ANTE REM STRUCTURALISM AND THE MYTH OF IDENTITY CRITERIA

A Thesis

by

HO KIN SIU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF ARTS

May 2008

Major Subject: Philosophy

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Approved by:	
Chair of Committee,	Christopher Menzel
Committee Members,	Robin Smith
	Ken Dykema
Head of Department,	Danial Conway

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ABSTRACT

Ante Rem Structuralism and the Myth of Identity Criteria. (May 2008)

Ho Kin Siu, B.A., The University of Hong Kong

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This thesis examines the connections between the motivations of *ante rem* structuralism and the problem of automorphism. *Ante rem* structuralists are led to the problem of automorphism because of their commitment to the thesis of structure-relative identity. Ladyman's and Button's solutions to the problem are both unsatisfactory. The problem can be solved only if *ante rem* structuralists drop the thesis of structure-relative identity. Besides blocking the problem of automorphism, there are further reasons why the thesis has to be dropped. (i) The purported metaphysical and epistemic purchase of adopting the thesis can be put into doubt. (ii) Primitive identity within a mathematical structure is more in line with *ante rem* structuralist's commitment to the *faithfulness constraint* and to the *ontological priority of structure over positions*. However, the cost of dropping the thesis is that *ante rem* structuralists cannot provide a satisfactory solution to Benacerraf's problem of multiple reductions of arithmetic.

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CHAPTER I

INTRODUCTION

This thesis is motivated by my interest to know whether there is a feasible way for *ante rem* structuralists to solve the *problem of automorphism*, a problem raised in Burgess [5] and articulated forcefully by Keranen in his [15] and [16]. Different attempted solutions were offered, yet none of them seems to successfully solve the problem. Jubien [13] brings to my attention the primitiveness of identity. This leads me to see the problem in a different way: the problem should be *dissolved* rather than solved. Given that identity is a primitive notion, problems are bound to arise if *ante rem* structuralists make it supervene on relations and properties excluding haecceities. This makes me ask the question: why *ante rem* structuralists are committed to the supervenience thesis in the first place? To answer this question, I attempt to make a thorough assessment of *ante rem* structuralism in order to identify the exact connections between its motivations and commitments on one hand and the problem of automorphism on the other. The result of this assessment is the followings which I try to establish in my thesis. (1) The problem of automorphism cannot be solved without doing severe damage to *ante rem* structuralism. (2) *Ante rem* structuralists are led to the problem of automorphism because of their commitment to *relative identity* — a thesis on which identity supervenes on relations and properties — in providing a solution to the problem of multiple reductions of arithmetic presented in Benacerraf [1]. (3) *Ante rem* structuralists are led to the thesis of *relative identity* by their commitment to the *faithfulness constraint*, which motivates them to take singular terms in mathematical sentences as denoting positions in some structured universals, and to their tenet that a mathematical object has no properties except its relations with other objects in the same structure taken to the extreme form as the *thesis of incompleteness*, according to which there is *no fact of the matter* as to whether one object from one mathematical structure is identical to an object from another structure. (4) The thesis of *relative identity* — at least in its involvement in *ante rem* structuralism — can be put into doubt for reasons independent of the problem of automorphism. (5) Taking identity to be primitive for mathematical objects in the same structure can be accommodated into the framework of *ante rem* structuralism by appealing to *ante rem* structuralists' commitment to the *faithfulness constraint* and the *ontological priority*

This thesis follows the style of *Journal of Philosophical Logic*.

of structure over positions. (6) Without relative identity, *ante rem* structuralists fail to provide a satisfactory solution to Benacerraf's problem.

In the first chapter, I seek to identify the motivations and commitments of *ante rem* structuralism and explain why there is a temptation for *ante rem* structuralists to propose the theses of ontological incompleteness of mathematical objects and relative identity as a solution to Benacerraf's problem of multiple reductions of arithmetic.

In the second chapter, I explain why *ante rem* structuralists' solution to the Benacerraf problem leads them into the problem of automorphism and argue that the problem cannot be solved without doing severe damage to their program. I also argue that whereas the thesis of relative identity can be put into doubt on independent grounds, primitive identity can be accommodated into the framework of *ante rem* structuralism by appealing to its commitment to the faithfulness constraint and the ontological priority of structure over positions.

In the third chapter, I seek to examine the consequences of dropping the thesis of relative identity. I argue that the most important consequence is that the Benacerrafian problem, which motivates various forms of structuralism, is left *unsolved* on *ante rem* structuralism. In the same chapter, I discuss several ways in which my project complements Shapiro's recent thoughts ([31], [32]) on the problem of automorphism.

CHAPTER II

STRUCTURALISM AND *ANTE REM* STRUCTURALISM

This chapter has two aims. The first aim is to answer two separate questions: (1) why structuralism, as articulated by Benacerraf in [1], is an appealing guiding notion in philosophy of mathematics and (2) why Resnik and Shapiro consider *ante rem* structuralism to be the best way of fleshing out the details of structuralism. The reason why these two questions are separate is that although *ante rem* structuralism is indeed inspired by Benacerraf's structuralist solution to the problem of multiple reductions of arithmetic, *ante rem* structuralism is only one among several possible strategies of formulating structuralism. Apart from *ante rem* structuralism, there are set theoretical structuralism and modal structuralism. A survey of these alternative forms of structuralism will make clear the desiderata which lead Resnik and Shapiro to propose *ante rem* structuralism as the best way of formulating structuralism.

The second aim of this chapter is to set the stage for the discussion of the next chapter by raising issues with *ante rem* structuralists' solution to the problem of multiple reductions of arithmetic — namely, the coupling of the theses of *ontological incompleteness of mathematical objects* and *relative identity.*

1. What Numbers Could Not Be?

The problem of multiple reductions of arithmetic

Benacerraf presents a serious challenge facing mathematical realism — the thesis that the truth of mathematical statements is grounded in the existence of mathematical objects independent of us — by telling a story of how two children, Ernie and Johnny, learn arithmetic in set theory.[1] Whereas Ernie is taught von Neumann's constructions of numbers — on which each number is the set of its predecessors — Johnny is taught Zermelo's, on which each number is the singleton of its immediate

[1] Mathematical realism can be characterized in a more fine-grained way. See Resnik [28] (p.11); Hellman [10] (p.2). According to Resnik's characterization, there are three conditions for mathematical realism: (1) to be committed to the existence of mathematical objects; (2) to hold that our current understanding of mathematics is true; (3) to hold that the existence of Xs and the truth of statements about Xs is independent of us. According to Hellman's characterization, one can be a mathematical realist without being committed to the existence of mathematical entities, if she holds that the truth of mathematical sentences is independent of us. Since this thesis is concerned with *ante rem* structuralism which is committed to all the conditions on Resnik's list, it suffices to use a more coarse-grained characterization of mathematical realism.

predecessor. Their opinions on numbers do not differ when asked about whether $1+2=3$ (yes), whether addition is associative (yes), or whether two numbers having the same successor can be different (no). However, their opinions differ when they are asked whether the successor x' of a number x is such that x' contains all of the members of x and x itself as its members, or whether a number x is smaller than a number y if and only if x is a member of y. To both questions, Ernie would answer: 'Yes, that is how the successor function and the smaller-than relation are defined', but Johny would demur: 'No. 3 is a successor of 2 but it does not contain its only member, namely 1. Also, 1 is smaller than 3 but it is not a member of 3.' How can the disagreement between them be settled? Whose number 3 is the *real* number 3?

The point behind Benacerraf's story is that if mathematical realists are to take numbers as sets, then they have to be able to say which sets they are, that is, they have to provide truth values for the identity statements $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ and $3 = \{\{\{\emptyset\}\}\}$. They cannot say that both statements are true, for the very simple reason that the two sets in question are different — Zermelo's 3 does not contain $\{\emptyset\}$, a member of von Neumann's 3. Nor can they say that both statements are false. If they claim that they are false, then they have to explain *why* they *know* that is the case. However, there is no evidence they can appeal to to make the judgment, because the difference between the two constructions is extraneous to arithmetic itself. We never in our ordinary use of number words talk about whether 2 belongs to 3, or whether 3 has three members. So, mathematical realists cannot provide the truth values for the two identity statements in a principled way; therefore, numbers cannot be sets.

It has to be noted that Benacerraf's argument is not restricted to sets. The same argument can be given to any kind of objects purported to be numbers: the even von Neumann numbers or Frege's classes, for example. Also, his argument is not restricted to natural numbers alone. Mathematical realists will face the same problem if they try to ask questions about the identity of a real number, a point on the Euclidean plane, or an element in a 3-member group. So, Benacerraf's conclusion should be taken to be much stronger: mathematical entities generally are not objects.

Benacerraf's structuralist solution to the problem

If numbers are not objects, there are two questions for Benacerraf to answer: (i) How should we treat the singular terms and quantifiers in mathematical sentences? (ii) How should we characterize the subject matter of mathematics?

Benacerraf's answer to the first question is that we should resist the temptation to treat singular terms and quantifiers in mathematical sentences as referring to extra-linguistic objects — in his own words: "There are not two kinds of things, numbers and number words, but just one, the words themselves" ([1], p.292). So, he seems to suggest that we should treat the following two kinds of sentences with different semantics:

(1) John and Mary give birth to May

(2) There is a dog between two cats

(3) $2 + 3 = 5$

(4) There is a natural number between 2 and 4

Whereas we should evaluate sentences (1) and (2) model-theoretically, that is to take the singular terms 'John', 'Mary' and 'May' in (1) and the quantifiers in (2) as referring to objects in the world, we should not do the same with (3) and (4); otherwise we commit ourselves to some extra-linguistic objects and run into the problem Benacerraf raises.

Benacerraf's answer to the second question is as follows. Any system of objects, be it sets or not, that forms a recursive progression satisfying the axioms of arithmetic is adequate. That different systems are adequate does not show that numbers can be identified with the elements of these systems. Rather, it shows that the "individuality of each element" in the systems is extraneous to arithmetic. What matters for arithmetic are the relations these objects hold to each other in the system they belong to, that is, "the structure which [these objects] jointly exhibit". So, the subject matter of arithmetic is the "abstract structure that all progressions have in common merely in virtue of being progressions"; it is "not concerned with particular objects — the numbers".

Focusing on the structural relations among mathematical entities rather than the intrinsic nature of each of them taken in isolation, Benacerraf's solution is structuralist in spirit. The appeal of structuralism is that it comports with mathematical practice. In [4], Bourbaki illustrates how the intrinsic nature of mathematical entities is left out in mathematical practice with the example of three apparently distinct mathematical structures: (a) real numbers under addition, (b) integers under multiplication modulo a prime number p, and (c) displacements in three-dimensional Euclidean space under composition. Despite their apparent differences, they all satisfy the group axioms — in virtue of having an associative operation, an identity element, and an inverse for each element in the structure — and can be treated in the same manner. For example, the cancellation law applies to the three structures in exactly the same way. In mathematics, there is no reason to give up a more economical way of talking about structures just because the intrinsic natures of the entities in them are different.

Assessment of the solution

Although Benacerraf's suggestions do seem to solve the problem he raises and comport with mathematical practice, they raise the following issues. First, his answers to questions (i) and (ii) seem to be jointly inconsistent. In answering (i), he claims that numbers are not objects. There he seems to be suggesting an eliminativist solution to the problem he raises, which goes in this way: since numbers are not objects, the reference of the number word '2' is eliminated. Therefore, there is no issue with whether $2 = \{\{\emptyset\}\}$ or not, for the simple reason that '2' has no denotation. However, in answering (ii), he claims that arithmetic elaborates an abstract structure, in which the " 'elements' have no properties other than those relating them to other 'elements' [in it]". It is not clear whether the abstract structure and the elements he refers to exist or not. If they do, Benacerraf is hard-pressed to answer the question of whether number words are indeed singular terms, now referring to the elements in the abstract structure, rather than to the various set theoretical reductions of them. Even if he is using metaphorical language there, he does not show how a "structuralism without abstract structures" can be worked out.

Second, if we have to pay a high price for evaluating mathematical sentences in the model-theoretic way and hence need to adopt a different semantics for mathematics, what would the semantics look like? Do we have to pay an even higher price in adopting an alternative semantics?

Third, could a mathematical realist, following Benacerraf's suggestion, treat mathematical objects as having no intrinsic nature other than their relations with other objects in the same structure and hence rule out questions about their intrinsic nature — whether they are identical to particular sets — from the start? This seems to be an option that Benacerraf fails to consider in length.

These are questions that any follower of structuralism has to answer. Different ways of answering these questions, as we will see, lead to the different ways of solving Benacerraf's problem in the structuralist spirit.[2]

[2]For a survey of different forms of structuralism, see: Parsons's [21]; Shapiro's [30] (pp. 84-90); Reck and Price's [26]; Hellman's [12].

2. Set Theoretical Structuralism and Modal Structuralism

In the following, I discuss two forms of structuralism other than *ante rem* structuralism: set theoretical structuralism and modal structuralism. By comparing *ante rem* structuralism with these alternative accounts, I seek to highlight the main desiderata of *ante rem* structuralism.

Set theoretical structuralism (STS)

There are two versions of this form of structuralism: the relative version and the universalized version. On both accounts there is no fact of the matter as to what is *the* natural number structure or what is *the* number 2. No structure is privileged over the others; any structures that satisfy the axioms of arithmetic, that is, having the correct structure, are adequate. It is in this sense that they are both structuralist in spirit. They both do not countenance the existence of an abstract structure suggested by Benacerraf. On both accounts, a structure is a set theoretical structure, a set or k-tuple with "a domain of objects together with certain functions and relations on the domain, satisfying certain given conditions [e.g. certain axioms]" ([21], p. 274). The truth of mathematical sentences is explained in terms of truth in set theoretical structures, rather than in some abstract structures whose nature is not spelled out in detail by Benacerraf. They differ only at the issue of whether mathematical sentences should be treated with a different semantics. On the relative version, a sentence of arithmetic is taken at face value relative to a *particular* structure; no novel semantics is called for. On the universalized version, a sentence of arithmetic is not taken at face value; rather it is taken to be a more complex second-order universal quantification sentence about *all* structures of arithmetic.

STS, Relative Version

This version is a structuralist articulation of our usual model-theoretic understanding of a sentence. It provides a deflationary solution to Benacerraf's problem. A mathematical sentence is true if it is true in a *particular* structure of an isomorphic type. For example, suppose we use the language $\{0, +, \cdot, S, <\}$, the sentence $2+3=5$ can be translated as $0'' + 0''' = 0'''''$, which can then be taken at face value in a particular structure satisfying the axioms of arithmetic. $0''$, $0'''$, and $0'''''$ are all singular terms denoting objects in the domain of that particular structure. So, the number 2 is the object denoted by $0''$ in that particular structure, and that particular structure is *a* natural number structure. There is no such thing as *the* number 2 or *the* natural number structure. There is no point in asking whether the number 2 is identical to the set $\{\emptyset, \{\emptyset\}\}$, without first fixing a particular structure. So, according to this view, Benacerraf's problem is not an issue in the first place.

STS, Universalized Version

This version is different from the relative version in that it does not take a mathematical sentence at its face value. It eliminates the reference of number words by taking a sentence of arithmetic as a general statement about *all* structures satisfying the axioms of arithmetic. To describe this program in a more concrete way, I discuss how a sentence of arithmetic can be translated into a general statement about all structures of arithmetic using a translation scheme.[3] First, we need to define the sentence AX which will figure in any translation of a given sentence of arithmetic. It is defined as the conjunction of Robinson's axioms and the second order induction principle, which are taken as the axioms of the natural number structure.[4]

$$\text{Induction principle: } \forall X((X0 \wedge (\forall x)(Xx \rightarrow Xx')) \rightarrow \forall x Xx)$$

$$AX = \text{conjunction of Robinson's axioms} \wedge \text{induction principle}$$

Second, we translate a sentence of arithmetic ϕ into our language $\{0, +, \cdot, S, <\}$. For example, if the given sentence is $2 + 3 = 5$, we translate it into $0'' + 0''' = 0'''''$. If the sentence is "there is a natural number between 2 and 4", we translate it into $\exists x(0'' < x \wedge x < 0'''')$. Next, we turn the translated sentence into a general sentence about all structures of natural numbers of following canonical form:

$$\forall Pxfgjk[AX^P(x/0,\ f/+, g/\cdot,\ j/S, k/<) \rightarrow \phi^P(x/0,\ f/+, g/\cdot,\ j/S, k/<)] \qquad (\phi_{USTS})$$

where 'P' is a 1-place relation variable; 'x' is an individual variable ranging over the elements in P; 'f', 'g' are 2-place function variables; 'j' a 1-place function variable; and 'k' is a 2-place relation variable. The subscript ^{P} over AX and ϕ indicates that the quantifiers in them are relativized to P. This translation is intended to mean for a sentence of arithmetic ϕ to be true, it is true in *all* structures satisfying the axioms of arithmetic i.e. AX.

[3] For the details of the program, see Reck and Price's [26]; McCarthy's [19] (pp. 137-138).

[4] See [3], p. 208 for Robinson's axioms.

For the sentences $0'' + 0''' = 0'''''$ and $\exists x(0'' < x \wedge x < 0'''')$, once they are translated into the canonical form, there are no determinate objects for $0''$ and $0''''$ to refer to. They now act as bound variables rather than singular terms. The objects to which they are assigned vary across structures. The same applies to the quantifier in $\exists x(0'' < x \wedge x < 0'''')$, because different structures have different domains and its range is relativized to them. So, reference of number words and quantifiers is eliminated on this account. Since number words turn out to have no reference on this account, proponents of this account are not under the burden of providing truth values for such identity sentences as $2 = \{\emptyset, \{\emptyset\}\}$ and $2 =$ Caesar.

Problems with STS

There are several problems with both forms of set theoretical structuralism. First, while the relative version takes for granted the existence of set theoretical structures for arithmetic, the universalized version takes for granted the existence of functions and relations, which are ranged over by the quantifiers. Where do these objects come from? Suppose our ontology fails to contain these objects, then all sentences of arithmetic — including the ones that are false — are all vacuously true. On the relative version, if there does not exist any structure of arithmetic, all sentences of arithmetic are trivially true in *all* structures. On the universalized version, if there do not exist functions and relations that can satisfy the axioms of arithmetic, the translation of any given sentence of arithmetic ϕ is trivially true. This problem is called *the problem of non-vacuity.* The natural solution for both accounts is to use set theory to provide an abundant number of objects so that we have enough objects to ensure the non-vacuity of mathematical truth. The cost of treating set theory this way, however, is to treat it non-structurally. In explaining the truth of sentences of a given theory in terms of set theoretical structures satisfying its axioms, we relieve ourselves of the burden of giving an account of the absolute nature of the objects of the theory in question. Any structure that satisfies the axioms will do; there is no determinate class of objects that can be identified with a particular theory. However, if we take a particular set theory to define a determinate class of objects, namely sets, we cannot avoid answering questions about their nature. Do we know whether there is a set x such that $x =$ Caesar? How do we know that no set can be Caesar? Among the many set theories available, why are only the sets of the chosen set theory real? So, set theoretical structuralism, rather than solving Benacerraf's problem, seems to defer it to the level of sets.

Second, as Benacerraf argues in his [2], one desideratum of a philosophical account of mathematical truth is that there need to be cognitive connections between the truth conditions and our true beliefs. Both accounts seem to fail to honor this desideratum. It is not clear how we can have epistemic access to set theoretical structures, abstract entities with which we have no causal contact.

Third, the universalized version fails to honor another desideratum Benacerraf points out, that is, to use a uniform semantics for both mathematics and the rest of our language. It is not clear whether sentences of mathematics do have the logical form stipulated by the account, and how we can explain the asymmetry between mathematical language and the rest of our language.

Modal structuralism (MS)

Like the proponents of the universalized version of STS, modal structuralists do not respect the surface logical form of a mathematical sentence; they solve the Benacerrafian problem by eliminating the reference of singular terms and quantifiers in mathematical sentences. In light of the problems facing the proponents of the universalized version of STS — namely their failure to treat set theory structurally and the difficulty of accounting for our knowledge of sets — modal structuralists improve the universalized version of STS by modalizing the translation ϕ_{USTS} discussed above. The motivation for introducing modal notions is to eliminate reference to all abstract objects — including sets — so that the problems facing proponents of the universalized version of STS can be avoided.

I now discuss in further detail how this account works. Instead of taking a sentence of arithmetic ϕ as elliptical for "ϕ is true in any structure of arithmetic" as on the universalized version of STS, this account takes it to be elliptical for "if X were any structure of of arithmetic, ϕ would hold in X" ([10], p. 16), which modal structuralists express formally in the following sentence:

$$\Box \forall Pxfgjk[AX^P(x/0, f/+, g/\cdot,\ j/S, k/<) \rightarrow \phi^P(x/0, f/+, g/\cdot,\ j/S, k/<)] \qquad (\phi_{MS})$$

Despite the presence of a necessity operator in the sentence, modal structuralists do not interpret the sentence using possible world semantics, because to do so is to be committed to abstract objects such as possible worlds, which are what they seek to eliminate. The introduction of the necessity operator at the front of the sentence, rather, is intended to eliminate reference to all abstract mathematical objects (including sets). With the universal quantifiers placed within the scope of the necessity operator, "[t]here is literally no quantification over objects at all in [ϕ_{MS}]" ([11], p. 316).

Since ϕ_{MS} is intended to mean "if X were a structure of arithmetic, ϕ would hold in X", modal structuralists would be faced with a modal version of *the problem of non-vacuity* if the existence of a structure of arithmetic is not possible, that is, if the following sentence is not true:

$$\Diamond\exists Pxfgjk(AX^P)$$

Proponents of the universalized version of STS are forced to solve the problem of non-vacuity by taking set theory as specifying a class of objects which are numerous enough to guarantee the non-vacuity of mathematical truth. However, once they make this move, as we discussed above, they have to answer difficult questions about the identity of sets and about our knowledge of sets. So, this move does not seem to be promising. It is less so for modal structuralists because their motivation is to eliminate reference to all abstract objects — including sets. Due to these considerations, Hellman does not rely on set theory to establish the possibility of the existence of a structure of arithmetic. Rather, he seeks to establish the possibility along what he calls "quasi-constructive lines" ([10], p. 29). His idea seems to be that we can stipulate a rule which prescribes how a sequence of concrete marks, which is isomorphic to a structure of arithmetic, can be constructed. Even though such a sequence of concrete marks does not exist in the actual world, it is a *logical possibility* that it might ([11], p. 317). So, on MS, the existence of a structure of arithmetic can be shown to be possible without recourse to a non-structuralist interpretation of set theory.

Two issues can be raised with this program. First, as with the universalized version of set theoretical structure, the major problem with this account is that its translations, purported to capture the real logical forms of mathematical sentences, further deviate from the apparent ones. It is questionable whether modal notions do figure as prominently in mathematics as the translations seem to suggest. Second, it is not clear how we know that the existence of a sequence of concrete marks isomorphic to a structure of arithmetic is logically possible. It seems Hellman trades the epistemological problems concerning abstract objects for epistemological problems concerning what is logically possible. The issue of whether the latter kind of epistemological problems is more tractable than the former, however, is beyond the scope of this thesis.[5]

[5]For discussions about the epistemological problems facing modal structuralism, see Hale's [9].

2. *Ante rem* Structuralism

Desiderata of *ante rem* structuralism

Ante rem structuralism is the only structuralist program which rejects Benacerraf's conclusion that numbers are not objects and takes literally his suggestion that arithmetic is about an abstract structure, in which "the 'elements' ... have no properties other than those relating them to other 'elements' of the same structure" ([1], p. 291). *Ante rem* structuralists hold that numbers are what Benacerraf calls the "elements" in the abstract structure of arithmetic; they exist and are referred to by singular terms and quantifiers in mathematical sentences. So, their program is *non-eliminativist*. *Ante rem* structuralists extend this view to all mathematical objects — numbers of other number systems, objects in a group, sets ... All of them are taken as *positions* in abstract structures. As positions, they do not have any internal structure or intrinsic nature; they have no properties other than their relations with other positions in the same structure.

There are three desiderata which lead *ante rem* structuralists to propose this account as the way out for Benacerraf's problem. First, *ante rem* structuralists think that by taking mathematical sentences at their *face value* — that is , to take the singular terms and quantifiers in them as referring to objects in our ontology — they can reach an interpretation of mathematics that is more *faithful* to mathematical practice. Shapiro articulates the importance of this desideratum, which he calls the 'faithfulness constraint', in the following way: "the goal of philosophy of mathematics is to interpret mathematics, and articulates its place in the overall intellectual enterprise. One desideratum is to have an interpretation that takes as much as possible of what mathematicians say about their subject as literally true, understood at or near face value" ([31], p. 110). This desideratum is one of the reasons why *ante rem* structuralists do not find modal structuralism attractive. For modal structuralists, $2+3=5$ is true because, if there were any structure of arithmetic, the sentence would hold in it. It is not true because three objects in our ontology — namely the numbers **2**, **3**, and **5** — are in the relation described by the sentence; there are no such things as numbers according to modal structuralists. However, to so interpret the sentence, according to *ante rem* structuralists, is not to treat the sentence as *literally* true. The numerals 2, 3 and 5, taken literally, refer to three distinct mathematical objects, namely the numbers **2**, **3**, and **5**. The sentence $2+3=5$ is true because it says correctly of how the numbers **2**, **3**, and **5** are related to each other in the *ante rem* structure of arithmetic.

Second, *ante rem* structuralists are not satisfied with set theoretical structuralists' move of treating set theory non-structurally. Alongside different set theories, there are other theories which can provide a background ontology for mathematics; it is highly contentious which theory is taken to be the background theory ([30], p. 87). Also, set theory, like other theories in mathematics, is a study of a mathematical structure; the "set theoretic hierarchy ... is a pattern along with others" ([27], p. 539). There is no reason to think that other theories are dependent on it. Therefore, *ante rem* structuralists think it is important to not give any privilege to set theory. Set theory has to be accommodated by any structuralist accounts.

Third, *ante rem* structuralists think that the epistemology of mathematics is more tractable if a mathematical object is always dependent for its existence on the structure to which it belongs, and has no intrinsic properties other than its relations with other mathematical objects in the same structure. They believe that this construal of the nature of mathematical objects can be well-supported by novel epistemic strategies such as *pattern recognition* and *abstraction*.[6]

General outline of *ante rem* structuralism

Since Benacerraf, quite likely, is using metaphorical language when mentioning the abstract structure of arithmetic, he does not discuss in detail the nature of the abstract structure and of the elements in them. This job is taken up by *ante rem* structuralists. They agree with Benacerraf that the elements, which they call *positions*, in the abstract structure of arithmetic are "structureless ([27], p. 530)". "There is no more to the individual numbers 'in themselves' than the relations they bear to each other" ([30], p. 73).

Resnik and Shapiro go further than Benacerraf by providing further details of the the positions and of the abstract structure itself. A position in a structure, on *ante rem* structuralism, has a dual character. Seen from one perspective, a position from the natural number structure is an *object*. Seen from another perspective, a position is a *role* or what Shapiro calls an *office*. It can be occupied by a position from another structure — a set, for example — taken as an *object*.

Shapiro introduces the distinction between *system* and *structure* to show how the fact that there are different isomorphic set theoretical structures for arithmetic can be understood in the framework of *ante rem* structuralism ([30], pp. 73-74; 99). Any one of the set theoretical structures

[6]For details of *ante rem* structuralists' epistemological program, see chapter 4 of Shapiro's [30] and chapter 11 of Resnik's [28].

of arithmetic $\langle N, 0, S \rangle$ is now called a *system*, which according to Shapiro is a "collection of objects with certain relations". Objects from a *system* of arithmetic can occupy the positions of the *ante rem* structure of arithmetic — taken as *offices*. The system is then said to *exemplify* the natural number *structure*, which is "the abstract form of [the system], highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system". For Shapiro, the criterion of exemplification is highly permissive. As long as there are enough objects — be they abstract or concrete — and they are in the right relation — for example, ordered by the successor relation — they can be taken as a system which exemplifies a mathematical structure; no further restriction is imposed on them. This characteristic is what makes an *ante rem* mathematical structure different from other abstract structures such as a particular chess pattern. A pattern isomorphic to a chess pattern cannot be a chess pattern if the objects which exemplify it are not chess pieces or are chess pieces that do not stand in the correct distance from one another. The abstract structure of a chess pattern, therefore, will cease to exist if there are no longer chess pieces which stand in the correct distance from one another. An *ante rem* structure, in contrast, is *freestanding*. Take the *ante rem* structure of natural numbers. It is exemplified by itself, because it consists of positions ordered by the successor relation. It is also exemplified by the system containing its even numbers or the von Neumann ordinals (which are *positions* from a set theoretical *structure*) ordered by a successor relation. Even if the existence of a universal is parasitic on its exemplifications by particulars, given the way an *ante rem* structure is characterized, it is necessary that there is an abundant supply of positions that exemplify it. This explains why *ante rem* structuralism is so called.[7]

[7]In this exposition, I take Shapiro's views on structure to be the default rather than Resnik's. On Shapiro's view, a structure is an object, to be ranged over by a special quantifier for structures ([30], p. 93). On Resnik's view, however, a structure cannot be an object. One of the reasons why he thinks so is that in trying to provide identity conditions for structures, one has to establish relations between objects from one structure to those of another. For example, if isomorphism is used as a criterion of identity for structures as on Shapiro's account ([30], p. 93), we need to find a function that maps objects from one structure to those of another structure. This move is incompatible with the *ante rem* structuralists' tenet that a mathematical object has no property apart from its relations with other objects in the same structure, because if mapping from one structure to another is allowed, an object from one structure has the extra-structural property of being mapped to an object in another structure. Despite Resnik's caution, his view is incoherent. On one hand, mathematical objects, according to him, are structureless positions in a structure ([27], p. 530). On the other, he claims that a structure is not an 'individual' (which I take to be synonymous with 'object') — in his own words: "unless we make radical revisions in our logical notation, speaking of patterns as identical or distinct is to treat them as individuals, since identity is a relation between individuals ([28], pp. 209-210)". He leaves us in the dark as to whether there exists structures for mathematical objects to *exist-in*. Due to this inconsistency in Resnik's version of *ante rem* structuralism, I take Shapiro's version to be a more defensible account of *ante rem* structuralism.

Solution to the problem: ontological incompleteness of mathematical objects

How can this elaborate account of mathematical structures solve Benacerraf's problem? Before discussing their solution to the problem, it is important to recognize the shape of Benacerraf's problem. The problem he raises has to do with various identity statements such as $2 = \{\{\emptyset\}\}$; $2 = \{\emptyset,\{\emptyset\}\}$; $2 =$ Caesar. These identity sentences are all grammatical. In all of them, an identity symbol is placed between two singular terms. Benacerraf's question is to ask whether one can provide a truth value for them in a principled way. Mathematical realists run into problems because they do not have a principled way of providing truth values for the sentences. Benacerraf's solution is to eliminate the reference of '2' so that the identity sentences become ungrammatical and hence do not demand from us a truth value. On *ante rem* structuralism, 2,$\{\{\emptyset\}\}$,$\{\emptyset, \{\emptyset\}\}$ are all taken at face value. They are singular terms referring to positions in structures. '2' refers to the 2-position in the natural number structure. $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ refer to two distinct positions in a set theoretical structure. It is, however, not clear how characterizing a mathematical object as a structureless position in a structure can by itself solve the problem; the identity symbol does not seem to respect whether the objects denoted by the two singular terms it flanks are structureless or not. In the following, we will examine whether *ante rem* structuralists have a good strategy to tackle the problem.

The solution to the problem proposed by Resnik and supported by Shapiro [30] is the thesis of *ontological incompleteness of mathematical objects*.[8] The upshot of the thesis is that it is not that we do not know the truth values of $2 = \{\{\emptyset\}\}$; $2 = \{\emptyset,\{\emptyset\}\}$; $2 =$ Caesar; it is that there are no such facts to be known about the number **2**. This thesis is articulated in one way or the other by Resnik and Shapiro:

> Mathematical objects are incomplete in the sense that we have no answers within or without mathematics to questions of whether the objects one mathematical theory discusses are identical to those another treats; whether, for example, geometrical points are real numbers ([28], p. 90).
>
> I want to add a caveat to my epistemic turn. Characterizing my structuralism as epistemic may suggest to some that the incompleteness of mathematical objects is to

[8] In the literature, *incompleteness* is often taken to be synonymous with *ante rem* structuralism's slogan that a mathematical object has no intrinsic properties other than its relations it bears with other objects in the same structure. However, in this thesis, I take incompleteness to be a thesis restricted to cross-structural identity of mathematical objects.

> be understood as the thesis that there really is a fact of the matter as to whether, say, numbers are sets, but it is an unknowable fact. *This is not my view.* The only unknowable facts that I have recognized are those we can credit to bivalence, and I have proposed restricting bivalence to avoid the facts associated with the incompleteness of mathematical objects ([28], p. 270, emphasis mine).

> It makes no sense to pursue the identity between a place in the natural-number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures ([30], p. 79).

This thesis is claimed by Resnik to be an epistemic turn. The nature of a mathematical object is such that identity statements about two objects from two different mathematical structures either "make no sense" or are neither true nor false. The thesis seems to be in line with *ante rem* structuralism's tenet that a mathematical object has no properties other than its relations with other objects *in the same structure*. The sentences $2 = \{\{\emptyset\}\}$, $2 = \{\emptyset,\{\emptyset\}\}$, $2 =$ Caesar seem to demand us to concede that a mathematical object does have properties other than its relations with other mathematical objects in the same structure, which is unacceptable according to *ante rem* structuralists. So, there is a temptation for *ante rem* structuralists to treat these sentences as either senseless or neither true nor false. We will consider in detail whether the temptation should be resisted in the next chapter.

Two questions are in order. (i) What are Resnik's and Shapiro's arguments for this thesis? (ii) Can the *ante rem* structuralist slogan that a mathematical object has no properties other than its relations with other mathematical objects in the same structure be accepted without careful qualifications? We first answer the second question. The answer to the question is 'no'. If mathematical objects have only structural properties, it is not clear how we can ever have knowledge of them or apply them. Does the number 2 have the property of being known by most of us, and the property of being the the number of biological parents a human being has? Also, the property of being self-identical has nothing to do with structural relations. So, *ante rem* structuralism's tenet that a mathematical object has only structural properties cannot be accepted without careful qualifications. (i) is therefore an important question; it asks whether the tenet can be accepted with respect to cross-structural identity-facts about mathematical objects as the thesis of incompleteness demands.

One argument Resnik often uses in supporting the incompleteness of mathematical objects is the argument from mathematical practice. A typical example is the following: "mathematical realists are not committed to claims about mathematical objects beyond those they hold by virtue of endorsing the claims of mathematics. Since mathematics recognizes no facts of the matter in the puzzling cases, mathematical realists are free to develop solutions that do not recognize them either ([28], p. 92)."

However, it is not clear how appeal to mathematical practice alone can solve philosophical problems about the nature of mathematical objects, which is outside of mathematicians' inquiry. It is also not clear how Resnik can infer from the fact that mathematics recognizes no facts of the matter concerning the sentence $2 = \{\{\emptyset\}\}$ that there *is* no such fact of the matter. To so infer is to confuse an epistemological question with an ontological question.

The second argument for the thesis of incompleteness addresses directly the preliminary remarks I made at the beginning of this section. It seeks to argue for the meaninglessness of an identity statement like $2 = \{\{\emptyset\}\}$ in a more subtle way. It puts forward a thesis of *relative identity* which restricts the scope of identity symbol to objects of one particular kind, within which *criteria of identity* can be provided for the objects.[9] It was articulated in length by Benacerraf [1] and was endorsed by Shapiro [30] who then associates it with the Quinean dictum of "no entity without identity":

> I propose to deny that all identities are meaningful, in particular to discard all questions of the form of (c) above [e.g. $17 = \{\{\{\emptyset\}\}\}$; Caesar $= \{\{\emptyset\}\}$] as senseless or "unsemantical" ... Identity statements make sense only in contexts where there exists possible individuating conditions. If an expression of the form "x=y" is to have a sense, it can be only in contexts where it is clear that both x and y are of the some kind or category C, and that it is the conditions which individuate things as the same C which are operative and determine its truth value. (Benacerraf [1], pp. 286-287).

'Frege's preliminary account does not have anything to say about the truth-value of the

[9] It is important to clarify that they are not committed to *relative identity* in the following sense: (R) there are distinct properties F and G such that $a = b$ is true relative to the property F while $a = b$ is false relative to the property G, and either $G(a)$ or $G(b)$ is true. This form of relative identity violates the indiscernibility of identicals. If a and b refer to the same F, they refer to the same G, for any property G. For example, a and b refer to the same horse, it cannot be that a and b refer to two distinct white things. See chapter 1 of Wiggins [33] for details.

> identity "Julius Caesar = 2." This quandary has come to be called the *Caesar problem.* A solution to it should determine how and why each number is the same or different from *any object whatsoever.* The Caesar problem is related to the Quinean dictum that we need criteria to individuate items in our ontology. If we do not have an identity relation, then we do not have bona fide objects. The slogan is "no entity without identity" ([30], p. 78).
>
> The Frege-Benacerraf questions do not have determinate answers, and they do not need them ([30], p. 80).
>
> Quine's thesis is that within a given theory, language, or framework, there should be definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this ([30], p. 92).

Shapiro's proposal here is that the identity symbol is restricted to objects of some kind within which certain *individuating conditions* or *criteria of identity* can be provided for the objects. $2 = 3$ has a determinate truth value, because **2** and **3** are both in the natural number structure; we can tell one apart from the other by appealing to the different structural relations they exhibit in the structure. $2 = 3$ is false because, whereas **2** has two predecessors, **3** has three predecessors. The same cannot be done with respect to the identity $2 = \{\{\emptyset\}\}$. Neither the axiom of extensionality nor the successor relation can allow us to differentiate one from the other. So this identity statement is meaningless, or we have to be committed to that both the number **2** and the set $\boldsymbol{\{\{\emptyset\}\}}$ are not objects, which is unacceptable according to *ante rem* structuralism.

Appealing as this solution might seem, we will see in the next chapter that it should be dropped because it leads to serious difficulties for *ante rem* structuralism.

CHAPTER III

THE PROBLEM OF AUTOMORPHISM AND THE NOTION OF *IDENTITY CRITERIA*

To pave the way for the discussion of this chapter, I now describe, in a more schematic way, what Shapiro [30] is committed to in his adoption of Benacerraf's proposal of *relative identity*. Their proposal of putting restrictions on identity can be captured by the following logical form:

$$(IC)\ \forall x \forall y (Fx \land Fy \rightarrow (x = y \leftrightarrow \varphi_F(x, y)))$$

Its intended meaning is that for any two objects x and y belonging to a kind F, they are the same object if and only if they satisfy an equivalence relation φ_F suitable for kind F. The equivalence relation is intended to provide the *criteria of identity* for objects belonging to kind F. Appealing to the Quinean dictum of "no entity without identity", Shapiro seems to suggest in his [30] that the schema (IC) should be adopted for his structuralist account of mathematics. He takes objects belonging to the same *ante rem* structure as belonging to the same kind; the identity of the objects in each structure has to be governed by an equivalence relation which can be expressed in terms of their relations to each other. For example, objects in the natural number structure and objects in a set theoretical structure, according to the proposal, should have their identity governed by schemas like the followings:

$$(N)\ \forall x \forall y (Number(x) \land Number(y) \rightarrow (x = y \leftrightarrow \forall z (z < x \leftrightarrow z < y)))$$

$$(AE)\ \forall x \forall y ((Set(x) \land Set(y)) \rightarrow (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)))$$

The equivalence relation $\varphi(x, y)$ in (N) "having the same predecessors" i.e.$\forall z (z < x \leftrightarrow z < y)$, on this proposal, can be said to govern the identity of natural numbers. Similarly, the equivalence relation "having the same members" i.e.$\forall z (z \in x \leftrightarrow z \in y)$ governs the identity of sets.

The reason why Shapiro [30] adopts this strategy, as we have discussed in the last chapter, is to allow an *ante rem* structuralist to rule out identity statements involving two objects from different structures, by appealing to the fact that their identity is not appropriately governed by the schema. With all these identities ruled out, an *ante rem* structuralist can maintain the thesis of *incompleteness of mathematical objects* and hence block the Benacerrafian question of whether one object from one structure is identical to an object from another structure.

Several preliminary questions concerning this proposal are in order. First, if Shapiro's thesis of relative identity is invoked only to maintain the incompleteness of mathematical objects, it appears to be an *ad hoc* move. Can *ante rem* structuralists provide an independent justification for their thesis about identity? Second, it is not clear whether and why the identity relations among mathematical objects have to be governed by other relations. Any object is identical to itself and to no other objects, and necessarily so. This seems to be a logical fact that need not be grounded on any other facts. Can identity of mathematical objects within any given structure be analyzed in the way Shapiro suggests? Third, it is not clear whether identities not governed by the schema Shapiro endorses are really senseless. Take the identity statements (i) a chess piece is identical to the natural number 2 and (ii) every natural number is equal to the sum of two real numbers i.e. $\forall x(Natural(x) \rightarrow \exists y \exists z(Real(y) \wedge Real(z) \wedge x = y + z))$. The objects to be identified in these two sentences fall outside of any identity schema, because there is not an *ante rem* structure to which both the chess piece and the natural number 2 belong, or one to which both the natural numbers and the real numbers belong. However, it seems that we are able to reasonably explain why (i) is false and (ii) is true without appealing to any identity schema. Consider (i): the natural number 2 is *freestanding* on *ante rem* structuralism; it therefore exists necessarily, whereas the chess piece does not. So, the natural number 2 cannot be a chess piece. Consider (ii): for every natural number, take 0 and itself to be the required real numbers. The sentence, though not mathematically interesting, is obviously true.

It seems that *ante rem* structuralists have to pay a high price in demanding a structure relative grounding of identity. The main question this chapter tries to answer is whether the price is right. I will answer the question in the negative. I will argue that the *problem of automorphism* is an inevitable result of Shapiro's thesis of relative identity; that Ladyman's and Button's solutions to the problem are both unsatisfactory; and finally that the demand for kind-relative identity criteria, though well-entrenched in philosophy, can be put into doubt and therefore has little purchase in discussions of metaphysical and epistemological issues surrounding *ante rem* structuralism.

1. The Problem of Automorphism

The problem of automorphism is one of the most discussed issues on *ante rem* structuralism. It was raised by Burgess in his [5] and elaborated by Keranen in [15] and [16]. The upshot of the problem is that in appealing to structural relations to provide criteria of identity for mathematical objects, *ante rem* structuralists are committed to the view that structurally indiscernible objects are identical.

I now discuss in detail the shape of the problem. Shapiro's proposal of relative identity works perfectly well for natural numbers and sets. With them, *ante rem* structuralists can restrict their identity relations using the following two schemas:

$$(N)\ \forall x \forall y (Number(x) \land Number(y) \rightarrow (x = y \leftrightarrow \forall z (z < x \leftrightarrow z < y)))$$

$$(AE)\ \forall x \forall y ((Set(x) \land Set(y)) \rightarrow (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)))$$

Using these schemas, *ante rem* structuralists can provide a basis for the identities of natural numbers and sets by appealing to their structural relations alone. In doing so, they honor the Quinean dictum of 'no entity without identity' — or more precisely, 'no entity without *identity criteria*' — without attributing to numbers and sets any non-structural properties, a move that runs against their tenet that a mathematical object has no properties other than its relations with other objects in the same structure.

However, *ante rem* structuralists run into problems when they are asked to provide identity criteria for mathematical objects belonging to structures which admit of non-trivial automorphisms. We will see why if we appeal to the homomorphism theorem. Let $\mathfrak{A}$ be the structure of a language $\mathcal{L}$ and j be a homomorphism from $\mathfrak{A}$ into $\mathfrak{A}$. Let $a_1, ..., a_n$ be the objects in the domain of $\mathfrak{A}$. For any formula $\phi(x_1, ..., x_n)$ of $\mathcal{L}$, $\models_{\mathfrak{A}} \phi(a_1, ..., a_n) \Leftrightarrow \models_{\mathfrak{A}} \phi(j(a_1), ..., j(a_n))$. We can read from the biconditional that any relation satisfied by a sequence of objects in the structure is also satisfied by their corresponding images under the automorphism. This fact about automorphism poses a serious problem for *ante rem* structuralists. The reason is that, seen through the lens of an *ante rem* structuralist, mathematical objects are structureless, characterized solely by their relations with other mathematical objects. They are not labeled by any names, by which we can trace them from one automorphic structure to another. Suppose a_k and $j(a_k)$ are distinct objects in the structure $\mathfrak{A}$. How a_k is related with $a_1, ..., a_n$ in the original structure is exactly the same as how $j(a_k)$ is related with $j(a_1), ..., j(a_n)$ in the structure automorphic to the original one. There is no way to tell one apart from the other by their structural relations alone. Therefore, if *ante rem* structuralists are to use structural relations to provide criteria of identity for objects in a structure admitting of non-trivial automorphisms, they have to be committed to the view that two distinct mathematical objects that are structurally indiscernible are identical, which is absurd.

To explain the problem in a more concrete way, we consider a simple directed graph, **G1:** $(\{a, b, c\}; \{\langle a, b\rangle, \langle a, c\rangle\})$. By sending b to c, c to b, and a to a, we have an automorphism for the structure. Here is the problem. By appealing to the structural relations alone, we can only single out **a** from the other two objects, because it is the only object which points to two distinct objects, i.e. satisfying the 1-place relation defined by $\exists y \exists z (y \neq z \wedge xRy \wedge xRz)$. The other two objects, however, are indistinguishable, because each of them does not point to any objects and is pointed to by one object. So, if *ante rem* structuralists are committed to the proposal of relative identity, they have to hold that **b** and **c** are the same object.

This problem poses a serious challenge to *ante rem* structuralists. Insofar as they are committed to the thesis of relative identity, they have to accept results that are unfaithful to mathematical practice. $i = -i$ is true on the complex number structure because the two numbers share the same structural relations. $3 = 4$ is true on $\langle R, < \rangle$ because with the automorphism $f(x) = x + 1$, we can show that **3** and **4** share the same structural relations. Is there a way out for *ante rem* structuralists? In the following, we consider two attempted solutions to the problem.

2. Ladyman's Solution and Quine's Notion of Weak Discernibility

To solve the problem of automorphism, Ladyman invokes Quine's notion of *weak discernibility* ([22], pp. 61-64; [24]): "two objects are 'weakly discernible' just in case there is two-place irreflexive relation that they satisfy" ([17], p. 220). The upshot of his solution is that we can provide criteria of identity for mathematical objects by identifying irreflexive relations in the structure to which they belong. His solution can be given the following logical form:

$$(L)\ \forall x \forall y (Fx \wedge Fy \rightarrow (x = y \leftrightarrow \forall R (\forall z \sim Rzz \rightarrow \sim Rxy)))$$

What this schema means is that for any two objects x, y belonging to the same $\mathcal{L}$-structure (strictly speaking, the same *ante rem* structure exemplified by the $\mathcal{L}$-structure), they are identical if and only if x and y do not satisfy any irreflexive relation *definable* in the language $\mathcal{L}$. Ladyman does not mention explicitly that the irreflexive relation involved in the identity criteria has to be understood as one definable in the language $\mathcal{L}$. However, from the examples which he uses to illustrate the resourcefulness of this schema, we can reasonably assume that the irreflexive relation he refers to has to be so understood.

Ladyman suggests that with this schema, we can rescue *ante rem* structuralists from the problem of automorphism with respect to the complex number structure $\langle \mathbb{C}, +, \cdot, 0, 1\rangle$ and the group $\langle \mathbb{Z}, +\rangle$. For example, with i and -i, which are structurally indiscernible and are hence identical on Shapiro's schema, we can now differentiate one from the other by defining the irreflexive relation 'is the additive inverse of' by the formula $x + y = 0$. Since i and $-i$ satisfy an irreflexive relation definable in the language, they cannot be the same object according to (L). We can use the same relation 'is the additive inverse of' with the structure $\langle \mathbb{Z}, +\rangle$ to differentiate z from $-z$, for any $z \in \mathbb{Z}$. So, Ladyman's solution seems to solve the problem of automorphism.

However, two questions are in order. First, does his solution really work — even for complex numbers? Further, can the schema provide criteria of identity for structures where there is no definable irreflexive relations? Second, what are his motivations for adopting a Quinean notion of identity for a problem in philosophy of mathematics? Is Quine's project relevant to our project of making clear the nature of mathematical objects?

I answer the first question in the negative. It is far from clear why Ladyman takes a particular case as a solution of a more general problem of differentiating $a + bi$ from $a - bi$. Setting aside for now whether the relation $x + y = 0$ can be singled out while the identities of the relata are at issue, it is not clear how this relation can settle the identities for the pair $a+bi$ and $a-bi$, where $a \neq 0$.[10] For example, what is the suitable irreflexive relation for $2 + i$ and $2 - i$? Ladyman does not provide us with any. So, his solution lacks generality.

There may be, however, one way we can rescue his solution. Drawing inspirations from Quine's program of reduction of identity ([22], [24]), we can provide a trivial irreflexive relation for $a+bi$ and $a - bi$ for all $a, b \in \mathbb{R}$. The following irreflexive relation $\varphi(x, y)$ is true for any two distinct complex numbers: $\exists z_1 \exists z_2(x + z_1 = z_2 \wedge y + z_1 \neq z_2))$, because it trivially means that any two distinct complex numbers cannot give the same sum when operated under addition with the same complex

[10] For discussions as to whether a relation can be ontologically prior to its relata, see Hellman's [12] (pp.193-194).

number. It seems to be an irreflexive relation no less respectable than Ladyman's — $\varphi(x, x)$ is false for any $x \in \mathbb{C}$. It is more general than Ladyman's in that it can ground the identities for *all* complex numbers. It also does not attribute to complex numbers any non-structural properties. So, it should be acceptable as a solution to the problem of automorphism — if Ladyman's proposal is. However, we are faced with very difficult issues once we are committed to this generalized form of Ladyman's solution. (i) Does the relation exist before the identities of the relata are settled? Do we in using this relation to give a criterion of identity for the complex numbers already presuppose their distinctness? (ii) If one does say that this relation is ontologically prior to the relata and hence it is legitimate to use it to ground the identities of the relata, how should we make sense of that claim? (iii) Would one make more sense if she proposes a mutual dependence thesis on which neither the relata nor the relations are prior to each other? These are difficult questions that may make us desist from adopting this as a repair of Ladyman's solution.[11]

Fortunately, we need not answer these difficult questions to see that Ladyman's solution — whether it is generalized or not — is not going to work. The attraction of using an irreflexive relation to solve the problem of automorphism can be further weakened if we think about some simple graphs. Take **G1:** $(\{a, b, c\}; \{\langle a, b\rangle, \langle a, c\rangle\})$, an example Button [6] uses to undermine Ladyman's solution. It is not clear how we can find within this structure an irreflexive relation that is satisfied by **b** and **c**. We can go even further than Button does in using a structureless graph, analogous to Shapiro's oft-mentioned finite cardinal structures ([30], p. 115; [31], pp. 141-142), to undermine Ladyman's solution. Call it **G0**: $(\{a, b, c\}; \emptyset)$. It seems very few will be tempted to adopt Ladyman's solution or its generalized version and say that there is actually one point on the graph.

We now come to the second question: what are Ladyman's motivations for borrowing from Quine the notion of weak discernibility? His answer to the question is disappointing for two reasons. First, in justifying his use of the notion, he appeals to the works of philosophers of physics, highlighting the purchase of weak discernibility in quantum physics. It is, however, not clear what makes it legitimate for Ladyman to directly apply views concerning the identity conditions of physical entities to the identity conditions of mathematical objects. In so doing, he makes the nature of mathematical objects dependent on how the physical world is like. However, there seems to be

[11] A possible source of these difficulties is that *ante rem* structuralists do not make clear the dependence relation between relata and relations. They also do not make clear whether the structural relations they refer to include the definable ones or not, and if they do, what restrictions should be set on them — for example, whether the identity symbol can be used in defining a relation.

no reason to rule out mathematical structures that are unlike any structure exemplifiable in the physical world. Second, what is also unclear is why we need identity conditions for *all* mathematical objects in the first place. Ladyman does not provide us with any answer to that.

3. Button's Hybrid Solution

We now consider another attempted solution to the problem of automorphism which respects Shapiro's proposal of providing structure-relative identity conditions for mathematical objects. Button offers a hybrid solution to the problem, which he characterizes as "realistic at its core and eliminativistic at its limits" ([6], p. 216). His solution is to hold that *ante rem* structures exhibiting non-trivial automorphisms are not objects; they are *constructed* from positions of basic structures, the rigid structures that really exist. Since structures admitting of non-trivial automorphisms are not objects, sentences about positions in them have to be treated in an eliminativist way, along the lines of the universalized version of set theoretical structuralism we discussed in the second chapter.

This solution seems to be intended to take the best out of eliminativist structuralism and *ante rem* structuralism. First, in rejecting the existence of structures admitting of automorphism, we seem to achieve some success in ontological reduction. To have a diversity of mathematical objects, we only need to construct them out of positions from the basic structures. It is not necessary to include in our ontology more entities than is necessary for the diversity of mathematical objects. Our acceptance of *ante rem* structuralism at its core already ensures that we have an abundant number of objects to construct a multiplicity of mathematical structures. Second, with *ante rem* structuralism to provide the background ontology, we do not fall into the problem of non-vacuity facing set theoretical structuralists, which behooves them to remedy it by privileging set theory.

However, if we take seriously the desiderata of different forms of structuralism, it is not clear what Button is trying to achieve, except to provide us with a trivial solution to the problem of automorphism. First, assessed from the perspective of a set theoretical structuralist, Button's ontological reduction is not thorough-going. For a set theoretical structuralist, all mathematical objects are reduced to sets. She does not have to countenance the existence of a universe of abstract universals as *ante rem* structuralists do. She also does not need to answer difficult questions about the nature of individual numbers and of the natural number structure, or whether the natural number 2 is identical to the real number 2, because on her view such things do not exist. All the difficult matters with the nature of mathematical objects are deferred to the level of sets. So, in not opting for a thorough-going ontological reduction, that is, in accepting the existence of rigid *ante rem* structures, Button is under the burden of giving us a solution to Benacerraf's problem regarding positions in all rigid structures. This does not seem to be attractive from the perspective of a set theoretical structuralist. Also, being a *mathematical realist* about sets, a set theoretical structuralist can rightly point out that, on Button's proposal, it seems it is up to us to determine which mathematical objects should exist based on whether they belong to a structure which admits of non-trivial automorphism(s). However, is there really any real metaphysical difference between symmetric structures and the rigid ones?

Second, in adopting a hybrid solution, we need to have a hybrid semantics. How should we understand the logical forms of the following sentences:

(B1) There are at least three numbers in the natural number structure.

(B2) There are at least three numbers in the complex number structure.

(B3) So, there are at least six mathematical objects.

A universalized set theoretical structuralist can apply the translation procedure to (B1) and (B2) reducing away the apparent commitment to the existence of numbers in them and argue that although (B3) is true — for they countenance the existence of sets — it does not follow from (B1) and (B2). An *ante rem* structuralist, taking these sentences at face value, can pronounce immediately that the three sentences make up a valid argument. A hybrid structuralist will say (B2) is of a different kind than (B1) and (B3); they cannot be treated under the same semantics. This consequence is undesirable from the standpoints of both eliminative structuralists and *ante rem* structuralists — particularly so for *ante rem* structuralists who hold that it is preferable to be faithful to mathematical practice by taking mathematical sentences at their face value.

Third, it is not unfair to ask whether a hybrid solution needs a hybrid epistemology to sustain. Is the natural number structure, which exists as an abstract universal on Button's proposal, known differently than natural numbers under linear ordering, which are merely constructions? Is an asymmetric graph known differently than a graph with symmetries? It is not clear whether we have two modes of knowing mathematics — one for rigid structures and one for non-rigid structures.

So, Button does not provide us with a good solution to the problem of automorphism either. What routes should *ante rem* structuralists take in the face of the problem? I will argue that they should drop their thesis of relative identity as a convenient tool to solve Benacerraf's problem of multiple reductions of arithmetic, because the thesis not only leads to the problem of automorphism, which we seem to have good reasons to believe is insurmountable, it can also be put into doubt, despite the fact that it is well-entrenched in philosophy.

4. What Criteria of Identity Could Be

Primitive identity for mathematical objects in the same structure

The reasonable route for *ante rem* structuralists to take, I submit, is to drop the thesis of relative identity and take identities among mathematical objects in the same structure as primitive, that is, as facts that need not be given an account of using the schema (IC) I mention at the beginning of this chapter. This move, however, is not considered by Ladyman and is strongly opposed by Button for reasons we shall discuss shortly. Before I examine in detail the possible reasons for supporting the thesis of relative identity, I make the following two preliminary remarks.

First, judging from what we have discussed thus far, there is a strong *prima facie* case against the thesis of relative identity. The thesis of relative identity is attractive to *ante rem* structuralists because it supports the thesis of incompleteness of mathematical objects, which allows them to block Benacerraf's question of whether one object from one structure can be identical to an object from another structure. *Ante rem* structuralists gain from their commitment to the thesis of relative identity a clean and convenient solution to Benacerraf's problem. However, as I have suggested at the beginning of this chapter, the commitment to relative identity makes it difficult for us to account for the meaningfulness of identity statements relating an object inside a mathematical structure to an object outside of it. It is not clear why the statements (i) 'a chess piece is identical to the natural number 2' and (ii) 'every natural number is equal to the sum of two real numbers' should be considered senseless because of our commitment to the thesis of relative identity. I do not rule out that *ante rem* structuralists can come up with ingenious solutions to account for their meaningfulness in a way that is consistent with the thesis of relative identity.[12] So, we at best can make here a weak *prima facie* case against the thesis of relative identity. However, with the problem of automorphism established, we have good reason to think that adherents of the thesis of

[12]One possible solution is to come up with a kind which subsumes the two objects in question. However, the problem with this solution is that there does not seem to be a principled way to determine the point at which we cannot subsume two kinds under a more general kind.

relative identity are on the defensive.[13] *Ante rem* structuralists who appeal to the thesis of relative identity are under the obligation of explaining why, amidst various difficulties, we have to adopt the thesis.

Second, taking identity to be primitive need not be seen as a desperate move by *ante rem* structuralists to solve the problem of automorphism in a trivial way. I submit that it can be motivated by *ante rem* structuralists' official commitments. On *ante rem* structuralism, "[t]he structure is [ontologically] prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it" ([30], p. 78). Setting aside Shapiro's use of a perhaps misleading analogy, we consider the following conceptual possibility.[14] Given that an *ante rem* structure is ontologically prior to the objects that belong to it, instead of looking first at how the objects are individuated, we should first ask what is essential to an *ante rem* structure. The natural answer to the question is 'its mathematical properties', given *ante rem* structualists' commitment to the *faithfulness constraint*. **G0**, a structureless graph with three objects in it, cannot be what it is without three distinct points in it. The complex number structure cannot be what it is with any pair of $(a+bi, a-bi)$ taken as one object. The distinctness of mathematical objects in a structure is essential to its existence. Without the existence of an *ante rem* structure, there are no mathematical objects whose identities are to be questioned. It seems that adherents of relative identity neglect *ante rem* structuralists' view about the dependence relation between structures and positions. On *ante rem* structuralism, positions depend for their existence on the structure to which they belong. So, they are wrong in asking questions about the individuation of mathematical objects without first asking questions about what is essential to the existence of a structure.

Since an *ante rem* structuralist can motivate intra-structural primitive identities along the lines I discuss here, friends of criteria of identity opposed to this move should point out why this conceptual possibility is illusory.

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[13] I do not discuss in length other possible but less plausible solutions to the problem of automorphism. They include the (a) invocation of haecceities and (b) the appeal to extra-structural relations. The former runs against the *ante rem* structuralism's tenet that a mathematical object has only intra-structural properties. It also leads to an explosion of isomorphic *ante rem* structures with objects having different haecceities, which generates the same Benacerrafian problem on the level of *ante rem* structures. The second solution fails because the choice of extra-structural relations is completely arbitrary. There seems to be no principled way of explaining why a particular extra-structural relation applies only to one object but not to its image under a non-trivial automorphism.

[14] The reason why the analogy is misleading is that an organization fails to capture what a mathematical structure is like. An organization can be what it is if certain offices are removed from it, but it is not case with mathematical structures. For example, the natural number structure cannot be what it is if the 1-place is removed from it.

The well-entrenched notion of criteria of identity

Given that we have a *prima facie* case against the the provision of structure-relative criteria of identity and a case for adopting primitive identity for objects *within* the same mathematical structure, how can friends of criteria of identity sustain their thesis? In order not to trivialize their position, before assessing their arguments for the thesis I should note that the notion of *criteria of identity* does not come out of nowhere, stipulated on an *ad hoc* basis to solve the Benacerrafian problem. *Criteria of identity* is in fact a well-entrenched notion in philosophy. Philosophers seem to have no solace until they can find the identity conditions for the objects they admit into the ontology. Different kinds of objects — whether or not they are physical or mathematical — are given criteria of identity in a sentence of the form (IC) or its variants. The following is an assortment of these attempts:

$(D)\, \forall x \forall y((Line(x) \wedge Line(y)) \rightarrow d(x) = d(y) \leftrightarrow x \parallel y))$

"[T]he direction of line x is identical with the direction of line y if and only if lines x and y are parallel with one another" (Frege [8], p. 136; Lowe [18], p. 620).

$(N_F)\, \forall F \forall G(Nx : Fx = Nx : Gx \leftrightarrow \exists R(\{x : Fx\}1 - 1_R\{x : Gx\})$

"[T]he number of Fs is identical with the number of Gs if and only if the set of Fs is one to one correlated with the set of *Gs"* (Frege [8], p. 135; Lowe [18], p. 620).

$(E)\, \forall x \forall y((E(x) \wedge E(y)) \rightarrow (x = y \leftrightarrow \forall z(E(z) \rightarrow ((C(x, z) \leftrightarrow C(y, z)) \wedge (C(z, x) \leftrightarrow C(z, y))))))$

"[I]f x and y are events, then x is identical with y if and only if x and y cause and are caused by the same events "(Davidson [7], p.179; Lowe [18], p.621).

(Qc) Classes are identical if and only if their members are identical (Quine [23], p.100).

(Qo) Physical objects are identical if and only if they are coextensive (Quine [23], p. 101).

What is the notion of *criteria of identity* for? According to some philosophers, it allows us to distinguish between entities that are metaphysically respectable and those that are not. Only entities for which identity criteria exist can be admitted into the ontology. For example, according to Lowe,

> What is crucial to the status of 'thinghood' ... is ... the possession of determinate identity-conditions ... This is where the notion of a 'thing' or 'object' ties in with that of a criterion of identity, for one guarantee that something possesses determinate identity-conditions is that it falls under a general concept which supplies a criterion of identity for its instances ([18], p. 613).
>
> [W]e are entitled to deny the status of 'objects' to facts and propositions (on the grounds that they lack determinate identity-conditions) ([18], p. 618).

According to some philosophers, criteria of identity also have an epistemic role of allowing us to provide a truth value for an identity statement or of understanding the nature of the objects for which we provide criteria of identity. So, it seems the notion of identity criteria is generally regarded in philosophy to have both metaphysical and epistemological purchase. It is therefore a daunting task to argue that it has no purchase at all in philosophy. Given that the goal of this chapter is to answer the question of whether it is right for *ante rem* structuralists to adopt the thesis of relative identity, I am not going to examine whether there is any unity in different philosophers' uses of the notion of identity criteria beyond their formal similarity; which philosophical projects can justify the use of the notion and which cannot; or whether the dictum of "no entity without kind-relative identity criteria" should be taken as a first principle in answering ontological questions. These issues are of a depth that this thesis cannot adequately address. I only seek to answer a narrower question of whether there are overriding reasons for *ante rem* structuralists to be committed to the dictum. I argue that the purported purchase of the dictum can be put into doubt on both the metaphysical and the epistemological fronts.

Purported metaphysical purchase of criteria of identity

One purported metaphysical function of criteria of identity is that they provide properties and relations on which the identity of objects *supervene*. Button and Keranen articulate this view in the following way:

> [A]ccepting indistinguishables [as distinct objects] requires that identity facts are primitive, for there are no *properties* or *relations* upon which the distinctness of individuals

> could *supervene* (Button [6], p. 219, emphasis mine).
>
> To the best of our knowledge, all extant theories of ontology maintain that the identity of objects is *governed* by their *properties* ... '$a = b$' is true if and only if for all properties ϕ in some class Φ, $\phi(a)$ if and only if $\phi(b)$ (Keranen [15], p. 313, emphasis mine).
>
> I maintain that ... there are no distinct but essentially indiscernible objects ... [G]iven any domain of objects, I believe that there must be some fact that metaphysically underwrites the distinctness of any two distinct objects in that domain. Suppose you think that the objects a and b are essentially indiscernible and yet distinct; you must still think that there is *something* about the world that is responsible for the objects being two and not one. Suppose that you think that the non-essential properties of a and b are not up to the task; you must still think that there is *something* about the world that is. Suppose that you think that it is a primitive, 'brute' fact that a and b are two objects there rather than one; surely you must still think that there is something about *each one* that makes it the case that it is the object it is, and not the other. (Keranen [16], p. 156).

I have the following three remarks on this view. First, it has to be noted that the admission of "having a certain haecceity" as a property makes the view entirely trivial. If such a property is allowed, adherents of the supervenience view can always find relations and properties that ground — or in my view, bear the appearance of grounding — identity. By the "something about *each one*", Keranen means haecceity, "a property that can be possessed by one entity alone" ([15], p. 313) — an example is the property of 'being identical to itself'. Of course, two objects x and y are the same if they share the property of "having a haecceity H", the *something* in the world that grounds the identity fact. However, it is not clear what this grounding of identity can do for us, except to secure the metaphysical conviction of certain philosophers. It is not clear why one should stop at the level of having the same haecceity. Why is the sameness of haecceity not grounded by properties that are more basic? Why is the bruteness of sameness of haecceity better than the bruteness of identity?

Second, if we remove "having a certain haecceity" as an acceptable property from the supervenience thesis, it is not difficult to see that the thesis that identity can supervene on relations or properties does not work. Consider again our simple graphs **G0** and **G1**. Suppose they did exist without being embedded into others structures.[15] There are three points in each of them. For **G0**, it is obvious that there are no structural relations on which the identity of the three objects in it can supervene. For **G1**, if identity did supervene on relations, there can be only two objects, because **b** and **c** share the same relational properties on the graph and have to be taken as one object. So, identity cannot supervene on relations or properties in general.[16] Can one argue that the use of this example is unwarranted because these graphs, which admit of non-trivial automorphisms, do not exist? I do not know how to adequately answer this question. I can only retort in the following way: one must have understood my examples in proposing the objection. If she does think that identity has to supervene on relations, could she explain what she understands using relational properties alone?

Third, if my examples fail to convince the adherents of the supervenience view, I shall appeal to Kripke's arguments in "Identity and Necessity" ([14]). A name rigidly designates the same person in different counterfacutal situations in which she exists. If identity is to supervene on relations, we may end up talking about a person resembling to a high degree the person in question but not being the *same* person as the person in question. So, again, the moral is that identity does not supervene on relations or properties.

Purported epistemic purchase of criteria of identity

We now consider two possible epistemic functions of identity criteria: (i) identity criteria allow us to understand what it is to be an instance of a kind *K;* (ii) identity criteria allow us to have access to identity facts.

According to Keranen, "in some sense the Axiom of Extensionality tells us what makes a set, it codifies a key ingredient in our understanding of what sets are in the first place ([16], p. 328)". He seems to suggest here that the criteria of identity of sets as encapsulated in the axiom of extensionality allow us to understand what it is to be a set. Learning from Savellos's [29] discussion of criteria of identity of events and Merricks's [20] discussion of the criteria of identity over time, we can give Keranen the following response. Indeed, the axiom of extensionality does tell us something important about a set, namely, its membership relation. However, the axiom does not tell us other important things about sets. For examples, sets of a very large size, which cannot find

[15] I add this supposition here because I want to block the possible counter-argument that **G0** is an object because it is embedded in a structure within which the positions in **G0** are structurally discernible.

[16] One can use these examples to show that Quine's reduction of identity ([22], [24]) does not yield genuine identity.

concrete exemplications in our world, are essentially abstract; pure sets are essentially positions of a structure, etc. So, criteria of identity do not seem to have a central role in allowing us to understand what it is to be an instance of a kind K.

As to the second epistemic function, Button provides for us the following articulation:

> Suppose m and n have *at all times* all the same properties (including being at the same location, if they are located at all) and all the same relations to everything. I say that $m = n$, as does [Ladyman] ... *[I]t is unclear how we could have access to primitive identity facts* ... [I]f we cannot know whether m and n are one object or two, we also cannot know whether m is one object, or three, or four, ... ([6], p. 219, emphasis mine).

Two remarks are in order. First, his argument seems to be that if there are no criteria of identity, we have to take identity facts to be primitive; yet epistemic access to primitive identity facts cannot be established. Here he is trying to argue for the epistemic merit of criteria of identity *in general*. However, his argument seems to be motivated by a philosophical position on what counts as the same *physical* object. It is not clear whether mathematical objects have to be of the same nature as physical objects. I suspect Button makes the same mistake Ladyman does in applying directly the notion of physical objecthood to mathematical objects. This suspicion is confirmed as he says later in his paper that

> b_{G1} and c_{G1} bear no relations to each other even though they are objects in the same configuration. This is arguably inconceivable. Such objects could not be spatial, for distinct spatial objects are invariably at a distance from each other but not from themselves (i.e. an irreflexive relation obtains between them) ... [I]f b_{G1} and c_{G1} are objects, they are unlike any objects with which I am familiar ([6], p. 219).

He concludes that the two objects **b** and **c** on the graph **G1** cannot be objects on the grounds that they do not share any resemblance with physical objects. It is not clear why mathematical objects need to have any resemblance with spatial objects. Also, his remarks are not in keeping with the structuralist slogan that a mathematical object has only relations it has with other objects in the same structure. Why is its resemblance to spatial objects essential to its existence? Bourbaki's insights are helpful here.

Second, Button's conclusion that we cannot have epistemic access to *primitive identity facts* is questionable. He seems to suggest that we need criteria of identity to ground our knowledge of all identity facts. However, this suggestion is doubtful. Indeed, at times, we do need to appeal to criteria of identity to have knowledge of certain identity facts. For example, with the identity statement $A \cup (B \cup C) = (A \cup B) \cup C$, we have to appeal to the axiom of extensionality which supplies the criteria of identity of sets in order to ground this particular fact — by doing a proof, for example. However, this is certainly not the case for all our knowledge of mathematics. Take our learning of numbers as an example. Do we really need an identity criteria statement as stringent as (N) $\forall x \forall y (Number(x) \land Number(y) \rightarrow (x = y \leftrightarrow \forall z (z < x \leftrightarrow z < y)))$ to tell $1 \neq 2$? It may be more plausible to think that $1 \neq 2$ is epistemically prior to (N), rather than the other way round.

Conclusion

Participants in the debate on *ante rem* structuralism appeal to the notion of criteria of identity to urge *ante rem* structuralists to adopt their suggested positions — however unfavorable they are. With Keranen, given his commitment to *governance of identity by relations and properties*, he leaves *ante rem* structuralists the dilemma of either (a) accepting that structurally indiscernible objects should be taken as the same object or (b) explaining their distinctness by appealing to haecceities and hence violating the tenet of *no properties except structural properties.* His arguments amount to rejecting *ante rem* structuralism outright, because neither (a) nor (b) is acceptable to an *ante rem* structuralist. With Button, respecting the notion of criteria of identity, he offers a hybrid solution, which, as I have discussed above, is also not attractive to *ante rem* structuralists.

However, as our discussion shows, the purchase of the notion of criteria of identity in discussions of the nature and knowledge of mathematical objects can be doubted. So, both Keranen and Button are not entitled to urge *ante rem* structuralists to make the revisions they suggest by their appeal to the notion of criteria of identity. With the notion of criteria of identity cleared away, *ante rem* structuralists should adopt primitive identity with respect to objects in the same mathematical structure, which, as my preliminary remarks of this section suggest, can be motivated by their official commitments to the *ontological priority of structure over positions* and the *faithfulness constraint.* Also, adopting primitive identity within a structure can prevent *ante rem* structuralists from running into the problem of automorphism, which we have reason to believe is irresolvable.

CHAPTER IV

CONCLUSIONS: *ANTE REM* STRUCTURALISM WITHOUT IDENTITY CRITERIA

1. Conclusion: The Benacerrafian Problem Is Left Unsolved

Despite the strong grounds we have for urging *ante rem* structuralists to drop the thesis of relative identity, there is now the pressing issue of Benacerraf's problem of multiple reductions of arithmetic. The thesis of relative identity, according to our discussion in the second chapter, is to support the thesis of ontological incompleteness of mathematical objects, which is intended to block the Benacerrafian problem. However, with relative identity dropped due to the reasons we discussed in the last chapter, the thesis of incompleteness of mathematical objects is left with no support; so the Benacerrafian problem demands from *ante rem* structuralists an alternative solution.

However, Shapiro in his recent works ([31], [32]) does not provide us with a satisfactory alternative solution, although he is well aware that cross-structural identities — such as the example I gave in the last chapter "every natural number is the sum of two real numbers" — cannot be banned as senseless:

> [S]ome cross-structural identities are at least prima facie natural, and perhaps even inevitable. Suppose that during a lecture, a mathematician writes a closed integral on a blackboard, and asks the class to determine whether the number it denotes is a natural number. A philosophy student, fresh from a study of structuralism — having read part of Resnik's book and/ or part of mine — quickly gets the floor while the other students are busy trying to resolve the integral, and proudly announces the solution: there is no fact of the matter. He says that the integral denotes a place in the real number structure, and so it is indeterminate whether places from the real number structure are the same as or distinct from places in the natural number structure. Clearly, something has gone wrong; the philosophy student is confused ([31], p. 125).

His current solution to the Benacerrafian problem is as follows:

> [T]he natural number 2, the real number 2, and the set $\{\emptyset, \{\emptyset\}\}$ are tied to different structures and so they are distinct objects ([31], p. 128).
>
> The mathematician asked a sensible, substantive mathematical question. The job of the student was to answer it, and our job as philosophers is to interpret it. Following the faithfulness constraint, I wanted the mathematician's language to be understood literally, at face value. My solution ... was that the mathematical community had *stipulated* that non-negative whole numbers of the real number structure are indeed the natural numbers. This makes the mathematician's question a proper mathematical one ([31], p. 125, Shapiro's own emphasis).

Shapiro's solution to the problem has two components. First, Shapiro answers the Benacerrafian questions of whether $2_{natural}$ is identical to 2_{real} and whether $2_{natural}$ is identical to $\{\emptyset, \{\emptyset\}\}$ in the negative: strictly speaking, $2_{natural} = 2_{real}$ and, by extension, $2_{natural} = \{\emptyset, \{\emptyset\}\}$ are *false*, because they are identity statements relating distinct objects from different mathematical structures.[17] However, identity sentences involving objects from distinct *ante rem* structures abound in mathematics. To claim that they are false is to violate the *faithfulness constraint*. Hence the need for the second component of the solution, which is to concede that mathematicians can *by stipulation* make it the case that $2_{natural} = 2_{real}$ and $2_{natural} = \{\emptyset, \{\emptyset\}\}$.

The second component is problematic, because it runs against the commitment of mathematical realism. How can mathematicians make it the case that $2_{natural} = 2_{real}$ and $2_{natural} = \{\emptyset, \{\emptyset\}\}$ if in each of them an identity relation is established between two distinct objects from two distinct mathematical structures? On further reflection, the first component of the solution can also be cast into doubt. Why should we think that $2_{natural}$, 2_{real}, and $\{\emptyset, \{\emptyset\}\}$ denote three distinct objects belonging to three distinct *ante rem* structures?

Here is Shapiro's reason for why $2_{natural}$, 2_{real}, $\{\emptyset, \{\emptyset\}\}$ denote three distinct objects in three distinct mathematical structures:

> The natural number 2 and the real number 2 and the set $\{\emptyset, \{\emptyset\}\}$ enjoy different relations to different objects. It is part of the essence of the real number 2 that it has a square

[17] Unfortunately, Shapiro does not say explicitly that they are *false*. He says only that (i) natural number 2, real number 2, and von Neumann 2 are distinct objects ([31], p. 128) and that (ii) he withdraws his claim about the need to provide identity criteria for mathematical objects ([31], pp. 140-141; [32]). However, does (ii) suggest that (iii) he already abandons relative identity and hence withdraws his claim in [30] that identity statements involving objects from different mathematical structures 'make no sense'? If so, (i) and (iii) do suggest that the sentences $2_{natural} = 2_{real}$ and $2 = \{\emptyset, \{\emptyset\}\}$ are *false*.

> root, it is greater than this square root, and it is less than π. The *natural* number 2 does not have *those* properties ([31], p. 128).

The upshot of this explanation is that $2_{natural}$, 2_{real}, $\{\emptyset, \{\emptyset\}\}$ denote three distinct objects because the three objects denoted have different relational properties. This explanation, I suspect, is question-begging. It is not clear what precludes the following possibility. In the realm of *ante rem* structures, the natural number structure is embedded in the real number structure, and both the natural number structure and the real number structures are embedded in a set theoretical structure. In this case, $2_{natural}$, 2_{real}, and $\{\emptyset, \{\emptyset\}\}$ denote one and the same object in the universe of *ante rem* structures — call it **p**. **p** is situated in the natural number structure, the real number structure, and the set theoretical structure in which the natural number structure and the real number structured are embedded. **p** cannot be what it is without entering into the smaller-than relation with its square root in the real number structure, or entering into the smaller-than relation with π in the same structure. It cannot be what it is without having **3** as its successor in the natural number structure. Also, **p** cannot be what it is without entering into the membership relation with the **empty set** and its **singleton**. **p** is a position with multiple structural essences.

Since a mathematical object is merely a structureless point on *ante rem* structuralism, there seems to be no reason for *ante rem* structuralists to block the possibility of a mathematical object occupying multiple offices. One cannot argue that this proposal makes the existence of numbers *contingent* on the existence of sets and hence privilege set theory. The reason is that, on *ante rem* structuralism, each structure *self-exemplifies* and hence exist necessarily. So, even though the natural number structure is embedded in a set theoretical structure, it does not seem to make sense to say that numbers depends for their existence on the existence of the set theoretical structure.

Therefore, it appears that Shapiro's solution to the Benacerrafian problem is unsatisfactory. The failure of *ante rem* structuralism to solve the problem raises the question of whether *ante rem* structuralists are misguided right at the beginning in proclaiming that they can solve Benacerraf's problem. Given their commitment that mathematical entities are objects, various questions about the identity of these objects seem to be unavoidable. Would it be preferable for *ante rem* structuralists to concede that they cannot offer any satisfactory non-trivial solution to the problem and that the attractiveness of their program lies not in its provision of a novel solution to the problem but in its other merits such as its accommodation of set theory and its satisfaction of the faithfulness constraint?

2. Ending Remarks: Shapiro's Recent Thoughts on Identity Criteria

In this final section, I discuss what is in common between this project's diagnosis of the notion of identity criteria and Shapiro's recent views on it. I also discuss several ways in which this project is important, despite the similarity between this project's diagnosis of the notion and Shapiro's.

I have to concede that there happens to be a lot in common between this current project's diagnosis of the notion of identity criteria and Shapiro's recent views on it.[18] From his [32], we can identify several arguments against the notion of identity criteria which complement our analysis in the last chapter — two of them questioning the metaphysical purchase of the notion and one putting into doubt the epistemic purchase of the notion. First, he raises the same issue with haecceities we discuss in the last chapter. He argues that if the having of haecceity is acceptable as a property to be used in an account of identity, identity can be trivially defined by the following truth in second-order logic $\forall x \forall y(x = y \equiv \forall X(Xx \equiv Xy))$; however, to do so is not "very deep or interesting" ([32], p. 166). Second, he puts forward the an argument of regress, which is analogous to our earlier question of whether we need a further analysis for the sameness of haecceities. His argument goes in the following way. Suppose the identity of sets has to be governed by the schema:

$$(AE)\ \forall x \forall y((Set(x) \wedge Set(y)) \rightarrow (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)))$$

Shapiro's immediate question is whether we need a further analysis for the membership relation, that is, whether we need a schema like the following:

$$(MS)\ \forall x \forall y(x \in y \equiv \underline{\qquad})$$

Even if an account can be given for membership relation, we can ask further questions about the relations used to give an account of the membership relation. Hence an infinite regress.

Shapiro's third argument touches on the epistemic primitiveness of identity, an issue we discuss in response to Button's argument for the epistemic purchase of the notion of identity criteria. In understanding implicit definitions of 'natural number', 'successor', 'addition' and 'multiplication',

[18] As my project proceeds, I fail to take into full consideration of Shapiro's [32]. It is only when my project is almost complete that I realize the importance of Shapiro's remarks in his [32].

Shapiro argues, our understanding of identity is already presupposed.

Despite the similarity between Shapiro's and this project's diagnosis of the notion of identity criteria, my project is important for the following reasons. First, this project shows that Shapiro's solution to the Benacerrafian problem runs against the commitment of mathematical realism and neglects the possibility of embeddings among *ante rem* structures. I thereby identify an important consequence of dropping the thesis of relative identity: the Benacerrafian problem, which motivates structuralism, cannot be solved on *ante rem* structuralism.

Second, this project explains why there is a motivation for *ante rem* structuralists to adopt the thesis of relative identity with schema (IC) I discuss in the second chapter. Given *ante rem* structuralism's tenet that a mathematical object has no properties beyond its relations with other objects in the same structure, there is a temptation for *ante rem* structuralists to deny trivial extra-structural properties such as being distinct from an object from another mathematical structure or from the non-mathematical realm. *Ante rem* structuralists succumb to the temptation because in doing so they can provide a clean solution to the Benacerrafian problem. They can argue that given that the number 2 does not have trivial extra-structural properties such as being distinct from the set $\{\emptyset, \{\emptyset\}\}$ or the set $\{\{\emptyset\}\}$, there is *no fact of the matter* as to whether $2_{natural} = \{\{\emptyset\}\}$ or $2_{natural} = \{\emptyset, \{\emptyset\}\}$.

Third, this project provides a way by which *ante rem* structuralists can accommodate the primitiveness of identity for mathematical objects within a structure. By appealing to their official commitment to the ontological priority of structure over positions and to the faithfulness constraint, they can accommodate intra-structural primitive identity into their framework and hence block the problem of automorphism from the start.

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VITA

Name	Ho Kin Siu
Address	Department of Philosophy Texas A&M University, 4237 TAMU College Station, TX 77843-4237
Email Address	c-siu@philosophy.tamu.edu
Education	B.A., Philosophy, The University of Hong Kong, 2006