A STUDY OF THE SENSITIVITY OF TOPOLOGICAL DYNAMICAL SYSTEMS AND THE FOURIER SPECTRUM OF CHAOTIC INTERVAL MAPS

A Dissertation

by

MARCO A. ROQUE SOL

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Goong Chen
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	Suhail Zubairy
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ABSTRACT

A Study of the Sensitivity of Topological Dynamical Systems
and the Fourier Spectrum of Chaotic Interval Maps. (August 2006)

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Chair of Advisory Committee: Dr. Goong Chen

We study some topological properties of dynamical systems. In particular the relationship between spatio-temporal chaotic and Li-Yorke sensitive dynamical systems establishing that for minimal dynamical systems those properties are equivalent. In the same direction we show that being a Li-Yorke sensitive dynamical system implies that the system is also Li-Yorke chaotic. On the other hand we survey the possibility of lifting some topological properties from a given dynamical system (Y, S) to another (X, T). After studying some basic facts about topological dynamical systems, we move to the particular case of interval maps. We know that through the knowledge of interval maps, $f : I \to I$, precious information about the chaotic behavior of general nonlinear dynamical systems can be obtained. It is also well known that the analysis of the spectrum of time series encloses important material related to the signal itself. In this work we look for possible connections between chaotic dynamical systems and the behavior of its Fourier coefficients. We have found that a natural bridge between these two concepts is given by the total variation of a function and its connection with the topological entropy associated to the n-th iteration, $f^n(x)$, of the map. Working in a natural way using the Sobolev spaces $W^{p,q}(I)$ we show how the Fourier coefficients are related to the chaoticity of interval maps.

To My Mother: Amparo, who gave me the strength to succeed in life.

To My family: Teresa, Andrea, Marco, and Diego, who are the reason of my existence.

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CHAPTER I

INTRODUCTION

A. Topological dynamical systems

Chaotic systems have been studied for a long time by physicists and mathematicians. Landau and Lifschitz[29] gave an explanation about the transition to turbulence for a flow past to a solid obstacle, through the introduction of an infinite number of degrees of freedom. Ruelle and Takens [38] showed that a similar process just in five dimensions could lead to a more chatoic behavior. However, it is well known that three degrees of freedom for a differential equation are necessary to obtain a chaotic behavior as Lorenz [31], Rossler [36], Curry and Yorke [19], and Bowen [11] have shown. Several definitions have been formulated to define chaos, starting with the classical paper of Li and Yorke [30], where they introduced for the first time the idea of a scrambled set, continuing with the idea introduced in the paper by Auslander and Yorke [7], then Robert L. Devaney [20] introduced a definititon of chaos where we can see an extra element , namely, the idea of a regularity condition given by a dense set of periodic points. However, Robinson [35] in his book did not take into account this issue, instead he pays more attention to the transitivity and sensitive dependence on initial conditions, the latter first formulated by Guckenheimer [26] in his study on maps of the interval. I would like to mention as well that, J. Banks *et al* [9] proved that if a dynamical system is transitive and the set of periodic points is dense then the system has sensitive dependence on initial conditions. Thus, it seems like the idea of unpredictability, which is behind all of the above definitions and introduced through the concept of sensitivity, plays a central role in defining chaos for interval maps.

The journal model is *The American Mathematical Monthly*.

We would like to analyze the concept of chaos not only for interval maps but also in a more general setting: for topological dynamical systems theory. Actually, some people [25, 27, 42, 10] have taken this approach and produced excellent results, and it is with this idea we start our work. This work focuses on several concepts and results in topological dynamics. This is a huge task, so we will concentrate our attention only on some areas of this topic. Particulary, with standard concepts, definitions, and results related to topics such as sensitivity, topological transitivity, minimality, etc., and derive with some results in this area.

B. Fourier analysis

The idea to expand a function as a trigonometric series was born during the $1700s$, in connection with the study of vibrating strings and other similar physical phenomena. However, it was not until 1808 when Fourier first wrote his celebrated memoir on the theory of heat; *Théorie Analytique de le Chaleur*, which He published in 1822. In that work, he made a detailed study of trigonometric series, which he used to solve a variety of heat conduction problems. Now, as we mentioned above, the use of trigonometric series in the solution of differential equations problems had been around for quite time ago, then: what was the key idea behind Fourier's work ? It seems that the main point was that he asserted that an arbitratry function could be expanded in a trigonometric series whose coefficients could be computed in a particular way. This assertion led to questions of convergence of series and integration of arbitrary functions in connection with calculating the coefficients of the expansion, and, of course to questions about the meaning of a function. It is known that one of the most studied problems in scientific research deals with the processing of a time series $x_1, x_2, ...,$ which are nothing but an experimental data sequence, distributed in the time domain, sometimes evenly, sometimes not. Such a sequence is obtained by successively sampling over some dynamical observables, characterizing the dynamical system under investigation. As a result of processing this information, we expect to understand the behavior of the dynamical system generating such a time series. In the case we are dealing with linear systems, to obtain this information is relatively simple. However, for nonlinear systems the situation could be much more complicated. The behavior of the system is so rich as to warrant a profound study. One way to achieve this goal is through the discrete Fourier analysis, where all the information is processed and a spectrum of frequencies is obtained, allowing us to see clearly which properties thereof are playing a fundamental role inside the system. In the continuous case, that is, when, e.g., we are dealing with a signal given by a function $f(t)$ there exist many reasons for expanding it as a trigonometric sum (for example: a time-dependent electrical voltage), then a decomposition of the function into a trigonometric series gives us a description of its component frequencies. In general, given a signal of the form $sin(kt)$, it has period of $(2\pi)/k$. Thus, the sine wave

$$4sin(2t) - 35sin(5t) + 10sin(200t)$$

contains frequency components vibrating $2, 5,$ and 200 times per 2π-interval length.

Two common tasks in a signal analysis are:

Elimination of high frequency noise: this can be done expressing f as a trigonometric series

$$f(t) = a_0 + \sum_k a_k cos(kt) + b_k sin(kt)$$

and then set the high frequency-coefficients (the a_k and b_k for large k) equal to zero. Second, data compression: the idea is to send a signal in a way that it requires minimal data transmission, which can be done by expressing f as a trigonometric series, as above, and then send only those coefficients a_k, b_k, that are greater (in absolute value) than a particular tolerance. In other words, either we have a discrete sytem or a continuous one. Fourier analysis is a good tool to start with our study of a dynamical system. In particular for the continuous case, the knowledge of Fourier coefficients can give us enough information to understand and control the main components of a given signal. Thus, we want to apply this tool to analyze and understand chaotic interval maps, trying to find some relationship between those two concepts and particularly, the way Fourier coefficients rule, in some way, the dynamics of the map, being able to determine when the map is chaotic.

CHAPTER II

SENSITIVITY OF TOPOLOGICAL DYNAMICAL SYSTEMS

A. Preliminaries

We will start our discussion with some basic concepts of what a dynamical system is and how the dynamic is generated.

Definition 2.1. A *dynamical system* (X, T) consists of a topological compact metric space X and a surjective, continuous map $T : X \to X$.

The first thing we have to do at this point is understand what the dynamics of the map T will be. The idea is to introduce it through the iteration of the map. As usual $T^0 = 1_X$, identity map of X, and $T^n, \quad n \geq 0$ is the n-fold composition $\underbrace{T \circ T \circ \ldots \circ T}_{n-times}$. Now, for $x \in X$ we define the orbit

$$O_T(x) = \{T^n(x) : n \in \mathbb{N}\} \subset X$$

as we are interested in asymptotic, or long term properties of the system. Keeping this in mind, it is natural to consider the set of limit points of the orbit $O_T(x)$, denoted by $\omega_T(x)$ and given by

$$\omega_T(x) = \bigcap_{N \geq 0} \overline{\bigcup_{n \geq N} \{T^n(x)\}} \subset X.$$

Equivalently, since we are working in a metric space, we say $y \in \omega_T(x)$ if and only if there exists a subsequence $\{n_k\} \subset \mathbb{N}$ such that $n_k \to \infty$ and

$$T^{n_k}(x) \to y.$$

Since X is compact, $\omega_T(x) \neq \phi$ and

$$\overline{O_T(x)} = O_T(x) \cup \omega_T(x).$$

At this point we introduce the idea of a relation R on X, because as we will see later it will play a key role throughout the paper. In general, a relation R on X is a subset of $X \times X$ with the property

$$R(x) := \{y : (x, y) \in R\}.$$

In this way, we have the orbit and the limit point relations given by

$$O_T = \{(x, y) : y \in OT(x)\}$$

$$\omega_T = \{(x, y) : y \in \omega_T(x)\}.$$

In this context, although, $\omega_T(x)$ is a closed set in X, it is not necessarily closed in $X \times X$, so we define a closed relation by

$$\Omega_T := \bigcap_{N \geq 0} \overline{\bigcup_{n \geq N} T^n} \subset X \times X.$$

Again, since we are working in a metric space $y \in \omega_T(x)$ if and only if there exists a subsequence $\{n_k\} \subset \mathbb{N}$ and a sequence $\{\overline{x}_k\} \subset X$ such that $n_k \to \infty$, $\overline{x}_k \to x$, and $T^{n_k}(\overline{x}_k) \to y$. It is clear (as $X \times X$ is compact) that Ω_T is a closed relation containing ω_T and thus containing $\overline{\omega_T}$.

Once we have a map T acting on a space X, a natural question is finding invariant sets under the action of T. A subset A of X is called *positively invariant* if $T(A) \subset A$, *negatively invariant* if $T^{-1}(A) \subset A$, and *invariant* if $T(A) = A$. If $A \subset X$ is a closed and positively invariant set, then $(A, T|_A)$ is called a subsystem of (X, T).

Clearly $\omega_T(x)$ and $\Omega_T(x)$ are positively invariant, and since X is compact, they are in fact invariant. Now from the pointwise point of view, we say that x is a *recurrent point* for T if $x \in \omega_T(x)$. If x satisfies $x \in \Omega_T(x)$, then x is said to be a *nonwandering point*, that is, $x \in X$ is a nonwandering point if for any neighborhood U of x, there exists $n \neq 0$ such that

$$U \cap T^{-n}(U) \neq \phi.$$

Fixed points and periodic points are nonwandering.

1. Transitivity

Topological transitivity is a distinctive property of dynamical systems. One of the main reasons is its strong connection with various other topics such as Hamiltonian mechanics, ergodic theory, theory of chaos, geometry of attractors and fractals, etc. In some sense, transitive dynamical systems are a type of 'building blocks' through which more general dynamical systems are built, see for instance Smale's decomposition in [5] or John Banks' paper on topologically transitive maps [8]. The idea behind transitivity is that it represents some kind of complex global behavior of the system, guaranteeing the indecomposability of the phase space with respect to the function T. Roughly speaking, the idea of transitivity is to require any point in the phase space to visit every portion of the space in course of time. Since, the motion of a point is seldomly known completely, due to round-off errors, it is necessary to modify our requirement. Instead, we ask that every neighborhood of every point visits every region at some time or other. This leads to the precise definition of topological transitivity.

Definition 2.2. Let (X, T) be a dynamical system, then this system is called *topologically transitive*, or simply we say "T is transitive", if for every pair of open, nonmempty subsets $U, V \subset X$ there is a positive integer n such that

$$U \cap T^{-n}(V) \neq \phi.$$

Definition 2.3. Let A and B subsets of X, we define the *hitting time* set or visiting time set (the set of return times) by

$$n(A, B) := \{n \geq 0 : A \cap T^{-n}(B) \neq \phi\}.$$

Definition 2.4. [4] Given a subset $A \subset X$, we introduce two sets

$$T_{\#}A := \bigcap_{n=0}^{\infty} T^{-n}(A),$$

$$T^{\#}A := \bigcup_{n=0}^{\infty} T^{-n}(A).$$

The first set represents all the points such that their orbit is enterely inside A, on the other hand the second set represents those points such that their orbit intersects the set A.

Definition 2.5. A point $x \in X$ is said to be a *transitive point* if $\omega_T(x) = X$. In case (X, T) is a topologically transitive system we write $Trans_T$ for the set of transitive points.

Often times a dynamical system (X, T) is transitive if there is an $x_0 \in X$ such that $\overline{O_T(x_0)} = X$, that is, X has a dense orbit. It turns out that both of these definitions of transitivity are equivalent, in a wide class of spaces, including all connected compact metric spaces. Thus, we have the next result [39] :

Theorem 2.6. Let X be a second category and separable topological space with no isolated points. Then, X is transitive if and only if there is a point x_0 whose orbit is dense in X.

Another related result telling us about the nature of the set $Trans_T$ is given by [3]

Theorem 2.7. For a dynamical system (X, T) the following conditions are equivalent.

(a) T is transitive.

(b) For every pair of open, nonempty subsets U and V of X, $n(U, V)$ is infinite.

(c) For every open, nonempty subset U of X the open subset $T^{\#}U$ is dense in X.

(d) For some point $X \in X$ the orbit of x is dense in X.

(e) The set $Trans_T$ is a dense G_δ subset of X.

2. Sensitivity

Definition 2.8. We say that a system (X, T) has *sensitive dependence on initial conditions* or more briefly, is *sensitive*, if $\exists \quad \epsilon > 0$ such that $\forall \quad x \in X \quad and \quad \forall \quad U$ neighborhood of x, $\exists \quad y \in U$ and $\exists \quad n \in \mathbb{N}$ with $\rho(T^n(x), T^n(y)) > \epsilon$. Otherwise we say that (X, T) is *insensitive*.

When (X, T) violates the above property, then we have that $\forall \quad \epsilon > 0 \quad \exists \quad x \in X$ and $\exists \quad U$ neighborhood of x such that $\forall \quad y \in U$ and $\forall \quad n \in \mathbb{N} \quad \rho(T^n(x), T^n(y)) \leq \epsilon$.

Lemma 2.9. Let a topologically transitive system (X, T) have no isolated points and not be sensitive. Then the above is equivalent to the following property: $\forall \quad \epsilon > 0$ there exists a transitive point $x_0 \in X$ and a neighborhood U of x_0 such that $\forall \quad y \in U$ and $\forall \quad n \in \mathbb{N}, \quad \rho(T^n(x_0), T^n(y)) \leq \epsilon$.

Proof. Let ϵ be given and let x and U violate the property of sensitiveness. By transitivity, $\exists x_0 \in X$ whose orbit is dense in X. Let $n_0 \in \mathbb{N}$ with $T^{n_0}(x_0) \in U$, there exists $\delta > 0$ such that $B_\delta(x_0) \subset U$. Denote $x_1 := T^n(x_0)$ and $V = B_\delta(x_1)$, then it is clear that $\forall \quad y \in V$ and $\forall \quad n \in \mathbb{N} \quad \rho(T^n(x_1), T^n(y)) \leq \epsilon$. Since X has no isolated points, the point x_1 is also a transitive one, and the proof is complete.

$\square$

Recall that a system (X, T) is called *uniformly rigid* if there exists a sequence $n_k \nearrow \infty$ such that the sequence T^{n_k} tends uniformly to the identity map on X.

Lemma 2.10. If (X, T) is a topologically transitive system, insensitive and X does not have isolated points. Then (X, T) is uniformly rigid.

Proof. Given an $\epsilon > 0$, by the previous lemma there is a transitive point x_0 and a neighborhood U of x_0 such that $\rho(T^n(x_0), T^n(y)) \leq \epsilon$ for every $n \geq n_0$, and every $y \in U$. Let k now satisfy $T^k(x_0) \in U$, then $\rho(T^{n+k}(x_0), T^n(x_0)) \leq \epsilon$ for every n, and since x_0 is transitive, it follows that $\rho(T^k(z), z) \leq \epsilon$ for every $z \in X$. Applying this observation to a sequence of $e_i's$ that tend to zero, then this gives a sequence of $k_i's$ such that T^{k_i} tends uniformly to the identity.

$\square$

3. Minimality

Definition 2.11. A *minimal set* is a non-empty, closed, and (positively) invariant set, which contains no proper non-empty closed invariant subsets. A non-empty closed set M is minimal if and only if, for each $x \in M$, the orbit closure satisfies $\overline{O_T(x)} = M$. A point is called a *minimal point* or *almost periodic point* if it belongs to a minimal set.

We can show through Zorn's lemma that if (X, T) is a dynamical system then any non-empty closed (positively) invariant subset of X contains minimal subsets. It is clear that stationary and periodic orbits are minimal sets. From now on we call a minimal set M *nontrivial* if M is infinite. Thus, minimal subsets are exactly the members of the class of closed, nonempty invariant subsets which are minimal with repsect to the ordering by inclusion.

4. Equicontinuity

Definition 2.12. A dynamical system (X, T) is said to be *equicontinuous* if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in X$ with $\rho(x, y) < \delta$ then for any $n \in \mathbb{N}$ one has $\rho(T^n(x), T^n(y)) \leq \epsilon$. By compactness, if (X, T) is not equicontinuous there is an $\epsilon > 0$ and a point $x \in X$ such that for any $\delta > 0$ and $n \in \mathbb{N}$ such that $\rho(T^n(x), T^n(y)) > \epsilon$.

A point $x \in X$ is called an *equicontinuous point* if for any $\epsilon > 0 \quad \exists \quad \delta > 0$ such that if $y \in X$ with $\rho(x, y) < \delta$ then $\forall \quad n \in \mathbb{N}$ one has $\rho(T^n(x), T^n(y)) < \epsilon$; obviously a system is equicontinuous if all its points are equicontinuity points. Another way to introduce equicontinuity is through the following concept. A point $x \in X$ is called *Liapunov stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(x, y) < \delta$ implies

$\rho_T(x, y) < \epsilon$, where

$$\rho_T(x, y) := sup\{\rho(T^n(x), T^n(y)) : n \geq 0\}$$

This condition is tells us that the sequence of iterates $\{T^n : n \geq 0\}$ is equicontinuous at x. In this direction we can define the sets

$$\begin{aligned} \mathcal{E}q_\epsilon(T) &:= \bigcup\{U \subset X : \quad U \quad is \quad open \quad with \quad diam_T(U) \leq \epsilon\} \\ \mathcal{E}q(T) &:= \bigcap_{\epsilon>0} \mathcal{E}q_\epsilon(T) \end{aligned}$$

where $diam_T(A) := sup\{\rho_T(x, y) : x, y \in A\}$. Thus, $\mathcal{E}q(T)$ represents the set of equicontinuity points. In this way (X, T) is equicontinuous exactly when the sequence $\{T^n : n \geq 0\}$ is uniformly equicontinuous, that is, $\mathcal{E}q(T) = X$. If the G_δ set $\mathcal{E}q(T)$ is dense in X then the system is called *almost equicontinuous* . However, if for some $\epsilon > 0$, $\mathcal{E}q_\epsilon(T) = \phi$ then the system has sensitive dependence upon initial conditions.

B. Chaos

During the last 30 years the study of nonlinear systems has been improved by new computational methods, software, and hardware which has helped with the investigation of nonlinear phenomena. Indeed, numerical observations can give us important intuition and insights in our research, ending up with propositions and theorems which build the structure of the theory behind the system. In particular, it can help to understand or at least visualize chaotic phenomena appearing in some nonlinear problems. First of all, we would like to set what a chaotic system is througout this dissertation. Yet, this is a challenging task, instead of this we just state some widely used definitions and we will clearly indicate which one is used. First the famous Li-Yorke paper [30] gives a definition of what a chaotic system means. The math community has tried to give alternative definitions of chaos or chaotic system, which involves the main characteristics associated with that type of systems. However, we do not yet have a universally accepted definiton of chaos. One of the most popular definition is given by Devaney [20]. Let us discuss both of them, below.

Definition 2.13. A continuous map T on a compact metric space (X, ρ) is said to be chaotic on an invariant set X_0 in the sense of Li-Yorke provided there is an uncountable (scrambled) set $S \subset X_0$, such that:

$$\begin{aligned} (i) \qquad & \limsup_{n\to\infty} \rho(T^n(x), T^n(y)) > 0 \quad \forall x, y, x \neq y, \in S. \\ (ii) \qquad & \liminf_{n\to\infty} \rho(T^n(x), T^n(y)) = 0 \quad \forall x, y, \in S. \end{aligned}$$

Definition 2.14. Let X be a metric space with metric $\rho(\cdot)$, and let $T : X \to X$ be a continuous function. We say that T is chaotic on X, in the sense of Devaney, if

(i) T is topologically transitive on X.

(ii) The set of all periodic points of T is dense in x.

(iii) T has sensitive dependence on initial condition.

As Devaney has mentioned in his book [20], condition (i) means that a chaotic system is indecomposable, i.e., the system can not be decomposed into the disjoint sum of two subsystems. Condition (ii) implies that all systems with no periodic point are not chaotic (a regularity condition), and condition (iii) says that the system is unpredictable in the long run, that is, a small change of initial data may cause a large deviation after many iterations. However, the above conditions are not independent.

Banks *et al.* [9], proved that conditions (i) and (ii) imply condition (iii). Now, if we analyze the second condition in the above definition we find that according to it minimal systems can not be chaotic, since they don't have periodic points. To leave that door open we can follow Robinson [35] who gave a slightly different definiton of chaotic system.

Definition 2.15. A continuous map T, on a metric space (X, ρ), is said to be chaotic provided that
(i) T is topologically transitive, and
(ii) T has sensitive dependence on initial data.

Here, we see he is paying more attention to the transitivity and sensitive dependence on initial conditions properties of the system than to the periodicity. Moreover, in the case of an interval map $f : I \to I$ Vellekoop and Berglund [40] proved that to be transitive is equivalent to be chaotic. Therefore for interval maps the most relevant property, as a chaotic system, is transitivity.

There are other ways of quantitative measurement of the complex or chaotic nature of the dynamics. There are the Liapunov exponents, various concepts of fractal dimension including the well known box dimension and the Hausdorff dimension, and topological entropy. For instance, if a system has positive Liapunov exponent, we say that it is chaotic. This definition of chaos is perhaps the most computable (in an approximation sense). The reader may find some relevant material in the excelent classical work of Wolf *et al* [41] . On the other hand, the box and Hausdorff dimensions are two important concepts in fractal geometry [21] [23], with the former defined in a constructive way seeming to serve as a reasonable quantitavie measure of chaos [35]. The topological entropy introduced by Adler, Konheim and McAndrew [1] for compact dynamical systems and later Bowen [12] [13] gave a new, but equivalent, definition for uniformly continuous map on a metric space which is not necessarily compact. It is known that a system is complex if it has positive topological entropy. Just to mention one more thing in this direction, Chen *et al* [15] introduced another way to understand chaos through $V_I(f^n)$, the total variation of the $n-th$ iterates of a function f taking into account the oscillatory behavior of the function f^n. Here, given an interval map $f : I \to I$, they study the oscillatory behavior of f^n, the $n-th$ iterates of the function, as $n \to \infty$ they have found four distinctive cases of the growth of total variation of it. They study in detail these cases in relation to the well-known notions of sensitive dependence on inital data, topological entropy, homoclinic orbits, etc.

C. Li-Yorke sensitive dynamical systems

Let (X, T) be a topological dynamical system with metric ρ, then for each $x \in X$ we define the sets of proximal , $\epsilon-$asymptotic, and asymptotic points to x as follows:

$$Prox(T)(x) := \{y : \lim_{n \in \infty} inf \rho(T^n(x), T^n(y)) = 0\}.$$

$$Asym_\epsilon(T)(x) := \{y : \exists \quad n > 0, s.t. \quad \rho(T^i(x), T^i(y)) < \epsilon \quad \forall i \geq n\}.$$

$$Asym(T)(x) := \cup_{\epsilon > 0} Asym_\epsilon(T)(x).$$

It is clear that a point y is asymptotic to x if and only if $y \in Asym_\epsilon(T)(x), \quad \forall \epsilon > 0.$

Keeping these ideas in mind, we can introduce the concept of a Li-Yorke sensitive dynamical system as in [4]

Definition 2.16. A topological dynamical system, (X, T), is called *Li-Yorke sensitive* if there exists an $\epsilon > 0$ such that for all $x \in X$ the set $Prox(T)(x) \setminus Asym_\epsilon(T)(x)$ is dense in X.

Another type of dynamical systems involved in this theory are the so called *mixing systems.*

Definition 2.17. A topological dynamical system (Y, S) is called *topologically mixing* if given nonempty open $U, V \subset X$ there is an $n_0 \in \mathbb{N}$ such that $T^n(U) \bigcap V \neq \emptyset$ whenever $n \geq n_0$.

Definition 2.18. A topological dynamical system (X, T) is called *topologically weakly mixing* if the system $(X \times X, T \times T)$ is topologically transitive or equivalent if and only if given nonempty subsets U, V of X there is an $n > 0$ such that $T^{-n}(U) \bigcap U \neq \emptyset$ and $T^{-n}(V) \bigcap U \neq \emptyset$.

It is clear that the following relation holds
Mixing $\Longrightarrow$ Weakly Mixing $\Longrightarrow$ Transitivity.

Using the above ideas we can characterize a sensitive dynamical system in the following way [4]

Theorem 2.19. For a dynamical system (X, T) the following conditions are equivalent:

1) The system is sensitive;

2) There exists a positive ϵ such that for all $x \in X$ $Asym_{\epsilon}(T)$ is a first category set of $X \times X$;

3) There exists a positive ϵ such that for all $x \in X$ $Asym_{\epsilon}(T)(x)$ is a first category set of $X \times X$;

4) There exists a positive ϵ such that for every $x \in X$ the set $X \setminus Asym_{\epsilon}(T)(x)$ is dense in X;

5)There exists a positive ϵ such that the set of points

$$\{(x, y) \in X \times X : limsup_{n \to \infty} \rho(T^n(x), T^n(y)) > \epsilon\}$$

is dense in $X \times X$.

A relationship between Li-Yorke and topologically weakly mixing systems is given in [4]:

Theorem 2.20. If (X, T) is a *weakly mixing* dynamical system, then for every $x \in X$, the proximal cell $Prox(T)(x)$ is dense in X.

Corollary 2.21. If a nontrivial dynamical system (X, T) is *weakly mixing* then it is *Li-Yorke* sensitive.

Moreover, the relation between sensitivity and Li-Yorke sensitivity is established in the following:

Theorem 2.22. Let (X,T) be a dynamical system. If (X,T) is Li-Yorke sensitive, then it is sensitive. If (X,T) is sensitive and for every $x \in X$ the proximal cell $Prox(T)(x)$ is dense in X then (X,T), is Li-Yorke sensitive.

In the case of minimal dynamical systems we have [4] the following:

Theorem 2.23

For a minimal dynamical system, the following conditions are equivalent:

1) The system is weakly mixing.

2) For every $x \in X$, the proximal cell $Prox(T)(x)$ is dense in X.

3) For some $x \in X$, the proximal cell $Prox(T)(x)$ is dense in X.

4)$Prox(T)(x)$ is dense in $X \times X$.

Something related to the latter defintion is the concept of spatio-temporally chaotic system, introduced in [10]

Definition 2.24. A topological dynamical system (X,T) is called *spatio-temporally chaotic* if for $x \in X$ the set $Prox(T)(x) \setminus Asym(T)(x)$ is dense in X.

It is clear that Li-Yorke sensitive dynamical system is also spatio-temporally chaotic. The next result [4] gives us sufficient conditions to ensure that a sytem is temporally chaotic.

Theorem 2.25. Assume that a dynamical system (X,T) satisfies the following conditions:

1) The system is infinite and transitive.

2) Every point is recurrent.

3) Every minimal point is periodic.

Then the system is spatio-temporally chaotic.

Now, for this part of the work we would like to establish the goals to be accomplished:

(Q1) Investigate for minimal systems, if the concepts of spatio-temporal chaos and Li-Yorke sensitivity are equivalent.

(Q2) Investigate if the Li-Yorke sensitivity property of a dynamical system implies that it is Li-Yorke chaotic.

(Q2) Let (X, T) and (Y, S) be two dynamical systems and let $h : (X, T) \to (Y, S)$ be a conjugation map between them. If (Y, S) is a Li-Yorke sensitive dynamical system, what can we say about (X, T) ? Viceversa. What if we use an action map ?

D. Minimal dynamical systems and spatio-temporal chaos.

To answer **(Q1)** we start with the following result.

Lemma 2.26. Let (X, T) be a compact dynamical system such that for all $\alpha \in X$, the set $Prox(T)(\alpha)$ is dense in X. Then the set $Prox(T)$ is dense in $X \times X$.

Proof. Let (z, w) be a point in $X \times X$. Given $\epsilon > 0$, take a point $\bar{y}$ such that

$$\rho(\bar{y}, z) < \epsilon,$$

since T is onto, then $\exists y \in X$ such that $T(y) = \bar{y}$. Thus we have

$$\rho(T(y), z) < \epsilon$$

for the same reason explained above $\exists u \in X$ such that $T(u) = w$, and by continuity of T we know that $\forall \epsilon > 0 \quad \exists \delta > 0$ such that

$$if \quad \rho(x,u) < \delta, \quad then \quad \rho(T(x),T(u)) = \rho(T(x),w) < \epsilon.$$

Now, by hypothesis $Prox(T)(\alpha)$ is dense in X for all $\alpha \in X$, in particular so is $Prox(T)(y)$ for y given as above, then we can pick $x \in Prox(T)(y)$ such that

$$\rho(x,u) < \delta,$$

which implies

$$\rho(T(x),w) < \epsilon,$$

but $(x,y) \in Prox(T)$ implies $(T(x),T(y)) \in Prox(T)$. Then given $\epsilon > 0$ and $(z,w) \in X \times X$, we have found a point $(T(x),T(y)) \in Prox(T)$ such that

$$\rho(T(y),z) < \epsilon,$$

$$\rho(T(x),w) < \epsilon.$$

Therefore Prox(T) is dense in $X \times X$.

$\square$

Theorem 2.27. Let (X,T) be a minimal topological dynamical system. Then the system is spatio-temporally chaotic if and only if is Li-Yorke sensitive.

Proof. By definition, a dynamical system (X,T) is spatio-temporally chaotic if $\forall x \in X$ the set $Prox(T)(x) \setminus Asym(T)(x)$ is dense in X. Thus, if (X,T) is Li-Yorke sensitive then $\exists \quad \epsilon > 0$ such that the set $Prox(T)(x) \setminus Asym_{\epsilon}(T)(x)$ is dense in X. But, $\forall \epsilon > 0$

$$Prox(T)(x) \setminus Asym_\epsilon(T)(x) \subset Prox(T)(x) \setminus Asym(T)(x).$$

Therefore the system is spatio-temporally chaotic.

On the other hand, if (X, T) is spatio-temporally chaotic, then $\forall x \in X$ the set $Prox(T)(x) \setminus Asym(T)(x)$ is dense in X; in particular so is $Prox(T)(x)$. Then by Lemma 2.25, $Prox(T)$ is dense in $X \times X$. But for minimal systems the latter property implies (X, T) is a weakly-mixing system, and therefore Li-Yorke sensitive [4] .

■

E. Li-Yorke sensitivity implies Li-Yorke chaoticity

To solve **(Q2)** we have the following results:

Lemma 2.28. Let X be a compact metric topological space and let $\{A_k\}_{k=0}^{\infty}$ be a family of open dense subsets of $X \times X$. For each $x \in X$ define

$$\alpha^k(x) := \{y \in X : (x, y) \in A_k\},$$

$$\beta^k := \{x \in X : \alpha^k(x) \quad is \quad a \quad dense \quad set\}.$$

Then β^k is dense in X.

Proof. Since X is compact metric and therefore a second countable topological space, then we know that there exists a countable basis $\{U_i\}_{i=1}^{\infty}$for the topology of X, and that there exists a dense subset $\{z_i\}_{i=1}^{\infty}$ of X (every second countable space is separable [24]). Now, given a point z_j, consider all the possible sets U_i containing z_j, and let us denote such by $\{U_{r,j}\}_{r\in I}$ where I is a countable index, i.e.,

$$z_j \in U_{r,j} \forall r \in I.$$

In terms of the family $\{U_{r,j}\}_{r\in I}$ and the dense subset $\{z_i\}_{i=1}^{\infty}$, the set β^k can be written as

$$\begin{aligned}
\beta^k &= \{x \in X : \alpha^k(x) \cap U_{r,j} \neq \phi, \forall r \in I, \forall j \in \mathbb{N}\} \\
&= \bigcap_{r\in A}^{j\in\mathbb{N}} \{x \in X : \alpha^k(x) \cap U_{r,j} \neq \phi\}.
\end{aligned}$$

Now, taking the complement of β^k, we obtain

$$\begin{aligned}
(\beta^k)^c &= \bigcup_{r\in A}^{j\in\mathbb{N}} \{x \in X : \alpha^k(x) \cap U_{r,j} = \phi\} \\
&= \bigcup_{r\in A}^{j\in\mathbb{N}} \{x \in X : \{y \in X : (x,y) \in A_k\} \cap U_{r,j} = \phi\} \\
&= \bigcup_{r\in A}^{j\in\mathbb{N}} \{x \in X : (x,y) \in (A_k)^c, \quad \forall y \in U_{r,j}\} \\
&= \bigcup_{r\in A}^{j\in\mathbb{N}} \gamma^k_{r,j},
\end{aligned}$$

where

$$\gamma^k_{r,j} := \{x \in X : (x,y) \in (A_k)^c, \quad \forall y \in U_{r,j}\}.$$

It follows from the fact the set $\{x \in X : \exists \quad y \in U_{r,j} \quad s.t \quad (x,y) \in A_k\}$ is open that

$\gamma^k_{r,j}$ is a closed set. Thus, $(\beta^k)^c$ is a countable union of closed sets. We claim $\gamma^k_{r,j}$ is a nowhere dense set. For if not, let U^k be a nonempty set such that

$$U^k \subset \gamma^k_{r,j}.$$

This implies

$$U^k \times U_{r,j} \subset (A_k)^c,$$

a contradiction to the fact A_k is a dense open set and $(A_k)^c$ is a nowhere dense set. Therefore, we conclude that $\gamma^k_{r,j}$ is a nowhere dense set which implies $(\beta^k)^c$ is a first category set and β^k is a residual one.

□

Theorem 2.29. Let (X, T) be a topological dynamical system which is Li-Yorke sensitive. Then $\exists \quad \epsilon > 0$ such that the set $Prox(T) \setminus Asym_\epsilon(T)$ contains an uncountable set.

Proof. Since (X, T) is a Li-Yorke sensitive dynamical system, $\exists \quad \epsilon > 0$ such that for all $x \in X$ the set $Prox(T)(x) \setminus Asym_\epsilon(T)(x)$ is dense in X. In particular we have that $\forall x \in X \quad Prox(T)(x)$ is a dense set in X. In this way we have from Lemma 2.25 that

$$Prox(T) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} (T \times T)^{-n}(V_k) = \bigcap_{k=1}^{\infty} A_k$$

is a dense set in $X \times X$, where

$$A_k := \bigcup_{n=1}^{\infty} (T \times T)^{-n}(V_k),$$

$$V_k := \{(x,y) : \rho(x,y) < \frac{1}{k}\}.$$

Thus, $Prox(T)$ is a G_δ set with $\{A_k\}_{k=1}^{\infty}$ a family of dense open sets of $X \times X$. On the other hand, since (X,T) is Li-Yorke sensitive, it is sensitive and therefore $Asym_\epsilon(T)$ is of the first category [4]. In this way the relation

$$Prox(T) \setminus Asym_\epsilon(T) = \bigcap_{k=o}^{\infty} A_k$$

is satisfied, where $A_0 := [Asym_\epsilon(t)]^c$, is a G_δ set in $X \times X$ with $\{A_k\}_{k=0}^{\infty}$ a family of dense open sets in $X \times X$. Now, take the sets $\alpha^k(x)$ and β^k as in Lemma 2.27 and define β as follows

$$\beta := \bigcap_{k=0}^{\infty} \beta^k.$$

Then β is a G_δ dense set in X, and $\forall x \in \beta$ the set α_x defined by

$$\alpha_x := \bigcap_{k=0}^{\infty} \alpha^k(x)$$

which is another G_δ dense set in X, since $\alpha^k(x)$ is so for all $x \in X$ and $k = 0, 1, \ldots$ Moreover, $\forall x \in \beta$, if $y \in \alpha_x$, then $y \in \alpha^k(x) \quad \forall k = 0, 1, ...$.

$=> \quad (x,y) \in A_k \quad \forall k = 0, 1, ...$

$=> \quad (x,y) \in Prox(T) \setminus Asym_\epsilon(T)$

$=> \quad \{(x,y) : x \in \beta, y \in \alpha_x\} \subset Prox(T) \setminus Asym_\epsilon(T),$

therefore

$$\alpha_x \subset Prox(T)(x) \setminus Asym_\epsilon(T)(x) \quad \forall x \in \beta.$$

Now, consider the family $\mathcal{F}$ of all nonempty subsets of β such that if $G \in \mathcal{F}$ then $G \times G \setminus \Delta_X \subset Prox(T) \setminus Asym_\epsilon(T)$, where $\Delta_X := \{(x,x) : x \in X\}$. This family is nonempty, as for some $x_1 \in \beta$ we take $x_2 \in [Prox(T)(x_1) \setminus Asym_\epsilon(T)(x_1)] \bigcap \beta$ such that $x_2 \neq x_1$ (this is possible since the latter set contains α_{x_1}, a G_δ dense set). Then $\{x_1, x_2\} \in \mathcal{F}$. Furthermore, X is a compact metric space. Therefore there exists $\{U_i\}_{i=1}^\infty$ countable basis for the topology of X (true for any second countable space) and inductively we can construct (relabeling the sets U_i if necessary)

$x_1 \in U_1 \quad and \quad x_1 \in \beta,$

$x_2 \in \{\beta \quad \bigcap \quad [Prox(T)(x_1) \setminus Asym_\epsilon(T)(x_1)]\} \bigcap U_2, \quad x_2 \neq x_1,$

$x_3 \in \{\beta \quad \bigcap \quad [\bigcap_{i=1}^2 Prox(T)(x_i) \setminus Asym_\epsilon(T)(x_i)]\} \bigcap U_3, \quad x_3 \neq x_2 \neq x_1,$

$\vdots$

$x_{n+1} \in \{\beta \bigcap [\bigcap_{i=1}^n Prox(T)(x_i) \setminus Asym_\epsilon(T)(x_i)]\} \bigcap U_{n+1}, \quad x_{n+1} \neq x_n \neq x_{n-1} \ldots \neq x_1,$

such that $\{x_1, x_2, \ldots, x_n\} \in \mathcal{F}$ and it is a dense set.

We can introduce a partial order on $\mathcal{F}$, by inclusion, i.e., for $F_1, F_2 \in \mathcal{F}$

$$F_1 < F_2 \quad <=> \quad F_1 \subset F_2.$$

Then every linearly ordered $\mathcal{L} = \{F_s\}_{s \in I}$ subset of $\mathcal{F}$ has an upper bound,namely, $F^u := \bigcup_{s \in I} F_s$. Thus, by Zorn's lemma $\mathcal{F}$ has a maximal element F. This set satisfies $\{x_1, x_2, \ldots, x_n\} \subset F$, but $\{x_1, x_2, \ldots, x_n\}$ is a dense set, so we claim F has to be an uncountable set, otherwise let F be a countable set, say $F = \{w_i\}_{i=1}^\infty$, then the set

$$\beta \quad \bigcap \quad [\bigcap_{i=1}^{\infty} Prox(T)(w_i) \setminus Asym_\epsilon(T)(w_i)]$$

contains a dense G_δ subset, so we can pick $w \in \beta \quad \bigcap \quad [\bigcap_{i=1}^{\infty} Prox(T)(w_i) \setminus Asym_\epsilon(T)(w_i)]$ with $w \neq w_i \quad \forall i = 1, 2, \ldots$.

In this way, we have $(w, w_i) \in Prox(T) \setminus Asym_\epsilon(T) \quad \forall i = 1, 2, \ldots$ Let $F' := F \bigcup \{w\} \subset \beta$. Then $F' \times F' \setminus \Delta_X \subset Prox(T) \setminus Asym_\epsilon(T)$. But this is a contradiction to the fact F is the maximal element of $\mathcal{F}$. This prove our claim, that is, F is an uncountable set. Finally, $F \times F$ is an uncountable set in $X \times X$ satisfying

$$F \times F \setminus \Delta_X \subset Prox(T) \setminus Asym_\epsilon(T).$$

∎

Corollary 2.30. Let (X, T) be a topological dynamical system. If (X, T) is Li-Yorke sensitive then it is Li-Yorke Chaotic.

Proof. If the system is Li-Yorke sensitive, then by the previous theorem $\exists \epsilon > 0$ and an uncountable set F such that

$$F \times F \setminus \Delta_X \subset Prox(T) \setminus Asym_\epsilon(T).$$

But $\forall \epsilon > 0$ we have

$$Prox(T) \setminus Asym_\epsilon(T) \subset Prox(T) \setminus Asym(T).$$

Thus, F satisfies

$$F \times F \setminus \Delta_X \subset Prox(T) \setminus Asym(T),$$

and the system is Li-Yorke chaotic.

□

F. Lifting of some properties of topological dynamical systems.

For this section we start with the following basic concept.

Definition 2.31. Let (X,T) and (Y,S) be two dynamical systems. Assume that $h:(X,T)\to(Y,S)$ is a map such that the diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

commutes. We say that h is:

- A *conjugation map* , if h is a homeomorphism.
- A *factor map*, if h is continuous and onto.
- An *action map* , if h is continuous.

Now, let us see how things go for the case of conjugation maps.

Theorem 2.32. Let (X,T) and (Y,S) be two dynamical systems and let $h:(X,T)\to(Y,S)$ be a conjugation map between (X,T) and (Y,S). Then (Y,S) is a topologically transitive dynamical sytem if and only if (X,T) is.

Proof. $\Rightarrow$)(Assume (X,T) is a topologically transitive dynamical system.)

Let M and N be any two nonempty open subsets of the space Y. Then $\exists$ U and V

nonempty open subsets of X such that

$$h(U) = M,$$

$$h(V) = N.$$

Now, from the hypothesis we know $\exists \quad k \geq 0$ such that

$$T^k(U) \cap V \neq \phi$$

for all U, V nonempty open subsets of X. But this implies:

$$\begin{aligned} S^k(M) \cap N &= [h \circ T \circ h^{-1}]^k(M) \cap N \\ &= [h \circ T^k \circ h^{-1}](M) \cap N \\ &= [h \circ T^k](h^{-1}(M)) \cap N \\ &= [h \circ T^k](U) \cap N \\ &= h(T^k(U)) \cap N \\ &= h(T^k(U)) \cap h(V) \\ &= h(T^k(U) \cap V) \neq \phi \qquad (h \quad is \quad a \quad homeomorphism). \end{aligned} \tag{2.1}$$

$\Leftarrow$) (Assume (Y, S) is a topolgically transitive dynamical system.)

Using the same idea with H^{-1} the proof follows.

■

Now, let us state the following result:

Theorem 2.33. Let $h : (X, T) \to (Y, S)$ be a conjugation map between two minimal dynamical systems. Then (Y, S) is a nontrivial weakly mixing dynamical system iff (X, T) is Li-Yorke sensitive.

Proof

$\Rightarrow$) Since h is a homeomorphism between (X, T) and (Y, S), the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

i.e., $h \circ T = S \circ h$. Now, let (Y, S) be a nontrivial weakly mixing dynamical system, i.e., $(Y \times Y, S \times S)$ is a topologically transitive system. Define :

$$\mathcal{Y} := Y \times Y,$$

$$\mathcal{F}_S := S \times S,$$

$$\mathcal{X} := X \times X,$$

$$\mathcal{F}_T := T \times T.$$

Thus, $(\mathcal{Y}, \mathcal{F}_S)$ and $(\mathcal{X}, \mathcal{F}_T)$ are two dynamical systems. Now, let us define the function

$$\begin{array}{cccc} H: & \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & (x_1, x_2) & \longmapsto & (h(x_1), h(x_2)) \end{array}$$

where h is the conjugation map between (X, T) and (Y, S).

Claim. H is a conjugation map between $(\mathcal{X}, \mathcal{F}_T)$ and $(\mathcal{Y}, \mathcal{F}_S)$.

Proof of Claim.

1) H is continuous.

This follows from the fact that given M and N open substes of Y exits U and V open subsets of X

$$H^{-1}(M \times N) = U \times V$$

2) H is one-to-one and onto.

Follows directly from the fact that h is a homeomorphism.

3) H^{-1} is continuous.

Same proof as in 1).

4) The diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathcal{F}_T"] \arrow[d, "H"'] & X \arrow[d, "H"] \\ \mathcal{Y} \arrow[r, "\mathcal{F}_S"'] & \mathcal{Y} \end{tikzcd}$$

commutes.

$$\begin{aligned} (H \circ \mathcal{F}_T)(x_1, x_2) &= H(\mathcal{F}_T(x_1, x_2)) \\ &= H(T(x_1), T(x_2)) \\ &= (h(T(x_1)), h(T(x_2))) \\ &= (S(h(x_1)), S(h(x_2))) \end{aligned}$$

$$\begin{aligned} &= \mathcal{F}_S(h(x_1), h(x_2)) \\ &= (\mathcal{F}_S \circ H)(x_1, x_2). \end{aligned}$$

$\square$

Thus, H is a cojugation map between $(\mathcal{Y}, \mathcal{F}_S)$ and $(\mathcal{X}, \mathcal{F}_T)$, but we know that $(\mathcal{Y}, \mathcal{F}_S)$ is topologically transitive, so is (from *Theorem 2.32*) $(\mathcal{X}, \mathcal{F}_T)$. Therefore (X, T) is a nontrivial weakly mixing system and from *Corollary 2.21*, we conclude that (X, T) is a Li-Yorke sensitive dynamical system.

$\Leftarrow$) Let (X, T) be a Li-Yorke sensitive system then by definition $\forall x \in X$, $x \in \overline{Prox(T)(x) \backslash Asym_\epsilon(T)(x)}$, in particular since

$$Prox(T)(x) \backslash Asym_\epsilon(T)(x) \subset Prox(T)(x),$$

holds for all $x \in X$, $x \in \overline{Prox(T)(x)}$. Now, for minimal systems this implies (*Theorem 2.23*) that the system (X, T) is weakly mixing. Then again using the action map H between $(\mathcal{X}, \mathcal{F}_T)$ and $(\mathcal{Y}, \mathcal{F}_S)$ we conclude that $(\mathcal{Y}, \mathcal{F}_S)$ is a topologically transitive dynamical system. Therefore, (Y, S) is a nontrivial weakly mixing dynamical system.

$\blacksquare$

Theorem 2.34. Let (X, T) and (Y, S) be two minimal dynamical systems, and let $h : (X, T) \to (Y, S)$ be a conjugation map. Then (Y, S) is Li-Yorke sensitive if and only if (X, T) is.

Proof. $\Rightarrow$) Let (X, T)be a Li-Yorke sensitive system, that is,

$$\forall x \in X, \quad Prox(T)(x) \backslash Asym_\epsilon(T)(x)$$

is dense in X. But this implies that $\forall x, \in X, \quad Prox(T)(x)$ is dense in X. Now, for minimal systems this condition is equivalent (*Theorem 2.23*) to saying that (X, T) is a weakly mixing system ($(\mathcal{X}, \mathcal{F}_T)$ is a Topologically transitive system), but since h is a conjugation map between (X, T) and (Y, S), there exists a conjugation map H between $(\mathcal{X}, \mathcal{F}_T)$ and $(\mathcal{Y}, \mathcal{F}_S)$ which implies that $(\mathcal{Y}, \mathcal{F}_S)$ is a topologically transitive system. Thus, (Y, S) is a weakly mixing system and therefore (*Corollary 2.21*) is Li-Yorke sensitive.

$\Leftarrow$)

Assume (Y, S) is a Li-Yorke sensitive system, then takes the conjugation map H^{-1} between $(\mathcal{Y}, \mathcal{F}_S)$ and $(\mathcal{X}, \mathcal{F}_T)$, and the result follows.

$\blacksquare$

CHAPTER III

FOURIER SPECTRUM OF CHAOTIC INTERVAL MAPS

A. Fourier series

We begin this section with some basic concepts and results.

Definition 3.1 (Integrable Functions). The space $L^1_{loc}(\mathbb{R})$ of *locally integrable functions* and the space $L^1(\mathbb{R})$ of *integrable functions* on $\mathbb{R}$ are defined by

$$L^1_{loc}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \quad \forall a < b, \int_a^b |f(t)|dt < \infty \right\}$$

and

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \quad ||f||_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(t)|dt < \infty \right\}.$$

Definition 3.2. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is the function F defined as

$$F(k) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt}dt \quad k \in \hat{\mathbb{R}}(= \mathbb{R}).$$

Notation

$$f \leftrightarrow F \quad \hat{f} = F \quad f = \check{F}.$$

Definition 3.3. Let $f \in L^1(\mathbb{R})$ and let $\hat{f} = F$. The *Fourier transform inversion formula* is given by

$$f(t) = \int F(k)e^{2\pi itk}dk.$$

Now, let us move on directly to the Fourier series.

Definition 3.4. Let $\Omega > 0$, and let $f : \mathbb{R} \to \mathbb{C}$ be a function. f is 2Ω-periodic with period 2Ω if $f(k+2\Omega) = f(k)$ for all $k \in \mathbb{R}$. If f is defined a.e., then f is 2Ω-periodic if $f(k+2\Omega) = f(k)$ a.e. .

Definition 3.5. Let $f \in L^1_{loc}(\mathbb{R})$ be 2Ω-periodic. The *Fourier series* of f is the series

$$S(f)(\lambda) = \sum c_k^1 e^{\pi i \lambda k/\Omega}$$

where

$$\forall k \in \mathbb{Z}, \quad c_k^1 = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} f(\lambda) e^{-\pi i \lambda k/\Omega} d\lambda.$$

The numbers c_k^1 are called the *Fourier Coefficients* of f. If $\Omega > 0$ and $f \in L^1_{loc}(\mathbb{R})$ is a 2Ω-periodic, then we write $f \in L^1(\mathbb{T}_{2\Omega})$. Mathematically speaking $\mathbb{T}_{2\Omega} = \mathbb{R}/(2\Omega\mathbb{Z})$; which is a quotient group, is referred to as the *circle group* depending on Ω. The point here is the fact that $f \in L^1(\mathbb{T}_{2\Omega})$ can be thought of as being defined on *any* fixed interval $I \subset \mathbb{R}$ of length 2Ω. Thus, this periodicity, combined with the knowledge of f on any such interval, completely determines f on $\mathbb{R}$. After having introduced the latter ideas let us see how the fourier coefficients behave as $k \to \infty$.

Theorem 3.6 (Riemann-Lebesgue lemma). If $f \in L^1(\mathbb{T}_{2\Omega})$ and $\{c_k^1\}$ the sequence of Fourier coefficients of f, then

$$\lim_{|k|\to\infty} c_k^1 = 0.$$

Proof. Assume $f \in C^1(\mathbb{R})$, then $g = f' \in L^1(\mathbb{T}_{2\Omega})$ has the properties that

$$\int_{-\Omega}^{\Omega} g(\gamma)d\gamma = 0 \quad and \quad \forall k \in [-\Omega, \Omega), \quad f(\lambda) = \int_{-\Omega}^{\lambda} g(t)dt + f(-\Omega).$$

Thus, for $k \neq 0$ we get

$$\begin{aligned} c_k^1 &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} f(\lambda) e^{-\pi i \lambda k/\Omega} d\lambda \\ &= \frac{1}{2\Omega} \Big[- \frac{\Omega}{\pi i k} e^{-\pi i k \lambda/\Omega} f(\lambda) \Big|_{-\Omega}^{\Omega} + \frac{\Omega}{\pi i k} \int_{-\Omega}^{\Omega} g(\lambda) e^{-\pi i k \lambda/\Omega} d\lambda \Big] \\ &= \frac{1}{2\pi i k} \int_{-\Omega}^{\Omega} g(\lambda) e^{-\pi i k \lambda/\Omega} d\lambda \end{aligned}$$

and hence

$$|c_k^1| \leq \frac{\Omega}{\pi |k|} ||g||_{L^1(\mathbb{T}_{2\Omega})}.$$

Consequently, $\lim_{|k| \to \infty} c_k^1 = 0$.

Let $f \in L^1(\mathbb{T}_{2\Omega})$ and $\epsilon > 0$. There is $f_\epsilon \in C^1(\mathbb{R})$ that is 2Ω-periodic and for which $||f - f_\epsilon||_{L^1(\mathbb{T}_{2\Omega})} < \epsilon$, then the theorem is true with f_ϵ and $g_\epsilon = f_\epsilon' \in L^1(\mathbb{T}_{2\Omega})$ instead of f and g. Then for $k \neq 0$ we have

$$\begin{aligned} |c_k^1| &\leq |c_k^1 - c_{k,\epsilon}^1| + |c_{k,\epsilon}^1| \\ &\leq ||f - f_\epsilon||_{L^1(\mathbb{T}_{2\Omega})} + \frac{\Omega}{\pi |k|} ||g_\epsilon||_{L^1(\mathbb{T}_{2\Omega})}, \end{aligned}$$

where $\{c^1_{k,\epsilon}\}$ is the sequence of Fourier coefficients of f_ϵ and where we have used the first part of the proof in the second inequality. We know that

$$\overline{\lim_{|k|\to\infty}} a_k \le \overline{\lim_{|k|\to\infty}} b_k + \overline{\lim_{|k|\to\infty}} c_k,$$

in case $a_k \le b_k + c_k$ and $a_k, b_k, c_k > 0$. Consequently,

$$\overline{\lim_{|k|\to\infty}} |c^1_k| \le \overline{\lim_{|k|\to\infty}} ||f - f_\epsilon||_{L^1(\mathbb{T}_{2\Omega})} < \epsilon.$$

Since the lefthand side of the equation is nonnegative and independent of ϵ, we see that

$$\lim_{|k|\to\infty} |c^1_k| = 0.$$

■

Another relevant result is the following.

Theorem 3.7 (Dirichlet Theorem).

If $f \in L^1(\mathbb{T}_{2\Omega})$ and f is differentiable at λ_0, then $S(f)(\lambda_0) = f(\lambda_0)$ in the sense that

$$\lim_{M,N\to\infty} \sum_{k=-M}^{N} c^1_k e^{\pi i k \lambda_0/\Omega} = f(\lambda_0),$$

where $\{c^1_k\}$ is the sequence of Fourier coefficients of f.

Proof. Without loss of generality, assume $\lambda_0 = 0$ and $f(\lambda_0) = 0$. In fact if $f(\lambda_0) \ne 0$, consider the function $f - f(\lambda_0)$, which is also an element of $L^1(\mathbb{T}_{2\Omega})$, and then translate this function to the origin. Since $f(0)$ and $f'(0)$ exist, we can verify that

$$g(\lambda) = \frac{f(\lambda)}{(e^{\pi i \lambda/\Omega} - 1)}$$

is bounded in some interval centered at the origin. This fact plus the integrability of

f on $\mathbb{T}_{2\Omega}$, yields the integrability of g on $\mathbb{T}_{2\Omega}$. Therefore since $f(\lambda) = g(\lambda)(e^{\pi i\lambda/\Omega} - 1)$, we have $c_k^1 = d_{k-1}^1 - d_k^1$, where $\{d_k^1\}$ is the sequence of Fourier coefficients of g. Thus, the partial sum $S_{M.N}(f)(0)$ is the telescoping series

$$\sum_{k=-M}^{N} \left(d_{k-1}^1 - d_k^1\right) = d_{-M-1}^1 - d_N^1.$$

In this way, we can apply Riemann-Lebesgue Lemma to obtain

$$\lim_{M,N\to\infty} S_{M,N}(f)(0) = 0.$$

■

B. Topological entropy

Two ideas from the previous chapter; topological entropy and total variaton of the $n-th$ iterates of a map f, will be used often, therefore they will be discussed more throughly. Following Bowen [12] and Robinson [35] we have that the topological entropy is a way to measure quantitatively speaking the chaoticity of a system. The idea behind it is simple since we are interested in determining how many *different orbits* there are for a given map. For instance, assume we have two orbits of length n each, and suppose that your resolution to distinguish different orbits is given by $\epsilon > 0$. In this way, the two orbits can be distinguished provided there is a k with $0 \leq k \leq n$ for which they are distance greater than ϵ. Let $f : X \to X$ be a continuous map on the metric space X with metric d. Let $r(n, \epsilon, f)$ be the number of such orbits of length n that can be distinguished. The entropy $h(f, \epsilon)$ for a given ϵ is the growth of rate of $r(n, \epsilon, f)$ as n goes to infinity, and the topological entropy $h_{top}(f)$ is the limit when $\epsilon \to 0$. At this point a clarification is necessary.

Definition 3.8. Let $f : X \to X$ be a continuous map on the metric space X with metric d. A set $S \subset X$is called $(n, \epsilon) -$ *separated* for f provided $d_{n,f}(x, y) > \epsilon$ for every pair of distinct points $x, y \in S, x \neq y$, where

$$d_{n,f}(x,y) := \sup_{0 \leq j < n} d(f^j(x), f^j(y)).$$

Definition 3.9. Let $f : X \to X$ be a continuous map on the metric space X with metric d. The *entropy for a given* ϵ, $h(f, \epsilon)$ is defined by

$$h(\epsilon, f) := \limsup_{n \to \infty} \frac{log(r(n, \epsilon, f))}{n}.$$

Therefore, if $r(n, \epsilon, f) = e^{nr}$, then $h(\epsilon, f) = r$, that is, $h(\epsilon, f)$ represents the exponent of the way $r(n, \epsilon, f)$ is increasing respect to n. Finally, the concept of topological entropy is defined as follows.

Definition 3.10. Let $f : X \to X$ be a continuous map on the metric space X with metric d. Then define the topological entropy of f on X as

$$h(f) := \lim_{\epsilon \to 0, \epsilon > 0} h(\epsilon, f).$$

Some relevant results around this concept can be found. We can see that it is enough to know the behavior of f in its nonwandering set Ω to make a conclusion over the entropy of f in the whole space.

Theorem 3.11. Let $f : X \to X$ be a continuous map on a compact metric space X. Let $\Omega \subset X$ be the nonwandering set of f, then the entropy of f equals the entropy of f restricted to its nonwandering set, $h(f) = h(f|_\Omega)$.

It can be seen that if the nonwandering set consits of a finite number of periodic orbits, then $h(f) = 0$. In the same direction, if we have two different dynamical systems (X, f) and (Y, g), where we can find a conjugation between them, i.e. , a continuous function $k : X \to Y$ that is a homeomorphism. Can we say something about the entropy of both systems ? The answer is given by the following theorem.

Theorem 3.12. Assume $f : X \to X$ and $g : Y \to Y$ are continuous maps, where X and Y are compact metric spaces with metrics d and d' respectively. Assume $k : X \to Y$ is a conjugacy map from f to g, then $h(f) = h(g)$.

There are various relevant results around this concept, however we will concentrate on those related to the total variation of a function $f : I \to I$. We start with a definition and some propositions in this direction

Definition 3.13. Let f be a piecewise continuous function on the interval I. Let P any partition on it. The *total variation* of f on I, denoted by $V_I(f)$, is defined by

$$V_I(f) := \sup_{P \in \mathcal{P}} \left\{ \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| : a = x_0 < x_1 < \ldots < x_n = b, \quad x_i \in P, i = 1, 2, \ldots \right\},$$

where *sup* is taken on all the possible partitions $\mathcal{P}$ of the interval I.

The total variation of a constant function is zero and the total variation of a monotonic function is the absolute value of the difference between the function values at the end points. In general, the total variation of a function is giving us information on how oscillatory a function is. Therefore, the question is what kind of information we can obtain from the total variation related to its chaotic behavior. In [15] we can find important results in this direction and just for completness we would like to mention two of them.

Theorem 3.14. Let $f \in C^0(I, I)$. Suppose that f has a periodic point whose period is not a power of 2. Then the growth rate for the total variation of f^n on I is exponential as $n \to \infty$.

The converse of the above result is

Theorem 3.15. Let $f \in C^0(I, I)$. Suppose f is a piecewise monotone. If the growth rate for the total variation of f^n on I is exponential as $n \to \infty$. Then, f has a periodic point whose period is not a power of 2 .

However, as we saw above, one way to measure the chaoticity of a system is through the concept of topological entropy h_{top} of a continuous map $f : X \to X$. In particular,we have the next result [32] .

Theorem 3.16. Let $f : I \to I$ be a piecewise monotone continuous function and let $V_I(f^n)$ be the total variation of f^n the $n-th$ iterates of f on I. Then the topological entropy of f is given by

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} log[V_I(f^n)].$$

In this way, for piecewise monotone continuous interval maps, if we know the behavior of $V_I(f^n)$, we also find the behavior of the topological entropy h_{top}. Keep in mind this connection, which will be used later.

C. Numerical results

We would like to start this section with the following question: Given a continuous interval map $f : I- > I$, can we determine wether f is chaotic or not from its fourier coefficients? :

$$c_k = \frac{1}{2}\int_0^1 f(x)e^{-2\pi kx}dx, \quad f(x) = \sum_{-\infty}^{\infty} c_k(f)e^{2\pi kx} \quad .$$

In general, the answer is most likely No, since an $\epsilon - perturbation$ of a nonchaotic map can become chaotic, and viceversa, as we can illustrate with the next example.

Example 3.17. In the case of the symmetric triangular map

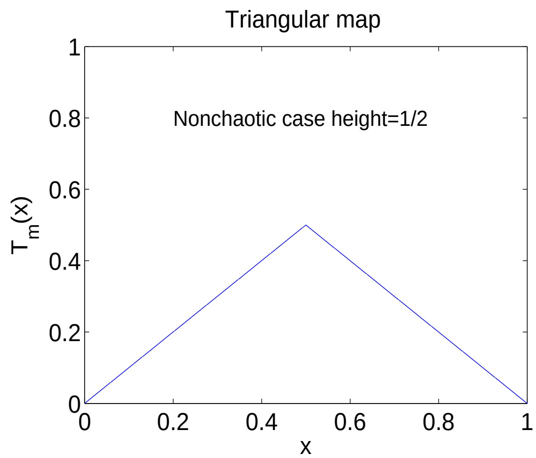

Fig. 1. Triangular map. Nonchaotic case.

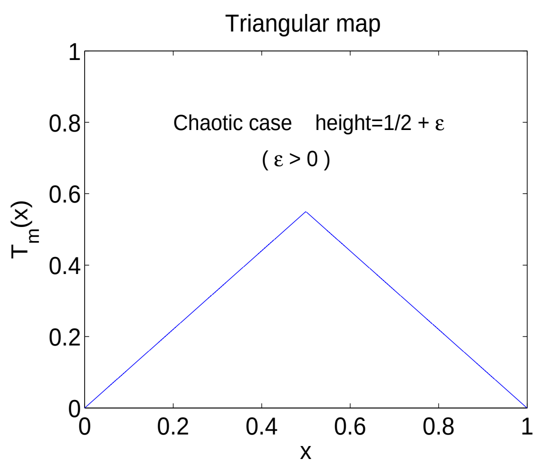

Fig. 2. Triangular map. Chaotic case.

The best hope one may have can only pin on information derived from the fourier coefficient of f^n where

$$f^n = f \circ f \circ \ldots \circ f \quad .$$

Let us look at some concrete numerical/graphical results, using the **tent map**

$$T_{m,h}(x) = \begin{cases} hmx, & \text{if } 0 \leq x < \frac{1}{m}. \\ \frac{hm}{1-m}(x-1), & \text{if } \frac{1}{m} \leq x \leq 1 \end{cases}$$

and the **quadratic map**

$$f_\mu(x) = \mu x(1-x)$$

families as models.

In the following graphs we will picture the graph of the $n-th$ iteration of the given function and the Fourier coefficients associated to it, as well. To this end, we use a partition of the interval $[0,1]$ of $1,000$ points and calculate the integrals numerically, using the trapezoidal rule. In all the case for the tent map we will use $h = 1$.

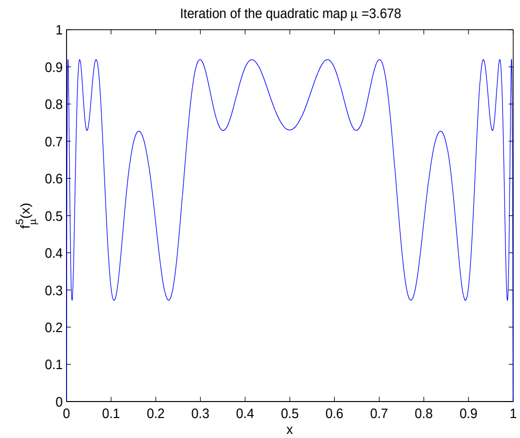

Fig. 3. 5^{th} iteration of the quadratic map, $\mu = 3.678$.

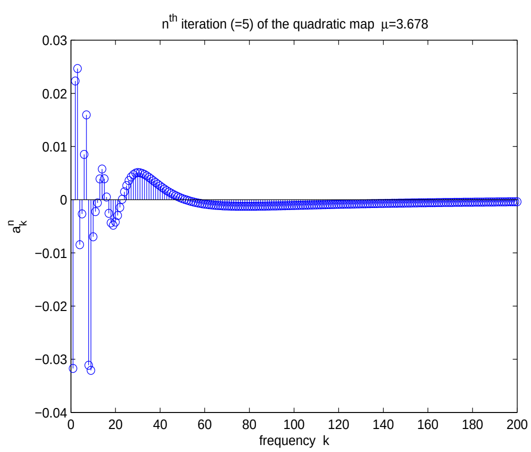

Fig. 4. Cosine Fourier coefficient for the 5^{th} iteration of the quadratic map, $\mu = 3.678$.

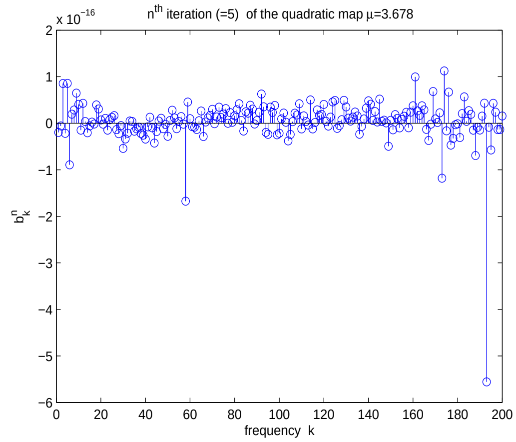

Fig. 5. Sine Fourier coefficient for the 5^{th} iteration of the quadratic map, $\mu = 3.678$.

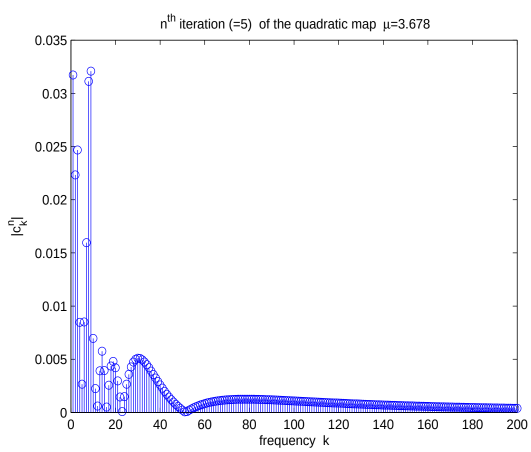

Fig. 6. Modulus of Fourier coefficients for the 5^{th} iteration of the quadratic map, $\mu = 3.678$.

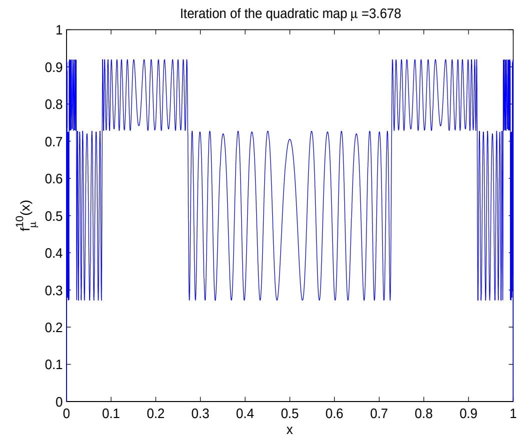

Fig. 7. 10^{th} iteration of the quadratic map, $\mu = 3.678$.

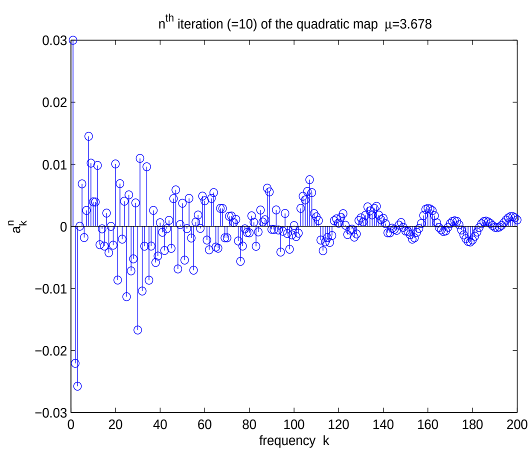

Fig. 8. Cosine Fourier coefficients for the 10^{th} iteration of the quadratic map, $\mu = 3.678$.

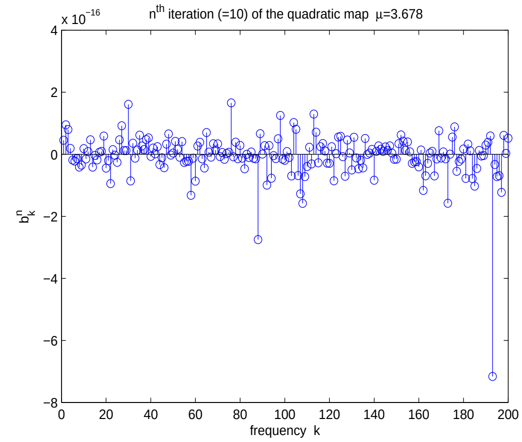

Fig. 9. Sine Fourier coefficients for the 10^{th} iteration of the quadratic map, $\mu = 3.678$.

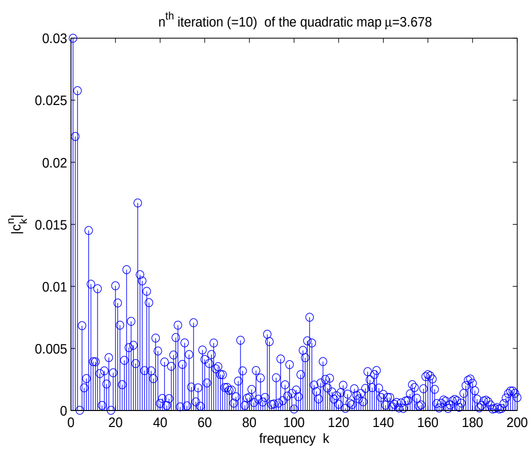

Fig. 10. Modulus of the Fourier coefficients for the 10^{th} iteration of the quadratic map, $\mu = 3.678$.

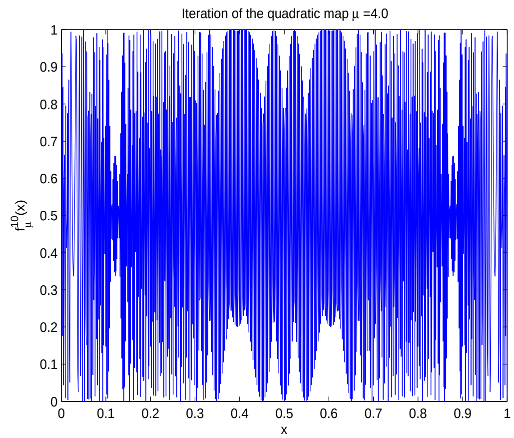

Fig. 11. 10^{th} iteration of the quadratic map, $\mu = 4$.

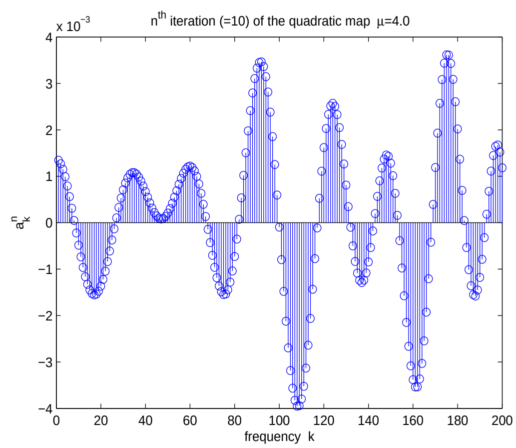

Fig. 12. Cosine Fourier coefficients for the 10^{th} iteration of the quadratic map, $\mu = 4$.

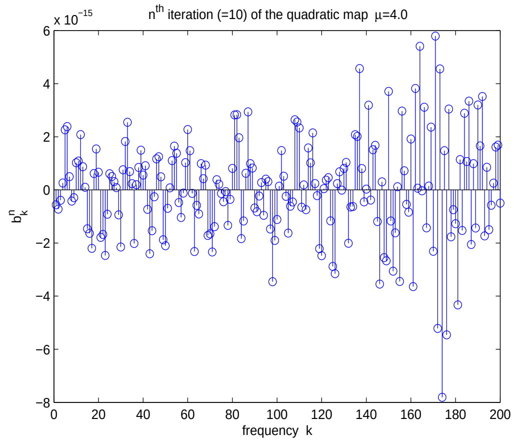

Fig. 13. Sine Fourier coefficients for the 10^{th} iteration of the quadratic map, $\mu = 4$.

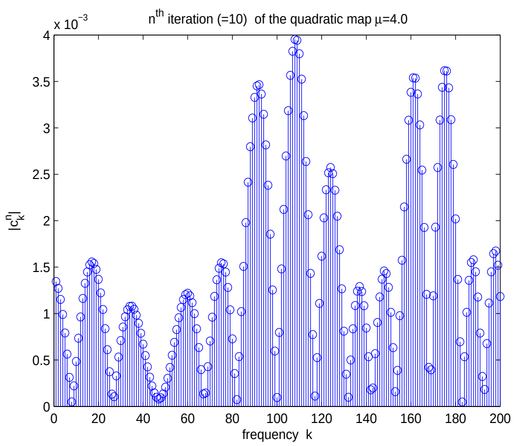

Fig. 14. Modulus of the Fourier coefficients for the 10^{th} iteration of the quadratic map, $\mu = 4$.

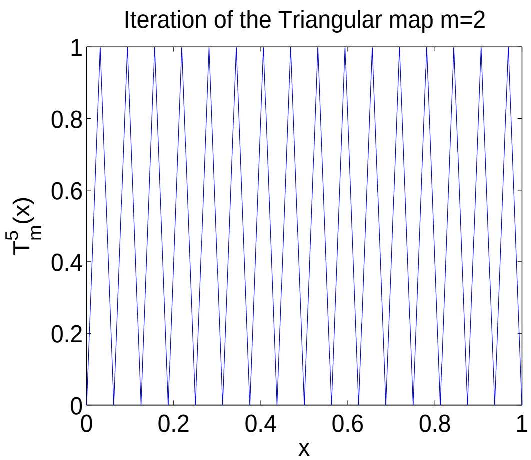

Fig. 15. 5^{th} iteration of the triangular map, $m = 2$.

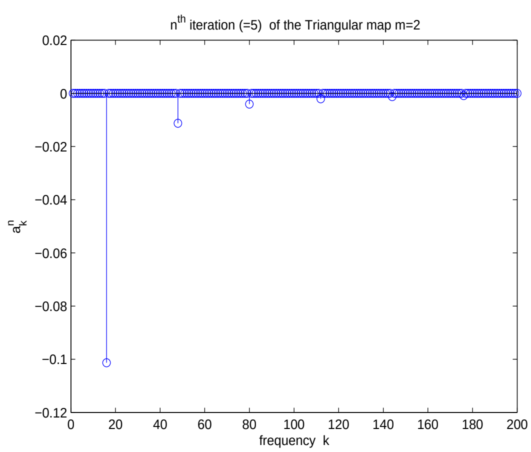

Fig. 16. Cosine Fourier coefficient for the 5^{th} iteration of the triangular map, $m = 2$.

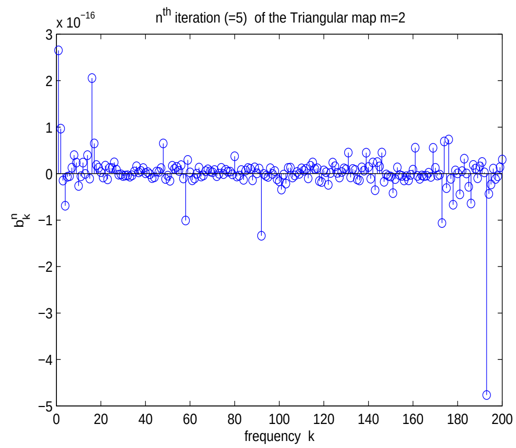

Fig. 17. Sine Fourier coefficient for the 5^{th} iteration of the triangular map, $m = 2$.

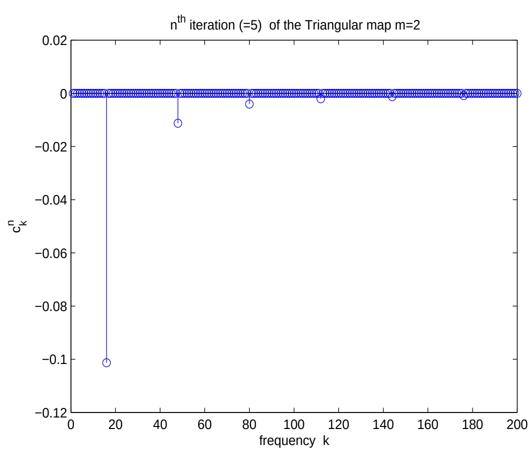

Fig. 18. Modulus of the Fourier coefficients for the 5^{th} iteration of the triangular map, $m = 2$.

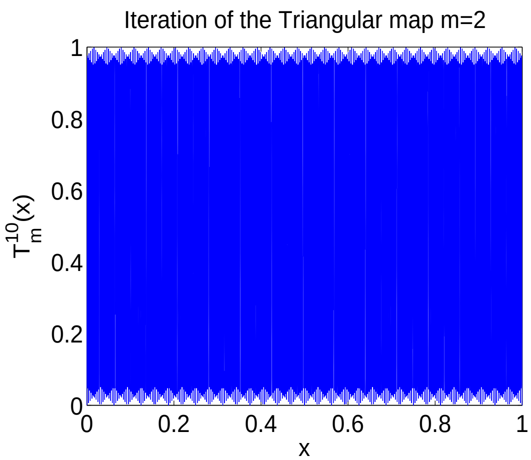

Fig. 19. 10^{th} iteration of the triangular map, $m = 2$.

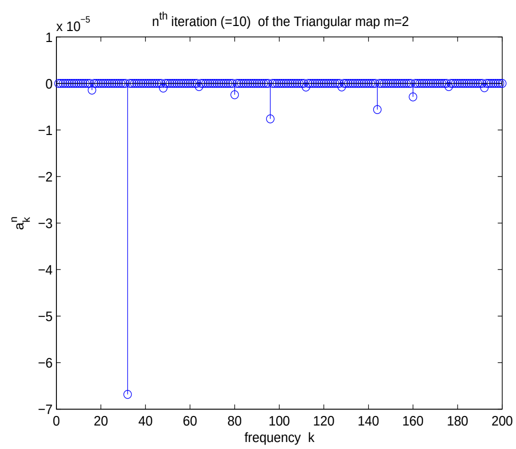

Fig. 20. Cosine Fourier coefficient for the 10^{th} iteration of the triangular map, $m = 2$.

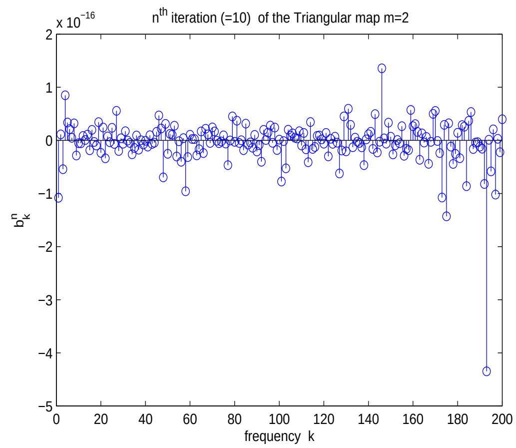

Fig. 21. Sine Fourier coefficient for the 10^{th} iteration of the triangular map, $m = 2$.

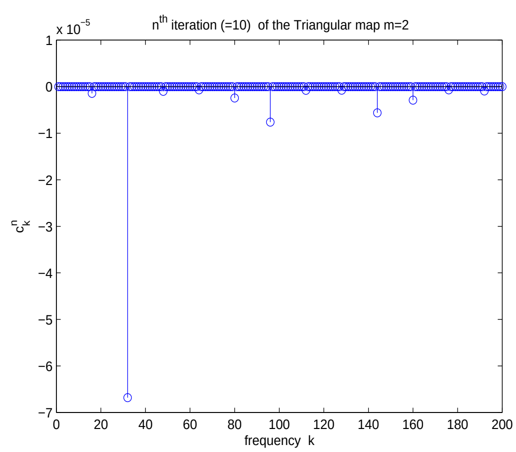

Fig. 22. Modulus of the Fourier coefficients for the 10^{th} iteration of the triangular map, $m = 2$.

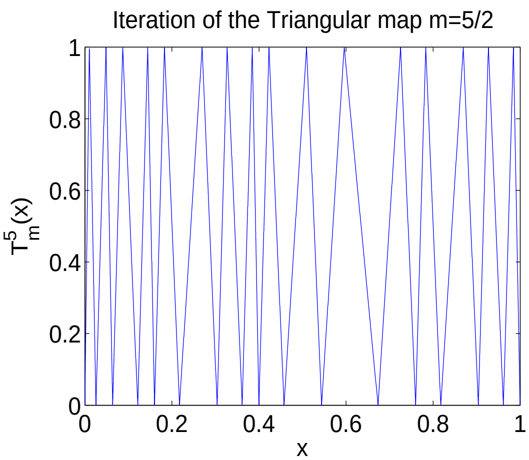

Fig. 23. 5^{th} iteration of the triangular map, $m = 5/2$.

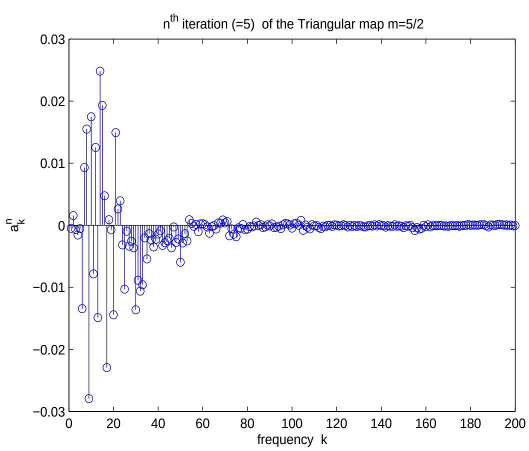

Fig. 24. Cosine Fourier coefficient for the 5^{th} iteration of the triangular map, $m = 5/2$.

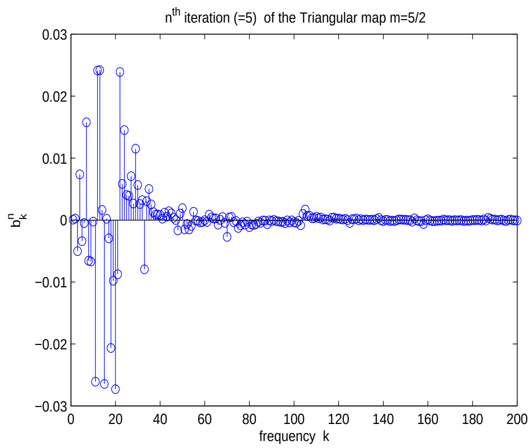

Fig. 25. Sine Fourier coefficient for the 5^{th} iteration of the triangular map, $m = 5/2$.

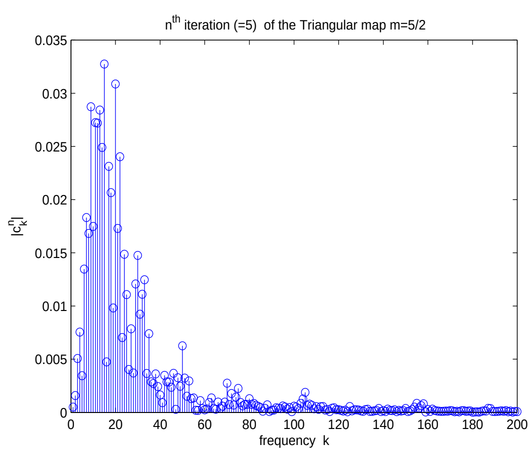

Fig. 26. Modulus of the Fourier coefficients for the 5^{th} iteration of the triangular map, $m = 5/2$.

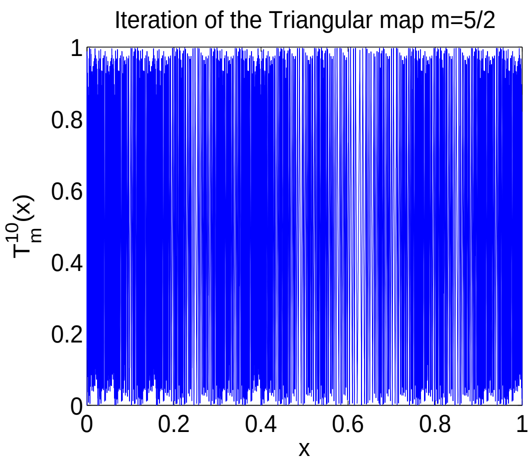

Fig. 27. 10^{th} iteration of the triangular map, $m = 5/2$.

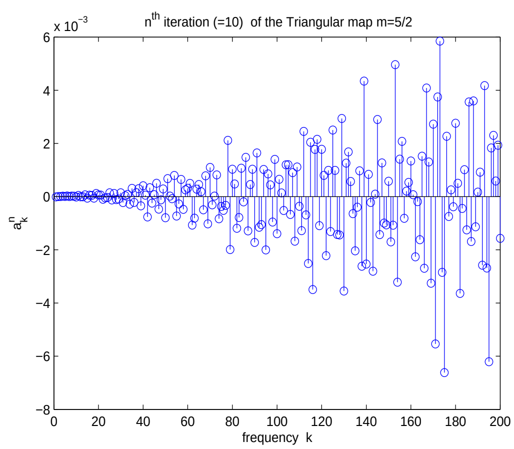

Fig. 28. Cosine Fourier coefficient for the 10^{th} iteration of the triangular map, $m = 5/2$.

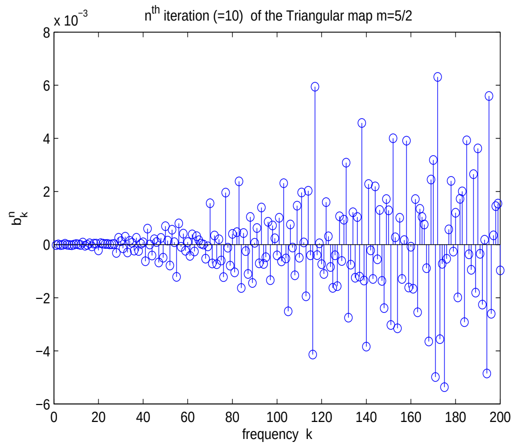

Fig. 29. Sine Fourier coefficient for the 10^{th} iteration of the triangular map, $m = 5/2$.

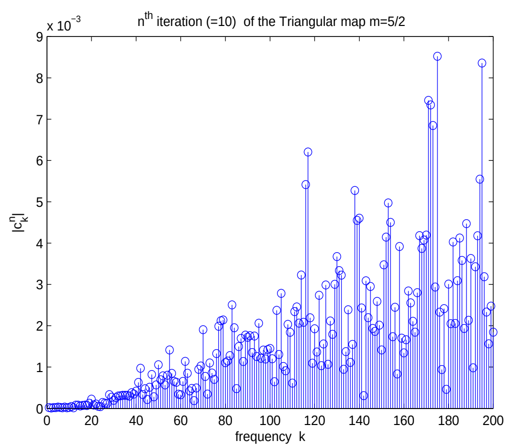

Fig. 30. Modulus of the Fourier coefficients for the 10^{th} iteration of the triangular map, $m = 5/2$.

Note: For the case m=2, the coefficients c_k^n are real, so we just need to plot $c_k^n \quad vs \quad k$, intead of its modulus.

Now, in the following set of images we will plot a $3-D$ pictures of $|c_k^n|$ as a function of k and n for some particular values of the parameter μ and m for the quadratic and triangular maps respectively.

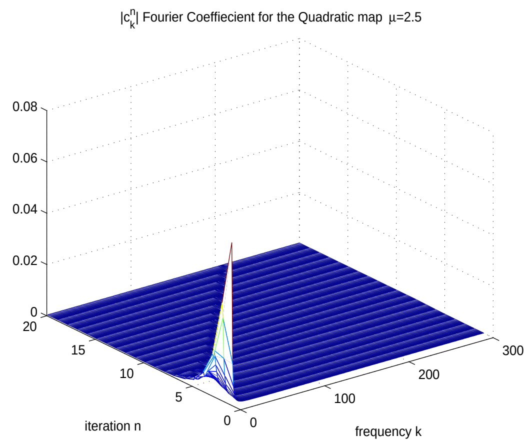

Fig. 31. $|c_k^n|$ coefficient for the quadratic map, $\mu = 2.5$.

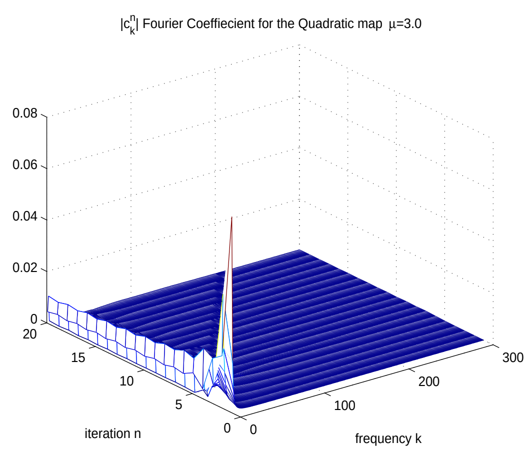

Fig. 32. $|c_k^n|$ coefficient for the quadratic map, $\mu = 3.0$

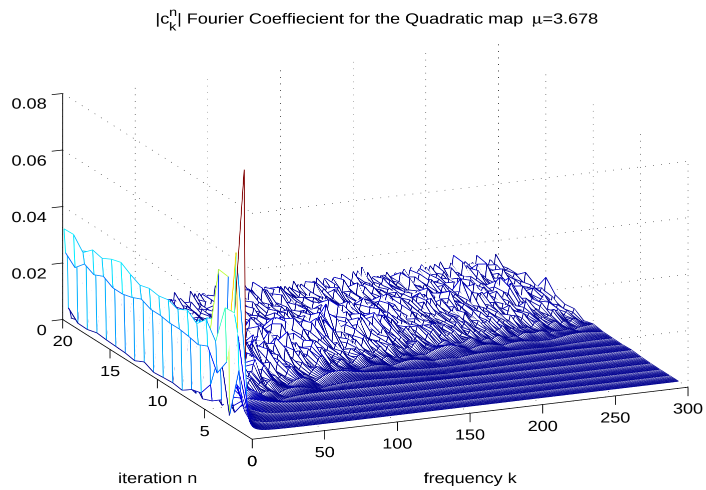

Fig. 33. $|c_k^n|$ coefficient for the quadratic map, $\mu = 3.678$

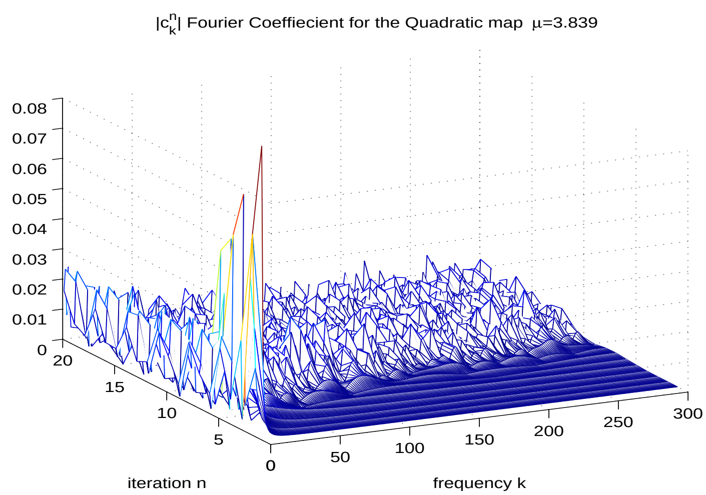

Fig. 34. $|c_k^n|$ coefficient for the quadratic map, $\mu = 3.839$

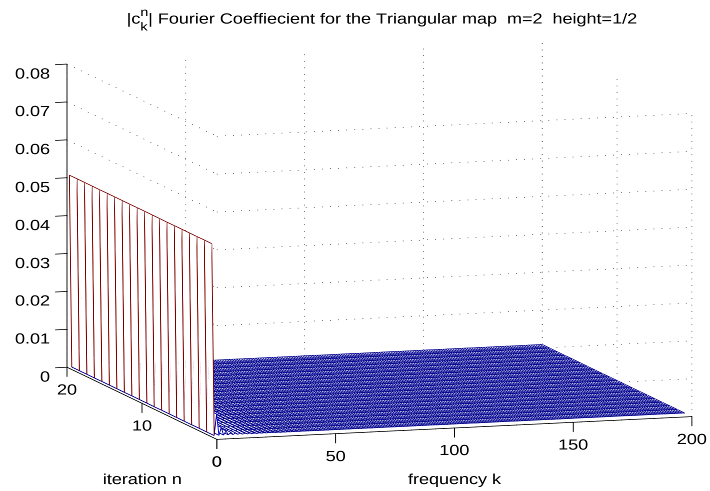

Fig. 35. $|c_k^n|$ coefficient for the triangular map, $m = 2$, $h = 1/2$

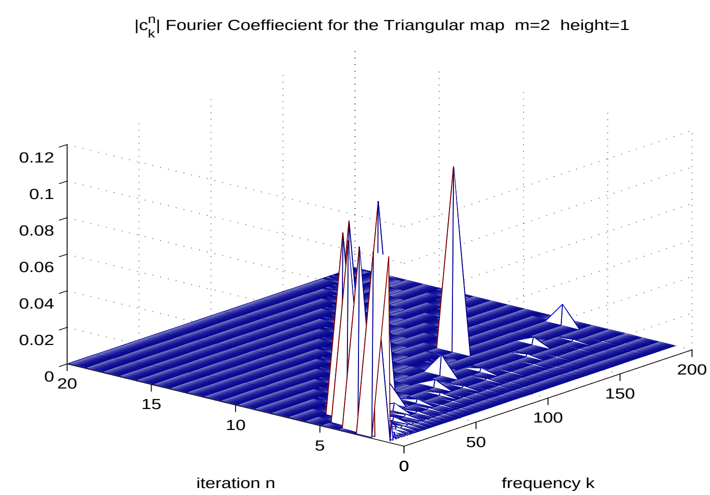

Fig. 36. $|c_k^n|$ Fourier coefficient for the triangular map, $m = 2$, $h = 1$

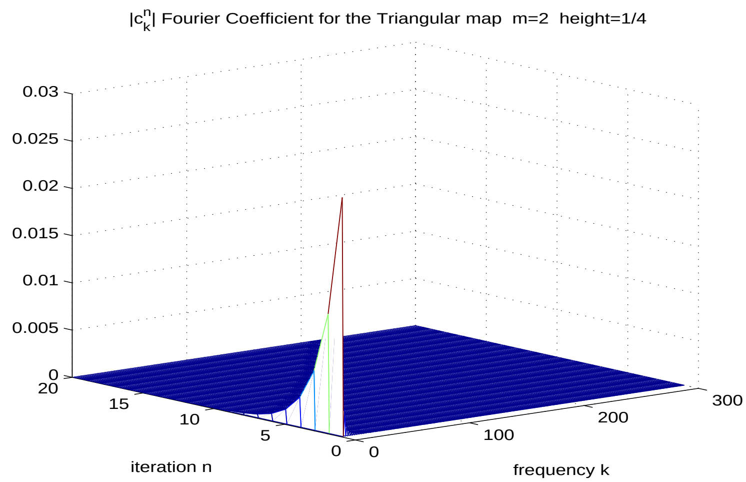

Fig. 37. $|c_k^n|$ Fourier coefficient for the triangular map, $m = 2$, $h = 1/4$

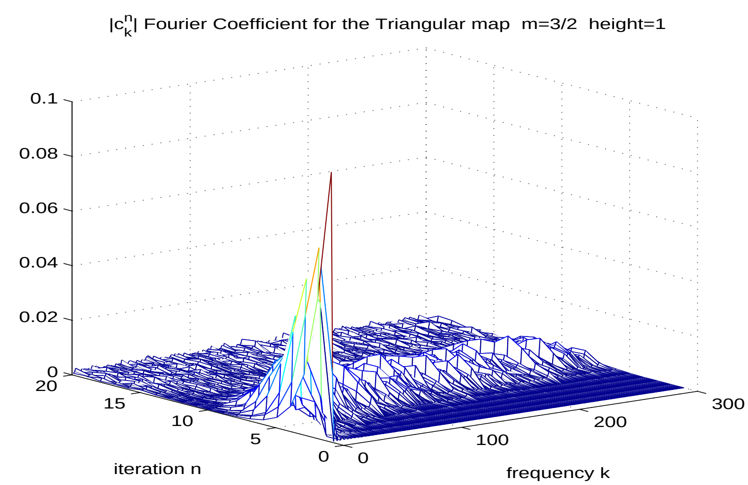

Fig. 38. $|c_k^n|$ Fourier coefficient for the triangular map, $m = 3/2$, $h = 1$

Information can not be exctracted immediately without further nalysis of the previous graphs. In the first set of pictures, when we have few iterations, the contribution to the spectrum is coming from low frequencies. However, as the number of iterations increases the contribution of high frequencies comes out, but the magnitud of the whole spectrum remains small compared to the initial situation. Consequently, it seems that as $n \to \infty$ we have more contribution of the high frequencies with a reduction of the magnitud of the whole spectrum. The last block of pictures gives us the quadratic case where we can see that when $\mu = 2.5$ (nonchaotic case)the graph is completely flat and different from the case $\mu = 4.0$ (chaotic case) where we can see some peaks. In the case of the triangular map when $m = 2$ and the height is equal to 1/2 (nonchaotic case), the plot is flat, but when $m = 2$ or $m = 5/2$ and height equals 1 (chaotic case) then the graph exhibits peaks again. In particular for the case $m = 2$ and height equals 1 it seems like some of the peaks posses the same value. As a result, we have the following conjectures:

Conjecture 1. In the nonchaotic case we have

$$\lim_{(n,k)\to\infty} c_k^n = 0.$$

Conjecture 2. In the chaotic case we have that there exists a subsequence $c_{k_j}^{n_i}$ such that

$$\lim_{(i,j)\to\infty} c_{k_j}^{n_i} \neq 0.$$

Conjecture 3. In the Chaotic case

$$\sup_{k\in\mathbb{Z}} c_k^{n_2} \leq \sup_{k\in\mathbb{Z}} c_k^{n_1} \quad n_1 \leq n_2.$$

Now, from a heuristic argument if f is chaotic, then f^n becomes more and more oscillatory as n goes to infinity. This highly oscillatory behavior should be reflected on the higher order harmonics of the Fourier series. The slope of f plays an important role as well. However, it is very complicated to evaluate the Fourier coefficients:

$$c_k^n = c_k^n(f) = \frac{1}{2}\int_0^1 f^n(x)e^{-2\pi kx}dx$$

for a given interval map. Most of the time, one must rely on numerical methods.

Example 3.18. The quadratic map $f_\mu(x) = \mu x(1-x)$, when $\mu = 4$:

$$f_4(x) = 4x(1-x)$$

satisfies the following simple recurrence relation.

Proposition 3.19. Let $f_4(x) = 4x(1-x)$ be the quadratic map. Then

$$f^{n+1}(x) = 4f^n(x)a_n(x), \quad n = 1, 2, 3, ...$$

where $\{a_n(x)\}$ satisfies

$$\begin{aligned} a_0(x) &= 1 - x \\ a_n(x) &= (2a_{n-1}(x) - 1)^2 \quad n = 1, 2, 3, ... \quad . \end{aligned}$$

Proof. By induction we have (througout the proof we will write $f(x)$ instead of $f_4(x)$) that for $n = 1$

$$\begin{aligned} f^2(x) &= 4f(x)(1 - f(x)) \\ &= 4[4x(1-x)](1 - [4x(1-x)]) \\ &= 4[4x(1-x)](1 - 4x + 4x^2)) \\ &= 4[4x(1-x)](1-2x)^2 \\ &= 4[4x(1-x)][2(1-x) - 1]^2 \\ &= 4[f(x)][2a_0(x) - 1]^2, \quad a_0 := 1 - x \\ &= 4f(x)[a_0(x) - 1]^2. \end{aligned}$$

Thus, the formula is true. Furthermore, for $n = 2, 3$ we obtain

$$f^3(x) = 4f^2[1 - f^2(x)]$$

but

$$\begin{aligned} 1 - f^2(x) &= 1 - 4f(x)a_1(x) \\ &= 1 - 4^2x(1-x)a_1(x) \\ &= 1 - 4a_1(x)[4x(1-x)] \\ &= 1 - 4a_1(x)[1 - a_1(x)] \\ &= 1 + 4a_1(x)^2 - 4a_1(x)] \\ &= [2a_1(x) - 1]^2, \end{aligned}$$

then

$$f^3(x) = 4f^2(x)[2a_1 - 1]^2 = 4f^2(x)a_2(x).$$

For $n = 4$ we have

$$f^4(x) = 4f^3[1 - f^3(x)],$$

but

$$\begin{aligned}
1 - f^3(x) &= 1 - 4f^2(x)a_2(x) \\
&= 1 - 4^3x(1-x)a_1(x)a_2(x) \\
&= 1 - 4a_2(x)(4a_1(x)[4x(1-x)]) \\
&= 1 - 4a_2(x)(4a_1(x)[1 - a_1(x)]) \\
&= 1 - 4a_2(x)(1 - a_2(x)) \quad 1 - a_2(x) = 4a_1(x)(1 - a_1(x)) \\
&= 1 - 4a_2(x) + 4a_2(x)^2 \\
&= [2a_2(x) - 1]^2,
\end{aligned}$$

then

$$f^4(x) = 4f^3(x)[2a_2 - 1]^2 = 4f^3(x)a_3(x).$$

Assuming that the equations

$$\begin{aligned}
f^k(x) &= 4f^{k-1}(x)a^{k-1}(x) \\
a_{k-1}(x) &= [2a_{k-2}(x) - 1]^2; \quad a_0 := 1 - x,
\end{aligned}$$

hold for all $k = 2, 3, \ldots n$, show that the above equations are true for $k = n + 1$. By definition we have

$$f^{n+1}(x) = 4f^n(x)[1 - f^n(x)],$$

but if $f^k(x)$, given by the above formula, is true for $k = 2, 3, \ldots$, then

$$\begin{aligned}
a_1(x) &= [2a_0(x) - 1]^2 = 4a_0(x)^2 - 4a_0(x) + 1 \\
1 - a_1(x) &= 4a_0(x) - 4a_0(x)^2 \\
&= 4a_0(x)(1 - a_0(x)) \\
&= 4(1 - x)x \\
&= f(x) \\
a_2(x) &= [2a_1(x) - 1]^2 = 4a_1(x)^2 - 4a_1(x) + 1 \\
1 - a_1(x) &= 4a_1(x) - 4a_1(x)^2 \\
&= 4a_1(x)(1 - a_1(x)) \\
a_3(x) &= [2a_2(x) - 1]^2 = 4a_2(x)^2 - 4a_2(x) + 1 \\
1 - a_2(x) &= 4a_2(x) - 4a_2(x)^2 \\
&= 4a_2(x)(1 - a_2(x)) \\
&\vdots \\
a_{n-1}(x) &= [2a_{n-2}(x) - 1]^2 = 4a_{n-2}(x)^2 - 4a_{n-2}(x) + 1 \\
1 - a_{n-2}(x) &= 4a_{n-2}(x) - 4a_{n-2}(x)^2 \\
&= 4a_{n-2}(x)(1 - a_{n-2}(x)).
\end{aligned}$$

Thus

$$f^{n+1}(x) = 4f^n(x)[1 - f^n(x)]$$

$$\begin{aligned}
1 - f^n(x) &= 1 - f^n(x)a_{n-1}(x) \\
&= 1 - 4a_{n-1}(x)[4f^{n-2}(x)a_{n-2}(x)] \\
&= 1 - [4a_{n-1}(x)][4a_{n-2}(x)]f^{n-2}(x) \\
&\vdots \\
&= 1 - [4a_{n-1}(x)][4a_{n-2}(x)]\dots[4a_1(x)][f(x)],
\end{aligned}$$

where

$$\begin{aligned}
4a_1(x)f(x) &= 4a_1(x)[1 - 4a_1(x)] = 1 - a_2(x) \\
(4a_2(x))(4a_1(x))f(x) &= 4a_2(x)(1 - a_2(x)) = 1 - a_3(x) \\
(4a_3(x))(4a_2(x))(4a_1(x))f(x) &= 4a_3(x)(1 - a_3(x)) = 1 - a_4(x) \\
&\vdots \\
(4a_{n-2}(x))(4a_{n-3}(x))\dots(4a_1(x))f(x) &= 4a_{n-2}(x)(1 - a_{n-2}(x)) = 1 - a_{n-1}(x).
\end{aligned}$$

Therefore,

$$1 - f^n(x) = 1 - 4a_{n-1}(x)[1 - a_{n-1}(x)] = 4a_{n-1}(x)^2 = 4a_{n-1}(x) + 1 = [2a_{n-1}(x) - 1]^2$$

and finally

$$f^{n+1}(x) = 4f^n(x)a_n(x)$$

$$a_n(x) = [2a_{n-1}(x) - 1]^2; \quad a_0(x) := 1 - x.$$

$\square$

However, we have yet to find an explicit formula for $c_k^n(f)$ for general $n = 1, 2, 3, \ldots,$ and $k = 0, \pm 1, \pm 2, \ldots$, which can be used to find the Fourier coefficients for the $n - th$ iterates of the quadratic map.

A different story can be told for the case of the triangular map with $m = 2$ and $h = 1$, where we can find explicit formulas for the $n - th$ iteration of the map $f^n(x)$, and the respective Fourier coefficients a_n^k, b_k^n, but this will be the beginning of the next section.

D. Spectral analysis and chaos

We start this section with the following result related to the triangular map.

Theorem 3.20. For the triangular map $T_2(x)$, we have the Fourier coefficients $c_k^n(T_2)$ are given by:

$$c_k^n(T_2) = \begin{cases} -\frac{1}{\pi^2 s^2} & \text{if } k = s2^{n-1} \quad s = 1, 3, 5 \ldots \\ 0 & \text{otherwise .} \end{cases}$$

Proof. The $n-th$ iteration of the triangular map $T_2(x)$ is given by:

$$T_2^n(x) = \begin{cases} 2^n x - 2(l-1) & \text{if } \frac{2(l-1)}{2^n} \le x \le \frac{2l-1}{2^n} \\ -2^n x + 2l & \text{if } \frac{2l-1}{2^n} \le x \le \frac{2l}{2^n} \end{cases}$$

for $l = 1, 2, \ldots, 2^{n-1}$, and

$$\begin{aligned}
c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^n(x) e^{-2\pi i k x} dx \\
&= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2(l-1)}{2^n}}^{\frac{2l-1}{2^n}} [2^n x - 2(l-1)] e^{-2\pi i k x} dx + \sum_{l=1}^{2^{n-1}} \int_{\frac{2l-1}{2^n}}^{\frac{2l}{2^n}} [-2^n x + 2l] e^{-2\pi k x} dx\Big\} \\
&= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2(l-1)}{2^n}}^{\frac{2l-1}{2^n}} [2^n x - 2(l-1)] e^{-2\pi i k x} dx\Big\} + \\
&\quad \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2l-1}{2^n}}^{\frac{2l}{2^n}} [-2^n x + 2l] e^{-2\pi i k x} dx\Big\} \\
&= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}} \int_{\frac{2(l-1)}{2^n}}^{\frac{2l-1}{2^n}} [2^n x - 2(l-1)] e^{-2\pi i k x} dx\Big\} \\
&= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}} \int_0^{\frac{1}{2}} 2t e^{-2\pi i k \frac{t+(l-1)}{2^{n-1}}} \frac{dt}{2^{n-1}}\Big\}; \quad 2t = 2^n x - 2(l-1), dt = 2^{n-1} dx \\
&= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}} \frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t[e^{-\frac{i\pi k t}{2^{n-2}}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}}] dt\Big\}; \\
&= \frac{1}{2}\Big\{\frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t e^{-\frac{i\pi k t}{2^{n-2}}} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}} dt\Big\} \\
&= \frac{1}{2}\Big\{\frac{1}{2^{n-2}} \int_0^{\frac{1}{2}} t e^{-\frac{i\pi k t}{2^{n-2}}} dt\Big\} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}} \\
&= \frac{1}{2^{n-1}}\Big\{\int_0^{\frac{1}{2}} t e^{-\frac{i\pi k t}{2^{n-2}}} dt\Big\} \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}}
\end{aligned}$$

$$= \Big\{\frac{2^{-2}}{-i\pi k}e^{-\frac{i\pi k}{2^{n-1}}} + \frac{2^{n-3}}{\pi^2 k^2}(e^{-\frac{i\pi k}{2^{n-1}}} - 1)\Big\}\sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k(l-1)}{2^{n-2}}}$$

and

$$\begin{aligned}
I_2 &= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}}\int_{\frac{2l-1}{2^n}}^{\frac{2l}{2^n}}[-2^n x + 2l]e^{-2\pi ikx}dx\Big\} \\
&= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}}\int_0^{\frac{1}{2}} 2te^{-2\pi ik\frac{l-t}{2^{n-1}}}\frac{dt}{2^{n-1}}\Big\}; \quad 2t = -2^n x + 2l, dt = -2^{n-1}dx \\
&= \frac{1}{2}\Big\{\sum_{l=1}^{2^{n-1}}\frac{1}{2^{n-2}}\int_0^{\frac{1}{2}} t[e^{\frac{i\pi kt}{2^{n-2}}}e^{-\frac{i\pi kl}{2^{n-2}}}]dt\Big\}; \\
&= \frac{1}{2}\Big\{\frac{1}{2^{n-2}}\int_0^{\frac{1}{2}} te^{\frac{i\pi kt}{2^{n-2}}}\sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi kl}{2^{n-1}}}dt\Big\}; \\
&= \frac{1}{2}\Big\{\frac{1}{2^{n-2}}\int_0^{\frac{1}{2}} te^{\frac{i\pi kt}{2^{n-2}}}dt\Big\}\sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi kl}{2^{n-2}}}; \\
&= \frac{1}{2^{n-1}}\Big\{\int_0^{\frac{1}{2}} te^{\frac{i\pi kt}{2^{n-2}}}dt\Big\}\sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi kl}{2^{n-2}}} \\
&= \Big\{\frac{2^{-2}}{i\pi k}e^{\frac{i\pi k}{2^{n-1}}} + \frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-1}}} - 1)\Big\}\sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi kl}{2^{n-2}}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^n(x)e^{-2\pi ikx}dx \\
&= I_1 + I_2 \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-2}}})(1 - e^{-\frac{i\pi k}{2^{n-1}}})^2\sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi kl}{2^{n-2}}}.
\end{aligned}$$

Now, if $k \neq s2^{n-1} \quad s = 1, 2, \ldots$, then

$$c_k^n(T_2) = \frac{1}{2}\int_0^1 T_2^n(x)e^{-2\pi ikx}dx = I_1 + I_2$$

$$
\begin{aligned}
&= -\frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-2}}})(1-e^{-\frac{i\pi k}{2^{n-1}}})^2 \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}} \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(1-e^{-i2\pi k})\left\{\frac{1-e^{-\frac{i\pi k}{2^{n-1}}}}{1+e^{-\frac{-i\pi k}{2^{n-1}}}}\right\} \\
&= 0.
\end{aligned}
$$

On the other hand, if $k = s2^{n-1} \quad s = 1, 3, 5, \ldots$, then

$$
\begin{aligned}
c_k^n(T_2) &= \frac{1}{2}\int_0^1 T_2^n(x)e^{-2\pi i k x}dx = I_1 + I_2 \\
&= -\frac{2^{n-3}}{\pi^2 k^2}(e^{\frac{i\pi k}{2^{n-2}}})(1-e^{-\frac{i\pi k}{2^{n-1}}})^2 \sum_{l=1}^{2^{n-1}} e^{-\frac{i\pi k l}{2^{n-2}}} \\
&= -\frac{1}{\pi^2 s^2}.
\end{aligned}
$$

$\square$

Now, one can quantify the oscillatory behavior of a function f through the total variation $V_I(f)$ of the function on the interval I, which is defined as

$$
\begin{aligned}
V_I(f) &= \textit{The total variation of a piecewise} \\
&\quad\ \textit{continuous function on an interval } I. \\
&= \sup_{p \in P}\left\{\sum_{i=0}^{n-1}|f(x_{i+1}) - f(x_i)| \quad : \quad a = x_0 < x_1 < \ldots < x_n = b,\right. \\
&\quad\ \left. x_i \in P, \quad i = 0, 1, \ldots\right\} \\
P &: \quad \textit{a finite partition of } I = [a, b].
\end{aligned}
$$

In this context keep in mind the following. Let $k = 0, 1, 2, 3, \ldots, \quad 1 \leq p \leq \infty$. For any distribution $f \in \mathcal{D}'(I)$, let $f^{(k)}$ the distributional derivative of f for $k = 0, 1, 2, 3, \ldots$, define the *Sobolev Space* $W^{p,q}(I)$ as

$$W^{k,p}(I) = \{f \in \mathcal{D}'(I) : ||f||_{k,p} = [\sum_{j=0}^{k} \int_I |f^{(j)}(x)|^p]^{\frac{1}{p}} < \infty\}$$

($p = \infty$ is interpreted in the sense of supremum a.e.). Then we have the following result.

Theorem 3.21. If $f \in W^{1,1}(I)$, then

$$V_I(f) = \int_a^b |f'(x)|dx \quad I = [a, b]$$

Subsequently, it is necessary to introduce the next concept.

Definition 3.22. A continuous map $f : I- > I$ is said to be *piecewise monote* on I, if f has infinitely many extremal points on I.

The following are facts and notation about monotone continuous functions that must be acknowledged prior to stating and proving the first Main Theorem 1.

1) If f is piecewise monotone on I, then [33]

$$\lim_{n \to \infty} \frac{1}{n} ln V_I(f^n) = h_{top} \quad (the \quad topological \quad entropy \quad of \quad f).$$

Consequently, if $V_I(f^n)$ grows exponentially with respect to n, i.e.

$$V_I(f^n) \geq Ce^{\alpha n} \quad for \quad some \quad C, \alpha > 0,$$

then $h_{top}(f) > 0$ and f is chaotic in the sense of Li-Yorke.

2) (Juang and Shieh) [28]

Let f be piecewise monotone on I. Then the following are equivalent:

(i) f has a periodic point whose period is not a power of 2;

(ii) f has a homoclinic point;

(iii) f has positive topological entropy;

(iv) $V_I(f^n)$ grows exponentially with respect to n.

3) (Chen, G. Y. Huang and T-W., Huang) [15]

Let $f \in C^0(I, I)$. If either f has two distinct fixed points and a periodic point with period 2, or if f has a periodic point with period 4, then

$$\lim_{n\to\infty} V_I(f^n) = \infty \quad (usually \quad polynomial \quad growth \quad w.r.t. \quad n).$$

4) (Chen, G. Y. Huang and T-W., Huang) [15]

If f is a piecewise monotone on I such that

$$\lim_{n\to\infty} V_J(f^n) = \infty$$

for every closed subinterval J of I, then f has sensitive dependence on initial data on I.

Notation

$$\begin{aligned}\mathcal{F}_1 \;:\; & The \quad set \quad of \quad all \quad functions \quad f \in C^0(I,I) \quad such \quad that \\ & f^n \quad has \quad bounded \quad total \quad variation for \quad n = 1, 2, \ldots\end{aligned}$$

$$\begin{aligned}\mathcal{F}_2 \;:\; & The \quad set \quad of \quad all \quad functions \quad f \in C^0(I,I) \quad such \quad that \\ & f^n \quad has \quad finitely \quad many \quad extremal \quad points.\end{aligned}$$

It is clear that $\mathcal{F}_2 \subset \mathcal{F}_1 \quad and \quad \mathcal{F}_2 \subset W^{1,1}(I)$.

Theorem 3.23 [22]. Let f be a periodic function on the interval I. If f is of bounded total variation on I, then

$$|c_k^1(f)| \leq \frac{1}{|k\pi|} V_I(f), \quad for \quad all \quad n \in \mathcal{Z}.$$

Proof. Assume that f is a continuous function of bounded variation on $I = [0, 2\pi]$. Consider the $k-th$ Fourier coefficient of f

$$\begin{aligned}c_k^1(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)d[\frac{-e^{-ikx}}{ik}] \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)d[g(x)], \qquad g(x) := \frac{-e^{-ikx}}{ik}.\end{aligned}$$

From the definition of the Riemman-Stieljes integral, the above equation implies that for all $\epsilon > 0$ there exists a partition $P = \{x_0 = 0, x_1, \ldots, x_n = 2\pi\}$ such that

$$|c_k^1(f) - \frac{1}{2\pi}\sum_{i=1}^{n} f(x_i)[g(x_i) - g(x_{i-1})]| < \epsilon$$

which implies

$$|c_k^1(f)| \leq \frac{1}{2\pi}|\sum_{i=1}^{n} f(x_i)[g(x_i) - g(x_{i-1})]| + \epsilon,$$

but

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)[g(x_i) - g(x_{i-1})] &= f(x_1)[g(x_1) - g(x_0)] \\ &- f(x_2)[g(x_2) - g(x_1)] \\ &- f(x_3)[g(x_3) - g(x_2)] \\ &\vdots \\ &- f(x_n)[g(x_n) - g(x_{n-1})]\end{aligned}$$

and since $g(x_0) = g(x_n)$, then

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)[g(x_i) - g(x_{i-1})] &= [f(x_1) - f(x_0)]g(x_0) \\ &- [f(x_2) - f(x_1)]g(x_1) \\ &- [f(x_3) - f(x_2)]g(x_2) \\ &\vdots \\ &- [f(x_n) - f(x_{n-1})]g(x_{n-1}) \\ &= [f(x_1) - f(x_n)]g(x_0) - \sum_{i=1}^{n-1}[f(x_{i+1}) - f(x_i)]g(x_i).\end{aligned}$$

Thus, we get

$$\begin{aligned}|c_k^1(f)| &\leq \frac{1}{2\pi}|f(x_1) - f(x_0)||g(x_1)| + \frac{1}{2\pi}\sum_{i=1}^{n-1}|f(x_{i+1}) - f(x_i)||g(x_i)| + \epsilon \\ &\leq \frac{1}{2\pi}V_I(f)\frac{1}{|k|} + \frac{1}{2\pi}\frac{V_I(f)}{|k|} + \epsilon \\ &\leq \frac{V_I(f)}{|\pi k|} \quad (\epsilon \to 0)\end{aligned}$$

However, if f is just a function of bounded total variation, then we can approximate it through $\{f_r\}$ a family of continuous functions of bounded total variation as

$$f_r(x) = r\int_x^{x+\frac{1}{r}} f(t)dt = r\int_0^{\frac{1}{r}} f(x+t)dt$$

where

$$\begin{aligned}
\sum_{i=1}^{n} |f_r(x_i) - f_r(x_{i-1})| &\leq r\int_0^{\frac{1}{r}} \sum_{i=1}^{n} |f(x_i+t) - f(x_{i-1}+t)|dt \\
\sum_{i=1}^{n} |f_r(x_i) - f_r(x_{i-1})| &\leq V_I(f) \\
V_I(f_r) &\leq V_I(f) \\
|c_k^1(f_r)| &\leq \frac{V_I(f_r)}{\pi k} \leq \frac{V_I(f)}{\pi k}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|c_k^1(f_r)| &= \frac{1}{2\pi}\int_0^{2\pi} f_r(x)e^{-ikx}dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} [r\int_0^{\frac{1}{r}} f(x+t)dt]e^{-ikx}dx \\
&= r\int_0^{\frac{1}{r}} \frac{1}{2\pi}\int_0^{2\pi} f(x+t)e^{-ikx}dxdt \\
&= r\int_0^{\frac{1}{r}} [\frac{1}{2\pi}\int_t^{t+2\pi} f(y)e^{-iky}dy]e^{ikt}dt \quad y = x+t. \\
&= r\int_0^{\frac{1}{r}} c_k^1(f)e^{ikt}dt \\
&= rc_k^1(f)[\frac{e^{ikt}}{ik}]|_0^{\frac{1}{r}} = rc_k^1(f)[\frac{e^{\frac{ik}{r}}-1}{ik}] \\
&= rc_k^1(f)e^{\frac{ik}{2r}}[\frac{e^{\frac{ik}{2r}} - e^{-\frac{ik}{2r}}}{ik}] = e^{\frac{ik}{2r}}c_k^1(f)\frac{sin(\frac{k}{2r})}{\frac{k}{2r}}.
\end{aligned}$$

Therefore,

$$\lim_{r\to\infty} |c_k^1(f_r)| = |c_k^1(f)|$$

and from the inequality for $|c_k^1(f_r)|$ we get

$$|c_k^1(f)| \leq \frac{V_I(f)}{|\pi k|}.$$

$\square$

Main Theorem 1. Let $f \in \mathcal{F}_1$, and denote c_k^n the $k-th$ Fourier coefficient of the $n-th$ iterates of f. If there exists a map $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfying

$$ln[\phi(n)] \geq \alpha_1 + \alpha_2 n, \quad for \quad some \quad \alpha_1 \in \mathbb{R}, \alpha_2 > 0,$$

such that

$$c_{\pm\phi(n)}^n \geq \delta \quad for \quad some \quad \delta > 0,$$

then

$$lim_{n\to\infty} \frac{1}{n} ln[V(f^n)] > 0.$$

Proof. We use the following basic fact from Fourier integrals that we already proved above, for any function g of bounded total variation on I=[0,1],

$$|kc_k^1(g)| \leq \frac{1}{\pi} V_I(g), \quad \forall k = 0, \pm 1, \pm 2, \ldots \quad .$$

Applying the above result to $g := f^n$ we have

$$|kc_k^n(f)| \leq \frac{1}{\pi} V_I(f^n), \quad \forall k = 0, \pm 1, \pm 2, \ldots \quad .$$

Now, let $|k| = \phi(n)$ then

$$V_I(f^n) \geq \pi |\phi(n) c_{\phi(n)}^n|$$

which implies

$$\lim_{n\to\infty}\frac{1}{n}ln[V_I(f^n)] \geq \lim_{n\to\infty}\frac{1}{n}ln[\pi|\phi(n)c^n_{\phi(n)}|] = \alpha_2 > 0.$$

Therefore, the proof is complete.

■

Example 3.24. For the full tent map $T_2(x)$

$$c^n_k(T_2) = \begin{cases} -\frac{1}{\pi^2 s^2} & \text{if } k = s2^{n-1} \quad s = 1,3,5\ldots \\ 0 & \text{otherwise} \end{cases}$$

if we choose

$$|k| = |s|2^{n-1} \equiv \phi(n) \quad s = 1,3,5\ldots$$

then

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}ln[|\phi(n)c^n_{\pm\phi(n)}(T_2)|] &= \lim_{n\to\infty}\frac{1}{n}ln[s2^{n-1}\frac{1}{\pi^2 s^2}] \\ &= \lim_{n\to\infty}\frac{1}{n}ln(2^{n-1}) \\ &= ln(2) > 0. \end{aligned}$$

Showing that the Main Theorem 1 applies.

Corollary I. Let $f \in \mathcal{F}_2$ be a function satisfying the conditions of the previous theorem. Then f has positive topological entropy, *i.e.*,

$$h_{top}(f) = \lim_{n\to\infty}\frac{1}{n}[V(f^n)] > 0.$$

Consequently, f is chaotic in the sense of Li-Yorke.

A somewhat generalized version of Main Theorem 1 may be given as follows.

Theorem 3.25. Let $f \in \mathcal{F}_2$. If there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n} ln[\phi(n) \sum_{k\in\mathbb{Z}} |c_k^n(f)|^2 sin^2(\frac{k\pi}{2\phi(n)})] > 0,$$

then

$$(**) \qquad \lim_{n\to\infty} \frac{1}{n} ln[V(f^n)] = \alpha' > 0 \quad for \quad some \quad \alpha'.$$

In particular, if

$$\lim_{n\to\infty} \frac{1}{n} ln[\phi(n)] \equiv \alpha > 0 \quad and \quad \sum_{k\in\mathbb{Z}} |c_k^n(f)|^2 sin^2(\frac{k\pi}{2\phi(n)}) > 0,$$

then $(**)$ holds.

Proof. To proof this theorem we need the following result.

lemma [22]. Suppose that $g \in L^2$, then

$$8r \sum_{k\in\mathbb{Z}} |c_k^1(g)|^2 sin^2(\frac{k\pi}{2r}) \leq \Omega_\infty g(\frac{\pi}{r}) V_I(g)$$

for any positive number r, where

$$\Omega_\infty g(a) = \sup_{0\leq\delta\leq a} ||T_\delta(g) - g||_{C^0}, \quad (T_\delta g)(x) = g(x-\delta)$$

and g is extended outside I by periodic extension.

Proof (lemma). $\forall r \in \mathbb{N}$

$$\begin{aligned}
h(x) &:= g(x + \frac{l\pi}{r}) - g(x + \frac{(l-1)\pi}{r}), \quad l = 1, 2, \ldots, 2r. \\
||h||_2^2 &:= \sum_{k\in\mathbb{Z}} |c_k^1(h)|^2 \\
c_k^1(h) &= \int [g(x + \frac{l\pi}{r}) - g(x + \frac{(l-1)\pi}{r})] e^{-ikx} dx
\end{aligned}$$

$$
\begin{aligned}
&= [e^{-\frac{ikl\pi}{r}} - e^{-\frac{ik(l-1)\pi}{r}}]c_k^1(g) \\
|c_k^1(h)|^2 &= 4sin^2(\frac{k\pi}{2r})|c_k^1(g)|^2 \\
\frac{1}{2\pi}\int[g(x+\frac{l\pi}{r}) - g(x+\frac{(l-1)\pi}{r})]^2dx &= 4\sum_{k\in\mathbb{Z}}|c_k^1(g)|^2 sin^2(\frac{k\pi}{2r}), \\
& \quad l = 1, 2, \ldots, 2r. \\
\frac{1}{2\pi}\int\sum_{l=1}^{2r}[g(x+\frac{l\pi}{r}) - g(x+\frac{(l-1)\pi}{r})]^2dx &= 8r\sum_{k\in\mathbb{Z}}|c_k^1(g)|^2 sin^2(\frac{k\pi}{2r})
\end{aligned}
$$

but

$$\frac{1}{2\pi}\int\sum_{l=1}^{2r}[g(x+\frac{l\pi}{r}) - g(x+\frac{(l-1)\pi}{r})]^2dx \leq \Omega_\infty g(\frac{\pi}{r})V_I(g).$$

□

By setting $r = \phi(n)$ and $g = f^n$, we may argue in the same way as in the proof of Main Theorem 1 and complete the proof.

■

Our next step is to show that given a function $f \in W^{1,2}(I)$ there are some relations between $V_I(f)$ and $||f||_{W^{1,2}(I)}$, which will allow us to state other results.

Proposition 3.26. Let $f \in W^{1,2}(I)$. Then

$$V_I(f^n) \leq 2\pi[\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2]^{\frac{1}{2}}.$$

Proof

Let $f \in W^{1,2}(I)$ with Fourier series expansion

$$f(x) = \sum_{k\in\mathbb{Z}} c_k^1(f)e^{i2\pi kx} \quad x \in I = [0,1]$$

then

$$V_I(f) \;=\; \int_0^1 |f'(x)|dx \leq \Big(\int_0^1 dx\Big)^{1/2}\Big(\int_0^1 |f'(x)|^2dx\Big)^{1/2}$$

$$
\begin{aligned}
&= \Big(\int_0^1 dx\Big)^{1/2}\Big(\int_0^1 |2i\pi \sum_{k\in\mathbb{Z}} kc_k^1(f)e^{i2\pi kx}|^2 dx\Big)^{1/2} \\
&\leq 2\pi\Big(\sum_{k\in\mathbb{Z}} |kc_k^1(f)e^{i2\pi kx}|^2\Big)^{1/2} \\
&= 2\pi\Big(\sum_{k\in\mathbb{Z}} |kc_k^1(f)|^2\Big)^{1/2}.
\end{aligned}
$$

Consequently for the case of the $n-th$ iterates f^n of f it follows that

$$V_I(f^n) \leq 2\pi\Big(\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2\Big)^{1/2}.$$

$\square$

Corollary 3.27. If $f \in C^1$ and $V_I(f^n)$ grows exponentially as $n \to \infty$, then

$$\sum_{k\in\mathbb{Z}} |kc_k^n(f)|^2$$

grows exponentially as $n \to \infty$.

Now, let us see how the Fourier coefficients of f behave when it is known that f has positive topological entropy and how f', the derivative of f plays an important role in this development.

Main Theorem 2. Let $f \in \mathcal{F}_2 \cap W^{1,2}(I)$. If f has positive topological entropy, then

$$\lim_{n\to\infty} \frac{1}{n} ln\Big[\sum_{k\in\mathbb{Z}} |kc_k^1(f)|^2\Big] > 0.$$

Proof. Since $f \in \mathcal{F}_2$ and f has positive topological entropy, we have

$$h_{top}(f) = \frac{1}{n} ln[V_I(f^n)] \geq \alpha > 0 \quad for \quad some \quad \alpha$$

but by the preceding proposition

$$2\pi\Big(\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big)^{1/2} \geq V_I(f^n)$$

$$\lim_{n\to\infty}\frac{1}{n}ln\Big[2\pi\Big(\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big)^{1/2}\Big] \geq \lim_{n\to\infty}\frac{1}{n}ln[V_I(f^n)] = \alpha > 0.$$

∎

Main Theorem 3. Let $f \in \mathcal{F}_2 \cap W^{1,\infty}(I)$ such that

$$|f'|_{L^\infty(I)} = \gamma > 0.$$

If

$$\lim_{n\to\infty}\frac{1}{n}ln\Big[\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big] - ln[\gamma] > 0,$$

then f has positive topological entropy and consequently, f is chaotic in the sense of Li-Yorke.

Proof. From Main Theorem 2, we have

$$V_I(f^n) \leq 2\pi\Big(\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big)^{1/2} = \Big[\int_I |f^{n'}(x)|^2 dx\Big]^2$$

If $|f'|_{L^\infty(I)} = \gamma$, then a.e. on I, we have

$$f^{n'}(x) = f'(f^{(n-1)'}(x))f'(f^{(n-2)'}(x))\ldots f'(f(x))f'(x)$$

and

$$|f^{n'}(x)| \leq \gamma^n \quad a.e. \quad on \quad I.$$

We combine the above and now obtain

$$V_I(f^n) \;\; \leq \;\; 2\pi\Big(\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big)^{1/2}$$

$$
\begin{aligned}
&= \Big[\int_I |f^{n'}(x)||f^{n'}(x)|dx\Big]^{1/2}\\
&\leq \Big[\int_I \gamma^n |f^{n'}(x)|dx\Big]^{1/2}\\
&\leq \gamma^{n/2}\Big[\int_I |f^{n'}(x)|dx\Big]^{1/2}\\
&\leq \gamma^{n/2}\Big[V_I(f^n)\Big]^{1/2}.
\end{aligned}
$$

Then

$$\frac{1}{n}ln\Big[\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big] \leq \frac{1}{n}ln[(\frac{1}{2\pi})^2\gamma^n V_I(f^n)\Big] \leq ln(\gamma) + \frac{1}{n}ln[V_I(f^n)\Big]$$

where, by assumption, we obtain

$$\lim_{n\to\infty}\frac{1}{n}ln[V_I(f^n)\Big] \geq \lim_{n\to\infty}\frac{1}{n}ln\Big[\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big] - ln(\gamma) > 0,$$

therefore

$$h_{top}(f) > 0.$$

■

To ilustrate the above results consider the next examples.

Example 3.28. Consider the triangular map

$$T_m(x) = \begin{cases} mx & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{m}{1-m}(x-1) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

with $m = \frac{1}{1-\frac{1}{\mu}}$ and $1 < \mu < 2$.

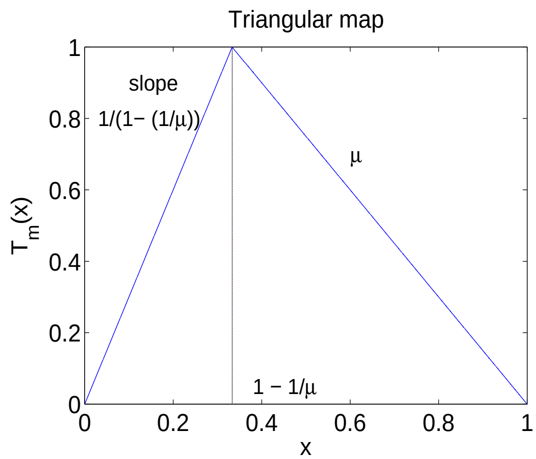

Fig. 39. Nonsymmetric triangular map.

After n iterations

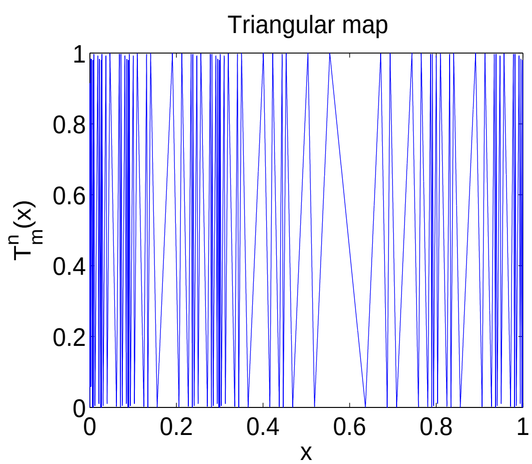

Fig. 40. Iteration of the nonsymmetric triangular map.

the total variation is $V(T_m^{(n)}) = 2^n$. Thus, we have

$$\begin{aligned}
a_{2l-1} &= a_{2l-1}^n \\
a_{2l} &= a_{2l}^n \\
a_{2l+1} &= a_{2l+1}^n
\end{aligned}$$

on

$$\begin{aligned}
(a_{2l-1}, a_{2l}) &: \quad f'(x) = \frac{1-0}{a_{2l} - a_{2l-1}} = \frac{1}{a_{2l} - a_{2l-1}} \\
(a_{2l}, a_{2l+1}) &: \quad f'(x) = \frac{0-1}{a_{2l+1} - a_{2l}} = -\frac{1}{a_{2l+1} - a_{2l}} \\
\int_0^1 |T_m^{(n')}|^2 dx &= \sum_{l=1}^{2^{n-1}} \Big[\int_{a_{2l-1}}^{a_{2l}} \frac{1}{(a_{2l} - a_{2l-1})^2} dx \quad \times \\
& \quad \int_{a_{2l}}^{a_{2l+1}} \frac{1}{(a_{2l+1} - a_{2l})^2} dx \Big] \\
&= \sum_{l=1}^{2^{n-1}} \Big[\frac{1}{a_{2l} - a_{2l-1}} + \frac{1}{a_{2l+1} - a_{2l}} \Big] \\
\mathit{but} & \\
a_{2l}^n - a_{2l-1}^n &= \lambda \lambda^{b_l} (1-\lambda)^{(n-1)-b_l} \\
a_{2l+1}^n - a_{2l}^n &= (1-\lambda) \lambda^{b_l} (1-\lambda)^{(n-1)-b_l} \\
\mathit{where} & \\
\lambda &= \frac{1}{\mu}, \quad 1 < \mu < 2. \\
l - 1 &= c_{n-2} 2^{n-2} + c_{n-3} 2^{n-3} + \ldots c_1 2^1 + c_0 \\
& \quad \mathit{the \quad binary \quad expansion \quad of} \quad l-1, \\
& \quad c_j = 0 \quad \mathit{or} \quad 1. \\
b_l &= \mathit{Number \quad of \quad zeroes \quad in \quad the \quad binary} \\
& \quad \mathit{coefficients:} \quad \{c_{n-2}, c_{n-3}, \ldots, c_1, c_0\}
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
\int_0^1 |T_m^{(n')}|^2 dx &= \sum_{l=1}^{2^{n-1}} \frac{1}{\lambda^{b_l+1}(1-\lambda)^{n-b_l}}\\
&= \sum_{b=0}^{n-1} \binom{n-1}{b} \frac{1}{\lambda^{b+1}(1-\lambda)^{n-b}} = \Big[\frac{1}{\lambda(1-\lambda)}\Big]^n\\
Thus\\
\frac{1}{n} ln\Big[\sum_{k\in\mathbb{Z}} |kc_k^1(f)|^2\Big] &= \frac{1}{n} ln\Big(\int_0^1 |T_m^{(n')}|^2 dx\Big)\\
&= \frac{1}{n} ln\Big[\frac{1}{\lambda(1-\lambda)}\Big]^n = ln\Big[\frac{1}{\lambda(1-\lambda)}\Big]\\
&= ln\Big[\mu\frac{1}{\mu-1}\Big] = 2ln(\mu) - ln(\mu-1)\\
and \quad if\\
\gamma &= |T'|_{L^\infty(I)} = max(\mu, \frac{\mu}{\mu-1}) = \frac{\mu}{\mu-1},\\
then\\
ln(\gamma) &= ln(\mu) - ln(\mu-1)\\
and\\
\lim_{n\to\infty} \frac{1}{n} ln\Big[\sum_{k\in\mathbb{Z}} |kc_k^1(f)|^2\Big] - ln(\gamma) &= [2ln(\mu) - ln(\mu-1)] - [ln(\mu) - ln(\mu-1)]\\
&= ln(\mu) > 0, \quad \forall \mu : 1 < \mu < 2.
\end{aligned}
$$

$\square$

Example 3.29. Consider the triangular map $T_q(x)$ with height $\frac{1}{2} < q < 1$.

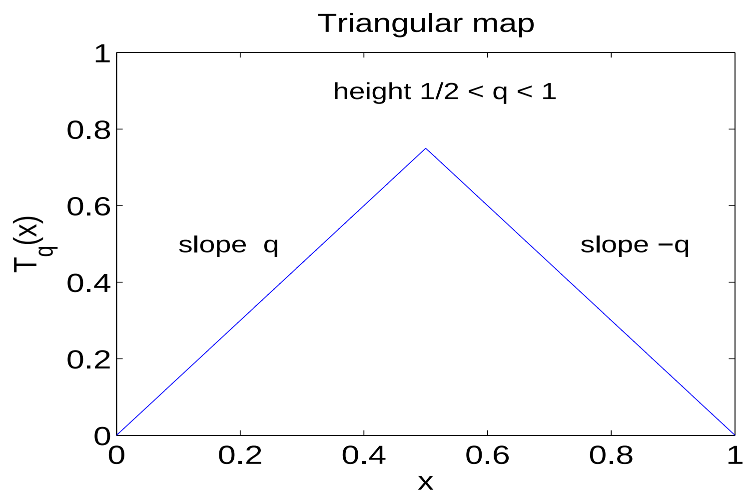

Fig. 41. Triangular map with height $\frac{1}{2} < q < 1$.

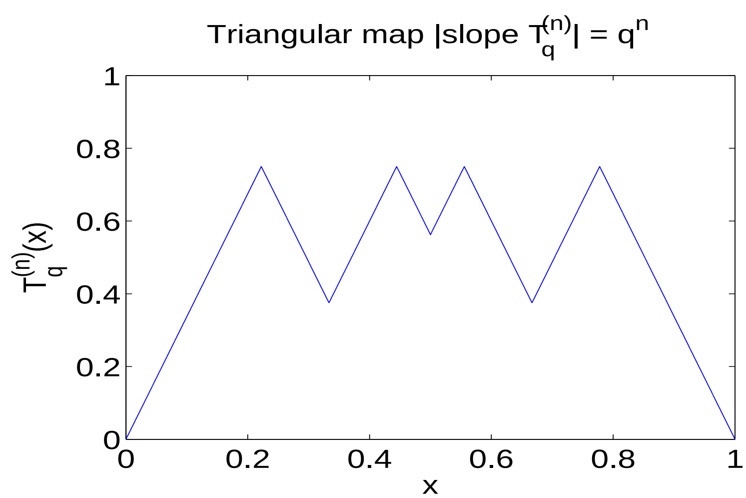

Fig. 42. Triangular map with —slope T_h^n—= q^n.

In this case, coefficients $c_k^n(T_q)$ are extremely hard to evaluate, but taking $\gamma = |T_q'|_{L^\infty} = q$ and using the last Main Theorem we obtain.

$$\begin{aligned}
\frac{1}{n}ln\Big[\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big] - ln(\gamma) &= \frac{1}{n}ln\Big[\sum_{k\in\mathbb{Z}}|kc_k^n(f)|^2\Big] - ln(q) \\
&= \frac{1}{n}ln\Big[\int_0^1 |T^{(n)'}(x)|^2dx\Big] - ln(q) \\
&= \frac{1}{n}ln(q^{2n}) - ln(q) \\
&= 2ln(q) - ln(q) = ln(q) > 0.
\end{aligned}$$

□

Main Theorem 4. Let $f, g \in W^{1,\infty}(I) \cap \mathcal{F}_2$ such that f and g are topologically conjugate

$$f = h \circ g \circ h^{-1}$$

Assume that $h^{-1} \in W^{1,p_1}(I)$, $\quad h \in W^{1,p_2}(I)$ for some p_1, p_2, $\quad 1 < p_1, p_2 < \infty$, satisfying either $p_2 > \frac{1}{(p_1-1)}$ or $p_1 > \frac{1}{(p_2-1)}$. Let $\gamma \equiv |f'|_{L^\infty(I)}$ such that

$$\lim_{n\to\infty}\Big[\frac{1}{n}ln[V_I(g^n)] - \frac{p_1+p_2}{p_1p_2}ln(\gamma)\Big] > 0.$$

Then

$$h_{top}(f) = \lim_{n\to\infty}\frac{1}{n}ln\Big[V_I(f^n)\Big] > 0.$$

Consequently, f is chaotic in the sense of Li -Yorke.

Proof. We have

$$\begin{aligned}
V_I(g^{(n)}) &= \int_I |g^n|dx = \int_I |h^{-1'}(f^n(h(x)))||f^{n'}(h(x))||h'(x)|dx \\
&\le \Big[\int_I |h^{-1'}|^{p_1}dx\Big]^{\frac{1}{p_1}}\Big[\int_I |f^{n'}(h(x))|^{q_1}|h'(x)|^{q_1}dx\Big]^{\frac{1}{q_1}}
\end{aligned}$$

$$
\begin{aligned}
& (\frac{1}{q_1}+\frac{1}{p_1}=1) \\
\leq\ & C_{h^{-1}}\Big[\int_I |f^{n'}(y)|^{q_1\frac{p_2}{p_2-q_1}}dy\Big]^{\frac{p_2-q_1}{p_2q_1}}\Big[\int_I |h'(x)|^{p_2}dx\Big]^{\frac{1}{p_2}} \\
=\ & C_hC_{h^{-1}}\Big[\int_I |f^{n'}(y)|^{\frac{p_2q_1}{p_2-q_1}}dy\Big]^{\frac{p_2-q_1}{p_2q_1}} \\
\leq\ & C^*\Big[\int_I |f^{n'}(y)|\gamma^{n(\frac{p_2q_1}{p_2-q_1}-1)}dy\Big]^{\frac{p_2-q_1}{p_2q_1}}, \quad (C^*\equiv C_hC_{h^{-1}}) \\
\leq\ & C^*\Big[V_I(f^n)\Big]^{\frac{p_2-q_1}{p_2q_1}}\gamma^{n(1-\frac{p_2-q_1}{p_2q_1})}.
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{p_2-q_1}{p_2q_1} &= \frac{p_1p_2-(p_1+p_2)}{p_1p_2} \\
1-\frac{p_2-q_1}{p_2q_1} &= \frac{p_1+p_2}{p_1p_2}
\end{aligned}
$$

and the above inequality gives

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}ln[V_I(f^n)] &\geq \frac{p_1p_2}{p_1p_2-(p_1+p_2)}\Big[\lim_{n\to\infty}\frac{1}{n}ln[V_I(g^{(n)})] \\
&- \quad ln(\gamma)\frac{p_1+p_2}{p_1p_2}\Big] > 0.
\end{aligned}
$$

Thus, the proof is complete.

■

Corollary IV. In the assumption of Main Theorem 4, if either $p_1=\infty$ or $p_2=\infty$ (or both), then

$$\lim_{n\to\infty}\frac{1}{n}ln[V_I(g^n)] > 0$$

and

$$\lim_{n\to\infty}\frac{1}{n}ln[V_I(f^n)] > 0$$

are equivalent.

Example 3.30. It is known that the quadratic map

$$f_4(x) = 4x(1-x)$$

and the triangular map $T_2(x)$ are topologically conjugate through

$$f_4 = h \circ T_2 \circ h^{-1},$$

where

$$h(x) = sin^2(\frac{\pi x}{2}), \quad h^{-1}(y) = \frac{2}{\pi} sin^{-1}(\sqrt{y}), \quad x, y \in [0,1].$$

We have

$$\begin{aligned} h'(x) &= \pi sin(\frac{\pi x}{2}) cos(\frac{\pi x}{2}) \\ &= \frac{\pi}{2} sin(\pi x) \in L^\infty(I) \Rightarrow p_2 = \infty \\ (h^{-1})'(y) &= \frac{2}{\pi} \frac{1}{\sqrt{y}} \frac{1}{\sqrt{1-y}} \Rightarrow (h^{-1})' \in L^{2-\delta}(I) \quad for \quad any \quad \delta > 0, \end{aligned}$$

therefore corollary IV applies. Indeed, we have

$$h_{top}(f_4) = h_{top}(T_2).$$

□

Example 3.31 (Application to PDEs). Now, we would like to show one application of the above results to the case of chaotic vibration of the wave equation with a Van Der Pol nonlinear boundary conditions, wich has been studied fby Chen *et al* [16, 17, 18]. Consider the wave equation

$$w_{tt}(x,t) - w_{xx}(x,t) = 0, \quad 0 < x < 1, \quad t > 0$$

with a nonlinear self-excitation boundary condition at the right end $x = 1$:

$$w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t)$$

$$0 \leq \alpha \leq 1, \quad \beta > 0$$

and a linear boundary condition at the left end $x = 0$

$$w_t(0,t) = -\eta w_x(0,t), \quad \eta > 0, \eta \neq 1, t > 0.$$

The remaining two conditions we require are the initial conditions

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \quad x \in [0,1].$$

Then using Riemman invariants

$$\begin{aligned} u &= \frac{1}{2}(w_x + w_t) \\ v &= \frac{1}{2}(w_x - w_t). \end{aligned}$$

the above system becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix},$$

where at the boundary $x = 0$ and $x = 1$ the reflection relations take place

$$\begin{aligned} v(0,t) &= \frac{1+\eta}{1-\eta}u(0,t) \equiv G(u(0,t)) \\ u(1,t) &= F(v(1,t)) \end{aligned}$$

and where $F(x) \equiv x + g(x)$ and $g(x)$ is the unique solution to the cubic equation

$$\beta g^3(x) + (1-\alpha)g(x) + 2x = 0, \quad x \in \mathbb{R}.$$

Solutions $w_x(x,t), w_t(x,t)$ of the wave equation display chaotic vibration behavior if $G \circ F$ (or equivalently $F \circ G$) is a chaotic interval map, when α, β, η lie in a certain region. We therefore deduce that for given $\alpha, \beta, \eta: \quad 0 < \alpha \leq 1, \quad \beta > 0$ and $\eta > 0, \quad \eta \neq 1$, the map $G \circ F$ is chaotic and the initial conditions $w_0(\cdot)$ and $w_1(\cdot)$ satisfy $w_0, w_1 \in \mathcal{F}_2$,

$$w_0 \in C^2([0,1]), \quad w_1 \in C^2([0,1]),$$

and the compatibility conditions

$$\begin{aligned} w_1(0) &= -\eta w_0^{'}(0) \\ w_0^{'}(1) &= \alpha w_1(1)\beta w_1^3(1) \\ w_0^{''}(0) &= -\eta w_1^{'}(0) \\ w_1^{'}(0) &= [\alpha - 3\beta w_1^2(1)]w_0^{''}(1) \end{aligned}$$

are satisfied. Thus, there exist $A_1 > 0, \quad A_2 > 0$ s.t. if

$$|w_0^{'}|_{C^0(I)}, \quad |w_1|_{C^0(I)} \leq A_1 \quad w_0^{'} \neq 0 \quad or \quad w_1 \neq 0,$$

then

$$|w_x|_{C^0(I)}, \quad |w_t|_{C^0(I)} \leq A_2$$

and

$$\lim_{n\to\infty} \frac{1}{n} ln\Big[\int_0^1 \Big(|w_{xx}(x, n+t_0)| + |w_{xt}(x, n+t_0)|\Big)dx\Big] > 0$$

$$\lim_{n\to\infty} \frac{1}{n} ln\Big[\int_0^1 \Big(|w_{xx}(x, n+t_0)|^2 + |w_{xt}(x, n+t_0)|^2\Big)dx\Big] > 0$$

for any $t_0 > 0$. Note that the previous equations have not been obtainable by any other methods (such as the energy multiplier method).

$\square$

Example 3.32 (Entropy and Hausdorff Dimension). Let X be a nonempty compact metric space and $f : X \to X$ a Lipschitz continuous map with Lipschitz constant L, that is, $\forall x, y \in X \quad f$ satisfies

$$||f(x) - f(y)|| \leq L \quad ||x - y||.$$

The topological entropy $h(f, Y)$ of f on an arbitrary subset $Y \subset X$, given by Bowen [11] can be constructed as follows. Let $\mathcal{A}$ be a finite open cover of X. For a set $B \subset X$ we write $B \prec \mathcal{A}$ if B is contained in some element of $\mathcal{A}$. Let $n_{f,\mathcal{A}}(B)$ be the largest nonegative integer such that $f^k(B) \prec \mathcal{A}$ for $k = 0, 1, 2, \ldots, n-1$. If $B \not\prec \mathcal{A}$, then $n_{f,\mathcal{A}}(B) = 0$, and if $f^k(B) \prec \mathcal{A}$ for all k, then $n_{f,\mathcal{A}}(B) = \infty$.

Now, we set

$$\begin{aligned}
diam_{\mathcal{A}}(B) &= exp(-n_{f,\mathcal{A}}(B)), \\
diam_{\mathcal{A}}(\mathcal{B}) &= exp(-n_{f,\mathcal{A}}(B)), \\
and & \\
D_{\mathcal{A}}(\mathcal{B}, \lambda) &= \sum_{B \in \mathcal{B}} (diam_{\mathcal{A}}(B))^{\lambda}
\end{aligned}$$

for a family $\mathcal{B}$ of subsets of X and a real number λ. Then

$$\mu_{\mathcal{A},\lambda}(Y) = \lim_{\epsilon \to 0} \inf \left\{ D_{\mathcal{A}}(\mathcal{B}, \lambda) : \mathcal{B} \quad is \quad a \quad cover \quad of \quad Y \quad and \quad diam_{\mathcal{A}}(B) < \epsilon \right\}$$

has similar properties as the classical Hausdorff measure:

$$\mu_\lambda(Y) = \liminf_{\epsilon\to 0}\Big\{\sum_{B\in\mathcal{B}}(diam(B))^\lambda : \mathcal{B} \quad is \quad an \quad open \quad cover \quad of \quad Y \quad and \quad diam(B) < \epsilon\Big\}$$

(where $diam(B) = \sup_{b\in\mathcal{B}} diam(B)$), that is, there exists $h(f, Y, \mathcal{A})$ such that

$$\mu_{\mathcal{A},\lambda}(Y) = \infty \quad for \quad \lambda < h(f, Y, \mathcal{A})$$
$$\mu_{\mathcal{A},\lambda}(Y) = 0 \quad for \lambda > h(f, Y, \mathcal{A}) \quad .$$

Finally we set

$$h(f, Y) = sup\Big\{h(f, Y, \mathcal{A}) : \mathcal{A} \quad is \quad a \quad finite \quad open \quad cover \quad of \quad Y\Big\}.$$

This number $h(f, Y)$ is the *topological entropy of f on the set Y*. If $Y = X$ then by proposition 1 of [11] we get

$$h(f, X) = h_{top}(f),$$

i.e., this concept is equal to the topological entropy of f, already defined before. From Misiurewicz [32] we have the following result.

Theorem 3.33. For any $Y \subset X$ the Hausdorff dimension of Y, for Lispchitz continuous map with Lipschitz constant $L > 1$ satisfies the inequality

$$H_d(Y) \geq \frac{h(f, Y)}{ln(L)}.$$

Corollary 3.34. The Hausdorff dimension of X satisfies

$$H_d(X) \geq \frac{h(f, X)}{ln(L)} = \frac{h(f)}{ln(L)}, \quad L > 1.$$

In this way, we will consider the case of an interval map $f : I \to I$ and let $\mathcal{L}$ the set defined by:

$$\begin{aligned}\mathcal{L} \;:\; & The \quad set \quad of \quad all \quad Lipschitz \\ & continuous \quad functions \quad f : I \to I, \quad with \\ & Lipschitz \quad constant \quad greater \quad than \quad 1.\end{aligned}$$

Theorem 3.35. Let $f \in W^{1,2}(I) \cap \mathcal{F}_1 \cap \mathcal{L}$ with Lipschitz constant $L > 1$. Let c_k^n be the $k-th$ Fourier coefficient of the $n-th$ iterates of f. If $\Omega(f)$ represents the set of nonwandering points of f, then

$$H_d(\Omega(f)) \geq \frac{1}{ln(L)} \lim_{n\to\infty} ln|kc_k^n|, \quad k = \pm 1, \pm 2, \ldots \quad .$$

Proof. Apply theorem 3.32 for $X = \Omega(f)$ which is an invariant, closed and therefore compact set

$$\begin{aligned} H_d(\omega(f)) &\geq \frac{h(f,\Omega(f))}{ln(L)} = \frac{h(\Omega(f))}{ln(L)} \\ &= \frac{h(f|_\Omega)}{ln(L)} \quad (Classical \quad topological \quad entropy) \\ &= \frac{h(f)}{ln(L)} \quad (true \quad for \quad classical \quad topological \quad entropy), \end{aligned}$$

but we alredy know that

$$V_I(f^n) \geq \pi|kc_k^n|,$$

then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} ln[V_I(f^n)] \geq \lim_{n\to\infty} \frac{1}{n} ln[|kc_k^n|]$$

which implies

$$H_d(\Omega(f)) \geq \frac{1}{ln(L)} \lim_{n\to\infty} \frac{1}{n} ln[|kc_k^n|.$$

□

Corollary 3.36. Let f be a function satisfying the hypotheses of theorem 3.34, and let $\phi : \mathbb{N} \to \mathbb{N}$ be an integer function such that grows exponentially and

$$\lim_{n\to\infty} |c_{\pm\phi(n)}^n| > 0.$$

Then the Hausdorff dimension of the nonwandering set $\Omega(f)$ is positive, i. e. ,

$$H_d(\Omega(f)) > 0$$

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} ln[|kc_k^n| &= \lim_{n\to\infty} \frac{1}{n} ln[|\phi(n) c_{\pm\phi(n)}^n|] \\
(\alpha_1, \alpha_2 > 0) &= \lim_{n\to\infty} \frac{1}{n} ln[|\alpha_1 e^{\alpha_2 n} c_{\pm\phi(n)}^n|] \\
&= \alpha_2 + \lim_{n\to\infty} \frac{ln(\alpha_1)}{n} + \lim_{n\to\infty} \frac{ln(c_{\pm\phi(n)}^n)}{n} \\
&= \alpha_2 > 0.
\end{aligned}$$

Therefore,

$$H_d(\Omega(f)) \geq \frac{1}{ln(L)} \lim_{n\to\infty} \frac{1}{n} ln[|kc_k^n| \geq \frac{\alpha_2}{ln(L)} > 0.$$

□

Thus, in the case of the full tent map

$$T_2(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ -2(x-1) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Let $\gamma = |f'|_{L^\infty(I)}$ then

$$
\begin{aligned}
|f(x) - f(y)| &= |f'(\psi)||x-y|, \quad for \quad some \quad x < \psi < y \\
&\leq \gamma |x-y| \\
=> & \\
H_d(\Omega(T)) &\geq \frac{\alpha_2}{\gamma}
\end{aligned}
$$

but in this case

$$
\begin{aligned}
\phi(n) &= s2^{n-1}, \quad s = 1, 3, \ldots \\
\phi(n) &= \frac{s}{2}2^n \quad s = 1, 3, \ldots
\end{aligned}
$$

therefore

$$
\begin{aligned}
H_d(\Omega(T)) &\geq \frac{ln(2)}{\gamma} \\
H_d(\Omega(T)) &\geq \frac{ln(2)}{2}.
\end{aligned}
$$

Now, we will move on to the application of the Sturm-hurwitz theorem to the theory we are developing here. Let X be a closed subset of the interval $I = [0, 1]$ and $f : X \to X$ a continuous mapping. Denote $\mathcal{J}$ the set of all possible subintervals of I, and for $\mathcal{J}|_Y$ the family of all subintervals of $I = [0, 1]$, each restricted to Y.

Definition 3.37. A cover A is called $f - mono$ if A is finite, $A \subset \mathcal{J}|_Y$, and for any $a \in A$ the map $f|_a$ is monotone.

To the light of the above definition we can see piecewise monotone functions in a slightly different way.

Definition 3.38. A map f is called piecewise monotone (p.m.), if there exists an $f-mono$ cover of X.

Definition 3.39. Let f be a p.m. continuous mapping from an interval I into itself. Denote

$$l_n = min\Big\{CardA : A \quad is \quad an \quad f^n - mono \quad cover\Big\}.$$

From [33] we have the next result.

Lemma 3.40. If $f : I \to I$ is a p.m. continuous map, then

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} ln[l_n].$$

As we already know the Sturm-Hurwitz theorem is telling us that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous $2\pi - periodic$ function and d_k be the $k-th$ Fourier coefficient of g, i.e. ,

$$d_k = \int_0^1 g(x)e^{-ikx}dx, \quad k = 0, \pm 1, \pm 2, \dots \quad .$$

If there exists a positve integer k_0 such that

$$d_k = \begin{cases} 0 & \text{if } |k| < k_0 \\ \neq 0 & \text{if } |k| \geq k_0 \end{cases},$$

then the function g has at least $2k_0$ distinct zeros in the interval $[0, 2\pi]$. A new proposition can be established by putting together the above two results.

Main Theorem 5. Let $f \in C^0(I), \quad I = [0,1]$, be a p.m. mapping with $f(0) = f(1)$. If there exists a map $\phi : \mathbb{N} \to \mathbb{N}$ satisfying

$$ln[\phi(n)] \geq \alpha_1 + \alpha_2 n, \quad for \quad some \quad \alpha_1 \in \mathbb{R}, \alpha_2 > 0,$$

such that

$$c_k^n = \begin{cases} 0 & \text{if } |k| < \phi(n) \\ \neq 0 & \text{for} \quad \text{some } |k| \geq \phi(n) \ . \end{cases}$$

Then

$$h_{top}(f) > 0.$$

Proof. For a given $n \in \mathbb{N}$ define

$$g_n(x) = f^n(\frac{x}{2\pi}) \quad x \in [0, 2\pi].$$

Thus, $g_n(0) = g_n(2\pi)$, so we can extend g_n to the whole line $\mathbb{R}$ continuously with period 2π.

Applying the Sturm-Hurwitz theorem, we have that $g_n(x)$ has at least $2\phi(n)$ zeros in the interval $[0, 2\pi]$. This implies that f^n has at least $2\phi(n)$ distinct zeros in $[0,1]$, therefore

$$l_n \geq 2\phi(n),$$

and it follows from lemma 3.39 that

$$\begin{aligned} h_{top}(f) &= \lim_{n\to\infty} \frac{1}{n} ln[l_n] \\ &\geq \lim_{n\to\infty} \frac{1}{n} \Big[ln[2] + ln[\phi(n)] \Big] \\ &\geq \lim_{n\to\infty} \frac{1}{n} \Big[ln[2] + \alpha_1 + \alpha_2 n \Big] \\ &= \alpha_2 > 0. \end{aligned}$$

$\square$

CHAPTER IV

CONCLUSIONS

We have answered the questions addressed in chapter two for topological dynamical systems. However, there is still an open problem related to the fact of lifting properties from one dynamical system to another one in the general case of action maps instead of conjugation maps. Some improvements could be done for the intermediate case of a factor maps or for an almost open action map. The main point in this analysis is that the use of an action map instead of a conjugation or factor map, reduces the possibility of pulling back convergence properties from one dynamical system to another in a natural way through the use of an inverse function or the surjectivity of the function.

For the second part, we are very satisfied with the work presented here. Starting from numerical analysis we have been able to find a set of results linking, in a basic level, two concepts of great importance in this time,namely, chaotic interval maps and Fourier coefficients. Throughout this work there are some basic relations between Fourier coefficients and the chaotic behavior of interval maps, and we have shown them through the main theorems stated and proved here where we have realized the important role played by the total variation $V_I(f)$ of the function f, which is a very intuitive concept but that has to be handled carefully to avoid misinterpretation, and which gave us the doorway to establish the main propositions. It is also important to remark that the numerical simulation had a great contribution to the global analysis. In particular, the 3D graphs, which give us a clue of the possibility of a chaotic situation. Thus, we were able to state some of the initial conjectures in chapter three, and after all the analysis done, we see that *conjecture 1*:

In the nonchaotic case we have

$$\lim_{(n,k)\to\infty} c_k^n = 0.$$

It is likely not true. We can argue that for the nonchaotic quadratic map $f_\mu(x) = \mu x(1-x)$ with $\mu = 3.2$ shown below (togeher with its $400-th$ iteration) :

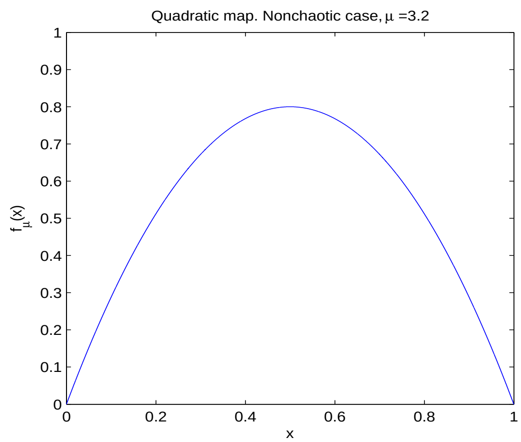

Fig. 43. Quadratic map. Nonchaotic case, $\mu = 3.2$.

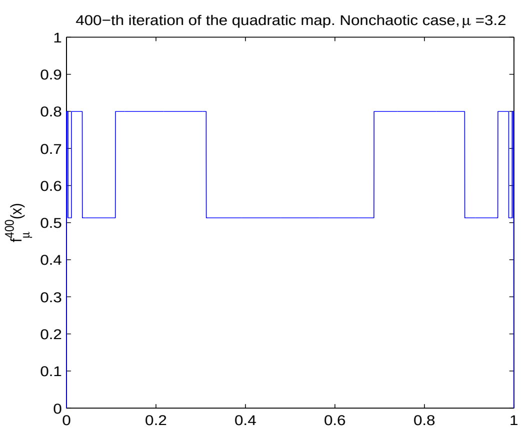

Fig. 44. $400-th$ iteration of the quadratic map. Nonchaotic case, $\mu = 3.2$.

It has the property that, as the number of iterations increases, the function behaves as a step function with well defined regions corresponding to the period-2 bifurcation curves. However, at the beginning and end of the interval we have a phenomenon which consists of high oscillatory behavior of the function, which implies Fourier coefficients of high frequency are presents all the time, and therefore our conjecture is not satisfied in this case.

For the second conjecture, we were able, through the analysis of the chaotic symmetric tent map, to prove a theorem which gives an affirmative answer to it. Indeed, we showed an explicit example where all the conditions are satisfied for a particular chaotic interval map.

Finally, for the last conjecture we did not find any proposition supporting this assertion or proving it is false, so more analysis is required to give a final answer to this case.

We need to mention the one difficulty in the theoretical part: the evaluation of f^n, the $n-th$ iterates of the function f, as it was clear that only in some simple cases we were able to find it explicitely. To avoid this problem perhaps it is necessary to generate new techniques to give, for instance, some estimates for f^n, instead of finding it explicitely, and using those results to apply our theorems and generate new ones. A natural way of continuation for this work is to study, the same problem but now from the point of view of *conituous Wavelet theory*. Some numerical results can be found for instance in [6], [34] where the authors analyze the bifurcation of a systems from order to chaos in a qualitative way. Perhaps using this technique it is possible to find new results without the necessity of finding f^n.

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