APPLICATIONS OF ALGEBRAIC GEOMETRY TO OBJECT/IMAGE RECOGNITION

A Dissertation

by

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ABSTRACT

Applications of Algebraic Geometry to Object/Image Recognition. (August 2007)

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In recent years, new approaches to the problem of Automated Target Recognition using techniques of shape theory and algebraic geometry have been explored. The power of this shape theoretic approach is that it allows one to develop tests for object/image matching that do not require knowledge of the object's position in relation to the sensor nor the internal parameters of the sensor. Furthermore, these methods do not depend on the choice of coordinate systems in which the objects and images are represented.

In this dissertation, we will expand on existing shape theoretic techniques and adapt these techniques to new sensor models. In each model, we develop an appropriate notion of shape for our objects and images and define the spaces of such shapes. The goal in each case is to develop tests for matching object and image shapes under an appropriate class of projections. The first tests we develop take the form of systems of polynomial equations (the so-called object/image relations) that check for *exact* matches of object/image pairs. Later, a more robust approach to matching is obtained by defining metrics on the shape spaces. This allows us in each model to develop a measure of "how close" an object is to being able to produce a given image. We conclude this dissertation by computing a number of examples using these tests for object/image matching.

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CHAPTER I

INTRODUCTION

A.   Target Recognition

A crucial first step in the problem of Automated Target Recognition (ATR) is to understand how data extracted from a single image of an object can be used to determine information about the geometry of the object. Unfortunately, without knowledge of the object's position in relation to the sensor and without knowledge of certain sensor parameters (such as focal length in the case of an optical camera) efficiently recognizing an object from this geometric information becomes a very difficult task, which has forced existing methods to rely upon this information.

What we would like is an approach that is invariant to the viewpoint and internal parameters of the sensor. The methods that are currently being used compare information taken from the image against templates that have been created for each possible viewpoint - approximating the infinite number of possibilities by a finite set of views [19]. In a strict sense, these methods are not viewpoint-invariant, and furthermore, are extremely computationally expensive.

Through the techniques of shape theory and algebraic geometry, an alternative has been made available which uses only information about the intrinsic geometry of the object and its image. This new approach does not require a priori knowledge of the sensor's viewpoint in relation to the object nor does it depend on the sensor's internal parameters. In this dissertation, we expand on existing shape theoretic methods in ATR and adapt these techniques to new target recognition models.

The journal model is SIAM Journal on Applied Mathematics.

B.   A Quick Review of Shape Theory

Since the reader may not be familiar with this branch of mathematics, we will give a brief introduction to shape theory before beginning our discussion of the mathematical aspects of target recognition. Additional details can be found in [16].

Shape theory has its beginnings in statistics with the work of David Kendall (see [9], [10]). Being concerned with archaeological applications, Kendall was interested in analyzing differences in shapes of artifacts. His approach was to represent an object (or an image of an object) by a finite set of points corresponding to prominent features called *landmarks*. In [16], landmarks are defined to be points chosen "to mark the location of important features and to give a partial geometric description of the image or object." For example, in Fig. 1 we see an Iron Age brooch represented by four feature points.
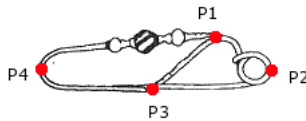
Fig. 1. Four landmarks on an Iron Age brooch (modified from [16]).

By representing objects and images by collections of feature points, Kendall reduced the problem of analyzing the overall shapes of objects and images to the problem of analyzing "shapes" of configurations of finitely many points.

What does Kendall mean by the "shape" of a configuration of points? Intuitively, we think of figures as having the same shape if they differ by a rotation, a translation or a dilation (scale factor). This is illustrated for triangles in Fig. 2.
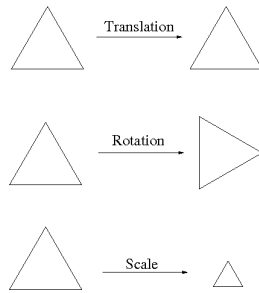
Fig. 2. All of these triangles have the same shape.

Kendall uses this natural concept to develop his idea of shape for configurations of points. In [10], Kendall informally defines the shape of a set of data points to be "what is left when the differences which can be attributed to translations, rotations, and dilations have been quotiented out." In other words, two configurations of points have the same shape if they differ by a rotation, translation, or dilation.

To make this more precise we make the following definition.

Definition B.1. A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *similarity transformation* if it has the form

$$(1.1) \qquad\qquad\qquad T(p) = \lambda A p + \underline{c}$$

where $\lambda > 0$ is a real number, $A \in SO(n)$ and $\underline{c} \in \mathbb{R}^n$. The group of similarity transformations on $\mathbb{R}^n$ is denoted $Sim(n)$.

Kendall's notion of shape is now given as follows.

Definition B.2. Two configurations $P_1, P_2, \ldots, P_k$ and $Q_1, Q_2, \ldots, Q_k$ of points in $\mathbb{R}^n$ *have the same shape* if there is a similarity transformation $T \in Sim(n)$ such that $T(P_i) = Q_i$ for $i = 1, \ldots, k$. The *shape* of a configuration of k points in $\mathbb{R}^n$ is its equivalence class under the action of $Sim(n)$ on $\mathbb{R}^n$.

One caveat that we should point out is that this definition of shape depends on the ordering of the points. For example, suppose we have two configurations $P_1 = (0,0), P_2 = (1,0), P_3 = (0,2)$ and $Q_1 = (0,0), Q_2 = (0,2), Q_3 = (1,0)$ so that the configurations consist of the same three points but with a different labeling of those points. Then, the similarity transformation that sends Q_1 to P_1 and Q_2 to P_2 is

$$(1.2) \qquad T(p) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p$$

However, we now see that $T(Q_3) = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \neq P_3$. Hence, by Kendall's definition, the configurations P_1, P_2, P_3 and Q_1, Q_2, Q_3 do not have the same shape even though they are equivalent as sets of points.

Some work has been done by Mirelle Boutin and Gregor Kemper in [2] and David Sepiashvili in [14] addressing the action of the permutation group $\mathfrak{S}(k)$ on configurations of k points in $\mathbb{R}^n$, but otherwise this problem remains to a large extent open.

Now that we have an understanding of Kendall's concept of shape, we would like to better understand the space of shapes of configurations of k points in $\mathbb{R}^n$. To avoid going into too much detail, we will here give Kendall's construction for configurations of k points in $\mathbb{R}^2$. The construction for the more general case of k points in $\mathbb{R}^n$ can be found in detail in [10] and [16].

Let $P_i = (x_i, y_i)$ for $i = 1, \ldots, k$ be a configuration of k feature points in $\mathbb{R}^2$ which we will represent as a single vector

$$(1.3) \qquad (P_1, P_2, \ldots, P_k) \in \left(\mathbb{R}^2 \right)^k .$$

To determine the shape of this configuration of landmarks, we must remove the information corresponding to rotation, translation, and scale.

To remove information coming from translations, we first compute the *centroid* of our data set which is given by

$$(1.4) \qquad \overline{P} = \frac{1}{n} \sum_{i=1}^{k} P_i.$$

We then standardize our configuration with respect to translation so that its centroid is $\underline{0} \in \mathbb{R}^2$ giving us the new configuration vector

$$(1.5) \qquad \left(P_1 - \overline{P}, P_2 - \overline{P}, \ldots, P_k - \overline{P}\right) \in \mathbf{F}^{2k-2}$$

which lies in the $(2k - 2)$-dimensional linear subspace

$$(1.6) \qquad \mathbf{F}^{2k-2} = \left\{ (P_1, P_2, \ldots, P_k) \in \mathbb{R}^{2k} \mid \sum_{i=1}^{k} P_i = \underline{0} \right\} \subset \mathbb{R}^{2k}.$$

We then standardize with respect to scale by dividing this new vector by the *centroid size* of the configuration. That is, we scale this vector so that it has length 1 in the usual norm on $\mathbb{R}^{2k}$ giving us the standardized configuration vector

(1.7)
$$\tau(P_1, P_2, \ldots, P_k) = \left(\frac{P_1 - \overline{P}}{\sqrt{\sum_{i=1}^{k} ||P_i - \overline{P}||^2}}, \frac{P_2 - \overline{P}}{\sqrt{\sum_{i=1}^{k} ||P_i - \overline{P}||^2}}, \ldots, \frac{P_k - \overline{P}}{\sqrt{\sum_{i=1}^{k} ||P_i - \overline{P}||^2}} \right).$$

It should be noted that for this construction to make sense, the feature points $P_1, P_2, \ldots, P_k$ must not be coincident, which will be a reasonable assumption in almost all applications. Therefore, we will henceforth exclude this degenerate case from our discussion.

Having made the assumption that at least two of the P_i are distinct, we can now

see that the vector τ lies in the $(2k-3)$-dimensional sphere

$$(1.8) \qquad\qquad \mathbf{S}^{2k-3}_* = \mathbf{F}^{2k-2} \cap \mathbf{S}^{2k-1}$$

We use the subscript $*$ to indicate that this is not the usual unit sphere in $\mathbb{R}^{2k-2}$, but rather is considered to be lying in a $(2k-2)$-dimensional linear subspace of $\mathbb{R}^{2k}$. In [10], Kendall makes the following definitions regarding τ and $\mathbf{S}^{2k-3}_*$.

Definition B.3. We will call the vector $\tau(P_1, \ldots, P_k)$ the *pre-shape* of the configuration $P_1, \ldots, P_k$ and we will refer to the sphere $\mathbf{S}^{2k-3}_*$ as the *pre-shape space* for configurations of k points in $\mathbb{R}^2$.

The final step is to remove information corresponding to rotations from our pre-shapes. To do this, we should observe that the group of rotations on $\mathbb{R}^2$ is $SO(2)$ and that the action of $SO(2)$ on the pre-shape space $\mathbf{S}^{2k-3}_*$ is simply the action induced by the usual action of $SO(2)$ on $\mathbb{R}^2$. In other words, for a configuration $P_1, \ldots, P_k$ in $\mathbb{R}^2$ and a transformation $A \in SO(2)$

$$(1.9) \qquad\qquad A \cdot \tau(P_1, \ldots, P_k) = \tau(AP_1, \ldots, AP_k).$$

We are now able to see that the shape space for configurations of k points in $\mathbb{R}^2$ (not all coincident) is the quotient space

$$(1.10) \qquad\qquad \Sigma^k_2 = \mathbf{S}^{2k-3}_* / SO(2)$$

under the action of $SO(2)$ on $\mathbf{S}^{2k-3}_*$. Small points out in [16] that the equivalence classes under this action are circles on $\mathbf{S}^{2k-3}_*$.

This definition of the shape space, however, has little meaning until we define a metric on it, which will be the primary tool for comparing shapes of configurations of feature points. Surprisingly, in [10] the natural metric that Kendall defines on the

shape space Σ_2^k is computed directly from pre-shape representatives without having to coordinatize Σ_2^k in any way.

Since the equivalence classes under the action of $SO(2)$ on the pre-shape space are circles, Kendall defines the distance between two shapes to be the minimum distance between equivalence classes in $\mathbf{S}_*^{2k-3}$ with it usual great circle metric.

The distance between two pre-shapes $\tau_1, \tau_2 \in \mathbf{S}_*^{2k-3}$ is given by

$$(1.11) \qquad\qquad d(\tau_1, \tau_2) = \cos^{-1} \langle \tau_1, \tau_2 \rangle \, ,$$

and so the distance between the shapes of these two pre-shapes then becomes

$$(1.12) \qquad d([\tau_1], [\tau_2]) = \inf\{\, d(\gamma_1, \gamma_2) \mid \gamma_1 \in [\tau_1], \gamma_2 \in [\tau_2] \,\}.$$

To evaluate this metric, we make the usual identification of $\mathbb{R}^2$ with $\mathbb{C}$. If for a configuration $P_1, \ldots, P_k$ of points in $\mathbb{R}^2$, we think of the P_i as complex numbers, the pre-shape of the configuration becomes

(1.13)

$$\tau(P_1, P_2, \ldots, P_k) = \left(\frac{P_1 - \overline{P}}{\sqrt{\sum_{i=1}^{k} |P_i - \overline{P}|^2}}, \frac{P_2 - \overline{P}}{\sqrt{\sum_{i=1}^{k} |P_i - \overline{P}|^2}}, \ldots, \frac{P_k - \overline{P}}{\sqrt{\sum_{i=1}^{k} |P_i - \overline{P}|^2}} \right)$$

viewed as a vector in $\mathbb{C}^k$ rather than $\mathbb{R}^{2k}$.

Now for two shapes $\sigma_1 = [\tau_1]$ and $\sigma_2 = [\tau_2]$, we write

$$(1.14) \qquad\qquad \tau_i = (\tau_{i1}, \tau_{i2}, \ldots, \tau_{ik}), \ i = 1, 2$$

where the τ_{ij} are complex entries, and as we see in [10] and [16], the distance between σ_1 and σ_2 is

$$(1.15) \qquad\qquad d(\sigma_1, \sigma_2) = \cos^{-1} \left| \sum_{i=1}^{k} \tau_{1i} \overline{\tau}_{2i} \right|$$

where $\overline{\tau}_{2i}$ is the complex conjugate of τ_{2i}. In [10], Kendall calls the metric given by

1.15 the *Procrustean metric* on Σ_2^k.

In [9], Kendall asserts (and later shows in more detail in [10]) that the shape space Σ_2^k of k-tuples of points in $\mathbb{R}^2$ is isomorphic to the complex projective space $\mathbb{P}_{\mathbb{C}}^{k-2}$. Moreover, the Procrustean metric on Σ_2^k is equivalent to the usual Fubini-Study metric on $\mathbb{P}_{\mathbb{C}}^{k-2}$.

C. Shape Theory and Object/Image Recognition

The techniques of shape theory apply in a very natural way to the problem of Automated Target Recognition. The approach is to first represent an object by a finite set of feature points in 3-space (in some cases $\mathbb{R}^3$ in others $\mathbb{P}_{\mathbb{R}}^3$) which we will call an *object configuration*. For example, a jet might be represented by choosing feature points corresponding to the nose, wingtips, and stabilizers as is shown in Fig 3.
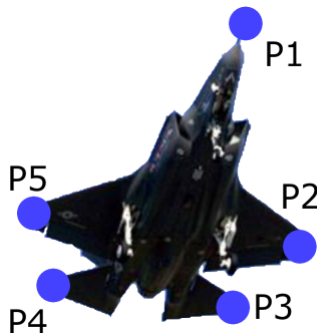
Fig. 3. An F-35 fighter jet represented by 5 landmarks (modified from [12]).

When an image is generated of the object, the object configuration is projected onto a plane in some fashion - the type of projection depending on the type of sensor used to produce the image. For our purposes we will be primarily concerned with the pinhole camera model, in which case we will consider the focal point projection illustrated in Fig. 4. A projection, T, of this type maps a point, P, in 3-space onto
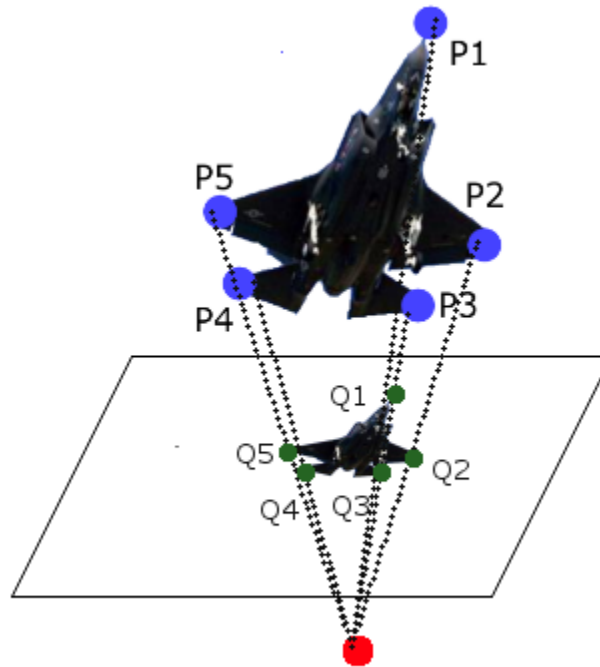
Fig. 4. An image of the jet in Fig. 3 generated by the focal point projection (modified from [12]).

a plane, L, by intersecting the line, l, that passes through the point P and the focal point F with L.

We will refer to the resulting configuration of points obtained by projecting the object configuration as an *image configuration*. More generally, we will use this term to refer to any configuration of points in the plane obtained as feature points on some image.

The goal is relate object configurations and image configurations under projections appropriate to the sensor being modeled. We want a method of target recognition that is invariant to changes in an object's position and orientation in relation to the sensor and invariant to the choice of coordinate system in which we represent

our object and image. To achieve this invariance, the techniques we will use relate the shapes of object and image configurations. As a result, the methods only use information coming from the intrinsic geometry of our configurations.

By introducing projections into our analysis, our notion of shape will vary with the type of sensor we are modeling. In this dissertation, we will be primarily concerned with configurations of points modulo the action of either the Affine group or the Projective General Linear group. However, in some cases we will be interested in the classical case involving the Similarity group.

It was David Jacobs in [8] first introduced the idea of matching shapes of 3D and 2D configurations under projections. The theory in the context of ATR was further developed by Asmuth, Stiller, and Wan in [20, 22, 21], Stiller in [17, 18], and Arnold and Stiller in [19]. This dissertation continues in their work and adapts the shape theoretic approach to new sensor models.

In Chapter II, we examine the Generalized Weak Perspective model for object/image recognition laid out in [18]. In this case we consider our configurations of points to be in affine space ($\mathbb{R}^n$ or $\mathbb{C}^n$) and analyze their shapes modulo the action of the group of affine transformations and the relationships that exist between object and image configurations under *generalized weak perspective* projections (which we will define in that chapter). We begin by reviewing the construction of the appropriate shape spaces. We then present the equations that relate the shapes of object and image configurations (given in [18]) and follow that by discussing the metrics on the shape spaces. We end the chapter by introducing three notions of distance between an object and an image shape and proving that they are all equivalent.

In Chapter III, we adapt the techniques of the Generalized Weak Perspective case to the Full Perspective case which more accurately models the production of an image by an optical camera. Here we consider our configurations to be in projective

space ($\mathbb{P}^n_{\mathbb{R}}$ or $\mathbb{P}^n_{\mathbb{C}}$) and investigate the relationship between the shapes of object and image configurations (modulo the action of the Projective General Linear group) under projection from a point in projective space. This projection is precisely the focal point projection used by an optical camera to produce an image. We begin this chapter by associating to each configuration of points a projective subvariety of a Grassmannian (embedded in some projective space) and then analyzing the structure of these varieties (called *shape varieties*). We follow this with a discussion of the matching object/image equations, presented in a manner that depends only on the shapes of the object and image configurations.

We conclude the chapter by investigating ways to embed our object and image shape spaces into some projective space. To do this, we need to construct a moduli space for our shape varieties - that is, we need to construct a map from the projective space containing the shape varieties to some other projective space that effectively collapses each shape variety to a single point and that sends distinct shape varieties to distinct points. We first consider the Chow embedding (see [15]). However, the dimension of the final projective space is so high that this method would be nearly useless in practice. We instead develop a rational map into a much lower dimensional projective space that has all the desired properties by composing a Veronese Map (see [6]) with a projection into a projective space of lower dimension.

In Chapter IV, we will briefly examine the Conformal case in which we consider shapes in the classical sense (that is, configurations modulo similarity transformations) and consider conformal projections (orthographic projection followed by a dilation). This case is much more useful in modeling radar image production. We will begin the chapter by constructing the appropriate shape spaces and follow that by computing the object/image equations for this case. We conclude the chapter with an analysis of the metrics on the shape spaces.

The dissertation will close with two final sections: Chapter V and and Appendix. Chapter V will give a summary our results and conclusions and will give a brief glimpse of avenues for further study. The Appendix will consist of detailed examples using code written for the computer algebra package Macaulay2 and will include the actual code used in these computations.

CHAPTER II

THE GENERALIZED WEAK PERSPECTIVE (AFFINE) MODEL

A.   Generalized Weak Perspective Projections

In this chapter, we will concern ourselves with the problem of identifying images which have been produced by an optical camera. As previously indicated, in our mathematical model of this problem, we represent an object by a configuration of feature points in 3-D, and we represent an image by a configuration of feature points in 2-D. For this first model, we will consider our points to be in affine space - that is, we will consider object configurations as points in $\mathbb{A}^3_{\mathbb{R}} = \mathbb{R}^3$ and we will consider image configurations as points in $\mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2$. For reasons that we will shortly see, we represent points $P \in \mathbb{A}^n_{\mathbb{R}}$ in the form

$$(2.1) \qquad P = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}.$$

Recall that in Chapter I, we noted that this type of sensor produces an image by a focal point projection (see Fig. 4). Unfortunately, this focal-point projection is nonlinear as a map from $\mathbb{A}^3_{\mathbb{R}}$ to $\mathbb{A}^2_{\mathbb{R}}$. For example, suppose the map $T : \mathbb{A}^3_{\mathbb{R}} \to \mathbb{A}^2_{\mathbb{R}}$ is a projection from the point $F = (0,0,2)$ onto the plane $R^2 = \{(x,y,z) \in \mathbb{A}^3_{\mathbb{R}} | z = 0\}$ (we identify R^2 with $\mathbb{A}^2_{\mathbb{R}}$ by dropping the third coordinate) and consider the points $P = (0,0,1)$ and $Q = (0,1,0)$. Then we see in Fig. 5 that $T(P) + T(Q) = (0,1,0)$ but $T(P+Q) = (0,0,2)$. Hence, T is not linear.

Another problem arises from the fact that for a given projection T from a point F onto a plane $L \subset \mathbb{A}^3_{\mathbb{R}}$, there are points in $\mathbb{A}^3_{\mathbb{R}}$ (other than P) for which T is undefined.
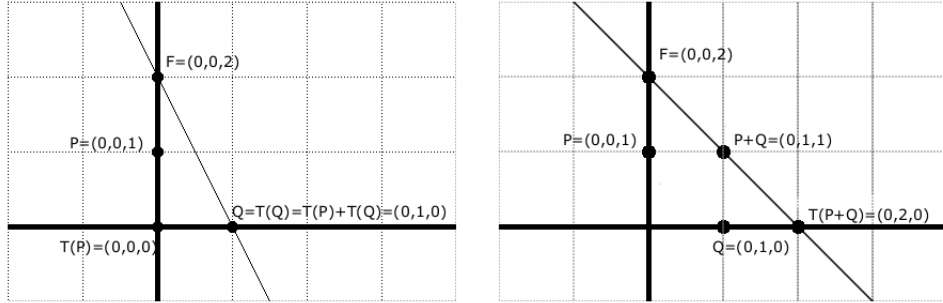
Fig. 5. $T(P) + T(Q)$ and $T(P + Q)$ drawn in the plane $A = \{(x, y, z)|x = 0\}$.

Namely, T is undefined for points lying in the plane $K \subset \mathbb{A}^3_{\mathbb{R}}$ passing through P that is parallel to L since the line m through F and any given point $P \in K$ will not intersect L. If we consider T to be the projection above from the point $F = (0, 0, 2)$ onto the plane R^2, then T is undefined for points lying in the plane $K = \{(x, y, z)|z = 2\}$ as shown in Fig. 6 below.
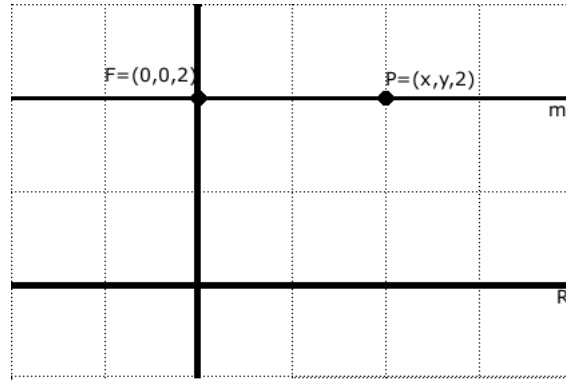
Fig. 6. The line m through the focal point F and a point $P \in K$ does not intersect R^2 (drawn in the plane $A = \{(x, y, z)|x = 0)$.

We avoid these problems by choosing to approximate focal point projections by *generalized weak perspective* (GWP) *projections*. When we represent points in $\mathbb{A}^n_{\mathbb{R}}$ in

the form 2.1, these projections (as maps from $\mathbb{A}^3_{\mathbb{R}}$ to $\mathbb{A}^2_{\mathbb{R}}$) may be written as matrices of the form

(2.2)
$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where T has maximal rank. The advantage to using this type of projection is that the map is now linear (and more importantly regular) and is well-defined on $\mathbb{A}^3_{\mathbb{R}}$ which allows us to attack the problem of object/image recognition using this model in an algebraic geometric context.

In the proceeding sections, we will follow the presentation in [18] to (1) develop the appropriate shape spaces for this model, (2) give necessary and sufficient conditions for an image configuration to be a GWP projection of an object configuration, and (3) define the natural metrics on the shape spaces. We will then define three separate notions of distance between an object shape and an image shape and prove that these three "metrics" are equivalent.

B.   Shape in the GWP Model

We now want to define an appropriate notion of shape for our object and image configurations in this model. That is, we want to define a group of transformations on $\mathbb{A}^n_{\mathbb{R}}$ whose action on configurations of k points in $\mathbb{A}^n_{\mathbb{R}}$ allows us to relate shapes of object and image configurations under GWP projections.

To begin, suppose the configuration $Q_1, Q_2, \ldots, Q_k \in \mathbb{A}^2_{\mathbb{R}}$ is the image of an object configuration $P_1, P_2, \ldots, P_k \in \mathbb{A}^3_{\mathbb{R}}$ under a GWP projection T i.e. $T(P_i) = Q_i$

for $i = 1, \ldots, k$. Now suppose that we move $Q_1, Q_2, \ldots, Q_k$ by a transformation

$$(2.3) \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

to another configuration $Q'_1, Q'_2, \ldots, Q'_k \in \mathbb{A}^2_{\mathbb{R}}$, and suppose that we move $P_1, P_2, \ldots, P_k$ to the configuration $P'_1, P'_2, \ldots, P'_k$ by a transformation

$$(2.4) \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

We should first note that since we want the matrix A to be a transformation on $\mathbb{A}^2_{\mathbb{R}}$, we must have that for every point $q = (q_1, q_2, 1) \in \mathbb{A}^2_{\mathbb{R}}$, Aq is of the form $(c_1, c_2, 1)$. In particular, the third entry must be equal to 1 giving us that

$$(2.5) \qquad q_1 a_{31} + q_2 a_{32} + a_{33} = 1$$

for all $q_1, q_2 \in \mathbb{R}$. It is then easy to see that $a_{31} = a_{32} = 0$ and $a_{33} = 1$. Thus, A must be of the form

$$(2.6) \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, since B must be a transformation on $\mathbb{A}^3_{\mathbb{R}}$, it must be of the form

$$(2.7) \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, since we want the set of allowable transformations to be a group, A and B should be invertible. We now define these groups of transformations in complete generality.

Definition B.1. Let K be a field. Then a map $M : \mathbb{A}^n_K \to \mathbb{A}^n_K$ is an *affine transformation* on $\mathbb{A}^n_K$ if it is of the form

$$(2.8) \qquad M(p) = Sp + \underline{v}$$

where S is an invertible $n \times n$ matrix over K and $\underline{v} \in \mathbb{A}^n_K$. The group of affine transformations on $\mathbb{A}^n_K$ is denoted $\mathrm{Aff}_K(n)$ or simply $\mathrm{Aff}(n)$ when the field is understood.

If we represent points in $\mathbb{A}^n_K$ in the form of 2.1 then affine transformations take the form

$$(2.9) \qquad M = \begin{pmatrix} & & & c_1 \\ & S & & \vdots \\ & & & c_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $S \in GL(n, K)$ and $c_i \in K$ for $i = 1, \ldots, n$. This is precisely the form that our transformations A and B take. Thus, we should consider two configurations (object or image) to have the same shape if they differ by an affine transformation.

Definition B.2. We will refer to the equivalence class of a configuration of k points

in $\mathbb{A}_{\mathbb{R}}^n$ under the action of $\text{Aff}_{\mathbb{R}}(n)$ as the *affine shape* of the configuration.

We should note that the transformation M in 2.9 acts on $\mathbb{A}_k^n$ by a change of basis by the matrix S followed by a repositioning of the origin to the point $(c_1, \ldots, c_n) \in \mathbb{A}_K^n$ i.e. an affine transformation corresponds to a change of affine coordinate system on $\mathbb{A}_K^n$. Since we want our method of object/image recognition to be independent of the choice of coordinate system in which we represent our points, we can further see that this is the "right" definition of shape for this model.

Returning to the setup at the beginning of this section, we observe that since $Q_1, \ldots, Q_k$ is the image of $P_1, \ldots, P_k$ under the GWP projection T, we must have that $Q_1', \ldots, Q_k'$ is the image of $P_1', \ldots, P_k'$ under the map ATB^{-1}. The important point to note here is that since A and B^{-1} are affine transformations, it is easy to see that ATB^{-1} is in fact a GWP projection. Thus we are able to relate affine *shapes* of object and image configurations (modulo affine transformations) under generalized weak perspective projections i.e. matching is well-defined on the level of equivalence classes.

C.   The Affine Shape Spaces

Having defined shape in the generalized weak perspective model, we will now construct the corresponding shape spaces. We will do this in complete generality for configurations of k points in $\mathbb{A}_{\mathbb{R}}^n$. The 3-D object and 2-D image shape spaces then become specific cases of the more general results. It should be noted that all of the following constructions are valid when working in $\mathbb{A}_{\mathbb{C}}^n$.

Let $P_i = (x_{1i}, x_{2i}, \ldots, x_{ni}, 1)$ for $i = 1, \ldots, k$ be a configuration of points in $\mathbb{A}_{\mathbb{R}}^n$. In our construction, we must assume that the P_i do not all lie in a single hyperplane. This is a reasonable assumption since a configuration of points lying in a hyperplane in $\mathbb{A}_{\mathbb{R}}^n$

could be considered to be a configuration of points in $\mathbb{A}_{\mathbb{R}}^{n-1}$. For instance, if an object

configuration lies in a single plane in $\mathbb{A}_{\mathbb{R}}^3$ then in some sense, that configuration is an

image and does not represent a real 3D object. Thus we will exclude this degenerate

case and henceforth assume that our configurations are noncoplanar. We will also

need to assume that $k \geq n + 1$. This leaves us with the task of understanding the

quotient space $\left((\mathbb{A}_{\mathbb{R}}^n)^k - V\right)/\mathrm{Aff}(n)$ where $k \geq n + 1$ and $V \subset (\mathbb{A}_{\mathbb{R}}^n)^k$ is the locus of

coplanar configurations.

We represent our configuration $P_1, \ldots, P_k$ as the $(n+1) \times k$ matrix

$$(2.10) \qquad M(P_1, \ldots, P_k) = \begin{pmatrix} x_{11} & x_{12} & & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nk} \\ 1 & 1 & & 1 \end{pmatrix}$$

obtained by letting the coordinates of our points be the columns of the matrix.

We will refer to this matrix as a *configuration matrix*. We then identify our

configuration with a $(k-n-1)$-dimensional subspace $K^{k-n-1} \subset \mathbb{A}_{\mathbb{R}}^k$. In particular,

K^{k-n-1} is the null space of the matrix $M(P_1, \ldots, P_k)$ viewed as a map from $\mathbb{A}_{\mathbb{R}}^k$ to

$\mathbb{A}_{\mathbb{R}}^{n+1}$. To see that K^{k-n-1} has dimension $k - n - 1$, note that since the points of

our configuration $P_1, \ldots, P_k$ are noncoplanar, the determinant of at least one of the

$(n+1) \times (n+1)$ minors of the configuration matrix, M, is nonzero. This means that

M has maximal rank $n+1$ and hence K^{k-n-1} has dimension $k - (n+1) = k - n - 1$.

The important thing to notice is that if we apply an affine transformation $A \in$

$\mathrm{Aff}_{\mathbb{R}}(n)$ to the configuration $P_1, P_2, \ldots, P_k$ we obtain a new $(n+1) \times k$

$$(2.11) \qquad M' = \begin{pmatrix} x'_{11} & x'_{12} & & x'_{1k} \\ x'_{21} & x'_{22} & \cdots & x'_{2k} \\ \vdots & \vdots & & \vdots \\ x'_{n1} & x'_{n2} & & x'_{nk} \\ 1 & 1 & & 1 \end{pmatrix} = AM,$$

but the null space of M' is exactly K^{k-n-1}, the null space of M. Thus we are able to associate to the affine *shape* of a configuration of k points in $\mathbb{A}_{\mathbb{R}}^n$ the $(k-n-1)$-plane K^{k-n-1}. Since $K^{k-n-1} \subset H^{k-1} = \left\{ (v_1, \ldots, v_k) \mid \sum_{i=1}^k v_i = 0 \right\}$, we are therefore able to identify affine shapes of configurations with points in the Grassmannian $Gr(k - n - 1, H^{k-1})$ of $(k-n-1)$-dimensional subspaces of the hyperplane $H^{k-1} \subset \mathbb{A}_{\mathbb{R}}^k$.

Theorem C.1. *The shape space* $\left((\mathbb{A}_{\mathbb{R}}^n)^k - V \right) / Aff(n)$ *for configurations of k points in $\mathbb{A}_{\mathbb{R}}^n$ is the Grassmannian $Gr(k - n - 1, H^{k-1})$.*

Proof. Define $\phi : \left((\mathbb{A}_{\mathbb{R}}^n)^k - V \right) / \mathrm{Aff}(n) \to Gr(k-n-1, H^{k-1})$ by sending the shape of a configuration to the null space of its corresponding configuration matrix. We have already seen that this map is well-defined on the quotient space $\left((\mathbb{A}_{\mathbb{R}}^n)^k - V \right) / \mathrm{Aff}(n)$.

To see that ϕ is surjective, let $K \in Gr(k - n - 1, H^{k-1})$. Then $K \subset \mathbb{A}_{\mathbb{R}}^k$ is the intersection of $n + 1$ independent hyperplanes given by the polynomial equations in

the variables $x_1, \ldots, x_k$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = 0$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k = 0$$

(2.12)
$$x_1 + x_2 + \cdots + x_k = 0.$$

In other words, K is the null space of the matrix

(2.13)
$$M_K = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Thus K is the image of the shape of the configuration $P_i = (a_{1i}, \ldots, a_{ni}, 1)$ under the map ϕ, and hence ϕ is surjective.

Now suppose we have two configurations $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ so that $\phi([P_1, \ldots, P_k]) = \phi([Q_1, \ldots, Q_k]) = K \in Gr(k - n - 1, H^{k-1})$. Since $K \in Gr(k - n - 1, H^{k-1})$, there is some $(n+1) \times (n+1)$ minor of M_K that has a nonzero determinant. For the remainder of this proof, we will assume this is the minor given by the first $n+1$ columns of M_K (the proof is the same no matter which minor you pick). Under this assumption, K may be uniquely represented as the null space of a matrix of the form

$$(2.14) \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & a_{1\,n+2} & \cdots & a_{1k} \\ 0 & 0 & 1 & 0 & \cdots & 0 & a_{2\,n+2} & \cdots & a_{2k} \\ 0 & 0 & 0 & 1 & \cdots & 0 & a_{3\,n+2} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{n\,n+2} & \cdots & a_{nk} \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Next, consider the configuration matrices

$$(2.15) \qquad M_P = M(P_1, \ldots, P_k) = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nk} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$(2.16) \qquad M_Q = M(Q_1, \ldots, Q_k) = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1k} \\ q_{21} & q_{22} & \cdots & q_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nk} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Since the null spaces of M_P, M_Q and M_K are the same, there are $(n+1) \times (n+1)$

invertible matrices A and B such that

$$(2.17) \qquad AM_P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & p'_{1\,n+2} & \cdots & p'_{1k} \\ 0 & 0 & 1 & 0 & \cdots & 0 & p'_{2\,n+2} & \cdots & p'_{2k} \\ 0 & 0 & 0 & 1 & \cdots & 0 & p'_{3\,n+2} & \cdots & p'_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & p'_{n\,n+2} & \cdots & p'_{nk} \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$(2.18) \qquad BM_Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & q'_{1\,n+2} & \cdots & q'_{1k} \\ 0 & 0 & 1 & 0 & \cdots & 0 & q'_{2\,n+2} & \cdots & q'_{2k} \\ 0 & 0 & 0 & 1 & \cdots & 0 & q'_{3\,n+2} & \cdots & q'_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & q'_{n\,n+2} & \cdots & q'_{nk} \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

By moving the first $n+1$ columns to this standard position, the p'_{ij} and q'_{ij} are uniquely determined. Thus we have that $p'_{ij} = a_{ij} = q'_{ij}$ for all $1 \le i \le n$, $n+2 \le j \le k$.

Moreover, since the $(n+1)^{\text{th}}$ rows of M_P and M_Q are fixed by A and B respectively, it is easy to see that A and B must be affine transformations. Thus, $AM_P = BM_Q$ and hence $M_P = A^{-1}BM_Q$. Therefore, the configurations, $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ differ by the affine transformation $A^{-1}B$ and so have the same shape thereby completing the proof that ϕ is injective. $\qquad \square$

Definition C.2. We will call the space $\mathcal{A}_{\mathbb{R}}(k,n) = Gr(k-n-1, H^{k-1})$ $(k \ge n+1)$ the *affine shape space* for configurations of k points in $\mathbb{A}_{\mathbb{R}}^n$. In the case where $n = 3$ we will call $\mathcal{O}_k = \mathcal{A}_{\mathbb{R}}(k,3) = Gr(k-4, H^{k-1})$ *affine object space* (or just *object space* when the context is understood). In the case where $n = 2$ we will call $\mathcal{I}_k = \mathcal{A}_{\mathbb{R}}(k,2) =$

$Gr(k-3, H^{k-1})$ *affine image space* (or simply *image space*).

In any case, the affine shape space, $\mathcal{A}_{\mathbb{R}}(k,n)$ is a well understood manifold of dimension $n(k-n-1)$, the structure of which we will use in the following sections to further our knowledge of the affine shapes.

D. Affine Shape Coordinates

To be able to compare shapes of configurations in a quantifiable way (in particular, to determine matching of object and image shapes under GWP projections) we will need to, in some way, assign coordinates to our shapes. Since $\mathcal{A}_{\mathbb{R}}(k,n)$ is a real manifold, it comes equipped with local coordinate charts, giving us a way define such coordinates. However, this only allows us to compare shapes in an open subset of our shape space rather than allowing us to consider all shapes under a single coordinate system. For computational purposes, it would be more convenient to be able to define *global* coordinates on our shape space. We are able to achieve this goal via the Plücker embedding of the Grassmannian into a real projective space.

In general, the Plücker embedding maps a Grassmannian $Gr(n, V^k)$ (n-dimensional subspaces of a k-dimensional vector space V^k) into the projective space $\mathbb{P}\left(\bigwedge^{k-n} V^k\right) \cong \mathbb{P}^{\binom{k}{k-n}-1} \cong \mathbb{P}^{\binom{k}{n}-1}$ in the following way. Let $K \in Gr(n, V^k)$. Then K is the intersection of $k-n$ hyperplanes in our vector space V^k given by

$$(2.19) \qquad \sum_{i=1}^{k} a_{ji} e_i^* = 0, \quad j = 1, \ldots, k-n$$

where $e_1^*, \ldots, e*_k$ is a basis for V^*, the dual vector space of V. More simply put, K

is the null space of the matrix

$$
(2.20) \qquad M_K = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-n\ 1} & a_{k-n\ 2} & \cdots & a_{k-n\ k} \end{pmatrix}.
$$

Now for each $1 \leq i_1 < i_2 < \ldots < i_{k-n} \leq k$ we define $m_{i_1,i_2,\ldots,i_{k-n}}$ to be the determinant of the $(k-n) \times (k-n)$ minor of M_K whose columns are the $i_1, i_2, \ldots, i_{k-n}$ columns of M_K, i.e.

$$
(2.21) \qquad m_{i_1,i_2,\ldots,i_{k-n}} = \det \begin{pmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_{k-n}} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_{k-n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-n\ i_1} & a_{k-n\ i_2} & \cdots & a_{k-n\ i_{k-n}} \end{pmatrix}.
$$

The Plücker embedding is now defined to be the map

$$
(2.22) \qquad \begin{aligned} \Phi_{n,k}: \quad G(n,k) &\longrightarrow \mathbb{P}^{\binom{k}{n}-1} \\ K &\longmapsto \left(m_{1,2,\ldots,k-n},\ \ldots\ , m_{n+1,n+2,\ldots,k} \right), \\ &\qquad \text{(all maximal minors)} \end{aligned}
$$

and the coordinates of $\Phi_{n,k}(K)$ are called the *Plücker coordinates* of K. We will assume that the minors $m_{i_1,i_2,\ldots,i_{k-n}}$ are ordered lexicographically.

It is important to note that this map does not depend on our choice of hyperplanes, but does depend on our choice of basis for V^k. We should also note that this map does in fact embed $G(n,k)$ as a closed projective variety in $\mathbb{P}^{\binom{k}{n}-1}$. In other words, $\Phi_{n,k}(G(n,k))$ is the zero locus of some system of polynomials $f_1, \ldots, f_s$ in the variables $x_{1,2,\ldots,k-n}, \ldots, x_{n+1,\ldots,k}$ with coefficients in the base field of V^k. We use the variables $x_{1,2,\ldots,k-n}, \ldots, x_{n+1,\ldots,k}$ to indicate that the $x_{i_1,i_2,\ldots,i_{k-n}}$ coordinate of $\Phi_{n,k}(K)$

is $m_{i_1,\ldots,i_{r-n}}$. The equations $f_i = 0$, $1 \le i \le s$ are known as the Plücker relations. For more on this, see [7] and [5].

The most obvious way to give global coordinates on $\mathcal{A}_{\mathbb{R}}(k, n)$ would be to embed this shape space in $\mathbb{P}_{\mathbb{R}}^{\binom{k-1}{k-n-1}-1}$ via the Plücker embedding $\Phi_{k-n-1,k-1}$. The problem with this approach is that it requires us to choose coordinates on H^{k-1} and then represent K as the intersection of n hyperplanes in H^{k-1}.

An alternative method of producing global coordinates on $\mathcal{A}_{\mathbb{R}}(k, n)$ is obtained by first observing that since H^{k-1} is a hyperplane in $\mathbb{A}_{\mathbb{R}}^k$, every $(k-n-1)$-dimensional linear subspace of H^{k-1} is also a $(k-n-1)$-dimensional linear subspace of $\mathbb{A}_{\mathbb{R}}^k$. Thus, we may view $\mathcal{A}_{\mathbb{R}}(k, n) = Gr(k-n-1, H^{k-1})$ as a submanifold of $Gr(k-n-1, k) = Gr(k-n-1, \mathbb{A}_{\mathbb{R}}^k)$, in which case $\Phi_{k-n-1,k}$ embeds $\mathcal{A}_{\mathbb{R}}(k, n)$ in $\mathbb{P}_{\mathbb{R}}^{\binom{k}{n+1}-1}$ as a subvariety of $\Phi_{k-n-1,k}(G(k-n-1, k))$. Under this map, a configuration $P_i = (x_{i1}, \ldots, x_{in})$, $i = 1, \ldots, k$ $(k \ge n+1)$ is mapped into $\mathbb{P}_{\mathbb{R}}^{\binom{k}{n+1}-1}$ by taking all maximal minors of the matrix

$$
(2.23) \qquad M = \begin{pmatrix} x_{11} & x_{21} & & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{n2} & & x_{kn} \\ 1 & 1 & & 1 \end{pmatrix}.
$$

Example D.1. Consider the following configuration of 4 points in $\mathbb{R}^2$

$$
P_1 = (0, 0), \ \ P_2 = (1, 0), \ \ P_3 = (0, 1), \ \ P_4 = (1, 1)
$$

These points give us the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

To find the point in $\mathbb{P}^{\binom{4}{2+1}-1} = \mathbb{P}^3$ corresponding to this configuration, we compute

$$m_{1,2,3} = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1$$

$$m_{1,2,4} = \det \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1$$

$$m_{1,3,4} = \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = -1$$

$$m_{2,3,4} = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = -1.$$

So the configuration P_1, P_2, P_3, P_4 corresponds to the point $(1 : 1 : -1 : -1) \in \mathbb{P}^3$.

Embedding our shape space $\mathcal{A}_{\mathbb{R}}(k, n)$ into $\mathbb{P}_{\mathbb{R}}^{\binom{r}{n+1}-1}$ in this fashion is in some sense a more natural way to give global coordinates on $\mathcal{A}_{\mathbb{R}}(k, n)$ than embedding it into $\mathbb{P}_{\mathbb{R}}^{\binom{k-1}{n}-1}$. This method allows us to work directly with the configuration matrix rather than forcing us to choose a basis for H^{k-1} and then rewrite our basis for K^{k-n-1} in terms of our chosen basis for H^{k-1}. Also, as we will see later in this paper, this method is also more closely related to the one that we will use in the full perspective

case.

Definition D.2. Given a configuration $P_1, \ldots, P_k \in \mathbb{A}_{\mathbb{R}}^n$ we will refer to the Plücker coordinates of K^{k-n-1} (the null space of $M(P_1, \ldots, P_k)$) viewed as a subspace of $\mathbb{A}_{\mathbb{R}}^k$ (rather than H^{k-1}) as the *shape coordinates* of the configuration $P_1, \ldots, P_k$.

In Example D.1, the homogeneous coordinates $(1 : 1 : -1 : -1)$ are the shape coordinates of the configuration P_1, P_2, P_3, P_4.

E. Dual Shape Coordinates

Before we can give a system of polynomial equation relating shapes of object and image configurations, we must first define the *dual shape coordinates* of a configuration $P_1, \ldots, P_k$ in $\mathbb{A}_{\mathbb{R}}^n$. It turns out that the general object/image relations can be written easily in terms of standard shape coordinates and dual shape coordinates.

Let K be a $(k-n-1)$-dimensional subspace of $\mathbb{A}_{\mathbb{R}}^k$. Then K is the null space of an $(n+1) \times k$ matrix

$$(2.24) \qquad A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1\ 1} & a_{n+1\ 2} & \ldots & l_{n+1\ k} \end{pmatrix},$$

and its Plücker coordinates are $(\ldots : m_{i_1,\ldots,i_{n+1}} : \ldots)$ where the $m_{i_1,\ldots,i_{n+1}}$ represent determinants of $(n+1) \times (n+1)$ minors of the matrix A.

Another way to associate K^{k-n-1} with a point in projective space is to first represent it as a $(k-n-1) \times k$ matrix whose rows form a basis for K^{k-n-1}. Let us

denote this matrix as

$$(2.25) \qquad B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1k} \\ b_{21} & b_{22} & \ldots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k-n-1\ 1} & b_{k-n-1\ 2} & \ldots & b_{k-n-1\ k} \end{pmatrix}.$$

We then assign to K^{k-n-1} the coordinates $(\ldots : m^*_{i_1,\ldots,i_{k-n-1}} : \ldots) \in \mathbb{P}_{\mathbb{R}}^{\binom{k}{k-n-1}-1} = \mathbb{P}_{\mathbb{R}}^{\binom{k}{n+1}-1}$, where $m^*_{i_1,\ldots,i_{k-n-1}}$ is the determinant of a $(k-n-1) \times (k-n-1)$ minor of the matrix B. These coordinates are the *dual Plücker coordinates* of K^{k-n-1}.

It is important to note that the row span of A is the orthogonal complement $K^{k-n-1^{\perp}}$ and that the null space of the matrix B is also $K^{k-n-1^{\perp}}$. The relationship this gives between the Plücker coordinates of K^{k-n-1} and its dual Plücker coordinates (which are the Plücker coordinates of $K^{\perp}$) is well understood and is given in the following theorem:

Theorem E.1. *Let $i_1,\ldots,i_k$ be a permutation of $1,\ldots,k$, and assume $1 \le i_1 < i_2 < \ldots < i_{n+1} \le k$ and $1 \le i_{n+2} < i_{n+3} < \ldots < i_k \le k$. Then*

$$(2.26) \qquad m_{i_1,\ldots,i_{n+1}} = c\epsilon_{i_1,\ldots,i_k} m^*_{i_{n+2},\ldots,i_k}$$

where c is a fixed constant and $\epsilon_{i_1,\ldots,i_k} = \pm 1$ depending on whether $i_1,\ldots,i_k$ is an even $(+1)$ or odd (-1) permutation.

Example E.2. Let K^1 be a linear subspace of dimension 1 of $\mathbb{R}^5$ (so $k = 5$ and

$n = 3$) with Plücker coordinates $m_{i_1 i_2 i_3 i_4}$. Then for some fixed constant c

$$m_{1234} = cm_5^*$$

$$m_{1235} = -cm_4^*$$

$$m_{1245} = cm_3^*$$

$$m_{1345} = -cm_2^*$$

$$(2.27) \qquad m_{2345} = cm_1^*.$$

We now define the dual shape coordinates in the obvious way.

Definition E.3. Let $P_1, \ldots, P_k$ be a configuration of points in $\mathbb{A}_{\mathbb{R}}^n$ and let $K \in \mathcal{A}_{\mathbb{R}}(k, n)$ be its affine shape (i.e. K is the null space of the configuration matrix $M(P_1, \ldots, P_k)$). Then the *dual shape coordinates* of the configuration $P_1, \ldots, P_k$ are the dual Plücker coordinates of K.

F. The Object/Image Relations

Given an object configuration $P_1, \ldots, P_k$ and an image configuration $Q_1, \ldots, Q_k$ we want to give necessary and sufficient conditions (the object/image relations) for the Q_i to be a generalized weak perspective projection of the P_i. Recall that we view our object space $\mathcal{O}_k$ as a subvariety of $\mathbb{P}_{\mathbb{R}}^{\binom{n}{4}-1}$ and our image space $\mathcal{I}_k$ as a subvariety of $\mathbb{P}_{\mathbb{R}}^{\binom{n}{3}-1}$. As such, we want to view the set V of pairs (K, L) where L is an image shape that comes from a generalized weak perspective projection of the object shape K (the so-called set of matching object/image pairs) as a subvariety $V \subset \mathcal{O}_k \times \mathcal{I} \subset \mathbb{P}_{\mathbb{R}}^{\binom{n}{4}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{n}{3}-1}$. Therefore, our object/image relations should be a system of bihomogeneous polynomials in the object and image shape coordinates whose zero locus is precisely V.

Recall that our object shapes are linear subspaces $K^{k-4} \subset \mathbb{A}_{\mathbb{R}}^k$ of dimension

$k-4$ and our image shapes are linear subspaces $L^{k-3} \subset \mathbb{A}_{\mathbb{R}}^k$ of dimension $k-3$. The following relates object and image shapes under generalized weak perspective projection.

Lemma F.1. *Let $P_1, \ldots, P_k$ be an object configuration with corresponding object shape K^{k-4} and let $Q_1, \ldots, Q_k$ be an image configuration with corresponding image shape L^{k-3}. Then the Q_i are a generalized weak perspective projection of the P_i if and only if*

(2.28) $$K^{k-4} \subset L^{k-3} \subset H^{k-1} \subset \mathbb{A}_{\mathbb{R}}^k$$

where $H^{k-1} = \left\{ (x_1, \ldots, x_k) \in \mathbb{A}_{\mathbb{R}}^k \mid \sum_{i=1}^k x_i = 0 \right\}$.

The preceding lemma follows easily from the observation that there is a GWP projection T such that $M(Q_1, \ldots, Q_k) = TM(P_1, \ldots, P_k)$ if and only if, the row span of $M(Q_1, \ldots, Q_k)$ is contained in the row span of $M(P_1, \ldots, P_k)$. The preceding lemma and the incidence relations given in [7] give us our object/image relations.

Theorem F.2. *Let $P_i = (x_i, y_i, z_i, 1)$, $1 \le i \le k$ be an object configuration with corresponding matrix*

$$M = \begin{pmatrix} x_1 & x_2 & & x_k \\ y_1 & y_2 & \cdots & y_k \\ z_1 & z_2 & & z_k \\ 1 & 1 & & 1 \end{pmatrix}$$

and let $Q_i = (u_i, v_i, 1)$, $1 \le i \le k$ be an image configuration with corresponding matrix

$$N = \begin{pmatrix} u_1 & u_2 & & u_k \\ v_1 & v_2 & \cdots & v_k \\ 1 & 1 & & 1 \end{pmatrix}.$$

For $1 \leq i_1 < i_2 < i_3 < i_4 \leq k$ and $1 \leq j_1 < j_2 < j_3 \leq k$ define the object shape coordinates

$$m_{i_1,i_2,i_3,i_4} = \det \begin{pmatrix} x_{i_1} & x_{i_2} & x_{i_3} & x_{i_4} \\ y_{i_1} & y_{i_2} & y_{i_3} & y_{i_4} \\ z_{i_1} & z_{i_2} & z_{i_3} & z_{i_4} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and the image shape coordinates

$$n_{j_1,j_2,j_3} = \det \begin{pmatrix} u_{i_1} & u_{i_2} & u_{i_3} \\ v_{i_1} & v_{i_2} & v_{i_3} \\ 1 & 1 & 1 \end{pmatrix}.$$

Then the points $Q_1, \ldots, Q_k$ are the images of $P_1, \ldots, P_k$ under a generalized weak perspective projection if and only if

$$(2.29) \qquad \sum_{1 \leq \lambda_1 < \lambda_2 \leq n} m_{\alpha_1,\alpha_2,\lambda_1,\lambda_2} n^*_{\lambda_1,\lambda_2,\beta_1,\ldots,\beta_{k-5}} = 0$$

for all choices of α_1, α_2 and $\beta_1, \ldots, \beta_{k-5}$ where $1 \leq \alpha_1 < \alpha_2 \leq k$ and $1 \leq \beta_1 < \beta_2 < \ldots < \beta_{k-5} \leq k$. The expressions $m_{\alpha_1,\alpha_2,\lambda_1,\lambda_2}$ and $n^*_{\lambda_1,\lambda_2,\beta_1,\ldots,\beta_{k-5}}$ should be treated as skew-symmetric in the entries of the indices.

Since we may write dual shape coordinates in terms of standard shape coordinates, we may write the relations completely in terms of the shape coordinates. The relations then become

$$(2.30) \qquad \sum_{1 \leq \lambda_1 < \lambda_2 \leq k} \epsilon_{\lambda_1,\lambda_2} m_{\alpha_1,\alpha_2,\lambda_1,\lambda_2} n_{\gamma_1,\gamma_2,\gamma_3} = 0$$

for all choices of $1 \leq \alpha_1 < \alpha_2 \leq k$ and $1 \leq \beta_1 < \beta_2 < \ldots < \beta_{k-5} \leq k$ where $1 \leq \gamma_1 < \gamma_2 < \gamma_3 \leq k$ is the complement of $\{\lambda_1, \lambda_2, \beta_1, \ldots, \beta_{k-5}\}$ in $\{1, \ldots, k\}$ when $\lambda_1, \lambda_2, \beta_1, \ldots, \beta_{k-5}$ are distinct (otherwise $n^*_{\lambda_1,\lambda_2,\beta_1,\ldots,\beta_{k-5}} = 0$) and $\epsilon_{\lambda_1,\lambda_2}$ is the sign of

the permutation

$$\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2, \beta_1, \ldots, \beta_{k-5}$$

of the numbers $1, \ldots, k$.

Example F.3. In the case of configurations of 5 points $(k = 5)$, each of the bilinear polynomials in 2.29 contains two α_i and no β_i giving us a total of $\binom{5}{2} = 10$ equations. For $\alpha = 1$, $\alpha = 2$, we have the equation

$$\sum_{1 \leq \lambda_1 < \lambda_2 \leq 5} m_{12\lambda_1\lambda_2} n^*_{\lambda_1\lambda_2} = 0.$$

Since the $m_{i_1 i_2 i_3 i_4}$ are skew-symmetric in the indices, if $\lambda_i = \alpha_j$ for any i, j, then $m_{\alpha_1\alpha_2\lambda_1\lambda_2} = 0$. Thus the preceding equation becomes,

$$m_{1234} n^*_{34} + m_{1235} n^*_{35} + m_{1245} n^*_{45} = 0.$$

Rewriting the $n^*_{\lambda_1\lambda_2}$ in standard shape coordinates according to Theorem E.1 gives us the equation

$$cm_{1234} n_{125} - cm_{1235} n_{124} + cm_{1245} n_{123} = 0.$$

Since $c \neq 0$, we may divide by c to leave us with the equation

$$m_{1234} n_{125} - m_{1235} n_{124} + m_{1245} n_{123} = 0.$$

The full system of object/image equations for 5 points is

$$m_{1234}n_{125} - m_{1235}n_{124} + m_{1245}n_{123} = 0$$

$$m_{1234}n_{135} - m_{1235}n_{134} + m_{1345}n_{123} = 0$$

$$m_{1234}n_{145} - m_{1245}n_{134} + m_{1345}n_{124} = 0$$

$$m_{1235}n_{145} - m_{1245}n_{135} + m_{1345}n_{125} = 0$$

$$m_{1234}n_{235} - m_{1235}n_{234} + m_{2345}n_{123} = 0$$

$$m_{1234}n_{245} - m_{1245}n_{234} + m_{2345}n_{124} = 0$$

$$m_{1235}n_{245} - m_{1245}n_{235} + m_{2345}n_{125} = 0$$

$$m_{1234}n_{345} - m_{1345}n_{234} + m_{2345}n_{134} = 0$$

$$m_{1235}n_{345} - m_{1345}n_{235} + m_{2345}n_{135} = 0$$

$$m_{1245}n_{345} - m_{1345}n_{245} + m_{2345}n_{145} = 0.$$

It should be noted that, as the system of matching equations indicate, given an object shape K, there are multiple image shapes that object could generate and given an image shape L, there are multiple object shapes capable of producing that image. See the Appendix for examples. In fact, these loci are linear "slices" of the object and images spaces.

G.   Metrics

While Theorem F.2 gives us a way to test for *exact* matches of object and image shapes under GWP projection, it is not an effective test for matching in practical application. Interference from external sources during image production and limited precision in extracting image data can cause some error in constructing our point

configurations. To deal with these problems, a more flexible method of matching is needed. A more robust approach is made available to us by defining metrics on our shape spaces. Using these metrics, we can create a measure of "how close" an image and an object are to matching, rather than just testing for an exact match.

1. Metrics on the Shape Spaces

Let L^l be a linear subspace of $\mathbb{A}_\mathbb{R}^n$ of dimension l and let K^k be a linear subspace of $\mathbb{A}_\mathbb{R}^n$ of dimension k with $k \geq l$. Define $\theta_1 \in \left[0, \frac{\pi}{2}\right]$ and unit vectors $u_1 \in L^l$ and $v_1 \in K^k$ to be such that

$$(2.32) \qquad \cos(\theta_1) = \max \left\{ u^T v | u \in L^l, v \in K^k, \|u\| = \|v\| = 1 \right\} = u_1^T v_1$$

where $\|\cdot\|$ is the usual Euclidean norm. In other words, we choose unit vectors $u_1 \in L^l$ and $v_1 \in K^k$ so that the angle θ_1 between them is a minimum.

Now for $2 \leq j \leq l$, let U_j be the orthogonal complement of $\mathrm{span}(u_1, \ldots, u_{j-1})$ in L^l and let V_j be the orthogonal complement of $\mathrm{span}(v_1, \ldots, v_{j-1})$ in K^k. We define $\theta_j \in \left[0, \frac{\pi}{2}\right]$ and unit vectors $u_j \in U_j \subset L^l$ and $v_j \in V_j \subset K^k$ to be such that

$$(2.33) \qquad \cos(\theta_j) = \max \left\{ u^T v | u \in U_j, v \in U_j, \|u\| = \|v\| = 1 \right\} = u_j^T v_j$$

Definition G.1. We call the angles $\theta_1, \theta_2, \ldots, \theta_l$ the *principal angles* between L^l and K^k. The corresponding vectors $u_1, \ldots, u_l$ and $v_1, \ldots, v_l$ are called *principal vectors*.

We should note that by this recursive construction, we have that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_l$. We will for the remainder of this chapter assume that the principal angles between two subspaces are listed in this ascending order. Another key point to observe is that for each i, the principal vector u_i is a unit vector in the span of the orthogonal projection of the principal vector v_i onto L and vice versa.

To actually compute the principal angles between L^l and K^k, we first choose orthonormal bases for L^l and K^k. We then arrange the basis for L^l as the columns of an $n \times l$ matrix L, and we arrange the basis for K^k as the columns of an $n \times k$ matrix K. Next, we compute the singular values $\lambda_1, \ldots, \lambda_l$ of the $l \times k$ matrix $L^T K$. The values $\theta_i = \arccos(\lambda_i)$ are the principal angles between L^l and K^k. For more on computing principal angles, see [1].

We now use these principal angles to obtain a metric on the affine shape space $\mathcal{A}_{\mathbb{R}}(k, n)$.

Definition G.2. $P_1, \ldots, P_k$ and $\tilde{P}_1, \ldots, \tilde{P}_k$ are two configurations of k points in $\mathbb{A}_{\mathbb{R}}^n$ whose affine shapes are $K^{k-n-1}, \tilde{K}^{k-n-1} \in \mathcal{A}_{\mathbb{R}}(k, n)$ respectively. Then the *affine shape distance* between K^{k-n-1} and $\tilde{K}^{k-n-1}$ is

$$(2.34) \qquad d\left(K, \tilde{K}\right) = \sqrt{\sum_{i=1}^{k-n-1} \theta_i^2}$$

where $\theta_1, \ldots, \theta_{k-n-1}$ are the principal angles between K^{k-n-1} and $\tilde{K}^{k-n-1}$.

This metric is more commonly known as the *Fubini-Study* metric on the Grassmannian $Gr(k-n-1, k)$. To avoid confusion as to which shape space we are working in, we will in the case of object space ($n = 3$) denote this metric by d_{Obj} and in the case of image space ($n = 2$) denote this metric by d_{Img}.

2. A "Distance" Between Object Shapes and Image Shapes

Now we would like to develop a notion of distance between an object shape and an image shape to measure that images failure to be a GWP projection of the object. There are three natural ways that we may do this.

Let $K \in \mathcal{O}_k$ be an object shape and let $L \in \mathcal{I}_k$ be an image shape (not necessarily matching). Let $V_K \subset \mathcal{I}_k$ be the locus of image shapes that K could produce under

GWP projection and let $U_L \subset \mathcal{O}_k$ be the locus of objects capable of producing the image L. The first way to compute a distance between K and L is to find the minimum distance between the object shape K and the locus U_L in $\mathcal{O}_k$. Explicitly, we compute

$$(2.35) \qquad d_1(K, L) = \min_{K' \subset L} \left(d_{Obj}(K, K') \right).$$

Similarly, we may compute a distance between K and L by minimizing the distance between the image shape L and the locus V_K in $\mathcal{I}_k$. We compute this explicitly as

$$(2.36) \qquad d_2(K, L) = \min_{L' \supset K} \left(d_{Img}(L', L) \right).$$

The third way to compute a distance between K and L is much more concise. As we saw in Lemma F.1, an image shape L is a GWP projection of K if and only if $K \subset L$, so what we want is a measure of the failure of K to be contained in L. This can be achieved using the principle angles. Simply put, the smaller the principal angles between K and L, the closer K is to being contained in L. In particular, when $K \subset L$, the principal angles between K and L are all zero. Thus, we compute this third distance to be

$$(2.37) \qquad d(K, L) = \sqrt{\sum_{i=1}^{k-4} \theta_i}$$

where $\theta_1, \ldots, \theta_{k-4}$ are the principal angles between K and L. Note that there are $k - 4$ principal angles because $k - 4 = \dim(K) < \dim(L) = k - 3$.

The important thing to note is that all three of these "metrics" are equal, which we shall now prove through a series of lemmas. The key step in this proof is to show that for two linear subspaces $V, W \subset \mathbb{A}_{\mathbb{R}}^k$ (not necessarily of the same dimension), the

principal angle distance $d(V, W) = \sqrt{\sum_{i=1}^{m} \theta_i^2}$ is equal to the principal angle distance $d(V^\perp, W^\perp) = \sqrt{\sum_{i=1}^{\tilde{m}} \tilde{\theta}_i^2}$. Here $\theta_1, \ldots, \theta_m$ are the principal angles between V and W and $\tilde{\theta}_1, \ldots, \tilde{\theta}_{\tilde{m}}$ are the principal angles between $V^\perp$ and $W^\perp$.

Let V and W be linear subspaces of $\mathbb{A}_{\mathbb{R}}^k$ with $\dim(V) = n$ and $\dim(W) = m$ with $n \leq m$. We should first observe that if $V \cap W$ has dimension greater than zero, then the principal angles $\theta_1, \ldots, \theta_{\dim(V \cap W)}$ are all zero. Thus, we may assume that $V \cap W = \{0\}$. Similarly, we can assume that $V^\perp \cap W^\perp = \{0\}$ so that $V + W = \mathbb{A}_{\mathbb{R}}^k$ (in particular, $k = n + m$).

Let $\theta_1, \ldots, \theta_n$ be the principal angles between V and W, and for each θ_i let $v_i \in V$ and $w_i \in W$ be the principal vectors corresponding to θ_i. Remember that by definition, $\|v_i\| = \|w_i\| = 1$ for all i. Then $v_1, \ldots, v_n$ forms an orthonormal basis for V, and we may choose $w_{n+1}, \ldots, w_m$ in W so that $w_1, \ldots, w_m$ is an orthonormal basis for W since $w_1, \ldots, w_n$ is an orthonormal set of vectors in W. Note also that since $\mathbb{A}_{\mathbb{R}}^k = V \oplus W$, $v_1, \ldots, v_n, w_1, \ldots, w_m$ form a basis for $\mathbb{A}_{\mathbb{R}}^k$.

For each v_i let $\tilde{v}_i = \frac{\mathrm{proj}_{W^\perp}(v_i)}{\|\mathrm{proj}_{W^\perp}(v_i)\|}$ and for each w_i let $\tilde{w}_i = \frac{\mathrm{proj}_{W^\perp}(w_i)}{\|\mathrm{proj}_{V^\perp}(w_i)\|}$. Now, note that for each i, $w_i = a_i \mathrm{proj}_W(v_i)$ for some a_i and that $v_i = b_i \mathrm{proj}_V(w_i)$ for some b_i. Thus, we have that for some scalars $\alpha_i, \ \beta_i, \gamma_i, \lambda_i$

$$(2.38) \qquad\qquad v_i = \alpha_i w_i + \beta_i \tilde{v}_i$$

and

$$(2.39) \qquad\qquad w_i = \gamma_i v_i + \lambda_i \tilde{w}_i.$$

Since we are assuming that $V \cap W = \{0\}$ we may assume that β_i and λ_i are nonzero.

Now define a linear map $\phi : \mathbb{A}_{\mathbb{R}}^k \to \mathbb{A}_{\mathbb{R}}^k$ by $\phi(v_i) = \tilde{v}_i$ and $\phi(w_i) = -\tilde{w}_i$. We will show that ϕ is an isomorphism that preserves the usual inner product, $\langle \ , \ \rangle$ on $\mathbb{A}_{\mathbb{R}}^k$.

Lemma G.3. $\tilde{v}_1, \ldots, \tilde{v}_n$ *forms an orthonormal basis for* $W^{\perp}$ *and* $\tilde{w}_1, \ldots, \tilde{w}_m$ *form an orthonormal basis for* $V^{\perp}$.

Proof. We first prove that $\tilde{v}_1, \ldots, \tilde{v}_n$ forms an orthonormal basis for $W^{\perp}$. Since $k = n + m$, $\dim(W^{\perp}) = n$ so it is enough to show that $\tilde{v}_1, \ldots, \tilde{v}_n$ is an orthonormal set. We have already defined $\tilde{v}_1, \ldots, \tilde{v}_n$ so that $\langle \tilde{v}_i, \tilde{v}_i \rangle = 1$. So suppose $i \neq j$. Then $\langle w_i, w_j \rangle = 0$ since we know $w_1, \ldots, w_m$ form an orthonormal set and $\langle w_i, \tilde{v}_j \rangle = \langle \tilde{v}_i, w_j \rangle = 0$, since $\tilde{v}_i, \tilde{v}_j \in W^{\perp}$ and $w_i, w_j \in W$. This gives us that

$$\langle v_i, v_j \rangle = \langle \alpha_i w_i + \beta_i \tilde{v}_i, \alpha_j w_j + \beta_j \tilde{v}_j \rangle$$

$$= \alpha_i \alpha_j \langle w_i, w_j \rangle + \alpha_i \beta_j \langle w_i, \tilde{v}_j \rangle + \alpha_j \beta_i \langle \tilde{v}_i, w_j \rangle + \beta_i \beta_j \langle \tilde{v}_i, \tilde{v}_j \rangle$$

$$(2.40) \qquad = \beta_i \beta_j \langle \tilde{v}_i, \tilde{v}_j \rangle$$

But $\langle v_i, v_j \rangle = 0$ and thus $\langle \tilde{v}_i, \tilde{v}_j \rangle = 0$ since $\beta_k \neq 0$ for any k. Thus $\tilde{v}_1, \ldots, \tilde{v}_n$ is an orthonormal set and hence is an orthonormal basis for $W^{\perp}$. A similar argument shows that $\tilde{w}_1, \ldots, \tilde{w}_m$ form an orthonormal basis for $V^{\perp}$. $\square$

Proposition G.4. ϕ *is an isomorphism.*

Proof. Since $\mathbb{A}_{\mathbb{R}}^k = V \oplus W$ we have that $\mathbb{A}_{\mathbb{R}}^k = V^{\perp} \oplus W^{\perp}$ which gives us that $\tilde{v}_1, \ldots, \tilde{v}_n, -\tilde{w}_1, \ldots, -\tilde{w}_m$ form a basis for $\mathbb{A}_{\mathbb{R}}^k$. Since ϕ maps the basis $v_1, \ldots, v_n, w_1, \ldots, w_m$ to the basis $\tilde{v}_1, \ldots, \tilde{v}_n, -\tilde{w}_1, \ldots, -\tilde{w}_m$ in a one-to-one fashion, ϕ is an isomorphism. $\square$

Now we will show by a series of lemmas that ϕ preserves the inner product.

Lemma G.5. *If* $i \neq j$, *then* $\langle v_i, w_j \rangle = \langle \tilde{v}_i, \tilde{w}_j \rangle = 0$.

Proof. We know that $\langle w_i, w_j \rangle = 0$ and since $\tilde{v}_i \in W^\perp$, $\langle \tilde{v}_i, w_j \rangle = 0$. Thus

$$\langle v_i, w_j \rangle = \langle \alpha_i w_i + \beta_i \tilde{v}_i, w_j \rangle$$

$$= \alpha_i \langle w_i, w_j \rangle + \beta_i \langle \tilde{v}_i, w_j \rangle$$

$$(2.41) \qquad\qquad = 0.$$

Now, note that

$$(2.42) \qquad\qquad \tilde{v}_i = \frac{1}{\beta_i} v_i - \frac{\alpha_i}{\beta_i} w_i$$

$$(2.43) \qquad\qquad \tilde{w}_j = \frac{1}{\lambda_j} w_j - \frac{\gamma_j}{\lambda_j} w_j.$$

For simplicity, we will write

$$(2.44) \qquad\qquad \tilde{v}_i = A_i v_i + B_i w_i$$

$$(2.45) \qquad\qquad \tilde{w}_j = C_j w_j + D_j v_j.$$

Now, since we have already seen that for $i \neq j$, $\langle v_i, w_j \rangle = 0$ we have that

$$\langle \tilde{v}_i, \tilde{w}_j \rangle = \langle A_i v_i + B_i w_i, C_j w_j + D_j v_j \rangle$$

$$= A_i C_j \langle v_i, w_j \rangle + A_i D_j \langle v_i, w_j \rangle + A_i C_j \langle v_i, w_j \rangle + B_i D_j \langle w_i, v_j \rangle$$

$$(2.46) \qquad = 0.$$

$\square$

Lemma G.6. $\langle v_i, w_i \rangle = - \langle \tilde{v}_i, \tilde{w}_i \rangle.$

Proof. As seen in the proof of the previous lemma,

$$(2.47) \qquad\qquad \tilde{v}_i = A_i v_i + B_i w_i$$

$$(2.48) \qquad\qquad \tilde{w}_i = C_i w_i + D_i v_i$$

Hence, $v_i, w_i, \tilde{v}_i$, and $\tilde{w}_i$ are all coplanar. Let θ_i be the angle between v_i and w_i, and let $\tilde{\theta}_i$ be the angle between $\tilde{v}_i$, and $\tilde{w}_i$. Since by definition, $\theta_i \leq \frac{\pi}{2}$, we may draw the vectors $v_i, w_i, \tilde{v}_i$, and $\tilde{w}_i$ in the plane as in Fig. 7. We see that since the angle
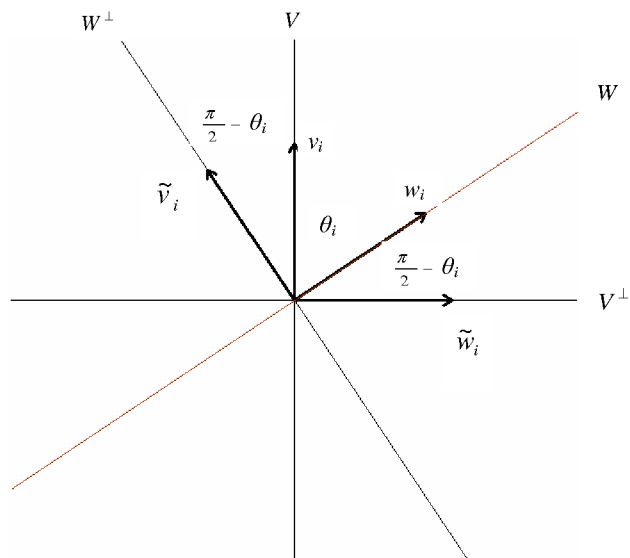
Fig. 7. The angle between $\tilde{v}_i$ and $\tilde{w}_i$ is $\pi - \theta_i$.

between v_i and $\tilde{w}_i$ is $\frac{\pi}{2}$, the angle between w_i and $\tilde{w}_i$ is $\frac{\pi}{2} - \theta_i$. Similarly, since the angle between w_i and $\tilde{v}_i$ is $\frac{\pi}{2}$, the angle between v_i and $\tilde{v}_i$ is $\frac{\pi}{2} - \theta_i$.

From this we see that $\tilde{\theta}_i = \pi - \theta_i$, and thus we have that

$$\langle \tilde{v}_i, \tilde{w}_i \rangle = \cos(\tilde{\theta}_i)$$
$$= \cos(\pi - \theta_i)$$
$$= -\cos(\theta_i)$$
$$(2.49) \qquad\qquad\qquad = -\langle v_i, w_i \rangle.$$

$\square$

Proposition G.7. *ϕ preserves the inner product.*

Proof. Let $x, y \in \mathbb{A}_{\mathbb{R}}^k$. Then we have

$$(2.50) \qquad x = \sum_i (a_i v_i) + \sum_i (b_i w_i)$$

$$(2.51) \qquad y = \sum_j (c_j v_j) + \sum_j (d_j w_j).$$

This gives us that

$$\langle x, y \rangle = \left\langle \sum_i (a_i v_i) + \sum_i (b_i w_i), \sum_j (c_j v_j) + \sum_j (d_j w_j) \right\rangle$$

$$= \sum_{i,j} a_i c_j \langle v_i, v_j \rangle + \sum_{i,j} (a_i d_j + c_i b_j) \langle v_i, w_j \rangle + \sum_{i,j} b_i d_j \langle w_i, w_j \rangle$$

$$= \sum_i a_i c_i + \sum_i (a_i d_i + c_i b_i) \langle v_i, w_i \rangle + \sum_i b_i d_i$$

$$(2.52) \qquad = \sum_i a_i c_i - \sum_i (a_i d_i + c_i b_i) \langle \tilde{v}_i, \tilde{w}_i \rangle + \sum_i b_i d_i.$$

We also have

$$\langle \phi(x), \phi(y) \rangle = \left\langle \sum_i (a_i \phi(v_i)) + \sum_i (b_i \phi(w_i)), \sum_j (c_j \phi(v_j)) + \sum_j (d_j \phi(w_j)) \right\rangle$$

$$= \left\langle \sum_i (a_i \tilde{v}_i) - \sum_i (b_i \tilde{w}_i), \sum_j (c_j \tilde{v}_j) + \sum_j (d_j \tilde{w}_j) \right\rangle$$

$$= \sum_{i,j} a_i c_j \langle \tilde{v}_i, \tilde{v}_j \rangle - \sum_{i,j} (a_i d_j + c_i b_j) \langle \tilde{v}_i, \tilde{w}_j \rangle + \sum_{i,j} b_i d_j \langle \tilde{w}_i, \tilde{w}_j \rangle$$

$$= \sum_i a_i c_i - \sum_i (a_i d_i + c_i b_i) \langle \tilde{v}_i, \tilde{w}_i \rangle + \sum_i b_i d_i$$

$$(2.53) \qquad = \langle x, y \rangle.$$

$\square$

Theorem G.8. *The principal angle distance $d(V^\perp, W^\perp)$ between $V^\perp$ and $W^\perp$ is equal to the principal angle distance $d(V, W)$ between V and W.*

Proof. Since ϕ is an isomorphism that preserves the inner product, principal angles between subspaces are preserved under ϕ. Thus, the principal angles between $\phi(V) = W^\perp$ and $\phi(W) = V^\perp$ are exactly the principal angles between V and W which gives us that $d(V^\perp, W^\perp) = d(V, W)$. $\qquad\square$

Now from Theorem G.8, we are in a position to prove the main result of this chapter.

Theorem G.9 (Object/Image Metric Duality)**.** *Given an object shape $K \in \mathcal{O}_k$ and an image shape $L \in \mathcal{I}_k$, the distances*

$$(2.54) \qquad\qquad d_1(K, L) = \min_{K' \subset L} d_{Obj}(K, K')$$

$$(2.55) \qquad\qquad d_2(K, L) = \min_{L' \supset K} d_{Img}(L', L)$$

$$(2.56) \qquad\qquad d(K, L) = \sqrt{\sum_{i=1}^{k-4} \theta_i^2}$$

where $\theta_1, \ldots, \theta_{k-4}$ are the principal angles between K and L and d is the principal angle distance between subspaces.

Proof. We first note that by the construction of the principal angles, if $v_1, \ldots, v_{k-4}$ are the principal vectors contained in L and $W = \mathrm{span}(v_1, \ldots, v_{k-4})$, then

$$d_1(K, L) = \min_{K' \subset L} d(K, K')$$

$$= d(K, W)$$

$$(2.57) \qquad\qquad\qquad = d(K, L).$$

Now we observe that

$$d_2(K, L) = \min_{L' \supset K} d(L', L)$$

(2.58)
$$= \min_{L'^{\perp} \subset K^{\perp}} d(L'^{\perp}, L^{\perp}).$$

Now by 2.57, we see that

(2.59)
$$d_2(K, L) = d(K^{\perp}, L^{\perp}),$$

and by Theorem G.8, we have

(2.60)
$$d_2(K, L) = d(K, L).$$

$\square$

This now gives us an explicit method for computing a distance between an object shape and an image shape to determine "how close" a given object configuration is to being capable of producing a given image configuration. To recap, the process for computing the distance between an object shape K^{k-4} and an image shape L^{k-3} is

1. Compute orthonormal bases for K^{k-4} and L^{k-3}

2. Make these basis vectors the columns of two matrices K and L

3. Compute the singular values $\lambda_1, \ldots, \lambda_{k-4}$ of $L^T K$

4. Compute the principal angles $\theta_i = \arccos(\lambda_i)$ between K^{k-4} and L^{k-3}

5. The distance between K^{k-4} an L^{k-3} is $d([K], [L]) = \sqrt{\sum_{i=1}^{k-4} \theta_i^2}$

For examples, see the Appendix.

CHAPTER III

THE FULL PERSPECTIVE (PROJECTIVE) CASE

A. Full Perspective Projection

In this chapter, we continue to address the problem of identifying optical camera images. Recall, that in our previous model, we chose to consider configurations of points (object and image) in $\mathbb{A}_{\mathbb{R}}^n$ up to affine transformation and that by doing this, we were only able to approximate the focal-point projections by generalized weak perspective projections. In this model, we will instead choose to consider our object configurations to be in projective space, $\mathbb{P}_{\mathbb{R}}^3$, and our image configurations to be in the projective plane,$\mathbb{P}_{\mathbb{R}}^2$. The advantage choosing to work in projective space is that a focal point projection used by an optical camera is a projective linear map. Namely, this map is the projection from a point $P \in \mathbb{P}_{\mathbb{R}}^3$ (here P is our focal point) which has the form

$$
(3.1) \qquad T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{pmatrix}
$$

where T has maximal rank 3. We call this type of map a *full perspective projection*. Suppose that $Q = (R : S : T) \in \mathbb{P}_{\mathbb{R}}^2$ is the image of $P = (X : Y : Z : W) \in \mathbb{P}_{\mathbb{R}}^3$ under a full perspective projection T (so $Q = TP$ up to a scaling of the homogeneous coordinates of P and Q). Then since we may scale the homogeneous coordinates of P and Q, we have that for any 3 scalar matrix A and any 4×4 scalar matrix B, $Q = (ATB)P$. Thus, the set of full perspective projections is equivalent to the set of 3×4 matrices of rank 3 up to multiplication on the left or right by a scalar matrix.

In addition to now being able to accurately view the focal point projection as

a projective linear map, we also have that projection from a point $P \in \mathbb{P}^3_{\mathbb{R}}$ is well defined for all points in $\mathbb{P}^3_{\mathbb{R}}$ except for P. To see this, let T be a projection of $\mathbb{P}^3_{\mathbb{R}}$ onto $\mathbb{P}^2_{\mathbb{R}}$ from a point $P \in \mathbb{P}^3_{\mathbb{R}}$, and let v be any point in $\mathbb{P}^3_{\mathbb{R}}$. Then $T(v)$ is well-defined unless v is in the null space T. Since T has maximal rank, the null space of T is a one dimensional subspace of $\mathbb{A}^4_{\mathbb{R}}$ which is a single point in $\mathbb{P}^3_{\mathbb{R}}$. However, we already know that T is not defined at the point P from which we are projecting. Thus, we see that the only point at which T is undefined is P. This causes us no problems because in the optical camera model, we will be able to assume that the focal point is not a point on our object.

B. Projective Shapes

Let T be a full perspective projection. Let A be a 3×3 matrix with $\det(A) \neq 0$ and let B be a 4×4 matrix with $\det(B) \neq 0$. Then $A^{-1}TB$ is again a 3×4 matrix of rank 3 i.e. $A^{-1}TB$ is again a full perspective projection. Note that, as previously observed, if we multiply A and B by scalar matrices, the projection $A^{-1}TB$ remains unchanged as a map between projective spaces. Thus, we should view A as an element of $PGL(3)$ and B as an element of $PGL(4)$. In general, $PGL(k)$ is the quotient $GL(k)/S$ where S is the subgroup of scalar matrices.

The impact of this observation is that the best we can hope to do is to relate object configurations up to a $PGL(4)$ transformation with image configurations up to a $PGL(3)$ transformation. As such, we should consider two configurations in $\mathbb{P}^n_{\mathbb{R}}$ to have the same shape if they differ by a $PGL(n+1)$ transformation.

Ideally, we would like to have the space of shapes of configurations of k points in $\mathbb{P}^n_{\mathbb{R}}$ be equal to the quotient space $(\mathbb{P}^n_{\mathbb{R}})^k/PGL(n+1)$. However, when we quotient $(\mathbb{P}^n_{\mathbb{R}})^k$ by $PGL(n+1)$, we do not arrive at a reasonable moduli space for our shapes.

Thus, we will be required to restrict our attention so some open set of configurations $U \subset (\mathbb{P}^n_\mathbb{R})^k$. At a very minimum, we should assume that for a configuration, $P_1, \ldots, P_k \in \mathbb{P}^n_\mathbb{R}$, the points do not all lie in a single hyperplane.

In the affine case, we were able to assign to each shape a distinct point in a Grassmannian viewed as a subvariety of a projective space. In the full perspective case, our ability to scale the homogeneous coordinates of the points of our configurations complicates matters so that no convenient analogue of the affine shape coordinates is available. We circumvent this problem by instead identifying the shape of a configuration with a natural projective variety.

Although ultimately we want to consider configurations of k points in $\mathbb{P}^2_\mathbb{R}$ and $\mathbb{P}^3_\mathbb{R}$, let us begin by examining configurations of 4 points in $\mathbb{P}^1_\mathbb{R}$. Let $P_i = (x_i : y_i) \in \mathbb{P}^1_\mathbb{R}$ for $1 \leq i \leq 4$ and assume that at least two of these points are distinct. In the spirit of the affine case we place this configuration with these homogeneous coordinates in a matrix

$$(3.2) \qquad M(P_1, P_2, P_3, P_4) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}.$$

As in the affine case, we associate to the configuration P_1, P_2, P_3, P_4 the point

$$(3.3) \qquad (m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}) \in Gr(2,4) \subset \mathbb{P}^{\binom{4}{2}-1}_\mathbb{R} = \mathbb{P}^5_\mathbb{R}$$

where

$$(3.4) \qquad m_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}.$$

Note that since the points are not all coincident, at least one of the m_{ij} is nonzero so that we have a well-defined point in projective space. Note also that the point 3.3 is invariant when we act on the configuration matrix from the left by a $PGL(2)$

transformation.

There is an indexing convention that we should observe. Suppose, $1 \leq i, j \leq 4$. Then since,

$$(3.5) \qquad m_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \qquad \text{and} \qquad m_{ji} = \det \begin{pmatrix} x_j & x_i \\ y_j & y_i \end{pmatrix}$$

we have that $m_{ij} = -m_{ji}$. In particular $m_{ii} = 0$ for all i.

Now, if for each $1 \leq i \leq 4$ we scale the homogeneous coordinates of P_i by a nonzero constant a_i, we have the same configuration of points in $\mathbb{P}^1_{\mathbb{R}}$, but our configuration matrix is now

$$(3.6) \qquad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

which corresponds to the point

$$(3.7) \quad (a_1 a_2 m_{12} : a_1 a_3 m_{13} : a_1 a_4 m_{14} : a_2 a_3 m_{23} : a_2 a_4 m_{24} : a_3 a_4 m_{34}) \in Gr(2,4) \subset \mathbb{P}^5_{\mathbb{R}}.$$

Thus for a given configuration of 4 points in $\mathbb{P}^1_{\mathbb{R}}$ with some fixed homogeneous coordinates we have a map $\Phi : (\mathbb{R}^*)^4 \to Gr(2,4)$ given by

(3.8)

$$\Phi(a_1, a_2, a_3, a_4) = (a_1 a_2 m_{12} : a_1 a_3 m_{13} : a_1 a_4 m_{14} : a_2 a_3 m_{23} : a_2 a_4 m_{24} : a_3 a_4 m_{34})$$

(here $\mathbb{R}^*$ is the multiplicative group of nonzero elements of $\mathbb{R}$). Notice however that

$$(3.9) \qquad \begin{aligned} \Phi(a, a, a, a) &= a^2(m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}) \\ &= (m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}). \end{aligned}$$

So we have in fact a well defined map $\overline{\Phi} : (\mathbb{R}^*)^4/\mathbb{R}^* \cong (\mathbb{R}^*)^3 \to Gr(2,4)$ whose image is a quasiprojective variety which we will denote $\mathcal{V}(P_1, P_2, P_3, P_4) \subset Gr(2,4) \subset \mathbb{P}_{\mathbb{R}}^5$ (or simply $\mathcal{V}$ when the configuration we are working with is understood). Thus, to each configuration we may assign a projective variety $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4)$, the closure of $\mathcal{V}$ in $\mathbb{P}_{\mathbb{R}}^5$.

Definition B.1. We will call the projective variety $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4)$ the *shape variety* of the configuration P_1, P_2, P_3, P_4.

Theorem B.2. *Every configuration P_1, P_2, P_3, P_4 is assigned a unique shape variety $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4)$, and two configurations P_1, P_2, P_3, P_4 and P_1', P_2', P_3', P_4' have the same shape variety if and only if they differ by a $PGL(2)$ transformation (and hence have the same shape).*

Proof. The fact that every configuration is assigned a unique variety is obvious. It is also clear from our construction of the shape varieties that if two configurations have the same shape, then they also have the same shape variety.

So suppose that the two configurations $P_i = (x_i : y_i)$, $1 \leq i \leq 4$ and $P_i' = (x_i' : y_i')$, $1 \leq i \leq 4$ have the same shape variety. Since $\mathcal{V}$ is the image of the irreducible variety, $(\mathbb{R}^*)^4/\mathbb{R}^*$, $\mathcal{V}$ must be irreducible. So we have that $\overline{\mathcal{V}}$ must also be irreducible. From this, we see that $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4) = \overline{\mathcal{V}}(P_1', P_2', P_3', P_4')$ if and only if $\mathcal{V}(P_1, P_2, P_3, P_4) = \mathcal{V}(P_1', P_2', P_3', P_4')$. Thus, for some $a_1, a_2, a_3, a_4 \in \mathbb{R}^*$

$$(m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}) =$$

$$(a_1 a_2 m_{12}' : a_1 a_3 m_{13}' : a_1 a_4 m_{14}' : a_2 a_3 m_{23}' : a_2 a_4 m_{24}' : a_3 a_4 m_{34}')$$

where $m_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$ and $m_{ij}' = \det \begin{pmatrix} x_i' & x_j' \\ y_i' & y_j' \end{pmatrix}$. So we have that the matri-

ces

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 x_1' & a_2 x_2' & a_3 x_3' & a_4 x_4' \\ a_1 y_1' & a_2 y_2' & a_3 y_3' & a_4 y_4' \end{pmatrix}$$

give the same point under the Plücker embedding and hence are in fact the same linear subspace of $\mathbb{R}^4$. Thus the matrices differ by the left action of a $GL(2)$ matrix from which we see that the configurations P_1, P_2, P_3, P_4 and P_1', P_2', P_3', P_4' differ by a $PGL(2)$ transformation. $\qquad\square$

Now, having placed shapes of configurations $P_1, P_2, P_3, P_4 \in \mathbb{P}^1_{\mathbb{R}}$ (up to a $PGL(2)$ transformation) in one-to-one correspondence with the projective varieties $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4)$, we would like understand the relations that the points in $\overline{\mathcal{V}}$ must satisfy. So let $P_1, P_2, P_3, P_4 \in \mathbb{P}^1_{\mathbb{R}}$. Compute $m_{12}, \ldots, m_{34}$ for some fixed homogeneous coordinates of P_1, P_2, P_3, P_4 and let $(x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34})$ be a point in $\mathcal{V} = \mathcal{V}(P_1, P_2, P_3, P_4)$. Then for some $a_1, a_2, a_3, a_4 \in \mathbb{R}^*$ the following must hold

$$x_{12} - a_1 a_2 m_{12} = 0$$

$$x_{13} - a_1 a_2 m_{13} = 0$$

$$x_{14} - a_1 a_4 m_{14} = 0$$

$$x_{23} - a_2 a_3 m_{23} = 0$$

$$x_{24} - a_2 a_4 m_{24} = 0$$

$$x_{34} - a_3 a_4 m_{34} = 0.$$

Using Gröebner bases, we eliminate the a_i's from this system and obtain the following Theorem.

Theorem B.3. $\overline{\mathcal{V}} \subset Gr(2,4)$ *is the zero locus of three polynomials in the variables*

$x_{12}, \ldots, x_{34}$

$$f_1 = m_{12}m_{34}x_{13}x_{24} - m_{13}m_{24}x_{12}x_{34}$$

$$f_2 = m_{12}m_{34}x_{14}x_{23} - m_{14}m_{23}x_{12}x_{34}$$

(3.10) $$f_3 = m_{13}m_{24}x_{14}x_{23} - m_{14}m_{23}x_{13}x_{24}.$$

These same relations can also be obtained by observing that if i_1, i_2, i_3, i_4 and j_1, j_2, j_3, j_4 are two appropriate permutations of 1,2,3,4 then

(3.11) $$\frac{m_{i_1i_2}m_{i_3i_4}x_{j_1j_2}x_{j_3j_4}}{m_{j_1j_2}m_{j_3j_4}x_{i_1i_2}x_{i_3i_4}} = \frac{m_{i_1i_2}m_{i_3i_4}(a_{j_1}a_{j_2}m_{j_1j_2})(a_{j_3}a_{j_4}m_{j_3j_4})}{m_{j_1j_2}m_{j_3j_4}(a_{i_1}a_{i_2}m_{i_1i_2})(a_{i_3}a_{i_4}m_{i_3i_4})} = 1.$$

Notice that in each of the monomials of f_1, f_2, and f_3, the numbers $1, 2, 3$, and 4 each appear once as entries of the indices of the m_{ij}. Thus, if we were to choose different homogeneous coordinates for P_1, P_2, P_3, P_4, we would have a new system of polynomials

(3.12) $$f_1' = (a_1a_2m_{12})(a_3a_4m_{34})x_{13}x_{24} - (a_1a_3m_{13})(a_2a_4m_{24})x_{12}x_{34}$$

$$f_2' = (a_1a_2m_{12})(a_3a_4m_{34})x_{14}x_{23} - (a_1a_4m_{14})(a_2a_3m_{23})x_{12}x_{34}$$

$$f_3' = (a_1a_3m_{13})(a_2a_4m_{24})x_{14}x_{23} - (a_1a_4m_{14})(a_2a_3m_{23})x_{13}x_{24},$$

but the zero locus of f_1', f_2', f_3' is precisely the zero locus of f_1, f_2, f_3 since $f_i' = a_1a_2a_3a_4f_i$. This tells us that the polynomials f_1, f_2, f_3 define our shape variety as a subvariety of $Gr(2, 4)$ independent of our choice of homogeneous coordinates for P_1, P_2, P_3, P_4.

We should also note that since $(m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34})$ and $(x_{12} :$

$x_{13} : x_{14} : x_{23} : x_{24} : x_{34})$ are points in $Gr(2,4) \subset \mathbb{P}^5_{\mathbb{R}}$, the Plücker relations

$$p_1 = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 0$$

$$p_2 = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

are satisfied. It is easily seen that as ideals in $\mathbb{R}[m_{12}, \ldots, m_{34}, x_{12}, \ldots, x_{34}]$,

$$
\begin{aligned}
\langle f_1, f_2, f_3, p_1, p_2 \rangle &= \langle f_1, p_1, p_2 \rangle \\
&= \langle f_2, p_1, p_2 \rangle \\
&= \langle f_3, p_1, p_2 \rangle.
\end{aligned}
$$
(3.13)

From this we see that $V(f_1) = V(f_2) = V(f_3)$ as subvarieties of $Gr(2,4) \subset \mathbb{P}^5_{\mathbb{R}}$ and hence $\overline{\mathcal{V}}$ is defined as the zero locus of any *one* of f_1, f_2, f_3. In particular $\overline{\mathcal{V}}$ is a hypersurface in $Gr(2,4)$ and so has dimension $\dim(\overline{V}) = \dim(Gr(2,4))$-1=3.

The preceding discussion can be easily generalized to the case of k points in $\mathbb{P}^n_{\mathbb{R}}$ where $k \geq n + 1$. Two configurations have the same shape if they differ by a $PGL(n + 1)$ transformation. For each configuration $P_i = (x_{0i}, \ldots, x_{ni})$, $1 \leq i \leq k$ of k points in $\mathbb{P}^n_{\mathbb{R}}$, we have a map $\overline{\Phi} : (\mathbb{R}^*)^k/\mathbb{R}^* \to Gr(n+1, k)$ obtained by constructing the configuration matrix

$$
(3.14) \qquad M(P_1, \ldots, P_k) = \begin{pmatrix} x_{01} & x_{02} & & x_{0k} \\ x_{11} & x_{12} & \cdots & x_{1k} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nk} \end{pmatrix}
$$

whose columns are homogeneous coordinates of $P_1, \ldots, P_k$ in $\mathbb{P}^n_{\mathbb{R}}$ and then scaling the columns of that matrix. We denote the image of $\overline{\Phi}$ by $\mathcal{V}(P_1, \ldots, P_k)$. We call the projective variety $\overline{\mathcal{V}}(P_1, \ldots, P_k)$ the *shape variety* of the configuration $P_1, \ldots, P_k$ and

we have the following theorem.

Theorem B.4. *Two configurations $P_1, \ldots, P_k$ and $P'_1, \ldots, P'_k$ of k point in $\mathbb{P}^n_\mathbb{R}$ have the same shape (up to a $PGL(n+1)$ transformation) if and only if they have the same shape variety.*

Proof. The proof of this result is exactly the same as that of Theorem B.2 but with more complicated notation. $\qquad\square$

Explicitly, the map $\overline{\Phi} : (\mathbb{R}^*)^k/\mathbb{R}^* \to Gr(n+1, k)$ is given by

$$(3.15) \qquad \overline{\Phi}(a_1, \ldots, a_r) = (a_{I_1} m_{I_1} : \ldots : a_{I_N} m_{I_N})$$

where $I_1, \ldots, I_N$ ($N = \binom{k}{n+1}$) are the $(n+1)$-subsets of $\{1, \ldots, k\}$ (ordered lexico-graphically), $a_{I_j} = \prod_{l \in I_j} a_l$, and m_{I_j} is the determinant of the $(n+1) \times (n+1)$ minor of $M(P_1, \ldots, P_k)$ whose columns are given by the elements of I_j.

As in the case of 4 points in $\mathbb{P}^1_\mathbb{R}$, we enforce an indexing convention on the m_I. If $I = (i_1, i_2, \ldots, i_{n+1})$ and if σ is a permutation of $1, 2, \ldots, n+1$, then

$$(3.16) \qquad m_{i_{\sigma(1)} \ldots i_{\sigma(n+1)}} = \det \begin{pmatrix} x_{0\sigma(1)} & x_{0\sigma(2)} & \cdots & x_{0\sigma(n+1)} \\ x_{1\sigma(1)} & x_{1\sigma(2)} & \cdots & x_{1\sigma(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n\sigma(1)} & x_{n\sigma(2)} & \cdots & x_{n\sigma(n+1)} \end{pmatrix}$$

Thus $m_{i_1 i_2 \ldots i_{n+1}} = \epsilon m_{i_{\sigma(1)} \ldots i_{\sigma(n+1)}}$ where ϵ is the sign of the permutation σ. Note that if $i_s = i_t$ for some $s \neq t$, then $m_{i_1 i_2 \ldots i_{n+1}} = 0$.

Since we have only made the assumption that the points of our configurations do not lie in a single hyperplane, it is conceivable that there is a configuration $P_1, \ldots, P_k \in \mathbb{P}^n_\mathbb{R}$ such that $m_{I_1} = 1$ and $m_{I_j} = 0$ for $2 \leq j \leq \binom{k}{n+1}$. In this case

$$(3.17) \qquad \overline{\mathcal{V}}(P_1, \ldots, P_k) = \{(1 : 0 : 0 : \cdots : 0)\}.$$

Now let $Q_1, \ldots, Q_k \in \mathbb{P}^n_\mathbb{R}$ be a configuration with $m_{I_1} \neq 0$. Then since we may obtain some points of $\overline{\mathcal{V}}(Q_1, \ldots, Q_k)$ from $\mathcal{V}(Q_1, \ldots, Q_k)$ by allowing some (but not all) of the a_i to be equal to zero, we see that $(1 : 0 : 0 : \cdots : 0) \in \overline{\mathcal{V}}(Q_1, \ldots, Q_k)$ by letting $a_i = 1$ for $i = 1, \ldots, n+1$ and letting $a_i = 0$ for $i = n+2, \ldots, k$. This gives us that $\overline{\mathcal{V}}(P_1, \ldots, P_k) \subset \overline{\mathcal{V}}(Q_1, \ldots, Q_k)$.

To avoid having one shape variety wholly contained in another and to ensure that the shape varieties have similar structure, we should restrict our attention to configurations whose shape variety has maximal dimension. In other words, we only want to consider configurations for which the map $\overline{\Phi}$ is injective (so that $dim(\overline{\mathcal{V}}) = k - 1$), rather than allowing all noncoplanar configurations.

Theorem B.5. *Suppose $P_1, \ldots, P_k$ is a configuration of k points in $\mathbb{P}^n_\mathbb{R}$ so that there is a subset $P_{i_1}, \ldots, P_{i_{n+2}}$ of $n + 2$ points in this configuration having the following properties:*

1. *for every subset $J = \{j_1, \ldots, j_{n+1}\} \subset \{i_1, \ldots, i_{n+2}\}$ the points $P_{j_1}, \ldots, P_{j_{n+1}}$ do not lie in a single hyperplane (i.e. $m_J \neq 0$)*

2. *there is some subset $S = \{s_1, \ldots, s_n\} \subset \{i_1, \ldots, i_{n+2}\}$ such that for all P_t not in the set $\{P_{i_1}, \ldots, P_{i_{n+2}}\}$ we have that the points $P_{s_1}, \ldots, P_{s_n}, P_t$ do not all lie in a single hyperplane (i.e. $m_{s_1 \ldots s_n t} \neq 0$).*

Then, the map $\overline{\Phi}$ is injective.

Proof. We will show that under these conditions,

$$(a_{I_1} m_{I_1} : \ldots : a_{I_N} m_{I_N}) = (m_{I_1} : \ldots : m_{I_N}) \qquad \Leftrightarrow \qquad a_i = a_j \text{ for all } i, j.$$

Note that

$$(a_{I_1} m_{I_1} : \ldots : a_{I_N} m_{I_N}) = (m_{I_1} : \ldots : m_{I_N})$$

if and only for all $i \neq j$,

$$\frac{a_{I_i} m_{I_i}}{a_{I_j} m_{I_j}} = \frac{m_{I_i}}{m_{I_j}}$$

assuming of course that $m_{I_j} \neq 0$.

First, let $\alpha, \beta \in \{1, \ldots, k\}$ be such that α, β are not in the set $\{i_1, \ldots, i_{n+2}\}$. Then by condition 2, if we let $A = (s_1, \ldots, s_n, \alpha)$ and let $B = (s_1, \ldots, s_n, \beta)$ we have that $m_A \neq 0$ and $m_B \neq 0$. Thus since

$$\frac{a_A m_A}{a_B m_B} = \frac{m_A}{m_B}$$

we have that

$$\frac{a_\alpha}{a_\beta} = 1$$

and hence $a_\alpha = a_\beta$.

Now suppose $\alpha, \beta \in \{i_1, \ldots, i_{n+2}\}$ with $\alpha \neq \beta$, and let $\{j_1, \ldots, j_n\} = \{i_1, \ldots, i_{n+2}\} - \{\alpha, \beta\}$. Let $A = (j_1, \ldots, j_n, \alpha)$ and let $B = (j_1, \ldots, j_n, \beta)$. Then m_A and m_B are nonzero and hence

$$\frac{a_A m_A}{a_B m_B} = \frac{m_A}{m_B}$$

from which we see that

$$\frac{a_\alpha}{a_\beta} = 1.$$

Thus we again see that $a_\alpha = a_\beta$.

Finally, suppose $\alpha, \beta \in \{1, \ldots, k\}$ are such that $\alpha \in \{i_1, \ldots, i_{n+2}\}$ but β is not. Let $\gamma \in \{i_1, \ldots, i_{n+2}\} - \{s_1, \ldots, s_n\}$ so that $a_\alpha = a_\gamma$ by the above case. Let $A = (s_1, \ldots, s_n, \gamma)$ and $B = (s_1, \ldots, s_n, \beta)$. Then m_A and m_B are both nonzero and once again

$$\frac{a_A m_A}{a_B m_B} = \frac{m_A}{m_B}.$$

Now we have that

$$\frac{a_\gamma}{a_\beta} = 1$$

which gives us that $a_\alpha = a_\gamma = a_\beta$.

Thus, for configurations satisfying conditions 1 and 2, the map $\overline{\Phi}$ is injective. $\square$

Up to this point, our constructions in the general case of k points in $\mathbb{P}^n_{\mathbb{R}}$ ($k \geq n+1$) have been identical to our constructions in the case of 4 points in $\mathbb{P}^1_{\mathbb{R}}$. We do see a slight variation when we compute the defining equations of the shape varieties of configurations of k points in $\mathbb{P}^n_{\mathbb{R}}$ for $k > 4$. For example, consider the case of 5 points $P_1, \ldots, P_5$ in $\mathbb{P}^1_{\mathbb{R}}$. Then as in the case of 4 points $\mathbb{P}^1_{\mathbb{R}}$ the quadratic relations

$$(3.18) \qquad m_{i_1 i_2} m_{i_3 i_4} x_{\sigma(i_1)\sigma(i_2)} x_{\sigma(i_3)\sigma(i_4)} - m_{\sigma(i_1)\sigma(i_2)} m_{\sigma(i_3)\sigma(i_4)} x_{i_1 i_2} x_{i_3 i_4} = 0$$

must hold for every $\{i_1, i_2, i_3, i_4\} \subset \{1,2,3,4,5\}$ (i_1, i_2, i_3, i_4 distinct) and for every appropriate permutation σ of $\{i_1, i_2, i_3, i_4\}$. We also observe that for an arbitrary point $(x_{12} : \ldots : x_{45})$ in

$\mathcal{V}(P_1, \ldots, P_5)$ there exist $a_1, \ldots, a_5 \in \mathbb{R}^*$ such that

$$(3.19) \qquad \frac{m_{13} m_{23} m_{45} x_{12} x_{34} x_{35}}{m_{12} m_{34} m_{35} x_{13} x_{23} x_{45}} = \frac{m_{13} m_{23} m_{45} (a_1 a_2 m_{12})(a_3 a_4 m_{34})(a_3 a_5 m_{35})}{m_{12} m_{34} m_{35} (a_1 a_3 m_{13})(a_2 a_3 m_{23})(a_4 a_5 m_{45})} = 1$$

giving us the relation

$$(3.20) \qquad f = m_{13} m_{23} m_{45} x_{12} x_{34} x_{35} - m_{12} m_{34} m_{35} x_{13} x_{23} x_{45} = 0.$$

The important point here is that for 4 points in $\mathbb{P}^1_{\mathbb{R}}$ each of the numbers 1,2,3, and 4 appeared exactly once in each monomial as an entry of an index of some m_{ij}, but now we have the number 3 appearing in each monomial *twice* as an entry of an index of an m_{ij}. However we if choose new homogeneous coordinates for $P_1, \ldots, P_5$ by scaling our current homogeneous coordinates of P_i by a_i, we get a new polynomial

$f' = a_1 a_2 a_3^2 a_4 a_5 f$ whose zero locus is exactly the same as that of the polynomial f. In particular, we see that the ideal of the shape variety of a configuration of 5 points in $\mathbb{P}_{\mathbb{R}}^1$ is generated by quadratic *and* cubic polynomials rather than just quadratic polynomials as we had in the case of 4 points in $\mathbb{P}_{\mathbb{R}}^1$.

In general we see that for configurations of k points in $\mathbb{P}_{\mathbb{R}}^n$, the defining equations are given by the following theorem.

Theorem B.6. *For a configuration $P_1, \ldots, P_k$ of k points in $\mathbb{P}_{\mathbb{R}}^n$, the variety $\overline{\mathcal{V}}(P_1, \ldots, P_k)$ as a subvariety of the Grassmannian $Gr(n+1, k) \subset \mathbb{P}_{\mathbb{R}}^{\binom{k}{n+1}-1}$ is the zero locus of the following system of polynomials*

$$(3.21) \qquad m_{I_1} m_{I_2} \cdots m_{I_r} x_{J_1} x_{J_2} \cdots x_{J_r} - m_{J_1} m_{J_2} \cdots m_{J_r} x_{I_1} x_{I_2} \cdots x_{I_r}$$

where $I_1, \ldots, I_r, J_1, \ldots, J_r$ are $n+1$-subsets of $\{1, \ldots k\}$ with the property that $\bigcup_{i=1}^r I_i = \bigcup_{i=1}^r J_i$ as multisets and r ranges from 2 to some positive integer $N(k, n)$.

Proof. For a multiset $I \subset \{1, \ldots, k\}$ and $a_1, \ldots, a_k \in \mathbb{R}^*$, define $a_I = \prod_{i \in I} a_i$. Now for an arbitrary point $(x_{12 \cdots n+1} : \cdots : x_{k-n \cdots k}) \in \mathcal{V}(P_1, \ldots, P_k)$ we have that for some $a_1, \ldots, a_k \in \mathbb{R}^*$, $x_I = a_I m_I$ for every $(n+1)$-subset I of $\{1, \ldots, k\}$. Now if for some $r \geq 1$, $I_1, \ldots, I_r, J_1, \ldots, J_r$ are $n+1$-subsets of $\{1, \ldots k\}$ with the property that $\bigcup_{i=1}^r I_i = \bigcup_{i=1}^r J_i$ as multisets, then

$$\frac{m_{I_1} m_{I_2} \cdots m_{I_r} x_{J_1} x_{J_2} \cdots x_{J_r}}{m_{J_1} m_{J_2} \cdots m_{J_r} x_{I_1} x_{I_2} \cdots x_{I_r}} = \frac{m_{I_1} m_{I_2} \cdots m_{I_r} (a_{J_1} m_{J_1})(a_{J_2} m_{J_2}) \cdots (a_{J_r} m_{J_r})}{m_{J_1} m_{J_2} \cdots m_{J_r} (a_{I_1} m_{I_1})(a_{I_2} m_{I_2}) \cdots (a_{I_r} m_{I_r})} = 1$$

giving us the relation

$$m_{I_1} m_{I_2} \cdots m_{I_r} x_{J_1} x_{J_2} \cdots x_{J_r} - m_{J_1} m_{J_2} \cdots m_{J_r} x_{I_1} x_{I_2} \cdots x_{I_r}.$$

It is easy to see that the only other relations that points in $\overline{\mathcal{V}}(P_1, \ldots, P_k)$ must satisfy are the Plücker relations. Thus the polynomials given in 3.21 generate an ideal whose

zero locus is the variety $\overline{\mathcal{V}}$ noting that since the polynomial ring $\mathbb{R}[x_{12\cdots n+1}, \ldots, x_{k-n\cdots k}]$ is Noetherian, there is an upper bound $N(k,n)$ on the degrees of the polynomials needed to generate the ideal. $\qquad\square$

We do not know the exact value of $N(k,n)$, but computing the equations of the shape varieties for small k and n using a Gröebner basis elimination seems to indicate that $N(k,n) = k - 2$. There is also some evidence to indicate that the ideal of a shape variety is in fact generated by the quadratic relations

$$(3.22) \qquad m_{i_1 i_2} m_{i_3 i_4} x_{\sigma(i_1)\sigma(i_2)} x_{\sigma(i_3)\sigma(i_4)} - m_{\sigma(i_1)\sigma(i_2)} m_{\sigma(i_3)\sigma(i_4)} x_{i_1 i_2} x_{i_3 i_4}$$

together with the Plücker relations. See the Appendix for examples.

C. Projective Object/Image Equations

Let $P_1, \ldots, P_k \in \mathbb{P}^3_{\mathbb{R}}$ be an object configuration consisiting of k points in projective 3-space, and let $Q_1, \ldots, Q_k \in \mathbb{P}^2_{\mathbb{R}}$ be an image configuration consisting of k points in the projective plane. We want to (as in the affine case) find necessary and sufficient conditions for the Q_i to be a full perspective projection of the P_i. Since every choice of homogeneous coordinates for $P_1, \ldots, P_k$ gives a unique point in $Gr(4, k) \subset \mathbb{P}^{\binom{k}{4}-1}_{\mathbb{R}}$ and every choice of homogeneous coordinates for $Q_1, \ldots, Q_k$ gives a unique point in $Gr(3, k) \subset \mathbb{P}^{\binom{k}{3}-1}_{\mathbb{R}}$, the set $V \subset Gr(4, k) \times G(3, k)$ of matching object/image pairs should be a projective variety defined by a system of bihomogeneous polynomials in the Plücker coordinates $m_{1234}, \ldots, m_{k-3\cdots k}$ on $Gr(4, k)$ and the Plücker coordinates $n_{123}, \ldots, n_{k-2\cdots k}$ on $G(3, k)$. These relations should be satisfied independent of our choice of homogeneous coordinates for our object and image configurations. In other words, we should have that if an image configuration $Q_1, \ldots, Q_n$ is a full perspective projection of an object configuration $P_1, \ldots, P_n$ then the product variety

$\overline{\mathcal{V}}(P_1, \ldots, P_n) \times \overline{\mathcal{V}}(Q_1, \ldots, Q_n)$ should be completely contained in V.

Our first approach to computing the projective object/image relations is to adapt the methods we used in deriving the affine object/image equations. Let $P_i = (x_i : y_i : z_i : w_i) \in \mathbb{P}^3_{\mathbb{R}}$, $i = 1, \ldots, k$ be an object configuration and let $Q_i = (r_i : s_i : t_i) \in \mathbb{P}^2_{\mathbb{R}}$, $i = 1, \ldots, k$ be an image configuration. Then the image configuration $Q_1, \ldots, Q_k$, is a full perspective projection of the object configuration $P_1, \ldots, P_k$ if there is a 3×4 matrix $A = (a_{ij})$ of rank 3 and nonzero scalars $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ such that

$$
(3.23) \qquad
\begin{pmatrix} r_1 & r_2 & & r_k \\ s_1 & s_2 & \cdots & s_k \\ t_1 & t_2 & & t_k \end{pmatrix}
\begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_k \end{pmatrix}
$$
$$
=
\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}
\begin{pmatrix} x_1 & x_2 & & x_k \\ y_1 & y_2 & \cdots & y_k \\ z_1 & z_2 & & z_k \\ w_1 & w_2 & & w_k \end{pmatrix}
\begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix}.
$$

Since the diagonal matrices are invertible, we may rewrite equation 3.23 as

$$
(3.24) \qquad
\begin{pmatrix} r_1 & r_2 & & r_k \\ s_1 & s_2 & \cdots & s_k \\ t_1 & t_2 & & t_k \end{pmatrix}
\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}
$$
$$
=
\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}
\begin{pmatrix} x_1 & x_2 & & x_k \\ y_1 & y_2 & \cdots & y_k \\ z_1 & z_2 & & z_k \\ w_1 & w_2 & & w_k \end{pmatrix}.
$$

We now have that $Q_1, \ldots, Q_k$ is a full perspective projection of $P_1, \ldots, P_k$ if and

only if for some nonzero $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^*$ the null space of

$$
\begin{pmatrix}
x_1 & x_2 & & x_k \\
y_1 & y_2 & \cdots & y_k \\
z_1 & z_2 & & z_k \\
w_1 & w_2 & & w_k
\end{pmatrix}
$$

is contained in the null space of

$$
\begin{pmatrix}
r_1 & r_2 & & r_k \\
s_1 & s_2 & \cdots & s_k \\
t_1 & t_2 & & t_k
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_k
\end{pmatrix}.
$$

If we let $m_{1234}, \ldots, m_{k-3\cdots k}$ be the Plücker coordinates of the null space of $M(P_1, \ldots, P_k)$ in $Gr(4, k)$ and let $n_{123}, \ldots, n_{k-2\cdots k}$ be the Plücker coordinates of the null space of $M(Q_1, \ldots, Q_k)$ in $Gr(3, k)$, then the above incidence relation is given by the system of polynomial equations

$$
(3.25) \qquad \sum_{1 \leq \zeta_1 < \zeta_2 \leq k} \epsilon_{\zeta_1, \zeta_2} \lambda_{\gamma_1} \lambda_{\gamma_2} \lambda_{\gamma_3} n_{\gamma_1, \gamma_2, \gamma_3} m_{\alpha_1, \alpha_2, \zeta_1, \zeta_2} = 0
$$

for all choices of $1 \leq \alpha_1 < \alpha_2 \leq k$ and $1 \leq \beta_1 < \beta_2 < \ldots < \beta_{k-5} \leq k$ where $1 \leq \gamma_1 < \gamma_2 < \gamma_3 \leq k$ is the complement of $\{\zeta_1, \zeta_2, \beta_1, \ldots, \beta_{k-5}\}$ in $\{1, \ldots, k\}$ when $\zeta_1, \zeta_2, \beta_1, \ldots, \beta_{k-5}$ are distinct and $\epsilon_{\zeta_1, \zeta_2}$ is the sign of the permutation

$$
\gamma_1, \gamma_2, \gamma_3, \zeta_1, \zeta_2, \beta_1, \ldots, \beta_{k-5}
$$

of the numbers $1, \ldots, k$(see Chapter II and [7]).

Letting $c_{\gamma_1, \gamma_2, \gamma_3} = \lambda_{\gamma_1} \lambda_{\gamma_2} \lambda_{\gamma_3}$ we then have a system of equations

$$
(3.26) \qquad \sum_{1 \leq \zeta_1 < \zeta_2 \leq k} \epsilon_{\zeta_1, \zeta_2} c_{\gamma_1 \gamma_2 \gamma_3} n_{\gamma_1, \gamma_2, \gamma_3} m_{\alpha_1, \alpha_2, \zeta_1, \zeta_2} = 0
$$

which is linear in the $c_{i_1 i_2 i_3}$, the $n_{i_1 i_2 i_3}$, and the $m_{i_1 i_2 i_3 i_4}$.

Since $c_{i_1 i_2 i_3} = \lambda_{i_1} \lambda_{i_2} \lambda_{i_3}$, we see that

$$(3.27) \qquad c_{i_1 i_2 i_3} c_{j_1 j_2 j_3} = \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} \lambda_{j_1} \lambda_{j_2} \lambda_{j_3} = c_{\sigma(i_1)\sigma(i_2)\sigma(i_3)} c_{\sigma(j_1)\sigma(j_2)\sigma(j_3)}$$

for all $i_i, i_2, i_3, j_1, j_2, j_3 \in \{1, \ldots, k\}$ and for all permutations σ of $i_i, i_2, i_3, j_1, j_2, j_3$. This gives us the relations

$$(3.28) \qquad c_{i_1 i_2 i_3} c_{j_1 j_2 j_3} - c_{\sigma(i_1)\sigma(i_2)\sigma(i_3)} c_{\sigma(j_1)\sigma(j_2)\sigma(j_3)} = 0.$$

In fact, it is true that if $I_l = (i_{l1}, i_{l2}, i_{l3})$ for $l = 1, \ldots, N$ with $i_{l1}, i_{l2}, i_{l3} \in \{1, \ldots, k\}$ and if σ is a permutation of $i_{11}, i_{12}, i_{13}; \ldots; i_{N1}, i_{N2}, i_{N3}$, then

$$(3.29) \qquad c_{I_1} c_{I_2} \cdots c_{I_N} - c_{\sigma I_1} c_{\sigma I_2} \cdots c_{\sigma I_N} = 0$$

where $\sigma I_l = (\sigma(i_{l1}), \sigma(i_{l2}), \sigma(i_{l3}))$.

It is possible to use the quadratic relations 3.28 to rewrite equation 3.29 in the form

$$(3.30) \qquad c_{I_1} c_{I_2} \ldots c_{I_N} = c_{I_1} c_{I'_2} \ldots c_{I'_N}$$

so that

$$(3.31) \qquad c_{I_2} \ldots c_{I_N} = c_{I'_2} \ldots c_{I'_N}.$$

Continuing this process inductively, we will eventually arrive at one of the quadratic relations 3.28. Thus, we see that in the polynomial ring $\mathbb{R}[c_{123}, \ldots, c_{k-2\ldots k}]$, the ideal generated by the polynomials $c_{I_1} c_{I_2} \cdots c_{I_N} - c_{\sigma I_1} c_{\sigma I_2} \cdots c_{\sigma I_N}$ is actually generated by the quadratic relations $c_{i_1 i_2 i_3} c_{j_1 j_2 j_3} - c_{\sigma(i_1)\sigma(i_2)\sigma(i_3)} c_{\sigma(j_1)\sigma(j_2)\sigma(j_3)}$. Let $\mathcal{I}$ be this ideal.

So to find the locus of pairs of matching object and image shapes, we should begin by looking in $\mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{k}{4}-1}$ with coordinates $c_{123}, \ldots, c_{k-2\cdots k}, n_{123} \ldots, n_{k-2\cdots k},$

$m_{1234}, \ldots, m_{k-3\cdots k}$. As we have seen the $c_{i_1i_2i_3}$ should satisfy the quadratic relations $c_{i_1i_2i_3}c_{j_1j_2j_3} - c_{\sigma(i_1)\sigma(i_2)\sigma(i_3)}c_{\sigma(j_1)\sigma(j_2)\sigma(j_3)} = 0$, and the $n_{i_1i_2i_3}$ and $m_{j_1j_2j_3j_4}$ should satisfy the Plücker relations on $Gr(3,k)$ and $Gr(4,k)$ respectively. Thus we want to only consider points in $V(\mathcal{I}) \times Gr(3,k) \times Gr(4,k) \subset \mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{k}{4}-1}$ where $V(\mathcal{I})$ is the zero locus of the ideal $\mathcal{I}$.

Let $\tilde{V}$ be the zero locus of the linear polynomials

$$\sum_{1 \leq \zeta_1 < \zeta_2 \leq k} \epsilon_{\zeta_1,\zeta_2} c_{\gamma_1,\gamma_2,\gamma_3} n_{\gamma_1,\gamma_2,\gamma_3} m_{\alpha_1,\alpha_2,\zeta_1,\zeta_2}$$

in $V(\mathcal{I}) \times Gr(3,k) \times Gr(4,k) \subset \mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1} \times \mathbb{P}_{\mathbb{R}}^{\binom{k}{4}-1}$. Then the variety V of matching object/image pairs is the projection

$$
\begin{array}{c}
\tilde{V} \subset V(\mathcal{I}) \times Gr(3,k) \times Gr(4,k) \\
\downarrow \\
Gr(3,k) \times Gr(4,k)
\end{array}
$$

(3.32)

of $\tilde{V}$ onto $Gr(3,k) \times Gr(4,k)$.

To give necessary and sufficient conditions for an image shape (given in the Plücker coordinates $n_{i_1i_2i_3}$) to be a full perspective projection of an object shape (given in the Plücker coordinates $m_{i_1i_2i_3i_4}$) is simply to give a generating set for the ideal of V. Such a generating set may be obtained by using Gröebner bases or resultants to compute a generating set for the elimination ideal $(\mathcal{J} + \mathcal{I}) \cap \mathbb{R}[n_{123}, \ldots, n_{k-2\cdots k}, m_{1234}, \ldots, m_{k-3\cdots k}]$ where $\mathcal{J}$ is the ideal of the polynomials

(3.33)
$$\sum_{1 \leq \zeta_1 < \zeta_2 \leq k} \epsilon_{\zeta_1,\zeta_2} c_{\gamma_1,\gamma_2,\gamma_3} n_{\gamma_1,\gamma_2,\gamma_3} m_{\alpha_1,\alpha_2,\zeta_1,\zeta_2}.$$

This elimination however, can be very computationally expensive, and as such, we are unable to give the complete set of matching equations here.

A more manageable method is seen in [18], but here some more stringent assumptions are made concerning the position of the points of our configurations. Consider an object configuration $P_i = (x_i : y_i : z_i : w_i) \in \mathbb{P}^3_{\mathbb{R}}$, $i = 1, \ldots, k$ with P_1, P_2, P_3, P_4, P_5 in general position. We may then move the configuration by a projective linear transformation so that $P_1 = (1 : 0 : 0 : 0), P_2 = (0 : 1 : 0 : 0), P_3 = (0 : 0 : 1 : 0), P_4 = (0 : 0 : 0 : 1)$ and $P_5 = (1 : 1 : 1 : 1)$. Assume also that for all $i \geq 6$, P_i does not lie in the plane spanned by P_1, P_2, P_3 so that $P_i = (p_{3i-17} : p_{3i-16} : p_{3i-15} : 1)$. These p_j form a fundamental set of invariants for the shape of our object configuration.

The projective linear map T moving P_1, P_2, P_3, P_4, P_5 to this standard position is given by

$$
T(x : y : z : w) = \left(\det \begin{pmatrix} x_2 & x_3 & x_4 & x \\ y_2 & y_3 & y_4 & y \\ z_2 & z_3 & z_4 & z \\ w_2 & w_3 & w_4 & w \end{pmatrix} m_{1345} m_{1245} m_{1235} : \right.
$$

$$
\det \begin{pmatrix} x_1 & x_3 & x_4 & x \\ y_1 & y_3 & y_4 & y \\ z_1 & z_3 & z_4 & z \\ w_1 & w_3 & w_4 & w \end{pmatrix} m_{2345} m_{1245} m_{1235} :
$$

$$
\det \begin{pmatrix} x_1 & x_2 & x_4 & x \\ y_1 & y_2 & y_4 & y \\ z_1 & z_2 & z_4 & z \\ w_1 & w_2 & w_4 & w \end{pmatrix} m_{2345} m_{1345} m_{1235} :
$$

$$(3.34) \qquad \det \begin{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ z_1 & z_2 & z_3 & z \\ w_1 & w_2 & w_3 & w \end{pmatrix} m_{2345} m_{1345} m_{1245} \end{pmatrix}.$$

In particular, we have that for all $i = 1, \ldots, k$

$$T(P_i) = (m_{234i} m_{1345} m_{1245} m_{1235} : m_{134i} m_{2345} m_{1245} m_{1235} :$$

$$(3.35) \qquad m_{124i} m_{2345} m_{1345} m_{1235} : m_{123i} m_{2345} m_{1345} m_{1245}).$$

Since none of $P_6, \ldots, P_k$ lie in the span of P_1, P_2, P_3, the values m_{123i} are nonzero for $i \geq 6$. Furthermore, by our general position hypothesis, $m_{2345}, m_{1345}, m_{1245}$ are also nonzero. Thus

$$p_{3i-17} = \frac{m_{234i} m_{1235}}{m_{123i} m_{2345}}$$

$$p_{3i-16} = \frac{m_{134i} m_{1235}}{m_{123i} m_{1345}}$$

$$(3.36) \qquad p_{3i-15} = \frac{m_{124i} m_{1235}}{m_{123i} m_{1245}}.$$

Note that the p_j are defined independent of our choice of homogeneous coordinates for $P_1, \ldots, P_k$ for if we scale the homogeneous coordinates of each P_i by a nonzero constant a_i, we get

$$p_{3i-17} = \frac{(a_2 a_3 a_4 a_i m_{234i})(a_1 a_2 a_3 a_5 m_{1235})}{(a_1 a_2 a_3 a_i m_{123i})(a_2 a_3 a_4 a_5 m_{2345})} = \frac{m_{234i} m_{1235}}{m_{123i} m_{2345}}$$

$$p_{3i-16} = \frac{(a_1 a_3 a_4 a_i m_{134i})(a_1 a_2 a_3 a_5 m_{1235})}{(a_1 a_2 a_3 a_i m_{123i})(a_1 a_3 a_4 a_5 m_{1345})} = \frac{m_{134i} m_{1235}}{m_{123i} m_{1345}}$$

$$(3.37) \qquad p_{3i-15} = \frac{(a_1 a_2 a_4 a_i m_{124i})(a_1 a_2 a_3 a_5 m_{1235})}{(a_1 a_2 a_3 a_i m_{123i})(a_1 a_2 a_4 a_5 m_{1245})} = \frac{m_{124i} m_{1235}}{m_{123i} m_{1245}}.$$

Similarly let $Q_i = (r_i, s_i, t_i) \in \mathbb{P}^2_{\mathbb{R}}$, $i = 1, \ldots, k$ be an image configuration with Q_1, Q_2, Q_3, Q_4 in general position such that for $i \geq 5$, Q_i is not on the line defined by

Q_1 and Q_2. We move the configuration by a projective linear transformation so that $Q_1 = (1:0:0), Q_2 = (0:1:0), Q_3 = (0:0:1), Q_4 = (1:1:1)$ and for each $i \geq 5$, $Q_i = (q_{2i-9} : q_{2i-8} : 1)$. These q_j form a fundamental set of invariants for the shape of this image configuration.

In this case, the projective transformation S on $\mathbb{P}^2_{\mathbb{R}}$ sending Q_1, Q_2, Q_3, Q_4 to $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ is given by

$$
S\left(r:s:t\right)) = \left(\det \begin{pmatrix} r_2 & r_3 & r \\ s_2 & s_3 & s \\ t_2 & t_3 & t \end{pmatrix} n_{134}n_{124} : \right.
$$
$$
\det \begin{pmatrix} r_1 & r_3 & t \\ s_1 & s_3 & s \\ t_1 & t_3 & t \end{pmatrix} n_{234}n_{124} :
$$
$$
\text{(3.38)} \qquad \left. \det \begin{pmatrix} r_1 & r_2 & r \\ s_1 & s_2 & s \\ t_1 & t_2 & r \end{pmatrix} n_{234}n_{134} \right).
$$

This gives us that

$$
\text{(3.39)} \qquad S(Q_i) = (n_{23i}n_{134}n_{124} : n_{13i}n_{234}n_{124} : n_{12i}n_{234}n_{134})
$$

from which we see that

$$
q_{2i-9} = \frac{n_{23i}n_{124}}{n_{12i}n_{234}}
$$
$$
\text{(3.40)} \qquad q_{2i-8} = \frac{n_{13i}n_{124}}{n_{12i}n_{134}}.
$$

We note that (as in the case of object configurations) the projective invariants $q_1, \ldots,$ q_{2k-8} are defined independent of our choice of homogeneous coordinates for $Q_1, \ldots, Q_k$.

When we make the preceding assumptions about the positioning of our configurations, the object/image equations have been completely determined ([18]). For example, in the case where $n = 6$, we have only one object/image equation given in terms of the projective invariants:

$$(3.41) \quad -q_2q_3p_2p_3 + q_3p_2p_3 - q_3p_3 - q_1q_4p_1 - q_1p_1p_2 + q_1p_1$$
$$= -q_1q_4p_1p_3 + q_4p_1p_3 - q_4p_3 - q_2q_3p_2 - q_2p_1p_2 + q_2p_2.$$

Making the appropriate substitutions and then clearing denominators and removing a monomial factors we have an object/image relation in terms of the Plücker coordinates

$$n_{125}n_{136}n_{234}m_{1236}m_{1246}m_{1345}m_{2345} - n_{123}n_{136}n_{234}m_{1236}m_{1246}m_{1345}m_{2345}$$

$$-n_{126}n_{135}n_{234}m_{1236}m_{1245}m_{1346}m_{2345} + n_{124}n_{135}n_{236}m_{1236}m_{1245}m_{1346}m_{2345}$$

$$+n_{125}n_{134}n_{236}m_{1235}m_{1245}m_{1346}m_{2345} - n_{124}n_{135}n_{236}m_{1235}m_{1246}m_{1346}m_{2345}$$

$$+n_{126}n_{134}n_{235}m_{1236}m_{1245}m_{1345}m_{2346} - n_{124}n_{136}n_{235}m_{1236}m_{1245}m_{1345}m_{2346}$$

$$-n_{125}n_{136}n_{234}m_{1235}m_{1246}m_{1345}m_{2346} + n_{124}n_{136}n_{235}m_{1235}m_{1246}m_{1345}m_{2346}$$

$$+n_{126}n_{135}n_{234}m_{1235}m_{1245}m_{1346}m_{2346} - n_{126}n_{134}n_{235}m_{1235}m_{1245}m_{1346}m_{2346} = 0.$$

$$(3.42)$$

We should note that since the p_i and q_i are defined independent of our choice of homogeneous coordinates for the P_i and Q_i, the relation 3.42 will be satisfied independent of our choice of homogeneous coordinates. This can be verified by simply counting the number of times each of the numbers $1, \ldots, 6$ appear as entries of the indices of the $n_{i_1i_2i_3}$ and $m_{j_1j_2j_3j_4}$ in each monomial.

Now let σ be a permutation of $1, \ldots, k$. Suppose that in our object configuration $P_1, \ldots, P_k \in \mathbb{P}^3_{\mathbb{R}}$ the points $P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}, P_{\sigma(4)}, P_{\sigma(5)}$ are in general position and that for all $i \geq 6$, $P_{\sigma(i)}$ is not in the span of $P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}$. Then we may move our

configuration by a projective transformation so that $P_{\sigma(1)} = (1:0:0:0), P_{\sigma(2)} = (0:1:0:0), P_{\sigma(3)} = (0:0:1:0), P_{\sigma(4)} = (0:0:0:1), P_{\sigma(5)} = (1:1:1:1)$, and for $i \geq 6, \ P_{\sigma(i)} = (p'_{3i-17} : p'_{3i-16} : p'_{3i-15} : 1)$.

Similarly, let τ be a permutation of $1, \ldots, k$, and suppose that in our image configuration $Q_1, \ldots, Q_k \in \mathbb{P}^2_{\mathbb{R}}$ the points $Q_{\tau(1)}, Q_{\tau(2)}, Q_{\tau(3)}, Q_{\tau(4)}$ are in general position and that for all $i \geq 6$, $Q_{\tau(i)}$ is not in the span of $Q_{\tau(1)}$ and $Q_{\tau(2)}$. We now move $Q_1, \ldots, Q_k$ by a projective transformation so that $Q_{\tau(1)} = (1:0:0), Q_{\tau(2)} = (0:1:0), Q_{\tau(3)} = (0:0:1), Q_{\tau(4)} = (1:1:1)$ and for $i \geq 5, \ Q_{\tau(i)} = (q'_{2i-9} : q'_{2i-8} : 1)$.

We now have a new set of object invariants $p'_1, \ldots, p'_{3k-15}$ and a new set of image invariants $q'_1, \ldots, q'_{2k-8}$ which, as before, may be written in terms of Plücker coordinates

$$p'_{3i-17} = \frac{m_{\sigma(2)\sigma(3)\sigma(4)\sigma(i)} m_{\sigma(1)\sigma(2)\sigma(3)\sigma(5)}}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(i)} m_{\sigma(2)\sigma(3)\sigma(4)\sigma(5)}}$$

$$p'_{3i-16} = \frac{m_{\sigma(1)\sigma(3)\sigma(4)\sigma(i)} m_{\sigma(1)\sigma(2)\sigma(3)\sigma(5)}}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(i)} m_{\sigma(1)\sigma(3)\sigma(4)\sigma(5)}}$$

$$p'_{3i-15} = \frac{m_{\sigma(1)\sigma(2)\sigma(4)\sigma(i)} m_{\sigma(1)\sigma(2)\sigma(3)\sigma(5)}}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(i)} m_{\sigma(1)\sigma(2)\sigma(4)\sigma(5)}}$$

$$q'_{2i-9} = \frac{n_{\tau(2)\tau(3)\tau(i)} n_{\tau(1)\tau(2)\tau(4)}}{n_{\tau(1)\tau(2)\tau(i)} n_{\tau(2)\tau(3)\tau(4)}}$$

(3.43) $$q'_{2i-9} = \frac{n_{\tau(1)\tau(3)\tau(i)} n_{\tau(1)\tau(2)\tau(4)}}{n_{\tau(1)\tau(2)\tau(i)} n_{\tau(1)\tau(3)\tau(4)}}$$

keeping in mind that we view the $m_{j_1 j_2 j_3 j_4}$ and the $n_{i_1 i_2 i_3}$ as skew-symmetric in their indices.

Using the method of [18] we get a new set of object/image relations in terms of these new invariants which we may again write in terms of Plücker coordinates. We should notice that since our projective transformations are completely determined by sending $P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}, P_{\sigma(4)}, P_{\sigma(5)}$ to $(1:0:0:0), (0:1:0:0), (0:0:1:0), (0:0:0:1), (1:1:1:1)$ respectively and by sending $Q_{\tau(1)}, Q_{\tau(2)}, Q_{\tau(3)}, Q_{\tau(4)}$

to $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ respectively, we may assume that $\sigma(6) < \ldots < \sigma(k)$ and that $\tau(5) < \ldots < \tau(k)$. Taking all of these object/image relations as σ ranges over all permutations of $1, \ldots, k$ with $\sigma(6) < \ldots < \sigma(k)$ and as τ ranges over all permutations of $1, \ldots, k$ with $\tau(5) < \ldots < \tau(k)$ gives us a global system of object/image relations (we use global here to mean that all of our configurations satisfy the conditions in Theorem B.5). Even in the case of $k = 6$, this list of polynomials is too long to list here and so is omitted.

D. Projective Shape Spaces in $\mathbb{P}^N_{\mathbb{R}}$

One shortcoming of the shape variety approach that we have presented in this chapter is that it gives us no natural notion of distance between our projective shapes. At first glance it seems that we should be able to define the distance between two shapes to be the minimum distance between their respective shape varieties in $\mathbb{P}^N_{\mathbb{R}}$. However, even in the case of 4 points in $\mathbb{P}^1_{\mathbb{R}}$ this "metric" fails.

As we saw in section B, the shape variety of a generic configuration of 4 points in $\mathbb{P}^1_{\mathbb{R}}$ has dimension 3. Since all of the shape varieties in this case are subvarieties of the 4-dimensional variety $Gr(2,4) \subset \mathbb{P}^5_{\mathbb{R}}$ we see that in general, the shape varieties of two configurations of 4 points in $\mathbb{P}^1_{\mathbb{R}}$ will intersect.

For example, consider the configurations $P_1 = (1:0), P_2 = (0:1), P_3 = (1:1), P_4 = (1:2)$ and $Q_1 = (1:0), Q_2 = (0:1), Q_3 = (1:1), Q_4 = (1:3)$ in $\mathbb{P}^1_{\mathbb{R}}$. Since $P_i = Q_i$ for $i = 1, 2, 3$ and since $P_4 \neq Q_4$, these two configurations have distinct shapes. Both of these configurations satisfy the conditions of Theorem B.5 and thus their respective shape varieties have dimension 3. Since these two varieties are contained in $Gr(2,4)$ (which has dimension 4), they must intersect making the minimum distance between these two shape varieties zero. This gives us that the

distance between these two shapes is zero, which we certainly don't want to happen.

What we need is an embedding of our shape space into some higher dimensional projective space $\mathbb{P}^N_\mathbb{R}$ so that each projective shape will be represented by a single point in $\mathbb{P}^N_\mathbb{R}$. In doing this, we obtain a metric on our shape space induced by the Fubini-Study metric on $\mathbb{P}^N_\mathbb{R}$.

To achieve this representation of projective shapes as points in some projective space, we will need to restrict our attention to the open set $\mathcal{U}_{k,n} \subset (\mathbb{P}^n_\mathbb{R})^k$ consisting of configurations $P_1, \ldots, P_k \in \mathbb{P}^n_\mathbb{R}$ whose points are in general linear position. That is, we will assume that no $n + 1$ of them lie in a single hyperplane so that all of the $m_{i_1 \cdots i_{n+1}}$ are nonzero. Since none of the $m_{i_1 \cdots i_{n+1}}$ are zero, we see from Theorem B.6 that the shape varieties all have the same dimension and degree. We will then embed the space of projective shapes $\mathcal{U}_{k,n}/PGL(n + 1)$ into some real projective space $\mathbb{P}^N_\mathbb{R}$.

1. The Chow Embedding

The *Chow embedding* is a map which assigns to each projective variety in $\mathbb{P}^n_\mathbb{R}$ of dimension m and degree d a unique point in some higher dimensional projective space $\mathbb{P}^N_\mathbb{R}$ in the following way. Let V be a projective variety in $\mathbb{P}^n_\mathbb{R}$ of dimension m and degree d. Then the locus of projective linear subspaces of dimension $\mathrm{codim}(V) - 1 = n - m - 1$ that have nonempty intersection with V is a hypersurface in $Gr(n - m, n + 1) \subset \mathbb{P}^{\binom{n+1}{n-d}-1}_\mathbb{R}$. This hypersurface is the zero locus of a homogeneous polynomial F of degree d in the Plücker coordinates on $Gr(n - m, n + 1)$. The coefficients of the monomials of F are called the *Chow coordinates* of the variety V and the point in $\mathbb{P}^N_\mathbb{R}$ with these coordinates is called the *Chow point* of V. The Chow embedding is the map that sends an m-dimensional variety $V \subset \mathbb{P}^n_\mathbb{R}$ of degree d to its corresponding Chow point.

Since the shape varieties of configurations of k points in $\mathbb{P}^n_\mathbb{R}$ in general position

all have the same dimension and degree, we can use the Chow embedding to assign to each projective shape a unique point in $\mathbb{P}_{\mathbb{R}}^N$. In the case of 4 points in $\mathbb{P}_{\mathbb{R}}^1$, the Chow forms of the shape varieties have been computed by Jody Wilson, Peter Stiller, and Amit Khetan using the methods of [11]. For a configuration of $P_1, P_2, P_3, P_4 \in \mathbb{P}_{\mathbb{R}}^1$, the Chow form of $\overline{V}(P_1, P_2, P_3, P_4)$ is

$$-m_{14}m_{23}m_{12}m_{34}x_{56}x_{35}x_{24}x_{12} + m_{14}^2m_{23}^2x_{56}x_{26}x_{15}x_{12}$$

$$-m_{14}m_{23}m_{24}m_{13}x_{56}x_{25}x_{24}x_{13} - m_{14}m_{23}m_{12}m_{34}x_{56}x_{25}x_{24}x_{13}$$

$$+m_{14}m_{23}m_{24}m_{13}x_{46}x_{36}x_{15}x_{12} + m_{24}^2m_{13}^2x_{46}x_{36}x_{14}x_{13}$$

$$-m_{14}m_{23}m_{12}m_{34}x_{46}x_{35}x_{25}x_{12} + m_{24}m_{13}m_{12}m_{34}x_{46}x_{35}x_{23}x_{14}$$

$$+m_{14}m_{23}m_{12}m_{34}x_{46}x_{25}^2x_{13} - m_{14}m_{23}m_{12}m_{34}x_{45}x_{25}x_{23}x_{15}$$

$$+m_{24}m_{13}m_{12}m_{34}x_{45}x_{36}x_{24}x_{13} + m_{12}^2m_{34}^2x_{45}x_{35}x_{24}x_{23}$$

$$(3.44) \qquad -m_{14}m_{23}m_{12}m_{34}x_{45}x_{26}x_{25}x_{13} + m_{14}m_{23}m_{12}m_{34}x_{45}x_{26}x_{23}x_{15}.$$

The drawback to this approach is that the projective space containing the Chow points is of extremely high dimension. In the simplest nontrivial case of 4 points in $\mathbb{P}_{\mathbb{R}}^1$ we have one Chow coordinate for each degree 4 monomial in the $\binom{6}{2} = 15$ variables $x_{12}, x_{13}, \ldots, x_{56}$. There are $\binom{15+4}{4} = 3876$ such monomials so the target space of our embedding is $\mathbb{P}_{\mathbb{R}}^{3875}$.

In general, for k points in $\mathbb{P}_{\mathbb{R}}^n$ the shape varieties are $k-1$ dimensional varieties in $\mathbb{P}_{\mathbb{R}}^{\binom{k}{n+1}-1}$. So for a given configuration $P_1, \ldots, P_k$ we are looking for projective linear subspaces of $\mathbb{P}_{\mathbb{R}}^{\binom{k}{n+1}-1}$ of dimension $\binom{k}{n+1}-1-(k-1)-1 = \binom{k}{n+1}-k-1$ that intersect $\overline{V}(P_1, \ldots, P_k)$. The locus of such subspaces is a hypersurface on $Gr\left(\binom{k}{n+1} - k, \binom{k}{n+1}\right)$ which lies in a projective space of dimension

$$\begin{pmatrix} \binom{k}{n+1} \\ \binom{k}{n+1} - k \end{pmatrix} - 1 = \begin{pmatrix} \binom{k}{n+1} \\ k \end{pmatrix} - 1.$$

In the case of 6 points in $\mathbb{P}_{\mathbb{R}}^2$ ($k = 6, n = 2$), this space will have dimension

$$\left(\begin{matrix} \binom{6}{3} \\ 6 \end{matrix} \right) - 1 = \binom{20}{6} - 1 = 38759.$$ If the Chow form then has degree d, the target

space for this "simple" case is $\mathbb{P}_{\mathbb{R}}^{\binom{38760+d}{d}-1}$. In this large of a space, it is nearly

impossible use the Chow coordinates to compute the distance between two shapes.

Notice, that in the case of 4 points in $\mathbb{P}_{\mathbb{R}}^1$, the Chow points are contained in

$\mathbb{P}_{\mathbb{R}}^{12} \subset \mathbb{P}_{\mathbb{R}}^{3876}$ where all but twelve of the coordinates are zero. This gives us some

hope that in the general case of k points in $\mathbb{P}_{\mathbb{R}}^n$ we may be able to compose the Chow

embedding with a projection onto a lower dimensional projective space so that we

may view our shapes as points in a more manageable dimension. However, this still

leaves us with the task of computing the Chow Forms for our shape varieties which

is, in general, extremely difficult. For more on computing these polynomials see [3],

[4], and [11].

2. An Alternative to the Chow Embedding

Since computing the Chow form is so difficult, let us instead try to find another map

that embeds the shape space $\mathcal{U}_{k,n}/PGL(n+1)$ in a projective space $\mathbb{P}_{\mathbb{R}}^N$ of lower

dimension. We begin by considering 4 points, in $\mathbb{P}_{\mathbb{R}}^1$ in general position (i.e. all 4

points are distinct).

Let $\mathcal{U}_{4,1}' \subset Gr(2,4)$ be the open set of points whose Plücker coordinates come from

a configuration in $\mathcal{U}_{4,1} \subset (\mathbb{P}_{\mathbb{R}}^1)^4$ (i.e. a configuration in general position). Consider

the configuration $P_i = (x_i : y_i)$, $i = 1, 2, 3, 4$ in $\mathcal{U}_{4,1} \subset (\mathbb{P}_{\mathbb{R}}^1)^4$. As we have seen, with

these homogeneous coordinates this configuration corresponds to a point

$$(3.45) \qquad (m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}) \in \mathcal{U}_{4,1}' \subset Gr(2,4) \subset \mathbb{P}_{\mathbb{R}}^5$$

where $m_{ij} = x_i y_j - x_j y_i$. Scaling these homogeneous coordinates of each P_i by a

$a_i \neq 0$ gives us a new point

(3.46) $(a_1 a_2 m_{12} : a_1 a_3 m_{13} : a_1 a_4 m_{14} : a_2 a_3 m_{23} : a_2 a_4 m_{24} : a_3 a_4 m_{34}) \in \mathcal{U}'_{4,1}.$

What we want is a map $\phi_{4,1} : \mathcal{U}'_{4,1} \subset Gr(2,4) \rightarrow \mathbb{P}^N_{\mathbb{R}}$ that sends all of the points $(a_1 a_2 m_{12} : a_1 a_3 m_{13} : a_1 a_4 m_{14} : a_2 a_3 m_{23} : a_2 a_4 m_{24} : a_3 a_4 m_{34})$ as a_i ranges over $\mathbb{R}^*$ to the same point in $\mathbb{P}^N_{\mathbb{R}}$. In other words, $\phi_{4,1}$ should collapse each $\mathcal{V}(P_1, P_2, P_3, P_4)$ to a single point. Moreover, $\phi_{4,1}$ should send distinct $\mathcal{V}(P_1, P_2, P_3, P_4)$ to distinct points in $\mathbb{P}^N_{\mathbb{R}}$. Notice that since we are working on $\mathcal{U}'_{4,1}$ we need only concern ourselves with $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4) \cap \mathcal{U}'_{4,1} = \mathcal{V}(P_1, P_2, P_3, P_4)$ rather than the entire shape variety $\overline{\mathcal{V}}(P_1, P_2, P_3, P_4) \subset \mathbb{P}^5_{\mathbb{R}}$.

Consider the map $\phi_{4,1} : \mathcal{U}'_{4,1} \rightarrow \mathbb{P}^2_{\mathbb{R}}$ given by

(3.47) $\phi_{4,1}(m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}) = (m_{12}m_{34} : m_{13}m_{24} : m_{14}m_{23}).$

When we scale the homogeneous coordinates $(x_i : y_i)$ by $a_1, a_2, a_3, a_4 \in \mathbb{R}^*$ we have

$$\phi_{4,1}(a_1 a_2 m_{12} : a_1 a_3 m_{13} : a_1 a_4 m_{14} : a_2 a_3 m_{23} : a_2 a_4 m_{24} : a_3 a_4 m_{34})$$
$$= (a_1 a_2 a_3 a_4 m_{12}m_{34} : a_1 a_2 a_3 a_4 m_{13}m_{24} : a_1 a_2 a_3 a_4 m_{14}m_{23})$$
$$= a_1 a_2 a_3 a_4 (m_{12}m_{34} : m_{13}m_{24} : m_{14}m_{23})$$
(3.48) $$= (m_{12}m_{34} : m_{13}m_{24} : m_{14}m_{23}).$$

Thus, $\phi_{4,1}$ maps all configurations in $\mathcal{U}'_{4,1}$ of the same shape to the same point in $\mathbb{P}^2_{\mathbb{R}}$ and so induces a well defined map $\overline{\phi}_{4,1} : \mathcal{U}_{4,1}/PGL(2) \rightarrow \mathbb{P}^2_{\mathbb{R}}$.

Now for a configuration $P_1, P_2, P_3, P_4 \in \mathbb{P}^1_{\mathbb{R}}$ in general position, we may move the points by a projective transformation so that $P_1 = (1 : 0)$, $P_2 = (0 : 1)$, $P_3 = (1 : 1)$, $P_4 = (t : 1)$. The value $t \neq 0, 1$ (as with the p_i and q_i in section C) is the fundamental invariant of the shape of the configuration. In other words, distinct

values of t yield configurations with distinct shapes.

When we move our configuration to this standard position, the Plücker coordinates become

$$m_{12} = 1 \qquad\qquad m_{23} = -1$$

$$m_{13} = 1 \qquad\qquad m_{24} = -t$$

(3.49) $\qquad\qquad m_{14} = 1 \qquad\qquad m_{34} = 1 - t.$

In terms of the invariant t, the map $\phi_{4,1}$ is given by

$$\phi_{4,1}(m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}) = (1 - t : -t : -1)$$

(3.50) $$= (t - 1 : t : 1)$$

We can now see that $\phi_{4,1}$ sends distinct shapes to distinct points in $\mathbb{P}_{\mathbb{R}}^2$ and hence the induced map $\overline{\phi}_{4,1} : \mathcal{U}_{4,1}/PGL(2) \to \mathbb{P}_{\mathbb{R}}^2$ is in fact an embedding of our shape space.

Definition D.1. Let P_1, P_2, P_3, P_4 be a configuration of points in $\mathbb{P}_{\mathbb{R}}^1$ and let m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34} be the Plücker coordinates corresponding to some choice of homogeneous coordinates for P_1, P_2, P_3, P_4. Then we call the coordinates of $\phi_{4,1}(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})$ the *projective shape coordinates* of the configuration P_1, P_2, P_3, P_4.

Notice that the map $\phi_{4,1}$ is the composition of the degree 2 Veronese map $\nu_{5,2}|_{\mathcal{U}'_{4,1}} : \mathbb{P}_{\mathbb{R}}^5 \to \mathbb{P}_{\mathbb{R}}^{\binom{7}{2}-1}$ (restricted to $\mathcal{U}'_{4,1}$) with a coordinate projection π onto $\mathbb{P}_{\mathbb{R}}^2$ defined by selecting degree 2 monomials in the m_{ij} where the values 1,2,3,4 each appear exactly once as an entry in the index of some m_{ij}. In general this composition is a rational map that is not defined when $m_{12}m_{34} = m_{13}m_{24} = m_{14}m_{23} = 0$. By restricting to the open set $\mathcal{U}'_{4,1}$, we force each m_{ij} to be nonzero so that $\phi_{4,1}$ is a well-defined regular map from $\mathcal{U}'_{4,1}$ into $\mathbb{P}_{\mathbb{R}}^2$.

We can extend this notion of shape coordinates to configurations $P_1, \ldots, P_k \in \mathbb{P}_{\mathbb{R}}^n$

whose points are in general position. We do this by defining a map $\phi_{k,n} : \mathcal{U}'_{k,n} \subset Gr(n+1,k) \to \mathbb{P}^N_{\mathbb{R}}$ that effectively collapses each $\mathcal{V}(P_1, \ldots, P_k)$ to a single point and sends distinct shape varieties to distinct points in $\mathbb{P}^N_{\mathbb{R}}$. Such a map will then induce an embedding $\overline{\phi}_{k,n} : \mathcal{U}_{k,n}/PGL(n+1) \to \mathbb{P}^N_{\mathbb{R}}$ of our shape space in $\mathbb{P}^N_{\mathbb{R}}$.

We define $\phi_{k,n}(m_{1\ldots n+1} : \ldots : m_{k-n\ldots k}) = (M_1 : M_2 : \ldots : M_N)$ where the M_i are monomials of degree d in the $m_{i_1 \ldots i_{n+1}}$ such that each of the numbers $1, \ldots, k$ appears t times as entries of indices of the $m_{i_1 \ldots i_{n+1}}$ in M_l, $l = 1, \ldots, N$. Here we want to choose t and d to be the smallest integers with $t \geq 2$ and $d(n+1) = kt$. So if $n+1$ does not divide k, we have $d = \frac{\operatorname{lcm}(k,n+1)}{n+1}$ and $t = \frac{\operatorname{lcm}(k,n+1)}{k}$. If $n+1$ divides k, then $\operatorname{lcm}(k, n+1) = k$ making $\frac{\operatorname{lcm}(k,n+1)}{k}$ equal to 1. In this case, we will let $t = 2$ making $d = \frac{2k}{n+1}$

Theorem D.2. *The map $\overline{\phi}_{k,n} : \mathcal{U}_{k,n}/PGL(n+1) \to \mathbb{P}^N_{\mathbb{R}}$ induced by the map $\phi_{k,n} : \mathcal{U}'_{k,n} \to \mathbb{P}^N_{\mathbb{R}}$ embeds the shape space $\mathcal{U}_{k,n}/PGL(n+1)$ in $\mathbb{P}^N_{\mathbb{R}}$ for some N.*

Proof. To show that $\overline{\phi}_{k,n}$ is an embedding, we need to show that for all configurations $P_1, \ldots, P_k$ in $\mathcal{U}_{k,n}$, $\phi_{k,n}(\mathcal{V}(P_1, \ldots, P_k))$ is a single point in $\mathbb{P}^N_{\mathbb{R}}$ and that $\phi_{k,n}$ maps distinct $\mathcal{V}(P_1, \ldots, P_k)$ to distinct points in $\mathbb{P}^N_{\mathbb{R}}$.

Let $P_1, \ldots, P_k$ be a configuration in $\mathcal{U}_{k,n}$ with Plücker coordinates $m_{i_1 i_2 \ldots i_{n+1}}$ and suppose that $\phi_{k,n}(\ldots : m_{i_1 i_2 \ldots i_{n+1}} : \ldots) = (M_1 : M_2 : \ldots : M_N)$. If we scale the homogeneous coordinates of each P_i by some $a_i \in \mathbb{R}^*$, then each M_i is scaled by $a_1^t a_2^t \cdots a_k^t$ since each of the numbers $1, \ldots, k$ appears t times in the indices of M_i. Thus we get

$$\phi_{k,n}(\ldots : a_{i_1} a_{i_2} \cdots a_{i_{n+1}} m_{i_1 i_2 \ldots i_{n+1}} : \ldots) = a_1^t a_2^t \cdots a_k^t (M_1 : M_2 : \ldots : M_N)$$

$$\text{(3.51)} \hspace{4cm} = (M_1 : M_2 : \ldots : M_N)$$

and hence $\phi_{k,n}(\mathcal{V}(P_1, \ldots, P_k))$ is the single point $(M_1 : M_2 : \ldots : M_N)$.

To see that $\phi_{k,n}$ sends distinct $\mathcal{V}(P_1, \ldots, P_k)$ to distinct points, we first observe that since $P_1, \ldots, P_k$ are in general position, we may move the P_i by a projective transformation and scale the homogeneous coordinates so that the configuration matrix is

$$(3.52) \qquad M(P_1, \ldots, P_k) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & p_1 & p_{n+1} & \cdots & p_{n(k-n-3)+1} \\ 0 & 1 & \cdots & 0 & 1 & p_2 & p_{n+2} & \cdots & p_{n(k-n-3)+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & p_n & p_{2n} & \cdots & p_{n(k-n-3)+n} \\ 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

As we saw in section C, the values $p_1, \ldots, p_{n(k-n-3)+n}$ form a fundamental set of invariants for the shape of the configuration. If we let $I_1, \ldots, I_{n+1}$ be the n-subsets of $\{1, 2, 3, \ldots, n+1\}$ ordered reverse lexicographically, then we can compute (as we did for configurations in $\mathbb{P}^2_\mathbb{R}$ and $\mathbb{P}^3_\mathbb{R}$) for $n + 3 \le i \le k$ and $1 \le j \le n$

$$(3.53) \qquad p_{n(i-n-3)+j} = \frac{m_{I_j \cup \{i\}} m_{I_{n+1} \cup \{n+2\}}}{m_{I_{n+1} \cup \{i\}} m_{I_j \cup \{n+2\}}}.$$

For instance, $p_{n(i-n-3)+1} = \frac{(m_{23\ldots n+1\ i})(m_{12\ldots n\ n+2})}{(m_{12\ldots ni})(m_{23\ldots n+1\ n+2})}$.

The important thing to observe here is that for each p_r, there are distinct monomials M_α and M_β such that $\frac{M_\alpha}{M_\beta} = p_r$. We note here that since our configurations are in general position, all of the $m_{i_1 \ldots i_{n+1}}$ are nonzero and hence all of the M_i are nonzero.

Now consider two configurations $P_1, \ldots, P_k$ and $P_1', \ldots, P_k'$ in $\mathcal{U}_{k,n}$ with projective invariants $p_1, \ldots, p_{n(k-n-2)}$ and $p_1', \ldots, p_{n(k-n-2)}'$ respectively. Suppose that $\phi_{k,n}(\mathcal{V}(P_1, \ldots, P_k)) = (M_1, \ldots, M_N)$ and that $\phi_{k,n}(\mathcal{V}(P_1', \ldots, P_k')) = (M_1', \ldots, M_N')$. If $P_1, \ldots, P_k$ and $P_1', \ldots, P_k'$ do not have the same shape, then for some r, $p_r \ne p_r'$. Hence for some $\alpha \ne \beta$, $\frac{M_\alpha}{M_\beta} \ne \frac{M_\alpha'}{M_\beta'}$ from which we see that $\phi_{k,n}(\mathcal{V}(P_1, \ldots, P_k)) \ne$

$$\phi_{k,n}(\mathcal{V}(P'_1, \ldots, P'_k)).$$ $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \Box$

Consider 6 points in $\mathbb{P}^3_{\mathbb{R}}$. Then the monomials M_i will be of degree $d = \frac{\text{lcm}(6,4)}{4} = \frac{12}{4} = 3$ and each of the numbers 1,2,3,4,5,6 will appear $t = 2$ times in the indices of the $m_{i_1 i_2 i_3 i_4}$ in each M_i. In this case, the map $\phi_{6,3}$ is given by

$$\phi_{6,3}(m_{1234} : \ldots : m_{3456}) = (m_{1234}m_{1256}m_{3456} : m_{1234}m_{1356}m_{2456} : m_{1234}m_{1456}m_{2356} :$$

$$m_{1235}m_{1246}m_{3456} : m_{1235}m_{1346}m_{2456} : m_{1235}m_{1456}m_{2346} :$$

$$m_{1236}m_{1245}m_{3456} : m_{1236}m_{1345}m_{2456} : m_{1236}m_{1456}m_{2345} :$$

$$m_{1245}m_{1346}m_{2356} : m_{1245}m_{1356}m_{2346} : m_{1246}m_{1345}m_{2356} :$$

(3.54) $\qquad \qquad m_{1246}m_{1356}m_{2345} : m_{1256}m_{1345}m_{2346} : m_{1256}m_{1346}m_{2345}).$

In particular, this map embeds our shape space in $\mathbb{P}^{14}_{\mathbb{R}}$ which is much more convenient for computations than the extremely high dimensional space we arrive at using the Chow forms.

Suppose now that we have 6 points in $\mathbb{P}^2_{\mathbb{R}}$. Then in this case $n = 2$ and $k = 6$ so that $n + 1$ divides k. If we do not make the assumption that $t \geq 2$, then we can define a map $\psi : \mathcal{U}'_{6,2} \subset Gr(3,6) \to \mathbb{P}^9_{\mathbb{R}}$ that collapses the $\mathcal{V}(P_1, \ldots, P_6)$ to points by

$$\psi(n_{123} : \ldots : n_{456}) = (n_{123}n_{456} : n_{124}n_{356} : n_{125}n_{346} : n_{126}n_{345} : n_{134}n_{256} :$$

(3.55) $\qquad \qquad n_{135}n_{246} : n_{136}n_{245} : n_{145}n_{236} : n_{146}n_{235} : n_{156}n_{234})$

Here the $n_{i_1,i_2,i_3}n_{j_1,j_2,j_3}$ are all of the degree 2 monomials in which each of the values $1, 2, 3, 4, 5, 6$ appears exactly *once*.

Consider the configuration

$$
\begin{array}{ll}
P_1 = & (1:0:0) \qquad P_4 = (1:1:1) \\
P_2 = & (0:1:0) \qquad P_5 = (2:4:1) \\
P_3 = & (0:0:1) \qquad P_6 = (1:3:1)
\end{array}
$$

(3.56)

with Plücker coordinates $(1:1:1:1:-1:-4:-3:-3:-2:1:1:2:1:1:0:-1:2:2:2:2)$ and the configuration

$$
\begin{array}{ll}
Q_1 = & (1:0:0) \qquad Q_4 = (1:1:1) \\
Q_2 = & (0:1:0) \qquad Q_5 = (-2:0:1) \\
Q_3 = & (0:0:1) \qquad Q_6 = (-3:-1:1)
\end{array}
$$

(3.57)

which has Plücker coordinates $(1:1:1:1:-1:0:1:1:2:1:1:-2:-3:-3:-4:-1:2:2:2:2)$. We can see that these two configurations have distinct shapes since they have different projective invariants (as defined in section C).

We now have that ψ identifies the configuration $P_1, P_2, P_3, P_4, P_5, P_6$ with the point

$$
\psi(1:1:1:1:-1:-4:-3:-3:-2:1:1:2:1:1:0:-1:2:2:2:2)
$$

(3.58)
$$
= (2:2:2:2:1:0:-3:-3:-4:1)
$$

and identifies the configuration $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ with the point

$$
\psi(1:1:1:1:-1:0:1:1:2:1:1:-2:-3:-3:-4:-1:2:2:2:2)
$$

(3.59)
$$
= (2:2:2:2:1:0:-3:-3:-4:1).
$$

Thus, under the map ψ, we have two distinct shapes identified with the same point in $\mathbb{P}^9_{\mathbb{R}}$. So we see that the assumption $t \geq 2$ is necessary.

CHAPTER IV

THE CONFORMAL CASE

In this chapter, we will return to the classical case in which we consider configurations of k points (at least 2 distinct) in $\mathbb{A}_\mathbb{R}^n$ up to a similarity transformation (definition B.1), and we will attempt to relate such shapes under *conformal projections*. This type of projection is an orthogonal projection followed by a translation and a dilation. When we represent points in $\mathbb{A}_\mathbb{R}^n$ in the form

$$(4.1) \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

such projections take the form

$$(4.2) \qquad T = \begin{pmatrix} & & & t_1 \\ & \lambda S & & \vdots \\ & & & t_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $\lambda > 0$, $(t_1, \ldots, t_n, 1) \in \mathbb{A}_\mathbb{R}^n$, and S is a $n \times m$ matrix whose rows are orthonormal vectors in $\mathbb{R}^m$ with the usual inner product. Here S gives us the orthogonal projection, λ is the scale factor of the dilation, and $(t_1, \ldots, t_n, 1)$ is the translation.

We choose to consider similarity shapes under conformal projections because they effectively model radar imaging. For now, we will be considering our objects to be configurations of points in the plane $\mathbb{A}_\mathbb{R}^2$ and our images to be configurations of points on the line $\mathbb{A}_\mathbb{R}^1$.

A. The Object and Image Shape Spaces

We have already seen in Chapter I that the space of shapes of configurations of k points (at least 2 distinct) in $\mathbb{A}_{\mathbb{R}}^2$ up to similarity transformations is the complex projective space, $\mathbb{P}_{\mathbb{C}}^{k-2}$. We will use a slightly different construction here than the one used in Chapter I. Consider k points $z_0, \ldots, z_{k-1}$ in $\mathbb{A}_{\mathbb{R}}^2$ which we will treat as a vector $(z_0, \ldots, z_{k-1}) \in \mathbb{A}_{\mathbb{C}}^k$ when we make the natural identification of $\mathbb{A}_{\mathbb{R}}^2$ with $\mathbb{C}$. We then identify configurations of k points in $\mathbb{A}_{\mathbb{R}}^2$ modulo translation with $\mathbb{A}_{\mathbb{C}}^{k-1} - \{0\}$ by moving $(z_0, \ldots, z_{k-1})$ to $(0, z_1 - z_0, z_2 - z_0, \ldots, z_{k-1} - z_0)$. Since we are assuming at least two of our points are distinct, $(z_1 - z_0, \ldots, z_{k-1} - z_0) \in \mathbb{A}_{\mathbb{C}}^{k-1}$ is nonzero.

Now having removed translation, we want to identify the space $\mathbb{A}_{\mathbb{C}}^{k-1} - \{0\}$ modulo rotation and scale. Since rotation and scale is simply multiplication by a nonzero complex number, we see that the shape space for k points in $\mathbb{A}_{\mathbb{R}}^2$ up to similarity transformation is the complex projective space $\mathbb{P}_{\mathbb{C}}^{k-2}$. For this chapter, we will refer to $\mathbb{P}_{\mathbb{C}}^{k-2}$ as object space.

Definition A.1. We call the homogeneous coordinates $(w_1 : \ldots : w_{k-1}) \in \mathbb{P}_{\mathbb{C}}^{k-2}$ of the shape of an object configuration the *shape coordinates* of that object configuration.

What then is the space of image shapes? In other words, we want to know what the shape space is for k points (at least 2 distinct) in $\mathbb{A}_{\mathbb{R}}^1$ up to similarity transformations. Notice that there are no rotations on $\mathbb{A}_{\mathbb{R}}^1$ since $SO(1)$ is the group of all 1×1 matrices of determinant 1 which consists only of the matrix (1). We should observe though, that if we identify $\mathbb{A}_{\mathbb{R}}^1$ with a line in $\mathbb{A}_{\mathbb{R}}^2$, then *reflecting* a configuration in $\mathbb{A}_{\mathbb{R}}^1$ is equivalent to rotating that configuration 180 degrees in $\mathbb{A}_{\mathbb{R}}^2$. In this light, we will stipulate that similarity transformations on $\mathbb{A}_{\mathbb{R}}^1$ (and only $\mathbb{A}_{\mathbb{R}}^1$) will include reflections.

So we will consider two image configurations $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ in $\mathbb{A}_{\mathbb{R}}^1$

equivalent if they differ by a translation, dilation, and/or reflection. Now consider the configuration $P_0, \ldots, P_{k-1} \in \mathbb{A}_{\mathbb{R}}^1$ which we will represent as a vector $(P_0, \ldots, P_{k-1}) \in \mathbb{A}_{\mathbb{R}}^k$. We identify configurations of k points in $\mathbb{A}_{\mathbb{R}}^1$ modulo translations with $\mathbb{A}_{\mathbb{R}}^{k-1} - \{0\}$ by moving $(P_0, \ldots, P_{k-1})$ to $(0, P_1 - P_0, \ldots, P_k - P_1)$.

So now we want to identify configurations in $\mathbb{A}_{\mathbb{R}}^{k-1} - \{0\}$ up to reflection and scale. Note that reflection and scale is simply multiplication by some $\lambda \in \mathbb{R}^*$. From this we see that the shape space of configurations of k points in $\mathbb{A}_{\mathbb{R}}^1$ up to similarity transformation is the projective space $\mathbb{P}_{\mathbb{R}}^{k-2}$.

Definition A.2. We call the homogeneous coordinates $(u_1 : \ldots : u_{k-1}) \in \mathbb{P}_{\mathbb{R}}^{k-2}$ of the shape of an image configuration the *shape coordinates* of that image configuration.

B.   The Object/Image Relations

As in the affine and projective models, we again want to find necessary and sufficient conditions for an image configuration $Q_1, \ldots, Q_k$ in $\mathbb{A}_{\mathbb{R}}^1$ to be a conformal projection of an object configuration $P_1, \ldots, P_k$. Let A be a similarity transformation on $\mathbb{A}_{\mathbb{R}}^1$, let B be a similarity transformation on $\mathbb{A}_{\mathbb{R}}^2$, and let T be a conformal projection from $\mathbb{A}_{\mathbb{R}}^2$ to $\mathbb{A}_{\mathbb{R}}^1$. Then A, B, and T take the forms

$$
(4.3) \qquad A = \begin{pmatrix} \gamma & s \\ 0 & 1 \end{pmatrix}
$$

$$
(4.4) \qquad B = \begin{pmatrix} e & -f & t_1 \\ f & e & t_2 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
(4.5) \qquad T = \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix}
$$

where $\gamma \in \mathbb{R}^*$, $e, f \in \mathbb{R}$ with e and f not both zero, and $a, b, c \in \mathbb{R}$ with a and b not both zero. Then

$$(4.6) \qquad ATB = \begin{pmatrix} \gamma(ea + fb) & \gamma(-fa + eb) & \tilde{s} \\ 0 & 0 & 1 \end{pmatrix}$$

where $\tilde{s} = \gamma a t_1 + \gamma b t_2 + \gamma c + s$.

Suppose that ATB is not a conformal projection. Then $ea + fb = -fa + eb = 0$. If $e = 0$, then $f \neq 0$ (since $\det \begin{pmatrix} e & -f \\ f & e \end{pmatrix} \neq 0$) and $fb = -fa = 0$. Thus, $a = b = 0$ in which case T is not a conformal projection. If $e \neq 0$ then since $ea + fb = 0$, $a = \frac{-fb}{e}$. From this we get that $-fa + eb = \frac{f^2 b}{e} + eb = b\left(\frac{f^2 + e^2}{e}\right) = 0$ which implies that $b = 0$ ($\frac{f^2 + e^2}{e} \neq 0$ since $e \neq 0$). Hence, $a = \frac{-fb}{e} = 0$ so that T is again not a conformal projection. In either case, we arrive at a contradiction, and so ATB must be a conformal projection.

This tells us that we may relate *shapes* of image configurations and *shapes* of object configurations under conformal projections. So the set V of matching objects and images should be the zero locus in $\mathbb{P}_{\mathbb{C}}^{k-2} \times \mathbb{P}_{\mathbb{R}}^{k-2}$ of some system of equations in the object and image shape coordinates.

We will begin by considering configurations of 4 points. Let $P_i = (x_i, y_i, 1)$, $i = 0, \ldots, 3$ be an object configuration and let $Q_i = (u_i, 1)$, $i = 0, \ldots, 3$ be an image configuration. Then the image Q_0, Q_1, Q_2, Q_3 is a conformal projection of the object P_0, P_1, P_2, P_3 if there exist $a, b, c \in \mathbb{R}$ with a and b not both zero such that

$$(4.7) \qquad \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

So if Q_0, Q_1, Q_2, Q_3 is the image of P_0, P_1, P_2, P_3 under a conformal projection, then

the system

(4.8) $$u_i = ax_i + by_i + c, \text{ for } i = 1, 2, 3, 4$$

is satisfied for some $a, b, c \in \mathbb{R}$ with a and b not both zero.

If we use Gröebner bases to eliminate a, b, and c from this system of polynomial equations, we are left with one single object/image relation

$$u_0(x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2)$$

$$-u_1(x_0y_2 - x_2y_0 - x_0y_3 + x_3y_0 + x_2y_3 - x_3y_2)$$

$$+u_2(x_0y_1 - x_1y_0 - x_0y_3 + x_3y_0 + x_1y_3 - x_3y_1)$$

(4.9) $$-u_3(x_0y_1 - x_1y_0 - x_0y_2 + x_2y_0 + x_1y_2 - x_2y_1) = 0.$$

Equation 4.9 is equivalent to the equation

(4.10) $$\det \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0.$$

If we assume that $(x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3)$, and $(1, 1, 1, 1)$ are linearly independent, then 4.10 simply means that (u_0, u_1, u_2, u_3) lies in the span of (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3), and $(1, 1, 1, 1)$. The vectors $(x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3)$, and $(1, 1, 1, 1)$ are linearly independent if and only if the configuration $P_i = (x_i, y_i, 1)$ is not collinear.

Suppose that the configuration $P_i = (x_i, y_i, 1)$, $i = 0, 1, 2, 3$ is collinear. Then we may rotate the configuration P_0, P_1, P_2, P_3 so that $y_i = 0$ for all i so that the

object/image equation 4.10 becomes

$$(4.11) \qquad \det \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ x_0 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0$$

which is satisfied for any configuration $Q_i = (u_i, 1)$ of four points in $\mathbb{A}_{\mathbb{R}}^1$. What this seems to suggest is that for every collinear object configuration P_0, P_1, P_2, P_3 in $\mathbb{A}_{\mathbb{R}}^2$ and for every image configuration $Q_0, Q_1, Q_2, Q3$, there is a conformal projection π so that $P_i = \pi(Q_i)$ for $i = 0, 1, 2, 3$, which is clearly not true. To avoid this problem, we will choose to only consider noncollinear configurations of points in $\mathbb{A}_{\mathbb{R}}^2$.

In the general case of k points, we can see that an image configuration $Q_i = (u_i, 1)$, $i = 0, \ldots, k-1$ is a conformal projection of an object configuration $P_i = (x_i, y_i, 1)$, $i = 0, \ldots, k-1$ if and only if $(u_0, \ldots, u_{k-1})$ lies in the span of $(x_0, \ldots, x_{k-1})$, $(y_0, \ldots, y_{k-1})$, and $(1, \ldots, 1)$. Since we are assuming $P_0, \ldots, P_{k-1}$ are not collinear, the object/image relations then become

$$(4.12) \qquad \det \begin{pmatrix} u_{i_0} & u_{i_1} & u_{i_2} & u_{i_3} \\ x_{i_0} & x_{i_1} & x_{i_2} & x_{i_3} \\ y_{i_0} & y_{i_1} & y_{i_2} & y_{i_3} \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0$$

for all $0 \leq i_0 < i_1 < i_2 < i_3 \leq k-1$.

How do we write these relations in terms of the shape coordinates? Let $P_i = (x_i, y_i, 1)$, $i = 0, \ldots, k-1$ be an object configuration, and let Let $Q_i = (u_i, 1)$, $i = 0, \ldots, k-1$ be an image configuration. We may move $P_0, P_1, \ldots, P_{k-1}$ by a translation so that $P_0 = (0, 0, 1)$ and $P_i = (\tilde{x}_i, \tilde{y}_i, 1)$ for $i = 1, \ldots, k-1$, and we may

move $Q_0, Q_1, \ldots, Q_{k-1}$ by a translation so that $Q_0 = (0, 1)$ and $Q_i = (\tilde{u}_i, 1)$ for $i = 1, \ldots, k-1$. Notice that $(\tilde{u}_1 : \ldots : \tilde{u}_{k-1})$ are the shape coordinates of the image configuration $Q_0, Q_1, \ldots, Q_{k-1}$, and $(z_1 : \ldots : z_{k-1})$ where $z_j = \tilde{x}_j + i\tilde{y}_j \in \mathbb{C}$ are the shape coordinates of the object configuration $P_0, P_1, \ldots, P_{k-1}$.

When we place our configurations in this standard position, we see that Q_0, $Q_1, \ldots, Q_{k-1}$ is a conformal projection of $P_0, P_1, \ldots, P_{k-1}$ if there exist $a, b \in \mathbb{R}$ not both zero such that

$$(4.13) \qquad \begin{pmatrix} \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_{k-1} \\ 1 & 1 & \ldots & 1 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_{k-1} \\ \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_{k-1} \\ 1 & 1 & & 1 \end{pmatrix}$$

or equivalently

$$(4.14) \qquad \begin{pmatrix} \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_{k-1} \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_{k-1} \\ \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_{k-1} \end{pmatrix}.$$

So we have that $Q_0, Q_1, \ldots, Q_{k-1}$ is a conformal projection of $P_0, P_1, \ldots, P_{k-1}$ if $(\tilde{u}_1, \ldots, \tilde{u}_{k-1})$ lies in the span of $(\tilde{x}_1, \ldots, \tilde{x}_{k-1})$ and $(\tilde{y}_1, \ldots, \tilde{y}_{k-1})$. By making the assumption that $P_0, \ldots, P_{k-1}$ is a noncollinear configuration, $(\tilde{x}_1, \ldots, \tilde{x}_{k-1})$ and $(\tilde{y}_1, \ldots, \tilde{y}_{k-1})$ are linearly independent Thus, in terms of the image shape coordinates $(\tilde{u}_1 : \ldots : \tilde{u}_{k-1})$ and the object shape coordinates $(z_0 : \ldots : z_{k-1})$ $(z_j = \tilde{x}_j + i\tilde{y}_j \in \mathbb{C})$, the object/image relations become

$$(4.15) \qquad \det \begin{pmatrix} \tilde{u}_{i_1} & \tilde{u}_{i_2} & \tilde{u}_{i_3} \\ \tilde{x}_{i_1} & \tilde{x}_{i_2} & \tilde{x}_{i_3} \\ \tilde{y}_{i_1} & \tilde{y}_{i_2} & \tilde{y}_{i_3} \end{pmatrix} = 0.$$

C. The Relationship with Affine Shapes

There are two important points to notice in our analysis so far in this chapter. First, every conformal projection

(4.16)
$$\begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix}$$

is necessarily a generalized weak perspective projection. This is only true for conformal projections of $\mathbb{A}_{\mathbb{R}}^k$ onto $\mathbb{A}_{\mathbb{R}}^1$.

Second, similarity transformations on $\mathbb{A}_{\mathbb{R}}^1$ (as we have defined them) and affine transformations on $\mathbb{A}_{\mathbb{R}}^1$ are equivalent. Thus, from our work with affine shapes from Chapter II we see that if $P_0, \ldots, P_{k-1}$ is an object configuration, if $Q_0, \ldots, Q_{k-1}$ is an image configuration, and if there is a conformal projection $\pi : \mathbb{A}_{\mathbb{R}}^2 \to \mathbb{A}_{\mathbb{R}}^1$ such that $Q_i = \pi(P_i)$ for all i, then for every affine transformation A on $\mathbb{A}_{\mathbb{R}}^2$, there is a conformal projection π_A such that $Q_i = \pi_A(AP_i)$.

Let W be the locus of shapes $(x_1 + iy_1 : \ldots : x_{k-1} + iy_{k-1}) \in \mathbb{P}_{\mathbb{C}}^{k-2}$ of collinear configurations $P_0, \ldots, P_{k-1}$. W then is the set of solutions to the system

(4.17) $$x_m y_n - x_n y_m = 0 \qquad \text{for } 1 \leq m < n \leq k - 1.$$

Now, a conformal projection π induces a projection $\overline{\pi}_{Sim}$

(4.18)
$$\begin{array}{c} \mathbb{P}_{\mathbb{C}}^{k-2} - W \\ \overline{\pi}_{Sim} \downarrow \\ \mathbb{P}_{\mathbb{R}}^{k-2} \end{array}$$

and induces a projection $\overline{\pi}_{Aff}$

(4.19)
$$\begin{array}{c} \mathcal{A}_{\mathbb{R}}(k, 2) \\ \overline{\pi}_{Aff} \downarrow \\ \mathbb{P}_{\mathbb{R}}^{k-2}. \end{array}$$

Since every similarity transformation is an affine transformation, there is a map $\phi : \mathbb{P}_{\mathbb{C}}^{k-2} - W \to \mathcal{A}_{\mathbb{R}}(k, 2)$ such that

(4.20)
$$
\begin{array}{ccc}
\mathbb{P}_{\mathbb{C}}^{k-2} - W & \xrightarrow{\phi} & \mathcal{A}_{\mathbb{R}}(k, 2) \\
\overline{\pi}_{Sim} \downarrow & & \overline{\pi}_{Aff} \downarrow \\
\mathbb{P}_{\mathbb{R}}^{k-2} & = & \mathbb{P}_{\mathbb{R}}^{k-2}
\end{array}
$$

commutes. Explicitly the map ϕ maps the shape $(z_1 : \ldots : z_{k-1}) \in \mathbb{P}_{\mathbb{C}}^{k-2} - W$ to $(m_{123} : \ldots : m_{k-2 \ k-1 \ k}) \in \mathcal{A}_{\mathbb{R}}(k, 2) \subset \mathbb{P}_{\mathbb{R}}^{\binom{k}{3}-1}$ where

(4.21)
$$
m_{j_1 j_2 j_3} = \det \begin{pmatrix}
\frac{1}{2}(z_{j_1} + \overline{z}_{j_1}) & \frac{1}{2}(z_{j_2} + \overline{z}_{j_2}) & \frac{1}{2}(z_{j_3} + \overline{z}_{j_3}) \\
\frac{1}{2i}(z_{j_1} - \overline{z}_{j_1}) & \frac{1}{2i}(z_{j_2} - \overline{z}_{j_2}) & \frac{1}{2i}(z_{j_3} - \overline{z}_{j_3}) \\
1 & 1 & 1
\end{pmatrix}.
$$

Notice that if $z_j = x_j + iy_j$, then $\frac{1}{2}(z_j + \overline{z}_j) = x_j$ and $\frac{1}{2i}(z_j - \overline{z}_j) = y_j$. Notice also that we must remove the set W of shapes of collinear configurations from $\mathbb{P}_{\mathbb{C}}^{k-2}$ in order for ϕ to be well-defined.

So the map $\overline{\pi}_{Sim}$ gives a fibering of the shape space $\mathbb{P}_{\mathbb{C}}^{k-2} - W$. In particular, if an image shape $(u_1 : \ldots : u_{k-1}) \in \mathbb{P}_{\mathbb{R}}^{k-2}$ is a conformal projection of an object shape $(z_1 : \ldots, z_{k-1}) \in \mathbb{P}_{\mathbb{C}}^{k-2} - W$, the fiber $\overline{\pi}_{Sim}^{-1}(u_1 : \ldots : u_{k-1})$ contains all shapes of configurations of k points in $\mathbb{A}_{\mathbb{R}}^2$ up to similarity transformation that differ from by $(z_1 : \ldots : z_{k-1})$ by an affine transformation of $\mathbb{A}_{\mathbb{R}}^2$.

D. Metrics

As in the affine case, we would like to have a measure of an image shape's failure to be a conformal projection of a given object shape. The natural way to do this is to begin with the metrics on the object and image shape spaces.

Since in this model, our shape spaces are projective spaces they come equipped with their respective Fubini-Study metrics. Let d_I denote the Fubini-Study metric on

our image space $\mathbb{P}_{\mathbb{R}}^{k-2}$, and let d_O denote the Fubini-Study metric on our object space $\mathbb{P}_{\mathbb{C}}^{k-2} - W$. Now consider an object shape $z \in \mathbb{P}_{\mathbb{C}}^{k-2} - W$ and an image shape $u \in \mathbb{P}_{\mathbb{R}}^{k-2}$. Define $O(u) \subset \mathbb{P}_{\mathbb{C}}^{k-2} - W$ to be the set of all object shapes capable of producing the image shape u under a conformal projection. Similarly, let $I(z) \subset \mathbb{P}_{\mathbb{R}}^{k-2}$ be the set of all possible images of z under conformal projection. We may then define two measures of distance d_1 and d_2 between z and u given by

$$(4.22) \qquad\qquad d_1(z, u) = \min_{z' \in O(u)} d_O(z, z')$$

$$(4.23) \qquad\qquad d_2(z, u) = \min_{u' \in I(z)} d_I(u', u).$$

Ideally, we would like to show that d_1 and d_2 are equal (perhaps up to some scale factor) as we did in the affine case. However, at this point we are unable to even verify this for specific examples. The problem we face comes from the fact that most methods of computing the distance d_1 (such as the method of Lagrange multipliers) require us to work in the compact space $\mathbb{P}_{\mathbb{C}}^{k-2}$ rather than $\mathbb{P}_{\mathbb{C}}^{k-2} - W$.

For a specific image shape $u \in \mathbb{P}_{\mathbb{R}}^{k-2}$, the set $O(u)$ as a subvariety of $\mathbb{P}_{\mathbb{C}}^{k-2} - W$ is defined by the object/image relations 4.12 (after substituting the homogeneous coordinates of u). However, as we noted earlier the subvariety of $\mathbb{P}_{\mathbb{C}}^{k-2}$ defined by the object/image relations is the union of $O(u)$ with the set W of all shapes of collinear configurations. Because of this, when we attempt to compute the distance $d_1(z, u)$ for any object shape $z \in \mathbb{P}_{\mathbb{C}}^{k-2} - W$ and any image shape $u \in \mathbb{P}_{\mathbb{R}}^{k-2}$, we arrive at $d_1(z, u) = 0$.

To address this problem, we should try to understand the set $O'(u) \subset \mathbb{P}_{\mathbb{C}}^{k-2}$ consisting of *all* shapes of k points in $\mathbb{A}_{\mathbb{R}}^2$ (at least two distinct) capable of producing the image shape $u \in \mathbb{P}_{\mathbb{R}}^{k-2}$. This should be the union of $O(u)$ and the set of all shapes $w \in W$ of collinear configurations capable of producing the shape u under conformal

projection. It is clear that there is only one shape $w \in W$ that can be conformally projected to u. Thus, $O'(u) = O(u) \cup \{w\}$.

What we should do, is redefine the distance $d_1(z, u)$ between an image shape u and an object shape $z \in \mathbb{P}_\mathbb{C}^{k-2}$ to be the minimum distance between z and $O'(u)$ in $\mathbb{P}_\mathbb{C}^{k-2}$ with the Fubini-Study metric i.e. $d_1(z, u) = min_{z' \in O'(u)} d_O(z, z')$. To be able to compute this distance, we will need to know all of the equations that define $O'(u)$. This system will include the object/image relations 4.12 together with some other set $\mathcal{J}$ of equations that eliminate the extraneous shapes in W that our first set of object/image relations left. Determining precisely the system $\mathcal{J}$ will be left for a future paper.

CHAPTER V

CONCLUSION

In this dissertation we have investigated several target recognition models in a shape theoretic and algebraic geometric context. Doing so has allowed us to consider matching of objects and images independent of the sensor viewpoint, internal sensor parameters and choice of coordinate systems in which we represent our objects and images.

We first extended the theory in the generalized weak perspective (GWP) model by introducing three notions of distance between an affine object shape and an affine image shape and proving that these three "metrics" are equivalent (the so called duality of the object and image shape metrics).

We followed this by adapting the shape theoretic techniques of the GWP model to the full perspective model. We defined an appropriate notion of shape for this model and gave a representation of shape as a projective subvariety of a Grassmannian. These projective varieties are given by systems of polynomials in the Plücker coordinates on our Grassmannian whose coefficients are monomials in the Plücker coordinates $m_{i_1 i_2 \ldots i_{n+1}}$ of the null space of the matrix of the configuration. We then gave ways to compute the object/image relations for this model and gave explicitly this matching equation for configurations of 6 points. We emphasize that these relations give a correspondence between object and image *shapes* under full perspective projection.

We concluded the discussion of this model by giving two ways to embed the shape space $\mathcal{U}_{k,n}/PGL(n+1)$ for configurations of k points in $\mathbb{P}^n_{\mathbb{R}}$ in general position into a projective space $\mathbb{P}^n_{\mathbb{R}}$. We first considered the Chow embedding, which is the natural geometric map to use for identifying the shape varieties with points in $\mathbb{P}^N_{\mathbb{R}}$. Due to the

difficulty in computing the Chow form of a variety and the high dimension of the target space $\mathbb{P}_{\mathbb{R}}^{N}$, this turns out not to be an effective method for representing projective shapes as single points in projective space. To avoid this problem we constructed another map $\phi_{k,n} : \mathcal{U}'_{k,n} \subset Gr(n+1,k) \to \mathbb{P}_{\mathbb{R}}^{N'}$ which induces an embedding of the shape space into projective space $\mathbb{P}_{\mathbb{R}}^{N'}$. This projective space has a much lower dimension than the target space of the Chow embedding making it much more useful for practical computations.

Finally, we examined the conformal model where we considered shapes of object configurations in $\mathbb{A}_{\mathbb{R}}^{2}$ and shapes of image configurations in $\mathbb{A}_{\mathbb{R}}^{1}$ up to similarity transformations, and investigated the relationship between such shapes under conformal projections. We were able to give necessary and sufficient conditions for an image configuration to be a conformal projection of a *noncollinear* object configuration in terms of their shapes. We then investigated the metrics on the object and image shape spaces and defined two notions of distance between an object shape and an image shape.

In this research there are several unanswered questions. In particular, the theory in the full perspective and conformal cases is incomplete. Firstly, the map $\phi_{k,n}$ induces an embedding of the shape space $\mathcal{U}_{k,n}/PGL(n+1)$ as a subset U of a projective space $\mathbb{P}_{\mathbb{R}}^{N}$, but we have only considered configurations whose points are in general position. It seems that the boundary points of U should correspond to shapes of degenerate configurations where some of the $m_{i_1 i_2 \ldots i_{n+1}}$ are zero. Instead of only considering configurations in $\mathcal{U}_{k,n}$ (those whose points are in general position), we should consider configurations in a larger open subset $\widetilde{\mathcal{U}}_{k,n} \subset (\mathbb{P}_{\mathbb{R}}^{n})^{k}$ so that the image of the induced map $\overline{\phi}_{k,n} : \widetilde{\mathcal{U}}_{k,n}/PGL(n+1) \to \mathbb{P}_{\mathbb{R}}^{N'}$ is $\overline{U}$, the Zariski closure of U in $\mathbb{P}_{\mathbb{R}}^{N'}$. In some sense, this set $\widetilde{\mathcal{U}}_{k,n}$ should be the largest set of configurations we can consider and still maintain a reasonable quotient space $\widetilde{\mathcal{U}}_{k,n}/PGL(n+1)$. These are likely to be

the stable or semi-stable configurations as defined in [13].

Secondly, as in the affine case, we would like to define a "distance" d between a projective object shape K and a projective image shape L. The natural way to do this would be to define a metric d_O on the projective object shape space and a metric d_I on the projective image shape space, and then compute $d(K, L)$ to be either $\min_{L'} d_O(K, L')$ where L' ranges over all object shapes capable of producing the image L or $\min_{K'} d_I(K', L)$ where K' ranges over the shapes of all possible images of the object K. If we define d_O to be the metric on our object space induced by the Fubini-Study metric on $\mathbb{P}_{\mathbb{R}}^{N_1}$ (after embedding the object space via the map $\phi_{k,3}$) and d_I to be the metric on our image space induced by the Fubini-Study metric on $\mathbb{P}_{\mathbb{R}}^{N_2}$ (after embedding the image space via the map $\phi_{k,3}$), are the "metrics", $d_1(K, L) = \min_{L'} d_O(K, L')$ and $d_2(K, L) = \min_{K'} d_I(K', L)$ equivalent? If not, can we define some other metrics on the projective object and image spaces so that this object/image metric duality does hold?

In the conformal case, we still need to determine the defining equations of the set $\emptyset'(u) \subset \mathbb{P}_{\mathbb{C}}^{k-2}$ of all object shapes capable of producing the image shape $u \in \mathbb{P}_{\mathbb{R}}^{k-2}$ under a conformal projection. Upon doing this, we would then like to show that the distances $d_1(z, u)$ and $d_2(z, u)$ between an object shape z and an image shape u are equal.

A natural next step in this research would be to consider our shape spaces modulo the action of the permutation group $\mathfrak{S}_k$ on the points of our configurations. So far we have considered our configurations to be *ordered* k-tuples of points in $\mathbb{A}_{\mathbb{R}}^n$ or $\mathbb{P}_{\mathbb{R}}^n$. In this way, we have been considering the configurations $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0), P_3 = (0 : 0 : 1)$ and $P_1' = (0 : 1 : 0), P_2' = (0 : 0 : 1), P_3' = (1 : 0 : 0)$ to be distinct. The difference in the two configurations is in the labeling that we have chosen to place on the points rather than the geometry of the configuration. Thus we

would ultimately like to study the shape spaces of configurations up to a permutation of the points and the action of some group of transformations.

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APPENDIX A

EXAMPLES

In this appendix, we will give a number of examples using the techniques presented in this dissertation and include the code used in these computations. The majority of the code is written for the computer algebra package Macaulay2. However, due to the fact that Macaulay2 cannot evaluate trigonometric functions, we will use MATLAB in our computations of the metrics in our target recognition models. In this appendix, the reader should assume we are using Macaulay2 unless otherwise noted.

A.   The Affine Case

In this first section, we will use the "affShapes" package to compute examples in the generalized weak perspective model.

1.   Affine Object Shapes

We first define three 3D objects.

```
i1 : Ob1= matrix {{1, 2, -3, 8, 0, 3}, {0, -2, -4, 6, 7, -5}, {-1,
5, 0, 1, -7, 10}, {1, 1, 1, 1, 1, 1}}


o1 = | 1   2  -3 8 0   3  |
     | 0  -2 -4 6 7  -5 |
     | -1 5   0  1 -7 10 |
     | 1   1  1  1 1   1  |
```

```
              4         6
o1 : Matrix ZZ  <--- ZZ


i2 : Ob2= matrix {{-3, 7, -10, 8, 1, 4}, {-9, 3, 1, 4, 6, -8}, {-4,

9, 4, 6, -10, -10}, {1, 1, 1, 1, 1, 1}}


o2 = | -3 7 -10 8 1    4   |

     | -9 3 1    4 6   -8  |

     | -4 9 4    6 -10 -10 |

     | 1  1 1    1 1    1  |


              4         6
o2 : Matrix ZZ  <--- ZZ


i3 : Ob3= matrix {{-4, 18, 7, -10, -36, 39}, {-2, 17, -3, 11, -21,

33},{3,9,-1,5,0,4}, {1, 1, 1, 1, 1, 1}}


o3 = | -4 18 7  -10 -36 39 |

     | -2 17 -3 11  -21 33 |

     | 3  9  -1 5   0   4  |

     | 1  1  1  1   1   1  |


              4         6
o3 : Matrix ZZ  <--- ZZ
```

We use the function "maxMinors" to compute the shape coordinates of the configurations (which are the maximal minors of the configuration matrices).

```
i4 : load "affShapes"

--loaded maxMinors

--loaded perm

--loaded affShapes


i5 : CoordsOb1=maxMinors Ob1


o5 = {20, 125, -37, -200, 60, 5, -95, 59, 193, -305, 210, -58, 26,

-50, -344}


o5 : List


i6 : CoordsOb2=maxMinors Ob2


o6 = {757, 3805, 1513, -461, 288, 2369, -3560, -1336, 400, 2168,

-51, -868, -4261, 548, 3992}


o6 : List


i7 : CoordsOb3=maxMinors Ob3


o7 = {-1960, -7, -2149, -1600, 4120, 1769, 2049, -2357, -2255, 2383,

5602, -6666, -6166, 6334, 232}
```

```
o7 : List
```

Notice that if we scale the shape coordinates so that the first entry is 1, the second coordinates become $\frac{25}{4}$, $\frac{3805}{757}$, and $\frac{1}{280}$. So we see that these three objects have distinct shapes.

Now we define an affine transformation on $\mathbb{A}^3_{\mathbb{R}}$.

```
i8 : affTrans=matrix {{5, 6, -2, -1}, {4, -3, 3, 2}, {1, 7, -4, 1},
{0, 0, 0, 1}}
```

```
o8 = | 5 6  -2 -1 |
     | 4 -3 3  2  |
     | 1 7  -4 1  |
     | 0 0  0  1  |
```

```
                4         4
o8 : Matrix ZZ  <--- ZZ
```

If we move object Ob1 by this affine transformation, we obtain a new object configuration having the same shape coordinates.

```
i9 : TransOb1=affTrans*Ob1
```

```
o9 = | 6 -13 -40 73 55  -36 |
     | 3 31  2   19 -40 59  |
     | 6 -31 -30 47 78  -71 |
     | 1 1   1   1  1   1   |
```

```
                    4          6
o9 : Matrix ZZ  <--- ZZ


i10 : coordsTransOb1=maxMinors TransOb1


o10 = {140, 875, -259, -1400, 420, 35, -665, 413, 1351, -2135, 1470,

-406, 182, -350, -2408}


o10 : List
```

In particular, we note that the shape coordinates of Ob1 and TransOb1 differ by a factor of det(affTrans).

```
i11 : det(affTrans)


o11 = 7


i12 : (1/7)*coordsTransOb1


o12 = {20, 125, -37, -200, 60, 5, -95, 59, 193, -305, 210, -58, 26,

-50, -344}


o12 : List
```

If we only need to test for matching but do not need the specific shape coordinates, we may use the "sameAffShape" function of the "affShapes" package. This function will take two configuration matrices and return "true" if the configurations

have the same shape and "false" otherwise. The determination is made by checking that the shape coordinates give the same point in projective space.

```
i13 : sameAffShape(Ob1,Ob2)
```

```
o13 = false
```

```
i14 : sameAffShape(Ob1,TransOb1)
```

```
o14 = true
```

2. Projections

We now define three generalized weak perspective projections from $\mathbb{A}^3_{\mathbb{R}}$ to $\mathbb{A}^2_{\mathbb{R}}$ and apply them to our object shapes.

```
i15 : proj1=matrix{{0,-2,3,-1},{1,0,3,0},{0,0,0,1}}
```

```
o15 = | 0 -2 3 -1 |
      | 1 0  3 0  |
      | 0 0  0 1  |

                3         4
o15 : Matrix ZZ  <--- ZZ
```

```
i16 : proj2=matrix{{-5,1,3,-2},{2,6,4,0},{0,0,0,1}}
```

```
o16 = | -5 1 3 -2 |
```

```
        | 2  6 4 0 |
        | 0  0 0 1 |


                3       4
o16 : Matrix ZZ  <--- ZZ


i17 : proj3=matrix{{1,5,0,2},{4,-1,0,3},{0,0,0,1}}


o17 = | 1 5  0 2 |
      | 4 -1 0 3 |
      | 0 0  0 1 |


                3       4
o17 : Matrix ZZ  <--- ZZ


i18 : Im1=proj1*Ob1


o18 = | -4 18 7  -10 -36 39 |
      | -2 17 -3 11  -21 33 |
      | 1  1  1  1   1   1  |


                3       6
o18 : Matrix ZZ  <--- ZZ


i19 : Im2=proj2*Ob1
```

```
o19 = | -10 1   9   -33 -16 8  |

       | -2  12 -30 56  14  16 |

       | 1   1  1   1   1   1  |


                3         6
o19 : Matrix ZZ  <--- ZZ



i20 : Im3=proj3*Ob3



o20 = | -12 105 -6 47  -139 206 |

       | -11 58  34 -48 -120 126 |

       | 1   1   1  1   1    1   |


                 3        6
o20 : Matrix ZZ  <--- ZZ



i21 : Im4=proj1*Ob3



o21 = | 12 -8 2 -8 41  -55 |

       | 5  45 4 5  -36 51  |

       | 1  1  1 1  1   1   |


                 3        6
o21 : Matrix ZZ  <--- ZZ
```

We use the "maxMinors" function to compute the shape coordinates of the four images we have just generated.

```
i22 : coordsIm1=maxMinors Im1
```

```
o22 = {-231, 400, 190, -47, 137, -241, 428, 530, -769, -303, -494,
-662, 244, 740, -322, -66, 908, -1060, -972, 996}
```

```
o22 : List
```

```
i23 : coordsIm2=maxMinors Im2
```

```
o23 = {-574, 960, 260, -54, 458, 136, 846, -20, -1458, -396, -1076,
-698, 326, 680, -444, -82, 302, -1846, -1106, 1042}
```

```
o23 : List
```

```
i24 : coordsIm3=maxMinors Im3
```

```
o24 = {4851, -8400, -3990, 987, -2877, 5061, -8988, -11130, 16149,
6363, 10374, 13902, -5124, -15540, 6762, 1386, -19068, 22260, 20412,
-20916}
```

```
o24 : List
```

```
i25 : coordsIm4=maxMinors Im4
```

```
o25 = {420, 800, -340, 1760, -20, 439, -527, 820, -920, -1413, -400,
 1199, -1867, 1960, -1880, -3513, 361, -413, -447, 327}
```

```
o25 : List
```

We should notice here that if we scale the shape coordinates of the images Im1, Im2, and Im4 so that the first coordinate is one, the second coordinates become $-\frac{400}{231}$, $-\frac{480}{287}$, and $\frac{40}{21}$ respectively. From this we see that Im1, Im2, and Im4 have distinct shapes. Notice that even though Im1 and Im2 are both generalized weak perspective projections of the object configuration Ob1, they do not have the same shape. This verifies our assertion in Chapter II that a single object can produce multiple image shapes under generalized weak perspective projection.

Similarly, we see that the images Im1 and Im3 have the same shape even though they are GWP projections of two objects having distinct shapes.

```
i26 : -1/21*coordsIm3
```

```
o26 = {-231, 400, 190, -47, 137, -241, 428, 530, -769, -303, -494,
-662, 244, 740, -322, -66, 908, -1060, -972, 996}
```

```
o26 : List
```

This confirms our assertion in Chapter II that given an image shape, there are multiple object shapes capable of producing that image shape under GWP projection.

3.   Object/Image Equations

Now we will evaluate the affine object/image relations for particular object and image shapes. We will generate the affine object/image relations using the "affOIrels" function (part of the "affShapes" package).

```
i27 : rels=affOIrels(6)
```

```
o27={-2}|n_{1,4,5}m_{0,1,2,3}-n_{1,3,5}m_{0,1,2,4}+n_{1,3,4}m_{0,1,2,5}
        +n_{1,2,5}m_{0,1,3,4}-n_{1,2,4}m_{0,1,3,5}+n_{1,2,3}m_{0,1,4,5} |
  {-2} |-n_{0,4,5}m_{0,1,2,3}+n_{0,3,5}m_{0,1,2,4}-n_{0,3,4}m_{0,1,2,5}
        -n_{0,2,5}m_{0,1,3,4}+n_{0,2,4}m_{0,1,3,5}-n_{0,2,3}m_{0,1,4,5} |
  {-2} | n_{0,1,5}m_{0,1,3,4}-n_{0,1,4}m_{0,1,3,5}+n_{0,1,3}m_{0,1,4,5} |
  {-2} |-n_{0,1,5}m_{0,1,2,4}+n_{0,1,4}m_{0,1,2,5}-n_{0,1,2}m_{0,1,4,5} |
  {-2} | n_{0,1,5}m_{0,1,2,3}-n_{0,1,3}m_{0,1,2,5}+n_{0,1,2}m_{0,1,3,5} |
  {-2} |-n_{0,1,4}m_{0,1,2,3}+n_{0,1,3}m_{0,1,2,4}-n_{0,1,2}m_{0,1,3,4} |
  {-2} | n_{2,4,5}m_{0,1,2,3}-n_{2,3,5}m_{0,1,2,4}+n_{2,3,4}m_{0,1,2,5}
        +n_{1,2,5}m_{0,2,3,4}-n_{1,2,4}m_{0,2,3,5}+n_{1,2,3}m_{0,2,4,5} |
  {-2} |-n_{0,2,5}m_{0,2,3,4}+n_{0,2,4}m_{0,2,3,5}-n_{0,2,3}m_{0,2,4,5} |
  {-2} |-n_{0,4,5}m_{0,1,2,3}+n_{0,3,5}m_{0,1,2,4}-n_{0,3,4}m_{0,1,2,5}
        +n_{0,1,5}m_{0,2,3,4}-n_{0,1,4}m_{0,2,3,5}+n_{0,1,3}m_{0,2,4,5} |
  {-2} |-n_{0,2,5}m_{0,1,2,4}+n_{0,2,4}m_{0,1,2,5}-n_{0,1,2}m_{0,2,4,5} |
  {-2} | n_{0,2,5}m_{0,1,2,3}-n_{0,2,3}m_{0,1,2,5}+n_{0,1,2}m_{0,2,3,5} |
  {-2} |-n_{0,2,4}m_{0,1,2,3}+n_{0,2,3}m_{0,1,2,4}-n_{0,1,2}m_{0,2,3,4} |
  {-2} | n_{3,4,5}m_{0,1,2,3}-n_{2,3,5}m_{0,1,3,4}+n_{2,3,4}m_{0,1,3,5}
        +n_{1,3,5}m_{0,2,3,4}-n_{1,3,4}m_{0,2,3,5}+n_{1,2,3}m_{0,3,4,5} |
  {-2} |-n_{0,3,5}m_{0,2,3,4}+n_{0,3,4}m_{0,2,3,5}-n_{0,2,3}m_{0,3,4,5} |
```

{-2} | n_{0,3,5}m_{0,1,3,4}-n_{0,3,4}m_{0,1,3,5}+n_{0,1,3}m_{0,3,4,5} |

{-2} |-n_{0,4,5}m_{0,1,2,3}-n_{0,2,5}m_{0,1,3,4}+n_{0,2,4}m_{0,1,3,5}

     +n_{0,1,5}m_{0,2,3,4}-n_{0,1,4}m_{0,2,3,5}-n_{0,1,2}m_{0,3,4,5} |

{-2} | n_{0,3,5}m_{0,1,2,3}-n_{0,2,3}m_{0,1,3,5}+n_{0,1,3}m_{0,2,3,5} |

{-2} |-n_{0,3,4}m_{0,1,2,3}+n_{0,2,3}m_{0,1,3,4}-n_{0,1,3}m_{0,2,3,4} |

{-2} | n_{3,4,5}m_{0,1,2,4}-n_{2,4,5}m_{0,1,3,4}+n_{2,3,4}m_{0,1,4,5}

     +n_{1,4,5}m_{0,2,3,4}-n_{1,3,4}m_{0,2,4,5}+n_{1,2,4}m_{0,3,4,5} |

{-2} |-n_{0,4,5}m_{0,2,3,4}+n_{0,3,4}m_{0,2,4,5}-n_{0,2,4}m_{0,3,4,5} |

{-2} | n_{0,4,5}m_{0,1,3,4}-n_{0,3,4}m_{0,1,4,5}+n_{0,1,4}m_{0,3,4,5} |

{-2} |-n_{0,4,5}m_{0,1,2,4}+n_{0,2,4}m_{0,1,4,5}-n_{0,1,4}m_{0,2,4,5} |

{-2} | n_{0,3,5}m_{0,1,2,4}-n_{0,2,5}m_{0,1,3,4}-n_{0,2,3}m_{0,1,4,5}

     +n_{0,1,5}m_{0,2,3,4}+n_{0,1,3}m_{0,2,4,5}-n_{0,1,2}m_{0,3,4,5} |

{-2} |-n_{0,3,4}m_{0,1,2,4}+n_{0,2,4}m_{0,1,3,4}-n_{0,1,4}m_{0,2,3,4} |

{-2} | n_{3,4,5}m_{0,1,2,5}-n_{2,4,5}m_{0,1,3,5}+n_{2,3,5}m_{0,1,4,5}

     +n_{1,4,5}m_{0,2,3,5}-n_{1,3,5}m_{0,2,4,5}+n_{1,2,5}m_{0,3,4,5} |

{-2} |-n_{0,4,5}m_{0,2,3,5}+n_{0,3,5}m_{0,2,4,5}-n_{0,2,5}m_{0,3,4,5} |

{-2} | n_{0,4,5}m_{0,1,3,5}-n_{0,3,5}m_{0,1,4,5}+n_{0,1,5}m_{0,3,4,5} |

{-2} |-n_{0,4,5}m_{0,1,2,5}+n_{0,2,5}m_{0,1,4,5}-n_{0,1,5}m_{0,2,4,5} |

{-2} | n_{0,3,5}m_{0,1,2,5}-n_{0,2,5}m_{0,1,3,5}+n_{0,1,5}m_{0,2,3,5} |

{-2} |-n_{0,3,4}m_{0,1,2,5}+n_{0,2,4}m_{0,1,3,5}-n_{0,2,3}m_{0,1,4,5}

     -n_{0,1,4}m_{0,2,3,5}+n_{0,1,3}m_{0,2,4,5}-n_{0,1,2}m_{0,3,4,5} |

{-2} | n_{1,2,5}m_{1,2,3,4}-n_{1,2,4}m_{1,2,3,5}+n_{1,2,3}m_{1,2,4,5} |

{-2} | n_{2,4,5}m_{0,1,2,3}-n_{2,3,5}m_{0,1,2,4}+n_{2,3,4}m_{0,1,2,5}

     -n_{0,2,5}m_{1,2,3,4}+n_{0,2,4}m_{1,2,3,5}-n_{0,2,3}m_{1,2,4,5} |

{-2} |-n_{1,4,5}m_{0,1,2,3}+n_{1,3,5}m_{0,1,2,4}-n_{1,3,4}m_{0,1,2,5}

     +n_{0,1,5}m_{1,2,3,4}-n_{0,1,4}m_{1,2,3,5}+n_{0,1,3}m_{1,2,4,5} |

{-2} |-n_{1,2,5}m_{0,1,2,4}+n_{1,2,4}m_{0,1,2,5}-n_{0,1,2}m_{1,2,4,5} |

{-2} | n_{1,2,5}m_{0,1,2,3}-n_{1,2,3}m_{0,1,2,5}+n_{0,1,2}m_{1,2,3,5} |

{-2} |-n_{1,2,4}m_{0,1,2,3}+n_{1,2,3}m_{0,1,2,4}-n_{0,1,2}m_{1,2,3,4} |

{-2} | n_{1,3,5}m_{1,2,3,4}-n_{1,3,4}m_{1,2,3,5}+n_{1,2,3}m_{1,3,4,5} |

{-2} | n_{3,4,5}m_{0,1,2,3}-n_{2,3,5}m_{0,1,3,4}+n_{2,3,4}m_{0,1,3,5}

　　　　-n_{0,3,5}m_{1,2,3,4}+n_{0,3,4}m_{1,2,3,5}-n_{0,2,3}m_{1,3,4,5} |

{-2} | n_{1,3,5}m_{0,1,3,4}-n_{1,3,4}m_{0,1,3,5}+n_{0,1,3}m_{1,3,4,5} |

{-2} |-n_{1,4,5}m_{0,1,2,3}-n_{1,2,5}m_{0,1,3,4}+n_{1,2,4}m_{0,1,3,5}

　　　　+n_{0,1,5}m_{1,2,3,4}-n_{0,1,4}m_{1,2,3,5}-n_{0,1,2}m_{1,3,4,5} |

{-2} | n_{1,3,5}m_{0,1,2,3}-n_{1,2,3}m_{0,1,3,5}+n_{0,1,3}m_{1,2,3,5} |

{-2} |-n_{1,3,4}m_{0,1,2,3}+n_{1,2,3}m_{0,1,3,4}-n_{0,1,3}m_{1,2,3,4} |

{-2} | n_{1,4,5}m_{1,2,3,4}-n_{1,3,4}m_{1,2,4,5}+n_{1,2,4}m_{1,3,4,5} |

{-2} | n_{3,4,5}m_{0,1,2,4}-n_{2,4,5}m_{0,1,3,4}+n_{2,3,4}m_{0,1,4,5}

　　　　-n_{0,4,5}m_{1,2,3,4}+n_{0,3,4}m_{1,2,4,5}-n_{0,2,4}m_{1,3,4,5} |

{-2} | n_{1,4,5}m_{0,1,3,4}-n_{1,3,4}m_{0,1,4,5}+n_{0,1,4}m_{1,3,4,5} |

{-2} |-n_{1,4,5}m_{0,1,2,4}+n_{1,2,4}m_{0,1,4,5}-n_{0,1,4}m_{1,2,4,5} |

{-2} | n_{1,3,5}m_{0,1,2,4}-n_{1,2,5}m_{0,1,3,4}-n_{1,2,3}m_{0,1,4,5}

　　　　+n_{0,1,5}m_{1,2,3,4}+n_{0,1,3}m_{1,2,4,5}-n_{0,1,2}m_{1,3,4,5} |

{-2} |-n_{1,3,4}m_{0,1,2,4}+n_{1,2,4}m_{0,1,3,4}-n_{0,1,4}m_{1,2,3,4} |

{-2} | n_{1,4,5}m_{1,2,3,5}-n_{1,3,5}m_{1,2,4,5}+n_{1,2,5}m_{1,3,4,5} |

{-2} | n_{3,4,5}m_{0,1,2,5}-n_{2,4,5}m_{0,1,3,5}+n_{2,3,5}m_{0,1,4,5}

　　　　-n_{0,4,5}m_{1,2,3,5}+n_{0,3,5}m_{1,2,4,5}-n_{0,2,5}m_{1,3,4,5} |

{-2} | n_{1,4,5}m_{0,1,3,5}-n_{1,3,5}m_{0,1,4,5}+n_{0,1,5}m_{1,3,4,5} |

{-2} |-n_{1,4,5}m_{0,1,2,5}+n_{1,2,5}m_{0,1,4,5}-n_{0,1,5}m_{1,2,4,5} |

{-2} | n_{1,3,5}m_{0,1,2,5}-n_{1,2,5}m_{0,1,3,5}+n_{0,1,5}m_{1,2,3,5} |

{-2} |-n_{1,3,4}m_{0,1,2,5}+n_{1,2,4}m_{0,1,3,5}-n_{1,2,3}m_{0,1,4,5}

$$-n_{\{0,1,4\}}m_{\{1,2,3,5\}}+n_{\{0,1,3\}}m_{\{1,2,4,5\}}-n_{\{0,1,2\}}m_{\{1,3,4,5\}} \mid$$

$$\{-2\} \mid n_{\{2,3,5\}}m_{\{1,2,3,4\}}-n_{\{2,3,4\}}m_{\{1,2,3,5\}}+n_{\{1,2,3\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,3,5\}}m_{\{0,2,3,4\}}+n_{\{2,3,4\}}m_{\{0,2,3,5\}}-n_{\{0,2,3\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid n_{\{3,4,5\}}m_{\{0,1,2,3\}}+n_{\{1,3,5\}}m_{\{0,2,3,4\}}-n_{\{1,3,4\}}m_{\{0,2,3,5\}}$$

$$-n_{\{0,3,5\}}m_{\{1,2,3,4\}}+n_{\{0,3,4\}}m_{\{1,2,3,5\}}+n_{\{0,1,3\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,4,5\}}m_{\{0,1,2,3\}}-n_{\{1,2,5\}}m_{\{0,2,3,4\}}+n_{\{1,2,4\}}m_{\{0,2,3,5\}}$$

$$+n_{\{0,2,5\}}m_{\{1,2,3,4\}}-n_{\{0,2,4\}}m_{\{1,2,3,5\}}-n_{\{0,1,2\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid n_{\{2,3,5\}}m_{\{0,1,2,3\}}-n_{\{1,2,3\}}m_{\{0,2,3,5\}}+n_{\{0,2,3\}}m_{\{1,2,3,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,3,4\}}m_{\{0,1,2,3\}}+n_{\{1,2,3\}}m_{\{0,2,3,4\}}-n_{\{0,2,3\}}m_{\{1,2,3,4\}} \mid$$

$$\{-2\} \mid n_{\{2,4,5\}}m_{\{1,2,3,4\}}-n_{\{2,3,4\}}m_{\{1,2,4,5\}}+n_{\{1,2,4\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,4,5\}}m_{\{0,2,3,4\}}+n_{\{2,3,4\}}m_{\{0,2,4,5\}}-n_{\{0,2,4\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid n_{\{3,4,5\}}m_{\{0,1,2,4\}}+n_{\{1,4,5\}}m_{\{0,2,3,4\}}-n_{\{1,3,4\}}m_{\{0,2,4,5\}}$$

$$-n_{\{0,4,5\}}m_{\{1,2,3,4\}}+n_{\{0,3,4\}}m_{\{1,2,4,5\}}+n_{\{0,1,4\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,4,5\}}m_{\{0,1,2,4\}}+n_{\{1,2,4\}}m_{\{0,2,4,5\}}-n_{\{0,2,4\}}m_{\{1,2,4,5\}} \mid$$

$$\{-2\} \mid n_{\{2,3,5\}}m_{\{0,1,2,4\}}-n_{\{1,2,5\}}m_{\{0,2,3,4\}}-n_{\{1,2,3\}}m_{\{0,2,4,5\}}$$

$$+n_{\{0,2,5\}}m_{\{1,2,3,4\}}+n_{\{0,2,3\}}m_{\{1,2,4,5\}}-n_{\{0,1,2\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,3,4\}}m_{\{0,1,2,4\}}+n_{\{1,2,4\}}m_{\{0,2,3,4\}}-n_{\{0,2,4\}}m_{\{1,2,3,4\}} \mid$$

$$\{-2\} \mid n_{\{2,4,5\}}m_{\{1,2,3,5\}}-n_{\{2,3,5\}}m_{\{1,2,4,5\}}+n_{\{1,2,5\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,4,5\}}m_{\{0,2,3,5\}}+n_{\{2,3,5\}}m_{\{0,2,4,5\}}-n_{\{0,2,5\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid n_{\{3,4,5\}}m_{\{0,1,2,5\}}+n_{\{1,4,5\}}m_{\{0,2,3,5\}}-n_{\{1,3,5\}}m_{\{0,2,4,5\}}$$

$$-n_{\{0,4,5\}}m_{\{1,2,3,5\}}+n_{\{0,3,5\}}m_{\{1,2,4,5\}}+n_{\{0,1,5\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,4,5\}}m_{\{0,1,2,5\}}+n_{\{1,2,5\}}m_{\{0,2,4,5\}}-n_{\{0,2,5\}}m_{\{1,2,4,5\}} \mid$$

$$\{-2\} \mid n_{\{2,3,5\}}m_{\{0,1,2,5\}}-n_{\{1,2,5\}}m_{\{0,2,3,5\}}+n_{\{0,2,5\}}m_{\{1,2,3,5\}} \mid$$

$$\{-2\} \mid -n_{\{2,3,4\}}m_{\{0,1,2,5\}}+n_{\{1,2,4\}}m_{\{0,2,3,5\}}-n_{\{1,2,3\}}m_{\{0,2,4,5\}}$$

$$-n_{\{0,2,4\}}m_{\{1,2,3,5\}}+n_{\{0,2,3\}}m_{\{1,2,4,5\}}-n_{\{0,1,2\}}m_{\{2,3,4,5\}} \mid$$

$$\{-2\} \mid n_{\{3,4,5\}}m_{\{1,2,3,4\}}-n_{\{2,3,4\}}m_{\{1,3,4,5\}}+n_{\{1,3,4\}}m_{\{2,3,4,5\}} \mid$$

```
{-2} |-n_{3,4,5}m_{0,2,3,4}+n_{2,3,4}m_{0,3,4,5}-n_{0,3,4}m_{2,3,4,5} |

{-2} | n_{3,4,5}m_{0,1,3,4}-n_{1,3,4}m_{0,3,4,5}+n_{0,3,4}m_{1,3,4,5} |

{-2} |-n_{2,4,5}m_{0,1,3,4}+n_{1,4,5}m_{0,2,3,4}+n_{1,2,4}m_{0,3,4,5}

         -n_{0,4,5}m_{1,2,3,4}-n_{0,2,4}m_{1,3,4,5}+n_{0,1,4}m_{2,3,4,5} |

{-2} | n_{2,3,5}m_{0,1,3,4}-n_{1,3,5}m_{0,2,3,4}-n_{1,2,3}m_{0,3,4,5}

         +n_{0,3,5}m_{1,2,3,4}+n_{0,2,3}m_{1,3,4,5}-n_{0,1,3}m_{2,3,4,5} |

{-2} |-n_{2,3,4}m_{0,1,3,4}+n_{1,3,4}m_{0,2,3,4}-n_{0,3,4}m_{1,2,3,4} |

{-2} | n_{3,4,5}m_{1,2,3,5}-n_{2,3,5}m_{1,3,4,5}+n_{1,3,5}m_{2,3,4,5} |

{-2} |-n_{3,4,5}m_{0,2,3,5}+n_{2,3,5}m_{0,3,4,5}-n_{0,3,5}m_{2,3,4,5} |

{-2} | n_{3,4,5}m_{0,1,3,5}-n_{1,3,5}m_{0,3,4,5}+n_{0,3,5}m_{1,3,4,5} |

{-2} |-n_{2,4,5}m_{0,1,3,5}+n_{1,4,5}m_{0,2,3,5}+n_{1,2,5}m_{0,3,4,5}

         -n_{0,4,5}m_{1,2,3,5}-n_{0,2,5}m_{1,3,4,5}+n_{0,1,5}m_{2,3,4,5} |

{-2} | n_{2,3,5}m_{0,1,3,5}-n_{1,3,5}m_{0,2,3,5}+n_{0,3,5}m_{1,2,3,5} |

{-2} |-n_{2,3,4}m_{0,1,3,5}+n_{1,3,4}m_{0,2,3,5}-n_{1,2,3}m_{0,3,4,5}

         -n_{0,3,4}m_{1,2,3,5}+n_{0,2,3}m_{1,3,4,5}-n_{0,1,3}m_{2,3,4,5} |

{-2} | n_{3,4,5}m_{1,2,4,5}-n_{2,4,5}m_{1,3,4,5}+n_{1,4,5}m_{2,3,4,5} |

{-2} |-n_{3,4,5}m_{0,2,4,5}+n_{2,4,5}m_{0,3,4,5}-n_{0,4,5}m_{2,3,4,5} |

{-2} | n_{3,4,5}m_{0,1,4,5}-n_{1,4,5}m_{0,3,4,5}+n_{0,4,5}m_{1,3,4,5} |

{-2} |-n_{2,4,5}m_{0,1,4,5}+n_{1,4,5}m_{0,2,4,5}-n_{0,4,5}m_{1,2,4,5} |

{-2} | n_{2,3,5}m_{0,1,4,5}-n_{1,3,5}m_{0,2,4,5}+n_{1,2,5}m_{0,3,4,5}

         +n_{0,3,5}m_{1,2,4,5}-n_{0,2,5}m_{1,3,4,5}+n_{0,1,5}m_{2,3,4,5} |

{-2} |-n_{2,3,4}m_{0,1,4,5}+n_{1,3,4}m_{0,2,4,5}-n_{1,2,4}m_{0,3,4,5}

         -n_{0,3,4}m_{1,2,4,5}+n_{0,2,4}m_{1,3,4,5}-n_{0,1,4}m_{2,3,4,5} |


                90              1

o27 : Matrix R2    <--- R2
```

First, let us consider the object Ob1 with shape coordinates CoordsOb1 and the image Im1 with shape coordinates coordsIm1. Since we already know that Im1 is the image of Ob1 under the GWP projection proj1, we expect all of these polynomials to evaluate to zero in this case.

```
i28 : substitute(rels,matrix {coordsIm1|CoordsOb1})
```

```
o28 = 0
```

```
                    90          1
o28 : Matrix ZZ    <--- ZZ
```

This computation confirms that Im1 is in fact a GWP projection of Ob1.

We may also use the function "OImatch" (also in the "affShapes" package) to evaluate the object/image relations. This function takes a list of object shape coordinates, a list of image shape coordinates, and the number of points k in our configurations and returns a value of "true" if the object/image pair is a match and "false" otherwise. This result is obtained by evaluating the object/image relations for the given object/image pair.

```
i29 : OImatch(CoordsOb1, coordsIm1,6)
```

```
o29 = true
```

Now consider the object Ob1 and the image Im4. Now we use the "OImatch" function to test for matching.

```
i30 : OImatch(CoordsOb1, coordsIm4,6)
```

```
o30 = false
```

Hence, there is no GWP projection that maps Ob1 to Im4.

4. Metrics

Since Macauly2 lacks the ability to work with trigonometric functions, we will use MATLAB to perform our computations in this section. MATLAB also provides us with the "null" command, which we use to compute orthonormal bases for the null spaces of our configuration matrices.

Let K1 be the shape of the object Ob1, and let K2 be the shape of the object Ob2. To compute the distance d_{Obj}(K1,K2) between our object shapes we first compute orthonormal bases for the null spaces of the matrices Ob1 and Ob2.

```
>> Ob1=[1, 2, -3, 8, 0, 3 ; 0, -2, -4, 6, 7, -5 ; -1, 5, 0, 1, -7, 10 ;
 1, 1, 1, 1, 1, 1]


Ob1 =


     1     2    -3     8     0     3
     0    -2    -4     6     7    -5
    -1     5     0     1    -7    10
     1     1     1     1     1     1


>> Ob2=[-3, 7, -10, 8, 1, 4 ; -9, 3, 1, 4, 6, -8 ; -4, 9, 4, 6, -10,
-10 ; 1, 1, 1, 1, 1, 1]


Ob2 =
```

```
   -3     7   -10     8     1     4

   -9     3     1     4     6    -8

   -4     9     4     6   -10   -10

    1     1     1     1     1     1
```

```
>> K1=null(Ob1)

K1 =


   -0.6512   -0.1709

    0.2005   -0.8126

    0.5482    0.2696

    0.3502    0.1573

   -0.1455    0.1077

   -0.3022    0.4488
```

```
>> K2=null(Ob2)

K2 =


   -0.1406   -0.6577

    0.6618   -0.1253

   -0.0145    0.3744

   -0.7204    0.0679

    0.0968   -0.2473

    0.1169    0.5881
```

Next we compute the singular values of $K1^T K2$.

```
>> [U,S,V]=svd(K1'*K2)


U =


   -0.4916   -0.8708

   -0.8708    0.4916




S =


    0.9073         0

         0    0.2540




V =


    0.5915   -0.8063

   -0.8063   -0.5915


>> singVals=diag(S)


singVals =


    0.9073
```

```
    0.2540
```

The principal angles between K1 and K2 are then the arccosines of the singular values.

```
>> princAngs=acos(singVals)


princAngs =


    0.4340
    1.3140
```

The distance $d_{Obj}(\text{K1,K2})$ is $\sqrt{\text{princAngs(1)}^2 + \text{princAngs(2)}^2}$ (here princAngs(i) is the ith entry of the vector princAngs) which is simply $\sqrt{(\text{princAngs})^T * \text{princAngs}}$.

```
>> dist=sqrt(princAngs'*princAngs)


dist =


    1.3838
```

So we have $d_{Obj}(\text{K1,K2})$=1.3838 (after rounding). Notice that since we have already determined that Ob1 and Ob2 do not have the same shape, we expect the distance $d_{Obj}(\text{K1,K2})$ to be nonzero.

Now let L1 and L3 be the shapes of the image configurations Im1 and Im3 respectively. Then we may use the same process to compute the distance $d_{Img}(\text{L1,L3})$, but we can do this much more consicely.

```
>> Im1=[-4,18,7,-10,-36,39 ; -2,17,-3,11,-21,33 ; 1,1,1,1,1,1]


Im1 =
```

```
 -4    18     7   -10   -36    39

 -2    17    -3    11   -21    33

  1     1     1     1     1     1
```

```
>> Im3=[-12,105,-6,47,-139,206 ; -11,58,34,-48,-120,126 ; 1,1,1,1,1,1]
```

```
Im3 =
```

```
 -12   105    -6    47  -139   206

 -11    58    34   -48  -120   126

   1     1     1     1     1     1
```

```
>> dist=sqrt((acos(svd((null(Im1))'*null(Im3))))'*
(acos(svd((null(Im1))'*null(Im3)))))
```

```
dist =
```

```
   2.5810e-08
```

As we have already observed, Im1 and Im3 have the same shape, so the distance dist=d_{Img}(L1,L3) between L1 and L3 should be zero which is what we have effectively computed here.

Finally, let K1 be the shape of the object Ob1, and let L1 and L4 be the shapes of the images Im1 and Im4 respectively. Let us compute the distances d(K1,L1) and d(K1,L4).

```
>> Ob1=[1, 2, -3, 8, 0, 3 ; 0, -2, -4, 6, 7, -5 ; -1, 5, 0, 1, -7, 10 ;
 1, 1, 1, 1, 1, 1]

Ob1 =

     1     2    -3     8     0     3
     0    -2    -4     6     7    -5
    -1     5     0     1    -7    10
     1     1     1     1     1     1

>> Im1=[-4,18,7,-10,-36,39 ; -2,17,-3,11,-21,33 ; 1,1,1,1,1,1]

Im1 =

    -4    18     7   -10   -36    39
    -2    17    -3    11   -21    33
     1     1     1     1     1     1

>> Im4=[12,-8,2,-8,41,-55 ; 5,45,4,5,-36,51 ; 1,1,1,1,1,1]

Im4 =

    12    -8     2    -8    41   -55
     5    45     4     5   -36    51
     1     1     1     1     1     1
```

```
>> distK1L1=sqrt((acos(svd((null(Im1))'*null(Ob1))))'
*(acos(svd((null(Im1))'*null(Ob1)))))


distK1L1 =


    2.1073e-08


>> distK1L4=sqrt((acos(svd((null(Im4))'*null(Ob1))))'
*(acos(svd((null(Im4))'*null(Ob1)))))


distK1L4 =


    1.2960
```

Notice that since Im1 is a GWP projection of Ob1, the distance distK1L1=d(K1,L1) is zero. We observe also that the distance distK1L4=d(K1,L4) between K1 and L4 is nonzero because Im4 is *not* a GWP projection of Ob1.

B. The Projective Case

In this appendix, we will use the "projShapes" package to perform computations involving shapes in the full perspective model. To maintain consistency with Macaulay2's indexing conventions, all indexing will begin with zero rather than 1. So a configuration of k points in $\mathbb{P}_{\mathbb{R}}^n$ will be represented as $P_0, \ldots, P_{k-1}$ rather than $P_1, \ldots, P_k$, and the corresponding Plücker coordinates will be represeted by $x_{01\ldots,n}, \ldots, x_{k-n-1\ldots k-1}$ rather than $x_{1\ldots n+1}, \ldots, x_{k-n\ldots k}$.

1. Projective Shape Varieties

To begin, let us define three object configurations.

```
i1 : Ob1= matrix {{1, 2, -3, 8, 0, 3}, {0, -2, -4, 6, 7, -5}, {-1,
5, 0, 1, -7, 10}, {1, 1, 1, 1, 1, 1}}
```

```
o1 = | 1   2  -3 8 0   3  |
     | 0  -2 -4 6 7  -5 |
     | -1 5  0  1 -7 10 |
     | 1  1  1  1 1  1  |
```

```
              4          6
o1 : Matrix ZZ  <--- ZZ
```

```
i2 : Ob2= matrix {{-3, 7, -10, 8, 1, 4}, {-9, 3, 1, 4, 6, -8}, {-4,
9, 4, 6, -10, -10}, {-1, 5, 0, 3, 2, 2}}
```

```
o2 = | -3 7 -10 8 1   4   |
```

```
        | -9 3 1   4 6   -8 |

        | -4 9 4   6 -10 -10 |

        | -1 5 0   3 2   2  |


              4         6
o2 : Matrix ZZ  <--- ZZ


i3 : Ob3= matrix {{3, 40, 28, -18, -54, 76}, {0, -38, -56, 74, 83,

-81}, {-16, 12, 8, -40, -59, 39}, {0, 14, 52, -82, -59, 37}}


o3 = | 3   40  28  -18 -54 76  |

     | 0   -38 -56 74  83  -81 |

     | -16 12  8   -40 -59 39  |

     | 0   14  52  -82 -59 37  |


              4         6
o3 : Matrix ZZ  <--- ZZ
```

Next we compute the shape varieties of these object configurations using the "ShapeVar" function. The shape varieties are obtained by computing the kernel of the map $\phi : k[x_{0123}, \ldots, x_{2345}] \to k[a_0, \ldots, a_5]$ defined by $\phi(x_{i_0 i_1 i_2 i_3}) = a_{i_0} a_{i_1} a_{i_2} a_{i_3} m_{i_0 i_1 i_2 i_3}$ where $m_{i_0 i_1 i_2 i_3}$ is the determinant of the $i_0 i_1 i_2 i_3$ minor of the configuration matrix.

```
i4 : load "projShapes"

--loaded diag

--loaded maxMinors

--loaded projShapes
```

```
i5 : V1=ShapeVar(Ob1)
```

$$o5 = \{-2\} \mid x_{\{0,3,4,5\}}x_{\{1,2,4,5\}}-x_{\{0,2,4,5\}}x_{\{1,3,4,5\}}$$
$$+x_{\{0,1,4,5\}}x_{\{2,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,3,4,5\}}x_{\{1,2,3,5\}}-x_{\{0,2,3,5\}}x_{\{1,3,4,5\}}$$
$$+x_{\{0,1,3,5\}}x_{\{2,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,2,4,5\}}x_{\{1,2,3,5\}}-x_{\{0,2,3,5\}}x_{\{1,2,4,5\}}$$
$$+x_{\{0,1,2,5\}}x_{\{2,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,1,4,5\}}x_{\{1,2,3,5\}}-x_{\{0,1,3,5\}}x_{\{1,2,4,5\}}$$
$$+x_{\{0,1,2,5\}}x_{\{1,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,3,4,5\}}x_{\{1,2,3,4\}}-x_{\{0,2,3,4\}}x_{\{1,3,4,5\}}$$
$$+x_{\{0,1,3,4\}}x_{\{2,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,2,4,5\}}x_{\{1,2,3,4\}}-x_{\{0,2,3,4\}}x_{\{1,2,4,5\}}$$
$$+x_{\{0,1,2,4\}}x_{\{2,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,1,4,5\}}x_{\{1,2,3,4\}}-x_{\{0,1,3,4\}}x_{\{1,2,4,5\}}$$
$$+x_{\{0,1,2,4\}}x_{\{1,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,2,3,5\}}x_{\{1,2,3,4\}}-x_{\{0,2,3,4\}}x_{\{1,2,3,5\}}$$
$$+x_{\{0,1,2,3\}}x_{\{2,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,1,3,5\}}x_{\{1,2,3,4\}}-x_{\{0,1,3,4\}}x_{\{1,2,3,5\}}$$
$$+x_{\{0,1,2,3\}}x_{\{1,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,1,2,5\}}x_{\{1,2,3,4\}}-x_{\{0,1,2,4\}}x_{\{1,2,3,5\}}$$
$$+x_{\{0,1,2,3\}}x_{\{1,2,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,1,4,5\}}x_{\{0,2,3,5\}}-x_{\{0,1,3,5\}}x_{\{0,2,4,5\}}$$
$$+x_{\{0,1,2,5\}}x_{\{0,3,4,5\}} \mid$$
$$\{-2\} \mid x_{\{0,1,4,5\}}x_{\{0,2,3,4\}}-x_{\{0,1,3,4\}}x_{\{0,2,4,5\}}$$

```
                  +x_{0,1,2,4}x_{0,3,4,5}                              |
{-2} | x_{0,1,3,5}x_{0,2,3,4}-x_{0,1,3,4}x_{0,2,3,5}
       +x_{0,1,2,3}x_{0,3,4,5}                                         |
{-2} | x_{0,1,2,5}x_{0,2,3,4}-x_{0,1,2,4}x_{0,2,3,5}
       +x_{0,1,2,3}x_{0,2,4,5}                                         |
{-2} | x_{0,1,2,5}x_{0,1,3,4}-x_{0,1,2,4}x_{0,1,3,5}
       +x_{0,1,2,3}x_{0,1,4,5}                                         |
{-2} | x_{0,2,4,5}x_{1,3,4,5}+12456x_{0,1,4,5}x_{2,3,4,5} |
{-2} | x_{0,2,3,5}x_{1,3,4,5}+11423x_{0,1,3,5}x_{2,3,4,5} |
{-2} | x_{0,2,3,4}x_{1,3,4,5}+5789x_{0,1,3,4}x_{2,3,4,5}  |
{-2} | x_{0,3,4,5}x_{1,2,4,5}+12457x_{0,1,4,5}x_{2,3,4,5} |
{-2} | x_{0,2,3,5}x_{1,2,4,5}+1332x_{0,1,2,5}x_{2,3,4,5}  |
{-2} | x_{0,2,3,4}x_{1,2,4,5}+13518x_{0,1,2,4}x_{2,3,4,5} |
{-2} | x_{0,1,3,5}x_{1,2,4,5}-11068x_{0,1,2,5}x_{1,3,4,5} |
{-2} | x_{0,1,3,4}x_{1,2,4,5}+11260x_{0,1,2,4}x_{1,3,4,5} |
{-2} | x_{0,3,4,5}x_{1,2,3,5}+11424x_{0,1,3,5}x_{2,3,4,5} |
{-2} | x_{0,2,4,5}x_{1,2,3,5}+1333x_{0,1,2,5}x_{2,3,4,5}  |
{-2} | x_{0,2,3,4}x_{1,2,3,5}+9626x_{0,1,2,3}x_{2,3,4,5}  |
{-2} | x_{0,1,4,5}x_{1,2,3,5}-11067x_{0,1,2,5}x_{1,3,4,5} |
{-2} | x_{0,1,3,4}x_{1,2,3,5}+12808x_{0,1,2,3}x_{1,3,4,5} |
{-2} | x_{0,1,2,4}x_{1,2,3,5}+3090x_{0,1,2,3}x_{1,2,4,5}  |
{-2} | x_{0,3,4,5}x_{1,2,3,4}+5790x_{0,1,3,4}x_{2,3,4,5}  |
{-2} | x_{0,2,4,5}x_{1,2,3,4}+13519x_{0,1,2,4}x_{2,3,4,5} |
{-2} | x_{0,2,3,5}x_{1,2,3,4}+9627x_{0,1,2,3}x_{2,3,4,5}  |
{-2} | x_{0,1,4,5}x_{1,2,3,4}+11261x_{0,1,2,4}x_{1,3,4,5} |
{-2} | x_{0,1,3,5}x_{1,2,3,4}+12809x_{0,1,2,3}x_{1,3,4,5} |
```

```
{-2} | x_{0,1,2,5}x_{1,2,3,4}+3091x_{0,1,2,3}x_{1,2,4,5}   |

{-2} | x_{0,1,3,5}x_{0,2,4,5}-4069x_{0,1,2,5}x_{0,3,4,5}   |

{-2} | x_{0,1,3,4}x_{0,2,4,5}+2810x_{0,1,2,4}x_{0,3,4,5}   |

{-2} | x_{0,1,4,5}x_{0,2,3,5}-4068x_{0,1,2,5}x_{0,3,4,5}   |

{-2} | x_{0,1,3,4}x_{0,2,3,5}-13113x_{0,1,2,3}x_{0,3,4,5} |

{-2} | x_{0,1,2,4}x_{0,2,3,5}-7088x_{0,1,2,3}x_{0,2,4,5}   |

{-2} | x_{0,1,4,5}x_{0,2,3,4}+2811x_{0,1,2,4}x_{0,3,4,5}   |

{-2} | x_{0,1,3,5}x_{0,2,3,4}-13112x_{0,1,2,3}x_{0,3,4,5} |

{-2} | x_{0,1,2,5}x_{0,2,3,4}-7087x_{0,1,2,3}x_{0,2,4,5}   |

{-2} | x_{0,1,2,4}x_{0,1,3,5}-75x_{0,1,2,3}x_{0,1,4,5}     |

{-2} | x_{0,1,2,5}x_{0,1,3,4}-74x_{0,1,2,3}x_{0,1,4,5}     |


                45           1

o5 : Matrix R1   <--- R1



i6 : V2=ShapeVar(Ob2)



o6 = {-2} | x_{0,3,4,5}x_{1,2,4,5}-x_{0,2,4,5}x_{1,3,4,5}
            +x_{0,1,4,5}x_{2,3,4,5}                        |

     {-2} | x_{0,3,4,5}x_{1,2,3,5}-x_{0,2,3,5}x_{1,3,4,5}
            +x_{0,1,3,5}x_{2,3,4,5}                        |

     {-2} | x_{0,2,4,5}x_{1,2,3,5}-x_{0,2,3,5}x_{1,2,4,5}
            +x_{0,1,2,5}x_{2,3,4,5}                        |

     {-2} | x_{0,1,4,5}x_{1,2,3,5}-x_{0,1,3,5}x_{1,2,4,5}
            +x_{0,1,2,5}x_{1,3,4,5}                        |

     {-2} | x_{0,3,4,5}x_{1,2,3,4}-x_{0,2,3,4}x_{1,3,4,5}
```

```
                 +x_{0,1,3,4}x_{2,3,4,5}                          |
{-2} | x_{0,2,4,5}x_{1,2,3,4}-x_{0,2,3,4}x_{1,2,4,5}
                 +x_{0,1,2,4}x_{2,3,4,5}                          |
{-2} | x_{0,1,4,5}x_{1,2,3,4}-x_{0,1,3,4}x_{1,2,4,5}
                 +x_{0,1,2,4}x_{1,3,4,5}                          |
{-2} | x_{0,2,3,5}x_{1,2,3,4}-x_{0,2,3,4}x_{1,2,3,5}
                 +x_{0,1,2,3}x_{2,3,4,5}                          |
{-2} | x_{0,1,3,5}x_{1,2,3,4}-x_{0,1,3,4}x_{1,2,3,5}
                 +x_{0,1,2,3}x_{1,3,4,5}                          |
{-2} | x_{0,1,2,5}x_{1,2,3,4}-x_{0,1,2,4}x_{1,2,3,5}
                 +x_{0,1,2,3}x_{1,2,4,5}                          |
{-2} | x_{0,1,4,5}x_{0,2,3,5}-x_{0,1,3,5}x_{0,2,4,5}
                 +x_{0,1,2,5}x_{0,3,4,5}                          |
{-2} | x_{0,1,4,5}x_{0,2,3,4}-x_{0,1,3,4}x_{0,2,4,5}
                 +x_{0,1,2,4}x_{0,3,4,5}                          |
{-2} | x_{0,1,3,5}x_{0,2,3,4}-x_{0,1,3,4}x_{0,2,3,5}
                 +x_{0,1,2,3}x_{0,3,4,5}                          |
{-2} | x_{0,1,2,5}x_{0,2,3,4}-x_{0,1,2,4}x_{0,2,3,5}
                 +x_{0,1,2,3}x_{0,2,4,5}                          |
{-2} | x_{0,1,2,5}x_{0,1,3,4}-x_{0,1,2,4}x_{0,1,3,5}
                 +x_{0,1,2,3}x_{0,1,4,5}                          |
{-2} | x_{0,2,4,5}x_{1,3,4,5}+13816x_{0,1,4,5}x_{2,3,4,5} |
{-2} | x_{0,2,3,5}x_{1,3,4,5}-9764x_{0,1,3,5}x_{2,3,4,5}  |
{-2} | x_{0,2,3,4}x_{1,3,4,5}-14044x_{0,1,3,4}x_{2,3,4,5} |
{-2} | x_{0,3,4,5}x_{1,2,4,5}+13817x_{0,1,4,5}x_{2,3,4,5} |
{-2} | x_{0,2,3,5}x_{1,2,4,5}-10269x_{0,1,2,5}x_{2,3,4,5} |
```

```
{-2} | x_{0,2,3,4}x_{1,2,4,5}+15137x_{0,1,2,4}x_{2,3,4,5} |

{-2} | x_{0,1,3,5}x_{1,2,4,5}+11568x_{0,1,2,5}x_{1,3,4,5} |

{-2} | x_{0,1,3,4}x_{1,2,4,5}+9033x_{0,1,2,4}x_{1,3,4,5}  |

{-2} | x_{0,3,4,5}x_{1,2,3,5}-9763x_{0,1,3,5}x_{2,3,4,5}  |

{-2} | x_{0,2,4,5}x_{1,2,3,5}-10268x_{0,1,2,5}x_{2,3,4,5} |

{-2} | x_{0,2,3,4}x_{1,2,3,5}-4530x_{0,1,2,3}x_{2,3,4,5}  |

{-2} | x_{0,1,4,5}x_{1,2,3,5}+11569x_{0,1,2,5}x_{1,3,4,5} |

{-2} | x_{0,1,3,4}x_{1,2,3,5}+1699x_{0,1,2,3}x_{1,3,4,5}  |

{-2} | x_{0,1,2,4}x_{1,2,3,5}+740x_{0,1,2,3}x_{1,2,4,5}   |

{-2} | x_{0,3,4,5}x_{1,2,3,4}-14043x_{0,1,3,4}x_{2,3,4,5} |

{-2} | x_{0,2,4,5}x_{1,2,3,4}+15138x_{0,1,2,4}x_{2,3,4,5} |

{-2} | x_{0,2,3,5}x_{1,2,3,4}-4529x_{0,1,2,3}x_{2,3,4,5}  |

{-2} | x_{0,1,4,5}x_{1,2,3,4}+9034x_{0,1,2,4}x_{1,3,4,5}  |

{-2} | x_{0,1,3,5}x_{1,2,3,4}+1700x_{0,1,2,3}x_{1,3,4,5}  |

{-2} | x_{0,1,2,5}x_{1,2,3,4}+741x_{0,1,2,3}x_{1,2,4,5}   |

{-2} | x_{0,1,3,5}x_{0,2,4,5}-7495x_{0,1,2,5}x_{0,3,4,5}  |

{-2} | x_{0,1,3,4}x_{0,2,4,5}-13394x_{0,1,2,4}x_{0,3,4,5} |

{-2} | x_{0,1,4,5}x_{0,2,3,5}-7494x_{0,1,2,5}x_{0,3,4,5}  |

{-2} | x_{0,1,3,4}x_{0,2,3,5}-9894x_{0,1,2,3}x_{0,3,4,5}  |

{-2} | x_{0,1,2,4}x_{0,2,3,5}-4706x_{0,1,2,3}x_{0,2,4,5}  |

{-2} | x_{0,1,4,5}x_{0,2,3,4}-13393x_{0,1,2,4}x_{0,3,4,5} |

{-2} | x_{0,1,3,5}x_{0,2,3,4}-9893x_{0,1,2,3}x_{0,3,4,5}  |

{-2} | x_{0,1,2,5}x_{0,2,3,4}-4705x_{0,1,2,3}x_{0,2,4,5}  |

{-2} | x_{0,1,2,4}x_{0,1,3,5}-156x_{0,1,2,3}x_{0,1,4,5}   |

{-2} | x_{0,1,2,5}x_{0,1,3,4}-155x_{0,1,2,3}x_{0,1,4,5}   |
```

```
                  45            1
o6 : Matrix R1    <--- R1



i7 : V3=ShapeVar(Ob3)


o7 = {-2} | x_{0,3,4,5}x_{1,2,4,5}-x_{0,2,4,5}x_{1,3,4,5}
            +x_{0,1,4,5}x_{2,3,4,5}                        |
     {-2} | x_{0,3,4,5}x_{1,2,3,5}-x_{0,2,3,5}x_{1,3,4,5}
            +x_{0,1,3,5}x_{2,3,4,5}                        |
     {-2} | x_{0,2,4,5}x_{1,2,3,5}-x_{0,2,3,5}x_{1,2,4,5}
            +x_{0,1,2,5}x_{2,3,4,5}                        |
     {-2} | x_{0,1,4,5}x_{1,2,3,5}-x_{0,1,3,5}x_{1,2,4,5}
            +x_{0,1,2,5}x_{1,3,4,5}                        |
     {-2} | x_{0,3,4,5}x_{1,2,3,4}-x_{0,2,3,4}x_{1,3,4,5}
            +x_{0,1,3,4}x_{2,3,4,5}                        |
     {-2} | x_{0,2,4,5}x_{1,2,3,4}-x_{0,2,3,4}x_{1,2,4,5}
            +x_{0,1,2,4}x_{2,3,4,5}                        |
     {-2} | x_{0,1,4,5}x_{1,2,3,4}-x_{0,1,3,4}x_{1,2,4,5}
            +x_{0,1,2,4}x_{1,3,4,5}                        |
     {-2} | x_{0,2,3,5}x_{1,2,3,4}-x_{0,2,3,4}x_{1,2,3,5}
            +x_{0,1,2,3}x_{2,3,4,5}                        |
     {-2} | x_{0,1,3,5}x_{1,2,3,4}-x_{0,1,3,4}x_{1,2,3,5}
            +x_{0,1,2,3}x_{1,3,4,5}                        |
     {-2} | x_{0,1,2,5}x_{1,2,3,4}-x_{0,1,2,4}x_{1,2,3,5}
            +x_{0,1,2,3}x_{1,2,4,5}                        |
     {-2} | x_{0,1,4,5}x_{0,2,3,5}-x_{0,1,3,5}x_{0,2,4,5}
```

```
                   +x_{0,1,2,5}x_{0,3,4,5}                            |

{-2} | x_{0,1,4,5}x_{0,2,3,4}-x_{0,1,3,4}x_{0,2,4,5}

        +x_{0,1,2,4}x_{0,3,4,5}                                       |

{-2} | x_{0,1,3,5}x_{0,2,3,4}-x_{0,1,3,4}x_{0,2,3,5}

        +x_{0,1,2,3}x_{0,3,4,5}                                       |

{-2} | x_{0,1,2,5}x_{0,2,3,4}-x_{0,1,2,4}x_{0,2,3,5}

        +x_{0,1,2,3}x_{0,2,4,5}                                       |

{-2} | x_{0,1,2,5}x_{0,1,3,4}-x_{0,1,2,4}x_{0,1,3,5}

        +x_{0,1,2,3}x_{0,1,4,5}                                       |

{-2} | x_{0,2,4,5}x_{1,3,4,5}+12456x_{0,1,4,5}x_{2,3,4,5} |

{-2} | x_{0,2,3,5}x_{1,3,4,5}+11423x_{0,1,3,5}x_{2,3,4,5} |

{-2} | x_{0,2,3,4}x_{1,3,4,5}+5789x_{0,1,3,4}x_{2,3,4,5}  |

{-2} | x_{0,3,4,5}x_{1,2,4,5}+12457x_{0,1,4,5}x_{2,3,4,5} |

{-2} | x_{0,2,3,5}x_{1,2,4,5}+1332x_{0,1,2,5}x_{2,3,4,5}  |

{-2} | x_{0,2,3,4}x_{1,2,4,5}+13518x_{0,1,2,4}x_{2,3,4,5} |

{-2} | x_{0,1,3,5}x_{1,2,4,5}-11068x_{0,1,2,5}x_{1,3,4,5} |

{-2} | x_{0,1,3,4}x_{1,2,4,5}+11260x_{0,1,2,4}x_{1,3,4,5} |

{-2} | x_{0,3,4,5}x_{1,2,3,5}+11424x_{0,1,3,5}x_{2,3,4,5} |

{-2} | x_{0,2,4,5}x_{1,2,3,5}+1333x_{0,1,2,5}x_{2,3,4,5}  |

{-2} | x_{0,2,3,4}x_{1,2,3,5}+9626x_{0,1,2,3}x_{2,3,4,5}  |

{-2} | x_{0,1,4,5}x_{1,2,3,5}-11067x_{0,1,2,5}x_{1,3,4,5} |

{-2} | x_{0,1,3,4}x_{1,2,3,5}+12808x_{0,1,2,3}x_{1,3,4,5} |

{-2} | x_{0,1,2,4}x_{1,2,3,5}+3090x_{0,1,2,3}x_{1,2,4,5}  |

{-2} | x_{0,3,4,5}x_{1,2,3,4}+5790x_{0,1,3,4}x_{2,3,4,5}  |

{-2} | x_{0,2,4,5}x_{1,2,3,4}+13519x_{0,1,2,4}x_{2,3,4,5} |

{-2} | x_{0,2,3,5}x_{1,2,3,4}+9627x_{0,1,2,3}x_{2,3,4,5}  |
```

```
{-2} | x_{0,1,4,5}x_{1,2,3,4}+11261x_{0,1,2,4}x_{1,3,4,5} |

{-2} | x_{0,1,3,5}x_{1,2,3,4}+12809x_{0,1,2,3}x_{1,3,4,5} |

{-2} | x_{0,1,2,5}x_{1,2,3,4}+3091x_{0,1,2,3}x_{1,2,4,5}  |

{-2} | x_{0,1,3,5}x_{0,2,4,5}-4069x_{0,1,2,5}x_{0,3,4,5}  |

{-2} | x_{0,1,3,4}x_{0,2,4,5}+2810x_{0,1,2,4}x_{0,3,4,5}  |

{-2} | x_{0,1,4,5}x_{0,2,3,5}-4068x_{0,1,2,5}x_{0,3,4,5}  |

{-2} | x_{0,1,3,4}x_{0,2,3,5}-13113x_{0,1,2,3}x_{0,3,4,5} |

{-2} | x_{0,1,2,4}x_{0,2,3,5}-7088x_{0,1,2,3}x_{0,2,4,5}  |

{-2} | x_{0,1,4,5}x_{0,2,3,4}+2811x_{0,1,2,4}x_{0,3,4,5}  |

{-2} | x_{0,1,3,5}x_{0,2,3,4}-13112x_{0,1,2,3}x_{0,3,4,5} |

{-2} | x_{0,1,2,5}x_{0,2,3,4}-7087x_{0,1,2,3}x_{0,2,4,5}  |

{-2} | x_{0,1,2,4}x_{0,1,3,5}-75x_{0,1,2,3}x_{0,1,4,5}     |

{-2} | x_{0,1,2,5}x_{0,1,3,4}-74x_{0,1,2,3}x_{0,1,4,5}     |


              45           1
o7 : Matrix R1   <--- R1
```

Now we use the "sameShape" function to compare the shapes of the objects Ob1, Ob2, and Ob3. This function takes two configuration and returns "true" if they have the same shape and "false" otherwise. This determination is made by testing for equality of the shape varieties.

```
i8 : sameShape(Ob1,Ob2)


o8 = false


i9 : sameShape(Ob1,Ob3)
```

```
o9 = true
```

Notice that Ob1 and Ob3 do in fact differ by a $PGL(4)$ transformation.

```
i10 : projTrans= matrix {{-1, -4, 5, 9}, {4, 9, -4, -8}, {-2, -3, 4,
-10}, {-4, -9, 0, 4}}


o10 = | -1 -4 5   9   |
      | 4   9  -4 -8  |
      | -2 -3 4   -10 |
      | -4 -9 0   4   |


                4         4
o10 : Matrix ZZ  <--- ZZ


i11 : det(projTrans)


o11 = -376


i12 : projTrans*Ob1


o12 = | 3    40   28   -18 -54 76  |
      | 0    -38 -56 74   83   -81 |
      | -16 12   8    -40 -59 39  |
      | 0    14   52   -82 -59 37  |
```

```
                 4          6
o12 : Matrix ZZ  <--- ZZ
```

2. Generic Shape Varieties

Here we will compute the ideal of the shape variety of an arbitrary configuration of k points in projective n-space using the "genShapeVar" function. This function computes the generators of the ideal of a shape variety by removing the variables $a_0, \ldots, a_{k-1}$ from the system of equations

$$\text{(A.1)} \qquad x_{i_0 \ldots i_n} - a_{i_0} a_{i_1} \cdots a_{i_{k-1}} m_{i_0 \ldots i_n} = 0$$

as $\{i_0, \ldots, i_n\}$ ranges over all $n+1$-subsets of $\{0, \ldots, k-1\}$. We compute these ideals for several small k and n (genShapeVar(k,n) is a generating set for the ideal of the shape variety of an arbitrary configuration of k points in $\mathbb{P}^n_{\mathbb{R}}$).

```
i13 : V41=genShapeVar(4,1)


o13={-4} | m_{0,1}m_{2,3}x_{0,2}x_{1,3}-m_{0,2}m_{1,3}x_{0,1}x_{2,3} |
   {-4} | m_{0,1}m_{2,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,1}x_{2,3} |
   {-4} | m_{0,2}m_{1,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,2}x_{1,3} |
   {-2} | m_{0,3}m_{1,2}-m_{0,2}m_{1,3}+m_{0,1}m_{2,3}        |
   {-2} | x_{0,3}x_{1,2}-x_{0,2}x_{1,3}+x_{0,1}x_{2,3}        |



                 5          1
o13 : Matrix S2  <--- S2
```

```
i14 : V51=genShapeVar(5,1)


o14={-4} | m_{1,2}m_{3,4}x_{1,3}x_{2,4}-m_{1,3}m_{2,4}x_{1,2}x_{3,4} |

    {-4} | m_{0,2}m_{3,4}x_{0,3}x_{2,4}-m_{0,3}m_{2,4}x_{0,2}x_{3,4} |

    {-4} | m_{1,2}m_{3,4}x_{1,4}x_{2,3}-m_{1,4}m_{2,3}x_{1,2}x_{3,4} |

    {-4} | m_{1,3}m_{2,4}x_{1,4}x_{2,3}-m_{1,4}m_{2,3}x_{1,3}x_{2,4} |

    {-4} | m_{0,2}m_{3,4}x_{0,4}x_{2,3}-m_{0,4}m_{2,3}x_{0,2}x_{3,4} |

    {-4} | m_{0,3}m_{2,4}x_{0,4}x_{2,3}-m_{0,4}m_{2,3}x_{0,3}x_{2,4} |

    {-4} | m_{0,1}m_{3,4}x_{0,3}x_{1,4}-m_{0,3}m_{1,4}x_{0,1}x_{3,4} |

    {-4} | m_{0,1}m_{2,4}x_{0,2}x_{1,4}-m_{0,2}m_{1,4}x_{0,1}x_{2,4} |

    {-4} | m_{0,1}m_{3,4}x_{0,4}x_{1,3}-m_{0,4}m_{1,3}x_{0,1}x_{3,4} |

    {-4} | m_{0,3}m_{1,4}x_{0,4}x_{1,3}-m_{0,4}m_{1,3}x_{0,3}x_{1,4} |

    {-4} | m_{0,1}m_{2,3}x_{0,2}x_{1,3}-m_{0,2}m_{1,3}x_{0,1}x_{2,3} |

    {-4} | m_{0,1}m_{2,4}x_{0,4}x_{1,2}-m_{0,4}m_{1,2}x_{0,1}x_{2,4} |

    {-4} | m_{0,2}m_{1,4}x_{0,4}x_{1,2}-m_{0,4}m_{1,2}x_{0,2}x_{1,4} |

    {-4} | m_{0,1}m_{2,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,1}x_{2,3} |

    {-4} | m_{0,2}m_{1,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,2}x_{1,3} |

    {-6} | m_{0,2}m_{1,2}m_{3,4}x_{0,1}x_{2,3}x_{2,4}

           -m_{0,1}m_{2,3}m_{2,4}x_{0,2}x_{1,2}x_{3,4}               |

    {-6} | m_{0,4}m_{1,2}m_{3,4}x_{0,3}x_{1,4}x_{2,4}

           -m_{0,3}m_{1,4}m_{2,4}x_{0,4}x_{1,2}x_{3,4}               |

    {-6} | m_{0,2}m_{1,4}m_{3,4}x_{0,4}x_{1,3}x_{2,4}

           -m_{0,4}m_{1,3}m_{2,4}x_{0,2}x_{1,4}x_{3,4}               |

    {-6} | m_{0,1}m_{2,3}m_{3,4}x_{0,3}x_{1,3}x_{2,4}

           -m_{0,3}m_{1,3}m_{2,4}x_{0,1}x_{2,3}x_{3,4}               |

    {-6} | m_{0,1}m_{2,4}m_{3,4}x_{0,4}x_{1,4}x_{2,3}
```

```
                   -m_{0,4}m_{1,4}m_{2,3}x_{0,1}x_{2,4}x_{3,4}                    |
{-6} | m_{0,2}m_{1,3}m_{3,4}x_{0,3}x_{1,4}x_{2,3}
                   -m_{0,3}m_{1,4}m_{2,3}x_{0,2}x_{1,3}x_{3,4}                    |
{-6} | m_{0,3}m_{1,2}m_{2,4}x_{0,2}x_{1,4}x_{2,3}
                   -m_{0,2}m_{1,4}m_{2,3}x_{0,3}x_{1,2}x_{2,4}                    |
{-6} | m_{0,3}m_{1,2}m_{3,4}x_{0,4}x_{1,3}x_{2,3}
                   -m_{0,4}m_{1,3}m_{2,3}x_{0,3}x_{1,2}x_{3,4}                    |
{-6} | m_{0,2}m_{1,3}m_{2,4}x_{0,4}x_{1,2}x_{2,3}
                   -m_{0,4}m_{1,2}m_{2,3}x_{0,2}x_{1,3}x_{2,4}                    |
{-6} | m_{0,1}m_{1,2}m_{3,4}x_{0,2}x_{1,3}x_{1,4}
                   -m_{0,2}m_{1,3}m_{1,4}x_{0,1}x_{1,2}x_{3,4}                    |
{-6} | m_{0,1}m_{1,3}m_{2,4}x_{0,3}x_{1,2}x_{1,4}
                   -m_{0,3}m_{1,2}m_{1,4}x_{0,1}x_{1,3}x_{2,4}                    |
{-6} | m_{0,1}m_{0,4}m_{2,3}x_{0,2}x_{0,3}x_{1,4}
                   -m_{0,2}m_{0,3}m_{1,4}x_{0,1}x_{0,4}x_{2,3}                    |
{-6} | m_{0,1}m_{1,4}m_{2,3}x_{0,4}x_{1,2}x_{1,3}
                   -m_{0,4}m_{1,2}m_{1,3}x_{0,1}x_{1,4}x_{2,3}                    |
{-6} | m_{0,1}m_{0,3}m_{2,4}x_{0,2}x_{0,4}x_{1,3}
                   -m_{0,2}m_{0,4}m_{1,3}x_{0,1}x_{0,3}x_{2,4}                    |
{-6} | m_{0,1}m_{0,2}m_{3,4}x_{0,3}x_{0,4}x_{1,2}
                   -m_{0,3}m_{0,4}m_{1,2}x_{0,1}x_{0,2}x_{3,4}                    |
{-2} | m_{1,3}m_{2,3}-m_{1,2}m_{2,4}+m_{0,3}m_{3,4}                               |
{-2} | m_{1,3}m_{1,4}-m_{0,4}m_{2,4}+m_{0,2}m_{3,4}                               |
{-2} | m_{1,2}m_{1,4}-m_{0,4}m_{2,3}+m_{0,1}m_{3,4}                               |
{-2} | m_{0,3}m_{1,4}-m_{0,2}m_{2,3}+m_{0,1}m_{2,4}                               |
{-2} | m_{0,3}m_{0,4}-m_{0,2}m_{1,2}+m_{0,1}m_{1,3}                               |
```

```
{-2} | x_{1,3}x_{2,3}-x_{1,2}x_{2,4}+x_{0,3}x_{3,4}            |

{-2} | x_{1,3}x_{1,4}-x_{0,4}x_{2,4}+x_{0,2}x_{3,4}            |

{-2} | x_{1,2}x_{1,4}-x_{0,4}x_{2,3}+x_{0,1}x_{3,4}            |

{-2} | x_{0,3}x_{1,4}-x_{0,2}x_{2,3}+x_{0,1}x_{2,4}            |

{-2} | x_{0,3}x_{0,4}-x_{0,2}x_{1,2}+x_{0,1}x_{1,3}            |


                    40          1
o14 : Matrix S2    <--- S2



i14 : V61=genShapeVar(6,1)


o14={-4} | m_{2,3}m_{4,5}x_{2,4}x_{3,5}-m_{2,4}m_{3,5}x_{2,3}x_{4,5} |

              .

            .(45 degree 2 polynomials)

              .

    {-4} | m_{0,2}m_{1,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,2}x_{1,3} |

    {-6} | m_{1,3}m_{2,3}m_{4,5}x_{1,2}x_{3,4}x_{3,5}

          -m_{1,2}m_{3,4}m_{3,5}x_{1,3}x_{2,3}x_{4,5}              |

              .

            .(150 degree 3 polynomials)

              .

    {-6} | m_{0,1}m_{0,2}m_{3,4}x_{0,3}x_{0,4}x_{1,2}

          -m_{0,3}m_{0,4}m_{1,2}x_{0,1}x_{0,2}x_{3,4}              |

    {-8} | m_{0,3}m_{1,3}m_{2,5}m_{4,5}x_{0,1}x_{2,4}x_{3,5}^2

          -m_{0,1}m_{2,4}m_{3,5}^2x_{0,3}x_{1,3}x_{2,5}x_{4,5}       |

              .
```

.(90 degree 4 polynomials)

.

{-8} | m_{0,1}^2m_{2,3}m_{4,5}x_{0,4}x_{0,5}x_{1,2}x_{1,3}

-m_{0,4}m_{0,5}m_{1,2}m_{1,3}x_{0,1}^2x_{2,3}x_{4,5} |

{-2} | m_{2,3}m_{3,4}-m_{1,5}m_{3,5}+m_{1,2}m_{4,5} |

{-2} | m_{2,3}m_{2,5}-m_{1,4}m_{3,5}+m_{0,5}m_{4,5} |

{-2} | m_{2,3}m_{2,4}-m_{1,3}m_{3,5}+m_{0,4}m_{4,5} |

{-2} | m_{1,5}m_{2,5}-m_{1,4}m_{3,4}+m_{0,3}m_{4,5} |

{-2} | m_{1,5}m_{2,4}-m_{1,3}m_{3,4}+m_{0,2}m_{4,5} |

{-2} | m_{1,4}m_{2,4}-m_{1,3}m_{2,5}+m_{0,1}m_{4,5} |

{-2} | m_{1,2}m_{2,5}-m_{0,5}m_{3,4}+m_{0,3}m_{3,5} |

{-2} | m_{1,2}m_{2,4}-m_{0,4}m_{3,4}+m_{0,2}m_{3,5} |

{-2} | m_{1,2}m_{1,4}-m_{0,5}m_{1,5}+m_{0,3}m_{2,3} |

{-2} | m_{1,2}m_{1,3}-m_{0,4}m_{1,5}+m_{0,2}m_{2,3} |

{-2} | m_{0,5}m_{2,4}-m_{0,4}m_{2,5}+m_{0,1}m_{3,5} |

{-2} | m_{0,5}m_{1,3}-m_{0,4}m_{1,4}+m_{0,1}m_{2,3} |

{-2} | m_{0,3}m_{2,4}-m_{0,2}m_{2,5}+m_{0,1}m_{3,4} |

{-2} | m_{0,3}m_{1,3}-m_{0,2}m_{1,4}+m_{0,1}m_{1,5} |

{-2} | m_{0,3}m_{0,4}-m_{0,2}m_{0,5}+m_{0,1}m_{1,2} |

{-2} | x_{2,3}x_{3,4}-x_{1,5}x_{3,5}+x_{1,2}x_{4,5} |

{-2} | x_{2,3}x_{2,5}-x_{1,4}x_{3,5}+x_{0,5}x_{4,5} |

{-2} | x_{2,3}x_{2,4}-x_{1,3}x_{3,5}+x_{0,4}x_{4,5} |

{-2} | x_{1,5}x_{2,5}-x_{1,4}x_{3,4}+x_{0,3}x_{4,5} |

{-2} | x_{1,5}x_{2,4}-x_{1,3}x_{3,4}+x_{0,2}x_{4,5} |

{-2} | x_{1,4}x_{2,4}-x_{1,3}x_{2,5}+x_{0,1}x_{4,5} |

{-2} | x_{1,2}x_{2,5}-x_{0,5}x_{3,4}+x_{0,3}x_{3,5} |

```
{-2} | x_{1,2}x_{2,4}-x_{0,4}x_{3,4}+x_{0,2}x_{3,5}            |

{-2} | x_{1,2}x_{1,4}-x_{0,5}x_{1,5}+x_{0,3}x_{2,3}            |

{-2} | x_{1,2}x_{1,3}-x_{0,4}x_{1,5}+x_{0,2}x_{2,3}            |

{-2} | x_{0,5}x_{2,4}-x_{0,4}x_{2,5}+x_{0,1}x_{3,5}            |

{-2} | x_{0,5}x_{1,3}-x_{0,4}x_{1,4}+x_{0,1}x_{2,3}            |

{-2} | x_{0,3}x_{2,4}-x_{0,2}x_{2,5}+x_{0,1}x_{3,4}            |

{-2} | x_{0,3}x_{1,3}-x_{0,2}x_{1,4}+x_{0,1}x_{1,5}            |

{-2} | x_{0,3}x_{0,4}-x_{0,2}x_{0,5}+x_{0,1}x_{1,2}            |


              315          1
o14 : Matrix S2     <--- S2


i15 : V52=genShapeVar(5,2)


o15 = {-4} | m_{0,1,4}m_{2,3,4}x_{0,2,4}x_{1,3,4}
            -m_{0,2,4}m_{1,3,4}x_{0,1,4}x_{2,3,4}                |

            .

            .(15 degree 2 polynomials)

            .

      {-4} | m_{0,1,3}m_{0,2,4}x_{0,1,4}x_{0,2,3}
            -m_{0,1,4}m_{0,2,3}x_{0,1,3}x_{0,2,4}                |
      {-6} | m_{0,1,4}m_{1,2,3}m_{2,3,4}x_{0,2,3}x_{1,2,4}x_{1,3,4}
            -m_{0,2,3}m_{1,2,4}m_{1,3,4}x_{0,1,4}x_{1,2,3}x_{2,3,4}  |

            .

            .(15 degree 3 polynomials)

            .
```

```
{-6} | m_{0,1,2}m_{0,1,3}m_{2,3,4}x_{0,1,4}x_{0,2,3}x_{1,2,3}

        -m_{0,1,4}m_{0,2,3}m_{1,2,3}x_{0,1,2}x_{0,1,3}x_{2,3,4}  |
{-2} | m_{1,2,3}m_{1,2,4}-m_{0,3,4}m_{1,3,4}+m_{0,2,4}m_{2,3,4} |
{-2} | m_{0,2,3}m_{1,2,4}-m_{0,1,4}m_{1,3,4}+m_{0,1,3}m_{2,3,4} |
{-2} | m_{0,2,3}m_{0,3,4}-m_{0,1,4}m_{1,2,3}+m_{0,1,2}m_{2,3,4} |
{-2} | m_{0,2,3}m_{0,2,4}-m_{0,1,3}m_{1,2,3}+m_{0,1,2}m_{1,3,4} |
{-2} | m_{0,1,4}m_{0,2,4}-m_{0,1,3}m_{0,3,4}+m_{0,1,2}m_{1,2,4} |
{-2} | x_{1,2,3}x_{1,2,4}-x_{0,3,4}x_{1,3,4}+x_{0,2,4}x_{2,3,4} |
{-2} | x_{0,2,3}x_{1,2,4}-x_{0,1,4}x_{1,3,4}+x_{0,1,3}x_{2,3,4} |
{-2} | x_{0,2,3}x_{0,3,4}-x_{0,1,4}x_{1,2,3}+x_{0,1,2}x_{2,3,4} |
{-2} | x_{0,2,3}x_{0,2,4}-x_{0,1,3}x_{1,2,3}+x_{0,1,2}x_{1,3,4} |
{-2} | x_{0,1,4}x_{0,2,4}-x_{0,1,3}x_{0,3,4}+x_{0,1,2}x_{1,2,4} |


             40          1
o15 : Matrix S2    <--- S2
```

Notice that in all of these cases, the maximum degree appearing in the generating set for the ideal of the shape variety $k - 2$ (where k is the number of points in our configurations). This seems to indicate that we must include polynomials of degree i for each $i = 2, \ldots, k-2$ as generators of the ideal of the shape variety of a configuration of k points in $\mathbb{P}^n_{\mathbb{R}}$.

Let us consider configurations of 5 points in $\mathbb{P}^1_{\mathbb{R}}$. The ideal of the shape variety for such a configuration is generated by V51.

We compute the quadratic relations 3.18 (from Chapter III) for 5 points in $\mathbb{P}^1_{\mathbb{R}}$ using the "quadRels" function.

```
i16 : Qrels51=quadRels(5,1)
```

```
o16={0}  | 0                                                        |
   {-4} | m_{0,1}m_{2,3}x_{0,2}x_{1,3}-m_{0,2}m_{1,3}x_{0,1}x_{2,3}  |
   {-4} | m_{0,1}m_{2,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,1}x_{2,3}  |
   {-4} | m_{0,1}m_{2,4}x_{0,2}x_{1,4}-m_{0,2}m_{1,4}x_{0,1}x_{2,4}  |
   {-4} | m_{0,1}m_{2,4}x_{0,4}x_{1,2}-m_{0,4}m_{1,2}x_{0,1}x_{2,4}  |
   {-4} | m_{0,1}m_{3,4}x_{0,3}x_{1,4}-m_{0,3}m_{1,4}x_{0,1}x_{3,4}  |
   {-4} | m_{0,1}m_{3,4}x_{0,4}x_{1,3}-m_{0,4}m_{1,3}x_{0,1}x_{3,4}  |
   {-4} | -m_{0,1}m_{2,3}x_{0,2}x_{1,3}+m_{0,2}m_{1,3}x_{0,1}x_{2,3} |
   {-4} | m_{0,2}m_{1,3}x_{0,3}x_{1,2}-m_{0,3}m_{1,2}x_{0,2}x_{1,3}  |
   {-4} | -m_{0,1}m_{2,4}x_{0,2}x_{1,4}+m_{0,2}m_{1,4}x_{0,1}x_{2,4} |
   {-4} | m_{0,2}m_{1,4}x_{0,4}x_{1,2}-m_{0,4}m_{1,2}x_{0,2}x_{1,4}  |
   {-4} | m_{0,2}m_{3,4}x_{0,3}x_{2,4}-m_{0,3}m_{2,4}x_{0,2}x_{3,4}  |
   {-4} | m_{0,2}m_{3,4}x_{0,4}x_{2,3}-m_{0,4}m_{2,3}x_{0,2}x_{3,4}  |
   {-4} | -m_{0,1}m_{2,3}x_{0,3}x_{1,2}+m_{0,3}m_{1,2}x_{0,1}x_{2,3} |
   {-4} | -m_{0,2}m_{1,3}x_{0,3}x_{1,2}+m_{0,3}m_{1,2}x_{0,2}x_{1,3} |
   {-4} | -m_{0,1}m_{3,4}x_{0,3}x_{1,4}+m_{0,3}m_{1,4}x_{0,1}x_{3,4} |
   {-4} | m_{0,3}m_{1,4}x_{0,4}x_{1,3}-m_{0,4}m_{1,3}x_{0,3}x_{1,4}  |
   {-4} | -m_{0,2}m_{3,4}x_{0,3}x_{2,4}+m_{0,3}m_{2,4}x_{0,2}x_{3,4} |
   {-4} | m_{0,3}m_{2,4}x_{0,4}x_{2,3}-m_{0,4}m_{2,3}x_{0,3}x_{2,4}  |
   {-4} | -m_{0,1}m_{2,4}x_{0,4}x_{1,2}+m_{0,4}m_{1,2}x_{0,1}x_{2,4} |
   {-4} | -m_{0,2}m_{1,4}x_{0,4}x_{1,2}+m_{0,4}m_{1,2}x_{0,2}x_{1,4} |
   {-4} | -m_{0,1}m_{3,4}x_{0,4}x_{1,3}+m_{0,4}m_{1,3}x_{0,1}x_{3,4} |
   {-4} | -m_{0,3}m_{1,4}x_{0,4}x_{1,3}+m_{0,4}m_{1,3}x_{0,3}x_{1,4} |
   {-4} | -m_{0,2}m_{3,4}x_{0,4}x_{2,3}+m_{0,4}m_{2,3}x_{0,2}x_{3,4} |
```

```
{-4} | -m_{0,3}m_{2,4}x_{0,4}x_{2,3}+m_{0,4}m_{2,3}x_{0,3}x_{2,4} |

{-4} | m_{1,2}m_{3,4}x_{1,3}x_{2,4}-m_{1,3}m_{2,4}x_{1,2}x_{3,4}  |

{-4} | m_{1,2}m_{3,4}x_{1,4}x_{2,3}-m_{1,4}m_{2,3}x_{1,2}x_{3,4}  |

{-4} | -m_{1,2}m_{3,4}x_{1,3}x_{2,4}+m_{1,3}m_{2,4}x_{1,2}x_{3,4} |

{-4} | m_{1,3}m_{2,4}x_{1,4}x_{2,3}-m_{1,4}m_{2,3}x_{1,3}x_{2,4}  |

{-4} | -m_{1,2}m_{3,4}x_{1,4}x_{2,3}+m_{1,4}m_{2,3}x_{1,2}x_{3,4} |

{-4} | -m_{1,3}m_{2,4}x_{1,4}x_{2,3}+m_{1,4}m_{2,3}x_{1,3}x_{2,4} |

{-2} | m_{1,3}m_{2,3}-m_{1,2}m_{2,4}+m_{0,3}m_{3,4}              |

{-2} | m_{1,3}m_{1,4}-m_{0,4}m_{2,4}+m_{0,2}m_{3,4}              |

{-2} | m_{1,2}m_{1,4}-m_{0,4}m_{2,3}+m_{0,1}m_{3,4}              |

{-2} | m_{0,3}m_{1,4}-m_{0,2}m_{2,3}+m_{0,1}m_{2,4}              |

{-2} | m_{0,3}m_{0,4}-m_{0,2}m_{1,2}+m_{0,1}m_{1,3}              |

{-2} | x_{1,3}x_{2,3}-x_{1,2}x_{2,4}+x_{0,3}x_{3,4}              |

{-2} | x_{1,3}x_{1,4}-x_{0,4}x_{2,4}+x_{0,2}x_{3,4}              |

{-2} | x_{1,2}x_{1,4}-x_{0,4}x_{2,3}+x_{0,1}x_{3,4}              |

{-2} | x_{0,3}x_{1,4}-x_{0,2}x_{2,3}+x_{0,1}x_{2,4}              |

{-2} | x_{0,3}x_{0,4}-x_{0,2}x_{1,2}+x_{0,1}x_{1,3}              |


                41          1
o16 : Matrix R    <--- R
```

We see in the following computation that Qrels51 and V51 generate the same ideal.

```
i17 : MM1=map(ring V51, ring Qrels51);


o17 : RingMap S2 <--- T1
```

```
i18 : ideal(MM1(Qrels51))==ideal(V51)
```

```
o18 = true
```

So the ideal of the shape variety of a configuration of 5 points in $\mathbb{P}^1_{\mathbb{R}}$ is generated by the quadratic polynomials

$$(A.2) \qquad m_{i_1 i_2} m_{i_3 i_4} x_{\sigma(i_1)\sigma(i_2)} x_{\sigma(i_3)\sigma(i_4)} - m_{\sigma(i_1)\sigma(i_2)} m_{\sigma(i_3)\sigma(i_4)} x_{i_1 i_2} x_{i_3 i_4}$$

with no need to include the higher degree polynomials in our generating set.

The same is true for 6 points in $\mathbb{P}^1_{\mathbb{R}}$ and 5 points in $\mathbb{P}^2_{\mathbb{R}}$.

```
i19 : Qrels61=quadRels(6,1);
```

```
                    121           1
o19 : Matrix T1     <--- T1
```

```
i20 : MM2=map(ring V61, ring Qrels61);
```

```
o20 : RingMap S2 <--- T1
```

```
i21 : ideal(MM2(Qrels61))==ideal(V61)
```

```
o21 = true
```

```
i22 : Qrels52=quadRels(5,2);
```

```
                    41           1
```

```
o22 : Matrix T1    <--- T1
```

```
i23 : MM3=map(ring V52, ring Qrels52);
```

```
o23 : RingMap S2 <--- T1
```

```
i24 : ideal(MM3(Qrels52))==ideal(V52)
```

```
o24 = true
```

Our computations of the shape varieties of Ob1, Ob2, and Ob3 seem to indicate that these quadratic polynomials generate the ideal of the shape variety in the case of 6 points in $\mathbb{P}^3_{\mathbb{R}}$ as well.

3. Embeddings of the Projective Shape Spaces in $\mathbb{P}^N_{\mathbb{R}}$

We will now compute the map $\phi_{k,n}$ which embeds the shape space $\mathcal{U}_{k,n}/PGL(n+1)$ of configurations of k points in $\mathbb{P}^n_{\mathbb{R}}$ (in general position) in some projective space $\mathbb{P}^N_{\mathbb{R}}$ for some small k and n. To do this, we use the "shapeEmbed" function of the "projShapes" package.

The map $\phi_{6,3} : \mathcal{U}'_{6,3} \subset Gr(4,6) \to \mathbb{P}^{14}_{\mathbb{R}}$ sends a point $(w_{0123} : \ldots, w_{2345}) \in \mathcal{U}'_{6,3}$ to the point $P \in \mathbb{P}^{14}_{\mathbb{R}}$ given below.

```
i25 : P=shapeEmbed(6,3)
```

```
o25 = {-3} | w_{0,1,2,5}w_{0,1,3,4}w_{2,3,4,5} |
      {-3} | w_{0,1,2,4}w_{0,1,3,5}w_{2,3,4,5} |
      {-3} | w_{0,1,4,5}w_{0,2,3,5}w_{1,2,3,4} |
```

```
{-3} | w_{0,1,3,5}w_{0,2,3,4}w_{1,2,4,5} |

{-3} | w_{0,1,2,3}w_{0,3,4,5}w_{1,2,4,5} |

{-3} | w_{0,1,3,4}w_{0,2,4,5}w_{1,2,3,5} |

{-3} | w_{0,1,4,5}w_{0,2,3,4}w_{1,2,3,5} |

{-3} | w_{0,1,2,3}w_{0,1,4,5}w_{2,3,4,5} |

{-3} | w_{0,1,2,5}w_{0,3,4,5}w_{1,2,3,4} |

{-3} | w_{0,1,3,4}w_{0,2,3,5}w_{1,2,4,5} |

{-3} | w_{0,1,3,5}w_{0,2,4,5}w_{1,2,3,4} |

{-3} | w_{0,1,2,5}w_{0,2,3,4}w_{1,3,4,5} |

{-3} | w_{0,1,2,4}w_{0,2,3,5}w_{1,3,4,5} |

{-3} | w_{0,1,2,3}w_{0,2,4,5}w_{1,3,4,5} |

{-3} | w_{0,1,2,4}w_{0,3,4,5}w_{1,2,3,5} |


            15          1
o25 : Matrix R   <--- R
```

The shapes of the configurations Ob1, Ob2, and Ob3 are then points in $\mathbb{P}_{\mathbb{R}}^{14}$ obtained by substituting some representative Plücker coordinates for the w_{ijkl} in P.

```
i26 : shapeOb1=substitute(P,matrix {maxMinors Ob1})


o26 = | -2545600 |

      | -2580000 |

      | 61950    |

      | -148200  |

      | -158600  |

      | 2238800  |
```

```
                   | 27550     |

                   | -34400    |

                   | 2369850   |

                   | -306800   |

                   | 2431800   |

                   | -175750   |

                   | -368750   |

                   | -193000   |

                   | 2211250   |


                 15          1
o26 : Matrix ZZ    <--- ZZ


i27 : shapeOb2=substitute(P,matrix {maxMinors Ob2})


o27 = | -57988879284 |

      | -68720542640 |

      | 19426518156  |

      | -57968286080 |

      | 4424544696   |

      | 4445137900   |

      | 8694854800   |

      | -10731663356 |

      | -8674261596  |

      | -53543741384 |

      | 10752256560  |
```

```
        | -66663140880 |

        | -72970259540 |

        | -6307118660  |

        | -4249716900  |


                  15        1

o27 : Matrix ZZ   <--- ZZ


i28 : shapeOb3=substitute(P,matrix {maxMinors Ob3})


o28 = | 135317416345600  |

      | 137146030080000  |

      | -3293099443200   |

      | 7877923123200    |

      | 8430759833600    |

      | -119008733388800 |

      | -1464485708800   |

      | 1828613734400    |

      | -125975007513600 |

      | 16308682956800   |

      | -129268106956800 |

      | 9342408832000    |

      | 19601782400000   |

      | 10259373568000   |

      | -117544247680000 |
```

```
               15            1
o28 : Matrix ZZ    <--- ZZ
```

If we scale each of these points by the greatest common divisor of their entries, we obtain new homogeneous coordinates.

```
i29 : sf1=gcd(flatten entries shapeOb1)


o29 = 50


i30 : shapeOb1=transpose matrix {(flatten entries shapeOb1)/sf1}


o30 = | -50912 |
      | -51600 |
      | 1239   |
      | -2964  |
      | -3172  |
      | 44776  |
      | 551    |
      | -688   |
      | 47397  |
      | -6136  |
      | 48636  |
      | -3515  |
      | -7375  |
      | -3860  |
      | 44225  |
```

```
                 15          1
o30 : Matrix QQ    <--- QQ


i31 : sf2=gcd(flatten entries shapeOb2)


o31 = 4


i32 : shapeOb2=transpose matrix {(flatten entries shapeOb2)/sf2}


o32 = | -14497219821 |

      | -17180135660 |

      | 4856629539   |

      | -14492071520 |

      | 1106136174   |

      | 1111284475   |

      | 2173713700   |

      | -2682915839  |

      | -2168565399  |

      | -13385935346 |

      | 2688064140   |

      | -16665785220 |

      | -18242564885 |

      | -1576779665  |

      | -1062429225  |
```

```
                  15            1
o32 : Matrix QQ    <--- QQ


i33 : sf3=gcd(flatten entries shapeOb3)


o33 = 2657868800


i34 : shapeOb3=transpose matrix {(flatten entries shapeOb3)/sf3}


o34 = | 50912  |
      | 51600  |
      | -1239  |
      | 2964   |
      | 3172   |
      | -44776 |
      | -551   |
      | 688    |
      | -47397 |
      | 6136   |
      | -48636 |
      | 3515   |
      | 7375   |
      | 3860   |
      | -44225 |

                  15            1
```

```
o34 : Matrix QQ    <--- QQ
```

Notice that after scaling, we see that shapeOb1 and shapeOb3 are the same point in $\mathbb{P}_{\mathbb{R}}^{14}$ which indicates that the configurations Ob1 and Ob3 have the same shape. This agrees with previous result obtained by comparing the shape varieties of Ob1 and Ob3. We also see from this computation that Ob1 and Ob2 have distinct shapes which confirms our earlier result obtained by comparing their shape varieties.

4.   Metrics

Here we will compute the distances between our shapes thought of as points in real projective space with the Fubini-Study metric. If Z, W are two points in a real projective space given as column vectors in homogeneous coordinates so that $\|Z\| = \|W\| = 1$, then the distance between Z and W is

(A.3) $$d(Z, W) = \arccos \sqrt{|Z^T W|}.$$

We use this metric to compute the distances between the shapes of Ob1, Ob2, and Ob3 in MATLAB.

```
>> shapeOb1=[-2545600, -2580000, 61950, -148200, -158600, 2238800,
27550, -34400, 2369850, -306800, 2431800, -175750, -368750, -193000,
2211250]'


shapeOb1 =


   -2545600
   -2580000
      61950
```

```
        -148200

        -158600

        2238800

          27550

         -34400

        2369850

        -306800

        2431800

        -175750

        -368750

        -193000

        2211250
```

```
>> shapeOb2=[-57988879284, -68720542640, 19426518156, -57968286080,
4424544696, 4445137900, 8694854800, -10731663356, -8674261596,
-53543741384, 10752256560, -66663140880, -72970259540, -6307118660,
-4249716900]'
```

```
shapeOb2 =

   1.0e+10 *


   -5.7989

   -6.8721

    1.9427

   -5.7968
```

```
      0.4425

      0.4445

      0.8695

     -1.0732

     -0.8674

     -5.3544

      1.0752

     -6.6663

     -7.2970

     -0.6307

     -0.4250


>> shapeOb3=[135317416345600, 137146030080000, -3293099443200,
7877923123200, 8430759833600, -119008733388800, -1464485708800,
1828613734400, -125975007513600, 16308682956800, -129268106956800,
9342408832000, 19601782400000, 10259373568000, -117544247680000]'


shapeOb3 =

   1.0e+14 *


      1.3532

      1.3715

     -0.0329

      0.0788

      0.0843
```

```
    -1.1901

    -0.0146

     0.0183

    -1.2598

     0.1631

    -1.2927

     0.0934

     0.1960

     0.1026

    -1.1754


>> shapeOb1=(1/norm(shapeOb1))*shapeOb1


shapeOb1 =


    -0.4308

    -0.4366

     0.0105

    -0.0251

    -0.0268

     0.3789

     0.0047

    -0.0058

     0.4010

    -0.0519

     0.4115
```

```
    -0.0297

    -0.0624

    -0.0327

     0.3742


>> shapeOb2=(1/norm(shapeOb2))*shapeOb2


shapeOb2 =


    -0.3672

    -0.4352

     0.1230

    -0.3671

     0.0280

     0.0281

     0.0551

    -0.0680

    -0.0549

    -0.3391

     0.0681

    -0.4221

    -0.4621

    -0.0399

    -0.0269


>> shapeOb3=(1/norm(shapeOb3))*shapeOb3
```

shapeOb3 =

     0.4308

     0.4366

    -0.0105

     0.0251

     0.0268

    -0.3789

    -0.0047

     0.0058

    -0.4010

     0.0519

    -0.4115

     0.0297

     0.0624

     0.0327

    -0.3742

>> distOb1Ob2=acos(sqrt(abs(shapeOb1'*shapeOb2)))

distOb1Ob2 =

     0.8602

>> distOb1Ob3=acos(sqrt(abs(shapeOb1'*shapeOb3)))

```
distOb1Ob3 =


    1.4901e-08


>> distOb2Ob3=acos(sqrt(abs(shapeOb2'*shapeOb3)))


distOb2Ob3 =


    0.8602
```

As we have already observed, Ob1 and Ob3 have the same shape, which is why the distance distOb1Ob3 between Ob1 and Ob3 is zero. The distance distOb1Ob2 is nonzero because Ob1 and Ob2 have distinct shapes (as previously noted).

C. The Code

Here we give the code for the "affShapes" and "projShapes" packages as well as the
code for the "perms" and "maxMinors" packages.

1. The "affShapes" Package

This package includes functions necessary for computations in the affine target recognition model.

```
load "maxMinors"

load "perms"

affOIrels=(k)->(ImInd=sort(subsets(k,3));

DualImInd=sort(subsets(k,k-3));

ObInd=sort(subsets(k,4));

Nvars={};

for i from 0 to #DualImInd-1 do Nvars=Nvars|{N_(DualImInd#i)};

mvars={};

for i from 0 to #ObInd-1 do mvars=mvars|{m_(ObInd#i)};

R1=ZZ/31991[Nvars,mvars];

Nvars={};

for i from 0 to #DualImInd-1 do Nvars=Nvars|{N_(DualImInd#i)};

mvars={};

for i from 0 to #ObInd-1 do mvars=mvars|{m_(ObInd#i)};

Alist=sort(subsets(k,2));

Blist=sort(subsets(k,k-5));

Llist=Alist;

for i from 0 to #Alist-1 do {
```

```
for j from 0 to #Alist-1 do{

p_{i,j};

};

};

OIeqns1={};

for i from 0 to #Alist-1 do {

for j from 0 to #Blist-1 do{

p#{i,j}=0;

for l from 0 to #Llist-1 do{

ALlist=(Alist#i)|(Llist#l);

BLlist=(Blist#j)|(Llist#l);

em=permsign(sort(ALlist),ALlist);

eN=permsign(sort(BLlist),BLlist);

if #(unique(ALlist))<4 or #(unique(BLlist))<k-3 then p#{i,j}=p#{i,j}

else p#{i,j}=p#{i,j}+em*eN*m_(sort(ALlist))*N_(sort(BLlist));

};

OIeqns1=OIeqns1|{p#{i,j}};

};

};

nvars={};

for i from 0 to #ImInd-1 do nvars=nvars|{n_(ImInd#i)};

R2=ZZ/31991[nvars,mvars];

mvars={};

for i from 0 to #ObInd-1 do mvars=mvars|{m_(ObInd#i)};

nvars={};

for i from 0 to #ImInd-1 do nvars=nvars|{n_(ImInd#i)};
```

```
mapList={};

for i from 0 to #DualImInd-1 do {

j=#DualImInd-1;

L=(DualImInd#i)|(ImInd#(j-i));

sortedL=sort(L);

e=permsign(sortedL,L);

mapList=mapList|{e*n_(ImInd#(j-i))};

};

mapList=mapList|mvars;

mm=map(R2,R1,mapList);

OIeqns=mm(transpose matrix {OIeqns1})

)
```

--return the affine object/image relations for configurations of k

--points.

```
OImatch=(O,I,k)->(rels=affOIrels(k);

if #O != (k!/((4!)*(k-4)!)) or #I != (k!/((3!)*(k-3)!)) then end

else eval=substitute(rels,matrix {I|O});

if ideal(eval)==0 then test=true

else test=false;

test

)
```

--takes a list O of object shape coordinates, a list I of image shape

--coordinates, and an integer k (the number of points in the

--configurations) and returns true if the object/image pair is a match

--and false otherwise.

```
sameAffShape=(M,N)->(kM=#(flatten entries M);

kN=#(flatten entries N);

nM=#(entries M);

nN=#(entries N);

if kM!=kN or nM!=nN then test=false

else(

coordsM=maxMinors(M);

coordsN=maxMinors(N);

L={};

for i from 0 to #coordsM-1 do {

L=L|{(coordsM#i)/(coordsN#i)};

};

if #unique(L)==1 then test=true

else test=false;

);

test

)
```

--takes two configuration matrices M and N and returns true if the

--configurations have the same shape and false otherwise.

2. The "projShapes" Package

This package includes functions necessary for computations in the projective target recognition model.

```
load "diag"

load "maxMinors"

load "perms"

ShapeVar=(M)->(n=#(entries M);

k=#(entries transpose M);

Ind=sort(subsets(k,n));

xvars={};

for i from 0 to #Ind-1 do xvars=xvars|{x_(Ind#i)};

R1=QQ[xvars];

xvars={};

                for i from 0 to #Ind-1 do xvars=xvars|{x_(Ind#i)};

R2=QQ[y_0..y_(k-1)];

D=diag genericMatrix(R2,y_0,1,k);

mapList=maxMinors(M*D);

mm=map(R2,R1,mapList);

eqns=transpose mingens kernel mm;

zvars={};

for i from 0 to #Ind-1 do zvars=zvars|{z_(Ind#i)};

R3=QQ[zvars];

R4=QQ[t_{0,0}..t_{k-1,n}];

tmat=genericMatrix(R4,t_{0,0},n,k);

tlist={};
```

```
for i from 0 to #Ind-1 do tlist=tlist|{det(submatrix(tmat,Ind#i))};

pluckMap=map(R4,R3,tlist);

plucks=kernel(pluckMap);

Xmap=map(R1,R3,xvars);

xplucks=Xmap(plucks);

I=transpose gens (xplucks+ideal(eqns))

)


--takes a configuration matrix M and returns the matrix of

--generators of the ideal of its shape variety (this ideal

--includes the Plucker relations).


sameShape=(M,N)->(m1=#(entries M);

m2=#(entries transpose M);

n1=#(entries N);

n2=#(entries transpose N);

if m1!=n1 or m2!=n2 then TorF=false

else(VM=ShapeVar(M);

VN=ShapeVar(N);

mm=map(ring VM, ring VN);

if ideal(VM)==ideal(mm(VN)) then TorF=true

else TorF=false

);

TorF

)
```

```
--returns true if the configurations M and N have the same
--shape and false otherwise.


genShapeVar=(p,n)->(Ind=subsets(p,n+1);

Ind=sort Ind;

k= #Ind;

wvars={};

for i from 0 to k-1 do wvars= wvars|{w_(Ind#i)};

S1=QQ[A_0..A_(p-1),wvars];

wvars={w_(Ind#0)};

for i from 1 to k-1 do wvars=wvars|{w_(Ind#i)};

alist=sequence(A_(Ind#0#0));

for i from 1 to (#Ind#0)-1 do alist=append(alist,A_(Ind#0#i));

monomialList={(times alist)*w_(Ind#0)};

for i from 1 to k-1 do (

alist=sequence(A_(Ind#i#0));

for j from 1 to (#Ind#i)-1 do(

alist=append(alist,A_(Ind#i#j)));

mon=(times alist)*w_(Ind#i);

monomialList=monomialList|{mon});

xvars={};

for i from 0 to k-1 do xvars= xvars|{x_(Ind#i)};

mvars={};

for i from 0 to k-1 do mvars= mvars|{m_(Ind#i)};

S2=QQ[mvars,xvars];

xvars={};
```

```
for i from 0 to k-1 do xvars= xvars|{x_(Ind#i)};

mvars={};

for i from 0 to k-1 do mvars= mvars|{m_(Ind#i)};

mapList=wvars|monomialList;

mm=map(S1,S2,mapList);

eqnideal=ideal mingens kernel mm;

zvars={};

for i from 0 to #Ind-1 do zvars=zvars|{z_(Ind#i)};

S3=QQ[zvars];

S4=QQ[t_{0,0}..t_{k-1,n}];

tmat=genericMatrix(S4,t_{0,0},n+1,k);

tlist={};

for i from 0 to #Ind-1 do tlist=tlist|{det(submatrix(tmat,Ind#i))};

pluckMap=map(S4,S3,tlist);

plucks=kernel(pluckMap);

Mmap=map(S2,S3,mvars);

Xmap=map(S2,S3,xvars);

mplucks=Mmap(plucks);

xplucks=Xmap(plucks);

eqns=transpose gens(eqnideal+mplucks+xplucks)

)


--takes an integer p (the number of points in the configuration)

--and an integer n (the dimension of the projective space containing

--the configuration) and returns the matrix of generators of the

--ideal of the shape variety of a generic configuration of p points
```

```
--in projective n-space.


quadRels=(k,n)->(Ind=sort(subsets(k,n+1));

mvars={};

xvars={};

for i from 0 to #Ind-1 do{

mvars=mvars|{m_(Ind#i)};

xvars=xvars|{x_(Ind#i)};

};

T1=QQ[mvars,xvars];

mvars={};

xvars={};

for i from 0 to #Ind-1 do{

mvars=mvars|{m_(Ind#i)};

xvars=xvars|{x_(Ind#i)};

};

L={};

Perms=perms(toList(sequence(0..(2*(n+1)-1))));

for i from 0 to #Ind-2 do{

for j from i+1 to #Ind-1 do{

I=(Ind#i)|(Ind#j);

J={};

for i from 0 to #Perms-1 do J=J|{I_(Perms#i)};

for l from 0 to #J-1 do {

I1=sort(take(J#l,n+1));

I2=sort(drop(J#l,n+1));
```

```
if #(unique I1)==(n+1) and #(unique I2)==(n+1) then

L=unique (L|{m_(Ind#i)*m_(Ind#j)*x_(I1)*x_(I2)-m_(I1)

*m_(I2)*x_(Ind#i)*x_(Ind#j)});

};

};

};

eqns=transpose gens ideal L;

zvars={};

for i from 0 to #Ind-1 do zvars=zvars|{z_(Ind#i)};

T2=QQ[zvars];

T3=QQ[t_{0,0}..t_{k-1,n}];

tmat=genericMatrix(T3,t_{0,0},n+1,k);

tlist={};

for i from 0 to #Ind-1 do tlist=tlist|{det(submatrix(tmat,Ind#i))};

pluckMap=map(T3,T2,tlist);

plucks=kernel(pluckMap);

Mmap=map(T1,T2,mvars);

Xmap=map(T1,T2,xvars);

mplucks=Mmap(plucks);

xplucks=Xmap(plucks);

Eqns=transpose gens(ideal(eqns)+mplucks+xplucks)

)


--takes an integer k  and an integer n and returns the quadratic

--relations that must be satisfied by the points in the shape

--variety of an arbitrary configuration of k points in projective
```

```
--n-space.



shapeEmbed=(k,n)->(Ind=sort(subsets(k,n+1));

d=k/(gcd(k,n+1));

t=(n+1)/(gcd(k,n+1));

if t==1 then (t=2;

d=2*d;

);

toInt=map(ZZ,QQ);

t=toInt(t);

d=toInt(d);

tester={};

for i from 0 to k-1 do {

L=toList(t:i);

tester=tester|L;

};

out=set Ind;

for i from 1 to d-1 do {

out=set apply(toList (out**set Ind),j->flatten toList j);

};

IndList=unique apply(toList out, j->(if sort(flatten(j))==tester

then j));

IndList=unique apply(IndList, j->sort(pack(n+1,j)));

wvars={};

for i from 0 to #Ind-1 do wvars=wvars|{w_(Ind#i)};
```

```
R=QQ[wvars];

monList={};

for i from 0 to #IndList-1 do {

wlist=apply(IndList#i,j->w_j);

monList=monList|{times toSequence wlist};

};

transpose matrix {monList}

)
```

--computes the point in projective N-space corresponding to the

--shape of an arbitrary configuration of k points in projective

--n-space.


```
shapeSpace=(k,n)->(Target=shapeEmbed(k,n);

L=flatten entries Target;

N=#L-1;

R=QQ[x_0..x_N];

mm=map(ring Target, R, L);

SS=transpose gens kernel mm

)
```


--computes the generators of the ideal of the image of the map

--shapeEmbed(k,n).

3. The "perms" Package

This package includes functions for working with permutations of lists.

```
perms = (L)->(n=#L;

if n==1 then L else

(tester = sort(L);

out=set L;

for i from 1 to n-1 do

(out=set apply(toList (out**set L),j->flatten toList j));

permlist=unique apply(toList out, k->(if sort(k)==tester then k)));

sort(permlist)

)
```

```
--produce permutations of input list L (thanks go to
--Henry Schenk for this function).
```

```
perms2 = (L)->(r=#L;

if r==1 then L else

(tester = sort(L);

out=L;

for i from 1 to r-1 do {

    out=apply(toList ((set out)**(set L)),j->flatten toList j);

            };

permlist={};

for i from 0 to #out-1 do{

if sort(out#i)==tester then permlist=unique permlist|{out#i};

};

);
```

```
sort(permlist)

)
```

```
--produce permutations of input list L (use if output

--of "perms" has null entries)
```

```
permdeg = (L)->(newperm=L;

I=sort L;

ddd=1;

while newperm != I do {

newperm = L_newperm;

ddd=ddd+1;

};

ddd

)
```

```
--return the degree of a permutation of a list {0 .. n}
```

```
permpower = (L,n)->(newperm=L;

for i from 2 to n do {

newperm=L_newperm;
```

```
};

newperm

)
```

--return the nth power of a permutation of a list {0..k}

```
perminv = (L)->(r=permdeg(L);

permpower(L,r-1)

)
```

--return the inverse of a permutation of a list {0..n}

```
posi=(a,L) -> (r=#L;

scan(r,i-> if (L#i)==a then posit = i);

posit

)
```

--return the position of the element a in a list L
--(thanks go to Henry Schenk for this function).

```
permsign=(P,Q)->(l=apply(P,i->posi(i,Q));

r=#P;

det map(ZZ^r,r,(i,j)->if l#i==j then 1 else 0)
```

```
)
```

```
--returns the sign of a permutation P of a list Q
--(thanks go to Henry Schenk for this function).
```

4. The "maxMinors" and "diag" Functions

```
maxMinors=M->(rows = #(entries M);  --number of rows
columns = #((entries M)#0);  --number of columns
if rows > columns then M=transpose M
else M=M;
end=sort subsets(columns,rows);
Mminors={};
for i from 0 to #Ind-1 do Mminors = Mminors|{det submatrix(M,Ind#i)};
Mminors
)
```

```
--Compute the determinants of the maximial minors of a matrix M.
--Returns the values in a list.
```

```
diag=M->(R=ring M;
 map(R^(numgens source M), source M,(i,j) -> if i === j then M_(0,i)
 else 0)
 )
```

```
--takes a 1 by n matrix M and returns a n by n diagonal matrix whose
--diagonal entries are the entries of M.
```

VITA

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