COGNITIVE ANALYSIS OF STUDENTS' ERRORS AND MISCONCEPTIONS IN VARIABLES, EQUATIONS, AND FUNCTIONS

A Dissertation

by

XIAOBAO LI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2006

Major Subject: Curriculum & Instruction

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Approved by:

Co-Chairs of Committee,	Gerald Kulm
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	Dianne Goldsby
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ABSTRACT

Cognitive Analysis of Students' Errors and Misconceptions in Variables, Equations, and Functions. (December 2006)

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M.S., Nanjing Normal University

Co-Chairs of Advisory Committee: Dr. Gerald Kulm
Dr. Yeping Li

The fundamental goal of this study is to explore why so many students have difficulty learning mathematics. To achieve this goal, this study focuses on why so many students keep making the same errors over a long period of time. To explore such issues, three basic algebra concepts - variable, equation, and function – are used to analyze students' *errors*, possible *buggy algorithms*, and the conceptual basis of these errors: *misconceptions*. Through the research on these three basic concepts, it is expected that a more general principle of understanding and the corresponding learning difficulties can be illustrated by the case studies.

Although students' errors varied to a great extent, certain types of errors related to students' misconceptions occurred frequently and repeatedly after one year of additional instruction. Thus, it is possible to identify students' misconceptions through working on students' systematic errors. The causes of students' robust misconceptions were explored by comparing high-achieving and low-achieving students' understanding of these three concepts at the *object* (structural) or *process* (operational) levels. In addition, high-achieving students were found to prefer using *object* (structural) thinking to solve problems even if the problems could be solved through both algebra and arithmetic approaches. It was also found that the relationship between students' misconception and object-process thinking explained why some misconceptions were particularly difficult to change. Students' understanding of concepts at either of two stages (*process* and *object*) interacted with either of two aspects (correct conception and misconception). When students had understood a concept as a *process* with misconception, such misconception was particularly hard to change.

In addition, other concerns, such as rethinking the misconception of the "equal sign," the influence of prior experience on students' learning, misconceptions and recycling curriculum, and developing teachers' initial subject knowledge were also discussed. The findings of this study demonstrated that the fundamental reason of misconception of "equal sign" was the misunderstanding of either side of equation as a *process* rather than as an *object*. Due to the existence of robust misconceptions as stated in this study, the use of recycling curriculum may have negative effect on students' understanding of mathematics.

DEDICATION

In memory of my grandmother

For her love that is transcending time and space

To my parents Zhangrong Li and Jumei Wu

For their support and sacrifices they made

To my wife Meixia Ding and my son Muzi

For their encouragement, patience, and understanding

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My appreciation goes especially to Dr. Gerald Kulm, my advisor and co-chair of the committee. Without Dr. Kulm, it would not have been possible for me to study at Texas A&M University. What I learned from Dr. Kulm is not only about research and the attitudes to research, but also the way to cultivate students. Dr. Kulm always understands, encourages and helps students from the needs and perspectives of students. It is really hard for me to express my gratefulness for Dr. Kulm by using the English language. I would also like to thank Dr. Yeping Li, my co-chair of the committee. Dr. Li cares about my progress and helps me even beyond my study. I also really appreciate Dr. Wilson and Dr. Goldsby for their insightful suggestions of improving my study.

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Most of all, I am grateful for the support from my wife, Meixia Ding. Without her help, it would have been hard for me to finish this dissertation on time. Thanks for her love, patience and encouragement. I also want to thank to my son, Muzi. He never disturbed me when I worked on my dissertation at home. One of my greatest motivations to finish this study was to open the door and play with him for a whole day.

I am thankful for my parents. They cannot read these words even written in Chinese; however, their courage to overcome difficulties influences and benefits me. This is one of the best educations that I have received in my life. Finally, I want to thank my grandmother, who had all the excellent virtues of traditional Chinese women. She was born and grew up in another dynasty that was totally different from mine. However, her insight of life and education greatly affected me. I believe that her love for me transcends time and space.

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1. INTRODUCTION

1.1 The Evolution of Research Questions

When I was a middle school mathematics teacher in China, many students in my class struggled with learning mathematics. They had nearly identical family backgrounds, and they an adequate time learning mathematics. My teaching quality was good by any reasonable standard, in terms of evaluations and feedback from my students, colleagues, and experts outside the school. However, after one semester, one year, or even after two years, these low-achieving students made essentially no improvement, in spite of the fact that I paid special attention to their learning. They were still not good at solving even simple problems or understanding certain fundamental concepts. When I discussed this phenomenon with my colleagues, I found it was a common situation. The explanations from different teachers were surprisingly consistent. One explanation was that some students were born with a certain innate ability in math while others lacked such innate abilities. Another explanation was that the problem results from the abstractness of mathematics. I was not satisfied with such responses because they only described phenomena through introducing more complicated concepts and myths to be answered. For example, if abstractness was the cause of learning difficulties, how did the abstractness cause the difficulties?

This dissertation follows the style of *Journal for Research in Mathematics Education.*

Several years later, I discovered another explanation during my graduate study in mathematics education in China. That is, those low-achieving students only memorized a few facts, formulas, and algorithms, without deep understanding of them. The lack of understanding prevented them from applying mathematics knowledge to new contexts in a flexible way. This explanation was better than the previous ones but still led to other issues. For example, what did "understanding mathematics" mean? I knew what memorizing a formula meant; that is, I could write it down without referring to any books or notes. However, how could I know whether I understood a formula?

When I continued my study and research in the U.S., I read more articles and found that I was not alone in my concerns. I also realized that the more important problem was how to improve mathematics understanding. Hibert and Carpenter (1992) define understanding as connections. So understanding a new concept means to construct a relationship between the new concept and the old conceptual network. Hibert and Carpenter also proposed ways to facilitate understanding through: reflecting and communicating, working on proper problems, or communicating with partners. The following comments about understanding clearly show the importance and difficulties of promoting students' understanding:

> One of the most widely accepted ideas within the mathematics education community is the idea that students should understand mathematics. The goal of many research and implementation efforts in mathematics education has been to promote learning with understanding. But achieving this goal has been like searching for the Holy Grail. There is a persistent belief in the merits of the goal,

> but designing school learning environments that successfully promote understanding has been difficult (p. 65).

In past studies about mathematics understanding, the direct approach was often used. That is, researchers mainly focused on what understanding means and how to improve it. If researchers can know and describe the mechanism of students' understanding in a detailed and characterized way, for example, like knowing how to make a sandwich, it will be extremely easy for teachers and researchers to design effective instruction to improve students' understanding, just like making a sandwich using a recipe. However, many phenomena in this world are extremely complicated and thus hard to explain based on current tools and knowledge; it is even harder to explore human learning and understanding issues compared to other issues, because we need to use the brain to explore "brain activities." Educational researchers are very careful to select important topics, whose solutions can contribute to the field. However, another standard for selecting a topic is equally important; that is, it is possible to solve the chosen topic using current tools and knowledge (Schoenfeld, 1999).

Based on the above considerations, this study narrows the topics little by little (see Figure 1.1). In a mathematics proof, it is sometimes impossible to prove a problem directly; however, it is relatively easy to use "proof by contradiction." If the reasons that students misunderstand mathematics concepts can be well understood, it should be easiedr to improve students' understanding. Students' errors are the "symptom" of misunderstanding. Among many different types of errors, systematic errors occur to many students over a long time period and it is relatively easy and thus possible to research with current knowledge and tools. The cause of systematic errors may relate to student' procedure knowledge, conceptual knowledge, or links between these two types of knowledge. In this study, I will focus primarily on systematic errors due to flawed conceptual knowledge (misconceptions).

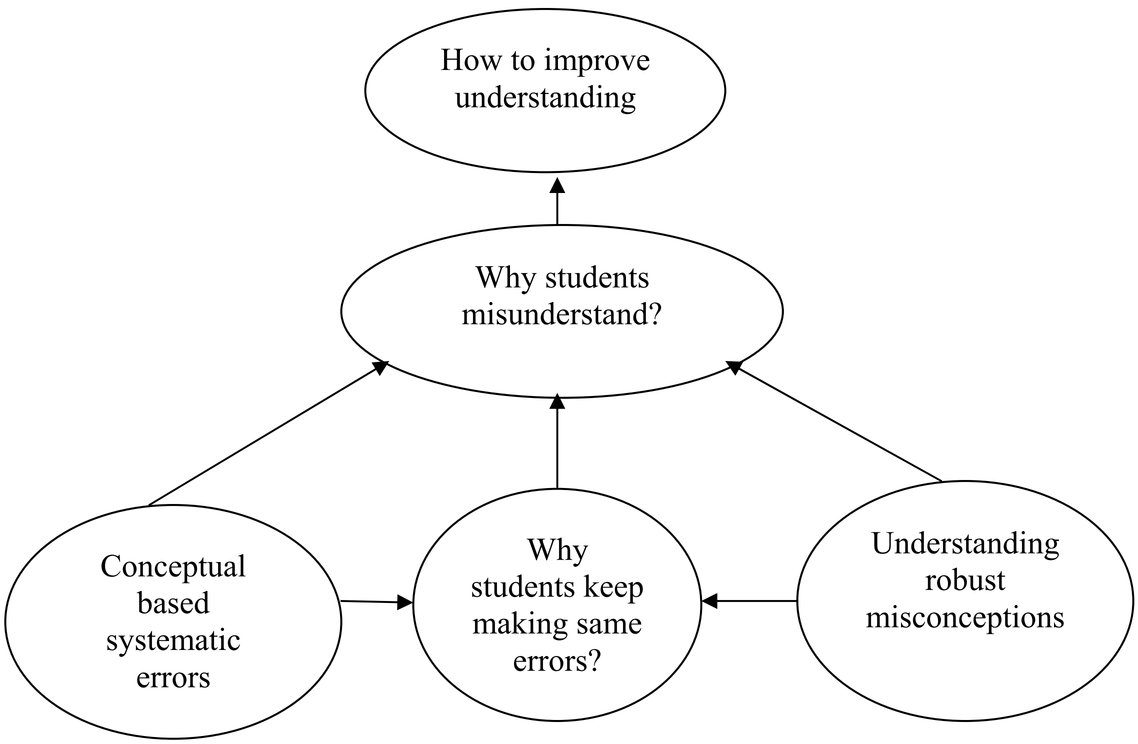


Figure 1.1. Flow chart for the evolution of the research topic.

1.2 Rationality and Feasibility of the Dissertation

Mathematics is important for both individual and country. "For students, it opens doors to careers. For citizens, it enables informed decisions, for nations, it provides knowledge to compete in a technological economy" (National Research Council [NRC], 1989, p. 1). However, the NRC reported little good news of mathematics education in U.S. in 1989: three of every four Americans gave up studying mathematics before completing career or job prerequisites. Most students leave school without sufficient preparation in mathematics to cope with the demands of the current job or for further academic advancement. Efforts to improve students' learning has continued through designing high quality curriculum or improving teaching quality, However, as of 2003, the situation of students' inadequate mathematics preparation had not improved. "Despite massive effort, relatively little is accomplished by remediation programs. No one—not educators, mathematicians, or researchers--knows how to reverse a consistent early pattern of low achievement and failure" (Ball, 2003, p.13). Although the improvement of mathematics learning is an extremely complex activity which requires coordinated efforts from multiple resources, good research support is particularly important. For example, one main reason for giving up on learning mathematics is learning difficulty. The use of the "equal sign" is a basic topic in elementary mathematics; nevertheless, researchers found that even college students had trouble understanding and using the equal sign (Barcellos, 2005). Thus, it is understandable why so many students give up learning mathematics if they keep making mistakes. However, there is little research about why students keep making such errors, even with "good" teaching in term of reasonable standards. Why are students' misconceptions of such a "simple" concept so robust, or resistant to change, over so many years? Is it because of characteristics of mathematics? As Poincare (1952, also cited by Sfard, 1991) stated:

> One…. fact astonishes us, or rather would astonish us if we were not too much accustomed to it. How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds …. how does it happen that there are so many people who are entirely impervious to it? (Poincare, 1952, p.49)

As stated at the beginning, the research on students' systematic errors can provide a good lens to see why students have difficulty learning mathematics. Examining students' wrong answers provides one way to demonstrate students' understanding of a concept. On the other hand, students' correct answers may not necessarily indicate a good conceptual understanding of related knowledge because students could have solved the problem correctly by just memorizing procedures or definitions without true understanding. Besides, students' correct answers are generally uniform, which does not provide an appropriate research setting. "Research on students' errors makes it possible to identify specific deficits in the way students' knowledge is connected so that instruction can be designed to address the specific connections students lack or to point out why certain connections are inappropriate" (Hiebert & Carpenter, 1992, p. 89).

Past research on students' errors regarding subtraction and addition has produced powerful results. For example, "buggy" theory (Brown & Burton, 1978) can predict about 50% of students' errors even before students actually do the calculation (Resnick, et al., 1989). Other studies (Chi, 2005; Slotta & Chi, 2006) in the science education field also provide necessary research methods and theoretical frameworks about robust misconceptions and conceptual change, which offer clues for researching errors and misconceptions in mathematics education.

1.3 The Purpose of the Dissertation

The final goal of this study is to explore why so many students fail to learn mathematics and why mathematics is so difficult for many students. In order to reach this goal, this study focuses on the nature of students' learning basic concepts by analyzing their errors in solving well-designed problems used to assess those concepts. The causes of students' errors are complicated; for example, students' errors may be due to carelessness, no understanding at all, confusing different concepts or failing to transition from object-oriented thinking to process-oriented thinking. Thus, this study focuses on errors made frequently by many students over long periods of time; that is, why many students keep making the same errors despite "good teaching."

Resnick (1982) attributed students' learning difficulties to concepts learning: "Difficulties in learning are often a result of failure to understand the concepts on which procedures are based" (p. 136). In order to explore such fundamental issues, three basic algebra concepts: variable, equation, and function were chosen to analyze students' *errors*, possible *buggy algorithms*, and the conceptual basis of these errors and buggy algorithms. Through focusing the research on these three fundamental concepts, it is expected that more general principles of understanding and learning difficulties can be illustrated by these cases.

This study examines two areas of inquiry. First, it recognizes differences between students' procedural errors, bugs, and misconceptions in the domain of algebra. Then, students' errors in solving problems related to variables, equations, and functions are reported and analyzed in detail.

Second, this study investigates the conceptual basis of students' errors. Past studies focused more on students' errors caused by correctly using buggy algorithms or incorrectly selecting algorithms in elementary arithmetic (Brown & Burton, 1978; Brown & VanLehn, 1982) and in elementary algebra (Matz, 1982; Sleeman, 1982). There are not enough studies about where buggy algorithms originate.

Compared to learning other subjects, students' misconceptions about mathematics particularly affect further learning, due to the hierarchy of mathematics knowledge structure; therefore, it is necessary to change students' early misconceptions, especially robust misconceptions. Thus, this study also explores why some misconceptions are particularly robust to change based on cognitive theory (Chi, 2005) and theory of nature of mathematics knowledge (Sfard, 1991, Sfard & Linchevski, 1994). Students' strategies in solutions and *predicate words* in students' verbal responses are coded to indicate whether students understand a concept as an *object* or a *process* (I will elaborate these two terms later). High and low level students' ontological differences in understanding these concepts were compared to verify the existence of understanding of a concept as *object* or *process*.

1.4 Statement of the Problem

Even very basic mathematics concepts or operations, like whole number addition and subtraction, may involve extremely complicated cognitive processes. Because teachers are already very familiar with those basic concepts or operations, they tend to ignore or underestimate their complexity and thus take a naïve approach to teaching mathematics concepts or operations (Schoenfeld, 1985). In the mathematics educational field, research on students' misconceptions is not well documented, especially compared with that of the science education field. Thus, in this study, three problems were investigated. First, past error analysis in the mathematics education field focused more on procedural analysis and less on misconception analysis. Although the analysis of procedural errors explains what and how students make errors in mathematics, it tells little about the origins of these bugs and procedural errors. Second, students' errors and misconceptions about variable, equation, and function, which are fundamental concepts in the learning of algebra, especially lack of systematic research. Finally, existing research on misconceptions in students' mathematics learning pays little attention to why some misconceptions are particularly robust to change and to how they could be changed.

1.5 Research Questions

Based on the Project *Improving Mathematics Teacher Practice and Students Learning through Professional Development* (IMTPSL) database, this study investigates the following questions:

1. What are students' error patterns in solving problems related to variable, equation, and function during a pretest and posttest of algebra knowledge? What misconceptions of variable, equation and function underlie those errors?
2. How do students change their understanding of variable, equation, and function after instruction? Are those misconceptions robust to change?
3. What ontological differences are demonstrated by high and low ability students in solving problems related to concepts?

1.6 Operational Definitions

Concept (mathematics): A theoretical construct of a mathematics idea (Sfard, 1991).

Conception (mathematics): The whole cluster of internal representations and associations evoked by the Concept, the subjective "universe of human knowledge" (Sfard, 1991).

Misconception: Misconceptions (1) are strongly held, stable cognitive structures; (2) differ from expert conceptions; (3) affect in a fundamental sense how students understand natural scientific explanations; and (4) must be overcome, avoided, or eliminated for students to achieve expert understanding (Hammer, 1996, p. 99).

Concept as object: A static structure, existing somewhere in space and time. It also means being able to recognize the idea " at a glance" and to manipulate it as a whole, without going into details. It is also called structural conception, and is characterized as static, instantaneous, and integrative (Sfard, 1991, p.5).

Concept as a process: A potential rather than actual entity, which comes into existence upon request in a sequence of actions. It is also called operational conception and is characterized as dynamic, sequential, and detailed.

Variable: A general purpose term in mathematics for an entity which takes various values in any particular context. The domain of the variable may be limited to a particular set of numbers or algebraic quantities (Schoenfeld & Arcavi, 1988, p. 422).

Equation: Used to model a change or situation.

Ontological attribute: An attribute that a category member may plausibly have (Chi, 1997; Chi & Roscoe, 2002) but not characteristically nor necessarily have. An entity, such as a bit of glass, can be colored even though it is colorless (Sommers, 1971), whereas an event, such as a baseball game, cannot be colorless (Chi, 2005, p. 164).

Low achieving student: Student whose total score is below ten percent during posttest.

High achieving student: Student whose total score is above top ten percent during posttest.

1.7 Significance of the Dissertation

The knowledge of teaching mathematics has been emphasized and studied by many researchers. However, the knowledge of students' learning of certain specific mathematics knowledge, such as variable, equation and function, is not well-documented. For example, variable is a fundamental concept and is especially important for students' transition from arithmetic to algebra, but few studies focus on it (Graham & Thomas, 2000; Schoenfeld & Arcavi, 1988). Without adequate knowledge about students' learning of basic mathematics concepts or operations, the teacher may underestimate the complexity of the learning. For example, during students' learning of variable, equation, and functions, especially at the middle school level, it is still not clear what errors and how often students tend to make them, where the errors are from, and how the errors could be remediated. Not being cognizant of students' misconceptions in these concepts could hinder teachers in using proper strategies to help students. As Brown and Burton (1978) pointed out "one of the greatest talents of teachers is their ability to synthesize an accurate 'picture,' or model, of a student's misconceptions from the meager evidence inherent in his errors" (p.155-156). As a result, detailed information about students' misconceptions in learning variable, equation, and function provided by this study can contribute to teachers' classroom instruction.

Students' superficial understanding of important mathematics concepts prevents them from applying proper algorithms or strategies (Schoenfeld, 1986). Moreover, improper application of algorithms may reinforce students' misconceptions (Woodward & Howard, 1994). Because the teaching strategies for correcting misconceptions and "buggy algorithms" are different, it is important and useful to distinguish between them. This study provides a method to distinguish misconceptions from "buggy algorithms" or errors. It is important to identify students' misconception since "a student need no longer be evaluated solely on the number of errors appearing on his test, but rather on the fundamental misconceptions which he harbors" (Brown & Burton, 1978, p.156).

Learning issues were recognized by Schoenfeld (1999) as one of the six fundamental problems in the 21st century that need to be addressed for the education field. He pointed out, that “The central question here is: Is it possible to build robust theories of learning--theories that provide rigorous and detailed characterizations of how people come to understand things, and develop increased capacities to do the things they want or need to do?” (p.6). It is a challenging task for educational researchers since learning is a mental activity hidden from direct observation. By analyzing students’ errors regarding variable, equation, and function, this study may provide a general principle of learning and understanding. “Theoreticians have long recognized that important insights into the nature of cognitive skills and its acquisition can be gained by examining errors” (Payne & Squibb, 1990, p. 445).

2. REVIEW OF LITERATURE

Although there are many causes of students' difficulties in learning mathematics, the lack of enough support from research fields for teaching and learning is an important one. If research could characterize students' learning difficulties, it would be possible to design effective instructions to help students' learning. The research on students' errors and misconceptions is a way to provide such support for both teachers and students. As Booth (1988) pointed out, "one way of trying to find out what makes algebra difficult is to identify the kinds of errors students commonly make in algebra and then to investigate the reasons for these errors" (p. 20). In this section, I will review the literature from three aspects: (1) Students' errors: the different interpretations of error resources are identified. In addition, errors, bugs, and misconceptions are also differentiated; (2) Conception research: the research on preconception, misconceptions, and conceptual change is reviewed to show the influence on students' learning. The theories about why some misconceptions are particularly robust to change is also reviewed; (3) Misconceptions in mathematics: students' errors in three algebra topics--variable, expression and function--will be addressed. The research on students' understanding of these three concepts at the *object* and *process* level are discussed.

2.1 Research on Students' Errors

This section begins by reviewing the most-used research method on errors, specifically, classifying students' errors based on the steps of solving problems or the sources of difficulties in solving problems. After that, "buggy algorithm" theory is reviewed in explaining sources of errors. This theory maintains that students correctly follow wrong algorithms, which is contrary to many teachers' views that students wrongly follow an algorithm. Then, the review focuses on where the bugs originate, which is believed to relate to misconceptions or links between conceptual and procedural knowledge. Finally, I compare and differentiate errors, bugs, and misconceptions in terms of past research.

2.1.1 Categories of errors

One of the main methods used to analyze students' errors is to classify them into certain categorizations based on an analysis of students' behaviors. Through using a cognitive information-processing model and considering the specialties of mathematics, Radatz (1979) classified the errors in terms of (1) language difficulties. Mathematics is like a "foreign language" for students who need to know and understand mathematical concepts, symbols, and vocabulary. Misunderstanding the semantics of mathematics language may cause students' errors at the beginning of problem solving; (2) difficulties in processing iconic and visual representation of mathematical knowledge; (3) deficiency in requisite skills, facts, and concepts; for example, students may forget or be unable to recall related information in solving problems; (4) incorrect associations or rigidity; that is, negative transfer caused by decoding and encoding information; and (5) application of irrelevant rules or strategies. Other researchers (Newman, 1977; Watson, 1980) have also used the classifying method but based theirs on the model of problem solving. Watson used Newman's (1977) model of the sequence of steps in problem solving: reading and comprehension, transformation, process skills, and encoding, to identify students' possible errors. He thought that students' errors may be due to deficiency in one or several of the above steps. In order to verify those hypotheses about students' errors, Watson designed both word and computation problems to compare errors made by two groups of students, with lesser and greater abilities. He found that most initial errors made by the more able group were at the stage of reading and comprehension. However, the less able group students made many more errors when applying and selecting mathematics processes. The above classification method was simply used to describe students' errors, but lacked detailed analysis of why students were unable to perform well in some steps. For example, why did students not select correct mathematics processes or operations? What strategies effectively helped students make correct decisions? Why did students have special difficulty in understanding mathematics language?

Being aware of the shortcomings of classification methods, Ashlock (2002) not only categorized students' errors in computation, geometry, and algebra, but also tried to attribute errors to overgeneralizing or overspecializing. For example, given the equation, $2y = 20 + y$, some students may overgeneralize the equation as $23=20+3$. Clearly, those students applied the learned rules of arithmetic fields (old and familiar situations) to algebra fields (new and unfamiliar situations). An example of overspecializing is that students may restrict the fraction addition or subtraction only to fractions with the same denominators. The overgeneralizing or overspecializing partly explains why students make certain mistakes; however, the remaining problem is determining why students overgeneralize or overspecialize. Matz (1982) provided an explanation of why students tend to make misgeneralizations in high school algebra. He said, “errors are the results of reasonable, though unsuccessful, attempts to adapt previously acquired knowledge to new situation” (p. 25-26). For example, many students made such errors as $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. Matz maintained one of the causes was “linearity” which was “a way of working with a decomposable object by treating each of its parts independently” (p. 29). He thought it was human nature to treat most mathematics operations as “linearity” because their past experiences were compatible with the above hypothesis of “linearity.”

2.1.2 Buggy algorithms

In addition to attributing students’ errors to misgeneralization, other researchers have attributed them to buggy algorithms. What are bugs? Where do bugs originate? The following subsection reviews these questions.

***What is a bug*?** Some researchers have tried to explain and diagnose students’ errors in the domain of arithmetic through focusing on certain faulty algorithms (or bugs) frequently held by students. Their main assumption is that student’ errors are caused by following faulty algorithms rather than wrongly following a correct one or the lack of necessary knowledge (Brown & Burton, 1978; Young & O’Shea, 1981).Thus, a main way to analyze students’ errors is to identify students’ faulty algorithms. Brown and Burton (1978) found that students were very good at following certain procedures but they often followed wrong procedures. By analyzing students’ errors in subtraction and addition, they referred to a computer term "bug" to describe students' faulty algorthims. For example, students were found to have the following "buggy algorithms" during the subtraction process: (1) students subtract the smaller digit in each column from the larger digit without considering which is on top; (2) whenever the top digit in a column is 0, the student writes the bottom digit in the answer. Many similar bugs existed in the students' solutions in whole addition and subtraction problems. VanLehn (1980) offered a detailed description about these bugs:

> Once we look beyond what kinds of exercises the student misses and look at the actual answers given, we find in many cases that these answers can be precisely predicated by computing the answers to the given problems using a procedure which is a small perturbation in the fine structure of the correct procedure. Such perturbations serve as a precise description of the errors. We call them "bugs" (p.7).

Moreover, those bugs may interact with each other. In order to diagnosis students' bugs in solving problems, Brown and Burton (1978) designed a computer program, "Buggy," to simulate students' behaviors. Many students' errors could be predicted by "Buggy." Attributing students' errors to "bugs" and interactions between "bugs," it is easier for teachers to predict students' possible errors, thus assisting them helping students. This method makes great progress compared to classification methods for researching students' errors.

Where do bugs originate? "Bug" theory provides a good explanation of students' errors in subtraction. However, it is more important to know where the bugs are from than to know how to describe errors by using bugs (Payne & Squibb, 1990; Resnick & Omanson, 1987). "We know they (bugs) are inventions by children-----because no one teaches incorrect procedures. But what is the process of invention and on what specific knowledge ---or lack of it---are bugs based?" (Resnick & Omanson, 1987, p.44).

In order to explain where bugs originate, VanLehn (1990) differentiated systematic errors and slips. For VanLehn, systematic errors meant consistent application of faulty methods, algorithms, or rules, which usually happened to novices. Slips were typically due to carelessness and happened to both experts and novices. "Bugs" partially explain students' systematic errors; however, many questions remained "mysteries" as pointed out by VanLehn (1990, p.16):

> Why are there so many different bugs? What caused them? What caused them to migrate or disappear? Why do certain bugs migrate only into certain other bugs? Often a student has more than one bug at a time ---- why do certain bugs almost always occur together? Do cooccurring bugs have the same cause? Most important, how is the educational process involved in the development of bugs?

The main explanation of bug source is the *repair* theory proposed by Brown and VanLehn (1980). They identified four types of impasses: decision impasse, reference impasse, primitive impasse, and critic impasse. According to their theory, an impasse occurs when students are unable to perform an action. Thus, when students reach an impasse in solving a problem, they frequently skip or *repair* it in order to continue execution of the procedure. For example, *Always-Borrow-Left* is a bug found in many students' subtraction operation. Students with that bug always borrow from the left-most column. Because students initially learned subtraction of two-digit numbers, they tend to interpret “borrow from the left-adjacent column” as from the leftmost column, which is the same process for two-digit number subtraction. “Procedures that lead to bugs are the results of generalization of examples rather than, say memorization of verbal or written recipes. Evidence for this claim comes from the fact that this and many other bugs depend on accidental, visual characteristic of the examples” (VanLehn, 1980, p.27). Therefore, when students face three-digit number subtraction, they may have the problem of determining from which column they should borrow. At this time, students may reach an *impasse*; they try to fix the problem by using flawed strategies. As a result, errors are produced. Accepting *repair* theory, Young and O’Shea (1981) suggested that students’ subtraction bugs appeared when students forgot or had not learned relative algorithms.

The above explanations of bug sources were recognized as superficial because these researchers were only aware of the surface structure of procedures without considering mathematical principles involved in subtraction, or the conception of place value (Resnick & Omanson, 1987). Silver (1986) further commented on the above studies of the source of “bugs” as more descriptive than explanatory:

> In recent years, the analysis of systematic procedural flaws, or “bugs,” has received increased attention. The seminal work of Brown and Burton (1978) on multidigit column subtraction errors suggests that a pure view of procedural bugs can be productive. Nevertheless, neither their analysis nor other analysis inspired by their work has explained the basis for a large percentage of the errors that

> children make, nor has it directly addressed the remediation of the errors, with reference to the total knowledge base-----both conceptual and procedural-----that the child possesses or with reference to the total curriculum that is being taught. (p. 187).

In summary. Though the theory (or the method) of *buggy algorithm* reflects substantial progress in exploring students' errors, it does not provide the underlying reasons of why students invent bugs. *Repair* theory which attempts to explain bug source is also superficial in that the conceptual basis of students' bugs and errors is still unclear.

2.1.3 The conceptual basis of students' bugs and errors: Misconceptions

What is the conceptual basis of students' bugs and errors? Research on the nature of mathematics knowledge provides a clue to explore this issue. Hiebert and Lefevre (1986) differentiated between conceptual and procedural knowledge in the domain of mathematics. They defined procedural knowledge as rules, algorithms, formal language of mathematics or procedures used to solve mathematical tasks. Conceptual knowledge was thought of as connections among information, a network of mathematics facts and propositions. They argued:

> The result is that students' mathematical behavior often consists of looking at surface features of problems and recalling and applying memorized symbol manipulation rules. Mathematically unreasonable answers often produced, and performance is low across a range of problems, even on those directly instructed and frequently practiced (p. 200).

Based on such classifications of mathematics knowledge, students' bugs had a conceptual source. Silver (1986, p. 187) stated, "systematic bugs in procedures can often be traced to flaws in conceptual knowledge or to the lack of conceptual/procedural knowledge linkages." An example of research on the conceptual basis of bugs can be found in Resnick et al.'s (1989) study. In their study, the researchers tried to identify students' three wrong rules (bugs) in subtraction: whole-number-rule, fraction-rule and zero-rule in comparing decimal fractions. The whole-number-rule was that the larger string means large value (this is true for positive whole numbers). The student with the fraction-rule bug tended to think the short string meant larger. Zero-rule was associated with the misunderstanding of "0" for decimal fraction. As misconception was usually hidden from direct observation, the identification of misconception was mainly based on reasonable inferences by using well designed instruments. The tasks in Resnick et al.'s study were well designed and administered to students in three countries. The frequency of students' errors related to each rule was recorded. Then students' responses were analyzed to determine students' conceptual bases of wrong rules. For example, the whole-number-rule bug was thought of as "confusion about the zero's place holder function" (p. 21). Resnick et al.'s study highlights one way to explore misconceptions underlying students' bugs. Their study also showed that it is important to pay attention to the deeper causes of students' errors or bugs.

2.1.4 Errors, bugs, and misconceptions: The similarities and differences

The terms of errors, bugs, and misconceptions form the foundation of this study, although they are often used inconsistently or incorrectly in different studies. Therefore, it is necessary to delineate the differences among them. Young & O'Shea (1981) provided an excellent and clear interpretation of *errors* and *faulty algorithms (bugs)*: "The ambiguity of problems highlights the need to distinguish carefully between, on the one hand, *errors*, i.e. actual wrongly answered problems, and on the other, *faulty algorithms* (or "*bugs*"), i.e., flaws in the program that generates the answers" (p. 156). On the other hand, *misconceptions* were students' naïve explanations of concepts which were stable, robust, and resistant to instruction (Anderson & Smith, 1987, also cited by Chi, 2005). This view is consistent with that of Hammer (1996) who thought students' misconceptions:

1. are strongly held, stable cognitive structures;
2. differ from expert understanding;
3. affect in a fundamental sense how students understand natural phenomena and scientific explanations; and
4. must be overcome, avoided, or eliminated for students to achieve expert understanding (p. 99).

In this study, I accept those four properties as basic characteristics of misconceptions. Regarding the relationship between "bugs" and "misconceptions", it is reasonable to assume that misconceptions are one of the deep reasons underlying bugs (Resnick & Omanson, 1987; Silver, 1986). Another difference between bugs and misconceptions is that misconceptions are more stable cognitive structures across different problem contexts, while bugs, on the other hand, tend to change or migrate across different problem contexts. Bugs are not as stable as misconceptions. Silver (1986) analyzed a case in Erlwanger's study (1973) where a student correctly answered problems such as "0.2 + 0.4 =?" and "2.0 + 4.0 =?" but answered the problem "0.2 + 4.0 =?" with the wrong answer, 0.6. Silver commented, "What can be said about Benny's procedural bug? It is interesting that Benny's procedural bug appears to have conceptual aspects. If his conceptual knowledge of place value and the decimal point were flawed, then it might provide the needed support for the procedural error" (p. 187). What follows is another example used to illustrate the subtle differences among errors, bugs, and misconceptions. Falkner, Levi, and Carpenter (1999) found that all sixth-grade students incorrectly fill the box in "8 + *4* = □+ 5" with 12 or 17. Such wrong answers clearly indicate that children have a partial understanding of equality and the equals sign (Falkner, et al , 1999). According to the explanations of errors, bugs, and misconceptions in this study, 12 or 17 is thought of as an *error*. Since the correct algorithm for this problem normally involves the sum of 8 and 4, then subtracting 5 from 12; the *faulty algorithm* therefore is that students only conduct the first step and get the definite value "12." Another *faulty algorithm* in this problem is to add all the numbers and get "17." Underlying all these faulty algorithms may be students' *misconceptions* of the equal sign, that is, interpreting "=" as "to do something." Another probable *misconception* is that they only understand "8 + 4" as a computation process without understanding "8 + 4" as a sum, an *object*. In later sections, this issue will be explored in detail.

In Summary. The above review documents the main efforts to identify the causes of students' errors. Misconceptions are one of the main causes of students' bugs and errors. Without a sound understanding of basic mathematics concepts, it is almost impossible for students to develop advanced thinking and succeed in further mathematics learning. In the following section, the importance and function of correct conceptions will be reviewed.

2.2 Research on Student Conception

In this section, at first, the importance of conceptual research is highlighted. Then I focus on the influence of preconception and misconception on students' learning and the necessity of conceptual change. Last, I also review why some misconceptions are particularly robust to change.

2.2.1 The importance of conceptual research

Most researchers hold the common assumption that students possess some informal knowledge before formal school learning or learning new content. As Confrey (1990) stated:

> Researchers in this tradition are united in (a) their rejection of the tabula rasa assumption that students enter instruction with no preconceptions about a topic before it is taught, and (b) their belief that these naïve ideas cannot easily be ignored or replaced through direct instruction or lecture (Gilbert, Osborne, & Fensham, 1982, p.5).

The research of the effects of earlier knowledge on students' learning has received more attention. Shulman (1999) emphasized that learning is a dual process, with forces both internal and external to the individual interacting with each other. He further argued that it was the learning already present in the learner, rather than the teaching, that had primary influence on new learning. Ausubel (1968) commented "If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (also cited by Shulman, 1999). There are three research approaches to students' conceptions: Piagetian approaches (Piaget, 1970) with a focus on the development of conceptions over time; the application of the philosophy of science on education research with a focus on students' perception, misconception, and conceptual change; and research on systematic errors (Confrey, 1990). In the following two subsections, I mainly focus on the effects of misconceptions on systematic errors and why some misconceptions are robust to change.

2.2.2 Preconception, misconception, and conceptual change

Preconception is usually recognized as the prior knowledge held by students which influences students' learning (Bruner, 1960). Chi and Roscoe (2002) referred to "preconception" as naïve knowledge which could be easily revised and removed, while "misconception" was naïve knowledge that was robust to change. In order to be consistent, I use Chi's definition of preconception and misconception, which are also consistent with Hammer's, as mentioned earlier in this study.

Regarding the influence of preconception and misconception on students' learning, some researchers have attributed students' learning difficulties to lack of necessary proficiency or knowledge (Anderson, 2002; Haverty, 1999). In contrast, other researchers have claimed that "earlier learning constrains later learning" (McNeil & Alibali, 2005, p. 884).

As pointed out before, students enter classrooms with different conceptions due to life experience or prior instruction. An important task for teachers is to identify students' misconceptions in order to correct them. The process of correcting students' misconceptions is called "conceptual change" (Chi, 2005; Chi & Roscoe, 2002).

Since misconceptions usually resist change, "any theory of learning must explain not only how people change, but also why people resist change" (McNeil & Alibali, 2005). Slota et al. proposed an explanation of why people resist change in the science education field. They assumed that concepts are associated with ontological categories. Since students already classify certain concepts according to their ontological attributes, they must revise or modify their categorization of the concept ontology. If their initial categories are not ontologically different from the actual classification, the process of change is not difficult. Otherwise, the revision process will be more difficult. By examining a science concept like *current*, Slotta et al. (1995) thought students initially classify current as a *material substance* which is something like water. The actual category of current is a *process* of interaction. As a result, there is an ontological difference between students' initial conception of current and its actual attributes. Thus, it is usually much harder to change students' misconception of *current* during formal schooling.

Students' difficulties of conceptual change may occur in the domain of mathematics. McNeil & Alibali (2005) explored the change difficulties by looking at students' prior knowledge and found that "the patterns with which people initially gain experience become entrenched, and learning difficulties arise when to-be-learned information overlaps with, but does not map directly onto, entrenched patterns" (p. 884). They found that there were three operational patterns that may hinder students' understanding complex equations, that is, "perform all operations"; "operations=answers"; and "understanding equal sign as total."

2.2.3 Robustness of misconception: Toward the framework

In this study, I will employ Chi's framework (2005) of robust misconception analysis. According to Chi, students' ontological knowledge and the actual ontological categories may or may not correspond. Many robust misconceptions are caused by a mismatch between students' conception and reality at the ontological level. The process of correcting students' misconceptions is called "conceptual change" (Chi, 2005; Chi & Roscoe, 2002). Therefore, "robust misconceptions are mis-categorizations across ontological boundaries in that a member of one ontological category is misrepresented as a member of another ontological category" (Chi, 2005, p. 164).

With regard to mathematics, Sfard (1991) pointed out mathematics concepts could be conceived in two fundamental ways: structurally and operationally which respectively results in “objects” and “process.” She distinguished those two conceptions in the following way:

> There is a deep ontological gap between operational and structural conceptions….Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing—a static structure, existing somewhere in space and time. It also means being able to reorganize the idea “at a glance” and to manipulate it as a whole, without going into details….. In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed. (p. 4)

In another article, Sfard & Linchevski (1994) maintained that students need to transition from *process* to *object* in order to understand concepts. She specified three stages in the transition: interiorization, condensation, and reification. Therefore, Sfard’s theory about understanding concepts is startlingly consistent with Chi’s (2002, 2005). This is why I could reasonably use Chi’s framework to explore why some students’ misconceptions in mathematics are usually robust to change. The transition of *process* to *object* is also consistent with Piaget’s theory of “reflective abstraction” (Simon, Heinz, & Kinzel,

2004) which has two phases: “a projection phase in which the actions at one level become the objects of reflection at the next and a reflection phase in which a reorganization takes place” (p. 313, 2004).

In this study, I assume that students' robust misconceptions in mathematics are also caused by mis-categorizations across *object* and *process*. However, to be able to understand mathematics concepts as *object* does not mean there is no misconception. It means that such a misconception will be easy to correct since it has no ontological difference from reality (the correct conception). "Alternative conceptions within an ontological category should be less entrenched and robust, meaning that they should be more readily resolved through learning, than misconceptions *across* ontological categories" (p. 164). Sfard (1991) expressed a similar perspective:

> The problem will seem less puzzling if we remind ourselves that reification is an ontological shift, a qualitative jump. Such conceptual upheaval is always a rather complex phenomenon, especially when it is accompanied by subtle alternations of meanings and applications…… The difficulties arising when a *process* is converted into an *object* are, in a sense, like those experienced during transition from one scientific paradigm to another; (p. 30)

In Summary. In this section, the negative effects of preconception and misconception on students' learning are reviewed. The theoretical explanations of why some misconceptions are robust to change are also provided. Thus, teachers need to identify students' preconceptions or misconceptions to help them learn mathematics effectively and efficiently. Ignoring students' misconceptions may have negative effects on students' new learning and will also reinforce original misconceptions. In the following section, the research on students' conceptions or misconceptions about three fundamental concepts will be reviewed.

2.3 Research on Variable, Equation and Function

The learning of algebra has received more attention at the middle school level where the transition from arithmetic to algebra occurs. Compared with the goal of arithmetic, which is to find the answer, the focus of algebra is to find the general method and use algebraic symbols to express these in a general form (Booth, 1988). The reasons for difficulties during the transition were investigated from the viewpoints of cognitive development (Hart, 1981), the use of algebra notations (Booth, 1984, 1988, Herscovics, 1989; MacGregor and Stracey, 1997), and understanding of fundamental concepts like variable and function (Usiskin, 1988).

In the next three subsections, I start with misconceptions about variables, equations, and functions. Then, I review several methods of researching the nature of mathematical knowledge, which are related to mathematics learning difficulties and understanding. Sfard pointed out the dual nature of knowledge, *object* and *process*, which is consistent with my framework. Last, I will elaborate what it means to understand variable, equation, and function as *object* or *process*.

2.3.1 What does it mean to understand or misunderstand variable, equation, and function?

I begin with equations and equal signs. The misconceptions, the strategies to change them, and the robustness of the change will be reviewed. Then I continue with variables and functions. The developmental trajectories, the relationship between variable and function and common misconceptions will be reviewed.

Equations and equal signs. The misconception of the equal sign as "to do something" is well documented and studied (Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Stacey & MacGregor, 1997). One of the most cited articles about the equal sign and equation is Falkner, Levi and Carpenter's (1999) study. They asked teachers from grade 2 to grade 6 to give their students the following problem:

$$8 + 4 = \square + 5$$

Surprisingly, most students solved this problem with the wrong answer of 12 or 17. Especially, all 145 sixth-graders were wrong. Among the sixth-graders, about 84% of them answered this problem with 12 while 14% answered 17. Those wrong answers clearly showed that students had no problem with computation. Thus, it was inferred that misunderstanding of the "=" as "to do something" was the cause of the students' uniform errors.

Behr, Erlwanger, and Nichols's (1980) study confirmed that students' misconception of the equal sign was the cause of the above students' errors. However, they mentioned another possibility of students' misconception of "2 + 4", that is, students tended to understand "2 + 4" as something to be done even without "equal sign". Students knew the addends represented numbers but were unwilling to accept "2 + 4" was another name for 6.

Regarding how to change students' misconception of "equal sign," Falkner et al. (1999) used several ways to develop students' understanding of equal signs, for example, through story problems and discussions of true or false problems such as 4 + 5 =9, 12- 5

= 9, 7= 3 + 4, 8 + 2 = 10 + 4, 7 + 4 = 15 – 4, and 8=8 (p. 234). One student acknowledged that 3 + 4 =7 was true but thought 7 = 3 + 4 was false. To that student, it was wrong to write an equation backwards. Some students were uncomfortable with 8 = 8. Although they thought eight equals eight, they thought it was wrong to write in that way. After one and one-half years, of the16 sixth-graders who participated in the pilot study for the same problem 8 + 4 = □ + 5, 14 of them correctly answered 7. The researchers then believed the instruction was effective and the students had gained a sound understanding of the equal sign, which laid a strong foundation for their later algebra study. However, the problem here was whether the correct solutions really showed students' solid understanding of equal signs. These students might have simply learned how to solve this type of problem. So it is much better to know whether students can answer problems related to equal signs in novel contexts.

Kieran (1981) found that even students who received an appropriate instructional method, which emphasized "equal sign" as a relationship, were still unable to accept the correct conception of "equal sign." Such misconceptions persisted at the high school and college levels (Clement, Lochhead, & Monk, 1981). The following solutions of equations demonstrated high school students' misconceptions about equal signs (Kieran, 1981, p.323):

$$\begin{aligned} & 2x+3=5+x \\ & 2x+3-3=5+x-3 \\ \text{Solve for } x: \quad & 2x=5+x-x-3 \\ & 2x-x=5-3 \\ & x=2 \end{aligned}$$

And

$$
\begin{aligned}
&x+3=7\\
&=7-3\\
&=4
\end{aligned}
$$

Clement(1982, p.7, also cited by Kieran) showed that even college students still use equal sign as a link between steps. They found the derivative of a function:

$$
\begin{aligned}
&f(x)=\sqrt{x^2+1}\\
&=(x^2+1)^2\\
&=\tfrac{1}{2}(x^2+1)^{-1/2}D_x(x^2+1)\\
&=\tfrac{1}{2}(x^2+1)^{-1/2}(2x)\\
&=x(x^2+1)^{-1/2}\\
&=\frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

As Falkner et al. (1999) pointed out, the correct understanding of "equal sign" lays a strong foundation for learning algebra because one important and fundamental algebra concept related to equal sign is "equation." Matz (1982) commented that equation was not a complete new concept but only an extension of the existing concept of arithmetic equity. Thus, equation provides researchers with good opportunities to investigate critical and fundamental learning issues: for example, how does students' prior knowledge affect their later learning of related advanced knowledge? What adjustments will be needed for students to learn new concepts based on their prior knowledge? What special difficulties will occur to students in such adjustments?

McNeil and Alibali (2005) found that the degree of students' adherence to the operational pattern of "equal sign" was strongly correlated with whether they can generate a correct strategy to solve equations. They found that students' entrenched conceptions of operational patterns constrained the learning of equation. Students' misconceptions and errors of solving equation have been documented in many studies across different grade levels. Their research approach is to identify the underlying bugs behind students' errors of solving equations (Matz, 1982; Payne & Squibb, 1990; Sleeman, 1984). The key is what specific traits equations caused difficulties for solving equations for students? Sleeman found that if there were multiple Xs in equations, students were often unable to solve them correctly. They may attempt to guess values for the Xs. Morevoer, they might give different values for the Xs. For example, for equation 3 * X + 2 * X = 12, one student gave 2 to the first X and 4 to the second X. Her solution in her worksheet follows:

3 * 2 + 2 + 4 =12

X = 2

X = 4 (Sleeman, 1984, p. 398)

Sleeman called such a faulty algorithm a *manipulative* mal-rule, which "is a variant on a correct rule which has one sub-stage either omitted or replaced by an inappropriate or incorrect operation" (p. 403). He listed the students' mal-rules in detail in solving equations. However, he found that it was not enough to use *manipulative* mal-rules to explain the following typical wrong answer in solving this equation. The answers did not reflect "mal-rules" and were unreasonable.

$6*x=3*x+12$

$6*x=3*x-12$	$6*x=3*x+12$
$9*x=12$	$x+x=12+3-6$
$x=12/9$	$2*x=9$
$x=4/3$	$x=9/2$

According to the interviews conducted by the author, the first student explained he tried to move the $3*x$ term to left side. The second student replaced the * operator by the + operator which exposed this students' profound misunderstanding of algebra. Sleeman called it a *parsing* mal-rule. In a word, *manipulative* mal-rules are due to incompletely or wrongly executing procedures, but *parsing* mal-rules relate to misconceptions of algebra notations.

Another interesting finding by Sleeman (1984) was that students might solve the equations correctly but fail to point out the interviewer's wrong solutions, that is, the student was able to solve the following equation:

2 * x + 3 = 9

2 * x = 9-3

But the student could not explain why the following solution was wrong:

X = 9-3 +2

The above situations demonstrate that even though students could solve the equations correctly, they might not know the rationales behind these solutions. Assigning different values to the Xs in the same equations demonstrates their misconception of variables. They just strictly followed the correct procedures without understanding them.

Variables and functions. The development of algebra was related to the changing meaning of variable over time. Harper (1979, 1987) pointed out that there were three stages in the development of function. The first stage is the period before Diophantus when there was no symbol to represent *unknown*; during the second stage (3^{rd}-16^{th} centuries), a letter was used only for *unknown* quantities; in the 1500s, the third stage, a letter was used to represent *given* as well as *unknown* quantities. Only after the establishing of variable as the above meanings, the concept of function appeared and algebra could be used to solve general problems. Harper (1979) thought the use of variable to distinguish "unknown" number from given number indicated the demarcation line between two distinct domains of mathematics: one in terms of the known or unknown quantities and the other in terms of variable and constant quantities. An example was used by Sierpinska (1992) to illustrate such an ontological difference:

> Two companies rent photocopiers. The first takes $300 for the location of the machine per month and $0.04 for each copy. The second takes $250 for the location and 0.06 per copy. 1. For what number of copies per month would the price be the same? 2. If you are a bigger user of photographers which company is preferable? (p. 36)

For the first problem, students only need to think this problem in terms of equation and unknown. That is, they need to write an equation like $300 + 0.04x = 250 + 0.06x$. In this equation, x is only an unknown, representing a special value. However, for the second problem, students need to think it in terms of function and variable. That is, they need to write two functions, for the first case, the function is y = 300 + 0.04x; for the second, the function is y= 250 + 0.06x. Here, x is a variable, representing a range of value. Therefore, it is important to have a correct understanding of variables in order to master functions. The misconceptions of variables are often the main obstacles to understanding functions.

Understanding and misunderstanding of variable. The invention of variable indicates the appearance of modern mathematics (Rajararnam, 1957). At the beginning, variable was closely related to the concept of function. "Related numbers that change together, like x and y in the above equation, are called variables. When one variable depends on another for its value, we say that it is a function of other" (Upton, 1936, p. 239. as also cited by Philipp, 1999, p. 157). Variables and constants were distinguished during the first half of the twentieth century. Constant represented only one value but the variable could represent many values. A typical example was given by Osborne (1909) to illustrate such a distinction. The equation $x^2 + y^2 = a^2$ was usually used to represent a circle, where x and y represented coordinates of the points in the circle. Since the radius is certain, "a" is a constant. In the latter half of the twentieth century, variable in the textbooks was separated from the function.

Usiskin (1988) provided detailed explanations of variable meanings. According to Usiskin, based on the views of algebra, there are four possible meanings of variables. (1) If algebra is viewed as generalized arithmetic, then variables were thought of as pattern generalizers. For example, the commutative characteristic of addition can described as a + b = b + a; (2) If algebra is viewed as the procedures for solving problems, variable is viewed as unknown, which is clearly related with equation; (3) If algebra is viewed as the study of relationship, the variable is understood as *argument* (i.e. "stands for a domain value of a function" (p. 10) or *parameter* ("Stands for a number on which other numbers depend" p. 10). (4) If algebra is thought of as study of structure, then variable is thought of as an arbitrary symbol. The first three cases relate to school algebra.

Schoenfeld and Arcavi (1989) investigated mathematicians, mathematics educators, and computer scientists for their understandings of the concept of variable by asking them to choose one word from this list: "symbol, placeholder, pronoun, parameter, argument, pointer, name, identifier, empty space, void, reference, instance" (p. 151). They found that even the experts described this fundamental concept in different ways. Furthermore, they examined different literatures for their explanations of variable. They listed ten different definitions which typically showed the complexity of such a fundamental concept. The core and common thing across these explanations was the recognition that the use and understanding of variable was related to problem contexts. For example, Philipp (1999, p. 160) used several examples to illustrate the usages of letters:

1. Labels f, y in 3f=1y (3 feet in 1 yard)
2. Constants π, e, c
3. Unknowns x in $5x-9=11$
4. Generalized numbers a,b in $a+b=b+a$
5. Varying quantities x,y in $y=9x-2$

6. Parameters m, b in $y = mx + b$

7. Abstract symbols e, x in $e * x = x$

With regard to students' possible misconceptions of variable, students may misunderstand variable as a label. Clement (1982) found that many students had difficulties in using algebraic expression to represent a relationship in this problem: "A university has six times as many students as professors. If S is the number of students at the university and P is the number of professors at the university, then write an equation expressing a relationship between S and P." Clement (1982) found that 37 percent of first-year college engineering majors and 57 percent of social science students at the college level answered by 6S=P rather than S=6P. Such reverse errors may result from misunderstanding variables as labels, that is, some students misunderstand S as "students" and P as "professor". Another misconception of variable by students is that different letters mean different values (Booth, 1988; Stephens, 2005). For example, many students thought $h + m + n = h + p + n$ was never true because "m" was different from "n" (Stephens, 2005).

2.3.2 Understanding the difficulties of algebra concepts: Perspectives based on the dual nature of mathematics knowledge

In this subsection, I review the research on the nature of mathematical knowledge. I begin from several philosophical perspectives about mathematics and mathematics education. Then I will review the *dichotomy* method concerning properties of mathematics knowledge. At last, Sfard's view of the dual nature of mathematical knowledge is reviewed.

Philosophical views of mathematics. The understanding of the nature of mathematics knowledge contributes to a deep understanding of learning difficulties of mathematics. Ernest (1991) introduced several different views of mathematics knowledge from philosophical perspectives. Absolutists maintain that mathematics knowledge consisted of certain and unchallengeable truths. However, such a view of mathematics knowledge cannot explain some contractions at the beginning of the twentieth century. Social constructivism holds another view which has already had strong effects on current mathematics education. The main points are as follows (Ernest, 1991, p.42):

(i) The basis of mathematical knowledge is linguistic knowledge, conventions and rules, and language is a social construction.

(ii) Interpersonal social process is required to turn an individual's subjective mathematical knowledge, after publication, into accepted objective mathematical knowledge.

(iii) Objective itself is understood to be social.

Such views of mathematical knowledge are echoed by current mathematics education principles which advocate cooperative learning, communication, explanations, and justifications (National Council of Teachers of Mathematics [NCTM], 2000). Because philosophical views of mathematics knowledge mainly provide general points about how to teach and learn mathematics, it is necessary to have a more detailed analysis of mathematics knowledge.

Dichotomy method for mathematics knowledge. The typical method used to research mathematics knowledge is to divide knowledge into conceptual knowledge and procedural knowledge (Hiebert & Lefevre, 1986), declarative and procedural knowledge (Anderson, 1976), or abstract and algorithmic knowledge. This is a *dichotomy* method to analyze mathematical knowledge. However, the drawback for the *dichotomy* method is that it is not easy to state the relationship clearly between those two types of knowledge. "The types of knowledge themselves are difficult to define; the core of each is easy to describe, but the outside edges are hard to pin down" (Hiebert & Lefevre, 1986, p.3).

Duality nature of mathematics knowledge. Sfard (1991) proposed another method to research mathematics knowledge: *the dual nature of mathematics knowledge*. She emphasized the fundamental difference between the *duality* and *dichotomy* methods. "Let me stress once more: unlike 'conceptual' and 'procedural', or 'algorithmic' and 'abstract', the terms 'operational' and 'structural' refer to inseparable, though dramatically different, facets of the same thing. Thus, we are dealing here with *duality* rather than *dichotomy*" (p. 9).

Most mathematics concepts embody such duality. The mathematics concept "number" will be discussed in detail to show the meaning of *process* and *object*. When children study the concept of "number," they start from "counting" which is natural and relatively easy for them. Sfard (1991) found that humans took over three thousand years to develop and recognize the concept of "number." "Number" over a long period of time was developed in the context of a measuring process. "Fraction" was thought of as the "ratio of two integers" to describe a measuring process and is thus hard for students to comprehend. The development of mathematics is basically consistent with and could be reflected by that of individual psychology. In Carpenter et al.'s (1980) study, the researchers found that 50% of 13-year-old students in their study were unable to represent a division problem by a fraction. For them, a fraction like 7/4 was not an acceptable final result but a computation process. Sfard (1991) also pointed out the finding of irrational numbers was due to the discovery that "in certain squares, the usual procedure for finding the length of the diagonal cannot be described in terms of integers and their ratios" (p. 12). After a long time, mathematicians broadened the number set to include irrational numbers. The concepts of "negative number" and "complex number" were the products of solving equations of the third and fourth orders. Jourdain (1956, p. 27 as also cited by Sfard, 1991) clearly showed that negative number was a type of process at first:

> Let a-b be c. To get c from a we carry out the operation of taking away b. This operation, which is the fulfillment of the order: "Subtract b" is a "negative number". Mathematicians call it a "number" and denote it by "-b" simply because of analogy: the same rules for calculation hold for "negative numbers" and "positive numbers".

From this development of "number", Sfard thought there were two stages: the operational and the structural. "To sum up, the history of numbers has been presented here as a long chain of transitions from operational to structural conceptions: again and again, processes performed on the already accepted abstract objects have been converted into compact wholes" (p. 14).

The duality nature of mathematics knowledge can be found in most mathematics concepts. In the following section, the variable, equation, and function will be explored at the *process* and *object* levels based on past studies.

2.3.3. What does understanding variable, equation, or function as process and object mean?

About variable. As the historical development of algebra revealed, variable was first used to represent *unknown* quantities and then to represent both *given* and *unknown*. In school algebra, the variable usually means something with multiple or varying value, which is a little different from an unknown in that the unknown is usually a fixed value but humans do not know what it is. In school algebra, Weinberg (2005) maintained that to understand variable as a *process* usually means to substitute it with a specific value; for example, students tend to refer to a specific number when they use variables to represent a relationship. To understand a variable as an *object* is to understand it as a placeholder or a given number. In this study, Weinberg's understanding of variable as *object* or *process* will be used. Another standard about students' understanding of variable as *object* is to see whether students can operate on or with variables. That is, if students can understand the variable as an *object*, they should be able to operate on or with it. The variable becomes the object of reflection or operation at a higher level by students. Students who understand variable as a *process* are usually uncomfortable or unable to operate with or on variables. For example, they may simplify an algebraic expression “T+1” into “T1”. They do not think that “T+1” is an acceptable or final answer. Students at the middle school level should have the ability to reach *object*-oriented thinking about the concept of variable.

About equation. Schoenfeld (1987) provided a good example about understanding an equation. If students have difficulty in making a judgment about whether two expressions are equal without computation, such as (235+ □) + (679-122) = 235 + 679, students may understand “equation” as a *process*, that is, the arithmetic approach of computation. On the other hand, students with *object*-oriented thinking tend to use the property of equation to figure out the unknown value or make a judgment about the equality without referring to computation.

With regard to understanding “equation” as *object*, Kieran (1992, p. 393) thought that “Algebraic equations are structural representations that involve a non-arithmetic perspective on both the use of the equal sign and the nature of the operations that are depicted.” That is, students should understand that the equal sign “is precisely that of respecting the symmetric and transitive character of equality” (Vergnaud, 1984, 1986, also cited by Kieran, 1992, p. 393). “Structural representation,” “symmetric and transitive character” means “understanding equation as *object,*” which should become the goal of school algebra and understood at the middle school level.

About function. Kieran (1992, p.391) thought “The early concept of function as an input-output procedural notion was soon replaced by more structural conceptions. …. Bourbaki, who defined function as a relation between two sets” As a result, if students only recognized a function as a way of computation or relationship between dependent and independent variable, then they only understand it as a *process*. Initially, such understanding of function is compatible with the human cognitive level and is also consistent with the historical development of the function concept; thus, it is acceptable at the very early stage.

If students understand a function as a set of ordered pairs, they understand a function as *object*. For high school or university students, to understand a function as an *object* means to understand the definition of a set of ordered pairs without referring to variables. They should be able to operate on or with functions, for example, the composition of functions or derivative of functions. According to Sfard (1991), if students can use a graph correctly to represent a function, to identify linear or nonlinear functions, it means that they also understood function as an *object*, or at least, these students had *object*-oriented thinking of mathematics concepts. However, the ability to understand a function as an *object* does not mean that students have no misconceptions about the function. They may have certain misconceptions but have no ontological difference from experts' conceptions of the function. Therefore, such misconceptions are not robust to change. For example, in this study, high achieving students interpret graphs or/and linearity well but are still unable to use symbolic representation of functions to solve the problem. For middle school students, they are required to use a graph to interpret or represent a function and to understand linear and nonlinear relationships by NCTM (2000). In this study, students' understanding of functions as object means to be able to use a graph to explain or represent functions or to understand linear or nonlinear functions. In a word, they are able to explain or solve the problems about functions by using the properties of function without referring to the beginning definition of functions: the input-output process.

2.4 The Shortage of Error and Misconceptions Research on Algebraic Concepts: Variable, Equation, and Function

Misconceptions are widely studied in science education. There were over 6000 studies about misconception or alternative conceptions of science concepts (Chi, 2005). However, recent studies of error and misconception analysis in mathematics education are rare (Barcellos, 2005). This is mainly because mathematics education emphasizes the logical relationship between concepts over the concept itself (Dubinsky, 1995 also cited by Simon, Tzur, Heinz, & Kinzel, 2004). As Thompson (1985) pointed out, “Little attention has been given to the issue of the development of mathematical objects in people’s thinking” (p. 232). Specifically for algebra, there is little firm evidence to support students’ errors caused by their mental representations or misconceptions (Payne & Squibb, 1990). Much less known is information about students’ errors on specific and fundamental mathematics concepts, especially variable, equation, and function. The concept of variable, which is a foundation of advanced mathematics and a basis for the transition from numbers to algebra, is overlooked by most researchers and even textbooks (Graham & Thomas, 2000; Schoenfeld & Arcavi, 1988). McNeill and Alibali (2005) found that the mechanisms underlying children’s difficulties with equations and the ultimate emergence of correct strategies were not well documented. Likewise, although research on function has been conducted by some researchers (Sfard, 1992; Dubinsky & Harel, 1992), few studies about functions at the middle school level were conducted. Given that middle school students are in the critical stage of transition from arithmetic to algebra, it is important to know the difficulties, the errors or misconceptions that middle school students harbor.

Students' robust change in understanding the "equal sign" has been studied for a long time and very good results have been published (Falkner, et al, 1999; McNeill and Alibali, 2005; Knuth, Stephens, McNeil, & Alibali, 2006). However, those researchers all thought students' misconception of "equal sign" as "to do something" was the mainsource of errors. One piece of evidence was that many students fill in "12" in the equation 8 + 4 = □ + 5. Furthermore, those studies documented the robust changing of misconception about the "equal sign," but they did not explore why students are resistant to changing their misconceptions. It is too simple to attribute students' robust misconceptions to teachers or teaching methods. Very few studies (of which I am aware) tried to figure out the causes of the wrong answer "12" from other perspectives. It is doubtful that the misconception of "=" as "to do something" is the main cause. This is because many students still calculated 8 + 4 and obtained results even if there is no "equal sign." For example, while students may understand the algebra expression "8 + 4" as a *process* rather than an *object*, such an alternative understanding of "8+4" may cause the errors. These students are unable to use an expression to represent a "quantitative number." There are very few studies about this possible error source. In this study, understanding the misconception of “=” based on *object* and *process* is explored.

3. METHODOLOGY

The data for this study is from a funded project: Improving Mathematics Teacher Practice and Students Learning through Professional Development (IMTPSL), which is a 5-year longitudinal study. Researchers at the University of Delaware and Texas A&M University, working in partnership with Project 2061 of the American Association for the Advancement of Science (AAAS), are investigating the interactions of teaching practices, selected curriculum materials, and professional development to understand the ways they can be optimized to improve student learning. Key lessons were carefully selected and videotaped by the project researchers. Students took pretests and posttests during three cohort school years (2003-04, 2004-05, and 2005-06).

3.1 Participants

For the three cohort years' (2003-2004, 2004-2005, and 2005-2006) data, only students' 2004-05 algebra pre- and posttest data were used in this study. Students from two states, Texas and Delaware, participated in the pretest in Fall 2004 (N = 456) and the posttest of Spring 2005 (N = 502). A total of 317 (171 grade 7 students and 146 grade 6 students) students participated in both the pre and posttests. Both the teachers and students had a choice to participate in the project or not. As a result, the sample was based on the convenience principle.

3.2 Instrumentation

3.2.1 Test items

The development of test items by the project IMTPSL. The algebra test was developed by researchers at AAAS in collaboration with researchers at Texas A&M University and the University of Delaware. The algebra test includes seven multiple-choice items and nine short-response items. The test content was aligned with the Principles and Standards for School Mathematics (NCTM, 2000) guidelines for objectives of middle school algebra. Three mathematics constructs, that is, change (function), variable, and equation were developed by project researchers; the main test contents for each construct were also developed (see Appendix 1for assessment map). The project researchers also specified how each test item was related to these three target constructs (see Appendix 2). All items were carefully developed through piloting and field-testing by the researchers of project IMTPSL. Pre and posttests with identical contents were administered during each school year with the pretest in the fall and posttest in spring, for each cohort.

The reliability and effectiveness of test items. The researchers of the project employed Confirmatory Factor Analysis method to evaluate whether the chosen algebra items adequately assessed the three concepts: change (function), variables, and equality and equations. The data was from students' achievement as measured by the algebra test of seventh and eighth graders in Delaware (N=339) and Texas (N=574) in fall 2003. The graders were strictly trained by AAAS researchers and their scoring reliability was tested by AAAS. According to project researchers, the test items adequately measure the constructs (Capraro, et al, 2004).

3.2.2 Task analysis

Since the goal of this study is to analyze students' misconceptions of function, variable, and equation, it is clearly important to evaluate whether these test items aligned well with intended concepts. In above reliability and effectiveness tests, the project researchers used quantitative methods to evaluate whether the items adequately measured students' understanding of the three concepts. In this section, each item is analyzed qualitatively via the framework of mathematics and science alignment developed by AAAS (Kulm, 2004). According to AAAS, analysis of a task involves three main categories: (1) groundwork, (2) content analysis, and (3) likely effectiveness. Each category contains several indicators. For groundwork, five indicators are used: (a) task completeness, (b) task clarification, (c) candidate goals, (d) goal clarification, and (e) potential alignment. For content analysis, two indicators are used: (a) necessary and (b) sufficiency. For likely effectiveness, four indicators are used: (a) comprehensibility, (b) clear expectations, (c) context, and (d) test wiseness (for detailed information, please refer to http://msmp.tamu.edu/project_papers/AERA).

The second category "Content analysis" was provided by the project researchers (Kulm, 2004). The third category "likely effectiveness" is also important. Simple and accurate English language was used in test items. For example, most items employed everyday life contexts such as raising a flag (Q11), cell phone plan (Q13), bricks and stones (or pine and apple trees) (Q8), and the changed value of a used car (Q15) (see Appendix 4 for test items). U.S. middle school students should be familiar with these contexts. Since the project assessment experts and researchers carefully designed the algebra items over time as I mentioned earlier, it was reasonable to assume that these items well qualify for the standards of content analysis and likely effectiveness.

Regarding the first category "groundwork", I developed indicator b, "task clarification," which is directly related to my study and is not provided by the project researchers. The other indicators of groundwork such as the detailed scoring rubric (indicator a), the learning goals (indicators c, d), and the potential alignment (indicator e) have been carefully considered and provided by the researchers.

According to AAAS, task clarification means: "Identify requisite concepts, skills needed for response, possible misconceptions, and multiple solution strategies" (Kulm, 2004, p. 2). Because the "multiple solution strategies" has already been offered by the project in the form of scoring rubrics, I will focus on (1) the concepts, (2) the skills (procedures), and (3) the possible misconceptions to analyze each item. Since the goal of this study is to identify students' misconceptions underlying their errors, I will also justify why students' wrong responses can expose their misconceptions. What follows are the detailed task classifications. I begin with the general description and then continue with the analysis of each item in depth.

Q1 to Q7 (see Appendix 4) are multiple-choice problems. Multiple-choice problems do not require students to show their solution process. It provides students, especially those who are not adept at representing their ideas by using mathematical symbols and formulas, the opportunity to demonstrate their understanding. Therefore, to include multiple-choice items improves the validity and reliability of the instrument. The shortcoming of such a form is that students may guess the answer without real understanding.

Q8 to Q16 (see Appendix 4) are short-response problems. In answering problems, students not only provide answers but also need to justify their answers by using everyday or mathematics language or other representations. Therefore, the strength or weakness of students' conceptual understanding of certain concepts can be identified from these problems. Students' misconceptions, which are the focus of this study, can also be found by analyzing students' responses.

Q1 (see Appendix 4) is a multiple-choice and one step problem. This item is used to examine students' understanding of "equation" and "equal sign." Past research showed that students were comfortable with the form, 8+□=12, because they viewed the right side of an equation as a definite result (Carpenter, Franke, & Levi, 2003). In this problem, the task of computation is trivial but students need to understand the "equal sign" as the relationship of equality. Carpenter et al. (2003) divided students' understanding or difficulties with equations into four levels: (1) able to solve a problem like 8 + 4 = □ + 5; (2) able to accept the form of an equation, like 8= 5+3, as true; (3) able to understand the "equal sign" as relationship and (4) able to compare the mathematical expressions on each side of an equation without actual calculation. Thus, students are expected to have at least a level 2 understanding of equation to solve this problem. If students really understand the semantic meaning of "equation" or "equal sign," the process (skill) of solving this problem will become very simple: a basic computational task for a seventh or eighth grader. Even if students calculate incorrectly, they should have the ability to find their error by substituting the wrong answer in the equation, if they have sound conceptual understanding of equation or equal sign. As a result, it is reasonable to assume that students do not have a solid understanding of equation or equal sign if they do not choose a correct answer; at least, the meaning of "equal" is not apparent to them.

Q2 (see Appendix 4) can be used to assess students' ability in translating word problems into symbolic representations. Students are required to use an equation to represent the relationship as stated, using the everyday language in the problem. The possible errors and misconceptions have been documented in past studies: that is, misunderstanding a letter as representing things rather than a quantitative number (Clement, 1982; Sims-Knight & Kaput, 1983). The requisite skills for solving this item include: (1) able to understand x representing the number of trading cards that Mary has; (2) able to use algebra expressions to represent the number of trading cards that Julie has; and (3)able to uncover the mathematics meaning of "they have 36 trading cards in all." The strategies used most often by students are either to replace the key English word sequentially with mathematical symbols or use key words blindly (MacGregor & Stacey, 1993). When students see the key word "in all," they tend to just add some or all items without considering actual relationships among them. For this item, students may also have difficulty adding two algebra expressions, which may make no sense for them. This is because students are used to adding two definite numbers and getting a definite result from their arithmetic training. Students who choose the answer $3x=36$ may indicate such a cognitive obstacle of adding two algebra expressions.

Q3 (see Appendix 4) is used to assess students' understanding of variables. In this item, students are expected to understand variable as a "given number" which indicates an ontological difference compared with the meaning of variable in Q2. Students tend to misunderstand variable as a "label" or represent a "thing" rather than "the number of a thing." Such a misconception is believed to be related to students' prior experiences. For example, students in elementary level use letters such as "f" to label "foot" and "i" to label "inch", thus, they usually write the relationship between foot and inch as "1f=12I". In this item, if students are unable to use and understand "n" as representing "the number of Girl Scouts," it is difficult for them to write an algebra expression representing the "the number of rows." Another possible cognitive obstacle is that students are more familiar with "multiplication" with given numbers. That is, it is relative easy for them to know the number of rows and the number of girls in each row to calculate the total. However, students may have difficulty operating on letters. This item is designed as "low" level complexity according to the project assessment specialists. It is a one-step problem.

Q4 (see Appendix 4) is about the characteristics of variable. According to Booth (1988), students who understand algebra as "generalized number" can understand a variable representing a "generalized number." If students are able to select the correct answer B, "order doesn't matter when adding two numbers," they are headed in the right direction transitioning from arithmetic to algebra thinking. This is a one-step problem.

Q5 (see Appendix 4) is used to assess students' understanding of a pattern implicated in the table. Students met similar problems at the elementary level, so this problem is relatively simple for them. It is low level complexity as claimed by project assessment specialists. Students who were familiar with the arithmetic approach should be able to solve this problem.

Q6 and Q7 (see Appendix 4) require that students understand function as the relationship between dependent and independent variables. Q7 is more complicated and confusing compared to Q6 in that this function has an intercept 5 but students are more familiar with the function whose graph crosses (0, 0), the original point. Moreover, students need to find the value change of y corresponding to the change of x for the function y=2x +5 but for Q6, students only need to point out the change between x and y. Finally, the possible cognitive obstacle is due to the prior experience of "addition." Students at the elementary level tend to think of "addition" as "more." Because there is a table in Q 7 to assist students, it is more likely for students to make a correct choice based on table value without considering the symbolic form of the function. In fact, the preliminary examination of students' answers showed that many students indeed just worked from the table. If students could look at the symbolic form of the functions in both items to make a choice, these students employed *object*-oriented thinking. On the other hand, if students used table values to find the pattern, they utilized *process*-oriented thinking.

Q8 (see Appendix 4)**: Apple trees/pine trees and stones/bricks**. There are minor differences in this problem between administering the pre and posttest instruments, but the main structures are the same. The only change is to use apple/pine trees to replace stones/bricks. There are four sub-problems labeled as A, B, C, and D. For A, students need no more than counting skills. For B, students are expected to find a pattern based on the pictures.

For 8C, students are asked to find the value of n for which the number of stones equals the number of bricks (or the number of pine trees equals the number of apple trees). This one is harder than both A and B. There are two approaches solving this problem. One is to list the values of the apple/pine trees with n=1, 2, 3, in the form of a table. Then the values of apple/pine trees were compared to find the value of n. Or students may simply guess and check their answers using the formulas to see whether they are equal. Both of these are arithmetic approaches without using the algebra formulas provided at the beginning of the problem. An algebra approach to solving the equation is $n \times n = 8 \times n$. It is interesting to see why so many students did not use these formulas to find the solutions but chose to use arithmetic approach, or put another way, why did students totally ignore this information of algebra formulas? Why was the activation of the algebra approach so hard for some students?

For 8D, the level of complexity is “high” as assigned by the project researchers. First, students need to understand the mathematical meaning of “quick.” Second, students should be able to distinguish “quick” and “more.” Similar to 8C, there are two ways to solve the problem. One is to use the table values, which is an arithmetic approach. Another is an algebra approach. Each time the number of rows (n) increases by 1, the number of bricks (pine trees) increases by 8, while the number of stones (apple trees) increases by $n^2 - (n-1)^2 = 2n - 1$. As a result, when 2n-1 > 8 (n is equal to or greater than 5), the stones increase quicker. The possible misconception is that " more means quicker." Such a misconception may cause this bug: using $n^2 > 8n$ to determine when the number of stones will increase more quickly than that of the bricks, or using the table values to find the same amount.

Q9 (see Appendix 4)*:* ***Tachi and Bill Problem*** is similar to the famous "professor-students-problem" (Clement, 1982). For this problem, students using syntax translations may produce the reverse error, T+1=B. The deeper reason for such errors is related to students' misunderstanding of equation and variables (Clement, 1982). MacGregor and Stacey (1993) hold a similar position that students' errors were more likely related to semantic, rather than syntax, translations. Compared to the multiple-choice problems (e.g., Q2 and 3), students must write an equation by themselves, which caused particular difficulties for students in that they were not good at operating on "generalized numbers." It is hard for them to use an algebra expression to represent a number. This is a one-step problem and should be challenging for students. The main difficulty will be the understanding of variable and operating on or with the variable rather than understanding the relationship between Tachi's and Bill's ages.

Q10 (see Appendix 4)***, a=b-2***, is used to assess students' understanding of function and equation and a "middle" level complexity was assigned to this item by the project specialists. Students are required to find a pair of values. This problem involves only simple computational skills. It is a "function machine" where the students input a number and get an output number. This problem assesses student's primary knowledge of function as an input-output process.

Q11 (see Appendix 4): ***Small boy raises a flag***. For this problem, students need to learn how to model a real life situation by using mathematics symbolism, a graph. It is important for students to know how to represent a function by using a graph which indicates students' understanding of a function as an object. Thus, it is extremely difficult for students to draw or choose a correct graph. Students need to think of the "function" as a mental object without considering the "input-output" process. Students' errors in this problem will be interfered with by their real life experiences. Students are so familiar with raising a flag that when they deal with this problem, the activated part of their knowledge structure would be the life situation rather than the function and graph which is relatively new for them. This situation is more likely to happen to students with no deep understanding of graph and function. The requisite skills for solving this problem are students' understanding of linear function and viewing a function as an object.

Q12 (see Appendix 4)***, missing number problem,*** is similar to Q5 and 9. The additional work of this problem is for students to find a pattern and to use this pattern to find the missing number. The skills to solve this problem concern finding the pattern. This is a typical input and output function machine problem, and students who only understand function as a *process* should be able to solve this problem.

Q13 (see Appendix 4)***, car value***, is used to assess students' understanding of the linearity of a function. If students only focus on one variable or using input-output process, they may make a wrong judgment. This problem was rated as middle level complexity by the project assessment specialists. Students who understand a linear relationship of function very well are able to answer this problem. The possible misconception is that students misunderstand "linearity" as "constant ratio" rather than the "constant difference over equal time".

Q14 (see Appendix 4)***, Stell's phone plan,*** will not be used in this study.

Q15 (see Appendix 4)***, name a variable****,* asks students to find a variable. Variable is an elusive but fundamental concept for them. Students need to know that a variable can represent many different values. An example of variable has been provided for students in this problem. This item is rated as "low" level by the project assessment specialists. Students need to understand variable as "placeholder" rather than only "specific numbers" or as a "label" for something.

Q***16*** (see Appendix 4,***: find the value of y***, is a little different from the common equation in that the left side of this equation is a number and the right side is an algebra expression. Students who can understand the transitivity of equations should be able to solve it. There are several different strategies for solving this problem. One is arithmetic approach: guess and check. Another is the algebra approach: do the same operations on both sides or use change side and change sign. The difficulty is understanding the structure of "3+4y." For example, some students understood "3+4y" as 3+4+y or (3+4)* y. The understanding of "y" presents another trouble for some students. For example, students thought of "y" as representing "$\times y$," the combination of operation sign and variable.

3.3 Data Coding

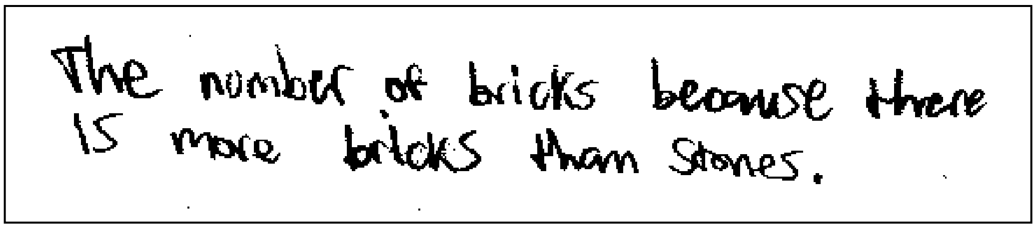
3.3.1 Data coding for the multiple choice items

Each student's answers for the multiple choice items were graded by the project trained graders. Available data includes how many students chose A, B, C, or D. The frequency of each choice was recorded and compared from pretest to posttest to see whether students' errors had changed in general, if necessary.

3.3.2 Data coding for misconceptions and errors

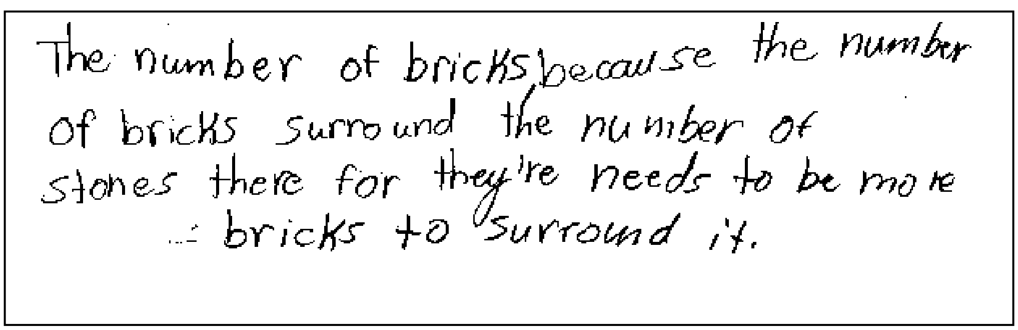
The development of rubrics for coding students' errors. As mentioned earlier, the project assessment experts had developed a scoring rubric, which provides detailed categories for rating various answers. Trained graders completed the scoring of students' answers. Part of the wrong answers were simply classified by graders in terms of the scoring rubric. These classifications may act as a reference for coding the students' errors in this study.

To analyze students' errors and misconceptions, I developed a new rubric (see Appendix 1). The creation of this rubric was mainly drawn from the pilot analysis of students' answers. One hundred students' answers were randomly selected from the pretest, and were recorded and classified. A rubric was developed according to the error types. For a reliability check, another graduate student independently selected one hundred students' test sheets and created a rubric by using the same procedure described above. Finally, the two rubrics were compared for inconsistency. Except for a few, most items were consistently classified. For example, students' responses to 8D "which will increase more quickly, the number of stones or the number of bricks?" (See Figure 3.1 and Figure 3.2)



The number of bricks because there is more bricks than stones.

Figure 3.1. Student 1 response to 8D.



The number of bricks because the number of bricks surround the number of stones there for they're needs to be more bricks to surround it.

Figure 3.2. Student 2 response to 8D.

Initially, there were different opinions about whether it was necessary to use two categories. These answers represent two misconceptions that students may have. One misconception is that the border is always larger than the inside; the other misconception is that more means quicker. At last, the two coders reached an agreement that two categories should be used to code the above answers. Although the main goal of this problem is to assess students' understanding of "faster" through using mathematics symbols and formula, the misconception of the relationship between surrounding and inside parts deserves to be carefully analyzed. Other inconsistent items were negotiated in the same manner to reach final agreement.

Coding of misconceptions. According to the rubric, each student' errors in pre and posttests were recorded. According to each category, the frequency of students' errors in pre and posttest was compared to see whether there was significant change. In addition to this quantitative analysis, the misconceptions underlying students' errors were identified and justified through qualitative analysis.

Students' misconceptions underlying students' errors are usually not observable, so the identification of misconceptions is mainly based on two categories which are developed from the literature of this study

(1) This type of error should be found consistently in different problems or contexts.

(2) This type of error should appear consistently in different items across pre and posttests and across grade levels. As mentioned earlier, misconceptions of mathematics knowledge are held by many people over a long time. It is expected that errors caused by misconceptions should not occur haphazardly.

Possible error sources, other than misconceptions, should be either eliminated or at least identified. The task analysis earlier showed that all the test items are one-step problems, which should be solved by recalling the corresponding concept knowledge. The execution of the procedure may become a possible error source even for a one-step problem. However, since the problems in this study mainly asked students to justify their answers, little procedural knowledge is needed. Thus, it is possible to determine a difference between errors due to misconception or execution of a procedure.

First, the most frequent errors of different problems were identified in terms of the error coding. Then, the misconceptions behind the most frequent errors were analyzed. At last, such misconceptions were also verified by looking for the same types of errors in other problems. For example, many students make a mistake in ***Q9*** (see Appendix 4) by writing an answer like Tachi's age = one year more than Bill. The frequency of such errors is high. So the possible misconception is that students understand "equal sign" as "association." The same type of error was also often found in **Q16** (see Appendix 4), where students usually wrote running equations, like 4+4 = 16+3 = 19. If the same type of error occurs frequently in different problems, the misconception behind them is identified. The assessment map and the form detailing how the test item relates to variable, equation, and function has been provided by the researchers of the project (see Appendix 1 and 2).

3.3.3 Coding of understanding of algebra concepts as *object* or *process*

The feasibility of coding students' understanding as process or object. The most important and challenging task is to not only identify misconceptions held by students but to seek how to change students' misconceptions. Most misconceptions were not only resistant to change but were also reinforced by improper instruction (Kilpartick, Swafford, & Findell, 2001). Knowing why students' misconceptions resist change is a prerequisite to altering students' misconceptions. As mentioned earlier, one of the particular difficulties of learning mathematics concepts is to transition from an understanding of concepts as *process* to *object*. One way to explore the mechanism of misconceptions caused by the transition from process to object is to compare experts' and novice's (or high achieving and low achieving students') differences in their understanding of concepts. Slotta, Chi, and Joram (1995) developed a method to find whether students understood a concept as a process or an object. Their rationale is "if novices have classified a concept as a *material substance*, their explanations should contain verbal predicates that correspond to the ontological attribute of that category" (Slotta et al, 1995, p.378). In the most recent article about students' misconceptions, Slotta and Chi (2006) employed the same method and gave an example to show how this method works:

> For example, if a subject said, "The current comes down the wire and gets used up by the first bulb, so very little of it makes its way to the second bulb, then these four (underlined) predicates were taken as evidence that subjects conceptualized current as a substance-like entity with attribute of (1) "moving," (2)"can be consumed," (3) "can be quantified," and (4) "moves" (Slotta & Chi, 2006, p.6).

For this study, I will refer to their method to code students' verbal explanations. Concepts as *object*s are characterized by static, instantaneous, and integrative words while concepts as *process* are characterized by dynamic, sequential, and detailed words.

However, mathematics differs from science or other subjects in that there is a symbol system, the special "mathematics language." Therefore, students' responses in science may mainly use verbal representation. In contrast, students in mathematics may mainly use mathematics language--notations and symbols. As a result, when I code students' understanding of a mathematical concept as a process or an object, I mainly focus on the strategies students used, such as arithmetic or algebra approaches. Generally, a student who understands a concept as *object* tends to use the algebra approach to solve a problem which can be solved through either an algebra or arithmetic means. This is because the algebra approaches usually involve only a few steps. On the other hand, students with *process* thinking tend to use arithmetic approaches due their inability to use algebra thinking. At the same time, students' verbal explanations will also be analyzed to improve the validity of the analysis.

The selected problems used in object and process analysis. Only open-endedproblems are chosen to analyze the difference between the high and low achieving students' understandings of mathematics concepts. The problems will be chosen according to three categories: (1) can be solved using multiple strategies; (2) directly reflect students' understanding of concepts; and (3) have proper complexity level. What follows are the detailed elaborations:

As to category 1, elementary students use arithmetic approaches to solve problems but students at the middle school level use either algebra or arithmetic approach. Different strategies reflect students' different level of understanding. As a result, only short-response problems will be used in analyzing students' object-like and process-like understanding.

As to category 2, students' understanding of concepts can be reflected in their problem solving. If a problem requires several steps to be solved, students' errors may occurr in either of these steps. The task analysis earlier showed that the short-response problems (see Appendix 4, Q8-16) meet this category because students just need one or two steps in solving these problems.

As to category 3, the complexity of the problem should not be so high that very few students can answer it, or so low that almost everyone can solve it correctly. There will be no way to do error analysis if there is no response or if all responses are correct. In a word, students' solutions should vary enough to enable the qualitative analysis. The task analysis earlier shows that the short-response problems meet this standard. Based on the above three categories, Q8 C and D, Q9, Q10, Q11, Q12, Q14, Q15, and Q16 (ee Appendix 4) will be used.

Rubric of understanding of variable, equation, and function as object or process. Students with process-like understanding usually demonstrate little algebraic thinking and their goal of solving problems is to get the "answer," usually a specific number. For example, these students tend to write the answer as "T1" instead of "T + 1" to represent "one more than T." This is because they are unable to see "T+1" representing an object, a final result. They cannot accept that there may still be operation signs in the final result. In contrast, students with object-like understanding tend to use an algebra approach to solve problems. Table 3.1 and Table 3.2 comprise the rubric for judging students' understandings of variable, equation, and function as objects or processes. The possible consequent strategies for each case are also provided based on the earlier literature review. The understanding level used in this rubric is consistent with middle school students' cognitive capabilities in terms of NCTM (2000) standards and American Association for the Advancement of Science (AAAS) (1993) benchmarks.

Table 3.1
Viewing concept as process

	Process (operational)	Possible strategies
Variable	Changing numbers	1. Referring to or listing specific numbers
Function	Computational process like function machine, the input-output process	1. Using tables to list possible pairs or solution
Equation	Interpreting "=" as "to do something" or associations	1. Referring to number facts 2. Guessing or counting 3. Changing side and sign 4. Using running equations

Table 3.2
Viewing concept as object

	Object (structural)	Possible strategies in solving problem
Function	1. Able to represent function by using graph or symbolic method; 2. Able to identify linear or nonlinear relationship.	1. Using the algebra formula to solve problems 2. Using the characters of function, such as slope or linearity to answer problem
Equation	1. Understanding "=" as a symbol of identity; 2. Understanding the transitivity and symmetry of equation	1. Able to add or multiply same number to the equation 2. Able to judge whether the equation is true without actual calculation
Variable	1. Understanding variable as a placeholder 2. Understanding variable as a given number or generalized number	1. Able to solve the problem without referring to specific numbers

Some examples of coding students' object-like or process-like understanding. Question 9 is the Tachi and Bill problem. Students' responses for this problem varied a great deal. Among the approximately 400 answers from the pretest, there were about 180 different responses. Below are some responses:

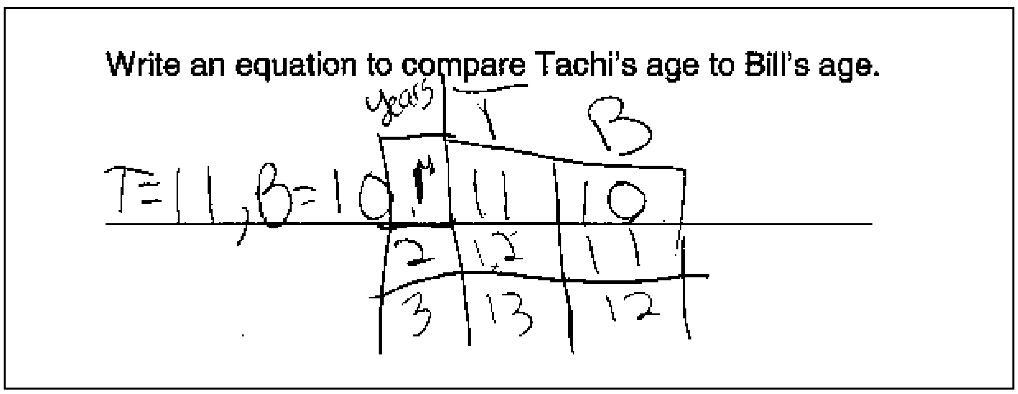


Figure 3.3. Student 1 response to Q9.

Student 1 in Figure 3.3 referred to more than one pair of numbers. Such an answer was coded as *process*-like thinking because this student used arithmetic strategy. That is, this student still thought of a variable as a changing number rather than as a "generalized number" or "given number." In other words, this student is still unable to operate on or with variables. In contrast, those students whose answers were T=B+1demonstrated their *object*-like understanding of variable. This is because they can operate on or with variables in that they write the correct equation without referring to specific numbers

Q10, a = b-2: This problem only needs process-like thinking of function to solve it since it only needs input and output process. It is expected that both high and low achieving students would do well on it.

Q11: This problem asks students to select a graph to model a real life situation. Student with *process*-like thinking of function as input and output process will have extremely difficulty because (1) there is no specific pairs of numbers for those students to figure out the relationship that tables of values usually provide; (2) the mathematics meaning was negatively and strongly interfered with by the real life situation. If students chose wrong answers B or D, those students were coded as process-like understanding of function since they are unable to find a proper graph to represent the relationship. Even if students chose the correct answers, A or C, their response will be coded to see whether they really understand the function as an object. The predicate words will be "steadily" or "linearity" or other similar words for choosing answer A, or "pause" or "break" for choosing answer C. If their responses had such predicate words, it means that these students used the properties of function to answer the problems.

The following two students' (Student 2 and 3) responses showed their understanding of functions as *object* (See Figure 3.4 and Figure 3.5). When students use these words, it means students consider the situation from the characteristics of function without referring to input-output process.

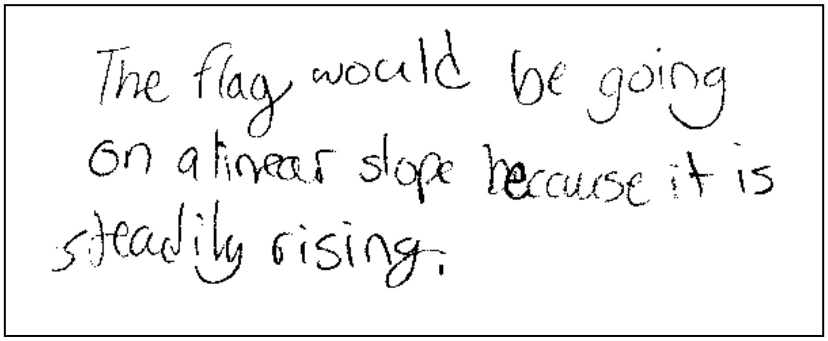
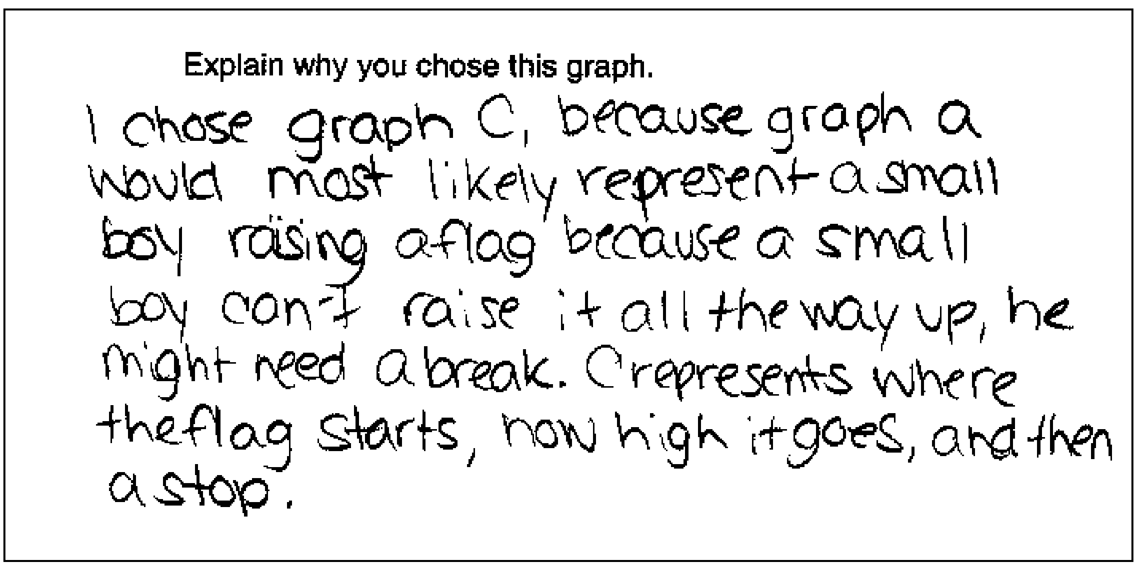
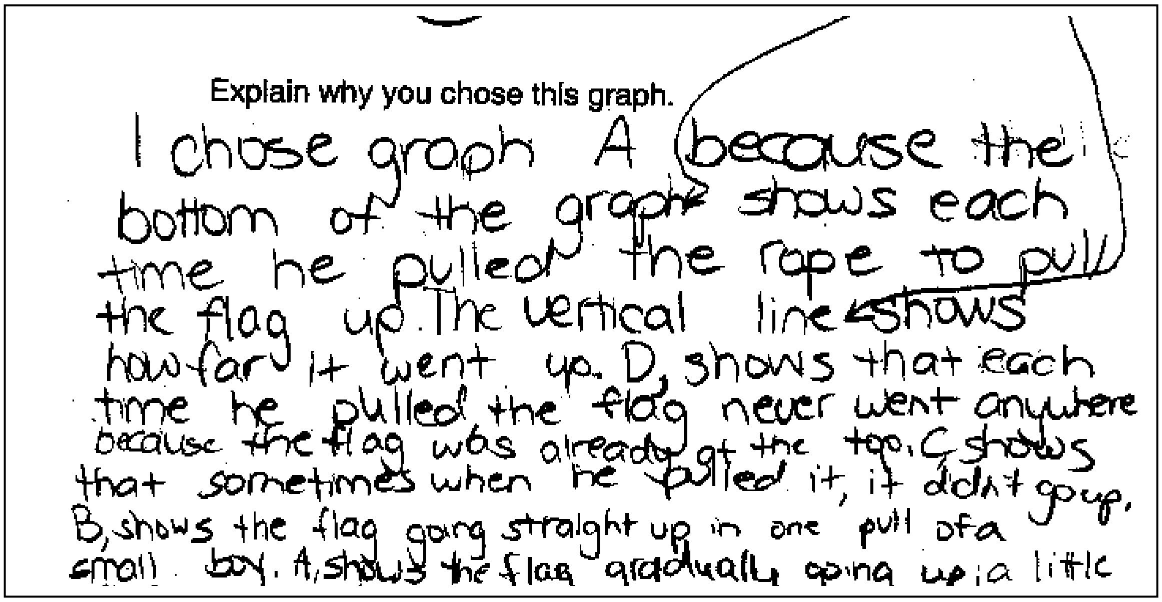

The flag would be going on a linear slope because it is steadily rising.

Figure 3.4. Student 2 response to Q11.


Explain why you chose this graph.

I chose graph C, because graph a would most likely represent a small boy raising a flag because a small boy can't raise it all the way up, he might need a break. C represents where the flag starts, how high it goes, and then a stop.

Figure 3.5: Student 3 response to Q11.



Explain why you chose this graph.

I chose graph A because the bottom of the graph shows each time he pulled the rope to pull the flag up. The vertical line shows how far it went up. D, shows that each time he pulled the flag never went anywhere because the flag was already at the top. C shows that sometimes when he pulled it, it didn't go up. B, shows the flag going straight up in one pull of a small boy. A, shows the flag graduall going up a little

Figure 3.6: Student 4 response to Q 11.

For Student 4 (See Figure 3.6), the response showed clearly that this student understood the meaning of vertical and horizontal lines. However, he used "gradually" to describe the movement of the flag. This student was not aware that the graph is linearity and the flag should grow "steadily" but not necessary "gradually." On the other hand, the use of "gradually" also demonstrates that this student was aware of the relationship between height and time.

Q16 C: This problem involved two functions which were initially represented by two tables. Students' strategies were coded in terms of algebra or arithmetic approach. If students used symbolic representation to solve the problem, it is thought students understood the function as an object.. If students just used the table, it is thought that students understood the function as a process.

Q16 D: In the task analysis, an algebra solution has been demonstrated.

3.3.4 The reliability of coding data

The reliability of data coding is important for this type of study. After developing a reliable coding rubric, the coding of students' responses followed the rubric strictly. The researcher completed all data coding. The coding sheet was developed according to the rubric (see Appendix 1). Every student was labeled using the first letter of their first and last names. Every student's code was put into a corresponding category if this student made a mistake. Thus, the check of the coding would be more effective. The third coder, who is an education major but talented at mathematics, was invited by the author to code part of teh test items. First, the coder was trained by the author. The effectiveness of training was tested. The trained coder was tested by coding Q8 (C) and Q8 (D) from ten students. The training process continued until the coder consistently coded above 90% correctly. Then the third author independently and randomly selected 10% of all students but just coded Q8 (C), Q8 (D), and Q9. For Q8 (C), the reliability is above 95% and 90% for Q8 (D) and Q9. I selected Q8 (C), (D) and Q9 to be tested because these items are difficult to code due to the variety and complexity of students' responses.

3.4 Procedure and Data Analysis

Misconceptions were usually were caused by outside of school knowledge (life experience) or prior instruction. For example, because students normally use "part of the whole" as definition and pattern blocks to learn fractions, they may have a misconception that all fractions are less than 1. Misconceptions usually produce systematic errors, which appear among different students and across different problems, consistently before and after instruction. I used such characteristics of misconceptions to identify the misconceptions.

3.4.1 For research question 1: Error patterns and misconceptions

Students' errors for the multiple choice and open-ended items were categorized. The frequency of each category of errors was reported. Students' misconceptions underlying systematic errors contexts were explored in terms of the standards developed earlier in this section.

3.4.2 For question 2: Robust misconceptions

The number of errors in the pretest and posttest related to the same misconceptions was tested to see whether there is any change after one year by using a statistical test for the difference between two proportions (Ott & Longnecker, 2001). If the misconception is robust, it is expected that the frequency of errors related to that misconception should not decrease greatly or should increase.

3.4.3 For *question 3*: *Ontological differences*

Students understanding of variable, equation, and function as *process* or *object* will be coded in terms of Table 3.1 and Table 3.2. It is expected that both low and high ability students would show a significant difference in "*Understanding a concept as an object*". Excerpted explanations from low and high ability students from these two groups are provided.

4. RESULTS

The results of this study are reported in three sections: (1) Students' error patterns in the selected short response problems (Q8, Q9, Q11, Q13, Q15, and Q16) in the pre and post tests; (2) Possible misconceptions underlying these errors. The robustness of these misconceptions will be analyzed; and (3) Comparison between the high and low achieving students in understanding the three fundamental algebra concepts (variable, equation, function) at the ontological level: object-oriented and process-oriented thinking. The ontological differences between the two groups of students were used to support the explanations in the prior section, that is, students' robust misconceptions are due to the transition from *process* thinking to *object* thinking.

4.1 Results of the Quantitative Analysis

In this subsection, I report students' error patterns reflected in each short-response problem (Q 8 (C), Q8 (D), Q9, Q11, Q13, Q15, Q16; see Appendix 4 for the items). Error types and frequencies are described first. Examples related to each error type will then be provided to justify the categorization of errors.

4.1.1 Tachi and Bill Problem (Q9)

This problem assesses students' abilities to translate a word problem into a symbolic algebra expression. It is generally assumed that this ability largely depends upon students' recognition of the relationship expressed in everyday language form.

Students' wrong answers varied widely in this problem. Wrong answers numbered approximately 180 (N=456, total) in the pretest and 110 (N=506) in the posttest for this problem (for detailed error results, see Appendix 7). It was hard to explain every error and find the misconceptions for each one. The most often used errors were coded and classified in terms of possible misconceptions. Students' most frequent errors were reverse errors, which is consistent with previous studies (Clement Lochhead, & Monk, 1981). Except for reverse errors, this study also uncovered other students' errors which directly supports the claim that some students misunderstood variables as "labels" or "specific numbers." The classification of students' errors in this study reports students' possible misunderstanding of variables and equations in detail. The results for this problem are reported in two ways. One is in terms of which forms did students use in expressing the relationships: algebra expressions, equations, or inequalities. Another is from students' possible misconceptions related to variables and equations. Table 4.1 shows the first result:

Table 4.1
Percentage of error types related to forms: Expressions, equations, or inequities

Error Type	Pretest (N= 456)		Posttest (N=506)	
	Frequency	Percent	Frequency	Percent
1.Using expressions	35	7%	23	4.6%
2.Using inequalities	11	2%	6	1.2%
3.Using everyday language	22	4.8%	7	1.4%
4.Reversed equations	54	11.8%	220	43.8%
5. Other wrong equations	136	29.8%	85	16.9%
6. No response	76	16.7%	42	8.36%
7. Total incorrect percent	334	73.24%	383	76.29%

This problem clearly required students to "write an equation to compare Tachi's age to Bill's age." In Table 4.1 above, students' error types 1, 2, and 3 reflect students' misunderstanding of the equation form. In the pretest, about 14% of students (sum of errors 1, 2, and 3) did not know what an equation looked like. These students used algebra expressions (7%), inequalities (2%) or just everyday language (4.8%) to stand for an equation expressing the relationship between Tachi's age and Bill's age. The situation improved in the posttest where only about 7% of students (sum of errors 1, 2, and 3) did not use equations to express the relationship. Error types 4 and 5 reflect that students already know the equation form though they still made some mistakes. For example, some students wrote a reversed equation such as writing T+1=B (wrong answer) as B-1=T (wrong answer). The percentage of students using reversed equations in pre and post tests increased from 11.84% to 43.82% which means students made some progress in understanding equation forms. As mentioned earlier, in the pretest, there were about 180 different answers for this simple problem and there were 110 different students' responses in the posttest. What cognitive difficulties did these students encounter in solving this problem? Why did these students have such varied answers for this simple, one-step problem? Were there common misconceptions underlying these errors? In order to explore the potential cognitive obstacles and miscomputations, a new category of students' responses were used. The previous studies revealed that students' errors on this type of question may stem from a misunderstanding of variable and equation. Thus, students' errors were categorized as shown in Table 4.2. The category "Others" includes: using everyday language, algebra expressions without simplifying T-B, or T-1 into TB or T-1. If the students wrote an algebra expression but wrongly simplified it into the form of T1, it was coded as "simplify algebra express as T1." "Others" also includes responses that were not classified into any categories. For example, some students' answers were B+B=B or L=1xB. It is unfeasible to code them in terms of cognitive obstacles.

Table 4.2
Error types related to understanding of variable and equation

Error type	Pretest (N = 456)		Posttest (N = 506)	
	Frequent	Percent	Frequent	Percent
1. Refer to specific values	38	8.3%	27	5.3%
2. Variables as labels	19	4.1%	10	2%
3. Simplifying algebra express as T1	22	4.8%	16	3.2%
4. Using "=" as association	12	2.6%	2	4%
5. Using special letters	17	3.7%	19	3.8%
6. T/B=1 or T over B	15	3.3%	2	0.4%
7. T- B = Difference	9	2%	19	3.75
8. Reversed equations	54	11.8%	220	43.5%
9. no response	76	16.67%	42	8.3%
10. Others *	96	21%	35	7%

From Table 4.2, except for the reversed errors, the most frequent errors in both pre and posttest were errors 1, 2, 3, and 5: students referring to variables as specific numbers, using variables as labels, "simplifying" an algebra expression into a "combination form" like T1 or using special letters. What follows are some students' typical responses and the analysis of possible misconceptions.

Students assign specific values to variables or use specific values to substitute for variables. Their responses showed that they clearly understood the relationship between Tachi's age and Bill's age, that is, Tachi is one year older than Bill (see part of the students' responses in Table 4.3). However, those students had difficulty operating on or with letters. As "repair theory" claims, when students had an "obstacle" which they did not know how to pass, they usually found an old strategy to tackle the new situation. In this problem, students needed to represent a number by using variable and algebra expressions. For them, the arithmetic approach was a natural one because they had worked with numeric values for long time. Thus, a "bug algorithm" was invented by students: using a specific value to substitute for the variable. Those students could not understand or accept a letter used as a "place holder."

Table 4.3
Students' error related to specific value

Student	Answer
1	B= 13, T= 14
2	T= 6, B = 5, T=B * 6=s
3	T=2, B=1
4	T=11, B=10 (table: T, B — 1: 11, 10; 2: 12, 11; 3: 13, 12)

	T	B
1	11	10
2	12	11
3	13	12

The response from student 5 (see Figure 4.1) provided some clues about why these students used specific values, although the use of T and B to represent ages was clearly emphasized in this question. Student 5 said he (she) could not work on this problem because he (she) did not know Bill's age. For him, the variable is no meaning if it had not been assigned or related to some specific numbers.

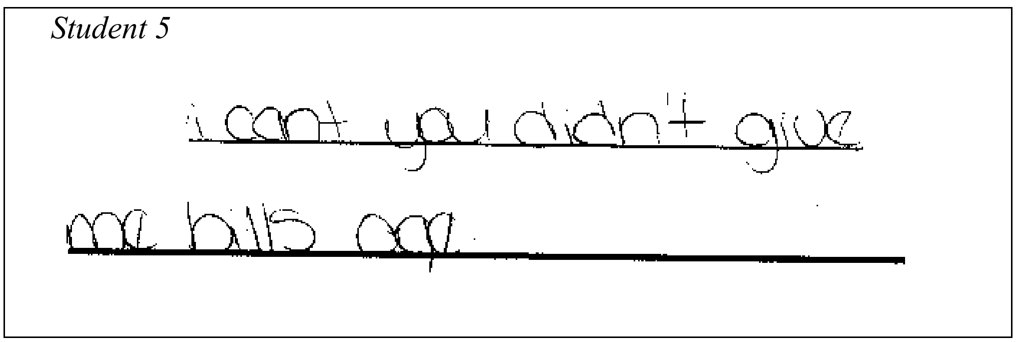
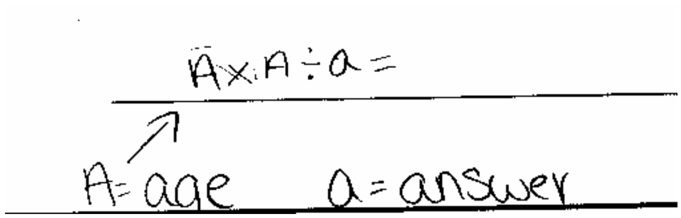


Figure 4.1. Student response related to errors using specific values.

Students are still in the transition process from arithmetic to algebra thinking. One typical difficulty is to use a letter to represent "generalized numbers" and operate on these letters. They are not sure what the result of the operation of variables should be. For example, one student used "one" rather than "1" in his equation, that is, T-B = One. This student might assume that the result of an operation on letters should also be letters. To write an equation for a real situation, students must reach the level that they are able to represent quantities by using letters and to operate on these letters. It is not easy for students who are familiar with and good at using specific numbers to make this transition.

Students misunderstand variables as labels. Some students were believed to misunderstand letters as labels. Students' reversed errors in the famous Students-and Professor Problems (Clement, Lochhead, & Monk, 1982) were interpreted as revealing students'naïve conceptions. In that problem, students used "*p"* to represent professors rather than the number of professors. However, this conclusion is mainly based on the assumption that if college students clearly understand the meaning of "*p*" and "*s,*" the number of students with reversed errors should not be so high. In this study, except for the reversed errors made by students in pretest and posttest, some students' errors in misunderstanding of variables as "labels" are more clearly and directly illustrated in Figure 4.2.

Student 1:



Student 2:

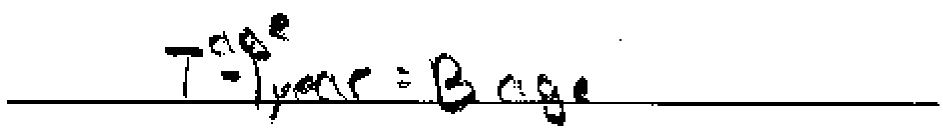


Figure 4.2. Student’s misunderstanding variables as labels.

Student 1 used “A” to represent age instead of T and B. He tried to use two As to represent both Tachi and Bill’s age. His using of “a” as a representation of “answer” more clearly showed that this student only understood a “letter” as a label. Student 2 added the units behind each letter. This representation reflects that this studen misunderstood “T” as “Tachi” and “B” as “Bill.” As a result, “T age” means “Tachi’s age” while “B age” means “Bill’s age.” Other students’ responses in Table 4.4 support this judgment:

Table 4.4
More examples of students' misunderstanding a variable as a "label"

Student	Answer
3	T is 365 days old than B
4	T is 1 year ahead of B
5	T= exactly one year older than B
6	T + B = Tachi + Bill
7	B = T + 1Y B= Bill's age, T = Tachil's age, Y= Years

Students' errors of "variables as labels" are believed to be related to students' prior experiences such as using "f" to represent "foot" and "p" to represent "pound." In our study, the reversed errors (that is, students write the equation as T+1= B or B-1=T or T=B-1 rather than the correct answer T=B+1 or T-1=B) are still the most frequent errors found in both pre and posttests. Moreover, the reversed errors increased greatly in the posttest. Rosnick (1981) conducted a study using the same problem but a different design by asking students to choose correct answers for the question “what does the letter *p* stand for?” (p. 314). Over 40% of 152 students did not pick the correct answer “the number of professors.” In this study, students’ responses to Q15 “Maria and Jinko’s donut sales,” which asked students to name a variable, also support this finding (see Table on pages 91-92). In that problem, 50% of the students did not answer correctly with “the number of donut sales” or other correct answers.

Students using other letters. Although the problem had clearly stated that “T” stands for Tachil’s age and “B” stands for Bill’s age, students used many other letters in their answers. The most often used letters are mainly x, y, A, B, or N. Table 4.5 shows some examples:

Table 4.5
Using special letters as variables

Student	Answer
1	T1 + B = N
2	X = X, you don't have any data to compare
3	T + N =1
4	2004-y + 2004-y=-1y,
5	Y=TX+13; Y=BX+ 11

It is not easy to know why students introduced new letters. Students might be more comfortable using familiar letters. They may think only certain letters could be used to represent variables. That is, these students misunderstood the variable as represented by special letters. It is similar to many students assuming only x and y can be unknowns; thus, it is hard to accept other letters as unknowns.

Students misunderstanding "equal sign" as an association. Some students' answers showed that they were unable to know "equal sign" means precisely an equivalence. They tended to understand "equal sign" as "is." Table 4.6 shows students' understanding of "equal sign" as an association.

Table 4.6
Example of students' answers in understanding "equal sign" as association

Student	Answer
1	T-B = answer
2	T-B= Age between
3	Tx1xB= 369 days older
4	T-B = 1 year older
5	T-B = how many years apart
6	T- B = Age difference

These typical answers show that students knew that they should add an "equal sign" between the two parts. Their answers also indicate that they understood the result of T-B means the age difference. Such errors might stem from teachers misusing the "equal sign" in classroom practice.

4.1.2 Maria and Jinko's donut sales (Q15)

The analysis of Q15 was used here because this problem is related to Q9 in a test of students' understanding of variable. Students were asked to write another variable by providing the example, "The number of donuts Maria sells is a variable." Students' error type and frequency are reported in Table 4.7:

Table 4.7
Students' error types in Q15

	Pretest (N = 456)		Posttest (N = 502)	
Error type	Frequency	percentage	Frequency	percentage
1. Five times as many as Maria (5xK, 25K, K= J * 5)	27	5.95%	24	4.78%
2. Jinko's donuts, donuts	18	3.94%	18	3.46%
3. Jinko sells five times as many as Maria	37	8.11%	38	7.57%

Table 4.7 (Continued)

Error type	Pretest (N = 456) Frequency	Pretest (N = 456) percentage	Posttest (N = 502) Frequency	Posttest (N = 502) percentage
4. Five times as many as Maria (5xK, 25K, K= J * 5)	27	5.95%	24	4.78%
5. Jinko's donuts, donuts	18	3.94%	18	3.46%
6. Jinko sells five times as many as Maria	37	8.11%	38	7.57%
7. 25 cents, price, or constant number	44	9.46%	73	14.54%
8. Others /don't know	15	3.29%	15	2.98%
9. No response	81	17.77%	77	15.34%
Total incorrect answers	222	40.41%	245	48.67%

Students' responses for this problem were consistent with the findings of other studies. In this study, 40.41% of students in the pretest and 48.67% in the posttest answered either incorrectly or with no response. The most frequent error was to use a constant number as a variable (9.46% in pretest and 14.54% in posttest) which is a similar pattern seen in the *Tachi and Bill Problem,* in which students tended to assign specific numbers to variables. Some students also misunderstood the variable as a concrete item, such as donuts, which appeared frequently in Q9 where the variable was understood as a person or a label.

4.1.3 Small boy raises a flag problem (Q11)

Raising a flag is a real life situation with which most students are quite familiar. This problem asks students to choose a graph representing the change of the height of raising a flag over time. There are four different graphs provided by the problem and two of them are correct. Students needed to explain why they chose a certain graph in terms of their understanding of the graph used to model the situation. According to the project scoring rubrics, only students who showed evidence of correct understanding could receive full credit. Two explanations were acceptable: (1) the height is steadily rising over time (corresponding to choice A); or (2) the flag will pause during some time (corresponding to choice C). As mentioned before, being able to use a graph to represent the relationship between two variables indicates a student's understanding of a function as an *object* rather than an input and output *process*. Some students chose a correct answer but they were unable to justify their choice.

This study addressed students' explanations of why they chose certain answers. Students' wrong explanations are classified into three categories and the possible misconceptions underlying these errors are analyzed below:

(1) Students did not find the mathematical meaning behind this life situation. They just mainly described the real life situation about raising a flag. For example, "the flag is going up"; "the flagpole goes up only", "the flag goes up straightly" or "the flag is

staying horizontally". Based on this information, the students may choose graph B or D which looks like a static flag (D) or rising flag (C). The vertical line was thought of as "flagpole" and the horizontal line as "ground." Students may understand the real life picture of the flag rising as the graph of the change of height of the flag over time. Thus, the students who chose B or D may not be aware of another variable: time. They simply understand "change" as the position of the flag over the ground without thinking of it as height (the distance above the ground) over time.

(2) Students used unrelated information. For example, this small boy is too small and has to pause for some time. Such explanations have something in common with the first one. The difference is, the key words such as "stop" or "pause" in students' explanations reflected students' consideration of time. In this case, students might choose a correct answer A or C.

(3) The students provided somewhat related but inaccurate information such as the speed of the flag. The main key words they used were "gradually" or "slowly" instead of "steadily." Such responses indicated that these students realized the height of the flag changed over time but they did not accurately state the relationship between the two variables. The expected clear and complete explanation for this problem should include key words like "steady growth" or "linear change." At this situation, students might also choose a correct answer A or C. Table 4.8 shows detailed information about students' errors:

Table 4.8
Students' types of errors in "small boy raises a flag"

Error Type	Pretest (N=456) Frequent	Pretest (N=456) Percent	Posttest (N=506) Frequent	Posttest (N=506) Percent
1. Describing a life pictures (such as: flag goes straight up)	201	34.87%	217	35.26%
2. Using irrelevant information (Such as: small boy is too small)	59	12.9%	68	13.54%
3. Describing the flag's speed (Such as gradually or slowly)	27	5.92%	29	5.78%
4. Other/I don't know	30	6.58%	29	5.78%
5. No response	68	17.11%	36	7.11%

The most frequent error made by students is Error 1: "Describing a life picture of flag." The next frequent ones are Error 2: "Using other irrelevant information" and Error 3 "Describing the flag's speed.". What follows is the detailed analysis:

Describing a life picture of flag. Many students (34.87% in the pretest and 35.26% in the posttest) chose answer B or D and explained the graph directly based on their real life experience. As a result, they only considered the variable of "height" without considering the independent variable "time". It is worth-noting that the "vertical" line is obviously consistent with the real life situation "flag pole" or "the way the flag goes." Therefore, many students just thought of the vertical line as the flag pole and the horizontal line as the ground. That is why so many students chose B.

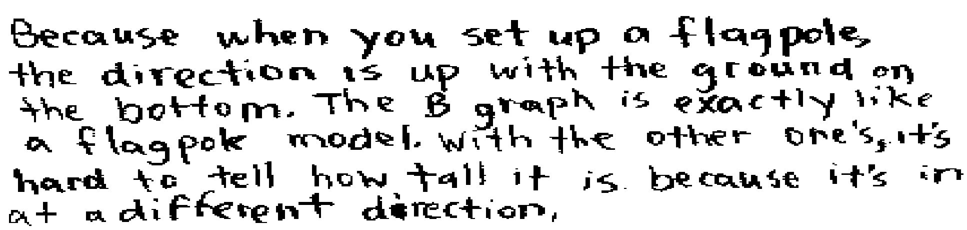
Numerous articles have documented the effects of prior experience and informal knowledge on students' learning (Davis & Vinner, 1986). Students' responses to this problem clearly indicate that students were influenced by their prior knowledge. 34.87% of students' errors were related to their understanding of real life experience in the pretest and 35.62% in the posttest. They exactly described how to raise a flag (Graph B) or what the flag looked like when it was raised (Graph D) without paying attention to the mathematical meaning. In other words, those students confused a graph the with the real life pictures.

One possible reason behind these errors is that students are frequently asked to generate table values (Swan, 1982). Another explanation was provided by Davis and Vinner (1986, p. 284):

> The error is a retrieval or choice error, quite akin to reaching for an old telephone directory instead of a new one. Thus the presentation by a student of an old (and incorrect) idea cannot be taken as evidence that the students does not know the correct idea. In many cases the student knows both, but has retrieved the old idea.

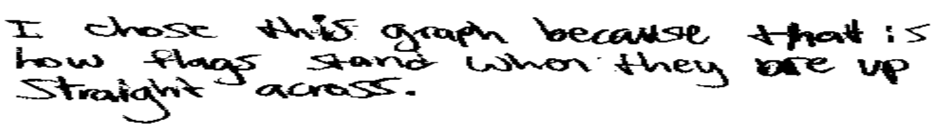
Graph is a new concept which is a contrast to common sense about the relationship between two things for middle school students. Flag-raising is a situation with which students are well acquainted. When the problem comes to them, it is easier and more natural for many students to retrieve their familiar life knowledge to explain what happened. The following answers provide evidence for the above claims. Both Student 1 and 2 in Figure 4.3 emphasized the direction of raising a flag while Student 3 referred to the position that the flag stands to explain why that graph was chosen. Because this kind of error happens often, teachers should be aware of the influence of students' prior knowledge and experiences on new learning.

Student 1:



This student confused the height of a flag in real life and the height of a flag represented by a graph. The student only considered the dependent variable without considering time: the independent variable.

Student 2:



This student chose graph D. This graph looks like a static flag when it reaches the highest point and stays horizontally. This type of response showed that students did not understand the meaning of the problem.

Student 3:

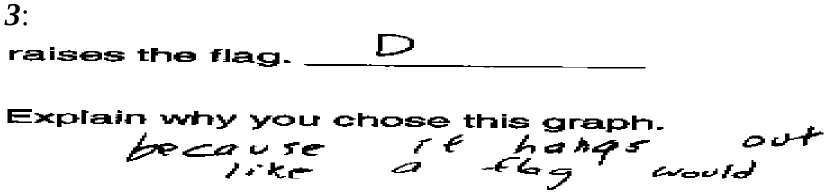
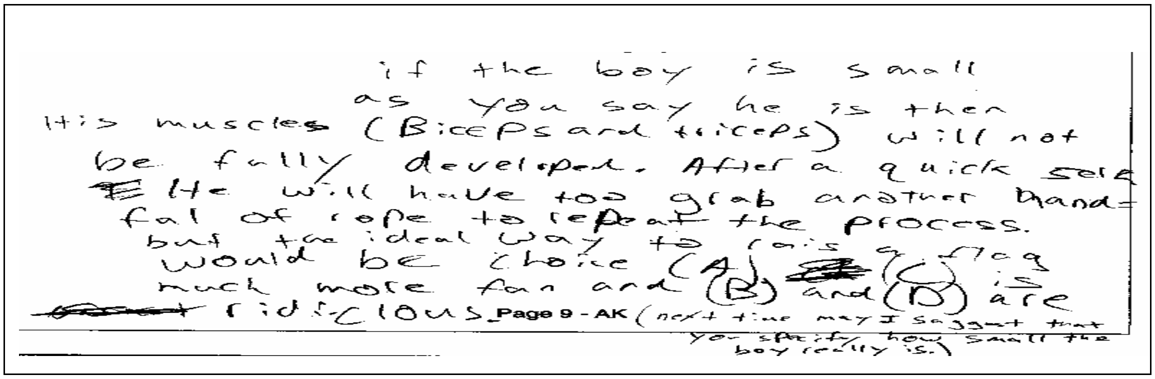


Figure 4.3. Typical responses based on students' life experiences.

Using irrelevant information. Most problems afford adequate information for students. Some information is necessary and important for solving a problem while other information is useless or can be replaced. The ability to grasp important information demonstrates students' understanding of mathematical concepts. High-achieving students or experts can quickly identify important information and mathematical meaning from a problem. In contrast, low-achieving students may have trouble distinguishing useful information from irrelevant information. Regarding this problem, many students paid attention to totally irrelevant information such as "small boy." Based on this irrelevant information, it was hard for students to give a reasonable mathematical explanation. Student 4 (see Figure 4.4) answered correctly with "A," but he did not provide enough explanation about why he chose "A." He talked about how the muscles of this small boy might affect the graph selection. This student's further suggestion, "next time may I suggest that you specify how small the boy really is," more clearly states that such information about "how small" was very important for him to choose his answer.



if the boy is small as you say he is then His muscles (Biceps and triceps) will not be fully developed. After a quick sele He will have too grab another hand-ful of rope to repeat the process. but the ideal way to rais a flag would be choice (A) (C) is much more fun and (B) and (D) are ridiculous Page 9 - AK (next time may I suggest that you specify how small the boy really is.)

Figure 4.4. Students' answer based on irrelevant information.

Describing the flag's speed . According to the project scoring rubric, it is evaluated as an incomplete explanation if students were unable to provide evidence of understanding: (1) the height changed over time, and (2) the change is steady. Students should recognize that graph A reflects a linear relationship between height and time, which means the speed of raising this flag is constant although the height of the flag is changing over time. Students need to show such understanding of graph A by using key words like "steadily" rather than "gradually" or "slowly." For C, students need to mention that the small boy stopped pulling the flag for a while during which the height of flag did not increase, and then the small boy raised it steadily again. Only a few students (10.01%) used "steadily" to explain why they chose graph A or C.

One student chose C and gave the following explanation (see Figure 4.5):

Student 5

The little boy would have to stop accdtually to raise the flag. The bottom part of the graph represents the time. The side access of the graph shows how high it was going. the graph would stop raising every time the boy had to stop.

Figure 4.5. Students' interpretation of the meaning of graph.

This student correctly understands the horizontal line on the graph as "time." He also clearly shows his understanding of what the vertical and horizontal lines mean. This explanation demonstrated students' awareness of mathematical meanings and the ideas behind life pictures. This student was able to use an accurate mathematics language-graph to describe and represent a life situation. Another student used more formal mathematics language to explain why he selected graph A (see Figure 4.6):

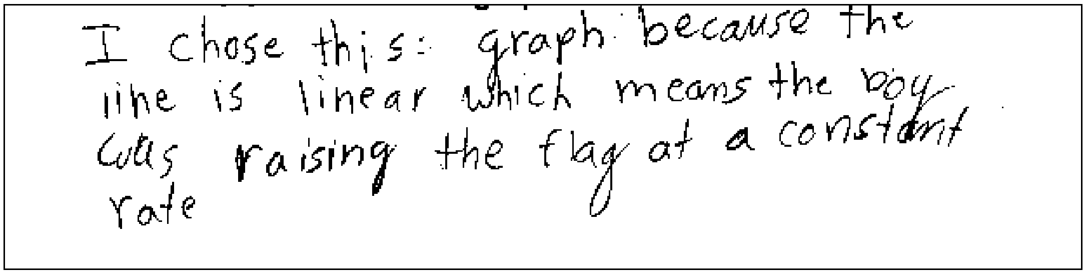


Figure 4.6. Using formal mathematics language to explain graph.

This student realized and understood that graph A represented a linear relationship between height and time. He also gave correct explanations of what “linear” means regarding this problem, that is, “the boy was raising the flag at a constant rate.”

4.1.3 Value of car not linear (Q13)

This problem is to assess students’ understandings of a linear relationship. Most students are very familiar with the real life situation between a car’s value and its age. Similar to the last problem, it is valuable to see whether students can find the mathematical meaning behind the real life situation, and whether they can retrieve correct mathematics knowledge to explain this problem. If the students correctly retrieve correct mathematics knowledge, it is important to know whether they have a sound understanding of such knowledge.

The ability to use property of function to solve problems is an indication of understanding function as an *object*. On the other hand, finding a pattern in terms of table values only requires *process*-oriented thinking. According to the project assessment experts, students need to show the understanding of linearity as a constant change in order to be given a full score. If students only saw a regular pattern without clearly stating whether there was a constant difference and what the constant difference meant, these responses were considered to be incomplete.

Students' wrong or incomplete explanations of this problem were classified into the following categories and the frequencies of errors were reported in Table 4.9:

(1) The students only described what the real situation looks like without considering the mathematics meaning behind such a real situation. For example, the car will become cheaper if it was used too long.

(2) Misunderstanding "linearity" as a constant ratio rather than "a constant difference" over equal intervals. Many students found the car value decreased by ½ comparing with that of the last year. They misunderstood the "1/2" as a constant change. As a result, a linear relationship was recognized by those students based on the pattern they generalized from the values of the car.

(3) Misunderstanding "linearity" as a certain direction.

(4) Students knew the definition of linearity but failed to answer this problem.

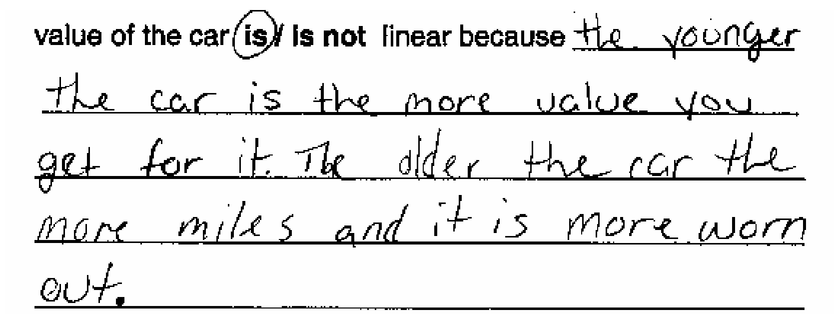
Table 4.9
S*tudents' errors in understanding linearity relationship* (car value)

Error Type	Pretest (N=456)		Posttest (N=506)	
	Frequency	Percentage	Frequency	Percentage
1. Describing a real life	68	14.9%	66	13.04%
2. Considering linearity as ratio	57	12.5%	101	19.96%
3. Considering linearity as direction	28	6.1%	21	4.15%
4. Write a definition without explanation	12	2.6%	13	2.57%
5. I don't know	53	11.6%	41	8.1%
6. No response	179	39.25%	95	18.77%

In Table 4.9, except for Error 5 (no response) and Error 6 (I do not know), the most frequent error is "Describing a real life," "Considering linearity as ratio," and "Considering linearity as direction" respectively. It is a little surprising the percentage of students who made Error 2 "considering linearity as ratio" increased from 12.55% in the pretest to 20.11% in the posttest. This increase demonstrated that students' ability to find the pattern improved but they still failed to understand the meaning of "linearity." What follows are detailed information and analysis of these errors.

Describing a real life. For the second type of errors, students were distracted by the unrelated information such as how expensive the car was. Many students provided explanations based on their real life knowledge about cars without paying attention to the mathematics meaning. The typical students' response is included in Figure 4.7:

Student 1:

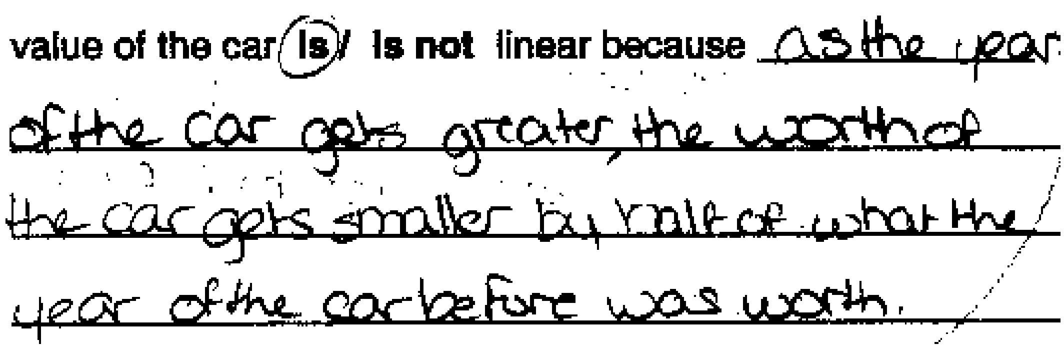

value of the car is / is not linear because the younger the car is the more value you get for it. The older the car the more miles and it is more worn out.

Comment: This student only described a real life picture for what happened to a new car over time. He did not grasp any mathematical meaning behind this real life situation.

Figure 4.7. Students' response to Q15 using life experience only.

Considering linearity as ratio. Mathematics meaning of linearity means constant difference over equal intervals of an independent variable. The graph representation is a line with a constant slope and its symbolic form is a linear equation. Many students misunderstand the constant ratio as the meaning of linearity. Some considered the change of independent variables without considering whether such change is over same equal interval (see the Figure 4.8 and Figure 4.9). For example, one student found that the car values decreased by half compared to the car's value of the last year. Such explanations demonstrated that students had some understanding of linearity by relating linearity to "constant." However, they did not really know what "constant" means.

Student 1:

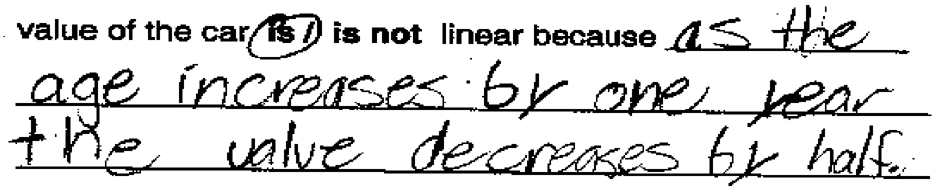


value of the car Is / Is not linear because as the year of the car gets greater, the worth of the car gets smaller by half of what the year of the car before was worth.

Comment: This student showed the linear relationship means constant change. Another variable, age of car, was also mentioned by the student. However, this student is wrong at calculating the amount of change.

Figure 4.8. Understanding linearity as ratio without considering another variable.

Student 2:


value of the car (is / is not) linear because as the age increases by one year the value decreases by half.

Comment: This student had the same errors as the above student. However, this student realized that it was necessary to consider the change over another variable---time. “Increase by one year” demonstrated “equal time”.

Figure 4.9. Misunderstanding linearity as ratio with considering another variable.

Considering linearity as inverse relationship. Some students pointed out in their responses that as the age of the car increased, the values of the car decreased, so the relationship between them was linear (or nonlinear). Those students misunderstood negative (or positive) correlation as a linear or nonlinear relationship.

Write a definition without explanation. Students could clearly state what the linearity meant but they failed to use this to explain why the relationship was non-linear.

4.1.5 “19 = 3 + 4y” (Q16)

This problem assesses students’ ability to solve a linear equation. There are several potential difficulties for students. First, the structure of this equation is different from the ones that students are familiar with, that is, the left side is the number while the right side contains an unknown number. Students should be able to understand that the order in the equation does not matter because the equation has a symmetry property. Second, this problem uses "y" rather than "x" to represent an unknown variable. Some students assume that only "x" could be used to represent an unknown quantity. Such misunderstanding might cause some errors in students' solutions. Finally, this problem is related to the understanding of the algebra expression. Some students may not understand what the "4y" represents, or not understand the structure of 3 + 4y. For example, some students in this study misunderstood 3 + 4y as 3 + 4 + y, (3 + 4) y, 3 x 4 + y, etc. Students' errors in their solution were classified into six types in Table 4.10.

Table 4.10
S*tudents' errors in Q16: 19=3+4y*

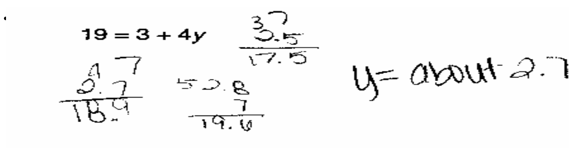
	Pretest (N=456)		Posttest (N=506)	
Types of errors	Frequency	Percent	Frequency	Percent
1. Add 3 and 4 first (19=7y, y=2.7)	5	1.1%	9	1.77%
2.Running equation: (4 y = 4 x 4 = 16+3 = 19)	34	7.45%	96	18.97%
3. Divided by 4 (4.75 = 3 + y)	1	0.2%	1	0.19%
4. y=12 (3+4=7, 19-7=12)	31	6.8%	78	15.41%
5. y=2 (4y = 4^y)	2	0.6%	3	0.59%
6. y=16 (confuse 4y with y)	9	1.96%	8	1.58%
7. y= x 4	4	0.66%	3	0.59%

Table 4.10 (Continued)

Types of errors	Pretest (N=456)		Posttest (N=506)	
	Frequency	Percent	Frequency	Percent
8. others/ I don't know	37	8.1%	31	6.1%
9. No response	272	59.6%	143	28.26%

Error 1(add 3 and 4 first) and Error 4 (y = 12). These two types of errors were similar and most often found because in both situations, students first added "3" and "4." The difference is that the structure of "3+4y" was misunderstood as "(3+4) y" in Error 1, while as "3+4+y" in Error 4. These students followed algorithms about solving linear equations very well in the remaining steps (see Figure 4.10 for typical students' response). That is, except for the first step of misunderstanding "3+4y" as "(3+4)y" or "3+4+y", the remaining steps of solving the equation were completely correct.

Student1: Example for Error 1.



Student 2: Example for Error 4.

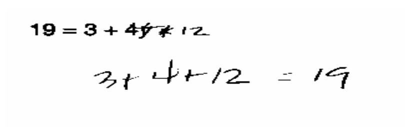


Figure 4.10. Examples of students' answers for error 1 and error 4.

It is easier to underestimate learning complexity by attributing students' errors to wrongly following an algorithm (Schoefeld, 1986). For this problem, the standard algorithm is to add and/or multiply the same number to both sides to simplify this equation. For example, students can add "-3" or multiply "1/4" to both sides first. However, in the above examples, both students added 3 and 4 first. Student 1 used the method of "trial and error" to find a solution while Student 2 subtracted 7 from 19. Though the two students were aware of the algorithm of solving this equation, they had trouble understanding the algebra expression of 3+4y. Therefore, they arrived at incorrect answers 2.7 and 12 respectively. Figure 4.11 clearly shows a typical error response.

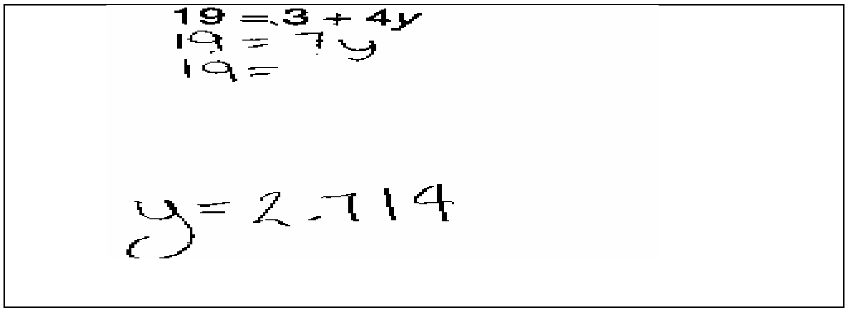


Figure 4.11. Example of adding 3 and 4 first.

When students' responses contained more than one error, their answers would be coded in terms of the major error type. Figure 4.12 shows a typical example. In this situation, the student's response is coded as Error 4. The main mistake here was that this student misunderstood the structure of 3+4y as 3+4+y as reflected in above figures. In fact, this student quite well understood the process of solving a simple linear equation. For example, he correctly solved the equation: 12=4y. This student was not aware that "3+4y" was the same as "$3+4\times y$".He might think of the omitted sign "x" as "+," which caused his error.

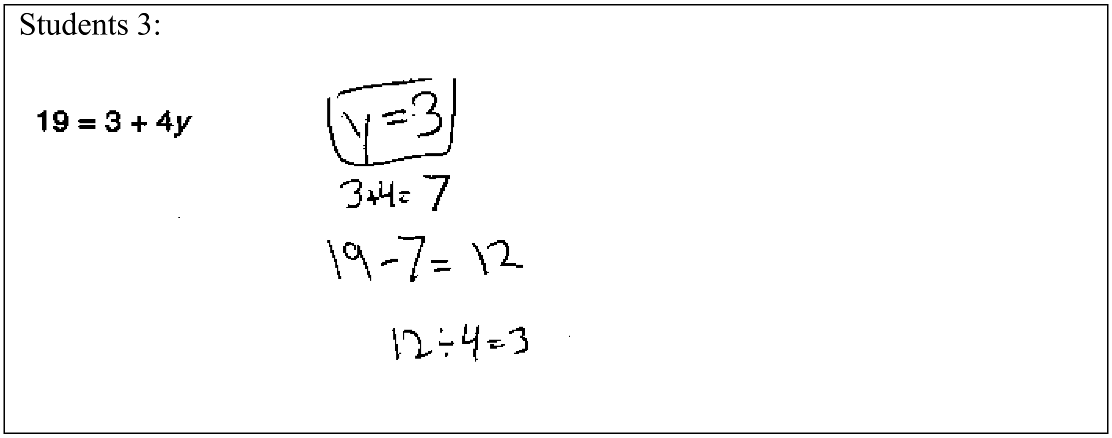


Figure 4.12. Student's response contains more than one error.

Error 2: Running equation. Regarding this type of error, students got the correct answer but explained why y = 4 by using running equations. In the pretest, 34 students used this approach while 96 students did so in the posttest. The increased number of students who got the correct answer, 4, demonstrated that students made progress in solving equations (at least their ability to reasonably guess). However, the error of running equations might reflect other problems about students' understanding of equations. Such errors are ignored or endured by many teachers. Moreover, some teachers even write such forms themselves (Ding, Li, Capraro, & Kulm, 2006). Convenience was thought as an explanation why both teachers and students tend to make such mistakes. On the other hand, this inaccurate written form is one of the important reasons why students tend to misunderstand "equal sign" as "to do something" or "association." This form is strongly against the fundamental properties of equation such as equivalence, symmetry, or transitivity (Kieran, 1992). As we mentioned before, there were more students in the posttest using a running equation than that of the pretest. Except for the reason of students' improved skills in solving equations (e.g., more students got the correct answer, 4), it might also be due to students' being exposed to instructions with inaccurate symbolic representations (e.g., more explanations were problematic). Because of this, some researchers believe misconceptions might be reinforced by incorrect instructions. Teachers need to set good examples of how to use mathematics symbols as accurately as possible.

Error 3: Divided part of the terms in the equation by 4. The third type of error is rare in both pretest and posttest. The main reason is because few students used an algebra approach. Students should perform the same operations on each side, but the principles behind the operations may not be easily understood by students. They need to understand that the same operation on both sides of an equation maintains the "equality" of the equation.

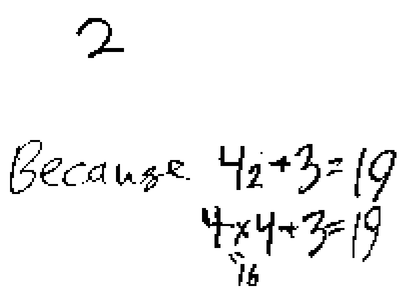
The main error in this type was that some students did the division on both sides but not on each item of both sides. For example, some students divided 19 by 4 on the left side and 4y by 4 on the right side. They forgot the item "3"; thus, they got the equation: $19/4 = 3 + y$. Actually, the standard algorithm is as simple and direct as "the algorithm of whole number addition and subtraction." The rule is to divide every item by the same number. Resnick (1982) maintained that students actually have excellent abilities to follow procedures. However, why do students "forget" to divide every term in the equation? Why do they use the wrong procedures? A reasonable explanation here is that students might have a "bug algorithm" when they try to solve this equation by using "divide by 4."

The "bug algorithm" in this problem might be caused by students' confusion of "dividing by the same number on both sides" with the similar algorithm of "subtracting the same number from both sides." Students usually learn how to solve an equation first by subtracting the same number from both sides. In this problem, they need to subtract 3 from "19" in the left side and subtract 3 from "3 + 4y" on the right side. If students are unable to see structure "3+4y" as an object, they may misunderstand the actual subtraction process as 19-3 = (3-3) +4y, rather than 19 -3 = (3 + 4y) -3. Therefore, they might invent an algorithm: "to operate on some items in each side rather than on algebra expressions." Therefore, when they learn how to use division to solve such an equation, they may tend to divide only one term in the right side rather than each term.

In a few words, there are two mathematics principles behind the algorithms of solving the linear equation. Students will have trouble if they only memorize the rules without understanding the underlying mathematics principles.

Error 5, Error 6, and Error 7. Error 5 is misunderstanding 4y as 4^y. This type of error can be traced to the understanding of "3+4y." Students misunderstood 4y as 4^y partly because they knew an expression such as 4^y contains no operation sign between the number and the letter. In contrast, the common expression "$4\times y$" contains a "x" sign. Therefore, some students might be uncomfortable with the expression "4y" when the “x” is omitted. Such inferences were confirmed by Error 7: y = x 4. Figure 4.13 provide students’ errors of type 5 and type 7:

Student : Example for Error 5.



Student   Example for Error 7.

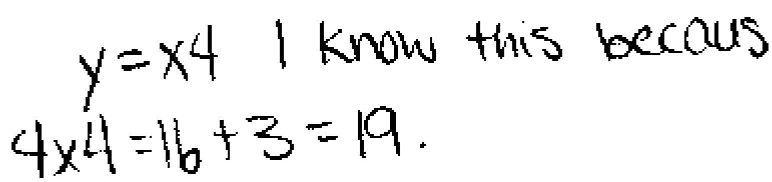


Figure 4.13. Example for students’ responses for error 5 and 7.

The above errors showed that students’ difficulties were not from the procedures of solving equations but from the understanding of variable and algebra expression. Some students’ errors showed more clearly that they had difficulties in understanding “4y” as

“$4\times y$”. This type of error also showed that students experienced difficulty understanding algebra expressions as a representation of a number even if they could use a letter to represent the number. Such difficulties were also reflected in *Error 6: confused 4y with y* (see Figure 4.14).

Student example

Figure 4.14. Example of student’s response in error 6.

Another interesting finding is that many students did not respond to this problem. In the pretest, 272 students (59.65%) did not answer this question while 143 (28.49%) did not answer in the posttest. Students might not be familiar with the equation form when the left side is a specific number and the right side is an algebraic expression. If students knew the transitive property of equations, it would be easy for them to change the unfamiliar form into the familiar one: 3+4y=19, and some students did that. The use of the running equation to justify their answers actually partly showed these students’ understanding of the transitivity of equation. This is because these students normally wrote the equation this way: 4 x 4 =16 + 3 =19 without rewriting it as 19 = 3 + 4 x 4.

4.1.6 Apple Trees/Pine Trees and Stones/Bricks (Q 8)

This problem contains four sub-questions. These questions are related to students' real life. Students were asked to understand the patterns demonstrated by the figures in this problem. The first sub-problem is very easy for most students because they only need to count the number of bricks. The second problem is a little harder. Students need to fill in the missing numbers based on available information. To find and write the pattern is thought to be an important step for the transition from arithmetic to algebra thinking (NCTM, 2000) because "the crux of algebraic thinking is the recognition of patterns" (Moses, 1999, p.98). In this study, we only analyze the third and fourth sub-problems: Q8(C) and Q8(D) in detail.

Q8 (C) is harder than the first two questions for many students. Students need the following abilities: (1) able to interpret what "n" represents (2) able to understand that $n \bullet n$ and $8 \bullet n$ could stand for the number of pines or apple trees (bricks and stones) (3) able to retrieve a proper algorithm to solve this problem. Another possible approach is that students can list the table value (arithmetic approach) to solve this problem. Regarding the solving strategies, there are mainly two types: algebra and arithmetic. If students used a formula to get the answer 8, the strategy is assumed to be algebraic. If students only extended the table to get the answer, the method is claimed as an arithmetic method even if they substitute the value for the n in the formula. Students' errors in this question were classified into five categories as shown in Table 4.11:

Table 4.11
Students' errors in solving apple tree (stone) problems

Types of errors	Pretest (N=456)		Posttest (N=506)	
	Frequency	Percentage	Frequency	Percentage
1.Guessing without explanations	64	14.03%	62	12.25%
2. Misunderstanding "n"	36	7.89%	40	7.9%
3. Adding all numbers	25	5.48%	26	5.13%
4. Describing unrelated patterns	9	1.97%	7	1.38%
5. Don't know	61	13.37%	66	13.4%
6. No response	117	15.65%	87	17.19%

These errors are explained in detail in the following paragraph in terms of their solution strategies or possible error sources.

Guessing without any explanation. Some students might solve this problem intuitively, but did not provide a proper explanation;

Misinterpreting "n"* as *the number of pine or apple trees / the number of bricks or stones (see Figure 4.15). Since mathematics is a language, the correct understanding of the meaning of symbols is a prerequisite to learning mathematics, especially algebra. In this problem, "n" was clearly stated as "number of rows of stones" and students were asked to find the value for "n." Students' misunderstanding of "n" includes two situations: (a) misunderstanding n as the number of apple trees/stones, and (b) relating their answers to "equality." Some students responded to this problem by looking at the incomplete table in the problem and found that there were two equal numbers in the table. When n=4, the number of stones is 16; when n=2, the number of bricks is also 16, as a result, the students concluded that n was equal to 2.

Students may just add all numbers to get the answer (see Figure 4.16). Some students may just add all numerical values when they did not figure a way to solve the problem.

Some students just describe a pattern (see Figure 4.17). Such as, "n" increases by 1. The possible reason may be due to their prior experience. Students are asked to write a pattern in terms of table values. Such exercises appear frequently in many textbooks.

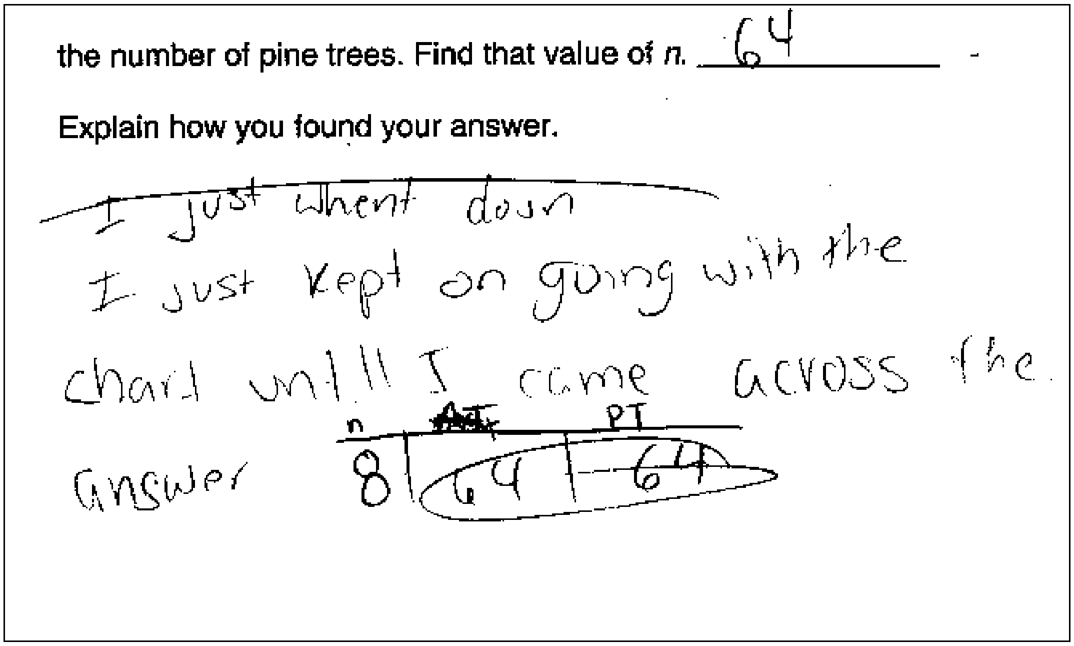


Figure 4.15. Example for misunderstanding “n” as the number of apple trees/stones.

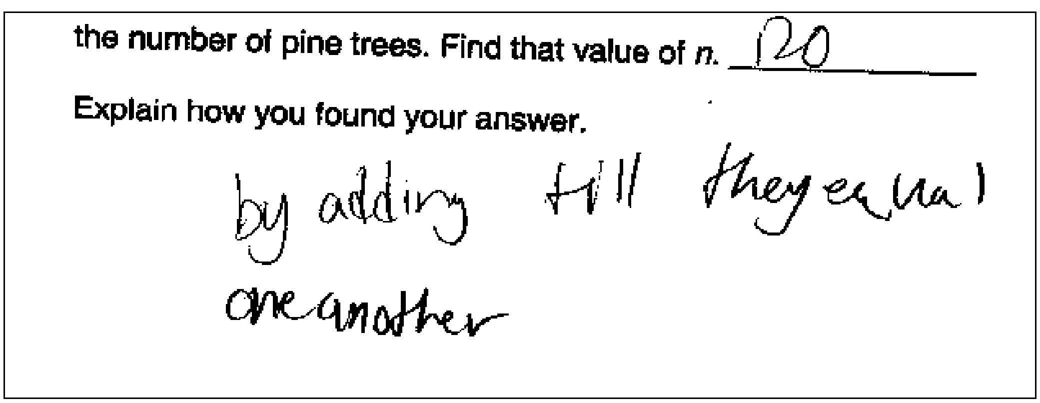
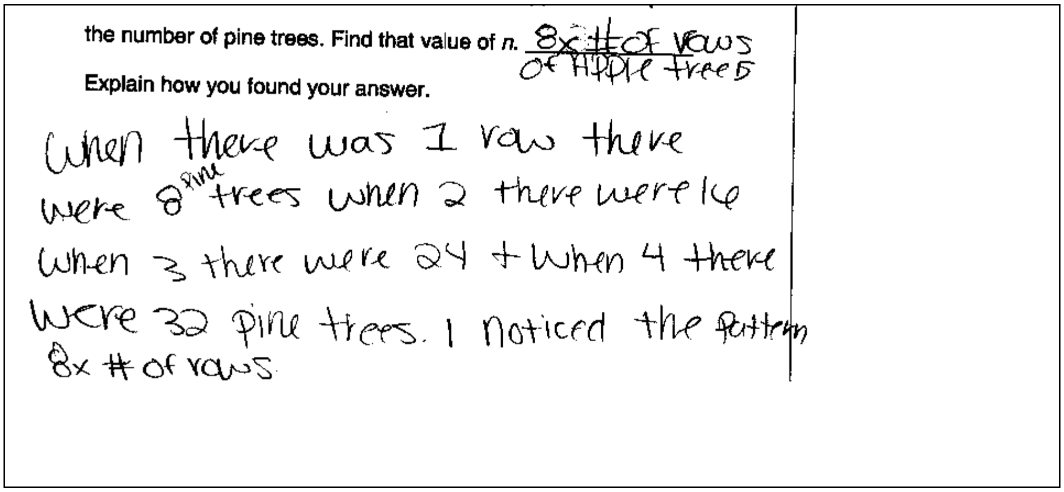


Figure 4.16. Example for adding all numbers.


the number of pine trees. Find that value of *n*. 8x # of raws of Apple trees

Explain how you found your answer.

When there was 1 raw there were 8 pine trees when 2 there were 16 when 3 there were 24 + when 4 there were 32 pine trees. I noticed the pattern 8x # of raws.

Figure 4.17. Students used unrelated patterns.

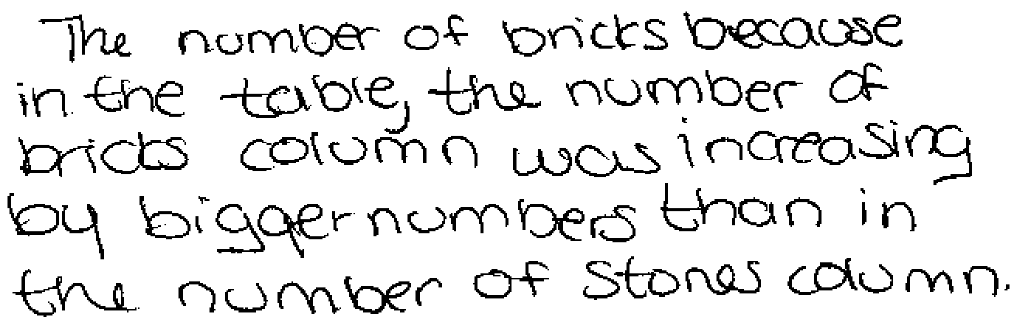
Q8 (D) is the toughest problem in the entire test. Students should be able to understand the meaning of "increase quickly." They need to distinguish between "increase quicker" and "larger amount." To increase quicker means the amount differences are larger in the same interval. The "larger amount" is the result of increasing. The main errors follow: (1) Using the existing table to give an incomplete explanation. Students make a judgment based only on the current table without extending the table to include more cases; (2) Border is large, thus pine or stone in the border will increase faster. Some students thought the border was larger because the border surrounds the inside parts; (3) Brick or pine is small so it takes more bricks (pines) for every stone (apple trees); (4) Using wrong patterns, such as: for every one pine tree, there are eight apple trees (for every one stone, there are eight bricks); and (5) Misunderstanding "quick" and "more." Table 4.12 shows students' errors in solving this problem

Table 4.12
Students' errors in solving 8 (D)

Types of errors	Pretest (N=456)		Posttest (N=506)	
	Frequency	Percentage	Frequency	Percentage
1. Incomplete table	24	5.26%	44	8.69%
2. Border explanation	51	11.18%	39	7.70%
3. Brick (pine) is small	20	4.38%	16	3.16%
4. Wrong pattern	62	13.59%	52	10.27%
5. Misunderstand "quick" and "more"	81	17.76%	108	21.34%
6. Others /don't know	52	11.40%	58	11.46%
7. No response	99	21.70%	60	11.86%

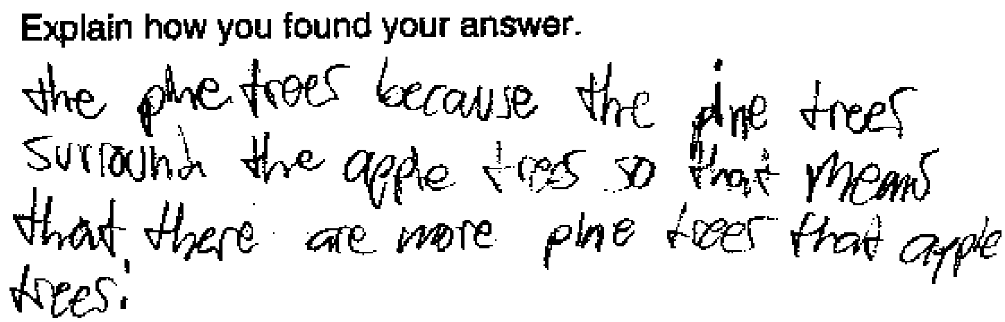
Figure 4.18, Figure 4.19, and Figure 4.20 show students' typical responses corresponding to the above errors:

Example for Error1


The number of bricks because in the table, the number of bricks column was increasing by bigger numbers than in the number of stones column.

Comment: This student looks at an incomplete table to obtain the wrong answer

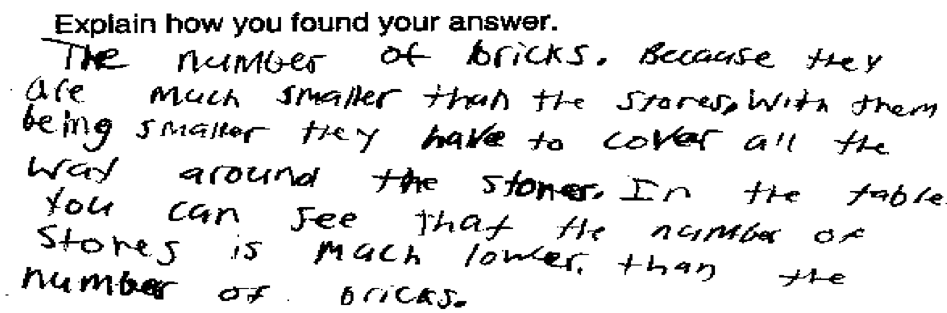
Example for Error 2


Explain how you found your answer.
the pine trees because the pine trees surround the apple trees so that means that there are more pine trees that apple trees.

Comment: This student wrongly assumes the border will need more bricks (pines) to cover up. The response also showed that the student misunderstood "quick" and "more," too. "The border is larger" is a wrong common sense assumption.

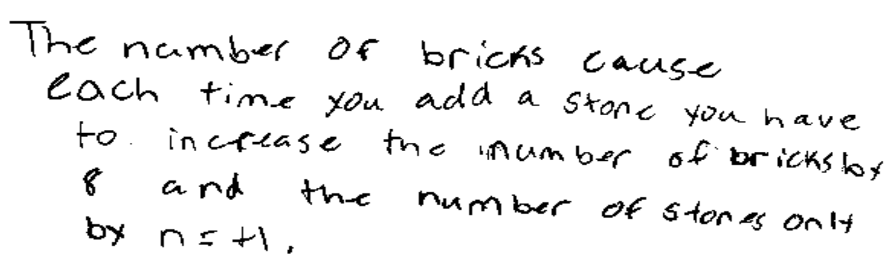
Figure 4.18. Typical students' response for error 1 and Error 2.

Example for Error 3


Explain how you found your answer.
The number of bricks. Because they are much smaller than the stones, With them being smaller they have to cover all the way around the stones. In the table you can see that the number of stones is much lower than the number of bricks.

Comment: Students wrongly used unrelated information that the brick is small so more bricks are needed to take space. Again, many students who explained this way also misunderstood “quick” and “more.”

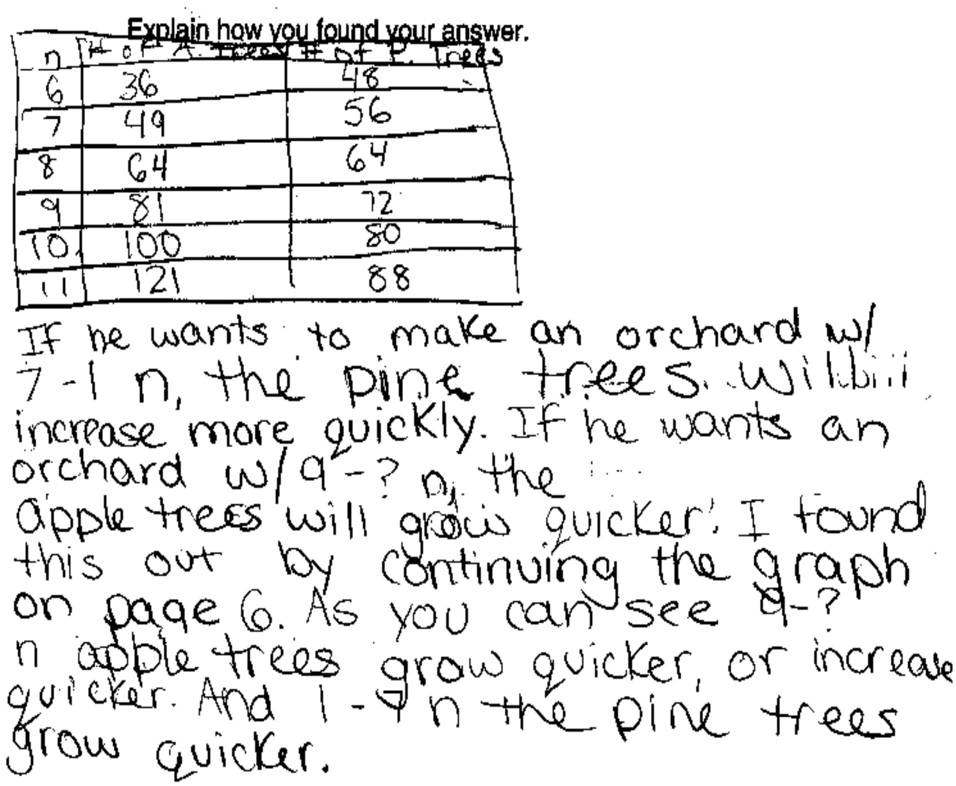
Example for Error 4


The number of bricks cause each time you add a stone you have to increase the number of bricks by 8 and the number of stones only by n=+1.

Comment: Students invented the wrong patterns for the relationship between bricks and stones. Such a response also shows that this student misunderstood “quick” and “more.”

Figure 4.19. Typical students’ responses for error 3 and error 4.

Example for Error 5


Explain how you found your answer.

n	# of A. Trees	# of P. Trees
6	36	48
7	49	56
8	64	64
9	81	72
10	100	80
11	121	88

If he wants to make an orchard w/ 7-1 n, the pine trees will increase more quickly. If he wants an orchard w/ 9-? n, the apple trees will grow quicker. I found this out by continuing the graph on page 6. As you can see 9-? n apple trees grow quicker, or increase quicker. And 1-9 n the pine trees grow quicker.

Comment: This example clearly illustrates the student's misunderstanding of " more" and "quick." This student found that the number of pine trees and apple tress were equal when n=8. This student pointed out that in 1-7 the pine trees grew quicker and apple trees would increase when n was greater than 9.

Figure 4.20. Typical student's response for error 5.

4.2 The Results of Identified Misconceptions

As the previous task analysis shows, most items can be solved in only one step. Students normally simply apply their understanding of the corresponding conceptual knowledge to the problems. That is, if students had a sound understanding of variable, equation, and function, they should perform well on these items.

In this section, I address the misconceptions underlying the errors identified in the prior section. The misconceptions related to variable, equation, and function will be reported in order. In each subsection, I begin with analyzing the most common errors in both pretest and posttest, across different problems. After that, I will identify possible misconceptions behind these errors.

4.2.1 What are the misconceptions of "variable"?

Students' errors related to the misconceptions of "variable" across different problems are reported in Table 4.13

Table 4.13
Frequency of students' errors across Q9, Q15 and Q16

Types of Errors	Q 9 Pre	Q 9 Post	Q 15 Pre	Q 15 Post	Q 16 Pre	Q 16 Post
1. Refer to specific	38	27	44	73		
2. Wrong combination	19	10			70	83
3. Label	17	19	18	18		
4. Reversed equations	54	220				
5 No response	76	42	81	77	276	141

Note. N_{pre}=456; N_{pst}=506.

The data in the above table illustrate two misconceptions of variable: (1) the misconception of a variable as a specific value, and (2) the misconception of a variable as a label. What follows is a detailed analysis of misconceptions based on Table 4.13:

The misconception of a variable as a specific value. Students tend to relate a variable to a specific value (Error 1 in Table 4.13). Q9 asked students to write an equation representing the relationship between Tachi and Bill's ages, and many students (38 in pretest and 27 in posttest) referred to specific values. The same type of error also happened in Q15, where students were asked to name a variable under the condition that an example of a variable was provided. The number of this type of error increased in students' posttest (14.42%, 73 students) compared with the pretest (9.64%, 44 students).

Students also tend to combine an algebra expression such as T+1 into T1 or 3+4y into 7y (Error 2 in above table). Such errors are believed to have the same conceptual basis as Error 1, that is, students are unable to operate on or with variables. According to Kieran (1992), students' preference to interpret a variable as a specific value and do arithmetic operations on them caused such types of errors. This type of errors occurred in both Q9 and Q16, with an especially high frequency in Q16.

According to our standards of misconceptions, the conceptual–based errors should occur in different contexts (different problems) and at different times (both pre and posttest). Therefore, "understanding a variable as a specific unknown value," as reflected by students' errors in both Q 9 and Q15 as well as Q16 in pre and posttest, is a misconception held by many students.

The misconception of a variable as a label. As shown in the above table, Error 3, "understanding a variable as a label," was also very common. In Q9, 17 students in the pretest and 19 in the posttest were found to misuse "variable" as label. This type of error was also found in Q15, where 18 students in pretest and 18 in posttest clearly answered another variable is "donuts" or "Jack's donuts" (see Table 4.13). When students view a variable as a label, they might be uncomfortable operating on it as a number. This inference is supported by students' answers in multiple choice problem Q2 (for Q2, see Appendix 4), where students were required to choose an equation to represent the relationship between Mary's and Julie's cards. The correct answer is C (see Table 4.14). The big difference among the wrong answers is that there is no operational sign in expression A (3x=36), while there is a "+" sign in both B(x+3=36), and D (3x+36=x).

Though the problem itself clearly states the key word "in all" which corresponds to the additional signs in both B and D, most students still selected A:

Table 4.14
Students in Q2: Equation and variable problems

	Q 2	
Answer	Pretest (N=456)	Posttest (N=506)
A. 3x=36	154	150
B. x+3=36	49	32
C. x+3x=36	211	284
D. 3x+36=x	52	40

One reasonable interpretation is that, these students might misunderstand variable as "label," thus they were uncomfortable seeing a variable operating with a number. Since there is no operational sign in expression A, most students in both pre (154 students) and posttest (150 students) selected this one.

In a word, two major misconceptions with variables were identified, one is misunderstanding a variable as a specific value and the other is misunderstanding a variable as a label.

4.2.2 How did students misunderstand "equation"?

As mentioned earlier, students' misconceptions of the equal sign and equity identified by most studies is "misunderstanding equal sign as to do something." The problem used most often is one in the form, $8 + 4 = \square +5$ (Carpenter, et al., 2003; Falkner, et al., 1999). In prior studies, most students filled the box with "12." According to the assumption of this misconception, the misconception of "equal sign" should be found in different contexts across different people. Was such a misconception also found in algebra? According to the assessment map (see Appendix 2), items Q1, Q4, Q9, and Q16 were used to analyze students' conceptions of equal sign and equality. The related errors are reported in Table 4.15.

Table 4.15
Frequency of students' errors related to "equal sign" across Q 9 and Q16

	Q 9 (Tachi and Bill)		Q16(19=3+4y)	
Error Type	Pretest	Posttest	Pretest	Posttest
Using equal sign as association	21	21		
Running equation			25	82

In responding to Q9, students wrote the equation like "T-B = difference" or "T = one year older than B." Students may just use the "=" as everyday language "is." Such a use of "=" was also found in the popular textbook: *Mathematics Application and*

Connections (Collins, Dritsas, Frey-Mason, & Howard, 1998), whereby students were required to finish the following item:

> Melanie made a long-distance call to her grandfather. The first 3 minutes cost $2, and each minute after that cost $0.5. How many minutes did they talk on the phone if the total cost of the call was $10? (p.473)

In the same page, this textbook also provides the following explanation about how to translate the word problem into symbolic form:

The cost of Phone call	is	$2 for the first 3 minutes	plus	$0.5 per minute	times	the number of minute
↓	↓	↓	↓	↓	↓	↓
10	=	2	+	0.5	x	m

This use of the "equal sign" as shown in the textbook may cause a misunderstanding of the equal sign as an association. In the posttest, more students used running equations. Many teachers tolerate them and may even use running equations in their own teaching (Ding, Li, Capraro, & Kulm, 2006). It is unknown whether this use of the equal sign is caused by the misunderstanding of "equal sign" or just writing convenience. According to Carpenter et al. (2003), equations like this, $12 = \square + 4$, are more difficult than $8 + 4 = \square + 5$. Students' performance on this kind of equation is reported below (see Table 4.16)

Table 4.16

Frequency of students' performance in equation problems

Q1 (43=□-28)				Q10 (a=b-2)		Q16(19=3+4y)	
Answers		Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
A	15	184	139	186	278	56	163
B	25	15	12				
C	61	18	9				
D	71	242	341				

Note. Q10, and Q16 only provide the frequency of students' correct performance.

The items in Table 4.16 have common characteristics. The first one is most simple and also most students answered it correctly. The second one is harder in that there are two variables. And the third one is most difficult because students need to understand the structure of algebra expression well. The earlier error analysis of the problems Q1 and Q16 showed that students have more difficulty understanding variables and algebra expression rather than the "equal sign."

4.2.3 How did students misunderstand function?

At the middle school level, students should understand how the change in one variable causes change in another. Moreover, students need to know linear or nonlinear functions, be able to use a graph to represent a function, and be able to compare different functions, for example, the speed of change. Table 4.17 reports the errors made by students in different problems.

Table 4.17
Students' responses for the multiple choice of function

Item	Q5		Q6		Q7	
	Pre	post	Pre	Post	pre	post
Correct	314	404	153	195	286	387

Note. $N_{pre} = 456$, $N_{post} = 506$.

Students did very well in both Q5 and Q7. For Q5, students need to find the pattern between two variables. For Q7, they must note how the value of y changes as that of x changes. For the Q6, students must answer whether there is a relationship and what the relationship is. Although in the multiple choice problems, students' are very well aware that function is about the relationship, they tended to consider function as the change of one variable which is the first misconception I identified from students' errors. Table 4.18 shows students' error responses about function in different problems:

Table 4.18
Students' error responses about function in Q11, and Q13

	Q11 (raising flag)		Q13 (car value)	
	Pretest	Posttest	Pretest	Posttest
Understanding function without considering another variable	159 (34.87%)	178 (35.26%)	57 (12.55%)	102 (20.11%)

For Q11, small boy raising a flag, many students chose B or C (159 students in pretest and 178 students in posttest (see Table 4.18). The typical reason is that, the flag leaves the ground. The choice of B or C for most students is due to the direction of the flag's raising. However, these students did not consider another variable: time. Only a few students used words such as "gradually," "slowly," "pause," "steadily," or "break." The use of such words is thought to be a sign of understanding the relationship between height and ground. Such a misunderstanding of function is more obviously demonstrated by Q13. Many students (57 students in the pretest and 102 students in the posttest, see Table 4.18) thought it was a linear relationship between the car's value and the age of the car. In fact, these students only compared the value of car difference without considering the time factor. That is, they did not mention that linear means a constant difference over equal time (equal interval).

Another misconception demonstrated by students in Q8 (D) is "the more, the quicker." Many students first compared the number of bricks (or pine trees) to the numbers of stones (apple trees), then they claimed that the number of stones (apple trees) will increase quicker than that of the bricks (pine trees). The typical reason held by these students is that there will be more bricks after n > 8. Such a confusing of "amount" and "rate" was clearly demonstrated in students' responses.

4.2.4 A statistical analysis of students' misconception: Test of the robustness of misconceptions.

A statistical test for the difference between two population proportions was used to test the change of students' errors related to misconceptions in the pre and posttests. "This test should be used only if $n_1\pi_1$, $n_1(1-\pi_1)$, $n_2\pi_2$, $n_2(1-\pi_1)$ are all at least 5" (Ott & Longnecker, 2001, p.486). This assumption was tested and met for this study. Table 4.19 shows the results of students' error changes:

Table 4.19
The change of students' errors related to misconceptions

		Percent of Error		Error Change	
Concept	Item	Pre (n=456)	Post(n=506)	Z	P (a <.05)
Variable	Q9	8%	5%	3.096348	.0008
	Q15	9.6%	14%	-2.85243	.002
Equation	Q16	5%	16%	-6.74945	<.001
Function	Q11	35%	35%	0	.4801
	Q13	13%	20%	-3.93653	<.001

As the above table shows, students made fewer errors in post test of Q9, which means students' understanding was significantly improved for this problem ($P_9 = .0008$). However, students' errors were not changed for Q11 from the pre to the posttest ($p_{11} = .4801$). Surprisingly, students made more errors in Q15, Q16, and Q13 in the posttest. The error increases were also significant for all three ($p_{15} = .002$; $p_{16} < .001$; and $p_{13} < .001$). In summary, students' errors related to certain misconceptions did not decrease but increased, which demonstrates the robustness of misconceptions.

4.3 Students' Understanding of Concepts as Object and Process

As mentioned earlier, mathematics knowledge has a dual nature. Sfard (1991) & Sfard & Linchevski (1994) explored the importance and difficulties of students' understanding concepts in the transition from *process* oriented to *object* oriented thinking. Chi (2005) investigated the robustness of misconceptions from a similar approach. She pointed out that the reason some misconceptions were extremely robust is because remediation of the misconceptions requires students' transition from a process-oriented to an object-oriented level. Therefore, it is reasonable to assume that high-achieving students (with few errors) understand some concepts at an object level while low-achieving students (with many errors) understand at a process level. The robustness to correction displayed by students' misconceptions underlying these errors is mainly due to the difficulties of transition from different ontological levels.

In this study, I examined the top 10% high achieving (HA, N=37) and top 10% low achieving (LA, N=40) students' understandings of variable, equation, and function in the posttest. Those students participated in both pre and posttest (N=320). Since some students' scores are the same, more than 32 students in each group were selected. Students' responses were coded at an ontological level according to the rubric developed in the methodology section. It is interesting to know whether there is a difference between the HA and LA students' understandings of algebra concepts. It is also interesting to investigate whether the high achieving students are more likely to use object oriented thinking in problem solving.

In this section, I begin with the report of general information of students' performance for two groups of items. I then report detailed information about students' solutions, which demonstrated ontological differences in their understanding of variable, equation, and function.

4.3.1 The general information

Report of students' object and process-oriented thinking. According to students' achievement (total score) on the posttest, 37 students were identified as high achieving (HA) students while 40 were identified as low achieving (LA) students. I first examined these students' performance on two groups of problems. To solve the first group of problems (Q5, Q6, Q7, Q10, Q12; see Appendix 4), students' understanding of algebra concepts must reach a process level, that is, process-oriented thinking is enough to solve those problems. To solve the second group of problems (Q8c, Q8d, Q9, Q11, Q13, and Q16; see Appendix 4), students' understanding of algebra concepts should reach an object level, that is, object-oriented thinking is necessary. Table 4.20 shows the number of students who correctly solved first and second group items.

Table 4.20 shows that the LA (Low Achieving) group performed poorly compared with the HA group on every item. They did worse on the questions which required *object* -oriented thinking.

Secondly, I examined and compared the two groups of students' understanding of the three algebra concepts at the ontological level in each short-response item (Q8c, Q8d, Q9, Q11, Q13, and Q16). Students' understanding was coded as *object*-oriented thinking or *process*-oriented thinking. Table 4.21 shows the results. The justification of coding a student's response as an object or process-thinking will be provided in subsection 4.3.2.

Table 4.20
The number of students who correctly solved the selected problems

Problems Selected		HA Students (N=37)	LA Students (N=40)
	Q5	37	17
***Process*-**	Q6	37	9
oriented	Q7	37	20
thinking needed	Q10	37	8
	Q12	37	7
	Q8(C)	32	0
***Object*-oriented**	Q8(D)	8	0
thinking needed	Q9	31	3
	Q11	17	0
	Q13	18	0
	Q16	34	0

Table 4.21
Students' object-oriented and process-oriented understanding

	HA students (N=37)			LA Students (N=40)		
	Object	Process	Below Process/ NA	Object	Process	Below Process/ NA
Q8(C)	0	32	5	0	0	40
Q8(D)	3	5	29	0	0	40
Q9	31	0	6	3	8	29
Q10	17	3	17	0	0	40
Q13	15	3	19	0	0	40
Q16	23	11	3	0	2	38

Interpretation of tables. There is a gap between HA and LA students' performance in solving process-oriented problems. Moreover, the gap becomes larger in solving object-oriented problems. Only three students out of the LA group correctly answered one problem (Q9) and demonstrated object-thinking about variables. Eight of the students from the LA group demonstrated process-oriented thinking about variables because they refer to specific values. That means they know variables can represent numerical values. Some of them refer to a variable as a label which is below the process-thinking level. HA students demonstrated more *object*-thinking about variable, equation, and function. However, most students at the middle school level had difficulties in using symbolic forms of function to explain which is in a higher level. In general, HA students' demonstrated much more *object*-oriented thinking than those in the LA group.

4.3.2 The detailed information about coding *object* and *process* thinking

In this subsection, I explain how students' responses in the HA and LA groups were coded at the object or process levels. Examples for each item are provided as evidence.

For Q5, Q6 and Q7, the solution required understanding how an independent variable changes over a dependent variable. Such a view of function is *process*-oriented. The performance difference between HA and LA group for these problems is significantly smaller than the gap for other problems: Q8 (C) (D), Q9, Q11, Q13 and Q16.

As we analyzed before, the solution for those problems needs object-oriented thinking. The indication of students' *object*-oriented or *process*-oriented thinking was demonstrated by students' solution strategies. In general, the algebra approach corresponds to *object*–oriented thinking while the arithmetic approach reflects students' *process*-oriented thinking. Some problems such as Q8 (C), Q8 (D), or Q16 can be solved by both arithmetic approach (e.g., listing table values) or algebraic (e.g., using symbolic formula). The method of listing table values is called arithmetic is because such an approach obtains a general conclusion by looking at limited cases.

About Q8 (c). In this problem, some students listed a table value and found that the number of bricks and stones is equal when n=8. This approach cannot eliminate other values, or justify why there are no other values which also met the requirements. No student used $n^2 = 8 \bullet n$ to get the solution directly. Some students listed a table to solve this problem. The following examples are students' typical arithmetic approaches (see Figure 4.21 and Figure 4.22).

Student 1.

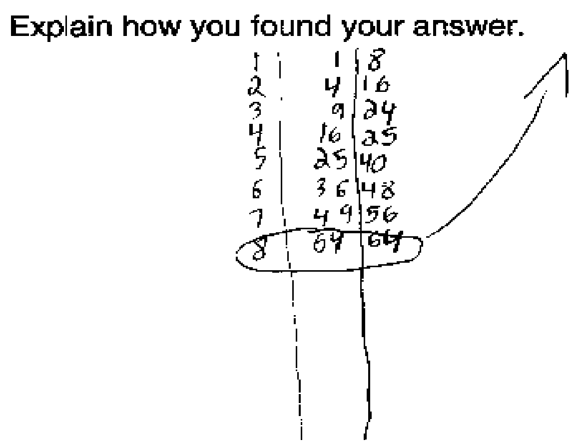


Figure 4.21. Using table values to find the answer n=8.

Student 2.

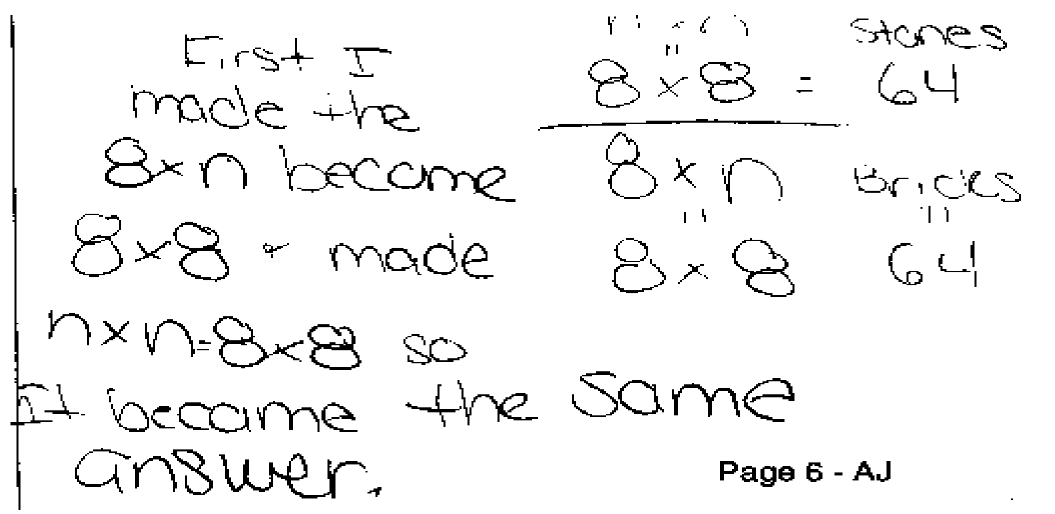


Figure4.22. First guessing and then checking the answer.

Though the above approaches are correct, they were still coded as *process*-oriented thinking. The reason is that, these approaches were arithmetic. Student 1 simply compared the outputs of a function. Student 2 used "guess and check." He/she made a comparison by referring to the formula. However, this student still used a specific value 8 (although it is correct in this problem) to substitute for the letters in the functions. Such an approach was ontologically different from the way of solving $n^2 = 8 \bullet n$, that is, to eliminate the other possible values. Thus, Student 2 's method only verified whether 8 was a correct answer. As a result, students who used the second approach were still coded as using *process*-oriented thinking.

About Q8 (D). In this problem, the main method used by students was to list a table and show how the number of stones increased more quickly. Figure 4.23 shows a typical answer.

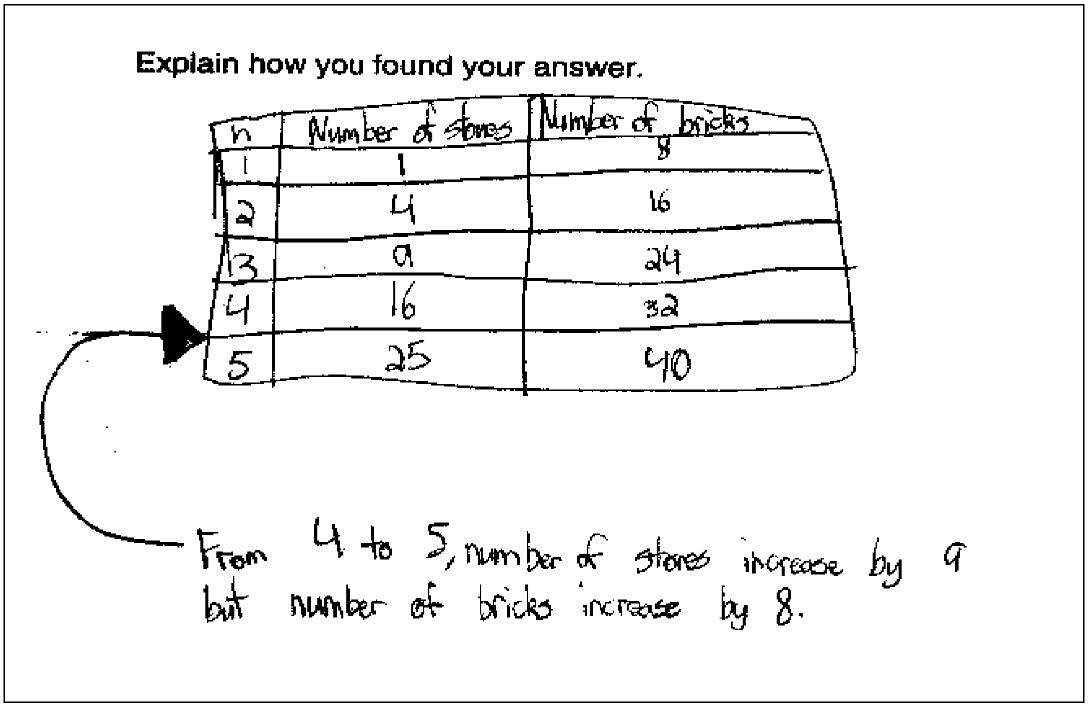


Figure 4.23. Students used the arithmetic approach to answer the problem.

Students who used such an approach were coded as *process*-thinking because they did not directly use a formula (such as: $n^2 - (n-1)^2 > 8$). When students listed the table values, they focused on the input and output process, which is an arithmetic method.

In contrast, students' understanding at an object level addressed a global pattern of function and went beyond the constituent levels---the input and output process. For example, they might draw a graph or point out the nonlinear characteristics of function. What follows are some typical answers (See Figure 4.24 and Figure 4.25)

Student 1

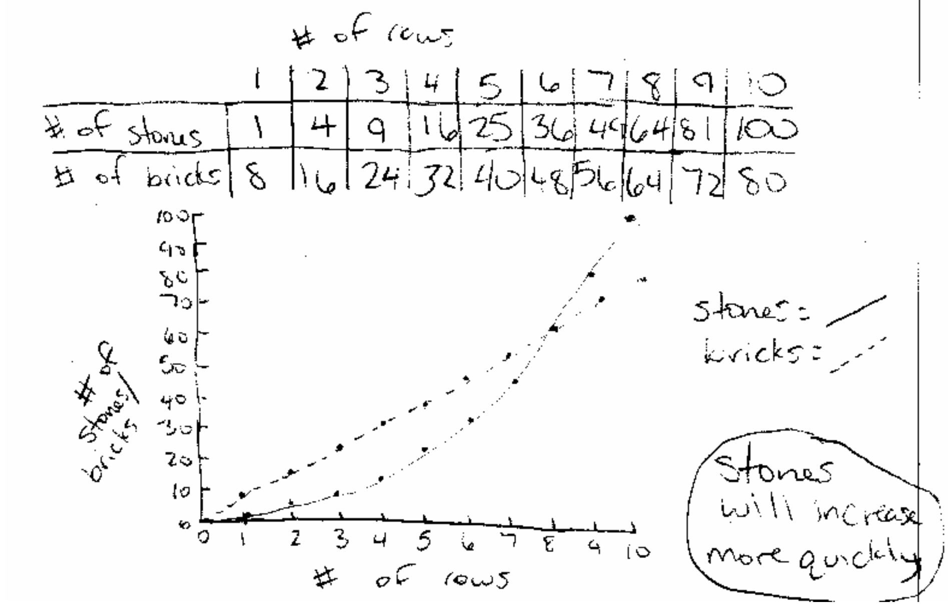
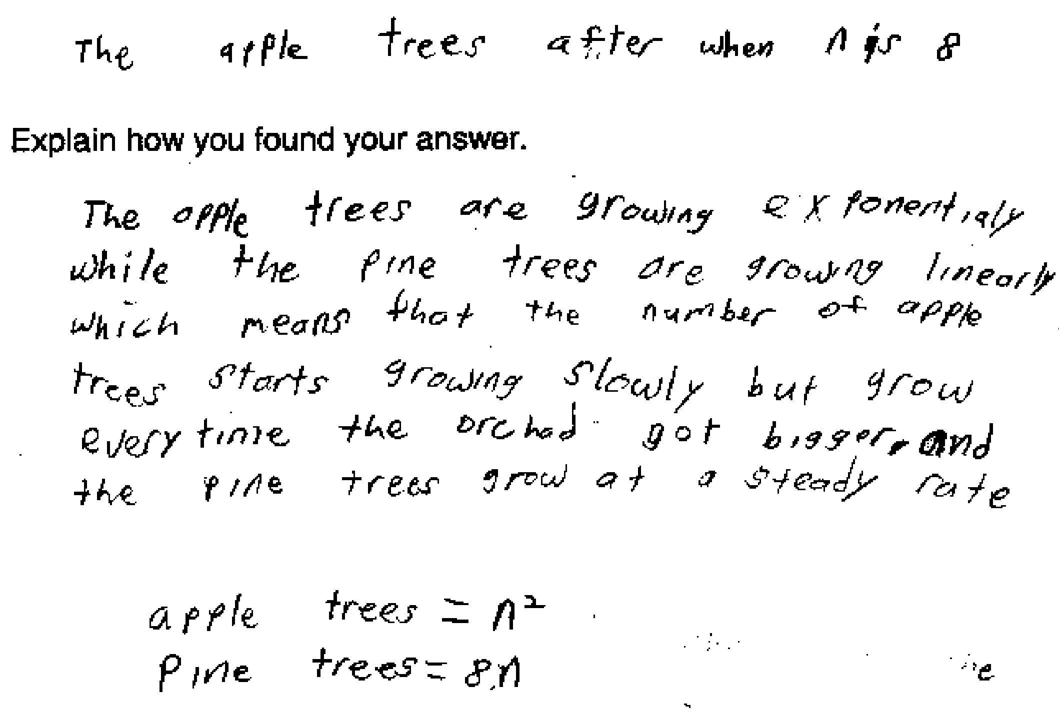


Figure 4.24. A graph response indicating an object-oriented thinking.

This student drew a graph. Although he/she did not clearly and completely explain why "stones will increase more quickly." this response was thought of as an algebra approach in that the student correctly drew two graphs and tried to compare their global properties; for example, which one is larger or quicker in the end. Such a comparison by using graphs does not need to go into the constituent level. Thus, this students' understanding reached an *object*-oriented level.

Student 2.



The apple trees after when n is 8

Explain how you found your answer.

The apple trees are growing exponentialy while the pine trees are growing linearly which means that the number of apple trees starts growing slowly but grow every time the orchad got bigger and the pine trees grow at a steady rate

apple trees $= n^2$
pine trees $= 8n$

Figure 4.25. Example of students' object-oriented thinking.

Student 3.

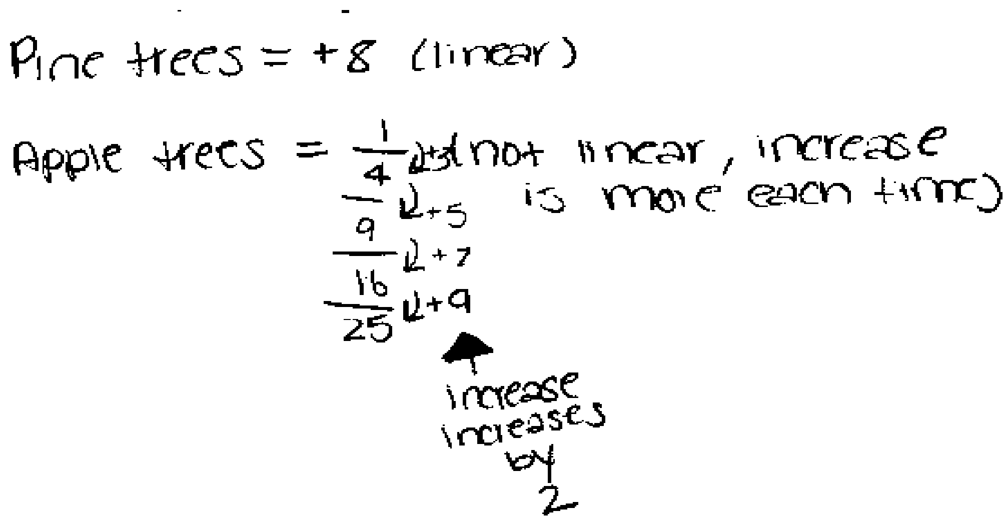


Figure 4.26. A response indicating object-oriented thinking.

Student 3 in Figure 4.26 explored the process of changing the number of apple trees over the number of rows. However, this student did not stop where the "amount of increase" changed from 3, 5, 7, 9, … He (she) gave an algebraic explanation of why apple trees increase more quickly from the linearity and nonlinearity of functions. If this student had not used the property of functions to explain his idea, it would have been coded as a *process*-oriented thinking.

About Q9. **This question is about understanding variables.** Students' approaches to using specific numbers was an indication of *process*-oriented thinking about variables (Weinberg, 2004). Moreover, if students incorrectly combined letters and numbers, such as T+1= T1, they were thought to be unable to operate on the variables. As a result, their understanding of variables was coded as *process*-oriented thinking. In contrast, if students could correctly answer the problem by T=B+1 without referring to any specific values, it would be thought of as *object*-oriented thinking. As a result, the correct form "T=B+1" indicated that students had operated well on variables; that is, variables thus become the "objects" to be operated.

About Q11 and Q13. Students who are not very aware of the meaning of linearity had difficulty responding to these problems completely. For Q11, students need to interpret the "graph" as "steadily," that is, be able to use mathematics language to interpret life situations. The students in the HA group used "steadily" or "linearity" to interpret why the graph they chose can describe the "raising flag." The example of students' responses thought of as *object*-thinking were provided in the methodology section. If students used key words like "gradually," or "slowly," such answers are thought of as *process*-thinking. Using these words showed that students understand that the height of the flag is changing. However, there is an ontological difference between recognition of a change and description of a change using mathematics language. If students only used key words like "go up" or "raise," such responses were thought of as separating height from time. The students thought only about the height above the ground, not over time. Such responses will not be coded as *object*-oriented thinking.

For Q13, it is about the linearity of function. The algebraic way is to find the symbolic way to represent the pattern as this student did (see Figure 4.27).

An example of the algebra approach and *object*-oriented thinking for Q13. The use of symbolic expression indicates that the student used the slope of linear function to support his answer. Such answers typically indicate that the student thinks of the function as an object by considering its global pattern without going into the constituent level.

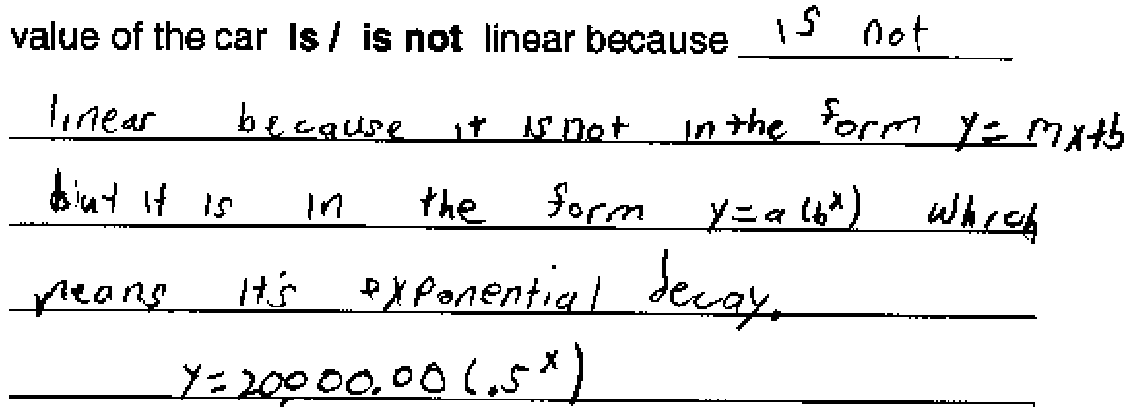

value of the car **Is / is not** linear because is not linear because it is not in the form y= mx+b but it is in the form y=a(b^x) which means it's exponential decay.
y=20000.00(.5^x)

Figure 4.27. Students' response indicating object-oriented thinking.

About Q16.Object and process thinking were mainly coded from their solution strategies. If students just guessed and checked to get the answer, they were categorized as process-thinking. This is because the guess and check process is mainly to compare the output value (by substituting y with the specific value) with 19. If students use an algebra approach, that is, they perform the same operation on the equation or use change side and change sign, then the students were coded as object-thinking about the equation in terms of the literature review.

In other words, if students thought about functions from the input-output perspective (for example, to answer Q5, Q6, and Q7 where students are asked to calculate the output value or describe how to calculate output) or used this approach to answer the problem (for example, to use table values to answer Q8 (C) (D)), those students were coded as *process*-oriented thinking. If students used the characteristics of functions to interpret or solve the problems, used or wrote graphs, used or wrote symbolic forms of functions, or operated with functions like comparison of function or subtraction between functions, all of those were coded as *objec*t-oriented thinking. For other cases, students' responses were not coded either *object* or *process.*

It was found that the high-achieving students are more likely to use object-oriented thinking in problem solving. In contrast, the low-achieving students who made many more errors either understand some concepts at a process level or, in some cases, below that level.

5. DISCUSSION AND CONCLUSION

The fundamental goal of this research is to explore why some students have difficulty learning school mathematics, including certain basic concepts. This study assumes that a better understanding of students' systematic errors in basic mathematics concepts leads to a better understanding of students' general mathematics learning principles, especially for those kids who fail to grasp basic mathematics concepts. This section includes discussion of misconceptions based on this study and further comments for future research and professional development.

5.1 Basic Concerns

5.1.1 Do students misunderstand "equal sign" as "to do something"?

For quite some time, researchers investigating students' misconceptions claimed that students tended to misunderstand "equal sign" as "to do something" and such misconceptions prevented students from further mathematics learning (Behr, Erlwanger, & Nichols 1980; Carpenter, et al, 2003; Sáenz-Ludlow & Walgamuth, 1998; Thompson & Babcock, 1978). In this study, some students made similar mistakes. However, they were found to perform reasonably well in solving problems such as "43= □-28." About 53.10% of students correctly answered this question in the pretest while 67.39% did so in the posttest. For another problem, "a=b-2," students also performed well (40.78% in the pretest and 54.94% in the posttest answered correctly). Referring to the study conducted by Falkner, et al. (1999), they found that all 145 sixth-graders filled the box in

8+4=□+5 with 12 or 17. Moreover, less than 10% of students can answer this problem at each grade level (also cited by Capenter et. al., 2003). The main reason for such bad performance was attributed to “misconception of equal sign.” I disagree with this attribution based on the results of my study. First, the students did reasonably well in the problems as mentioned before. If students had such a misconception of equal sign as “to do something,” they would have difficulty solving the problems like “43=□-28” or “a=b-2”; however, their performance on these two problems was much better than that for the problem, 8+4=□+5. Furthermore, according to a study that investigated Chinese and U.S. elementary students’ performance on similar problems (Ding, et al., 2006), about 99% of Chinese sixth graders and about 88% of Chinese second graders correctly answered those problems. If students truly have the robust misconception of the equal sign as claimed by Carpenter et al, it is expected that Chinese students would not do so well in this problem and American seventh and eighth graders in our study should also not do so well on these problems. This is because a robust misconception should hold across different people (including students from America, China and other countries).

So what kind of misconception does underlie the errors for the problem 8+4=□+5? I argue that the misconception of “8+4” was the cause of students’ errors. There are two interpretations of “8+4”; it can be thought as an algebraic expression representing the same amount as “12,” or it indicates an addition between 4 and 8, and the result of that operation is 12. In solving an equation, it is necessary to understand the left and right side as an *object*: the algebraic expression rather than a computation process. As I analyzed students’ errors in 19=3+4y, students had more difficulty understanding the structure of "3+4y" rather than the meaning of "=." Students who used the "guess and check" method without using an algebra approach also know the way to check an equation. In other words, students' difficulties or errors about the problem such as $8+4 = \square+5$ should not mainly be attributed to the misunderstanding of "=" but the difficulty of understanding of "8+4" as an object. The understanding of algebraic expression as an *object* and the distributive law is the key to understanding equation; for example, the transitivity and symmetry of equation. I recommend that teachers should pay more attention to helping students understand these conception rather than the equal sign.

5.1.2 Are misconceptions of variable and function robust to change?

Students' most common error in solving the "Tachi and Bill problem" in this study is "reverse error" in both the pretest (11.84%) and the posttest (43.48%). Such a result is consistent with other studies (Clement, et al., 1981). In Clement et al.'s study in which college students were asked to write an equation to describe "there are six time as many students as professors," many students wrote a wrong equation, 6s=p. The authors attributed such errors to students' misconception of the variable as a "label," that is, "S" was misunderstood as "student" rather than "the number of students." As stated in Chapter 4, students' errors directly related to the misconception of variable as "label" did not decrease from pretest to posttest. In order to know whether this misconception was robust to change, a statistical analysis was conducted and the result showed that students' reverse errors related to the misconception of variable did not decrease but significantly increased (Z=-20.9084, p <0.001). Therefore, this misconception proved to be robust.

As mentioned before, one of the students' misconceptions of function was to consider only the dependent variable without the independent variable. In the "car value" problem, students only considered the change of car value without referring to the change of time. Similarly, in the "small boy raising flag" problem, students only considered the change in the height of the flag without referring to the change over time. Students' errors related to the above misconception did not change after one year instruction.

5.1.3 Why do students' misconceptions usually resist change?—An *objec*t and *process* perspective.

The robust misconceptions about science concepts have been well documented in science education. In mathematics education, there were also some studies about misconceptions in whole number of addition and subtraction area. However, research on misconception about algebra concepts is not well documented.

Although the causes of misconceptions are very complicated, they have common characteristics. That is, they are all "invented" by students. Thus, these misconceptions are meaningful for students and are easily activated in solving problems, and repeated misuse may reinforce the misconceptions. Another important cause is the complexity of mathematical concepts. Those concepts are usually developed over hundreds or thousands of years but students are expected to grasp them within several years. One big challenge for students is that they need to transition their understanding of mathematics concepts from *process* to *object,* which is consistent with historical development of most concepts. For the learning of variable, students have the challenge to transition from *process*-oriented to *object*-oriented thinking. They need to consider a variable as a "place holder" rather than a specific value or substituted by specific values. Without such a transition, students tend to make many errors such as: simplifying an algebra expression like T+1 to T1. The relationship between misconceptions, robust misconception, object and process-oriented thinking is stated in the Figure 5.1. Their different combinations may cause different learning

results:

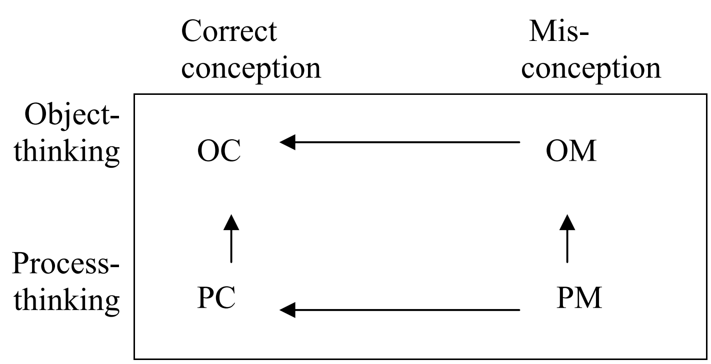


Figure 5.1. A matrix of the relationship of misconception and object-process thinking.

Note. ***OC:*** This is the best result. Students have *objec*t-oriented thinking and have no misconceptions

PC: Students have a sound understanding at the process level. For example, students thought of function as a relationship between dependent and independent variables. Such understanding has potential misconceptions about the function which might easily occur in a new situation.

OM: This misconception is easily to be changed because it is at the same stage as OC. For example, some students have such a misconception about variable: different letter should represent different values.

PM: This misconception is hard to change because it is at a lower stage of thinking than OC.

The above matrix shows outcomes of students' understanding of concepts at either of two stages (*process* and *object*) interacting with either of two aspects (correct conception and misconception). The arrows show the possible path of correcting students' misconceptions. When students' thinking is in the *process* stage, sometimes such thinking (PC) does not affect how to solve a problem. For example, when students understand the function from the change of dependent variable over the independent variable, students have no problem in solving problems like "look at the equation, y=2x+5, if x increases by 1, what happens to y?" Students' understanding of function as a relationship will not affect how to solve this problem because they can use table values (the *process*-oriented approach) to answer this type of question. However, if students can understand the function as an *object,* they can make a prompt judgment by just looking at the "slope" of the function without referring to the input and output process. On the other hand, if students are asked to answer the problem like "which function will increase more quickly: $y = 8x$ or $y = x^2$?" It does not help much if students still think of function as a relationship. For example, with *process*-oriented thinking, students usually compare these two functions by using table values. As a result, many students may develop alternative strategies or "bug strategies". For example, in Q8 (D), students used the invented "bug strategies"—they compared the amount rather than the change of amount to solve this problem. Due to understanding of function as process--that is, the relationship, students have difficulty using the properties of functions (linear or square) or operating on their symbolic forms to solve problems.

Another example is about students' understanding of algebra expression. For example, some students may only see subtracting from one item,"3," rather than from "3 + 4y." That is, they just saw 19-3 = (3-3) +4 y, rather than 19-3 = (3 + 4y)-3. At this time, it is extremely possible for these students to invent an algorithm: "operate on some items in the right side rather than the whole algebra expression." Later, when students learn how to use "dividing by the same number on both sides" to solve this equation, the invented algorithm may turn to a "bug algorithm," that is, "divide some items on each side rather than the whole algebra expression," which might then cause students' errors such as "19/4=3 + 4y/4" as demonstrated by students' solution (see Results section, 4.1.5). Therefore, only when students are able to see each side as an algebra expression, an "*object,*" can they solve this equation by firstly dividing the equation by 4, that is $\frac{(19)}{4} = \frac{(3+4y)}{4}$, then using the distributive law to finish the remaining steps. Therefore, when students' understanding is PC, teachers need to help students transition their thinking to reach OC even if there is no misconception at the *process* level (PC). Further, if students have PM, it is not enough for teachers to just tell them standard algorithms which are only correct procedures at the process level (PC), without correcting students' misconceptions of algebra expressions. Teachers need to help correct students' misconceptions toward an object-oriented understanding. Otherwise, students' misconceptions may be reinforced by the repeating instruction emphasizing only the "correct algorithms".

Misconceptions in the *object*-thinking stage are evident in that students still harbor some wrong misconceptions (OM). For example, students may have a sound understanding of "variable" as "placeholder." However, they may think that different letters should represent different numbers. Students with such misconceptions have difficulty understanding problems such as whether the equation c = r is true (Stephens, 2005). Such a misconception is relatively easy to be corrected because it is in the same stage as that of the OC. Stephens (2005, p. 97) found that students (27% of grade 6, 36.8% of grade 7, and 45.2% of grade 8) could correctly answer the problem "Is $h + m + n = h + p + n$ Always, Sometimes, or Never True" (N = 371). I interpreted the above data as evidence that students made good progress from grade six to grade eight.

The particularly robust misconception is the misconception at the stage of *process*-oriented thinking (PM). For example, students only consider one variable of a function, that is, students tend to ignore the independent variable. For example, in the "car value" problem, students only considered the difference between values of the car without mentioning the age. Such misconceptions are particularly robust in that students need to change their conception ontologically. For the PM, there are basically two paths to correct students' misconceptions. One is PM-OM-OC, that is, first help students transition to object thinking and then correct their misconceptions. Another is PM-PC-

OC, that is, to help students correct their misconception at the *process* level, then help them transition to *object* level. It is very important and also difficult to research on how to correct students' misconceptions, especially robust misconceptions which will be the focus of my future study.

In summary: The dual nature of mathematics concepts can be described as *object* or *process.* Some researchers use *structural* and *operational* to describe the dual nature. There is no necessary causal relationship between *object-process* understanding level and students' misconceptions. That is, whether students had the *object* or *process* thinking for a concept, they may have or have no misconceptions of that concept at the corresponding level. The relationship between *object-process* and misconception is in that if students had misconceptions at the process level, such misconception will be particularly robust. Otherwise, if students had the misconception at the *object*-level, it is still robust but relatively easy to be corrected compared to the PM (misconception at process-level). Last, there may be more than one misconception for a specific concept.

5.1.4 The difference between high-achieving and low-achieving students

The comparison between high achieving and low achieving students' understandings of concepts as *object* or *process* showed that a larger difference existed in solving problems that need *object*-oriented thinking. Another finding is that high achieving students prefer to use *object*-oriented thinking more often to solve problems which can also be easily solved using *process*-oriented thinking. For example, in solving the problem Q 10 ($a = b - 2$), the high-achieving students tended to use the property of equation rather than computation to solve this problem (See Figure 5.2 and Figure 5.3). Students 1 and 2 in Figure 5.2 used large numbers which partly showed that those students used the object-oriented thinking of equation. When the numbers are small, there is no big difference between the two levels of thinking. However, when the numbers are too large, for example, 98999+ 9809 = 98998+ □, object-oriented thinking will show the advantage. Student 3 in Figure 5.3 more clearly showed his/her object-thinking process: he used the property that adding the same quantity to both sides of equation would not change the equivalence. In contrast, the low achieving students preferred using the equation as a formula to produce a pair of numbers which are unusually small. Such a solution without any explanation partially shows the low achieving students' understanding of the concepts at a process level .

Student 1 and Student 2 used the big number. It is more likely those two students answer this problem in term of property of equation.

Figure 5.2. Student solution partially shows object thinking.

Student 3: this student used the property of equation, that is, add the same number to each side of this equation

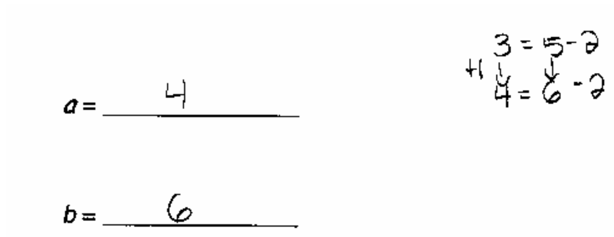


Figure 5.3. Student's solution clearly shows object-thinking.

Therefore, the gap between high-achieving and low-achieving students is not only in students' scores. The more important difference is in their understanding of concepts which causes differences in approaches to solving problems. This result clearly illuminates the necessity of transition from *process*-oriented thinking to *object*-oriented thinking. At the same time, such a difference also demonstrates the difficulties of transition and robustness of misconceptions.

5.2 Further Comments

5.2.1 Not underestimating the complexity of learning mathematics concepts

As mentioned earlier, compared to the research on conception, especially misconception, in science education, similar research is not well documented in mathematics education. Thompson (1985) claimed that "little attention has been given to the issue of the development of mathematical objects in people's thinking" (p. 232).

Besides, teachers also tend to underestimate the complexity of learning mathematics concepts.

This study shows that students have extreme difficulty understanding fundamental algebra concepts. In the pretest, approximately 180 different types of incorrect responses were found in answering the simple problem "Tachi is one year older than Bill" by using an equation. By analyzing students' answers, the typical difficulty for students is about how to operate on or with a variable. Thus, the fundamental problem is to understand "variable." The key to improve the understanding of variable for those students is the transition from *process* to *object* thinking. That is, students should not always refer to specific values or concrete material to solve such problems. Teachers should be aware that such a transition is an ontological change from arithmetic to algebra thinking. In the historical development of mathematics, the use and introduction of variable spans more than one thousand years. It is certainly not enough for teachers just to tell students the definition of variables. The learning of function is also difficult in that students need to transition from *process* to *object* thinking. The introduction of function usually begins with tables, computations of input and output values, or finding patterns. Then the property of functions is introduced and the multiple representations of functions are emphasized. Such a sequence is consistent with the development of function. Though it is not enough to stay in the *process* stage, it is difficult for students to understand a function at an *object* level as found by this study. Even high level students experienced difficulty in thinking about functions from the object perspective.

Essentially, the transition from process-thinking to object-thinking is the source of students' errors and difficulties. So far, there is still no effective way supported by research that can be used to help students facilitate such transitions. One of the most often used strategies is to use manipulative tools to help students understand these complex concepts. The negative aspect of such a method should be noted. Does the implementation of teaching using manipulatives really have a great impact on the learning and understanding of mathematics? Two ways were typically used by past researchers to explore this problem. One is to study the relationship between effectiveness of using manipulatives and students' achievement. Students were found to have higher scores when manipulatives were used in the classroom (Butler, Miller, Crehan, Babbit, & Pierce, 2003; Stein & Bovaino, 2001). Another way is through analyzing the cognitive foundation of using manipulatives which may go back to Piaget. According to Piaget, students have four stages of learning: sensory motor (infancy), preoperational (preschool), concrete operational (elementary and middle school), and formal operational (higher grades). During the third stage, children can process abstract concepts and symbols like fractions; however, they have no mental ability to learn these abstract concepts without referring to concrete materials (Piaget, 1952). As Uttal, Scudder, and Deloache (1997, p.38) state, "Concrete objects allow children to establish connections between their everyday experiences and their nascent mathematics concepts and symbols."

However, manipulatives are not magical. Ball (1992, p.47) argued that manipulatives were not the carrier of meaning because "although kinesthetic experience can enhance perception and thinking, understanding does not travel through the finger and up the arm." The reasons are also related to manipulative tools themselves. Moyer (2001, p.177) pointed out that "the manipulative is simply the manufacturer's representation of a mathematics concept that may be used for different purpose in various contexts with varying degrees of 'transparency'.'' Since students are neophytes to the abstract mathematics concepts, they have difficulty finding the relationships between the manipulative tools and correspondent mathematics concepts which may be obvious for teachers. Hiebert and Carpenter (1992) used the connection perspective to explain the reasons: "It is not easy for students to relate their interactions with materials to existing networks. They do not interpret the materials in the way that the teacher expects, and the use of concrete material is then likely to generate only haphazard connections" (p.71).

The results of this study partly support the statements of Moyer (2001) and Hiebert and Carpenter (1992). The students' life experiences interfered with students' understanding in that students are unable to find the mathematics meaning behind the real life situations. For example, in the problem about the flag raising, 201 (34.87%) students in the pretest and 217 (35.26%) students in the posttest interpreted the graph based on their life experience and composed a wrong explanation for the problem. The use of manipulatives is a way to use students' life experiences or familiar things to facilitate students' understanding or conceptual change. Thus, students may be negatively affected by irrelevant information contained in manipulative tools and thus fail to think in the way that teachers desire. For example, many students in this study were concerned with unimportant information, such as "small boy" and "the older, the less value of the car," in solving Q11 and Q13 respectively.

Does using manipulatives help students transition from PM to PC or OM to OC? From the historical development of mathematics, arithmetic can be thought of as the abstract of concrete material or life experience. For example, the number "1" is produced from different individual objects, one apple, one orange, or one pound; thus, the use of concrete material or manipulatives to help students develop "number" concepts seems to be useful. The learning of operation on numbers by using manipulatives looks a little harder because the "number" becomes an "*object*", that is, the conception of addition is at the higher level whose object is a mathematics concept rather than concrete material. For algebra, even its basic concepts are largely abstracted from arithmetic concepts and thus are further removed from concrete materials. The use of manipulatives to improve algebra understanding at least is not as effective as that of arithmetic. In algebra study, finding mathematical meanings carried by manuipulative tools will be more difficult for many students.

Another strategy to help students transition from process to object oriented thinking is to use confrontation and to cause a cognitive conflict. However, confronting only makes students realize the problem but not fix the problem. The importance of the confronting approach is that students realized the necessity of conceptual change. For example, when students made the errors in the problem like 8+4 =+5,. if they realized the unreasonable facets of their wrong answers like "12", it would be possible for them to reflect on their solution or thinking, which is an initial sequence of real conceptual change. This study did not contradict such an approach. Even more, it points out that the most important and difficult aspects of conceptual change are to help students transition from *process* to *object*-oriented thinking. However, the approaches to effectively complete such a transition need be given more attention in the future.

5.2.2 The influence of prior experience and knowledge on students' learning

The effect of prior experience and knowledge can be both positive and negative. In this study, 201 (34.87%) students in the pretest and 217 (35.26%) students in post test only drew on the everyday life experience to justify their answers about the "raising a flag" problem without referring to any mathematics knowledge or language. In another problem in this study, 68 (14.91%) students in the pretest and 66 (13.04%) students in the posttest only talked about the values of the car without mentioning the linearity issue. Since students' understanding or intuition caused by outside school knowledge is sometimes contrary to formal mathematics knowledge, teachers need to pay much more attention to identify and use students' everyday life experiences to help them understand mathematical concepts. For example, the "small boy raises flag" problem, many students were confused by the life pictures with the graphs. They interpreted the graph of "height over time" as the life picture of "position above ground". On the other hand, since everyday life experience is most familiar and also meaningful for students, the proper use of this informal knowledge can improve students' understanding of mathematics learning.

5.2.3 Students' robust misconceptions and curriculum development

Since students' errors can be attributed to flawed procedural or conceptual knowledge and it is difficult to change students' robust misconceptions despite "good" teaching, teachers should be careful when recycling curricula. This study shows that students' misconceptions are often robust to change. For example, some errors made by students did not change after one year of instruction as demonstrated by the comparison of students' pre and posttests. Such robust misconceptions are also demonstrated by the surprising posttest achievement gap between the seventh and eighth graders. Among the top 10% of students (see Appendix 8) in the total test, the number of seventh graders greatly exceeded that of the eighth graders (75% vs. 25%) though there were an almost equal number of participants in each grade level (171 vs. 151 respectively). I interpreted this strange phenomenon partly as the entrenched misconceptions being reinforced in students' ongoing studies. Therefore, it is important for students to gain correct understanding of concepts at the very beginning. "The best time to learn mathematics is when it is first taught; the best way to teach mathematics is to teach it well the first time" (Everybody Count, p .13). Based on the results of this study, I raise the concern about whether recycling curricula characterized by increasing complexity but repeating the same contents are good for students' learning of mathematics. East Asian countries do not present mathematics contents in such a repetitive way. Mathematics teachers in those countries usually make a judgment about what prior knowledge needs to be reviewed for new learning based on the real situation of students in their classrooms.

5.2.4 Developing teachers initial subject knowledge

Ma (1999) compared American and Chinese elementary teachers' mathematical knowledge. Although American elementary teachers normally complete sixteen years or more of education while Chinese elementary teachers only have a 12-year education, Chinese teachers demonstrate a better "profound understanding of fundamental mathematics (PUFM)" compared to their American counterparts. A more surprising result stated in Ma's research is that even Chinese high school students and pre-service teachers demonstrate better PUFM compared to American in-service teachers.

> All members of the Chinese groups (ninth graders and prospective elementary teachers) succeeded in computing $1\frac{3}{4} \div \frac{1}{2}$ and knew the formulas for calculating perimeter and area. However, only 43% of the U.S. teachers succeeded in the division by fractions calculation, and 17% of the U.S. teachers reported that they did not know the area and perimeter formulas. For the two more conceptually demanding questions, the difference was even greater. Eight-five percent of the Chinese prospective teachers and 40% of the Chinese ninth graders created a conceptually correct story problem to represent the meaning of division by fractions, but only 4% of the U.S. teachers did (Ma, 1999, p. 127-128).

Cuban (1984) pointed out that teacher education has weak influence on K-12 teaching and he also suggested part of the reason. "Teachers learn more about teaching from the thousands of hours they have spent as students in K-12 classrooms than they do from their relatively brief time under the tutelage of teacher educators" (Labaree, 1992, p. 139). Since many current students will become future K-12 teachers, their solid understanding on fundamental mathematical concepts will lend them a hand in their teaching. Since students' misconceptions are robust and even reinforced by continuing instruction (Resnick, et al., 1989), the effect of teacher education for pre-service teachers in universal course may not be as effective as expected. Therefore, to ensure sound recycled curricula, it is necessary for students to develop a deep mathematical understanding from the very beginning, the K-12 education.

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APPENDIX A. ASSESSMENT MAP-ALGEBRA

Symbolic equations can be used to summarize how the quantity of something changes over time or in response to other changes. Benchmarks 11C 6-8 #4	
TEKS The student uses letters as variables in mathematical expressions to describe how one quantity changes when a related quantity changes. Gr. 6 #4 The student represents a relationship in…symbolic form. Gr. 7 #4 The student makes connections among various representations of a numerical relationship.	Delaware Analyze functional relationships to explain how a change in one quantity results in a change in another. 10.61 Represent situations with…equations... 7.60 Model and solve real-world and mathematical problems using algebraic methods. 7.61

If a relationship is linear slope represents the rate of change.
AND
Rates of change can be computed from

Mathematical statements can be used to describe how one quantity changes when another changes.
AND

When two variables are related, the relationship is either linear or nonlinear.

Things change in steady, repetitive, or irregular ways--or sometimes in more than one way at the same time. AC1

Change
This strand includes concepts of change over time, as well as the change in one variable as it relates to change in another.

Variables can be used to represent a generalized rule or principle, e.g. $a + b = b + a$.

Variable expressions can represent a generalization of numeric or geometric patterns.

Quantities (known or unknown) can be represented by symbols other than numerals, e.g. Mary sells *k* donuts. Jon sells three times as many, i.e. 3 x *k* .

A letter stands for a specific object

Each letter has a specific numeric val

The choice of letters is arbi

Variables and constants are

Variable
Variables are seen in this strand as component parts of equations, as they are used to model

Equations can be used to find unknown values.
AE4

An equation containing a variable may be true for just one value of the variable. AND An equation containing

Equations can be used to model real-world situations.
AE2

The equals sign indicates an equivalence between two expressions.
AE1

Equality and Equations
Equations are being used to model notions of change, so this strand focuses on concepts of equality and the use of

DESCRIPTION OF "STRANDS" ("STORY LINES"):

Change – This strand includes concepts of change over time, as well as the change in one variable as it relates to change in another.

Variable – Variables are seen in this strand as component parts of equations, as they are used to model change.

Equality and Equations – Equations are being used to model notions of change, so this strand focuses on concepts of equality and the use of equations to model situations.

Color Key:
Black = targeted learning goals
Red = prerequisite ideas
Green = common student errors/misconceptions
Blue = strand (story line)

* This map is from MSMP website, please refer to http://msmp.tamu.edu

APPENDIX B. ITEMS USEDTO TEST EACH CONCEPT: VARIABLE, EQUATION, AND FUNCTION

Form A Item Number	Item Description	Item Type	Map Location	Level of Complexity	Score Points	Weighting Factor
1	43= x-28	mc	AE4*	L	1	1
2	Equation to represent trading cards	mc	AE2	L	1	1
3	Expression to repression represent Girl Scouts	mc	AV1*	L	1	1
4	Jacob's rule	mc	AV3	L	1	1
5	Rule for numbers in table	mc	AC5*	M	1	2
6	What's not true about y=2t	mc	AC5 AC3	M	1	2
7	What's true about y =2x + 5	mc	AC1 AC3 AC5	M	1	2
8	Tachi and Bill	scr	AC3 AE2	L	1	1
9	a = b – 2	scr	AE3	M	1	2
10	Small boy raises the flag	scr	AC1 AC3	M	2	1
11	Missing number in table	scr	AC3	M	1	2
12	Value of car not linear	scr	AC2	M	2	1
13	Stella's phone plan	scr	AC5 AC1	M	2	1
14	Maria and Jinko's donut sales	scr	AV4 AV1	L	2	.5
15	19 = 3 + 4x	scr	AE4	L	2	.5
16A	How many pine trees in 2 rows	Super	----	L	1	0
16B	Complete the table	Super	AC1 AC3	L	2	.5
16C	N x N = 8N	Super	AC1 AC5 AE1 AE2 AV2	M	2	1
16D	Compare apple trees and pine trees over time	Super	AC1 AC4	H	2	1.5

Note. AE: Algebra equation; AC: Algebra change (function); AV: Algebra variable

APPENDIX C. ALEGBRA RUBRIC FOR CODING STUDENTS' WRONG ANSWERS

Items	code	Type of errors	Comments	Examples
8 C	1	Gave an answers without any explanations	Some students just give an answer without any explanations.	*n x n ;* n^2 *; or 64*
	2	Misunderstand meaning of "n"	Students misunderstand "n" as the number of stones or bricks; students may also seek the pattern for n.	For the # of bricks, I just keeping on adding 8
	3	Add all or some numbers n the table	Students add the numbers appeared in the tables or part of the numbers, the typical answer is 120	I add all the numbers to get 120
	4	Describe a unrelated pattern	Students describe what happens in the problem	The n increase 1 every time
	5	Others/ or I do not know	Some students say they do not understand.	I do not know, sorry.
	6	No response	Students have no responses at all.	

Items	Code	Types of errors	Comments	Examples
8 D	1.	Using incomplete table	Some students just use values in the table provided by the prior problem 8C to explain their ideas.	Number of bricks because, well just look at the graph in C.
	2.	Border is always larger	Some students think that the border is larger, therefore, more bricks or pines are needed to cover or surround the inside part.	Bricks. Because it takes more bricks to surround the stones.
	3	Brick (pine) is small	Some students just look at the picture and think: Since the outside figures (bricks or pine trees) are small, more these stuffs are then needed.	The number of bricks increases quickly because the stones are bigger than the bricks in the diagram, so it takes more bricks to surround the stones.
	4.	Confuse the "quick" and "more"	Even some student chose correct answer, they may explain the stones (apple trees) will increase quickly because the number of these stuffs will be greater after n>8.	Number of stones. Once it passes 8 rows, there will be more stones because 9 x 8 is less than 9 x 9, 10 x 8 < 10 x 10, etc, etc.
	5	Others/I do not know	Some students say they do not understand this problem.	I do not understand what the question is asking.
	6	No response	Students have no response at all.	

Items	code	Types of errors	Comments	Examples
11	1	Flag go straight up	Some students who chose B may consider the raising flag situation without using any mathematics knowledge.	Because the flag goes straight up. .
	2	Small boy	Some students choose C by only considering the boy is small rather than the relationship between height and time.	Because the boy is small, he pulled the flag and stopped, and then continued.
	3	Slowly/gradually	Some students are aware the variable "time", however, they did not mention the linear relationship by words such as "steadily".	Because it goes up gradually.
	4	Others / I do not know	Students say they do not understand this problem.	I do not understand what the question is asking.
	5	No response		

Items	code	Types of errors	Comments	Examples
13	1	Linearity as constant ratio	Some students think the relationship here "Is" linear because they see the ratio of the car prices.	Each year the price of car is half of the price than the year before.
	2	Describe a real life	Some students just consider real life of experience of car's age and value.	As the car becomes older, it loses its' value.
	3	Constant difference without time	Some students see the differences of car values are not constant. But they may not consider another variable "time"	Because the car value decreased
	4	Linearity related to direction	Some students consider linearity just can be represented as a line goes up rather than go down. Another misconception is "linearity" just related to direction whatever the value change.	"Is not" linear. Because if you make a line, it goes down rather than going up. "Is" linear. Because the year is going up while the price is going down.
	5	Others / I do not know	Some students say they do not understand.	If I know what "linear" means, it will helps.
	6	No response	Students have no response at all.	

Items	code	Types of errors	Comments	Examples
15	1	Five times as many as	Students view "5 times" as a variable	X 5, five times as many
	2	Donuts, Jinko's donuts	Students view "donuts" as a variable	Donuts
	3	Jinko sells five times as	Students view the term "Jinko sells " something as a variable	Jinko sells five times as
	4	Price, 25 cents or constant numbers	Students write "price" a known number "25 cents" as varible	25 cents, Price
	5	Others / I do not know	Students do not understand this problem	I do not know what "variable" means
	6	No response	Students have no response	

Items	code	Types of errors	Comments	Examples
16	1	Y = 2.7	Some students view the equation as: 19 = (3 + 4)y	3 + 4 = 7, 19 = 7y Y = 2.7
	2	Y = 12	Some students view this equation as: 19 = 3 + 4 + y	3 + 4 = 7, Y = 19 – 7 Y = 12
	3	Divided by 4	Some students just divided 19 rather than every item by 4.	19 = 3 +4y 4.75 = 3 + y Y = 1.75
	4	Y = 2	Some students view 4y and 4^y as the same thing.	19=3+4y 16= 4y Y=2
	5	Y = 16	Some students view "y" and "4y" as the same thing.	19 = 3 + 4y Y = 19 -3 = 16
	6	Running equation	Some students get correct answer with running equation.	19=3 + 4y 4 x 4 = 16 + 3 = 19
	7	Y = x 4	Some students are more compatible with the expression 4 x y than 4y. So when they get correct answer 4, they thought it should be x 4.	19 = 3 + 4y 16 = 4y Y = 4 or x 4
	8	Others / I do not know	Some students say they do not understand.	I do not understand,
	9	No response	Some students have no response.	

APPENDIX D. THE ALGEBRA TEST ITEMS USED IN THE PROJECT AND THIS STUDY

1 2 6 9 6

PART 1 **MULTIPLE CHOICE SECTION**

DIRECTIONS: Circle the letter of the correct answer.

1. What is the value of □ in this equation?

   43 = □ - 28

   (A.) 15

   B. 25

   C. 61

   D. 71

2. Mary has some trading cards. Julie has 3 times as many trading cards as Mary. They have 36 trading cards in all.

Which of these equations represents their trading card collection?

A. $3x = 36$

B. $x + 3 = 36$

C. $x + 3x = 36$

D. $3x + 36 = x$

1 2 6 9 6

3. There are *n* Girl Scouts marching in a parade. There are 6 girls in each row. Which expression could you use to find out how many rows of Girl Scouts are marching in the parade?

A. $n - 6$

(B.) $\frac{6}{n}$

C. $6n$

D. $\frac{n}{6}$

---

4. Jacob writes the following rule:
If *a* and *b* represent any two numbers, $a + b = b + a$.

Which of the following describes Jacob's rule in words?

A. Equals added to equals are equal.

(B.) Order doesn't matter when adding two numbers.

C. The sum of two whole numbers is a whole number.

D. When adding three numbers, it doesn't matter how the numbers are grouped.

1 2 6 9 6

5.

A	B
12	3
16	4
24	6
40	10

What is a rule used in the table to get the numbers in column B from the numbers in column A?

A. Add 9 to the number in column A.

B. Subtract 9 from the number in column A.

C. Multiply the number in column A by 4.

D. Divide the number in column A by 4.

6. Which of the following statements is NOT TRUE about the equation $y = 2t$, if t is a positive number?

A. It shows how y changes for different values of t.

B. It shows a linear relationship between y and t.

C. It shows that the value of y is independent of the value of t.

D. It shows that as t increases, y also increases.

1 3 0 7 2

7. The table shows values for the equation $y = 2x + 5$.

x	y
1	7
2	9
3	11
4	13

Which sentence describes the change in the y values compared to the change in the x values?

A. The y values increase by 6 as the x values increase by 1.

B. The y values increase by 7 as the x values increase by 1.

C. The y values increase by 2 as the x values increase by 1.

D. The y values increase by 5 as the x values increase by 2.

END OF PART 1

GO ON TO PART 2

PART 2

DIRECTIONS: All of the questions in the next set are based on the same story. Some of the questions may seem very simple, while others are more difficult.

Read the story. Then turn the page to read the first question.

Trees

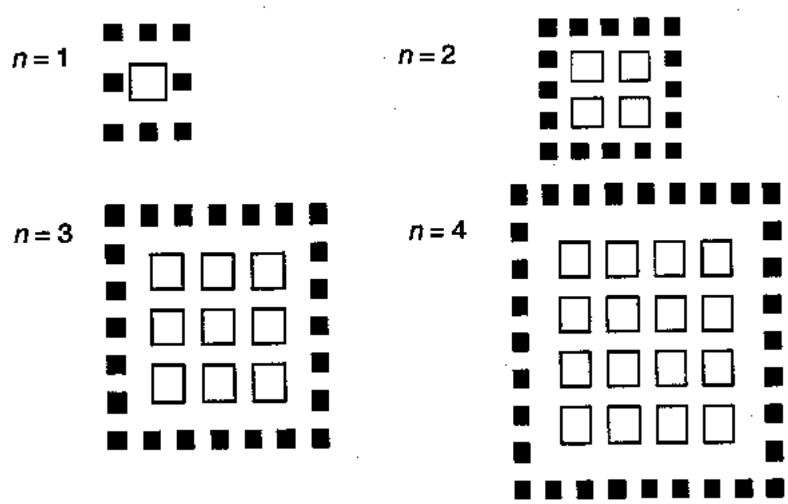
8. A farmer plants his orchard so that pine trees are all around the border and apple trees are in the center in a grid.

Here you see a diagram of this situation where you can see the pattern of apple trees and pine trees for any number of apple trees:

■ = pine tree

□ = apple tree

n = number of rows of apple trees



1 3 0 7 2

(A) How many pine trees are in an orchard with 2 **rows** of apple trees?

5

(B) Complete the table. (*n* = number of rows of apple trees)

n	Number of apple trees	Number of pine trees
1	1	8
2	4	11
3	7	15
4	10	17
5	13	20

(C) Look at the table. You might notice that the number of apple trees can be found by using the formula $n \bullet n$. The number of pine trees can be found by using the formula $8 \bullet n$. Remember, *n* is the number of rows of apple trees.

There is a value of *n* for which the number of apple trees equals the number of pine trees. Find that value of *n*. ______________

Explain how you found your answer.

1 3 0 2 2

(D) Suppose the farmer wants to make a **much larger** orchard with many rows of trees. As the farmer makes the orchard bigger, which will increase more quickly, the number of **apple trees** or the number of **pine trees**?

Explain how you found your answer.

END OF PART 2

GO ON TO PART 3

1 2 0 7 9

PART 3 SHORT RESPONSE SECTION

DIRECTIONS: Answer Questions 9 through 16 in this Test Booklet. Be sure to show all your work in the Test Booklet.

9. Tachi is exactly one year older than Bill.

Let T stand for Tachi's age and B stand for Bill's age.

Write an equation to compare Tachi's age to Bill's age.

T - B

---

10. $a = b - 2$ is a true statement when $a = 3$ and $b = 5$.

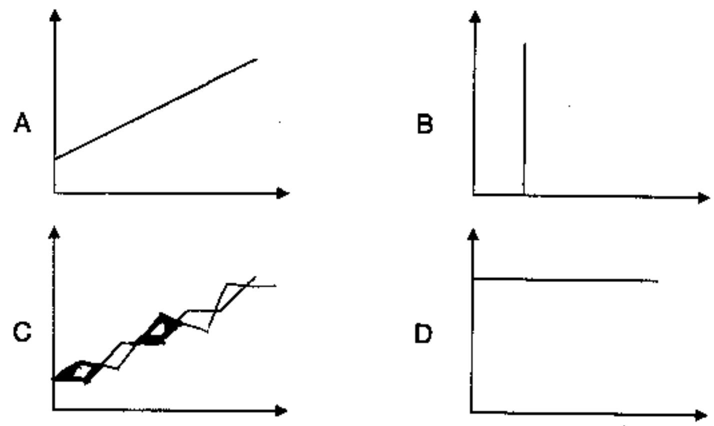
Find a **different** pair of values for a and b that **also** make this a true statement.

a = 2

b = ______________

1 3 0 7 2

11. A small boy was raising a flag up a flagpole.



Write the letter of the graph that best represents the height of the flag above the ground as the small boy raises the flag. D

Explain why you chose this graph.

it shows height by tallness

1 3 6 7 2

12. The table represents a relationship between A and B.

A	B
8	3
12	5
20	9
32	15
?	23

Based upon this relationship, what is the missing number in column A?________________

13.

Age of car (in years)	Value of car
0	$20,000.00
1	10,000.00
2	5,000.00
3	2,500.00

Circle the correct choice (**Is** or **is not**) BELOW and complete the statement.

The relationship between the age of the car and the value of the car **Is / Is not** linear because ____________

____

____

____

____

____

1 3 0 7 2

14. Stella has a phone plan. She pays $10.00 each month plus $0.10 each minute for long distance calls.

One month she made 100 minutes of long distance calls and her bill was $20.00.

The next month she made 300 minutes of long distance calls and her bill was $40.00.

Stella said, "If I talk three times as long it only costs me two times as much!"

Will Stella's rule **always** work? ___________________

Show or explain why or why not.

1 3 0 7 2

15. Maria sells k donuts. Jinko sells five times as many donuts as Maria. They sell the donuts for 25 cents each.

The number of donuts Maria sells is a variable.

A. Name another variable in the problem.________________

B. Name something in the problem that is NOT a variable.

____

16. Find the value(s) of y that make the equation true.
Show how you got your answer.

$19 = 3 + 4y$

END OF ALGEBRA TEST

APPENDIX E. STUDENTS DETAILED ERROR ANSWERS

S_ID	pre test
12450	A-B=1
12030	T+B=total + T1 = tache's older
12035	T+1 and B = -1
12040	T=1year old or B
12041	T+B=a, a = both the ages combined
12042	T=12 & B=13=exactly one year difference
12043	B+9=T
12060	T-B=One
12062	T=Tachi age, B= Bills age
12064	T-B
12071	T-B
12072	T-B=
12073	B?=T?
12074	T-B=7
12076	T is old than B
12077	1/T > 0/B
12078	T-2= B
12079	T-B
12081	T x 1 x B = 369 days older
12082	T=d years old / B = 1 year old
12084	T-B =age between
12086	x = x, you don't have any data to compare
12451	T + 1 = B age
12452	T + B
12456	T- B= answer
12460	T x 2=B
12463	15(Bill) -16 (Tachi) =1 year older; 15 (Bill) / 16 (Tachi)= 1
12464	T+B=25
12514	T+1=B
12515	T: 49+1=50; B: 48+1=49
12516	T+1year =B
12524	T+1=B or B+1=T
12600	T=a and B=1
12603	13(younger)-14 (older)
12604	T/B=1, 1 stands for 1 year, and "/" = over billy
12607	T-B
12608	B=1-T
12691	Billy is one year older than Tachi
12692	T>B
12693	T=19=Ba
12694	1= -14
12695	T>B

12698	L+4=10+B=17
12700	B=T1
12701	x = B/x1 = T
12702	T=8, B=7; Tis 8 years old, B is 7 years old
12703	a=1, B=6, Tis 7 and Bis 6
12704	T-5
12814	T1=B
12843	T+1=B or B+1=T
12844	T+1+B
12847	T-B=n
12851	T-B=difference
12852	T-B=age difference
12853	T+B=T-1
12854	T1-B=1year older
12931	T=B-1
12935	B/T=1
13051	year x 1 year
13054	T-age, B-age
13056	T/B
13057	Tis 365 days older than B
13059	T-B=
13060	B+T=x, T+B=( ), T-B=x, B-T= ( )
13061	T-B=age difference
13063	n-T+n=B
13064	3/4 + 6/4 = 9 (T) / 8 (B)
13080	Tx2=B
13082	T1=B
13087	B-xT
13090	B-xT
13093	B=T+1y; B=Bills age, T= Tacci's age, y=years
13149	T/B=1,
13150	n=how old they are; T: n+1=?; B: n=1=?-1=?
13154	tx1=b
13201	T+B=one year higher than B
13203	T=1=B
13204	a-b=2 (T) - 1 = 1(B)
13205	T1+B=one higher number
13206	T+B=TB
13207	T-B=____
13208	T20-1=B19
13210	T+B=T
13212	Tx1=B
13213	T11-1(1year older 10)=B10
13235	T=Bx1year
13236	T-1<than B
13237	T-B=
13239	T1<B / T-1=B
13261	T=1xB

13263	T=n+1
13264	B= -1T
13267	T=1B
13270	B=1T
13271	T=Ba, a=age
13283	B=1T
13284	T=-1B
13285	T-a=b
13288	T=Bx
13289	T=Bx
13290	T/B
13293	T>B by 1year
13302	T-B=
13303	T/B
13304	T-B=How much older T is than B
13323	T= -B1
13324	T+1 (2); B (1)
13440	t18, B17, B17+1=18
13443	T=-1; B=+2
13444	9(T) - 8(B) = 1 (T)
13447	B+T= or T-B=
13560	T-B
13562	T-B
13563	AxA/a= A=age; a=answer
13564	T/B=X/1
13568	T (13) - B (12) =1
13569	T-B
13571	B/T
13742	B/T
13743	1/B=T/B
13813	TXB=1
13814	Tachie is older than Bill
12045	T-365=B
12052	T=B+B
12057	Tachi was born one year before Bill, so tachi is one year older then Bill.
12465	T--13, B--12
12467	You can do = T-B= years apart (now many)
12469	Bis 14, so T would be 15 yrs. Old, B + 1= T
12472	T+B=Tachi and Bill
12473	T=11, B=12
12479	A-B=1
12529	B+1=T's Age
12617	T/B=yrs
12618	T+1-B=T1 / T=15, B=14
12619	T-n=B, n=1
12620	B/T
12622	B=13, T=14, so Bill is 13 and Tachi is 14.
12708	a=B-1

12710	I can't. You didn't give me Bill's age.
12712	T=21, B=20, 1=T-B
12713	19T+18B
12715	T=12, B=13, B-T=1
12716	T is 1 year ahead of B
12828	BN+1=TN
12829	T=nB, n=older
12830	B-T=comparision
12858	T<B
12946	T-B-1
12948	N=1
12949	B+T-B=T
13065	T=1=n, B=0=y; n= year older, y=year younger
13076	22
13068	T-B=B-T
13069	T=13, B=12
13071	T/B=
13074	T-B=n
13079	n-n=T
13155	tx1=B
13158	13=(1)-12
13160	T-8-1=b=B
13161	T is 10-1 = 9 - B's age
13162	T-B=T10+B11=1
13163	T=13, B=12, T=13-1=B
13165	TB
13167	B+1T-1
13168	Tachi was born 365days before Bill
13215	T=2, B=1, T+B=3, or B+T=3
13216	T/B=1T
13217	B=1year younger than T.
13221	T-B=0
13222	Tachi age - Bill age = 1 year
13223	T=3 yrs old, B= 2 yrs old, I don't really know!
13224	T=B=1
13225	T-B=B's age
13226	B-T=the year difference
13228	B10, T11
13229	T-B=n
13247	T-B=B's age; B+1=T's age
13248	B+360=T
13249	T>B
13250	T+N=1
13305	T-age=B
13307	T1=B-
13311	T=1year older than B
13312	Bx3=T or 3B=T
13313	T<B

13315	T1+B=N	
13316	T-B=amount of years older, TxB=Equals?	
13319	T x 1=B	
13455	B=T-1yr.	
13579	T=1year older than B	
13583	T+1=B+0	
13585	T-B	
13587	T+(B+1), T+(B1)	
13588	T-B=(y)	
13589	T-B	
13759	T1+B	
13760	T+n=B	
13761		4
13762	Tachi is older than Bill	
13763		4
13819	Tachi's age = Bill's age + 1 yr	
13825	2004-y+2004-y=-1y	
13827	T is 1 year older	
13828	B+1yr. = Tachi - 1yr.	
13829	T-B=?	
13875	Bill is 9 years old	
13995	T=9, B=10	
13996	T is one year older than B	
13997	T-B=A	
13998	Tachi is bigger than Bills	
13999	T-1year than Bill	
14000	year	
14003	T>b	
14004	x+B=T	
14005	T=B	
14006	T-B	
14161	T+B=14T	
14162	T+ -------B	
12080	T-B=1 year difference	

S_ID	posttest
13463	T2=B1
13460	T2=36s B
13582	T-b=1 13-12=1 N-n=1 N
13585	T-B=y y=age difference
13337	T-b=1 13-12=1 N-n=1 N
13340	12+11=23
13336	T*B=n
13338	T+B=age
15182	T*1=T age
13337	T-b=1 12-13=1
15178	T-B=age
13827	B+n=age
13825	T+1
13278	T=1x+1
13280	Tahi 1+T Bill B
13283	T=1x+-1
12076	T is older than B
12077	T>B T-b=1
12080	TXB=T+b
12084	B+T
15150	T-B=
15198	B+B=B
12081	T+b=1
13214	T=1year=B
13217	T= exactly one year older than B
13221	T-B=A
13223	B=1 T=11 B+1=11
13219	Tachi ten Bil 9
13220	Bx1=T or T-1=B
12825	T>B
12826	T is B-1
15194	TX1=B+2=one
12827	Tx1=B
12829	TXB=1
12832	T-1=b Tachi-1=Bill
15049	T age is increased 1 time than B's age
12618	y=T1x+B
12621	T=x+B+x
12629	Y=Tx+12 Y=Bx+11
15195	T-B=
12619	B=+-1
13067	12/32=2.6
13071	L=1XB
13072	T=1 year older

13065	T=+B=1x
13073	T=9 B=8
13074	Bx12 months=T
13076	B-T=How much older Tachi is than Bill
13079	T-B=Age difference
15064	B+T=1
15192	T+1=-1B
12529	B's age +1ys=T's age
12533	B+1+
15147	T+B=1
12867	T-B=age
12712	T-B=1 year older
12473	T=1 to B=0
12474	T>B
12479	B=A-2
12465	T=11 B=10 years
12467	T's age +1 = B's age
12471	T-B=difference
12472	B+1=T Age
12477	T-B=Age
12052	T is 1 than B
15144	T?
12053	T1=-B
12055	Tx+1=B
12057	Tage =B age
15076	T1=-B
13101	9/16=0.5 or 16/9=1.7
13247	How many years apart
13312	Tn+B
13319	age=T+1+B
13157	T=1B
13155	T=Bx1
13158	T-B =age difference
13162	T>B
13167	T=1y=B
13441	T-B=D D=difference in age
13444	T-B=x
13445	BT=1
13443	B-y=T B=Bill's y=year T=Tach's
13744	T1+B
13565	T+1-B
13571	T1=B
13322	T+B
15090	Tachi (12) is older than Bill by one
15080	T-B(T/B)=x
13813	(T)1-B
13800	T12=B
13802	B+365=T

13803	T>B
13814	T=10 B=9
13271	T=15 B=14 y=T-B
13273	4+1 (1)
12062	TX1+B
12071	TXB
12063	T1+B1
12064	T+1=B
15138	1T-B
13200	T=Bage
13225	B=1T
13227	B+1g=T
12810	T=6 B=5 T=Bx6=s
12814	nT+1-nB
12812	T X B=c (conpared answer)
15127	T=T1 B
12603	y=T*n+B*n-1
13054	t-b
13057	T=2=1years=B+1+ 1 2
15056	T=one year older /b=age
15122	B=1t
13291	T>1=B
15094	TA+TB=n
13249	B=T1
13243	Tx1=b
13259	T+B+1=age
13094	T+1-B
13083	T/B
13080	A=T-B
13090	T=1 B=0
13088	TxB=
15141	T=Bx1
12035	T+1=B-1=
12457	Bx1 =T
12456	T-B=answer
12453	Tx1-B
12462	B+1=T+1

APPENDIX F. TOP 10% AND BELOW 10% STUDENTS

Top 10% students according to posttest scores

Student ID	Teacher ID	Grade	Pretest	Posttest
1113459	42	7	13	23
1212050	99	8	17	24
1212059	99	8	15	22
1212512	4	7	18	20
1212520	4	7	5	21
1212521	4	7	17	21
1212532	4	7	15	22
1212846	6	7	18	20
1212850	6	7	15	22
1212851	6	7	13	21
1212866	6	7	9	20
1212868	6	7	11	21
1212931	5	7	20	22
1212934	5	7	16	21
1212941	5	7	11	24
1212942	5	7	14	22
1212943	5	7	16	25
1212944	5	7	14	20
1212951	5	7	15	20
1212953	5	7	9	21
1213059	88	8	6	20
1213239	101	8	16	21
1213266	8	8	7	21
2212044	99	8	11	20
2212517	4	7	17	22
2212519	4	7	16	20
2212522	4	7	17	20
2212523	4	7	12	22
2212524	4	7	15	21
2212531	4	7	15	26
2212847	6	7	13	21
2212858	6	7	12	22
2212862	6	7	13	24
2212865	6	7	9	20
2212940	5	7	13	21
2212945	5	7	26	22
2212946	5	7	11	20
2212947	5	7	17	26
2213142	103	7	8	21
2213164	103	7	12	20

2213296	102	8	17	25
2213311	102	8	17	20

Below 10% students according to posttest scores

Student ID	Teacher ID	Grade	Pretest	Posttest
1213064	88	8	6	1
2212699	24	7	6	2
1212450	59	7	5	2
1212052	99	8	8	2
2212827	69	7	6	3
1113445	42	7	4	3
1212454	59	7	2	3
1212704	24	7	1	3
1212035	99	8	5	3
1213067	88	8	5	3
1213057	88	8	4	3
1212062	10	8	4	3
1213301	102	8	4	3
1213217	65	8	3	3
1213063	88	8	2	3
1212474	59	7	6	4
1213071	88	8	6	4
2213219	65	8	5	4
1212073	10	8	5	4
2212063	10	8	5	4
1212046	99	8	4	4
1213054	88	8	4	4
2213082	110	8	4	4
2213091	110	8	3	4
2212057	99	8	3	4
1213088	110	8	2	4
2213101	110	8	1	4
1212473	59	7	8	5
1212815	69	7	8	5
2212716	24	7	6	5
2212812	69	7	4	5
2212829	69	7	3	5
1212071	10	8	5	5
1212040	99	8	5	5
2213319	102	8	4	5
2213315	102	8	3	5
1213102	110	8	2	5
2212053	99	8	2	5

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