. Define the cylindrical sets

$$G_1 := \{\{\epsilon_n\}_n \,:\, \epsilon_{n_i} = a_i,\, i = 1, \ldots, k\} \text{ and } G_2 := \{\{\epsilon_n\}_n \,:\, \epsilon_{n_i} = -a_i,\, i = 1, \ldots, k\}$$

it is clear that $G_1 \cap G_2 = \emptyset$ and that $\mu(G_1) = \mu(G_2) = \frac{1}{2^k}$. The function g_α does not separates this two cylindrical sets and hence $W^*(g_\alpha) \neq L^\infty(M,\mu)$. Therefore, $W^*(X_\alpha) \neq \mathcal{A}$. $\square$

F. Moments of A^*A

In this section we will give a combinatorial formula describing the moments of A^*A. Let $\{c_n\}_n \in l_1(\mathbb{N})$ and $A = \sum_{n=1}^{+\infty} c_n V_n$ where $V_n = I^{\otimes(n-1)} \otimes V$. Then given $p \geq 1$ we see that

$$(A^*A)^p = \sum_{1\leq n_1,m_1,\ldots,n_p,m_p} \overline{c_{n_1}} c_{m_1} \overline{c_{n_2}} c_{m_2} \ldots \overline{c_{n_p}} c_{m_p} \; V_{n_1}^* V_{m_1} V_{n_2}^* V_{m_2} \ldots V_{n_p}^* V_{m_p}.$$

For $p \geq 1$ consider p elements of color red and p of color white. Order them linearly and alternating the colors. Let $1 \leq k \leq p$ and $n_1 \geq n_2 \geq \ldots \geq n_k$ be such that $\sum_{i=1}^{k} n_i = p$. We define $\alpha(p\,; n_1, \ldots, n_k)$ the number of partitions of these $2p$ elements in k blocks $B_1, B_2, \ldots, B_k$ of size $2n_1, 2n_2, \ldots, 2n_k$ such that each block contains the same amount of element of each color and are alternating, i.e.: if we look at the elements of one block the colors are alternating.

Example F.1. For the case $p = 2$ we have that $\alpha(2\,;1,1) = 2$, $\alpha(2\,;2) = 1$. For $p = 3$ we have $\alpha(3\,;1,1,1) = 6$, $\alpha(3\,;2,1) = 6$ and $\alpha(3\,;3) = 1$. Some of the possibles partitions for $p = 3$ and $p = 4$ can be seen in Figure 6 and Figure 7.

Given $1 \leq n \leq p$ let us denote $\beta(p\,;n)$ the number of blocks of size $2n$ satisfying the alternating condition. We first choose the n elements of red color which will be located at the positions $1 \leq r_1 < r_2 < \ldots < r_n \leq 2p - 1$ (note that the red elements are located at the odd integers while the white at the even). Then we choose the n elements of white color. In order to satisfy the alternating condition, the positions $\{w_i\}_{i=1}^{n}$ of the white elements have to satisfy either

$$1 \leq r_1 < w_1 < r_2 < w_2 < \ldots < r_n < w_n \leq 2p$$

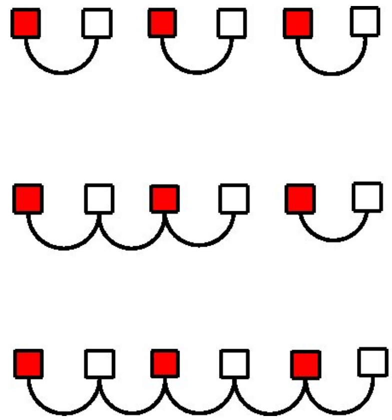

Fig. 6. An element of $\alpha(3\,;1,1,1)$ an element of $\alpha(3\,;2,1)$ and the only element of $\alpha(3\,;3)$

or

$$1 \leq w_1 < r_1 < w_2 < r_2 < \ldots < w_n < r_n \leq 2p-1.$$

If $r_1 = 1$ then we have

$$\frac{1}{2^n}(r_2-1)\cdot(r_3-r_2)\ldots(r_n-r_{n-1})\cdot(2p+1-r_n)$$

possibilities to choose the white elements. If $r_1 > 1$ we have the option of either start with white or with red. Starting with white we have

$$\frac{1}{2^n}(r_1-1)(r_2-r_1)\cdot(r_3-r_2)\ldots(r_n-r_{n-1})$$

and starting with red we have

$$\frac{1}{2^n}(r_2-r_1)\cdot(r_3-r_2)\ldots(r_n-r_{n-1})\cdot(2p+1-r_n).$$

Then

$$\beta(p\,;n) = \frac{1}{2^n}\Bigg(\sum_{1\le r_1<\ldots<r_n}(r_2-r_1)\cdot(r_3-r_2)\ldots(r_n-r_{n-1})\cdot(2p+1-r_n)$$
$$+\sum_{2\le r_1<\ldots<r_n}(r_1-1)(r_2-r_1)\cdot(r_3-r_2)\ldots(r_n-r_{n-1})\Bigg).$$

Note that $\beta(p\,;p-1)=2\binom{p}{p-1}$. The next Lemma provides us with some information about the combinatorial numbers $\alpha(p\,;n_1,\ldots,n_k)$.

Lemma 6. *Let $p\ge 1$ and $n_1\ge n_2\ge\ldots\ge n_k$ such that $\sum_{i=1}^k n_i=p$ let $\alpha(p\,;n_1,\ldots,n_k)$ then*

1. $\alpha(p\,;1,1,\ldots,1)=p!$
2. $\alpha(p\,;p)=1$
3. $\alpha(p\,;p-n,1,\ldots,1)=\beta(p\,;p-n)\cdot n!$

Proposition F.2. *Let $p\ge 1$ then*

$$\tau((A^*A)^p)=\sum_{k=1}^{p}\left[\frac{1}{2^k}\cdot\sum_{(n_1,\ldots,n_k)}\left(\alpha(p\,;n_1,\ldots,n_k)\cdot\sum_{p_1,\ldots,p_k}|c_{p_1}|^{2n_1}|c_{p_2}|^{2n_2}\ldots|c_{p_k}|^{2n_k}\right)\right]$$

where the second sum runs over all the k–tuples such that $n_1\ge n_2\ge\ldots\ge n_k$ with $\sum_{i=1}^k n_i=p$, and the last one over all the possible $1\le p_1,\ldots,p_k<+\infty$ with $p_i\ne p_j$ if $i\ne j$.

Proof. We have to observe that

$$(A^*A)^p=\sum_{1\le n_1,m_1,\ldots,n_p,m_p}\overline{c_{n_1}}c_{m_1}\overline{c_{n_2}}c_{m_2}\ldots\overline{c_{n_p}}c_{m_p}\;V^*_{n_1}V_{m_1}V^*_{n_2}V_{m_2}\ldots V^*_{n_p}V_{m_p}$$

and since $\tau(V)=\tau(V^*)=0$, $V^2=V^{*2}=0$, $VV^*=P$ and $V^*V=Q$ then $\tau(V^*_{n_1}V_{m_1}V^*_{n_2}V_{m_2}\ldots V^*_{n_p}V_{m_p})$ is going to be nonzero if all the V's are paired with V^*'s

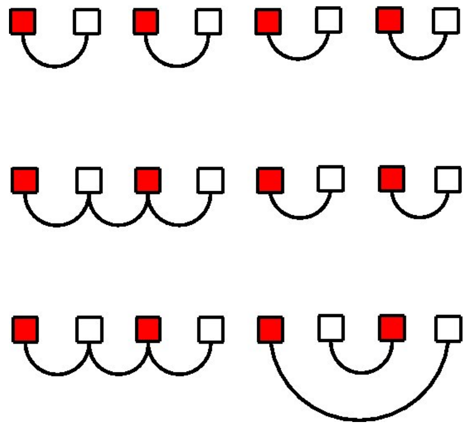

Fig. 7. An element of $\alpha(4\,;1,1,1,1)$ an two elements of $\alpha(4\,;2,1,1)$

in an alternating way. Using the definition of the numbers $\alpha(p\,;n_1,\ldots,n_k)$ it is not difficult to see that the formula in the Proposition follows.

$\square$

Question 7. Is there a nice formula, or recursive description of the numbers $\alpha(p\,; n_1, \ldots, n_k)$? If we fix k, can we at least compute recursively

$$s_p(k) := \sum_{(n_1, n_2, \ldots, n_k)} \alpha(p\,; n_1, \ldots, n_k) \quad \text{where} \quad n_1 \geq \ldots \geq n_k \quad \text{with} \quad \sum_{i=1}^{k} n_i = p \ ?$$

p	$\alpha(p\,; n_1, \ldots, n_k)$	$s_p(k)$
$p = 1$	$\alpha(1; 1) = 1$	$s_1(1) = 1$
$p = 2$	$\alpha(2; 2) = 1$	$s_2(1) = 1$
	$\alpha(2; 1, 1) = 2$	$s_2(2) = 2$
$p = 3$	$\alpha(3; 3) = 1$	$s_3(1) = 1$
	$\alpha(3; 2, 1) = 6$	$s_3(2) = 6$
	$\alpha(3; 1, 1, 1) = 6$	$s_3(3) = 6$
$p = 4$	$\alpha(4; 4) = 1$	$s_4(1) = 1$
	$\alpha(4; 3, 1) = 8$	$s_4(2) = 14$
	$\alpha(4; 2, 2) = 6$	$s_4(3) = 40$
	$\alpha(4; 2, 1, 1) = 40$	$s_4(4) = 24$
	$\alpha(4; 1, 1, 1, 1) = 24$	

CHAPTER IV

LIMITS LAWS FOR GEOMETRIC MEANS OF FREE RANDOM VARIABLES

A. Introduction

Denote by $\mathcal{M}$ the family of all compactly supported probability measures defined in the real line $\mathbb{R}$. We denote by $\mathcal{M}_+$ the set of all measures in $\mathcal{M}$ which are supported on $[0, \infty)$. On the set $\mathcal{M}$ there are defined two associative composition laws denoted by $*$ and $\boxplus$. The measure $\mu * \nu$ is the classical convolution of μ and ν. In probabilistic terms, $\mu * \nu$ is the probability distribution of $X + Y$, where X and Y are commuting independent random variables with distributions μ and ν, respectively. The measure $\mu \boxplus \nu$ is the free additive convolution of μ and ν introduced by Voiculescu [51]. Thus, $\mu \boxplus \nu$ is the probability distribution of $X + Y$, where X and Y are free random variables with distribution μ and ν, respectively.

There is a free analogue of multiplicative convolution also. More precisely, if μ and ν are measures in $\mathcal{M}_+$ we can define $\mu \boxtimes \nu$ the multiplicative free convolution by the probability distribution of $X^{1/2} Y X^{1/2}$, where X and Y are free random variables with distribution μ and ν, respectively.

In this paper we prove a multiplicative version of the Free Central Limit Theorem. More precisely, let $\{T_k\}_{k=1}^{\infty}$ be a family of $*$–free identically distributed operators in a finite von Neumann algebra. Let $B_n := T_1^* T_2^* \ldots T_n^* T_n \ldots T_2 T_1$, then B_n is a positive operator and $B_n^{1/2n}$ converges in distribution to an operator Λ. We completely determine the probability distribution ν of Λ from the probability distribution of $|T|^2$.

Our first remark is that it is enough to restrict ourselves to positive operators. In other words, let $a_k = |T_k|$ then $B_n = T_1^* T_2^* \ldots T_n^* T_n \ldots T_2 T_1$ has the same distribution as $b_n = a_1 a_2 \ldots a_n^2 \ldots a_2 a_1$ for all $n \geq 1$. Hence, to prove that $B_n^{1/2n}$ converges in distribution it is enough to prove that $b_n^{1/2n}$ converges in distribution.

Our main result is the following, let μ be the probability distribution of $|T_k|^2$ then

$$B_n^{1/2n} \longrightarrow \Lambda \quad \text{in distribution.}$$

Let ν be the probability distribution of Λ, then

$$\nu = \beta\delta_0 + \sigma \qquad \text{with} \qquad d\sigma = f(t)\,\mathbf{1}_{\left(\||T_1|^{-1}\|_2^{-1},\||T_1|\|_2\right]}(t)\,dt \tag{4.1}$$

where $\beta = \mu(\{0\})$, $f(t) = \left(F_\mu^{<-1>}\right)'(t)$ and $F_\mu(t) = S_\mu(t-1)^{-1/2}$ ($F_\mu^{<-1>}$ is the inverse with respect to composition of F_μ).

This gives us, naturally, a map

$$\mathcal{G} : \mathcal{M}_+ \to \mathcal{M}_+ \quad \text{with} \quad \mu \mapsto \mathcal{G}(\mu) = \nu.$$

The measure $\mathcal{G}(\mu)$ is a compactly supported positive measure with at most one atom at zero and $\mathcal{G}(\mu)(\{0\}) = \mu(\{0\})$.

We would like to mention that Vladislav Kargin in Theorem 1 of [29] proved an estimate in the norm of the positive operators b_n. More precisely, he proved that if $\tau(a_1^2) = 1$ there exists a positive constant $K > 0$ such that

$$\sqrt{n}\,\sigma(a_1^2) \le \|b_n\| \le K\,n\,\|a_1^2\|$$

where $\sigma^2(x) = \tau(x^2) - \tau(x)^2$.

It is interesting to compare this result with the analogous result in the classical case. Let $\{a_k\}_{k=1}^\infty$ be independent positive identically distributed commutative random variables with distribution μ. Applying the Law of the Large Numbers to the random variables $\log(a_k)$, in case $\log(a_k)$ is integrable, or applying Theorem 5.4 in

[13] in the general case, we obtain that

$$\Big(a_1 a_2 \ldots a_n\Big)^{1/n} \longrightarrow e^{\tau(\log(a_1))} \in [0, \infty)$$

where the convergence is pointwise.

The Lyapunov exponents of a sequence of random matrices was investigated in the pioneering paper of Furstenberg and Kesten [14] and by Oseledec in [36]. Ruelle [39] developed the theory of Lyapunov exponents for random compact linear operators acting on a Hilbert space. Newman in [33] and [34] and later Isopi and Newman in [21] studied Lyapunov exponents for random $N \times N$ matrices as $N \to \infty$. Later on, Vladislav Kargin [30] investigated how the concept of Lyapunov exponents can be extended to free linear operators (see [30] for a more detailed exposition).

In our case, given $\{a_k\}_{k=1}^{\infty}$ be free positive identically distributed random variables. Let μ be the spectral probability distribution of a_k^2 and assume that $\mu(\{0\}) = 0$. Then

$$\Big(a_1 a_2 \ldots a_n^2 \ldots a_2 a_1\Big)^{1/2n} \longrightarrow \Lambda$$

where Λ is a positive operator. The probability distribution of the Lyapunov exponents associated to the sequence $\{a_k\}_{k=1}^{\infty}$, is the spectral probability distribution γ of the selfadjoint operator $L := \ln(\Lambda)$. Moreover, γ is absolutely continuous with respect to Lebesgue measure and has Radon–Nikodym derivative given by

$$d\gamma(t) = e^t f(e^t)\, \mathbf{1}_{\left(\ln \|a_1^{-1}\|_2^{-1},\, \ln \|a_1\|_2\right]}(t)\, dt$$

where the function $f(t)$ is as in equation (4.1).

Now we will describe the content of this paper. In section §2, we recall some preliminaries as well as some known results and fix the notation. In section §3, we prove our main Theorem and study how the map $\mathcal{G}$ behaves with respect to additive and multiplicative free convolution. In section §4, we present some examples. Finally, in section §5, we derive the probability distribution of the Lyapunov exponents of the sequence $\{a_k\}_{k=1}^{\infty}$.

Acknowledgment: I thank my advisor, Ken Dykema, for many helpful discussions and comments.

B. Preliminaries and Notation

We begin with an analytic method for the calculation of multiplicative free convolution discovered by Voiculescu. Denote $\mathbb{C}$ the complex plane and set $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $\mathbb{C}^- = -\mathbb{C}^+$. For a measure $\nu \in \mathcal{M}_+ \setminus \{\delta_0\}$ one defines the analytic function ψ_ν by

$$\psi_\nu(z) = \int_0^\infty \frac{zt}{1-zt}\, d\nu(t)$$

for $z \in \mathbb{C} \setminus [0,\infty)$. The measure ν is completely determined by ψ_ν. The function ψ_ν is univalent in the half-plane $i\mathbb{C}^+$, and $\psi_\nu(i\mathbb{C}^+)$ is a region contained in the circle with center at $-1/2$ and radius $1/2$. Moreover, $\psi_\nu(i\mathbb{C}^+) \cap (-\infty, 0] = (\beta - 1, 0)$, where $\beta = \nu(\{0\})$. If we set $\Omega_\nu = \psi_\nu(i\mathbb{C}^+)$, the function ψ_ν has an inverse with respect to composition

$$\chi_\nu : \Omega_\nu \to i\mathbb{C}^+.$$

Finally, define the S–transform of ν to be

$$S_\nu(z) = \frac{1+z}{z}\chi_\nu(z)\ , \qquad z \in \Omega_\nu.$$

See [4] for a more detailed exposition. The following is a classical Theorem originally proved by Voiculescu and generalized by Bercovici and Voiculescu in [6] for measures with unbounded support.

Theorem B.1. *Let $\mu, \nu \in \mathcal{M}_+$. Then*

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z)$$

for every z in the connected component of the common domain of S_μ and S_ν.

It was shown by Hari Bercovici in [5] that the additive free convolution of probability measures on the real line tend to have a lot fewer atoms. To be more precise.

Theorem B.2. *Let μ and ν be two probability measures supported in $\mathbb{R}$. The number a is an atom for the free additive convolution of μ and ν if and only if a can be written as $a = b+c$ where $\mu(\{b\})+\nu(\{c\}) > 1$. In this case, $\mu \boxplus \nu\,(\{a\}) = \mu(\{b\})+\nu(\{c\})-1$.*

For measures supported on the positive half-line, an analogous result holds, with a difference when zero is an atom. The following Theorem was proved by Serban Belinschi in [3].

Theorem B.3. *Let μ and ν be two probability measures supported in $[0, \infty)$.*

1. *The following are equivalent*

   (a) $\mu \boxtimes \nu$ has an atom at $a > 0$

   (b) there exists u and v so that $uv = a$ and $\mu(\{u\}) + \nu(\{v\}) > 1$.
   Moreover, $\mu(\{u\}) + \nu(\{v\}) - 1 = \mu \boxtimes \nu\,(\{a\})$.

2. *$\mu \boxtimes \nu\,(\{0\}) = \max\{\mu(\{0\}), \nu(\{0\})\}$.*

In [35] Nica and Speicher introduced the class of R–diagonal operators in a non–commutative C*-probability space. An operator T is R–diagonal if T has the same $*$–distribution as a product UH where U and H are $*$–free, U is a Haar unitary, and H is positive.

The next Theorem and Corollary were proved by Uffe Haagerup and Flemming Larsen in [18] where they completely characterized the Brown measure of an R–diagonal element.

Theorem B.4. *Let (M, τ) be a non–commutative finite von Neumann algebra with a faithful trace τ. Let u and h be $*$–free random variables in M, u a Haar unitary, $h \geq 0$ and assume that the distribution μ_h for h is not a Dirac measure. Denote μ_T the Brown measure for $T = uh$. Then*

1. *μ_T is rotation invariant and*

$$\mathrm{supp}(\mu_T) = [\|h^{-1}\|_2^{-1}, \|h\|_2] \times_p [0, 2\pi).$$

2. *The S–transform S_{h^2} of h^2 has an analytic continuation to neighborhood of the interval $(\mu_h(\{0\}) - 1, 0]$, $S_{h^2}((\mu_h(\{0\}) - 1, 0]) = [\|h\|_2^{-2}, \|h^{-1}\|_2^2)$ and $S'_{h^2} < 0$ on $(\mu_h(\{0\}) - 1, 0)$.*

3. *$\mu_T(\{0\}) = \mu_h(\{0\})$ and $\mu_T(B(0, S_{h^2}(t-1)^{-1/2}) = t$ for $t \in (\mu_h(\{0\}), 1]$.*

4. *μ_T is the only rotation symmetric probability measure satisfying (3).*

Corollary B.5. *With the notation as in the last Theorem we have*

1. *the function $F(t) = S_{h^2}(t-1)^{-1/2} : (\mu_h(\{0\}), 1] \to (\|h^{-1}\|_2^{-1}, \|h\|_2]$ has an analytic continuation to a neighboorhood of its domain and $F' > 0$ on $(\mu_h(\{0\}), 1)$.*

2. μ_T has a radial density function f on $(0,\infty)$ defined by

$$g(s) = \frac{1}{2\pi s}(F^{<-1>})'(s)\,\mathbf{1}_{(F(\mu_h(\{0\})),F(1)]}(s).$$

Therefore, $\mu_T = \mu_h(\{0\})\delta_0 + \sigma$ with $d\sigma = g(|\lambda|)dm_2(\lambda)$.

C. Main Results

In this section we prove our main results. Let us first fix some notation. We say two operators A and B in a finite von Neumann algebra $(\mathcal{N},\tau)$ have the same $*$–distribution iff $\tau(p(A,A^*)) = \tau(p(B,B^*))$ for all non–commutative polynomials $p \in \mathbb{C}\langle X,Y\rangle$. In this case we denote $A \sim_{*d} B$. If A and B are self–adjoint we say that A and B have the same distribution and we denote it by $A \sim_d B$.

Lemma 8. *Let $\{T_k\}_{k=1}^{\infty}$ be a family of $*$–free identically distributed operators in a finite von Neumann algebra. Let $a_k = |T_k|$ be the modulus of T_k. Then the positive operators $B_n = T_1^*T_2^*\dots T_n^*T_n\dots T_2T_1$ and $b_n = a_1a_2\dots a_n^2\dots a_2a_1$ have the same distribution.*

Proof. Let $T_k = u_ka_k$ be the polar decomposition of the operator T_k. Since we are in a finite von Neumann algebra we can always assume that u_k are unitaries. We will proceed by induction on n. The case $n=1$ is obvious since $T_1^*T_1 = a_1^2$. Assume now that B_k has the same distribution as b_k for $k<n$. Then by $*$–freeness and the induction hypothesis

$$B_n = T_1^*T_2^*\dots T_n^*T_n\dots T_2T_1 \sim_d (u_1a_1)^*(a_2\dots a_n^2\dots a_2)(u_1a_1).$$

Hence

$$B_n \sim_d a_1u_1^*(a_2\dots a_n^2\dots a_2)u_1a_1 = u_1^*(u_1a_1u_1^*)(a_2\dots a_n^2\dots a_2)(u_1a_1u_1^*)u_1.$$

Since conjugating by a unitary does not alter the distribution we see that

$$B_n \sim_d (u_1 a_1 u_1^*)(a_2 \dots a_n^2 \dots a_2)(u_1 a_1 u_1^*).$$

Since the operators $\{T_k\}_{k=1}^{\infty}$ are $*$–free then $\{\{u_k, a_k\}\}_k^{\infty}$ is a $*$–free family and $a_1 \sim_d u_1 a_1 u_1^*$ and are free with respect to $\{a_k\}_{k\geq 2}$. Then, by freeness,

$$B_n \sim_d (u_1 a_1 u_1^*)(a_2 \dots a_n^2 \dots a_2)(u_1 a_1 u_1^*) \sim_d a_1 a_2 \dots a_n^2 \dots a_2 a_1$$

concluding the proof. □

Now we are ready to prove our main Theorem.

Theorem C.1. *Let $\{T_k\}_k$ be a sequence of $*$–free equally distributed operators. Let μ in $\mathcal{M}_+$ be the distribution of $|T_k|^2$ and let B_n be as in the previous Lemma. The sequence of positive operators $B_n^{1/2n}$ converges in distribution to a positive operator Λ with distribution ν in $\mathcal{M}_+$. Moreover,*

$$\nu = \beta\delta_0 + \sigma \qquad \text{with} \qquad d\sigma = f(t)\, \mathbf{1}_{\left(\||T_1|^{-1}\|_2^{-1}, \||T_1|\|_2\right]}(t)\, dt$$

where $\beta = \mu(\{0\})$, $f(t) = \left(F_\mu^{<-1>}\right)'(t)$ and $F_\mu(t) = S_\mu(t-1)^{-1/2}$.

Proof. From the previous Lemma it is enough to prove the Theorem for $a_k = |T_k|$. Let u a Haar unitary $*$–free with respect to the family $\{a_k\}_k$ and let $h = a_1$. Let T be the R–diagonal operator defined by $T = uh$. It is easy to see, by the freeness assumptions, that $(T^*)^n T^n$ and b_n have the same distribution. Moreover, by [19] the sequence $\left[(T^*)^n T^n\right]^{1/2n}$ converges in the strong operator topology to a positive operator Λ. Let ν be the probability measure distribution of Λ.

If the distribution of a_k^2 is a Dirac delta, $\mu = \delta_\lambda$, then $h = \sqrt{\lambda}$ and

$$\left[(T^*)^n T^n\right]^{1/2n} = \left[\lambda^n (u^*)^n u^n\right]^{1/2n} = \sqrt{\lambda}.$$

Therefore, $b_n^{1/2n}$ has the Dirac delta distribution distribution $\delta_{\sqrt{\lambda}}$ and $\nu = \delta_{\sqrt{\lambda}}$. If the distribution of a_k is not a Dirac delta, let μ_T the Brown measure of the operator T. By Theorem 2.5 in [19] we know that

$$\int_{\mathbb{C}} |\lambda|^p d\mu_T(\lambda) = \lim_n \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}} = \lim_n \tau\Big([(T^*)^n T^n]^{\frac{p}{2n}}\Big) = \tau(\Lambda^p) = \int_0^\infty t^p \, d\nu(t). \tag{4.2}$$

We know by Theorem B.4 and Corollary B.5 that

$$\mu_T = \beta\delta_0 + \rho \quad \text{with} \quad d\rho(r,\theta) = \frac{1}{2\pi} f(r) \, \mathbf{1}_{(F_\mu(\beta), F_\mu(1)]}(r) \, dr d\theta \tag{4.3}$$

where $f(t) = \left(F_\mu^{<-1>}\right)'(t)$ and $F_\mu(t) = S_\mu(t-1)^{-1/2}$.

Hence, using equation (4.2) we see that

$$\int_0^\infty r^p \, d\nu(r) = \int_0^{2\pi} \int_{F_\mu(\beta)}^{F_\mu(1)} \frac{1}{2\pi} r^p f(r) \, dr d\theta = \int_{F_\mu(\beta)}^{F_\mu(1)} r^p f(r) dr$$

for all $p \geq 1$. Using the fact that if two compactly supported probability measures in $\mathcal{M}_+$ have the same moments then they are equal, we see that

$$\nu = \beta\delta_0 + \sigma \quad \text{with} \quad d\sigma = f(t) \, \mathbf{1}_{(F_\mu(\beta), F_\mu(1)]}(t) \, dt.$$

By Corollary B.5, we know that

$$F_\mu(1) = \|a_1\|_2 \quad \text{and} \quad \lim_{t \to \beta^+} F_\mu(t) = \|a_1^{-1}\|_2^{-1}$$

concluding the proof.

$\square$

Note that the last Theorem gives us a map $\mathcal{G} : \mathcal{M}_+ \to \mathcal{M}_+$ with $\mu \mapsto \mathcal{G}(\mu) = \nu$. The measure $\mathcal{G}(\mu)$ is a compactly supported positive measure with at most one atom at zero and $\mathcal{G}(\mu)(\{0\}) = \mu(\{0\})$.

Since

$$\mathcal{G}(\mu) = \beta\delta_0 + \sigma \qquad \text{with} \qquad d\sigma = f(t)\,\mathbf{1}_{(F_\mu(\beta),F_\mu(1)]}(t)\,dt$$

and $f(t) = \left(F_\mu^{<-1>}\right)'(t)$ where $F_\mu(t) = S_\mu(t-1)^{-1/2}$ for $t \in (\beta, 1]$. The function $S_\mu(t-1)$ for $t \in (\beta, 1]$ is analytic and completely determined by μ. If $\mu_1, \mu_2 \in \mathcal{M}_+$ and $S_{\mu_1}(t-1) = S_{\mu_2}(t-1)$ in some open interval $(a, b) \subseteq (0, 1]$ implies that $\mu_1 = \mu_2$. Therefore, the map $\mathcal{G}$ is an injection.

Remark C.2. A measure μ in $\mathcal{M}_+$ is said $\boxtimes$-infinitely divisible if for each $n \geq 1$ there exists a measure μ_n in $\mathcal{M}_+$ such that

$$\mu = \mu_n \boxtimes \mu_n \ldots \boxtimes \mu_n \quad (n \text{ times}).$$

We would like to observe that the image of the map $\mathcal{G}$ is not contained in the set of $\boxtimes$-infinitely divisible laws since an $\boxtimes$-infinitely divisible law cannot have an atom at zero (see Lemma 6.10 in [6]).

The next Theorem investigates how the map $\mathcal{G}$ behaves with respect to additive and multiplicative free convolution.

Theorem C.3. *Let μ be a measure in $\mathcal{M}_+$ and $n \geq 1$. If $\mathcal{G}(\mu) = \beta\delta_0 + \sigma$ with $d\sigma = f(t)\,\mathbf{1}_{(F_\mu(\beta),F_\mu(1)]}(t)\,dt$ then*

$$\mathcal{G}(\mu^{\boxplus n}) = \beta_n\delta_0 + \sigma_n \quad \textit{with} \quad d\sigma_n = \sqrt{n}f(t/\sqrt{n})\,\mathbf{1}_{(\sqrt{n}F_\mu(\frac{\beta_n+n-1}{n}),\sqrt{n}F_\mu(1)]}(t)\,dt$$

where $\beta_n = \max\{0, n\beta - (n-1)\}$ and

$$\mathcal{G}(\mu^{\boxtimes n}) = \beta\delta_0 + \rho_n \quad \textit{with} \quad d\rho_n = \frac{1}{n}t^{\frac{1-n}{n}}f(t^{1/n})\,\mathbf{1}_{(F_\mu(\beta)^n,F_\mu(1)^n]}(t)\,dt.$$

Proof. Recall the relation between the R_μ and S_μ transform (see [18]),

$$\Big(zR_\mu(z)\Big)^{<-1>} = zS_\mu(z).$$

By the fundamental property of the R–transform we have $R_{\mu^{\boxplus n}}(z) = nR_\mu(z)$. Therefore,

$$\Big(znR_\mu(z)\Big)^{<-1>} = zS_{\mu^{\boxplus n}}(z).$$

Hence

$$\frac{z}{n}S_\mu(z/n) = zS_{\mu^{\boxplus n}}(z)$$

thus

$$S_{\mu^{\boxplus n}}(z) = \frac{1}{n}S_\mu(z/n). \tag{4.4}$$

Then

$$F_{\mu^{\boxplus n}}(t) = S_{\mu^{\boxplus n}}(t-1)^{-1/2} = \left(\frac{1}{n}S_\mu\Big(\frac{t-1}{n}\Big)\right)^{-1/2} = \sqrt{n}F_\mu\Big(\frac{t+n-1}{n}\Big)$$

it is a direct computation to see that

$$F^{<-1>}_{\mu^{\boxplus n}}(t) = nF^{<-1>}_\mu(t/\sqrt{n}) - n + 1. \tag{4.5}$$

By iterating Theorem B.2 we see that $\mu^{\boxplus n}(\{0\}) = \max\{0, n\beta - (n-1)\} = \beta_n$. Now using Theorem C.1 we obtain

$$\mathcal{G}(\mu^{\boxplus n}) = \beta_n\delta_0 + \sigma_n \quad \text{with} \quad d\sigma_n = \sqrt{n}f(t/\sqrt{n})\,\mathbf{1}_{(\sqrt{n}F_\mu(\frac{\beta_n+n-1}{n}),\sqrt{n}F_\mu(1)]}(t)\,dt.$$

Now let us prove the multiplicative free convolution part, let $\mu^{\boxtimes n}$ then

$$S_{\mu^{\boxtimes n}}(z) = S^n_\mu(z).$$

Then $F_{\mu^{\boxtimes n}}(t) = F_\mu^n(t)$ and therefore,

$$F_{\mu^{\boxtimes n}}^{<-1>}(t) = F_\mu^{<-1>}(t^{1/n}). \tag{4.6}$$

By Theorem B.3 we now that $\mu^{\boxtimes n}(\{0\}) = \mu(\{0\}) = \beta$. Therefore, using Theorem C.1 again we obtain

$$\mathcal{G}(\mu^{\boxtimes n}) = \beta\delta_0 + \rho_n \quad \text{with} \quad d\rho_n = \frac{1}{n} t^{\frac{1-n}{n}} f(t^{1/n})\, \mathbf{1}_{(F_\mu(\beta)^n,\, F_\mu(1)^n]}(t)\, dt.$$

□

D. Examples

In this section we present some examples of the image of the map $\mathcal{G}$.

Example D.1. (Projection) Let p be a projection with $\tau(p) = \alpha$. Then the spectral probability measure of p is $\mu_p = (1-\alpha)\delta_0 + \alpha\delta_1$. We would like to compute $\mathcal{G}(\mu_p)$. Recall that

$$S_p(z) = \frac{z+1}{z+\alpha}.$$

Therefore,

$$F_\mu(t) = \Big(\frac{t-1+\alpha}{t}\Big)^{1/2} \quad \text{and} \quad F_\mu^{<-1>}(t) = \frac{1-\alpha}{1-t^2}.$$

Hence,

$$\mathcal{G}(\mu_p) = (1-\alpha)\delta_0 + \sigma \quad \text{with} \quad d\sigma = \frac{2t(1-\alpha)}{(t^2-1)^2}\, \mathbf{1}_{(0,\sqrt{\alpha}]}(t)\, dt.$$

Example D.2. Let h be a quarter–circular distributed positive operator,

$$d\mu_h = \frac{1}{\pi}\sqrt{4-t^2}\, \mathbf{1}_{[0,2]}(t)\, dt.$$

A simple computation shows that

$$S_{h^2}(z) = \frac{1}{z+1}$$

hence by Theorem C.1 we see that

$$d\mathcal{G}(\mu_{h^2}) = 2t\,\mathbf{1}_{[0,1]}(t)\,dt.$$

Example D.3. (Marchenko – Pastur distribution)

Let $c > 0$ and let μ_c be the Marchenko Pastur or Free Poisson distribution given by

$$d\mu_c = \max\{1-c, 0\}\delta_0 + \frac{\sqrt{(t-a)(b-t)}}{2\pi t}\,\mathbf{1}_{(a,b)}(t)\,dt$$

where $a = \left(\sqrt{c}-1\right)^2$ and $b = \left(\sqrt{c}+1\right)^2$.

It can be shown (see for example [18]) that

$$S_{\mu_c}(z) = \frac{1}{z+c}.$$

Therefore,

$$F_{\mu_c}(t) = \sqrt{t-1+c} \quad \text{and} \quad F_{\mu_c}^{<-1>}(t) = t^2+1-c.$$

Hence,

$$\mathcal{G}(\mu_c) = \max\{1-c, 0\}\delta_0 + \sigma \quad \text{with} \quad d\sigma = 2t\,\mathbf{1}_{(\sqrt{\max\{c-1\,,0\}},\sqrt{c}]}(t)\,dt.$$

E. Lyapunov Exponents of Free Operators

Let $\{a_k\}_{k=1}^{\infty}$ be free positive identically distributed operators. Let μ be the spectral probability measure of a_k^2 and assume that $\mu(\{0\}) = 0$. Using Theorem C.1 we know that the sequence of positive operators

$$\left(a_1 a_2 \dots a_n^2 \dots a_2 a_1\right)^{1/2n}$$

converges in distribution to a positive operator Λ with distribution ν in $\mathcal{M}_+$. Since

$\mu(\{0\}) = 0$, this distribution is absolutely continuous with respect to the Lebesgue measure and has Radon–Nikodym derivative

$$d\nu(t) = f(t)\, \mathbf{1}_{(\|a_1^{-1}\|_2^{-1}, \|a_1\|_2]}(t)\, dt$$

where $f(t) = \left(F_\mu^{<-1>}\right)'(t)$ and $F_\mu(t) = S_\mu(t-1)^{-1/2}$.

Let L be the selfadjoint, possibly unbounded operator, defined by $L := \ln(\Lambda)$, and let γ be the spectral probability distribution of L. It is a direct calculation to see that γ is absolutely continuous with respect to Lebesgue measure and has Radon–Nikodym derivative

$$d\gamma(t) = e^t f(e^t)\, \mathbf{1}_{(\ln\|a_1^{-1}\|_2^{-1}, \ln\|a_1\|_2]}(t)\, dt.$$

The probability distribution γ of L is what is called the distribution of the Lyapunov exponents (see [33], [34] and [39] and [30] for a more detailed exposition on Lyapunov exponents in the classical and non–classical case).

Theorem E.1. *Let $\{a_k\}_{k=1}^\infty$ be free positive identically distributed invertible operators. Let μ be the spectral probability measure of a_k^2. Let γ be probability distribution of the Lyapunov exponents associated to the sequence. Then γ is absolutely continuous with respect to Lebesgue measure and has Radon–Nikodym derivative*

$$d\gamma(t) = e^t f(e^t)\, \mathbf{1}_{(\ln\|a_1^{-1}\|_2^{-1}, \ln\|a_1\|_2]}(t)\, dt.$$

where $f(t) = \left(F_\mu^{<-1>}\right)'(t)$ and $F_\mu(t) = S_\mu(t-1)^{-1/2}$.

Remark E.2. Note that if the operators a_k are not invertibles in the $\|\cdot\|_2$ then the selfadjoint operator L is unbounded. See in the next example the case $\lambda = 1$.

The following is an example done previously in [30] using different techniques.

Example E.3. (Marchenko – Pastur distribution) Let $\{a_k\}_{k=1}^{\infty}$ be free positive identically distributed operators such that a_k^2 has the Marchenko–Pastur distribution μ of parameter $\lambda \geq 1$. Then as we saw in the Example D.3, in the last section

$$d\nu(t) = 2t\,\mathbf{1}_{(\sqrt{\lambda-1},\sqrt{\lambda}]}(t)\,dt.$$

Therefore, we see that the probability measure of the Lyapunov exponents is γ with

$$d\gamma(t) = 2e^{2t}\,\mathbf{1}_{\left(\frac{1}{2}\ln(\lambda-1),\frac{1}{2}\ln(\lambda)\right]}(t)\,dt.$$

If $\lambda = 1$, this law is the exponential law discovered by C.M.Newman as a scaling limit of Lyapunov exponents of large random matrices. (See [33], [34] and [21]). This law is often called the "triangle" law since it implies that the exponentials of Lyapunov exponents converge to the law whose density is in the form of a triangle.

CHAPTER V

CONCLUSION

This dissertation consists of three more or less independent projects. In the first project, we find the microstates free entropy dimension of a large class of $L^\infty[0,1]$–circular operators, in the presence of a generator of the diagonal subalgebra.

In the second one, for each sequence $\{c_n\}_n$ in $l_1(\mathbb{N})$, we define an operator A in the hyperfinite II_1-factor $\mathcal{R}$. We prove that these operators are quasinilpotent and they generate the whole hyperfinite II_1-factor. We show that they have non-trivial, closed, invariant subspaces affiliated to the von Neumann algebra, and we provide enough evidence to suggest that these operators are interesting for the hyperinvariant subspace problem. We also present some of their properties. In particular, we show that the real and imaginary part of A are equally distributed, and we find a combinatorial formula as well as an analytical way to compute their moments. We present a combinatorial way of computing the moments of A^*A.

Finally, let $\{T_k\}_{k=1}^{\infty}$ be a family of $*$–free identically distributed operators in a finite von Neumann algebra. In this paper, we prove a multiplicative version of the Free Central Limit Theorem. More precisely, let $B_n = T_1^*T_2^* \dots T_n^*T_n \dots T_2T_1$ then B_n is a positive operator and $B_n^{1/2n}$ converges in distribution to an operator Λ. We completely determine the probability distribution ν of Λ from the distribution μ of $|T|^2$. This gives us a natural map $\mathcal{G} : \mathcal{M}_+ \to \mathcal{M}_+$ with $\mu \mapsto \mathcal{G}(\mu) = \nu$. We study how this map behaves with respect to additive and multiplicative free convolution. As an interesting consequence of our results, we illustrate the relation between the probability distribution ν and the distribution of the Lyapunov exponents for the sequence $\{T_k\}_{k=1}^{\infty}$ introduced by Vladismir Kargin.

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